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RESEARCH

The Tamagawa number conjecture and Kolyvagin's conjecture for motives of modular forms

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Abstract

Assuming specific instances of two general conjectures in arithmetic algebraic geometry (bijectivity of p-adic regulator maps, injectivity of p-adic Abel–Jacobi maps), we prove several cases of the p-part of the Tamagawa number conjecture (p-TNC) of Bloch–Kato and Fontaine–Perrin-Riou for (homological) motives of modular forms of even weight ≥ 4 in analytic rank 1. More precisely, we prove our results for a large class of newforms f and prime numbers p that are ordinary for f and such that the weight of f is congruent to 2 modulo $2(p-1)$. Inspired by work of W. Zhang in weight 2, which builds on the congruence method originally developed by Bertolini–Darmon, the key ingredient in our strategy is an analogue for p-adic Galois representations attached to higher (even) weight newforms of Kolyvagin's conjecture on the p-indivisibility of derived Heegner points on elliptic curves, which we prove via a p-adic variation method exploiting the arithmetic of Hida families. Along the way, we also prove (under similar assumptions) the p-TNC for modular motives in analytic rank 0 and the rationality conjecture of Beilinson and Deligne on the existence of zeta elements on the fundamental line in analytic ranks 0 and 1. Prior to this work, the only known results on (questions related to) the p-TNC for modular motives were in weight 2 and analytic rank ≤ 1 and in even weight and analytic rank 0. As further applications of our result on Kolyvagin's conjecture in higher weight, we deduce a structure theorem for Selmer groups, p-parity results, converse theorems and higher rank results for modular forms and modular motives.

Keywords: Modular forms, Tamagawa number conjecture, Kolyvagin's conjecture

Mathematics Subject Classification: 11F11, 14C15

Contents

1 Introduction

The Tamagawa number conjecture (TNC, for short) of Bloch and Kato [15] predicts formulas for special values of L-functions of motives and represents a vast generalization of the analytic class number formula and of the Birch–Swinnerton-Dyer conjecture for abelian varieties. The conjecture of Bloch–Kato, which was originally expressed (by analogy with the theory of algebraic groups) in terms of Haar measures and Tamagawa numbers, was later reformulated and extended by Fontaine and Perrin-Riou ([43]; *cf.* also [42]) using the language of determinants of complexes and Galois cohomology; similar ideas were developed also by Kato [72,73]. The Tamagawa number conjecture was then generalized by Burns and Flach to an equivariant setting that covers the case of motives with not necessarily commutative coefficients [21,22], thus giving birth to the so-called equivariant Tamagawa number conjecture.

The main result of the present paper is a proof of the p-part of the Tamagawa number conjecture (p-TNC) for the Grothendieck (*i.e.*, homological) motive of a modular form f in analytic rank 1; we assume very specific instances of two general conjectures in arithmetic algebraic geometry (bijectivity of p-adic regulator maps, injectivity of p-adic Abel–Jacobi maps) and some technical conditions on f and p. In the rest of this introduction we will describe our results; this will also give us an occasion to outline the structure of the article.

1.1 A reformulation of p-TNC for modular motives

The first result we describe is a reformulation of the p-part of the TNC for the motive of a higher, even weight modular form. Although a large portion (if not all) of what we say below applies to weight 2 forms as well (see, *e.g.*, [77] for the case of rational elliptic curves), in what follows we prefer to stick to the case of weight larger than 2, as in the weight 2 setting most of the notions that we introduce become significantly simpler and the notation that we use turns out to be unnecessary cumbersome when dealing with weight 2 newforms (whose motives are given by Jacobians of modular curves).

1.1.1 Modular motives and their arithmetic invariants

Let $N \geq 1$ be an integer, let $k \geq 4$ be an even integer and let $f \in S_k(\Gamma_0(N))$ be a normalized newform of weight k and level $\Gamma_0(N)$, whose q-expansion will be denoted by $f(q) = \sum_{n\geq 1} a_n(f)q^n$. Let $F := \mathbb{Q}\big(a_n(f) \mid n \geq 1\big) \subset \mathbb{C}$ be the totally real number field generated over $\mathbb{Q}$ by the Fourier coefficients of f and let $\mathcal{O}_F$ be its ring of integers. Put $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$; moreover, for a prime number p set also $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_p := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We attach to f (and a prime p) the following objects.

- The motive $\mathcal{M} = (X, \Pi, k/2)$ of f. This is a Grothendieck (*i.e.*, homological) motive defined over $\mathbb{Q}$ with coefficients in F, equipped with its étale realization V_p for each prime number p (which is an F_p-module), its Betti (*i.e.*, singular) realization V_{B} and its de Rham realization V_{dR} (which are F-vector spaces), and comparison isomorphisms between these realizations. Here X is the Kuga–Sato variety of level N and weight k,

while Π is a projector on the ring of correspondences of X; see Sects. 2.2 and 2.4 for details.

- The (Bloch–Kato) Shafarevich–Tate group $Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M})$ of $\mathcal{M}$ at p, which is defined as the quotient of the Bloch–Kato Selmer group of V_p/T_p by its maximal p-divisible subgroup, where T_p is a suitable Galois-stable $\mathcal{O}_p$-lattice in V_p (see Sect. 2.18). In our arguments, the interplay between the *finite* group $Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M})$ and the Shafarevich–Tate group $Ш^{\mathrm{Nek}}_p(\mathbb{Q}, \mathcal{M})$ of Nekovář, which is the quotient of the Bloch–Kato Selmer group of V_p/T_p by the image of a certain p-adic Abel–Jacobi map, will be crucial (see, *e.g.*, Sects. 5.5.2 and 5.7.1).
- For every place v of $\mathbb{Q}$ and prime p, a Tamagawa $\mathcal{O}_p$-ideal $\mathrm{Tam}^{(p)}_v(\mathcal{M})$, whose definition is recalled in Sect. 2.21 (in particular, $\mathrm{Tam}^{(p)}_v(\mathcal{M}) = \mathcal{O}_p$ for all but finitely many v).
- The p-torsion part $\mathrm{Tors}_p(\mathcal{M})$ of $\mathcal{M}$ (see Sect. 2.23.1).
- The period $\Omega_{\mathcal{M}} \in (F \otimes_{\mathbb{Q}} \mathbb{C})^\times$ coming from the comparison isomorphism between Betti and de Rham realizations (notice that, in practice, we actually work with a period $\Omega_\infty \in F_\infty^\times$, which is defined in Sect. 2.5, that takes care of an appropriate twist in the Betti realization and is related to $\Omega_{\mathcal{M}}$ by the equality $\Omega_{\mathcal{M}} = \Omega_\infty/(2\pi i)^{k/2}$).
- The motivic cohomology group $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$, defined in Sect. 2.6. This is a conjecturally finite-dimensional F-vector space; assuming this finite-dimensionality (see Conjecture 2.12), we set

$$r_{\mathrm{alg}}(\mathcal{M}) := \dim_F\big(H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})\big).$$

  The F_∞-module $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) \otimes_F F_\infty$ is equipped with a conjecturally non-degenerate height pairing *à la* Gillet–Soulé (see Sect. 2.7); we write $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$ for the determinant of this pairing with respect to an F-basis $\mathscr{B}$ of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$, so that $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) \neq 0$ if the pairing is non-degenerate. The F_p-module $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) \otimes_F F_p$ is endowed with a p-adic regulator map

$$\mathrm{reg}_p : H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) \otimes_F F_p \longrightarrow H^1_f(\mathbb{Q}, V_p)$$

  with values in the Bloch–Kato Selmer group $H^1_f(\mathbb{Q}, V_p)$ of V_p; this map is conjectured to be an isomorphism of F_p-modules (see Conjecture 2.50).
- The *completed* L-function $\Lambda(\mathcal{M}, s)$ of $\mathcal{M}$, which is an entire function on $\mathbb{C}$. We write $r_{\mathrm{an}}(\mathcal{M})$ (respectively, $\Lambda^*(\mathcal{M}, 0)$) for the order of vanishing (respectively, the leading term of the Taylor expansion) of $\Lambda(\mathcal{M}, s)$ at $s = 0$ (see Sect. 2.9).

All these invariants will appear in our reformulation of the p-part of the TNC for $\mathcal{M}$, which uses the language of determinants of (complexes of) projective modules, as proposed by Fontaine–Perrin-Riou in [43] (at least when the field of coefficients is $\mathbb{Q}$, the formulation of Fontaine–Perrin-Riou is indeed equivalent to the one originally given by Bloch–Kato: see, *e.g.*, [42,118] for details).

1.1.2 A reformulation of p-TNC for $\mathcal{M}$: assumptions

As above, p is a prime number. To prove the result below, we work under the following assumptions, for precise statements of which we refer to later sections:

(1) the Gillet–Soulé height pairing is non-degenerate (*i.e.*, Conjecture 2.18 holds true);
(2) the rationality conjecture of Beilinson and Deligne on the existence of zeta elements on the fundamental line (Conjecture 2.40) holds true;
(3) the p-adic regulator reg_p is an isomorphism (*i.e.*, Conjecture 2.50 over $\mathbb{Q}$ holds true);

If reg_p is an isomorphism for some prime p, then $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ has finite dimension over F (*i.e.*, Conjecture 2.12 over $\mathbb{Q}$ holds true). Note that the non-degeneracy condition in (1) is imposed only to force $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$ to be non-zero and thus can be removed once we know that, in the arithmetic situations we consider, $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) \neq 0$. Significant advances on (let alone complete proofs of) any of the conjectures above in a general setting would represent a major breakthrough in arithmetic geometry: in this paper we have nothing new to say about them and simply content ourselves with assuming their validity in specific instances whenever needed. However, it is worthwhile to remark that in the low rank contexts we are interested in (*i.e.*, when $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$) we know that $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) \neq 0$ either by definition (if $r_{\mathrm{an}}(\mathcal{M}) = 0$, in which case $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) := 1$) or as a consequence of S.-W. Zhang's formula of Gross–Zagier type for higher weight modular forms [165]. Furthermore, assuming the injectivity of certain p-adic Abel–Jacobi maps, we can also prove that if $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$, then the rationality conjecture of Beilinson and Deligne is true (Theorems 5.24 and 5.37). Therefore, conditions (1) and (2) will not be needed in the proofs of the main results of Section 5, where we assume that $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$.

1.1.3 A reformulation of p-TNC for $\mathcal{M}$: statement

In the following lines, for a finite $\mathcal{O}_p$-module M we denote by $\mathcal{I}(M)$ the $\mathcal{O}_p$-ideal such that

$$\mathrm{ord}_{\mathfrak{p}}\big(\mathcal{I}(M)\big) = \mathrm{length}_{\mathcal{O}_{\mathfrak{p}}}(M)$$

for each prime $\mathfrak{p}$ of F above p, where $\mathcal{O}_{\mathfrak{p}}$ is the completion of $\mathcal{O}_F$ at $\mathfrak{p}$ and $\mathrm{ord}_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation.

Theorem A *Under the assumptions in* Sect. 1.1.2*, the p-part of the TNC for $\mathcal{M}$ is equivalent to the equality*

$$\left(\frac{\Lambda^*(\mathcal{M}, 0)}{\Omega_{\mathcal{M}} \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}\right) = \frac{\mathcal{I}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q}, \mathcal{M})\big) \cdot \mathcal{I}_p(\gamma_f) \cdot \prod_{v \in S} \mathrm{Tam}_v^{(p)}(\mathcal{M})}{\big(\det(\mathrm{A})\big)^2 \cdot \mathrm{Tors}_p(\mathcal{M})}$$

of fractional $\mathcal{O}_p$-ideals.

The reader is referred to Sect. 2.23 for the terms $\mathcal{I}_p(\gamma_f)$ and $\mathrm{A} \in \mathrm{GL}_{r_{\mathrm{alg}}(\mathcal{M})}(F_p)$, the latter being denoted by $\mathrm{A}_{\tilde{\mathscr{B}}}$ later in the text. To sketchily explain their roles, we observe that the definitions of some of the objects appearing in Theorem A, which are introduced in Sect. 1.1.1, involve choices (not reflected in the notation above) of suitable bases; these are encoded in the terms $\mathcal{I}_p(\gamma_f)$ and A, and then it can be checked that the validity of the resulting formula is independent of such choices.

To the best of our knowledge, Theorem A, which corresponds to Theorem 2.83, offers the first reformulation of such an explicit kind of p-TNC for $\mathcal{M}$ in arbitrary analytic rank; a similar interpretation when $r_{\mathrm{an}}(\mathcal{M}) = 0$ was proposed by Dummigan–Stein–Watkins [38].

1.2 *p*-TNC for $\mathcal{M}$ in analytic rank 1

We can now describe our main result on the p-TNC for the modular motive $\mathcal{M}$.

1.2.1 *p*-TNC for $\mathcal{M}$ in analytic rank 1: assumptions

We prove our result under the following assumptions:

(1) an integral variant of reg_p is an isomorphism (see Sects. 3.4.3 and 3.4.4);
(2) certain p-adic Abel–Jacobi maps are injective on suitable modules of Heegner-type cycles (see Sect. 5.1.3).

Moreover, we also assume all the conditions described in Sect. 1.3.1 (or, rather, minor variations thereof), so as to be able to apply Theorem C. Observe, in particular, that the square-freeness of N forces f not to be CM; in addition, k and p must satisfy the congruence $k \equiv 2 \pmod{2(p-1)}$. Here we are deliberately vague about hypothesis (2), as the actual injectivity properties of p-adic Abel–Jacobi maps that are needed are too technical to state in this introduction: we just remark that, while it seems to be a "folklore" conjecture that such maps are always injective, in this article we need to impose this injectivity condition only in very specific cases (*cf.* Remark 5.23 for further comments).

1.2.2 *p*-TNC for $\mathcal{M}$ in analytic rank 1: statement

As before, $r_{\mathrm{alg}}(\mathcal{M})$ (respectively, $r_{\mathrm{an}}(\mathcal{M})$) denotes the algebraic (respectively, analytic) rank of $\mathcal{M}$.

Theorem B (p-TNC for $\mathcal{M}$) *Suppose that* $r_{\mathrm{an}}(\mathcal{M}) = 1$. *Under the assumptions in* Sect. 1.2.1, *the following results hold:*

(1) $r_{\mathrm{alg}}(\mathcal{M}) = 1$;
(2) $Ш_p^{\mathrm{BK}}(\mathbb{Q}, \mathcal{M}) = Ш_p^{\mathrm{Nek}}(\mathbb{Q}, \mathcal{M})$;
(3) *the p-part of the TNC for* $\mathcal{M}$ *is true.*

This result corresponds to Theorem 5.42. Albeit not available (as far as we know) in the literature, and never formulated for the motive $\mathcal{M}$, parts (1) and (2) were essentially already known, thanks to a combination of work of Nekovář on the arithmetic of Chow groups of Kuga–Sato varieties [110] and analytic results by Bump–Friedberg–Hoffstein [20], Murty–Murty [107] and Waldspurger [160]; thus, the novelty of Theorem B lies almost entirely in part (3). We prove the p-part of the TNC for $\mathcal{M}$ by showing that, under the assumptions described above, the equality in Theorem A is satisfied. In doing so, a key role is played by our proof of a higher weight counterpart of a conjecture due to Kolyvagin about the non-triviality of his system of "derived" Galois cohomology classes built out of Heegner points on elliptic curves ([82, Conjecture A]): in Sect. 1.3, we outline our arguments for proving this Kolyvagin-type conjecture. Among the several other ingredients that enter our proof of Theorem B, we would like to highlight fundamental results by Kato [74] and by Skinner–Urban [151] on the Iwasawa theory of modular forms, which led us to a proof of an analogue of Theorem B (in particular, of the p-part of TNC for $\mathcal{M}$) in analytic rank 0 (Theorem 5.29). Finally, we point out once more that, along our way to Theorem B, we prove (under the assumptions in Sect. 1.2.1) the rationality conjecture of Beilinson and Deligne both in analytic rank 0 (Theorem 5.24) and in analytic rank 1 (Theorem 5.37). Since we know, as a by-product of Zhang's formula of Gross–Zagier type for higher weight modular forms [165], that $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) \neq 0$ when $r_{\mathrm{an}}(\mathcal{M}) = 1$, the only assumption from

Sect. 1.1.2 that we need to impose is (an integral variant of) the one concerning reg_p (*cf.* condition (1) from Sect. 1.2.1).

1.3 Kolyvagin's conjecture in higher weight

Inspired by work of W. Zhang in weight 2 [166], which builds on the congruence method originally developed by Bertolini–Darmon to prove one divisibility in the anticyclotomic Iwasawa main conjecture for rational elliptic curves [9], the key ingredient in our proof of Theorem B is an analogue for p-adic Galois representations attached to higher (even) weight newforms of Kolyvagin's conjecture on the p-indivisibility of derived Heegner points on rational elliptic curves, which we prove via a p-adic variation method exploiting the arithmetic of Hida families of modular forms.

1.3.1 Kolyvagin's conjecture: assumptions

Let $\mathfrak{p}$ be a prime of F above the prime number p. Write D_F for the discriminant of F and c_f for the index of the order $\mathbb{Z}\big[a_n(f) \mid n \geq 1\big]$ in $\mathcal{O}_F$. We prove Kolyvagin's conjecture under the following assumptions on the pair $(f, \mathfrak{p})$:

(1) $N \geq 3$ is square-free;
(2) $p \nmid 6ND_F c_f$;
(3) $k \equiv 2 \pmod{2(p-1)}$;
(4) $a_p(f) \in \mathcal{O}_{\mathfrak{p}}^{\times}$;
(5) $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$.

We further require the p-adic Galois representation attached to f to have big image (by which we mean that the image of this representation contains all $g \in \mathrm{GL}_2(\mathcal{O}_p)$ such that $\det(g) \in (\mathbb{Z}_p^{\times})^{k-1}$, *cf.* Sect. 3.1.1) and impose suitable ramification conditions on residual representations at primes dividing N (*cf.* Sect. 4.5.2). With the exception of (3), which we briefly comment upon in Sect. 1.3.3, these assumptions are analogous to those appearing in weight 2 in [166]: at least in principle, they could be relaxed (*cf.* Remark 4.15 for the ordinariness condition (4)), but doing so would add extra technicalities to the proofs, while bringing at the same time no significant novelty to the main arguments. Finally, observe that the other assumptions in Sect. 1.1.2 and Sect. 1.2.1 play no role in the statement and proof of Kolyvagin's conjecture.

1.3.2 Kolyvagin's conjecture: statement

Let $p \nmid 6ND_F c_f$ be a prime number such that the p-adic Galois representation attached to f has big image: this rules out only finitely many p. Choose an imaginary quadratic field K where all the prime factors of Np split. Fix a prime $\mathfrak{p}$ of F above p. Using Heegner cycles on Kuga–Sato varieties [110] in place of Heegner points on elliptic curves, we mimic a recipe of Kolyvagin and define a set $\kappa_{f,\infty}$ of Kolyvagin-type "derived" Galois cohomology classes in $H^1(K, T_{\mathfrak{p}}/p^M T_{\mathfrak{p}})$ for suitable integers M, where $T_{\mathfrak{p}} := T_p \otimes_{\mathcal{O}_p} \mathcal{O}_{\mathfrak{p}}$. We call $\kappa_{f,\infty}$ the *Kolyvagin set* associated with f, K, $\mathfrak{p}$: see Sects. 4.2 and 4.3 for the detailed construction of $\kappa_{f,\infty}$.

Theorem C (Kolyvagin's conjecture) *Under the assumptions in* Sect. 1.3.1, $\kappa_{f,\infty} \neq \{0\}$.

Theorem C, which corresponds to Theorem 4.14, shows that the higher (even) weight counterpart of Kolyvagin's conjecture for elliptic curves that was first formulated in [97,

Conjecture A] holds true for ordinary primes p satisfying the conditions described above. Actually, in Theorem 4.14 we prove a stronger statement that implies Kolyvagin's conjecture. We remark that Kolyvagin's original conjecture was proved (at least in the ordinary case, under some technical assumptions) by W. Zhang for $p \nmid N$ and by Skinner–Zhang for $p \| N$. Other than a crucial role in our proof of Theorem B, Theorem C has also consequences on structure theorems for Selmer groups, p-parity results, converse theorems and higher rank results for modular forms and modular motives, some of which are outlined in Sect. 1.4.

1.3.3 Kolyvagin's conjecture: strategy of proof

Our strategy for proving Theorem C is based on a deformation-theoretic approach; in a nutshell, it goes as follows:

(1) we take the p-adic Hida family $\boldsymbol{f}$ passing through our p-ordinary form f (or, rather, through the p-stabilization of f);
(2) we consider big Heegner points $\mathfrak{X}_n \in H^1(K_n, \mathbb{T}^\dagger)$ *à la* Howard, where K_n is the ring class field of K of conductor n and $\mathbb{T}^\dagger$ is the critical twist of Hida's "big Galois representation" attached to $\boldsymbol{f}$;
(3) we define Kolyvagin-type classes $d(\boldsymbol{f}, n) \in H^1(K_n, \mathbb{T}^\dagger)$ built out of the $\mathfrak{X}_n$;
(4) finally, we combine results of Zhang [166] and Skinner–Zhang [152] on Kolyvagin's conjecture for (modular) abelian varieties and specialization results of Howard [64], Castella [27] and Ota [122] for big Heegner points to deduce, using the classes $d(\boldsymbol{f}, n)$, Kolyvagin's conjecture for f from the corresponding statement in weight 2.

As explained in Sect. 4.8.2, the congruence $k \equiv 2 \pmod{2(p-1)}$ allows us to directly compare residual *self-dual* representations attached to the specializations at weight k and at weight 2 of $\boldsymbol{f}$; in fact, we found it quite natural to use the results already available in weight 2 as a "bridge" to the general case. It might be possible to efficiently combine twisting techniques of Chida–Hsieh [29] with recent progress on the p-adic variation of Heegner points due to Büyükboduk–Lei [25], Disegni [37] and Jetchev–Loeffler–Zerbes [68] to get rid of this congruence condition by considering both $\boldsymbol{f}$ and a twisted Hida family of the form $\boldsymbol{f} \otimes \chi$ for an appropriately chosen character χ (in this sense, a combination of [29] and [37] seems to be particularly promising). In order to keep the size of the present paper under (reasonable) control, we preferred not to pursue this question here, but we would like to come back to it in the near future. Finally, we point out that a direct generalization of Zhang's strategy from [166] to the higher weight case (which avoids p-adic variation methods) has been developed, under the condition $p > k+1$, in our recent joint paper [90] with Pati. In particular, our arguments in [90] build crucially on results of Wang on the indivisibility of Heegner cycles over Shimura curves [161]; see Remark 4.16 for more details.

To further elaborate on our congruence assumption on k and p, notice that for a given f the condition $k \equiv 2 \pmod{2(p-1)}$ is clearly satisfied only by finitely many primes p. However, by arguing as follows we can offer infinitely many examples of pairs (f, p) enjoying this property. Let f be a newform of weight 2, level $\Gamma_0(N)$ and trivial character and let $p \nmid N$ be an ordinary prime for f, then take the Hida family passing through the p-stabilization of f. There are infinitely many cusp forms of weight k such that $k \equiv 2 \pmod{2(p-1)}$, level $\Gamma_0(Np)$ and trivial character appearing as specializations of the Hida

family at k: these forms are ordinary p-stabilizations of newforms of weight k, level $\Gamma_0(N)$ and trivial character to which our results apply.

1.4 Other consequences of Theorem C

As hinted above, we deduce from Theorem C, in addition to the p-TNC for $\mathcal{M}$, a structure theorem for Selmer groups (Theorem 6.4), a p-parity result (Theorem 7.1), converse theorems (see, *e.g.*, Theorem 8.4) and higher rank results (Theorem 9.2). The next theorem is a sample of these results.

Theorem D *Under suitable assumptions on height pairings and p-adic Abel–Jacobi maps, the following statements are true:*

(1) $r_{\mathrm{alg}}(\mathcal{M}) \equiv r_{\mathrm{an}}(\mathcal{M}) \pmod 2$*;*
(2) *if* $r_{\mathrm{alg}}(\mathcal{M}) = 1$*, then* $r_{\mathrm{alg}}(\mathcal{M}) = r_{\mathrm{an}}(\mathcal{M})$*;*
(3) *if* $r_{\mathrm{an}}(\mathcal{M}) > 1$ *is even, then* $r_{\mathrm{alg}}(\mathcal{M}) \geq 2$*;*
(4) *if* $r_{\mathrm{an}}(\mathcal{M}) > 1$ *is odd, then* $r_{\mathrm{alg}}(\mathcal{M}) \geq 3$*.*

Each result in Theorem D is proved under its own set of specific assumptions on the non-degeneracy of Gillet–Soulé height pairings and the injectivity of p-adic Abel–Jacobi maps: since their formulations are rather intricate, we do not attempt to describe these hypotheses here and simply refer to Sections 6–9 for all details.

1.5 Comments on the assumptions

We previously pointed out that the main results of this paper will be proved under (partially) different sets of assumptions. As a general rule, in each section we indicate the running assumptions under which we are working, in the hope that this will help readers to isolate the specific hypotheses that are needed for a given theorem. Inevitably, this policy leads to overlaps between all these lists of conditions. In order to try to remedy (at least in part) this inconvenience, in the lines that follow we collect the hypotheses in Assumption 5.38, which is in force in Theorem B, the main contribution of the present article. Explanations on these issues can already be found in this introduction and further details will be given at length in the main body of the paper, so here we refrain from illustrating the reasons why we need to impose every single condition appearing below.

1.5.1 Assumptions in arithmetic algebraic geometry

The general hypotheses in arithmetic algebraic geometry that we need are the following two:

- an integral variant of the p-adic regulator map is an isomorphism;
- p-adic Abel–Jacobi maps are injective on suitable modules of Heegner-type cycles.

These are specific instances of major conjectures that are expected to hold true for all primes p: in this paper, we simply assume their validity when necessary.

1.5.2 Assumptions on modular forms and Galois representations

As before, let $\mathfrak{p}$ be a prime of F above p; write ρ_p (respectively, $\bar{\rho}_{\mathfrak{p}}$) for the p-adic (respectively, mod $\mathfrak{p}$) representation attached to f. With notation as above, here is the list of assumptions on the pair (f, p) we work under:

- $N \geq 3$ is square-free;
- $p \nmid 6ND_F \mathrm{N}(\mathfrak{a}_{f,\Gamma_0(N)})c_f$;
- ρ_p has big image;
- f is p-isolated, *i.e.*, there are no non-trivial congruences modulo p between f and normalized eigenforms in $S_k(\Gamma_0(N))$;
- $a_p(f) \in \mathcal{O}_{\mathfrak{p}}^{\times}$ for all primes $\mathfrak{p}$ of F above p;
- $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of F above p;
- $\bar{\rho}_{\mathfrak{p}}$ is ramified at all primes dividing N for all primes $\mathfrak{p}$ of F above p;
- $k \equiv 2 \pmod{2(p-1)}$.

Here $\mathrm{N}(\mathfrak{a}_{f,\Gamma_0(N)})$ denotes the norm of a certain ideal $\mathfrak{a}_{f,\Gamma_0(N)}$ of $\mathcal{O}_F$ defined in Sect. 5.3.6. The first four conditions rule out only finitely many "exceptional" primes p, which can be explicitly characterized, though the available results in our higher weight setting seem to be much less refined than the corresponding ones in weight 2, especially for elliptic curves (*cf.* Remarks 3.5 and 3.10). Notice that, in this paper, we always use the "big image" property in the guise of Proposition 3.3, which essentially asserts that residual representations are irreducible (*cf.* Remark 3.4). It turns out that the fourth condition is satisfied for all but finitely many p, which can be explicitly characterized (*cf.* Sect. 3.2.1). In order to better understand the role of p-isolation, observe that the paper [9] by Bertolini–Darmon, whose congruence method was exploited by W. Zhang in [166], assumes that the prime p does not divide the degree of a modular parametrization of a given rational elliptic curve E, which implies that the weight 2 newform attached to E is indeed p-isolated; this p-isolation hypothesis was removed by Pollack–Weston in [126] and, in fact, Zhang uses results from [126] to get rid of this assumption. Higher weight generalizations of the work by Pollack–Weston have been studied by Kim–Ota in [76], so it could be tempting to try to relax (or even remove) the p-isolation assumption also in our proof of p-TNC for $\mathcal{M}$. Unfortunately, things appear to be a little trickier here, since we use this property to ensure that the integer $m \geq 0$ introduced right before [110, Proposition 3.1] is 0 (formally, this is justified by (3.9)). Presently, in general we cannot control the integer m effectively (although we suspect it is equal to the valuation at p of the congruence ideal), so we chose to stick to the p-isolation condition when needed; furthermore, as remarked earlier, p-isolation excludes only finitely many p, which is in line with the philosophy of this paper.

Coming back to our list of assumptions, the fifth condition is an ordinariness property at the primes of F above p: by analogy with what happens in weight 2, one expects it to hold for infinitely many primes p. On the other hand, the sixth condition plays a role in our proof of Kolyvagin's conjecture, more precisely, when dealing with weight 2 specializations of big Heegner points in our p-adic Hida family (this is ultimately due to the fact that we cannot *a priori* exclude that the weight 2 specialization of the Hida family has conductor Np), while the residual ramification properties at the prime divisors of N are the higher weight counterpart of an analogous condition for rational elliptic curves in [166]. The last assumption, requiring that $k \equiv 2 \pmod{2(p-1)}$, is a condition on the p-adic weight space of Coleman–Mazur [31] and forces the inequality $p < k$ (this was commented upon in Sect. 1.3.3). As is explained in Remarks 3.5 and 3.10, this suggests that, in particular, the big image and the p-isolation properties are genuine constraints we will need to impose on the pair (f, p): in particular, we cannot make sure they are satisfied by simply taking the prime p to be "large enough" with respect to the weight k of f.

Finally, we remark once again that, as already stressed, the big image, the p-isolation and the ordinariness assumptions on f should all be dispensable, although at the moment it is not completely clear to us how to remove them altogether from our arguments. In particular, following considerations of Lenstra [87], the condition of p-isolation is used in Sect. 3.2.4 to obtain a very useful splitting of Hecke algebras; however, this property plays no role, *e.g.*, in our proof of Kolyvagin's conjecture.

1.5.3 A general strategy to produce examples

Clearly, the most delicate assumptions on f as far as their compatibility is concerned are the third, the fourth and the last one in Sect. 1.5.2. Here we would like to describe a general procedure that allows us to produce infinitely many examples of pairs (f, p) as above satisfying all those three properties.

`Step 1.` Pick a semistable elliptic curve E over $\mathbb{Q}$ of conductor N and let $f_E \in S_2(\Gamma_0(N))$ be the newform associated with E.

`Step 2.` Choose a prime number $p \nmid N$ that is ordinary for E such that

- the image of the Galois representation attached to the p-adic Hida family $\boldsymbol{f}_E$ passing through (the p-stabilization of) f_E is big [39];
- f_E is p-isolated.

At the cost of taking p sufficiently large (which we can do, since there are infinitely many ordinary primes for E), these two conditions can be satisfied simultaneously (see, *e.g.*, [159, Theorem 4.15] and Sect. 3.2.1).

`Step 3.` Notice that, by standard facts in Hida theory, the conditions required in the previous step force all (classical) specializations of $\boldsymbol{f}_E$ at even weights to have irreducible residual Galois representations (see, *e.g.*, [159, §4.5]) and be p-isolated.

`Step 4.` Observe that, by `Step 3`, every (classical) specialization of $\boldsymbol{f}_E$ of weight k such that $k \equiv 2 \pmod{2(p-1)}$ satisfies the third, the fourth and the last assumptions in Sect. 1.5.2.

1.6 Relation to the existing literature

Prior to this work, the only known results on (questions related to) the p-TNC for modular motives were in weight 2 and analytic rank at most 1 and in even weight and analytic rank 0. More precisely, the p-part of the Birch and Swinnerton-Dyer formula for elliptic curves over $\mathbb{Q}$ (*i.e.*, for weight 2 newforms with rational Fourier coefficients) of analytic rank at most 1 has been the subject, under different arithmetic assumptions and in various degrees of generality, of intense study in recent years. Here we would like to mention, in (rough) chronological order, the papers by Kobayashi [79], Skinner–Urban [151], W. Zhang [166], Skinner–Zhang [152], Berti–Bertolini–Venerucci [7], Jetchev–Skinner–Wan [69], Castella [26]. None of these articles is written in a motivic language or refers to the Tamagawa number conjecture explicitly, but all of them prove *de facto* results on the p-TNC for the motives of elliptic curves over $\mathbb{Q}$, as it is known that the (complete) Birch–Swinnerton-Dyer conjecture for an elliptic curve is equivalent to the Tamagawa number conjecture for the corresponding motive (see, *e.g.*, [77] for a detailed explanation of this equivalence, which is highly non-trivial).

As for motives of higher weight modular forms, work of Dummigan–Stein–Watkins [38] deals with the analytic rank 0 case. Results in a rank 0 setting have been obtained also by Fouquet–Wan [44]. More recently, some of the results in weight 2 in the above-mentioned papers were partially extended to higher weights by Thackeray [155]. In particular, a formula was proved that relates the orders of Shafarevich–Tate groups to logarithms of generalized Heegner cycles *à la* Bertolini–Darmon–Prasanna [10]; this formula might be linked to ours and could perhaps be used to deduce, following our approach, the p-TNC for $\mathcal{M}$. Finally, we point out that, along different lines of investigation, the Tamagawa number conjecture for motives of modular forms was the subject of Gealy's PhD thesis [46], while Diamond–Flach–Guo studied the Tamagawa number conjecture for adjoint motives of modular forms [35,36].

1.7 Notation and conventions

We denote by $\bar{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ and write $\bar{\mathbb{Z}}$ for the ring of integers in $\bar{\mathbb{Q}}$ (*i.e.*, the integral closure of $\mathbb{Z}$ in $\bar{\mathbb{Q}}$). For every prime number ℓ we fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

For any number field K we denote by $G_K := \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group of K, where $\bar{K}$ is a fixed algebraic closure of K. For any continuous G_K-module M we write $H^i(K, M)$ for the i-th continuous cohomology group of G_K with coefficients in M in the sense of Tate ([154, §2]). Finally, if K/F is an extension of number fields, then

$$\mathrm{res}_{K/F} : H^i(F, M) \longrightarrow H^i(K, M), \quad \mathrm{cores}_{K/F} : H^i(K, M) \longrightarrow H^i(F, M)$$

denote the restriction and corestriction maps in cohomology, respectively.

2 The TNC for motives of modular forms

We describe the Tamagawa number conjecture (TNC, for brevity) of Bloch–Kato ([15, Conjecture 5.15]) in the case of motives of modular forms. As will be clear, our exposition follows [43,77] quite closely. We remark that the TNC for modular forms in analytic rank 0 was also considered by Dummigan–Stein–Watkins in [38], while Diamond–Flach–Guo studied in [36] the TNC for adjoint motives of modular forms. Results in rank 0 have also been obtained by Fouquet–Wan in [44].

Although we are chiefly interested in the TNC over $\mathbb{Q}$, we introduce some of the relevant notions (*e.g.*, motivic cohomology, L-functions, Selmer groups, Shafarevich–Tate groups) over arbitrary number fields; in particular, we will eventually need to work over certain imaginary quadratic fields. For details on the TNC for more general motives, the reader may consult, *e.g.*, [21,41,77].

2.1 Review of motives

We briefly review the basic definitions in the theory of motives; for details, the reader is referred, *e.g.*, to [2, Ch. 4], [96,140].

2.1.1 Pure motives

Let K be a field and write $\mathcal{V}_K$ for the category of smooth projective schemes over K. Given an object X of $\mathcal{V}_K$ and $d \in \mathbb{N}$, denote by $\mathcal{Z}^d(X)$ the group of cycles of codimension d on X, *i.e.*, the free abelian group generated by the irreducible subschemes of X of codimension

d. Let $\sim$ be an adequate equivalence relation on cycles (see, *e.g.*, [2, Définition 3.1.1.1]) and let R be a commutative ring. Set $\mathcal{Z}^d_\sim(X)_R := \big(\mathcal{Z}^d(X) \otimes_{\mathbb{Z}} R\big)\big/ \sim$; it will also be convenient to put $\mathcal{Z}^d_\sim(X)_R = 0$ for $d \in \mathbb{Z}_{<0}$. Let X, Y be objects of $\mathcal{V}_K$ and suppose that X is of pure dimension d; the group of correspondences modulo $\sim$ of degree r from X to Y with coefficients in R is

$$\mathrm{Corr}^r_\sim(X, Y)_R := \mathcal{Z}^{d+r}_\sim(X \times_K Y)_R.$$

If X is not of pure dimension, then $\mathrm{Corr}^r_\sim(X, Y)_R$ can be defined in terms of the irreducible components of X as in [140, §1.3]. Composition of correspondences furnishes, via intersection theory, a product structure

$$\mathrm{Corr}^r_\sim(X, Y)_R \times \mathrm{Corr}^s_\sim(Y, Z)_R \longrightarrow \mathrm{Corr}^{r+s}_\sim(X, Z)_R.$$

In particular, $\mathrm{Corr}^0_\sim(X, X)_R$ inherits a ring structure for every object X of $\mathcal{V}_K$. Now let $\mathcal{V}^0_{K,R}$ be the category whose objects are those of $\mathcal{V}_K$ and whose morphisms are given by degree 0 correspondences modulo $\sim$ with coefficients in R. By definition, the category $\mathscr{M}_\sim(K)_R$ of *pure $\sim$-motives (defined) over K with coefficients in R* is the pseudo-abelian completion of $\mathcal{V}^0_{K,R}$; see, *e.g.*, [2, §4.1], [77, Definition 1.1] for details; here the adjective "pure" refers to the fact that we consider only *smooth* projective schemes. More explicitly, a (pure) $\sim$-motive over K with coefficients in R is a triple

$$\mathcal{M} = (X, q, r)$$

where X is an object of $\mathcal{V}_K$, $q \in \mathrm{Corr}^0_\sim(X, X)_R$ is an idempotent and $r \in \mathbb{Z}$. If $r = 0$, then $\mathcal{M}$ is said to be *effective*. Furthermore, if $\mathcal{M}_1 = (X_1, q_1, r_1)$ and $\mathcal{M}_2 = (X_2, q_2, r_2)$ are motives, then

$$\mathrm{Hom}_{\mathscr{M}_\sim(K)_R}(\mathcal{M}_1, \mathcal{M}_2) = q_2 \cdot \mathrm{Corr}^{r_1-r_2}_\sim(X_1, X_2)_R \cdot q_1 \subset \mathrm{Corr}^{r_1-r_2}_\sim(X_1, X_2)_R.$$

Given motives $\mathcal{M}_i = (X_i, q_i, r_i)$ for $i = 1, 2$, the *product* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is

$$\mathcal{M}_1 \otimes_K \mathcal{M}_2 := (X_1 \times_K X_2, q_1 \times_K q_2, r_1 + r_2),$$

One can also define the *direct sum* of two motives (see, *e.g.*, [140, §1.14]), and it turns out that $\mathscr{M}_\sim(K)_R$ is an additive, R-linear, pseudo-abelian category ([140, Theorem 1.6]).

The *dual* of a motive $\mathcal{M} = (X, q, r)$ is

$$\mathcal{M}^\vee := \big(X, q^t, \dim(X) - r\big),$$

where q^t is the image of the idempotent q under the map that interchanges the factors of $X \times_K X$ (*cf.* [77, Definition 1.2]). A motive $\mathcal{M}$ is *self-dual* if $\mathcal{M} \simeq \mathcal{M}^\vee(1)$, where $\star(1)$ denotes Tate twist (see, *e.g.*, [2, §4.1.5]). Finally, if X is an object of $\mathcal{V}_K$ and Δ_X is the diagonal in $X \times_K X$, then the effective $\sim$-motive $(X, \Delta_X, 0)$ is the *$\sim$-motive of X*.

2.1.2 Chow motives

Taking $\sim$ to be rational equivalence (see, *e.g.*, [2, §3.2.2]), we obtain the category $\mathscr{M}_{\mathrm{rat}}(K)_R$ of *rational* (or *Chow*) *motives over K with coefficients in R*. As we will see in Sect. 2.2, the motive of modular forms of given weight and level is an object of $\mathscr{M}_{\mathrm{rat}}(\mathbb{Q})_R$ for R a suitable Hecke algebra. However, in order to define the motive of a single modular form one needs to pass to the category of Grothendieck (*i.e.*, homological) motives, which we introduce below.

2.1.3 Grothendieck motives

With notation as in Sect. 2.1.1, the category of *homological motives over K with coefficients in R* is $\mathscr{M}_{\mathrm{hom}}(K)_R$, where "hom" indicates homological equivalence (see, *e.g.*, [2, §3.3.4]). Following Scholl ([139, §1.2.3]), we shall call the objects of $\mathscr{M}_{\mathrm{hom}}(K)_R$ *Grothendieck motives*. Tensor products, duals and self-duality of Grothendieck motives are defined formally as for Chow motives.

Rational equivalence is finer than homological equivalence (in fact, rational equivalence is the finest of all adequate equivalence relations; see, *e.g.*, [2, Lemme 3.2.2.1]), so there is a natural functor

$$\mathscr{F}_K : \mathscr{M}_{\mathrm{rat}}(K)_R \longrightarrow \mathscr{M}_{\mathrm{hom}}(K)_R \tag{2.1}$$

that is the identity on objects and allows one to view a Chow motive as a Grothendieck motive.

2.2 Motives of modular forms

Let $N \geq 3$ be an integer and $k \geq 4$ be an even integer. We want to introduce, following Scholl [139], the (Grothendieck) motive of a fixed modular form of weight k and level N, whose realizations will be carefully described in Sect. 2.4. In order to do this, we need to introduce first the (Chow) motive of all modular forms of weight k and level N.

2.2.1 Anaemic Hecke algebras

Write $\mathfrak{h}_k(\Gamma(N)) \subset \mathrm{End}_{\bar{\mathbb{Q}}}\big(S_k(\Gamma(N); \bar{\mathbb{Q}})\big)$ for the $\mathbb{Z}$-algebra generated by the Hecke operators T_n with $(n, N) = 1$; it is often called the "anaemic" Hecke algebra of weight k and level $\Gamma(N)$. Analogously, denote by $\mathfrak{h}_k(\Gamma_0(N))$ the anaemic Hecke algebra of weight k and level $\Gamma_0(N)$. There is a natural surjection

$$\mathfrak{h}_k(\Gamma(N)) \twoheadrightarrow \mathfrak{h}_k(\Gamma_0(N)) \tag{2.2}$$

of $\mathbb{Z}$-algebras that is induced by the inclusion $S_k(\Gamma_0(N)) \subset S_k(\Gamma(N))$. Finally, for any $\mathbb{Z}$-algebra A, set $\mathfrak{h}_k(\Gamma_0(N))_A := \mathfrak{h}_k(\Gamma_0(N)) \otimes_{\mathbb{Z}} A$.

2.2.2 The motive of modular forms of weight k and level N

Denote by $\tilde{\mathcal{E}}_N^{k-2}$ the *Kuga–Sato variety* of level N and weight k, *i.e.*, the smooth projective $\mathbb{Q}$-scheme defined as the canonical desingularization of the $(k-2)$-fold product $\mathcal{E}_N^{k-2}$ of the universal generalized elliptic curve $\pi : \mathcal{E}_N \to X(N)$ over the compact modular curve $X(N)$ of level $\Gamma(N)$ (see, *e.g.*, [139, §1.2.0]). An idempotent Π_ϵ in the ring of correspondences of

degree 0 of $\tilde{\mathcal{E}}_N^{k-2}$ can be constructed as follows. Recall that if we set

$$\Gamma_{k-2} := \left((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \{\pm 1\}\right)^{k-2} \rtimes \mathfrak{S}_{k-2}, \tag{2.3}$$

where $\mathfrak{S}_{k-2}$ is the symmetric group on $k-2$ letters, then there is a canonical action of Γ_{k-2} on $\tilde{\mathcal{E}}_N^{k-2}$ ([139, §1.1.0, §1.1.1]); we define Π_ϵ to be the projector associated with the character $\epsilon : \Gamma_{k-2} \to \{\pm 1\}$ that is the sign character on $\mathfrak{S}_{k-2}$, the trivial character on $(\mathbb{Z}/N\mathbb{Z})^{2(k-2)}$ and the product character on $\{\pm 1\}^{k-2}$; see [139, §1.1.2] for details (note, in particular, that in order to define Π_ϵ we need to invert $2N(k-2)!$). Moreover, write Π_B for the idempotent attached to the quotient $\Gamma_0(N)/\Gamma(N)$, whose order t_N we need to invert in order to define Π_B. Let

$$\mathcal{M}_k(N) := \left(\tilde{\mathcal{E}}_N^{k-2}, \Pi_B\Pi_\epsilon, k/2\right)$$

be the Chow motive of modular forms of weight k and level N ([139, §1.2.2]). The motive $\mathcal{M}_k(N)$ is an object of $\mathscr{M}_{\mathrm{rat}}(\mathbb{Q})_{\mathfrak{H}_k(\Gamma(N))[\frac{1}{2Nt_N(k-2)!}]}$, where $\mathfrak{H}_k(\Gamma(N))$ is the Hecke algebra from Sect. 2.2.1. Note that $\mathcal{M}_k(N)$ is self-dual.

Remark 2.1 When $N \geq 5$, the motive of modular forms can also be constructed, following [10, Appendix], by means of the (compact) modular curve $X_1(N)$ of level $\Gamma_1(N)$ instead of $X(N)$ as in [139]. Sometimes working over the connected curve $X_1(N)$ can be more convenient; however, for the goals of this paper the two perspectives are essentially equivalent, since we will work exclusively with the various realizations of our modular motives.

2.2.3 The motive of a modular form

Let $f \in S_k(\Gamma_0(N))$ be a normalized newform of weight k and level $\Gamma_0(N)$, whose q-expansion will be denoted by $f(q) = \sum_{n\geq 1} a_n(f)q^n$. Let $F := \mathbb{Q}\big(a_n(f) \mid n \geq 1\big)$ be the Hecke field of f, which is a totally real number field, and let $\mathcal{O}_F$ be its ring of integers. There is a set-theoretic inclusion $F \subset \mathbb{R}$. Let $\mathcal{M}(f)$ be the Grothendieck motive over $\mathbb{Q}$ with coefficients in F attached to f by Scholl ([139, Theorem 1.2.4]). To construct $\mathcal{M}(f)$, consider the projector Ψ_f associated with f ([139, §4.2.0]) and define $\mathcal{M}(f)$ to be the kernel of Ψ_f, which is a submotive of $\mathcal{M}_k(N)$; in other words, $\mathcal{M}(f)$ is the object of $\mathscr{M}_{\mathrm{hom}}(\mathbb{Q})_F$ given by

$$\mathcal{M}(f) := \Big(\tilde{\mathcal{E}}_N^{k-2}, (1-\Psi_f)\circ(\Pi_B\Pi_\epsilon \otimes 1), k/2\Big).$$

Like $\mathcal{M}_k(N)$, the motive $\mathcal{M}(f)$ is self-dual.

Remark 2.2 The crux in Scholl's construction of $\mathcal{M}(f)$ is a decomposition of $\mathcal{M}_k(N)$ under the action of the Hecke algebra. This decomposition takes place in $\mathscr{M}_{\mathrm{hom}}(\mathbb{Q})_F$; that is, one needs to replace $\mathcal{M}_k(N)$ with its image via the functor $\mathscr{F}_K$ introduced in (2.1). As is pointed out in [139, Remark 1.2.6], it is reasonable to expect that the splitting of $\mathcal{M}_k(N)$ can be performed already in the category $\mathscr{M}_{\mathrm{rat}}(\mathbb{Q})_F$ (as in the $k=2$ case), but this seems very hard to achieve without assuming Grothendieck's standard conjectures.

Remark 2.3 In this article, we will need to consider also the motive $\mathcal{M}(f^K)$ of the twist f^K of f by the Dirichlet character associated with a suitable imaginary quadratic field K.

2.3 Notation

To simplify our notation, from now on we set $X := \tilde{\mathcal{E}}_N^{k-2}$, $\bar{X} := X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$, $\Pi := (1 - \Psi_f) \circ (\Pi_B \Pi_\epsilon \otimes 1)$, $\mathcal{M} := \mathcal{M}(f)$. Thus, $\mathcal{M}$ is the Grothendieck motive

$$\mathcal{M} = (X, \Pi, k/2)$$

defined over $\mathbb{Q}$ with coefficients in F. For any number field K, we also write

$$\mathcal{M}/K := (X \times_{\mathbb{Q}} K, \Pi \times_{\mathbb{Q}} K, k/2)$$

for the Grothendiek motive over K with coefficients in F obtained from $\mathcal{M}$ by extension of scalars, *i.e.*, base change (see, *e.g.*, [2, §4.2.3], [77, Remark 1.7]).

2.4 Realizations of $\mathcal{M}$

Here we consider only the $(k-1)$-st realizations of the motive $\mathcal{M}$, which correspond to the choice $i = 2r - 1$ in [77, Ch. 1, §2]. We also briefly describe the realizations of $\mathcal{M}/K$ obtained from those of $\mathcal{M}$ by base change.

2.4.1 f-isotypic submodules

With notation as in Sect. 2.2.1, let $\theta_f : \mathfrak{H}_k(\Gamma_0(N)) \to \mathcal{O}_F$ be the ring homomorphism associated with f; explicitly, $\theta_f(T_n) := a_n(f)$. Composing with (2.2), θ_f yields a ring homomorphism (denoted by the same symbol) $\theta_f : \mathfrak{H}_k(\Gamma(N)) \to \mathcal{O}_F$.

For every $\mathfrak{H}_k(\Gamma(N))$-module M, let us set

$$M[\theta_f] := \big\{ m \in M \mid T \cdot m = 0 \text{ for all } T \in \ker(\theta_f) \big\}. \tag{2.4}$$

We call $M[\theta_f]$ the *f-isotypic submodule* of M.

2.4.2 Betti realization

Fix a subring $R \subset \mathbb{C}$ and define $R(k/2) := (2\pi i)^{k/2} \cdot R$, which we view as an R-submodule of $\mathbb{C}$. In the following, we view $R(k/2)$ as a locally constant sheaf on $X(\mathbb{C})$. Define

$$T_{\mathrm{B}}(R) := \Pi \cdot H^{k-1}\big(X(\mathbb{C}), R(k/2)\big) = \Big(\Pi_B \Pi_\epsilon \cdot H^{k-1}\big(X(\mathbb{C}), R(k/2)\big)\Big)[\theta_f],$$

viewed as an R-submodule of $T_{\mathrm{B}}(\mathbb{C})$. In particular, with t_N as in Sect. 2.2.2, if $2Nt_N(k-2)!$ is not invertible in R, then $T_{\mathrm{B}}(R)$ may not be contained in $H^{k-1}\big(X(\mathbb{C}), R(k/2)\big)$.

Recall that the field F arises naturally as a subfield of $\mathbb{C}$.

Definition 2.4 The *Betti* (or *singular*) *realization* of $\mathcal{M}$ is the F-vector space $V_{\mathrm{B}} := T_{\mathrm{B}}(F)$.

The F-vector space V_{B} is also the Betti realization of the motive $\mathcal{M}/K$ for any number field K. Set

$$T_{\mathrm{B}} := T_{\mathrm{B}}(\mathcal{O}_F). \tag{2.5}$$

Complex conjugation τ induces involutions $\iota_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ and $F(k/2) \to F(k/2)$, the latter being given by multiplication by $(-1)^{k/2}$. Denote by $\phi_\infty : V_{\mathrm{B}} \to V_{\mathrm{B}}$ the composition of these two involutions and write

$$V_{\mathrm{B}}^+ := V_{\mathrm{B}}^{\phi_\infty = 1} \tag{2.6}$$

for the F-subspace of V_{B} on which ϕ_∞ acts trivially.

Now we describe the Betti realization more explicitly. Let $Y(N)$ be the open modular curve of level $\Gamma(N)$. We still denote by $\pi : \mathcal{E}_N \to Y(N)$ the universal elliptic curve over $Y(N)$ and, with R as before, set

$$\mathcal{F}_{\mathrm{B}}(R) := \mathbf{R}^1\pi_*(R(k/2)), \quad \mathcal{F}_{\mathrm{B}}^{k-2}(R) := \mathrm{Sym}^{k-2}\mathcal{F}_{\mathrm{B}}(R),$$

where R stands for the corresponding constant sheaf on $\mathcal{E}_N$, which we regard as a sheaf on $X(N)$ by pushforward via the canonical embedding $Y(N) \hookrightarrow X(N)$. Write

$$\mathbb{T}_{\mathrm{B}}(R) := H^1_{\mathrm{par}}\Big(Y(N), \mathcal{F}_{\mathrm{B}}^{k-2}(R)\Big)$$

for the parabolic cohomology group that is the image of the natural map

$$H^1_{\mathrm{cpt}}\Big(Y(N), \mathcal{F}_{\mathrm{B}}^{k-2}(R)\Big) \longrightarrow H^1\Big(Y(N), \mathcal{F}_{\mathrm{B}}^{k-2}(R)\Big),$$

where $H^1_{\mathrm{cpt}}(\heartsuit, \diamondsuit)$ denotes compactly supported cohomology. Set

$$\mathbb{V}_{\mathrm{B}} := \mathbb{T}_{\mathrm{B}}(F), \quad \mathbb{T}_{\mathrm{B}} := \mathbb{T}_{\mathrm{B}}(\mathcal{O}_F).$$

By [139, Theorem 1.2.1], V_{B} is related to parabolic cohomology by a canonical isomorphism

$$V_{\mathrm{B}} \simeq \Pi \cdot \mathbb{V}_{\mathrm{B}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{V}_{\mathrm{B}})[\theta_f] \tag{2.7}$$

(note that Π_B and Π_ϵ act canonically on $\mathbb{V}_{\mathrm{B}}$). Since f is a newform, Eichler–Shimura theory guarantees that V_{B} (respectively, V_{B}^+) has dimension 2 (respectively, 1) over F. Isomorphism (2.7) induces an isomorphism

$$T_{\mathrm{B}} \simeq \Pi \cdot \mathbb{T}_{\mathrm{B}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{T}_{\mathrm{B}})[\theta_f],$$

where the right-hand side is, in general, not contained in $\mathbb{T}_{\mathrm{B}}$, but only in $\mathbb{V}_{\mathrm{B}}$.

2.4.3 Étale realization

Let p be a prime number and R a $\mathbb{Z}_p$-algebra. For every $r \in \mathbb{Z}$, we denote by $R(r)$ the r-fold Tate twist of R and define the R-module

$$T_p(R) := \Pi \cdot H^{k-1}_{\text{ét}}\big(\bar{X}, R(k/2)\big) = \Big(\Pi_B \Pi_\epsilon \cdot H^{k-1}_{\text{ét}}\big(\bar{X}, R(k/2)\big)\Big)[\theta_f], \tag{2.8}$$

which is equipped with a natural action of $G_{\mathbb{Q}}$. As before, we view $T_p(R)$ as an R-submodule of $T_p(\bar{\mathbb{Q}}_p)$: if $2Nt_N(k-2)!$ is not invertible in R, then $T_p(R)$ is not necessarily contained in $H^{k-1}_{\text{ét}}\big(\bar{X}, R(k/2)\big)$.

From here on, set $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Definition 2.5 The *étale realization* of $\mathcal{M}$ at p is the F_p-module $V_p := T_p(F_p)$.

By [139, Theorem 1.2.4], V_p is equivalent to the self-dual twist $V_{f,p}^\dagger := V_{f,p}(k/2)$ of the representation $V_{f,p}$ attached by Deligne to f and p [33]; in particular, V_p is free of rank 2 over F_p. There is an identification $F_p = \prod_{\mathfrak{p}|p} F_\mathfrak{p}$, where the product ranges over all primes $\mathfrak{p}$ of F above p and $F_\mathfrak{p}$ is the completion of F at $\mathfrak{p}$. As a consequence, there is a splitting $V_p = \prod_{\mathfrak{p}|p} V_\mathfrak{p}$, where each $V_\mathfrak{p}$ is a 2-dimensional $F_\mathfrak{p}$-vector space on which $G_\mathbb{Q}$ acts. As above, $V_\mathfrak{p}$ is equivalent to the self-dual twist $V_{f,\mathfrak{p}}^\dagger = V_{f,\mathfrak{p}}(k/2)$ of the $\mathfrak{p}$-adic representation $V_{f,\mathfrak{p}}$ attached to f. Similarly, set $\mathcal{O}_p := \mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$ (notice that F_p is the total quotient ring of $\mathcal{O}_p$) and define

$$T_p := T_p(\mathcal{O}_p). \tag{2.9}$$

Denoting by $\mathcal{O}_\mathfrak{p}$ the completion of $\mathcal{O}_F$ at a prime $\mathfrak{p}$, there is a splitting $T_p = \prod_{\mathfrak{p}|p} T_\mathfrak{p}$, where each $T_\mathfrak{p} = T_p(\mathcal{O}_\mathfrak{p})$ is a $G_\mathbb{Q}$-stable $\mathcal{O}_\mathfrak{p}$-lattice inside $V_\mathfrak{p}$, *i.e.*, a free $\mathcal{O}_\mathfrak{p}$-submodule of rank 2 of $V_\mathfrak{p}$ on which $G_\mathbb{Q}$ acts. It follows that T_p is a $G_\mathbb{Q}$-stable self-dual $\mathcal{O}_p$-lattice inside V_p. In the rest of this paper, we fix an equivalence of representations $V_p \simeq V_{f,p}^\dagger$ and write $T_{f,p}^\dagger$ for the $G_\mathbb{Q}$-stable $\mathcal{O}_p$-lattice inside $V_{f,p}^\dagger$ corresponding under this equivalence to the distinguished $G_\mathbb{Q}$-stable $\mathcal{O}_p$-lattice that is obtained from (2.8) with $R = \mathcal{O}_p$ by omitting the $k/2$-fold Tate twist; an analogous convention applies to each $T_\mathfrak{p}$. Finally, set

$$A_p := V_p/T_p \tag{2.10}$$

and notice that A_p is (canonically isomorphic to) the Pontryagin dual of T_p as a $\mathbb{Z}_p$-module.

One can also describe the étale realizations V_p and T_p as follows. Define the p-adic sheaves

$$\mathcal{F}_{\text{ét}}(R) := \pi_*\big(R(k/2)\big), \quad \mathcal{F}_{\text{ét}}^{k-1}(R) := \mathrm{Sym}^{k-2}\big(\mathcal{F}_{\text{ét}}(R)\big).$$

on $Y(N)$; as before, we still denote by $\mathcal{F}_{\text{ét}}^{k-2}(R)$ the sheaf on $X(N)$ obtained by pushforward via $Y(N) \hookrightarrow X(N)$. Set

$$\mathbb{T}_{\text{ét}}(R) := H^1_{\text{ét}}\big(X(N), \mathcal{F}_{\text{ét}}^{k-2}(R)\big), \quad \mathbb{T}_{\text{ét}} := \mathbb{T}_{\text{ét}}(\mathcal{O}_p), \quad \mathbb{V}_{\text{ét}} := \mathbb{T}_{\text{ét}}(F_p).$$

The projector Π acts on $\mathbb{V}_{\text{ét}}$ and there is a canonical isomorphism

$$V_p \simeq \Pi \cdot \mathbb{V}_{\text{ét}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{V}_{\text{ét}})[\theta_f] \tag{2.11}$$

(see, again, [139, Theorem 1.2.1]). Isomorphism (2.11) induces an isomorphism

$$T_p \simeq \Pi \cdot \mathbb{T}_{\text{ét}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{T}_{\text{ét}})[\theta_f];$$

here, as above, the right-hand side is, in general, not contained in $\mathbb{T}_{\text{ét}}$, but only in $\mathbb{V}_{\text{ét}}$.

2.4.4 de Rham realization

Let R be a $\mathbb{Z}[1/N]$-algebra. We view $X(N)$ and X as schemes over $\mathbb{Z}[1/N]$. Define the R-module

$$T_{\mathrm{dR}}(R) := \Pi \cdot \Big(H^{k-1}_{\mathrm{dR}}(X) \otimes_{\mathbb{Z}[1/N]} R\Big) = \Big(\Pi_B \Pi_\epsilon \cdot \big(H^{k-1}_{\mathrm{dR}}(X) \otimes_{\mathbb{Z}[1/N]} R\big)\Big)[\theta_f].$$

Here $H^i_{\mathrm{dR}}(X) := \mathbb{H}^i\big(\Omega^\bullet_X\big)$ is the i-th hypercohomology group of the de Rham complex $\Omega^\bullet_X$ of X; thus, $T_{\mathrm{dR}}(R)$ is a finitely generated R-module. Recall that $H^i_{\mathrm{dR}}(X)$ is equipped with the filtration

$$\mathrm{Fil}^n\big(H^i_{\mathrm{dR}}(X)\big) := \mathrm{im}\Big(\mathbb{H}^i\big(\Omega^{\geq n}_X\big) \longrightarrow \mathbb{H}^i\big(\Omega^\bullet_X\big) = H^i_{\mathrm{dR}}(X)\Big).$$

Define a filtration on $T_{\mathrm{dR}}(R)$ by setting

$$\mathrm{Fil}^n\big(T_{\mathrm{dR}}(R)\big) := \Pi \cdot \Big(\mathrm{Fil}^{n+k/2}\big(H^{k-1}_{\mathrm{dR}}(X)\big) \otimes_{\mathbb{Z}[1/N]} R\Big).$$

Finally, put $V_{\mathrm{dR}} := T_{\mathrm{dR}}(F)$.

Definition 2.6 The *de Rham realization* of $\mathcal{M}$ is the filtered F-vector space given by the pair $\Big(V_{\mathrm{dR}}, \big(\mathrm{Fil}^n(V_{\mathrm{dR}})\big)_{n\in\mathbb{N}}\Big)$.

The *tangent space* of $\mathcal{M}$ is the F-vector space

$$t(\mathcal{M}) := V_{\mathrm{dR}} \big/ \mathrm{Fil}^0(V_{\mathrm{dR}}). \tag{2.12}$$

Following [36, §1.2.4], we offer an alternative description of de Rham cohomology, as we did for the étale and Betti realizations. We write $\bar\pi : \bar{\mathcal{E}}_N \to X(N)$ for the generalized universal elliptic curve over $X(N)$ and denote by $\omega^1_\star$ the sheaf of relative logarithmic differentials of $\star$ (see, *e.g.*, [71, §1.7]). Put $T := \mathrm{Spec}(R)$. By [71, Theorem 3.5 and Proposition 3.12], there is a short exact sequence of coherent locally free $\mathcal{O}_{\bar{\mathcal{E}}_N}$-modules

$$0 \longrightarrow \bar\pi^*\big(\omega^1_{X(N)/T}\big) \longrightarrow \omega^1_{\bar{\mathcal{E}}_N/T} \longrightarrow \omega^1_{\bar{\mathcal{E}}_N/X(N)} \longrightarrow 0.$$

Let us consider the locally free sheaves

$$\mathcal{F}_{\mathrm{dR}}(R) := \mathbf{R}^1\bar\pi_*\big(\omega^\bullet_{\bar{\mathcal{E}}_N/X(N)}\big), \quad \mathcal{F}^{k-2}_{\mathrm{dR}}(R) := \mathrm{Sym}^{k-2}\big(\mathcal{F}_{\mathrm{dR}}(R)\big)$$

of $\mathcal{O}_{X(N)}$-modules on $X(N)$, where $\omega^\bullet_{\bar{\mathcal{E}}_N/X(N)}$ is the complex $d : \mathcal{O}_{\bar{\mathcal{E}}_N} \to \omega^1_{\bar{\mathcal{E}}_N/X(N)}$. Set $\omega(R) := \bar\pi_*\big(\omega^1_{\bar{\mathcal{E}}_N/X(N)}\big)$; the sheaf $\mathcal{F}_{\mathrm{dR}}(R)$ has a decreasing filtration with $\mathrm{Fil}^2\,\mathcal{F}_{\mathrm{dR}}(R) = 0$, $\mathrm{Fil}^1\,\mathcal{F}_{\mathrm{dR}}(R) = \omega(R)$, $\mathrm{Fil}^0\,\mathcal{F}_{\mathrm{dR}}(R) = \mathcal{F}_{\mathrm{dR}}(R)$. In turn, this filtration produces a filtration on $\mathcal{F}^{k-2}_{\mathrm{dR}}(R)$. Now define

$$\mathbb{T}_{\mathrm{dR}}(R) := \mathbb{H}^1\big(X(N), \omega^\bullet(\mathcal{F}^{k-2}_{\mathrm{dR}}(R))\big), \quad \mathbb{T}_{\mathrm{dR}} := \mathbb{T}_{\mathrm{dR}}\big(\mathcal{O}_F[1/N]\big), \quad \mathbb{V}_{\mathrm{dR}} := \mathbb{T}_{\mathrm{dR}}(F),$$

where $\mathbb{H}^i(\heartsuit, \diamondsuit)$ denotes hypercohomology and $\omega^\bullet\big(\mathcal{F}^{k-2}_{\mathrm{dR}}(R)\big)$ is the complex associated with $\mathcal{F}^{k-2}_{\mathrm{dR}}(R)$ equipped with the logarithmic connection that is induced by the (logarithmic) Gauss–Manin connection $\nabla : \mathcal{F}_{\mathrm{dR}}(R) \to \mathcal{F}_{\mathrm{dR}}(R) \otimes_{\mathcal{O}_{X(N)}} \omega^1_{X(N)/T}$. The filtration

on $\mathcal{F}^{k-2}_{\mathrm{dR}}(R)$ yields a filtration on $\mathbb{T}_{\mathrm{dR}}(R)$ whose graded pieces can be described (up to isomorphism) as

$$\mathrm{gr}^i\big(\mathbb{T}_{\mathrm{dR}}(R)\big) \simeq \begin{cases} H^0\big(X(N), \omega^{k-2}(R) \otimes \omega^1_{X(N)/T}\big) & \text{if } i = k-1, \\ H^1\big(X(N), \omega^{2-k}(R)\big) & \text{if } i = k/2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the complex upper half-plane. Pulling back to $\coprod_{t\in(\mathbb{Z}/N\mathbb{Z})^\times} \mathcal{H}$ and trivializing by the differential form $(2\pi i)^{k-1} dz \wedge dz_1 \wedge \cdots \wedge dz_{k-2}$, where z is the coordinate on $\mathcal{H}$ and z_i is the coordinate on the i-th copy of the universal elliptic curve, we obtain an isomorphism

$$\mathrm{Fil}^{k-1}\big(\mathbb{T}_{\mathrm{dR}}(\mathbb{C})\big) \xrightarrow{\simeq} \bigoplus_{t\in(\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N)),$$

where $M_k(\Gamma(N))$ is the $\mathbb{C}$-vector space of modular forms of weight k and level $\Gamma(N)$. By the q-expansion principle, the map $\bigoplus_{t\in(\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N)) \to \mathbb{C}[\![q^{1/N}]\!]$ sending $g(e^{2\pi i\tau/N})$ to $g(q^{1/N})$ identifies $\mathrm{Fil}^{k-1}(\mathbb{T}_{\mathrm{dR}})$ with the subset of $\bigoplus_{t\in(\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N))$ consisting of those modular forms whose q-expansion at ∞ has coefficients in $\mathcal{O}_F[1/N, \mu_N]$, where $\mu_N \subset \bar{\mathbb{Q}}^\times$ is the group of N-th roots of unity.

By [139, Theorem 1.2.1] (see also [10, §2.1] and [138]), there is an isomorphism

$$V_{\mathrm{dR}} \simeq \Pi \cdot \mathbb{V}_{\mathrm{dR}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{V}_{\mathrm{dR}})[\theta_f], \tag{2.13}$$

with filtration on the left-hand side obtained by shifting by $k/2$ the obvious filtration on the right-hand side. Then $\mathrm{Fil}^{k/2-1}(V_{\mathrm{dR}})$ is spanned by (the image of) the differential form on X given (on each connected component) by

$$\omega_f := (2\pi i)^{k-1} \cdot f(z) dz \wedge dz_1 \wedge \cdots \wedge dz_{k-2}. \tag{2.14}$$

See, *e.g.*, [36, §1.2.1] and [75, Appendix 1] for details; *cf.* also the explanation offered in [19, Appendix] for the factor $(2\pi i)^{k-1}$ appearing in the right-hand term of (2.14). Isomorphism (2.13) induces an isomorphism

$$T_{\mathrm{dR}} \simeq \Pi \cdot \mathbb{T}_{\mathrm{dR}} = (\Pi_B \Pi_\epsilon \cdot \mathbb{T}_{\mathrm{dR}})[\theta_f],$$

where, as before, the right-hand side is, in general, not contained in $\mathbb{T}_{\mathrm{dR}}$, but only in $\mathbb{V}_{\mathrm{dR}}$.

2.5 The period map

From the comparison isomorphism between singular and de Rham cohomology we obtain a comparison isomorphism of $(F \otimes_{\mathbb{Q}} \mathbb{C})$-modules

$$\mathrm{Comp}_{\mathrm{B,dR}} : V_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} V_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

If ϕ_∞ is the involution from Sect. 2.4.2, then this isomorphism is equivariant with respect to the actions of $\phi_\infty \otimes \tau$ on the left and of $1 \otimes \tau$ on the right, so it induces an isomorphism

$$\mathrm{Comp}_{\mathrm{B,dR}} : (V_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C})^{\phi_\infty \otimes \tau = 1} \xrightarrow{\simeq} V_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R} \tag{2.15}$$

of $\mathbb{R}$-vector spaces.

2.5.1 Period map

Let $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$ and set

$$V^+_{\mathrm{B},\infty} := V^+_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{R} = V^+_{\mathrm{B}} \otimes_F F_\infty, \quad V_{\mathrm{dR},\infty} := V_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R} = V_{\mathrm{dR}} \otimes_F F_\infty.$$

Moreover, write $t(\mathcal{M})_\infty$ for $t(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{R} = t(\mathcal{M}) \otimes_F F_\infty$. The *period map* is the isomorphism

$$\alpha_{\mathcal{M}} : V^+_{\mathrm{B},\infty} \xrightarrow{\simeq} t(\mathcal{M})_\infty \tag{2.16}$$

of free F_∞-modules of rank 1 that is obtained by composing the natural inclusion $V^+_{\mathrm{B},\infty} \hookrightarrow (V_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C})^{\phi_\infty \otimes \tau = 1}$ with isomorphism $\mathrm{Comp}_{\mathrm{B,dR}}$ from (2.15) and the map $V_{\mathrm{dR},\infty} \to t(\mathcal{M})_\infty$ defined by tensoring with $\mathbb{R}$ over $\mathbb{Q}$ the canonical projection $V_{\mathrm{dR}} \twoheadrightarrow t(\mathcal{M})$.

Remark 2.7 In the language of motives, isomorphism (2.16) says that $\mathcal{M}$ is *critical* (see, *e.g.*, [158, §2.5]).

2.5.2 Embeddings and periods

Let Σ be the set of embeddings of F into $\mathbb{R}$ (equivalently, since F is totally real, into $\mathbb{C}$). For each $\sigma \in \Sigma$, let $f^\sigma(q) := \sum_{n\geq 1} a_n(f^\sigma)q^n \in S_k(\Gamma_0(N))$ be the newform of weight k and level N such that $a_n(f^\sigma) = \sigma\big(a_n(f)\big)$ for all $n \geq 1$ (the form f^σ is the *σ-conjugate* of f). Clearly, the Hecke field of f^σ is $\sigma(F)$. We write $\iota_F : F \hookrightarrow \mathbb{R}$ for the distinguished embedding corresponding to the inclusion $F \subset \mathbb{R}$.

Remark 2.8 There is a canonical isomorphism of $\mathbb{R}$-algebras $F_\infty \simeq \mathbb{R}^\Sigma = \prod_{\sigma\in\Sigma}(F \otimes_{F,\sigma} \mathbb{R})$ that induces an isomorphism of $\mathbb{C}$-algebras $F \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^\Sigma$. From here on, we shall usually not distinguish between F_∞ and $\mathbb{R}^\Sigma$; we consider the embedding

$$\iota_\Sigma : F \hookrightarrow F_\infty = \mathbb{R}^\Sigma, \quad x \longmapsto \big(\sigma(x)\big)_{\sigma\in\Sigma}$$

and identify any $x \in F$ with $\iota_\Sigma(x)$. With this convention in mind, when we say that an element $\alpha \in F_\infty$ belongs to F we really mean that there exists $x \in F$ such that $\iota_\Sigma(x) = \alpha$.

Observe that $t(\mathcal{M})_\infty$ is spanned by the image of T_{dR}. Fix $\gamma \in V^+_{\mathrm{B}} \smallsetminus \{0\}$.

Definition 2.9 The *period of f relative to γ* is the determinant $\Omega^{(\gamma)}_\infty$ of $\alpha_{\mathcal{M}}$ computed with respect to the basis $\{\gamma\}$ of $V^+_{\mathrm{B},\infty}$ and the image of ω_f in $t(\mathcal{M})_\infty$.

Remark 2.10 For our later purposes of investigating the p-part of the Tamagawa number conjecture for $\mathcal{M}$, an integral choice $\gamma \in T^+_{\mathrm{B}} \smallsetminus \{0\}$, which is in line with an analogous choice in [38, §4], will be preferable.

2.6 Motivic cohomology

We introduce (the first two groups of) motivic cohomology in a concise, utilitarian way, following [21, §3.1]; more precisely, with the (standard) notation of [21], we define motivic cohomology for the motive $h^{k-1}(X)(k/2)$. We remark that Kuga–Sato varieties do not possess, in general, proper, flat, regular models over $\mathbb{Z}$. Therefore, the broader definition

of motivic cohomology in low degrees that is given, in terms of (eigenspaces for Adams operators on) K-theory groups of varieties and of their integral models, in [77, Definition 1.19] does not apply in our case. When the relevant varieties do not admit such a "nice" integral model, (integral) motivic cohomology is defined in [142] by combining K-theory with the theory of alterations in the sense of de Jong [32]. It would be interesting to compare our definition with the one in [142], but in this paper we do not pursue this matter any further.

2.6.1 Motivic cohomology of $\mathcal{M}$

Let K be a number field. Denote by $\mathrm{CH}_0^{k/2}(X/K)$ the abelian group of codimension $k/2$ homologically trivial cycles on X defined over K. Let us write $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ for the subgroup of $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K)$ consisting of (the images of) the classes of those cycles that admit an integral model having trivial intersection with all the cycles of dimension k supported on special fibers; here $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K)$, and hence $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ too, should be viewed as a subgroup of $\mathrm{CH}_0^{k/2}(X/K)\otimes_{\mathbb{Z}}\mathbb{Q}$ (or, rather, should be identified with its natural image in $\mathrm{CH}_0^{k/2}(X/K)\otimes_{\mathbb{Z}}\mathbb{Q}$).

For any $\mathbb{Z}$-algebra R and $\star \in \{0, \mathrm{arith}\}$, set

$$\mathrm{CH}^{k/2}(X/K)_R := \mathrm{CH}^{k/2}(X/K)\otimes_{\mathbb{Z}} R, \quad \mathrm{CH}^{k/2}_\star(X/K)_R := \mathrm{CH}^{k/2}_\star(X/K)\otimes_{\mathbb{Z}} R. \tag{2.17}$$

With notation as in (2.4) and $i \in \{0,1\}$, let us define (following, *e.g.*, Burns–Flach ([21, §3.1]); *cf.* also Remark 2.11) the *i-th motivic cohomology group of $\mathcal{M}$ over K* as

$$H^i_{\mathrm{mot}}(K,\mathcal{M}) := \begin{cases} 0 & \text{if } i=0, \\ \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_F[\theta_f] & \text{if } i=1. \end{cases} \tag{2.18}$$

As these are the only motivic cohomology groups that play a role in our paper, we do not introduce $H^i_{\mathrm{mot}}(K,\mathcal{M})$ for $i \geq 2$.

Remark 2.11 Since we are considering null-homologous cycles only, every cycle (class) in $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ has trivial image in $H^k\big(X(\mathbb{C}),\mathbb{C}\big)$; in other words, the cycles in $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ satisfy conditions (a) and (b) in [165, §1.3], so there is a Gillet–Soulé height on $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ (*cf.* Sect. 2.7). This is the main reason for considering, in this article, $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$ instead of the larger group $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K)$. Notice, however, that, according to a conjecture of Beilinson ([4, Conjecture 2.2.5]), the $\mathbb{Q}$-vector spaces $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_{\mathbb{Q}}$ and $\mathrm{CH}_0^{k/2}(X/K)_{\mathbb{Q}}$ are expected to coincide: the reader is referred to the discussion in [141, §4.2] and also to [165, §1.3]. This justifies (at least conjecturally) our definition of $H^i_{\mathrm{mot}}(K,\mathcal{M})$ for $i \in \{0,1\}$ and the consistency of our Conjecture 2.50 on the bijectivity of the p-adic regulator map with analogous conjectures that are usually stated for $\mathrm{CH}_0^{k/2}(X/K)$ instead of $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$. See Remark 2.51 for further discussion.

2.6.2 A finiteness conjecture

The finiteness conjecture we are about to state is a special case of a classical conjecture predicting that Chow groups of smooth projective varieties over number fields are finitely generated (*cf.* Conjecture 2.14).

Conjecture 2.12 (Finiteness of H^1_{mot}) *$H^1_{\mathrm{mot}}(K,\mathcal{M})$ has finite dimension over F.*

The validity of Conjecture 2.12 is implied by assumption (3) in Sect. 1.1.2 (*cf.* Remark 2.52). In our route to the Tamagawa number conjecture for $\mathcal{M}$, in particular from here until Theorem 2.83, we shall assume that Conjecture 2.12 holds true, or even that the stronger condition (3) in Sect. 1.1.2 (which corresponds to the validity of Conjecture 2.43) is satisfied. Thus, we can give

Definition 2.13 The *algebraic rank of $\mathcal{M}$ over K* is $r_{\mathrm{alg}}(\mathcal{M}/K) := \dim_F\big(H^1_{\mathrm{mot}}(K,\mathcal{M})\big)$.

For simplicity, we also set $r_{\mathrm{alg}}(\mathcal{M}) := r_{\mathrm{alg}}(\mathcal{M}/\mathbb{Q})$ and call it the *algebraic rank of $\mathcal{M}$*.

2.6.3 Some remarks on Chow groups

We offer some motivation for Conjecture 2.12, which, as we pointed out in Sect. 2.6.2, is essentially a by-product of a more general conjecture on Chow groups of projective varieties over number fields.

Let Y be a smooth projective variety defined over a number field L. As usual, write $\mathrm{CH}^n(Y/L)$ for the Chow group of codimension n algebraic cycles on Y defined over L. The following well-known conjecture is wide open.

Conjecture 2.14 *The abelian groups $\mathrm{CH}^n(Y/L)$ are finitely generated for all $n \in \mathbb{N}$.*

As a consequence, the groups $\mathrm{CH}^n_0(Y/L)$ of codimension n homologically trivial cycles will be finitely generated as well. Clearly, Conjecture 2.14 is stronger than Conjecture 2.12.

Remark 2.15 The conjecture that the groups $\mathrm{CH}^n_0(Y/L)$ are finitely generated is attributed to Swinnerton-Dyer ([4, Conjecture 5.0]).

Remark 2.16 Little is known, as far as we are aware of, about Conjecture 2.14. From a broader point of view, it is a special case of the generalized (*i.e.*, motivic) Bass conjecture; for the convenience of the reader, we briefly recall why this is true. Let Y and L be as above. Let Θ be the finite set of primes of L at which Y has bad reduction (see, *e.g.*, [63, Proposition A.9.1.6, (i)]) and let $\mathcal{O}_{L,\Theta}$ be the ring of Θ-integers of L. By [63, Proposition A.9.1.6, (ii)]), there is a smooth model $\mathscr{Y}$ of Y over $\mathcal{O}_{L,\Theta}$. The ring $\mathcal{O}_{L,\Theta}$ is regular, so $\mathscr{Y}$ is regular. Since $\mathscr{Y}$ is a $\mathbb{Z}$-scheme of finite type, the motivic Bass conjecture (see, *e.g.*, [70, Conjecture 37, b)]) predicts that the motivic cohomology groups $H^i_{\mathscr{M}}\big(\mathscr{Y},\mathbb{Z}(n)\big)$ of Suslin–Voevodsky ([104, Lecture 3]) are finitely generated for all $i,n \in \mathbb{N}$. On the other hand, if $\mathrm{CH}^n(\mathscr{Y},i)$ denote, for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, Bloch's higher Chow groups [13], then

- $\mathrm{CH}^n(\mathscr{Y},0) = \mathrm{CH}^n(\mathscr{Y})$ for all $n \in \mathbb{N}$ ([13, p. 268]);
- $H^i_{\mathscr{M}}\big(\mathscr{Y},\mathbb{Z}(n)\big) \simeq \mathrm{CH}^n(\mathscr{Y},2n-i)$ for all $i,n \in \mathbb{N}$ ([104, Theorem 19.1]).

It follows that $H^{2n}_{\mathscr{M}}\big(\mathscr{Y},\mathbb{Z}(n)\big) \simeq \mathrm{CH}^n(\mathscr{Y})$, therefore the motivic Bass conjecture predicts, in particular, that the Chow groups $\mathrm{CH}^n(\mathscr{Y})$ are finitely generated for all $n \in \mathbb{N}$. Finally, taking scheme-theoretic closures of codimension n cycles on Y, one checks that for all $n \in \mathbb{N}$ the natural group homomorphism $\mathrm{CH}^n(\mathscr{Y}) \to \mathrm{CH}^n(Y/L)$ is surjective, which completes the argument.

2.7 The Gillet–Soulé height pairing for $\mathcal{M}$

We introduce the height pairing on our motivic cohomology groups and then define a regulator in terms of this pairing.

2.7.1 Gillet–Soulé height pairings

Let K be a number field. Denote by

$$\langle \cdot, \cdot \rangle_{\mathrm{GS}} : \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K) \times \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K) \longrightarrow \mathbb{R} \tag{2.19}$$

the height pairing defined by S.-W. Zhang ([165, §1.3]) using arithmetic intersection theory *à la* Gillet–Soulé [49]. For each $\sigma \in \Sigma$, pairing (2.19) yields an F-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{GS},\sigma} : H^1_{\mathrm{mot}}(K, \mathcal{M}) \times H^1_{\mathrm{mot}}(K, \mathcal{M}) \longrightarrow \mathbb{R} = F \otimes_{F,\sigma} \mathbb{R} \tag{2.20}$$

Let us consider the F_∞-module

$$\begin{aligned} H^1_{\mathrm{mot}}(K, \mathcal{M})_\infty &:= H^1_{\mathrm{mot}}(K, \mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{R} \\ &= H^1_{\mathrm{mot}}(K, \mathcal{M}) \otimes_F F_\infty = \prod_{\sigma \in \Sigma} H^1_{\mathrm{mot}}(K, \mathcal{M}) \otimes_{F,\sigma} \mathbb{R}, \end{aligned}$$

which is free of rank $r_{\mathrm{alg}}(\mathcal{M}/K)$. We can define an F_∞-bilinear height pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{GS},\infty} : H^1_{\mathrm{mot}}(K, \mathcal{M})_\infty \times H^1_{\mathrm{mot}}(K, \mathcal{M})_\infty \longrightarrow F_\infty \tag{2.21}$$

by the rule

$$\big((x_\sigma)_{\sigma\in\Sigma}, (y_\sigma)_{\sigma\in\Sigma}\big) \longmapsto \big(\langle x_\sigma, y_\sigma \rangle_{\mathrm{GS},\sigma}\big)_{\sigma\in\Sigma},$$

where $(x_\sigma)_{\sigma\in\Sigma}, (y_\sigma)_{\sigma\in\Sigma}$ belong to $H^1_{\mathrm{mot}}(K, \mathcal{M}) \otimes_{F,\sigma} \mathbb{R}$ and each pairing $\langle \cdot, \cdot \rangle_{\mathrm{GS},\sigma}$ in (2.20) has been extended $\mathbb{R}$-linearly over F (with respect to σ).

Remark 2.17 Given $x, y \in H^1_{\mathrm{mot}}(K, \mathcal{M})$ and $\sigma \in \Sigma$, it is easy to check that $\langle x, y \rangle_{\mathrm{GS},\iota_F} = 0$ if and only if $\langle x, y \rangle_{\mathrm{GS},\sigma} = 0$. It follows that $\langle \cdot, \cdot \rangle_{\mathrm{GS},\iota_F}$ is non-degenerate if and only if $\langle \cdot, \cdot \rangle_{\mathrm{GS},\sigma}$ is non-degenerate.

2.7.2 A non-degeneracy conjecture

The validity of the next conjecture is predicted by the arithmetic analogues of the standard conjectures proposed by Gillet–Soulé [50]; it is also a special case of general conjectures of Beilinson [4] and Bloch [12] on positive definiteness of height pairings.

Conjecture 2.18 (Non-degeneracy of height) *The pairing $\langle \cdot, \cdot \rangle_{\mathrm{GS},\infty}$ is non-degenerate.*

The validity of this conjecture corresponds to condition (1) from Sect. 1.1.2. In this paper, we often assume Conjecture 2.18 to hold true: this is imposed only to force the regulator $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$, which will be defined in Sect. 2.7.3, to be non-zero. Notice, however, that no condition of this sort will be needed in our main result on the p-part of the Tamagawa number conjecture for $\mathcal{M}$ (Theorem 5.42).

Remark 2.19 Of course, Conjecture 2.18 is true if and only if the $\mathbb{R}$-linear extension (with respect to σ) of $\langle \cdot, \cdot \rangle_{\mathrm{GS},\sigma}$ is non-degenerate for each $\sigma \in \Sigma$.

2.7.3 The $\mathscr{B}$-regulator of $\mathcal{M}$ over K

Set $r := r_{\mathrm{alg}}(\mathcal{M}/K)$. If $r > 0$, then fix a basis $\mathscr{B} = \{t_1, \dots, t_r\}$ of $H^1_{\mathrm{mot}}(K, \mathcal{M})$ over F; clearly, $\mathscr{B}$ is also a basis of $H^1_{\mathrm{mot}}(K, \mathcal{M})_\infty$ over F_∞.

Definition 2.20 If $r > 0$, then the *Gillet–Soulé $\mathscr{B}$-regulator of $\mathcal{M}$ over K* is

$$\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}/K) := \det\big(\langle t_i, t_j\rangle_{\mathrm{GS},\infty}\big)_{1\le i,j\le r} \in F_\infty.$$

If $r = 0$, then $\mathrm{Reg}(\mathcal{M}/K) := 1$.

For simplicity, we put $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) := \mathrm{Reg}_{\mathscr{B}}(\mathcal{M}/\mathbb{Q})$ and call it the *$\mathscr{B}$-regulator of $\mathcal{M}$*.

Remark 2.21 If Conjecture 2.18 holds true, then $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}/K) \in F_\infty^\times$.

Remark 2.22 If $\mathscr{B}$ and $\mathscr{B}'$ are two bases of $H^1_{\mathrm{mot}}(K, \mathcal{M})$ over F, then $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}/K)$ and $\mathrm{Reg}_{\mathscr{B}'}(\mathcal{M}/K)$ differ by multiplication by the square of the determinant of the transition matrix from $\mathscr{B}$ to $\mathscr{B}'$.

Assume $r > 0$. For each $\sigma \in \Sigma$, it is convenient to define

$$\mathrm{Reg}^\sigma_{\mathscr{B}}(\mathcal{M}) := \det\big(\langle t_i, t_j\rangle_{\mathrm{GS},\sigma}\big)_{1\le i,j\le r} \in \mathbb{R}, \tag{2.22}$$

so that we can write $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) = \big(\mathrm{Reg}^\sigma_{\mathscr{B}}(\mathcal{M})\big)_{\sigma\in\Sigma}$.

2.8 p-adic Galois representations

For every prime v of K, let K_v be the completion of K at v, fix an algebraic closure $\bar{K}_v$ of K_v and write $K_v^{\mathrm{unr}} \subset \bar{K}_v$ for the maximal unramified extension of K_v. Furthermore, set $G_{K_v} := \mathrm{Gal}(\bar{K}_v/K_v)$, denote by $I_v \subset G_{K_v}$ the inertia subgroup and let $\mathrm{Frob}_v \in \mathrm{Gal}(K_v^{\mathrm{unr}}/K_v) \simeq G_{K_v}/I_v$ be the arithmetic Frobenius. Finally, for every prime number p, fix a field embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

2.8.1 Dieudonné modules

Let v be a prime of K above p. If V is a p-adic representation of G_{K_v}, by which we mean that V is a finite-dimensional $\mathbb{Q}_p$-vector space that is equipped with a continuous action of G_{K_v}, then we denote by

$$\mathbf{D}_{\mathrm{cris}}(V) := H^0\big(G_{K_v}, V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}}\big)$$

and

$$\mathbf{D}_{\mathrm{dR}}(V) := H^0\big(G_{K_v}, V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}\big)$$

the crystalline and de Rham Dieudonné modules of V, where $\mathbf{B}_{\mathrm{cris}}$ (respectively, $\mathbf{B}_{\mathrm{dR}}$) is Fontaine's crystalline (respectively, de Rham) ring of periods ($\mathbf{B}_{\mathrm{dR}}$ is, in fact, a field). Recall that $\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with a filtration $\mathrm{Fil}^i\big(\mathbf{D}_{\mathrm{dR}}(V)\big)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ is endowed with a distinguished endomorphism ϕ, the so-called Frobenius endomorphism.

2.8.2 Tangent space

The *tangent space of* V is

$$t(V) := \mathbf{D}_{\mathrm{dR}}(V) \big/ \mathrm{Fil}^0\big(\mathbf{D}_{\mathrm{dR}}(V)\big), \tag{2.23}$$

which is a finite-dimensional $\mathbf{B}_{\mathrm{dR}}$-vector space.

Remark 2.23 Since $\mathbf{B}_{\mathrm{cris}}$ is a subring of $\mathbf{B}_{\mathrm{dR}}$, there is a natural injection $\mathbf{D}_{\mathrm{cris}}(V) \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V)$.

2.9 L-functions of $\mathcal{M}$

Let p be a prime number and let K be a number field. We can regard the $G_{\mathbb{Q}}$-representation V_p from (2.8) as a G_K-representation, and hence as a G_{K_v}-representation for every prime v of K.

2.9.1 Euler factors and the L-function

For every prime v of K, we define the Euler factor

$$L_v(V_p, x) := \begin{cases} \det_{F_p}\big(\mathrm{id} - \mathrm{Frob}_v^{-1}\, x,\, V_p^{I_v}\big) & \text{if } v \nmid p, \\ \det_{F_p}\big(\mathrm{id} - \phi x,\, \mathbf{D}_{\mathrm{cris}}(V_p)\big) & \text{if } v \mid p. \end{cases}$$

The polynomial $L_v(V_p, x)$ has coefficients in F and is independent of p. Denote by q_v the cardinality of the residue field of K_v and set

$$L_v(V_p, s) := L_v(V_p, q_v^{-s}).$$

Definition 2.24 The *L-function of* $\mathcal{M}$ *over* K is the formal Euler product

$$L(\mathcal{M}/K, s) := \prod_v L_v(V_p, s)^{-1},$$

where v varies over all primes of K.

As a shorthand, we write $L(\mathcal{M}, s)$ instead of $L(\mathcal{M}/\mathbb{Q}, s)$. As in [21, Remark 7], for every $s \in \mathbb{C}$ we regard $L_v(V_p, s)$ as an element of $F \otimes_{\mathbb{Q}} \mathbb{C}$. The function $L(\mathcal{M}/K, s)$ of the complex variable s admits a holomorphic continuation to $\mathbb{C}$, which takes values in $F \otimes_{\mathbb{Q}} \mathbb{C}$. Furthermore, by [21, Lemma 8], $L(\mathcal{M}/K, s) \in F_\infty$ if $s \in \mathbb{R}$.

Remark 2.25 Let $\mathcal{K}$ be a quadratic field, let $\epsilon_{\mathcal{K}}$ be the Dirichlet character attached to $\mathcal{K}$ and, as in Remark 2.3, denote by $f^{\mathcal{K}}$ the twist of f by $\epsilon_{\mathcal{K}}$, which is a newform satisfying $a_n(f^{\mathcal{K}}) = \epsilon_{\mathcal{K}}(n) \cdot a_n(f)$ for all $n \geq 1$. If $\mathcal{M}\big(f^{\mathcal{K}}\big)$ is the motive of $f^{\mathcal{K}}$, then

$$L(\mathcal{M}/\mathcal{K}, s) = L(\mathcal{M}, s) \cdot L\big(\mathcal{M}(f^{\mathcal{K}}), s\big).$$

This equality is a special case of a factorization that holds over any abelian extension of $\mathbb{Q}$.

2.9.2 The completed L-function

Let Γ be the classical complex Γ-function. The *completed L-function of* f is the complex-valued function

$$\Lambda(f,s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \cdot \Gamma(s) \cdot L(f,s).$$

It satisfies a functional equation (usually referred to as the functional equation for $L(f,s)$) of the form

$$\Lambda(f,s) = \varepsilon(f) \cdot \Lambda(f, k-s),$$

where $\varepsilon(f) \in \{\pm 1\}$ is the *root number* of f. In particular, $\Lambda(f,s)$ is holomorphic on (or, rather, can be holomorphically continued to) the whole complex plane, and the same is true of $L(f,s)$ (see, *e.g.*, [80, p. 141]). As a shorthand, put $L_\infty(f,s) := \left(\sqrt{N}/2\pi\right)^s \cdot \Gamma(s)$, so that $\Lambda(f,s) = L_\infty(f,s) \cdot L(f,s)$. We find it convenient to introduce the normalization of $\Lambda(f,s)$ given by

$$\tilde{\Lambda}(f,s) := \frac{\Lambda(f,s)}{\left(i\sqrt{N}\right)^{k/2}}. \tag{2.24}$$

It is well known that $\Gamma(n) = (n-1)!$ for every integer $n \geq 1$, so if $g^{(i)}$ denotes the i-th derivative of a complex function g, then there is an equality

$$\tilde{\Lambda}^{(r)}(f, k/2) = \frac{(k/2-1)! \cdot L^{(r)}(f, k/2)}{(2\pi i)^{k/2}},$$

where $r \in \mathbb{N}$ is the order of vanishing of $L(f,s)$ at $s = k/2$ (from Sect. 2.9.3 onwards, this integer will be called the analytic rank of f, *cf.* Definition 2.31). Our next goal is to define the completed L-function of $\mathcal{M}$; to do this, we introduce the archimedean factor

$$L_\infty(\mathcal{M}, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^{s+k/2} \cdot \frac{\Gamma(s+k/2)}{\left(i\sqrt{N}\right)^{k/2}} = \frac{\left(i\sqrt{N}\right)^s \cdot \Gamma(s+k/2)}{(2\pi i)^{s+k/2}},$$

where the equality on the right follows from a trivial computation.

Definition 2.26 The *completed L-function of* $\mathcal{M}$ *over* $\mathbb{Q}$ is

$$\Lambda(\mathcal{M}, s) := L_\infty(\mathcal{M}, s) \cdot L(\mathcal{M}, s).$$

This is the L-function in terms of which we shall prove our main results. Like $L(\mathcal{M}, s)$, the completed L-function $\Lambda(\mathcal{M}, s)$ has an $F \otimes_{\mathbb{Q}} \mathbb{C}$-valued holomorphic continuation to $\mathbb{C}$; moreover, $\Lambda(\mathcal{M}, s) \in F_\infty$ for all $s \in \mathbb{R}$. In particular, there is an equality

$$\Lambda^{(r)}(\mathcal{M}, 0) = \frac{(k/2-1)! \cdot L^{(r)}(\mathcal{M}, 0)}{(2\pi i)^{k/2}},$$

where now $r \in \mathbb{N}$ is the order of vanishing of $L(\mathcal{M}, s)$ at $s = 0$.

Remark 2.27 By carefully keeping track of Γ-factors, one can define $\Lambda(\mathcal{M}/\mathcal{K}, s)$ for any number field $\mathcal{K}$. Since we will have no use for it, we refrain from introducing this more general notion. Rather, in the special case where $\mathcal{K}$ is a quadratic field we set

$$\Lambda(\mathcal{M}/\mathcal{K}, s) := \Lambda(\mathcal{M}, s) \cdot \Lambda\big(\mathcal{M}(f^{\mathcal{K}}), s\big).$$

This equality, which we take as the definition of the left-hand term, is in fact a special case of a factorization that holds over any abelian extension of $\mathbb{Q}$ (*cf.* Remark 2.25).

2.9.3 Analytic ranks and leading terms

Recall the L-functions $L(\mathcal{M}/K, s)$ and $\Lambda(\mathcal{M}, s)$ from Definitions 2.24 and 2.26, respectively.

Definition 2.28 (1) The *analytic rank* $r_{\mathrm{an}}(\mathcal{M}/K)$ *of* $\mathcal{M}$ *over* K is the order of vanishing of $L(\mathcal{M}/K, s)$ at $s = 0$, *i.e.*, $r_{\mathrm{an}}(\mathcal{M}/K) := \mathrm{ord}_{s=0} L(\mathcal{M}/K, s)$.

(2) The *leading term* $L^*(\mathcal{M}/K, 0)$ *of* $L(\mathcal{M}/K, s)$ *at* $s = 0$ is the leading term of the Taylor expansion of $L(\mathcal{M}/K, s)$ at $s = 0$, *i.e.*, $L^*(\mathcal{M}/K, 0) := \lim_{s\to 0} s^{-r_{\mathrm{an}}(\mathcal{M}/K)} L(\mathcal{M}/K, s)$.

(3) The *leading term* $\Lambda^*(\mathcal{M}, 0)$ *of* $\Lambda(\mathcal{M}, s)$ *at* $s = 0$ is the leading term of the Taylor expansion of $\Lambda(\mathcal{M}, s)$ at $s = 0$, *i.e.*, $\Lambda^*(\mathcal{M}, 0) := \lim_{s\to 0} s^{-r_{\mathrm{an}}(\mathcal{M}/\mathbb{Q})} \Lambda(\mathcal{M}, s)$.

Observe that $L_\infty(\mathcal{M}, s)$ has neither a zero nor a pole at $s = 0$, so the orders of vanishing of $L(\mathcal{M}, s)$ and $\Lambda(\mathcal{M}, s)$ at $s = 0$ are equal: this justifies part (1) of Definition 2.28.

In line with notation that was introduced earlier, we set $L^*(\mathcal{M}, 0) := L^*(\mathcal{M}/\mathbb{Q}, 0)$ and $r_{\mathrm{an}}(\mathcal{M}) := r_{\mathrm{an}}(\mathcal{M}/\mathbb{Q})$. Therefore, there is an equality

$$\Lambda^*(\mathcal{M}, 0) = \frac{(k/2-1)! \cdot L^*(\mathcal{M}, 0)}{(2\pi i)^{k/2}} \tag{2.25}$$

that establishes a relation between the leading terms of $L(\mathcal{M}, s)$ and $\Lambda(\mathcal{M}, s)$.

Remark 2.29 For each $\sigma \in \Sigma$, write $L^*(f^\sigma, k/2)$ for the leading term of $L(f^\sigma, s)$ at $s = k/2$, where $L(f^\sigma, s)$ is the L-function of f^σ. Under the identification of Remark 2.8, the leading term $L^*(\mathcal{M}, 0)$ corresponds to $\big(L^*(f^\sigma, k/2)\big)_{\sigma\in\Sigma}$. It follows that $L^*(\mathcal{M}/K, 0) \in F_\infty^\times$.

A similar remark applies to $\Lambda^*(\mathcal{M}, 0)$. More precisely, denote by $\tilde{\Lambda}^*(f^\sigma, k/2)$ the leading term of $\tilde{\Lambda}(f^\sigma, s)$ at $s = k/2$, where $\tilde{\Lambda}(f^\sigma, s)$ is the normalized completed L-function of f^σ as in (2.24). Then $\Lambda^*(\mathcal{M}, 0)$ corresponds to $\big(\tilde{\Lambda}^*(f^\sigma, k/2)\big)_{\sigma\in\Sigma}$ and $\Lambda^*(\mathcal{M}, 0) \in F_\infty^\times$.

Recall the algebraic rank $r_{\mathrm{alg}}(\mathcal{M}/K)$ from Definition 2.13. The following conjecture can be seen as the rank part of the Beilinson–Bloch–Kato conjecture for the motive $\mathcal{M}$ over K.

Conjecture 2.30 (Equality of ranks) $r_{\mathrm{an}}(\mathcal{M}/K) = r_{\mathrm{alg}}(\mathcal{M}/K)$.

Quite generally, let g be an eigenform of weight k and level $\Gamma_0(N)$; as usual, let $L(g, s)$ be the (complex) L-function of g.

Definition 2.31 The *analytic rank of* g is $r_{\mathrm{an}}(g) := \mathrm{ord}_{s=\frac{k}{2}} L(g, s) \in \mathbb{N}$.

In an analogous way, given a number field K, one can define the *analytic rank* $r_{\mathrm{an}}(f/K)$ *of* g *over* K. As in Sect. 2.9.2, if g is a newform, then write $\varepsilon(g) \in \{\pm 1\}$ for the root number

of g, *i.e.*, the sign of the functional equation for $L(g, s)$. The root number controls the parity of $r_{\mathrm{an}}(g)$, in the sense that $\varepsilon(g) = (-1)^{r_{\mathrm{an}}(g)}$. Equivalently, there is a congruence

$$r_{\mathrm{an}}(g) \equiv \frac{1-\varepsilon(g)}{2} \pmod 2. \tag{2.26}$$

With notation as in Sect. 2.5, recall the σ-conjugate f^σ of f.

Remark 2.32 Under the isomorphism in Remark 2.8, the $F \otimes_{\mathbb{Q}} \mathbb{C}$-valued function $L(\mathcal{M}, s)$ corresponds to the $\mathbb{C}^\Sigma$-valued function $\big(L(f^\sigma, s+k/2)\big)_{\sigma\in\Sigma}$ of the complex variable s. This is the point of view of Deligne in [34, §2.2] (*cf.* also [123, §1]); it offers, in particular, a more explicit interpretation of the analytic rank $r_{\mathrm{an}}(\mathcal{M})$ from part (2) of Definition 2.28. Namely, there is an equality

$$r_{\mathrm{an}}(\mathcal{M}) = \min\big\{r_{\mathrm{an}}(f^\sigma) \mid \sigma \in \Sigma\big\}. \tag{2.27}$$

It was conjectured by Deligne and Gross ([34, Conjecture 2.7, (ii)]) that $r_{\mathrm{an}}(f^\sigma)$ is constant as σ varies in Σ (which, if true, would imply that $r_{\mathrm{an}}(\mathcal{M}) = r_{\mathrm{an}}(f)$), but we will not need this property in the rest of this paper. When $r_{\mathrm{an}}(f) \le 1$, this conjecture was proved by Gross–Zagier (in weight 2, see [55, Ch. V, Corollary 1.3]) and Zhang (in higher weight, see [165, Corollary 0.3.5]), while for $r_{\mathrm{an}}(f) > 1$ only very partial results in this direction are available (*cf.* [55] and [165]). Later on, it will sometimes be convenient to identify $L(\mathcal{M}, s)$ with $\big(L(f^\sigma, s+k/2)\big)_{\sigma\in\Sigma}$ and view $L(\mathcal{M}, s)$ as taking values in $\mathbb{C}^\Sigma$.

Remark 2.33 With notation as in Sect. 2.9.2, $L_\infty(f, k/2) \neq 0$, so $r_{\mathrm{an}}(f)$ is equal to the order of vanishing of $\Lambda(f, s)$ at $s = k/2$.

For later use, we record an auxiliary result.

Lemma 2.34 *Let $r \in \{0, 1\}$. If $r_{\mathrm{an}}(\mathcal{M}) = r$, then $r_{\mathrm{an}}(f^\sigma) = r$ for all $\sigma \in \Sigma$. Conversely, if $r_{\mathrm{an}}(f^\sigma) = r$ for some $\sigma \in \Sigma$, then $r_{\mathrm{an}}(\mathcal{M}) = r$.*

Proof In light of Remark 2.32, the lemma follows from [165, Corollary 0.3.5]. □

2.9.4 Leading term and periods

In the statement below, which should be interpreted as explained in Remark 2.8, the period $\Omega_\infty^{(\star)}$ is the one defined in Sect. 2.5.2.

Lemma 2.35 *Let $\gamma, \gamma \in V_{\mathrm{B}}^+ \smallsetminus \{0\}$ and let $\mathscr{B}, \mathscr{B}'$ be bases of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ over F. Then $L^*(\mathcal{M}, 0)\big/\big(\Omega_\infty^{(\gamma)} \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})\big) \in F^\times$ if and only if $L^*(\mathcal{M}, 0)\big/\big(\Omega_\infty^{(\gamma')} \cdot \mathrm{Reg}_{\mathscr{B}'}(\mathcal{M})\big) \in F^\times$.*

Proof Since the F-vector space V_{B}^+ is 1-dimensional, γ and γ' differ by multiplication by an element of $F^\times$, and then the same is true of the periods $\Omega_\infty^{(\gamma)}$ and $\Omega_\infty^{(\gamma')}$. On the other hand, as was pointed out in Remark 2.22, the regulators $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$ and $\mathrm{Reg}_{\mathscr{B}'}(\mathcal{M})$ differ by multiplication by an element of $F^\times$ as well, and the lemma is proved. □

2.10 The fundamental line of $\mathcal{M}$

We introduce the "fundamental line" of $\mathcal{M}$ in the formulation of Fontaine–Perrin-Riou [43], using the theory of determinants described in Appendix A, to which the reader is

referred for details. In this case, the projective modules that play a role are vector spaces over F.

In the definition that follows, V_B^+ is the F-subspace of V_B from (2.6), $t(\mathcal{M})$ is the tangent space of $\mathcal{M}$ introduced in (2.12) and $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ is the cohomology group from (2.18).

Definition 2.36 The *fundamental line of* $\mathcal{M}$ is

$$\Delta(\mathcal{M}) := \mathrm{Det}_F^{-1}\big(H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})\big) \cdot \mathrm{Det}_F\big(H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})^*\big) \cdot \mathrm{Det}_F\big(t(\mathcal{M})\big) \cdot \mathrm{Det}_F^{-1}\big(V_{\mathrm{B}}^+\big).$$

Note that, by construction, the F-vector space underlying $\Delta(\mathcal{M})$ is 1-dimensional.

Remark 2.37 In order to compare Definition 2.36 with [77, Definition 2.4], recall from (2.18) that $H^0_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) = 0$ and keep in mind that $\mathcal{M}^\vee(1) \simeq \mathcal{M}$.

Remark 2.38 As a motivation for introducing $\Delta(\mathcal{M})$, it may be helpful to consider the *fundamental exact sequence* of Fontaine–Perrin-Riou for a general motive $\mathscr{M}$ (see, *e.g.*, [77, Conjecture 5] and [158, Conjecture 3.3]). This is a conjectural exact sequence that takes place in the archimedean setting (as opposed to a p-adic setting) and formulates a deep relation between motivic cohomology and the kernel and the cokernel of the analogue for $\mathscr{M}$ of the period map $\alpha_{\mathcal{M}}$ from (2.16). In our case, the period map is an isomorphism, so the existence of such an exact sequence for $\mathcal{M}$ is obvious. See, *e.g.*, [42], [43], [77], [158] for details.

2.11 Rationality conjecture

For now, let us assume that

- Conjecture 2.18 holds true.

Define the $\mathbb{R}$-vector space

$$\Delta(\mathcal{M})_\infty := \Delta(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{R};$$

then $\Delta(\mathcal{M})_\infty \simeq \Delta(\mathcal{M}) \otimes_F F_\infty$ is a free F_∞-module of rank 1. Conjecture 2.18, which we are assuming, ensures that the Gillet–Soulé height pairing $\langle \cdot, \cdot \rangle_{\mathrm{GS},\infty}$ from (2.21) induces an isomorphism of F_∞-modules

$$H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_\infty \xrightarrow{\simeq} H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})^*_\infty.$$

Combining the base change formula (A.7) for determinants, the multiplicativity (A.2) of determinants in short exact sequences and isomorphism (2.16), we obtain an isomorphism

$$\theta_\infty : \Delta(\mathcal{M})_\infty \xrightarrow{\simeq} (F_\infty, 0) \tag{2.28}$$

of F_∞-modules.

Remark 2.39 If $r_{\mathrm{alg}}(\mathcal{M}) = 0$, then $\Delta(\mathcal{M}) = \mathrm{Det}_F\big(t(\mathcal{M})\big) \cdot \mathrm{Det}_F^{-1}\big(V_{\mathrm{B}}^+\big)$. Therefore, in this case θ_∞ reduces to the isomorphism $\alpha_{\mathcal{M}}$ from (2.16), which is induced by the comparison isomorphism $\mathrm{Comp}_{\mathrm{B,dR}}$ in (2.15). In other words, when $r_{\mathrm{alg}}(\mathcal{M}) = 0$ we are using the Betti–de Rham comparison isomorphism to trivialize the fundamental line of $\mathcal{M}$.

2.11.1 Rationality conjecture

Let θ_∞ be the isomorphism of F_∞-modules in (2.28). The following conjecture is essentially due to Beilinson [3] and Deligne ([34, Conjecture 1.8]).

Conjecture 2.40 (Rationality conjecture) *There exists $\zeta_f \in \Delta(\mathcal{M})$ such that the equality*

$$\theta_\infty(\zeta_f) = L^*(\mathcal{M}, 0)^{-1}$$

holds in $F_\infty^\times$.

The element ζ_f is called a *zeta element* and $\{\zeta_f\}$ is, of course, a basis of $\Delta(\mathcal{M})$ over F. Moreover, since θ_∞ is an isomorphism, such a ζ_f is unique if it exists. Under some technical conditions, later in this paper we will prove Conjecture 2.40 when $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$ (Theorems 5.24 and 5.37).

2.11.2 A variant of the rationality conjecture

Now we offer an alternative formulation of Conjecture 2.40 that involves the completed L-function $\Lambda(\mathcal{M}, s)$. In light of equality (2.25), a straightforward computation shows that Conjecture 2.40 is equivalent to

Conjecture 2.41 (Rationality conjecture, second version) *There exists $\zeta_f^* \in \Delta(\mathcal{M})$ such that the equality*

$$\theta_\infty(\zeta_f^*) = \Big((2\pi i)^{k/2}\Lambda^*(\mathcal{M}, 0)\Big)^{-1}$$

holds in $F_\infty^\times$.

One can switch between Conjecture 2.40 and Conjecture 2.41 by means of the relation $\zeta_f^* = \zeta_f \big/ (k/2-1)!$.

2.11.3 A reformulation of the rationality conjecture

The term "rationality" in Conjecture 2.40 is justified by the reformulation below, which involves the leading term $L^*(\mathcal{M}, 0)$ and the Gillet–Soulé $\mathscr{B}$-regulator $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$.

Remark 2.42 Let R be a ring, let M be a free R-module of finite rank, say r, and write $M^* := \mathrm{Hom}_R(M, R)$ for the R-linear dual of M. Let $\langle\cdot,\cdot\rangle : M \times M \to R$ be an R-bilinear pairing and let $f : M \to M^*$ be the R-linear map given by $t \mapsto \langle t, \cdot\rangle$. Choose a basis $\mathscr{B} = \{t_1, \dots, t_r\}$ of M over R, let $\mathscr{B}^*$ be the dual basis of M^*, set $A := \big(\langle t_i, t_j\rangle\big)_{1\le i,j\le r}$ and denote by $\det(f)_{\mathscr{B}^*}^{\mathscr{B}}$ the determinant of f computed with respect to $\mathscr{B}$ and $\mathscr{B}^*$. A straightforward calculation shows that $\det(f)_{\mathscr{B}^*}^{\mathscr{B}} = \det(A)$.

Like Lemma 2.35, the result we are about to state should be understood in terms of the embedding ι_Σ, as explained in Remark 2.8.

Proposition 2.43 *Conjecture 2.40 is equivalent to $L^*(\mathcal{M}, 0)\big/\big(\Omega_\infty^{(\gamma)} \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})\big) \in F^\times$ for all $\gamma \in V_{\mathrm{B}}^+ \smallsetminus \{0\}$ and all bases $\mathscr{B}$ of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ over F.*

Proof Thanks to Lemma 2.35, it is enough to prove the claim for fixed γ and $\mathscr{B}$ as above. Thus, let $\mathscr{B} = \{t_1, \dots, t_r\}$ be a basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ over F, where $r = r_{\mathrm{alg}}(\mathcal{M})$, and let

$\mathscr{B}^* = \{t_1^*, \dots, t_r^*\}$ be the dual basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})^*$. Define

$$\underline{t}_{\mathscr{B}} := t_1 \wedge \cdots \wedge t_r, \quad \underline{t}^*_{\mathscr{B}} := t_1^* \wedge \cdots \wedge t_r^*,$$

so that $\{\underline{t}_{\mathscr{B}}\}$ and $\{\underline{t}^*_{\mathscr{B}}\}$ are F-bases of $\bigwedge^r H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ and $\bigwedge^r H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})^*$, respectively. Pick $\gamma \in V_{\mathrm{B}}^+ \smallsetminus \{0\}$ and set

$$\zeta^{\gamma}_{\mathscr{B}} := \underline{t}^{-1}_{\mathscr{B}} \otimes \underline{t}^*_{\mathscr{B}} \otimes \gamma^{-1} \otimes \omega_f,$$

where $\omega_f \in t(\mathcal{M})_\infty$ is the differential form in (2.14). Then $\{\zeta^{\gamma}_{\mathscr{B}}\}$ is a basis of $\Delta(\mathcal{M})$ over F and, in light of Remark 2.42, there is an equality

$$\theta_\infty\big(\zeta^{\gamma}_{\mathscr{B}}\big) = \big(\Omega_\infty^{(\gamma)} \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})\big)^{-1} \tag{2.29}$$

(see, *e.g.*, [77, Example 1.30] for the computation of determinants). On the other hand, since any ζ_f as in Conjecture 2.40 differs from $\zeta^{\gamma}_{\mathscr{B}}$ by multiplication by an element of $F^\times$, Conjecture 2.40 is equivalent to the assertion that

$$a\theta_\infty\big(\zeta^{\gamma}_{\mathscr{B}}\big) = L^*(\mathcal{M}, 0)^{-1} \tag{2.30}$$

for some $a \in F^\times$. The desired result follows by combining (2.29) and (2.30). □

Since Conjectures 2.40 and 2.41 are equivalent, Proposition 2.43 offers a reformulation of Conjecture 2.41 as well.

Remark 2.44 When $r_{\mathrm{alg}}(\mathcal{M}) = 0$, the regulator of $\mathcal{M}$ is equal to 1 and Proposition 2.43 is equivalent to well-known rationality results due to Shimura. More precisely, Shimura's periods $u_\sigma^\pm \in \mathbb{C}^\times$, introduced in [148] and [149], are well defined up to scalar multiples in $F^\times$. They may be chosen to satisfy the following key property: $\Lambda(f^\sigma, s)/u_\sigma^\pm \in \sigma(F)$ and $\sigma\big(\Lambda(f, s)/u^\pm\big) = \Lambda(f^\sigma, s)/u_\sigma^\pm$ for all $\sigma \in \Sigma$, where we set $u^\pm := u^\pm_{\iota_F}$. The reader is referred especially to [149, Theorem 1] for more details on the various additional properties of these periods. Later in this paper, a specific choice of the periods $u_\sigma^\pm$ (to be denoted by $\Omega^\pm_{f^\sigma,\Gamma_0(N)}$ in Section 5) will be made using modular symbols (see, in particular, Sect. 5.3.1; *cf.* also Remark 5.13). A comparison between these periods and the period $\Omega_\infty^{(\gamma)}$ in Proposition 2.43 is carried out in Proposition 5.14 for a suitable choice of γ.

2.12 Local Galois cohomology

Let K be a number field, p a prime number, v a place of K and V a p-adic representation of G_{K_v}. For a continuous G_{K_v}-module M we write

$$\mathbf{R}\Gamma(K_v, M) := \mathcal{C}^\bullet(G_{K_v}, M)$$

for the complex of continuous cochains of G_{K_v} with values in M. Let $t(V)$ be as in (2.23), keep Remark 2.23 in mind and consider the complex

$$\mathbf{R}\Gamma_f(K_v, V) := \begin{cases} \Big(\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{(1-\phi,\mathrm{pr})} \mathbf{D}_{\mathrm{cris}}(V) \oplus t(V)\Big) & \text{if } v \mid p, \\ \mathbf{R}\Gamma(K_v, V) & \text{if } v \mid \infty, \\ \Big(V^{I_v} \xrightarrow{1-\mathrm{Frob}_v} V^{I_v}\Big) & \text{if } v \nmid p\infty, \end{cases}$$

where ϕ is, as above, the Frobenius of $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathrm{pr} : \mathbf{D}_{\mathrm{cris}}(V) \to t(V)$ is the canonical map (*cf.* Remark 2.23). Note that if $v \nmid \infty$, then $\mathbf{R}\Gamma_f(K_v, V)$ is concentrated in degrees 0 and 1. Denote by $H^\bullet_f(K_v, V)$ the cohomology of $\mathbf{R}\Gamma_f(K_v, V)$. In particular, if $v \nmid p\infty$, then

$$H^0_f(K_v, V) = H^0(K_v, V) \tag{2.31}$$

and

$$H^1_f(K_v, V) = H^1_{\mathrm{unr}}(K_v, V) := H^1\big(\mathrm{Gal}(K_v^{\mathrm{unr}}/K_v), V^{I_v}\big). \tag{2.32}$$

We also set

$$H^1_s(K_v, V) := H^1(K_v, V)\big/H^1_f(K_v, V)$$

and call it the *singular part of V at v*. The complex $\mathbf{R}\Gamma_f(K_v, V)$ is quasi-isomorphic to a subcomplex of the complex $\mathbf{R}\Gamma(K_v, V)$, and we define $\mathbf{R}\Gamma_s(K_v, V)$ to be the cokernel of the corresponding inclusion map.

Remark 2.45 If V is a p-adic representation of G_K, then we call $H^j_f(K_v, V)$ the *j-th finite cohomology group of V at v*.

2.13 Global Galois cohomology

Let K be a number field, write $\mathscr{P}_K$ for the set of (archimedean and non-archimedean) primes of K and let p be a prime number. The set

$$S := \big\{v \in \mathscr{P}_K \mid v \text{ divides } p\infty\big\} \cup \big\{v \in \mathscr{P}_K \mid V_p \text{ is ramified at } v\big\} \tag{2.33}$$

is clearly finite. Let us write $G_{K,S}$ for the Galois group over K of the maximal extension of K unramified outside S. Finally, for any continuous $G_{K,S}$-module M denote by

$$\mathbf{R}\Gamma(G_{K,S}, M) := \mathcal{C}^\bullet(G_{K,S}, M)$$

the complex of continuous cochains of $G_{K,S}$ with values in M.

Remark 2.46 For our later arguments, it would be equally fine to fix, in place of the set S defined in (2.33), any subset of $\mathscr{P}_K$ containing S.

2.13.1 Finite cohomology

The *finite complex of V_p* is

$$\mathbf{R}\Gamma_f(K, V_p) := \mathrm{Cone}\left(\mathbf{R}\Gamma(G_{K,S}, V_p) \longrightarrow \bigoplus_{v\in S} \mathbf{R}\Gamma_s(K_v, V_p)\right)[-1].$$

We denote by $H^\bullet_f(K, V_p)$ the cohomology of $\mathbf{R}\Gamma_f(K, V_p)$ and call it the *finite* (or *unramified*) *cohomology of V_p* over K. In particular, by [21, Lemma 19], we have $H^j_f(\mathbb{Q}, V_p) = 0$ for $j \notin \{0, 1, 2, 3\}$ and there are isomorphisms

$$H^j_f(\mathbb{Q}, V_p) \simeq H^{3-j}_f(\mathbb{Q}, V_p)^*, \tag{2.34}$$

where, as before, $(\cdot)^*$ denotes the $\mathbb{Q}_p$-linear dual (note that we are implicitly using the fact that $V_p \simeq V_p^*(1)$).

Remark 2.47 The reader can see, *e.g.*, [119] for a definition of cohomological functors that extend the definition of H^1_f of Bloch–Kato for crystalline p-adic representations of a local or global Galois group.

2.13.2 Cohomology with compact support

We introduce cohomology with compact support only for $K = \mathbb{Q}$; for simplicity, let us set $G_S := G_{\mathbb{Q},S}$. See, *e.g.*, [114, §5.3] for the case of a general global field.

Let M be a continuous G_S-module. The *compact complex of* M is

$$\mathbf{R}\Gamma_c(G_S, M) = \mathrm{Cone}\left(\mathbf{R}\Gamma(G_S, M) \longrightarrow \bigoplus_{v\in S} \mathbf{R}\Gamma(\mathbb{Q}_v, M)\right)[-1].$$

We denote by $H^\bullet_c(\mathbb{Q}, M)$ the cohomology of $\mathbf{R}\Gamma_c(G_S, M)$ and call it the *cohomology with compact support of* M. As explained, *e.g.*, in [21, §3.2], there is a short exact sequence of complexes

$$0 \longrightarrow \mathbf{R}\Gamma_c(G_S, V_p) \longrightarrow \mathbf{R}\Gamma_f(\mathbb{Q}, V_p) \longrightarrow \bigoplus_{v\in S} \mathbf{R}\Gamma_f(\mathbb{Q}_v, V_p) \longrightarrow 0 \tag{2.35}$$

(this is called a "true triangle" by Burns–Flach in [21]).

2.14 The p-adic étale regulator of $\mathcal{M}$

We introduce the p-adic étale regulator of the modular motive $\mathcal{M}$ over a number field.

2.14.1 Anaemic splittings in étale cohomology

Let p be a prime number, fix a prime $\mathfrak{p}$ of F above p and let $\iota_\mathfrak{p} : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ be an embedding that induces $\mathfrak{p}$. Let us define

$$W_{\bar{\mathbb{Q}}_p} := \Pi_B \Pi_\epsilon \cdot H^{k-1}_{\text{ét}}\big(\bar{X}, \bar{\mathbb{Q}}_p(k/2)\big).$$

If we set $\theta_\mathfrak{p} := \iota_\mathfrak{p} \circ \theta$ and let θ range over all homomorphisms $\theta : \mathfrak{H}_k(\Gamma_0(N))_{\bar{\mathbb{Q}}} \to \bar{\mathbb{Q}}$ of $\bar{\mathbb{Q}}$-algebras, then $\theta_\mathfrak{p}$ varies over all homomorphisms $\mathfrak{H}_k(\Gamma_0(N))_{\bar{\mathbb{Q}}} \to \bar{\mathbb{Q}}_p$ of $\bar{\mathbb{Q}}$-algebras. It follows that there is an "anaemic" splitting

$$W_{\bar{\mathbb{Q}}_p} = \bigoplus_\theta W_{\bar{\mathbb{Q}}_p}[\theta_\mathfrak{p}], \tag{2.36}$$

where $W_{\bar{\mathbb{Q}}_p}[\theta_\mathfrak{p}]$ is the $\theta_\mathfrak{p}$-eigenspace of $W_{\bar{\mathbb{Q}}_p}$ under the action of $\mathfrak{H}_k(\Gamma_0(N))_{\bar{\mathbb{Q}}}$. Recall from Sect. 2.4.1 that the ring homomorphism θ_f, which arises as a map $\mathfrak{H}_k(\Gamma_0(N)) \to \mathcal{O}_F$, can also be viewed as a map $\mathfrak{H}_k(\Gamma(N)) \to \mathcal{O}_F$ via the surjection $\mathfrak{H}_k(\Gamma(N)) \twoheadrightarrow \mathfrak{H}_k(\Gamma_0(N))$. Set

$$W_\mathfrak{p} := \Pi_B \Pi_\epsilon \cdot H^{k-1}_{\text{ét}}\big(\bar{X}, F_\mathfrak{p}(k/2)\big),$$

so that $V_\mathfrak{p} = W_\mathfrak{p}[\theta_f]$. By a slight abuse of notation, we adopt the same symbol for θ_f and $\iota_\mathfrak{p} \circ \theta_f$, which allows us to view $W_{\bar{\mathbb{Q}}_p}[\theta_f]$ as one of the direct summands appearing in (2.36).

There is a canonical injection $W_{\mathfrak{p}} \hookrightarrow W_{\bar{\mathbb{Q}}_p}$ that gives rise to a commutative square

$$\begin{array}{ccc} W_{\mathfrak{p}} & \hookrightarrow & W_{\bar{\mathbb{Q}}_p} \\ \uparrow & & \downarrow \\ V_{\mathfrak{p}} & \hookrightarrow & W_{\bar{\mathbb{Q}}_p}[\theta_f], \end{array} \tag{2.37}$$

where the right vertical arrow is the projection induced by (2.36) and the other maps are the obvious injections. Thus, we obtain from (2.37) a canonical surjection $\pi_{f,\mathfrak{p}} : W_{\mathfrak{p}} \twoheadrightarrow V_{\mathfrak{p}}$. Finally, set

$$W_p := \bigoplus_{\mathfrak{p}\mid p} W_{\mathfrak{p}}.$$

Taking sums over all $\mathfrak{p} \mid p$, the maps $\pi_{f,\mathfrak{p}}$ yield a canonical surjection $\pi_{f,p} : W_p \twoheadrightarrow V_p$.

2.14.2 p-adic étale regulator

Let K be a number field. For $\star \in \{p\} \cup \{\mathfrak{p} \mid p\}$, set

$$H^1_{\mathrm{mot}}(K, \mathcal{M})_\star := H^1_{\mathrm{mot}}(K, \mathcal{M}) \otimes_F F_\star. \tag{2.38}$$

Denote by

$$\mathrm{AJ}_{K,\mathbb{Z}_p} : \mathrm{CH}_0^{k/2}(X/K) \longrightarrow H^1\Big(K, H^{k-1}_{\text{ét}}\big(\bar{X}, \mathbb{Z}_p(k/2)\big)\Big) \tag{2.39}$$

the (integral) p-adic Abel–Jacobi map induced by the p-adic cycle class map (see, *e.g.*, [110, §4], [111, §1], [112, §1]). With notation as in (2.17), for each prime $\mathfrak{p}$ of F above p we obtain a map

$$\mathrm{AJ}_{K,F_{\mathfrak{p}}} : \mathrm{CH}_0^{k/2}(X/K)_{F_{\mathfrak{p}}} \longrightarrow H^1\Big(K, H^{k-1}_{\text{ét}}\big(\bar{X}, F_{\mathfrak{p}}(k/2)\big)\Big).$$

Therefore, taking sums over all $\mathfrak{p} \mid p$, we get from $\mathrm{AJ}_{K,F_{\mathfrak{p}}}$ a map

$$\mathrm{AJ}_{K,F_p} : \mathrm{CH}_0^{k/2}(X/K)_{F_p} \longrightarrow H^1\Big(K, H^{k-1}_{\text{ét}}\big(\bar{X}, F_p(k/2)\big)\Big). \tag{2.40}$$

Finally, applying the F_p-linear extension of $\Pi_B\Pi_\epsilon$ to (2.40), restricting the resulting map to $H^1_{\mathrm{mot}}(K, \mathcal{M})_p$ and applying the map induced by $\pi_{f,p}$ to its target, we get a map

$$\mathrm{reg}_{K,p} : H^1_{\mathrm{mot}}(K, \mathcal{M})_p \longrightarrow H^1(K, V_p) \tag{2.41}$$

that is called the *p-adic étale regulator* (or simply the *p-adic regulator*) *of* $\mathcal{M}$ *over* K. We also set $\mathrm{reg}_p := \mathrm{reg}_{\mathbb{Q},p}$. By construction, there is a splitting $\mathrm{reg}_{K,p} = \bigoplus_{\mathfrak{p}\mid p} \mathrm{reg}_{K,\mathfrak{p}}$, where

$$\mathrm{reg}_{K,\mathfrak{p}} : H^1_{\mathrm{mot}}(K, \mathcal{M})_{\mathfrak{p}} \longrightarrow H^1(K, V_{\mathfrak{p}}) \tag{2.42}$$

is the $\mathfrak{p}$*-adic regulator of* $\mathcal{M}$ *over* K. Again, we may set $\mathrm{reg}_{\mathfrak{p}} := \mathrm{reg}_{\mathbb{Q},\mathfrak{p}}$.

Remark 2.48 As a consequence of work of Saito on the weight-monodromy conjecture for compactified Kuga–Sato varieties [135,136] and of results of Nekovář [112] and Nizioł [120] on p-adic regulators, we know that $\mathrm{im}(\mathrm{reg}_{K,p}) \subset H^1_f(K, V_p)$, where $H^1_f(K, V_p)$ is the finite cohomology group from Sect. 2.13.1. See, *e.g.*, [93, Theorem 2.4] for details.

Remark 2.49 See, *e.g.*, [137, §5.1] for more general Abel–Jacobi maps in the context of motivic cohomology.

The next conjecture predicts a deep relation between motivic cohomology and (global) unramified cohomology.

Conjecture 2.50 (p-adic regulator) *For all primes p and all number fields K, the p-adic regulator in* (2.41) *induces an isomorphism*

$$\mathrm{reg}_{K,p} : H^1_{\mathrm{mot}}(K, \mathcal{M})_p \xrightarrow{\simeq} H^1_f(K, V_p) \tag{2.43}$$

of F_p-modules.

We refer to the statement of Conjecture 2.50 for fixed p and K as the *p-part of Conjecture 2.50 over K*. Later on (see, *e.g.*, Sects. 2.20, 2.23 and the main theorems of Section 5), we shall need to assume that (the p-part of) Conjecture 2.50 holds true over certain number fields.

Remark 2.51 Conjecture 2.50 is inspired by [15, Conjecture 5.3] and [77, Conjecture 7], which coincide in our case thanks to [104, Theorem 19.1]. Note that [77, Conjecture 7] is stated for motivic cohomology defined by means of the whole Chow group of homologically trivial cycles instead of our $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$; however, as observed in Remark 2.11, thanks to a conjecture of Beilinson, the motivic cohomology groups used in [77] and those defined in this paper are conjecturally equal when tensored with $\mathbb{Q}$, and therefore Conjecture 2.50 is (conjecturally) consistent with [15, Conjecture 5.3] and [77, Conjecture 7].

Remark 2.52 Let K be a number field and suppose that the p-part of Conjecture 2.50 over K holds true for a prime number p. It is well known (essentially a consequence of results of Tate, *cf.* [154, §2]) that $H^1_f(K, V_p)$ is finitely generated over F_p, so isomorphism (2.43) implies that $H^1_{\mathrm{mot}}(K, \mathcal{M})_p$ is finitely generated over F_p as well. It follows that $H^1_{\mathrm{mot}}(K, \mathcal{M})$ is finite-dimensional as an F-vector space: we conclude that Conjecture 2.50 for some prime p implies Conjecture 2.12. Note that isomorphism (2.43) ensures, in fact, that $H^1_f(K, V_p)$ is free (of finite rank) over F_p.

Remark 2.53 For the counterpart of Conjecture 2.50 for motives of elliptic curves, the reader is referred to [77, Example 2.16].

2.15 Projective $\mathcal{O}$-structures in $\mathcal{M}$

In the definition that follows, $\mathcal{O}$ is an order of F. Furthermore, given a prime p, we consider the semilocal ring $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$; notice that, as in Sect. 2.4.3, throughout this paper we reserve the symbol $\mathcal{O}_p$ for $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Denote by

$$\mathrm{Comp}_{\mathrm{B,ét}} : V_{\mathrm{B}} \otimes_F F_p \xrightarrow{\simeq} V_p$$

the comparison isomorphism between Betti and étale cohomology.

The following notion was introduced in [21, §3.3, Definition 1].

Definition 2.54 A *projective $\mathcal{O}$-structure* in $\mathcal{M}$ is a finitely generated projective $\mathcal{O}$-module $\mathcal{T}_{\mathrm{B}} \subset V_{\mathrm{B}}$ such that

(1) $\mathcal{T}_{\mathrm{B}} \otimes_{\mathcal{O}} F \simeq V_{\mathrm{B}}$;
(2) $\mathrm{Comp}_{\mathrm{B,ét}}\big(\mathcal{T}_{\mathrm{B}} \otimes_{\mathcal{O}} (\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p)\big)$ is a $G_{\mathbb{Q}}$-stable $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$-lattice in V_p for all primes p.

The remark below deals with the special case where $\mathcal{O}$ is the maximal order $\mathcal{O}_F$ of F.

Remark 2.55 The $\mathcal{O}_F$-module T_{B} that was defined in Sect. 2.4.2 is a projective $\mathcal{O}_F$-structure in $\mathcal{M}$. Moreover, if $T_p \subset V_p$ is the $G_{\mathbb{Q}}$-stable $\mathcal{O}_p$-lattice introduced in (2.9), then the integrality properties of $\mathrm{Comp}_{\mathrm{B,ét}}$ (see, *e.g.*, [1, Exp. XI, Théorème 4.4, (iii)]) ensure that

$$\mathrm{Comp}_{\mathrm{B,ét}}(T_{\mathrm{B}} \otimes_{\mathcal{O}_F} \mathcal{O}_p) = T_p. \tag{2.44}$$

We highlight this equality for future use.

2.16 The Tamagawa number conjecture for $\mathcal{M}$

We formulate the Tamagawa number conjecture of Bloch–Kato [15] and Fontaine–Perrin-Riou [43] in the case of the motive $\mathcal{M}$. We assume that

- the p-part of Conjecture 2.50 over $\mathbb{Q}$ holds true.

This will establish, in particular, an isomorphism between the F_p-linear duals of $H^1_f(K, V_p)$ and $H^1_{\mathrm{mot}}(K, \mathcal{M})_p$.

2.16.1 The isomorphism $\theta_{p,S}$

Define the F_p-module

$$t(\mathcal{M})_p := t(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{Q}_p = t(\mathcal{M}) \otimes_F F_p.$$

The comparison isomorphism between de Rham and étale cohomology induces an isomorphism

$$\mathrm{Comp}_{\mathrm{dR,ét}} : t(V_p) \xrightarrow{\simeq} t(\mathcal{M})_p.$$

Let v be place of $\mathbb{Q}$. Note that

$$\mathrm{Det}^{-1}_{F_p}\big(\mathbf{R}\Gamma_f(\mathbb{Q}_v, V_p)\big) \simeq \begin{cases} (F_p, 0) & \text{if } v \notin \{p, \infty\}, \\ \mathrm{Det}_{F_p}\big(t(\mathcal{M})_p\big) & \text{if } v = p, \\ \mathrm{Det}^{-1}_{F_p}\big(H^0(\mathbb{R}, V_p)\big) & \text{if } v = \infty. \end{cases} \tag{2.45}$$

Combining the base change property of determinants (*cf.* Sect. A.4), the multiplicativity of Det_{F_p} applied to (2.35) and, finally, isomorphism (2.45), we get an isomorphism

$$\mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_c(G_S, V_p)\big) \simeq \mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}\big(t(\mathcal{M})_p\big) \cdot \mathrm{Det}^{-1}_{F_p}(V_{\mathrm{B}}^+ \otimes_F F_p). \tag{2.46}$$

Define the F_p-module

$$\Delta(\mathcal{M})_p := \Delta(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{Q}_p = \Delta(\mathcal{M}) \otimes_F F_p.$$

Using Conjecture 2.50, the definition of the fundamental line $\Delta(\mathcal{M})$ (Definition 2.36) and (2.34), we get a conjectural isomorphism of F_p-modules

$$\theta_{p,S} : \Delta(\mathcal{M})_p \xrightarrow{\simeq} \mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_c(G_S, V_p)\big). \tag{2.47}$$

We want to provide some more details on this isomorphism. We are assuming the validity of the p-part of Conjecture 2.50 over $\mathbb{Q}$, so the regulator isomorphism yields an isomorphism of F_p-modules

$$\begin{aligned}\Delta(\mathcal{M})_p \simeq\ & \mathrm{Det}_{F_p}^{-1}\big(H^1_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}\big(H^1_f(\mathbb{Q}, V_p)^*\big) \\ & \cdot \mathrm{Det}_{F_p}\big(t_p(\mathcal{M})\big) \cdot \mathrm{Det}_{F_p}^{-1}(V_{\mathrm{B}}^+ \otimes_F F_p).\end{aligned}$$

On the other hand, the isomorphism $H^1_f(\mathbb{Q}, V_p)^* \simeq H^2_f(\mathbb{Q}, V_p)$, which uses the self-duality $V_p^*(1) \simeq V_p$, leads to yet another isomorphism of F_p-modules

$$\begin{aligned}\Delta(\mathcal{M})_p \simeq\ & \mathrm{Det}_{F_p}^{-1}\big(H^1_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}\big(H^2_f(\mathbb{Q}, V_p)\big) \\ & \cdot \mathrm{Det}_{F_p}\big(t_p(\mathcal{M})\big) \cdot \mathrm{Det}_{F_p}^{-1}(V_{\mathrm{B}}^+ \otimes_F F_p).\end{aligned} \tag{2.48}$$

Comparing (2.46) and (2.48), we see that to complete the construction of $\theta_{p,S}$ in (2.47) we need just to show that

$$\mathrm{Det}_{F_p}^{-1}\big(H^1_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}\big(H^2_f(\mathbb{Q}, V_p)\big) = \mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_f(\mathbb{Q}, V_p)\big).$$

To begin with, there is an isomorphism of F_p-modules

$$\begin{aligned}\mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_f(\mathbb{Q}, V_p)\big) \simeq\ & \mathrm{Det}_{F_p}\big(H^0_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}^{-1}\big(H^1_f(\mathbb{Q}, V_p)\big) \\ & \cdot \mathrm{Det}_{F_p}\big(H^2_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}^{-1}\big(H^3_f(\mathbb{Q}, V_p)\big)\end{aligned}$$

(*cf.* Sect. 2.13.1), so it remains to check that

$$\mathrm{Det}_{F_p}\big(H^0_f(\mathbb{Q}, V_p)\big) \cdot \mathrm{Det}_{F_p}^{-1}\big(H^3_f(\mathbb{Q}, V_p)\big) = F_p. \tag{2.49}$$

By duality, there is an isomorphism of F_p-modules

$$H^3_f(\mathbb{Q}, V_p) \simeq H^0_f(\mathbb{Q}, V_p)^*.$$

Now, by definition, $H^0_f(\mathbb{Q}, V_p) = 0$, so $H^3_f(\mathbb{Q}, V_p) = 0$ and equality (2.49) follows.

2.16.2 TNC for $\mathcal{M}$

We formulate the Tamagawa number conjecture (TNC, for short) for the motive $\mathcal{M}$ over $\mathbb{Q}$. Our previous notation is in force: S is the set of primes that was fixed in (2.33) and $\theta_{p,S}$ is the isomorphism in (2.47). Recall the zeta elements ζ_f and ζ_f^* appearing in Conjectures 2.40 and 2.41.

Conjecture 2.56 (TNC for $\mathcal{M}$) *Assume Conjectures* 2.40 and 2.50. *Let $\mathcal{T}_{\mathrm{B}}$ be a projective $\mathcal{O}$-structure in $\mathcal{M}$ for some order $\mathcal{O}$ of F. For every prime number p there is an equality*

$$\theta_{p,S}(\zeta_f^*)\cdot(\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p)=\mathrm{Det}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_c\Big(G_S,\mathrm{Comp}_{\mathrm{B,\acute{e}t}}\big(\mathcal{T}_{\mathrm{B}}\otimes_{\mathcal{O}}(\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\big)\Big)\Big) \tag{2.50}$$

of $(\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p)$-submodules of $\mathrm{Det}_{F_p}\big(\mathbf{R}\Gamma_c(G_S,V_p)\big)$.

Henceforth, equality (2.50) for a given p will be referred to as the *p-part of the TNC* for $\mathcal{M}$; we will sometimes indicate it as p-TNC for $\mathcal{M}$.

Remark 2.57 As in [77, Remark 2.21], one can show that Conjecture 2.56 is independent of the choice of the $\mathcal{O}$-projective structure $\mathcal{T}_{\mathrm{B}}$; furthermore, keeping Remark 2.46 in mind, it can also be checked that Conjecture 2.56 does not depend on the choice of S, in the sense explained in [77, Remark 2.22].

Although we formulated Conjecture 2.56 for any order $\mathcal{O}$ of F, we shall prove our main results for $\mathcal{O}=\mathcal{O}_F$ exclusively. In particular, in our computations we will take advantage of the fact that $\mathcal{O}_p$, which stands for $\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p$, splits as a direct product $\mathcal{O}_p=\prod_{\mathfrak{p}\mid p}\mathcal{O}_{\mathfrak{p}}$ of principal ideal domains. In light of Remark 2.57, when $\mathcal{O}=\mathcal{O}_F$ it is not restrictive to prove Conjecture 2.56 under the assumption that $\mathcal{T}_{\mathrm{B}}$ is the distinguished projective $\mathcal{O}_F$-structure T_B from Sect. 2.4.2 (*cf.* Remark 2.55). In this case, thanks to equality (2.44), equality (2.50) reads

$$\theta_{p,S}(\zeta_f^*)\cdot\mathcal{O}_p=\mathrm{Det}_{\mathcal{O}_p}\big(\mathbf{R}\Gamma_c(G_S,T_p)\big). \tag{2.51}$$

This is the expression of p-TNC for $\mathcal{M}$ that we shall work with in the rest of this article.

Remark 2.58 Let $\mathcal{O}$ be an order of F. It would be interesting to compare the validity of p-TNC for $\mathcal{M}$ relative to $\mathcal{O}$ and the validity of p-TNC for $\mathcal{M}$ relative to $\mathcal{O}_F$, which is a task we do not approach in this paper. For example, notice that the order $\mathcal{O}:=\mathbb{Z}[a_n:n\geq 1]$ of $\mathcal{O}_F$ generated by the Fourier coefficients of f does not coincide, in general, with $\mathcal{O}_F$. Betti cohomology groups are equipped with Hecke actions and $T_{\mathrm{B}}(\mathscr{R})$ has a canonical structure of projective $\mathscr{R}$-module for any subring $\mathscr{R}$ of $\mathcal{O}_F$ containing $\mathcal{O}$, so in Conjecture 2.56 one can take $\mathcal{T}_{\mathrm{B}}=T_{\mathrm{B}}(\mathscr{R})$ for any $\mathscr{R}$ as above. However, notice that, since $\mathcal{O}$ has finite index in $\mathcal{O}_F$, the equality $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p=\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p$ holds for all but finitely many primes p. Therefore, since our arguments force us to disregard an infinite set of prime numbers p for technical reasons, assuming $\mathcal{O}=\mathcal{O}_F$ is essentially not restrictive for our aims.

2.17 Bloch–Kato Selmer groups

Let K be a number field and let p be a prime number. Let V be a p-adic representation of G_K and let T be a $\mathbb{Z}_p$-lattice in V. Set $A:=V/T$. If T is endowed with a $\mathbb{Z}_p$-linear action

of an order $\mathcal{O}$ of F, then V inherits a structure of an F_p-module, while both T and A are equipped with a structure of $\mathcal{O}_p$-modules.

2.17.1 Finite local conditions

Let v be a place of K. The *finite local conditions* $H^\bullet_f(K_v, T)$ and $H^\bullet_f(K_v, A)$ at v are defined by propagation from the cohomology groups $H^\bullet_f(K_v, V)$ in Sect. 2.12 using the canonical maps $T \hookrightarrow V$ and $V \twoheadrightarrow A$ (see, *e.g.*, [101, §1.1]). In particular, it follows from (2.31) that if $v \nmid p\infty$, then

$$H^0_f(K_v, T_p) = H^0(K_v, T_p), \quad H^0_f(K_v, A_p) = H^0(K_v, A_p).$$

We denote by $H^\bullet_s(K_v, T)$ (respectively, $H^\bullet_s(K_v, A)$) the quotients of $H^\bullet(K_v, T)$ (respectively, $H^\bullet(K_v, A)$) by $H^\bullet_f(K_v, T)$ (respectively, $H^\bullet_f(K_v, A)$).

2.17.2 Bloch–Kato Selmer groups

In the following definition, let $M \in \{V, T, A\}$.

Definition 2.59 The *Bloch–Kato Selmer group of M over K* is

$$H^1_f(K, M) := \ker\left(H^1(K, M) \longrightarrow \prod_v H^1_s(K_v, M)\right),$$

where the product is taken over all places v of K.

One can check (see [77, Lemma 2.15] or [95, Lemma 5.1]) that

$$H^1_f(K, M) = \ker\left(H^1(G_{K,S}, M) \longrightarrow \bigoplus_{v\in S} H^1_s(K_v, M)\right), \tag{2.52}$$

where S is the finite set of places of K that is fixed in Sect. 2.13.

As in Sect. 2.4.3, for all primes $\mathfrak{p}$ of F above p we set $T_\mathfrak{p} := T_p \otimes_{\mathcal{O}_p} \mathcal{O}_\mathfrak{p}$, which is an $\mathcal{O}_\mathfrak{p}$-lattice inside $V_\mathfrak{p}$; there is a splitting $T_p = \prod_{\mathfrak{p}|p} T_\mathfrak{p}$. We also put $A_\mathfrak{p} := V_\mathfrak{p}/T_\mathfrak{p}$; if A_p is defined as in (2.10), then $A_p = \prod_{\mathfrak{p}|p} A_\mathfrak{p}$. There is a splitting

$$H^1_f(K, M_p) = \bigoplus_{\mathfrak{p}|p} H^1_f(K, M_\mathfrak{p}), \tag{2.53}$$

where the direct sum is taken over all primes $\mathfrak{p}$ of F above p.

2.18 Shafarevich–Tate groups of $\mathcal{M}$

Let p be a prime number. From now on, for a p-primary abelian group G we denote by G_{div} the maximal p-divisible subgroup of G. We introduce Shafarevich–Tate groups *à la* Bloch–Kato.

2.18.1 Shafarevich–Tate groups

Let K be a number field. For any prime $\mathfrak{p}$ of F above p, recall the Bloch–Kato Selmer group $H^1_f(K, A_\mathfrak{p})$ of $A_\mathfrak{p}$ over K from Sect. 2.17.2. The following definition of Shafarevich–Tate group is due to Bloch–Kato ([15, Remark 5.15.2]; *cf.* also [40]).

Definition 2.60 (1) The (*Bloch–Kato*) *Shafarevich–Tate group of* $\mathcal{M}$ *over* K *at* $\mathfrak{p}$ is

$$Ш^{\mathrm{BK}}_{\mathfrak{p}}(K,\mathcal{M}) := H^1_f(K,A_{\mathfrak{p}})/H^1_f(K,A_{\mathfrak{p}})_{\mathrm{div}}.$$

(2) The (*Bloch–Kato*) *Shafarevich–Tate group of* $\mathcal{M}$ *over* K *at* p is

$$Ш^{\mathrm{BK}}_{p}(K,\mathcal{M}) := H^1_f(K,A_{p})/H^1_f(K,A_{p})_{\mathrm{div}}.$$

(3) The (*Bloch–Kato*) *Shafarevich–Tate group of* $\mathcal{M}$ *over* K is

$$Ш^{\mathrm{BK}}(K,\mathcal{M}) := \bigoplus_{p} Ш^{\mathrm{BK}}_{p}(K,\mathcal{M}),$$

where p varies over all prime numbers.

There is a splitting $Ш^{\mathrm{BK}}_{p}(K,\mathcal{M}) = \bigoplus_{\mathfrak{p}|p} Ш^{\mathrm{BK}}_{\mathfrak{p}}(K,\mathcal{M})$, where the direct sum is taken over all primes $\mathfrak{p}$ of F above p. Therefore, we can write

$$Ш^{\mathrm{BK}}(K,\mathcal{M}) = \bigoplus_{\lambda} Ш^{\mathrm{BK}}_{\lambda}(K,\mathcal{M}),$$

where λ varies over all primes of F. Notice that $Ш^{\mathrm{BK}}_{\lambda}(K,\mathcal{M})$ is finite for all λ, and then the same is true of $Ш^{\mathrm{BK}}_{\ell}(K,\mathcal{M})$ for all prime numbers ℓ (see, *e.g.*, [43, Ch. II, Proposition 5.3.5]). We remark that in Sect. 5.5.1 we introduce also Shafarevich–Tate groups $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K,\mathcal{M})$ *à la* Nekovář: the interplay between $Ш^{\mathrm{BK}}_{\mathfrak{p}}(K,\mathcal{M})$ and $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K,\mathcal{M})$ will be crucial for our arguments (see, *e.g.*, Sects. 5.5.2 and 5.7.1).

2.18.2 A finiteness conjecture

By analogy with a classical conjecture for Shafarevich–Tate groups of abelian varieties over global fields, it is natural to propose

Conjecture 2.61 (Finiteness of Ш) *For all number fields* K*, the group* $Ш^{\mathrm{BK}}(K,\mathcal{M})$ *is finite.*

Clearly, Conjecture 2.61 (which will play no explicit role in the paper) is equivalent to the prediction that, for all number fields K, the group $Ш^{\mathrm{BK}}_{\ell}(K,\mathcal{M})$ is trivial for all but finitely many ℓ.

Remark 2.62 To be in line with terminology and notation introduced in Definition 2.60 for Shafarevich–Tate groups, we could alternatively set $\mathrm{Sel}_p(K,\mathcal{M}) := H^1_f(K,A_p)$ and call it the *Bloch–Kato Selmer group of* $\mathcal{M}$ *over* K *at* p. However, later on we shall reserve a symbol of this kind (at least when K varies over the finite layers of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$) for Selmer groups in the sense of Greenberg (*cf.* Sect. 5.6), so here we chose to adopt the notation that was originally used by Bloch and Kato in [15].

2.19 Local finite cohomology groups

We collect some basic facts on local cohomology groups of p-adic Galois representations.

2.19.1 Local Tate duality

Let V, T, A be as in Sect. 2.17. Define

$$T^* := \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$$

and recall that if v is a place of $\mathbb{Q}$, then under the local Tate duality pairing

$$(\cdot,\cdot)_v : H^1(\mathbb{Q}_v, T) \times H^1\big(\mathbb{Q}_v, T^* \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

the subgroups $H^1_f(\mathbb{Q}_v, T)$ and $H^1_f\big(\mathbb{Q}_v, T^* \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big)$ are exact annihilators of each other ([15, Proposition 3.8]). We remark that $H^1_f\big(\mathbb{Q}_v, T^* \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big)$ is defined by propagation from the corresponding local conditions for the representation $V^*(1)$, where $V^* := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$. Now we assume that there is an isomorphism $V^*(1) \simeq V$ under which $T^*(1) \simeq T$. It follows that $A \simeq T^* \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)(1)$, so $H^1_f\big(\mathbb{Q}_v, T^* \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big)$ is isomorphic to $H^1_f(\mathbb{Q}_v, A)$. Then the local Tate pairing at v yields a perfect pairing

$$(\cdot,\cdot)_v : H^1(\mathbb{Q}_v, T) \times H^1(\mathbb{Q}_v, A) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \tag{2.54}$$

under which the subgroups $H^1_f(\mathbb{Q}_v, T)$ and $H^1_f(\mathbb{Q}_v, A)$ are exact annihilators of each other. Since $(\cdot,\cdot)_v$ is perfect, this means that there are isomorphisms

$$\varphi_v : H^1_f(\mathbb{Q}_v, T) \xrightarrow{\simeq} H^1_s(\mathbb{Q}_v, A)^\vee \tag{2.55}$$

and

$$\psi_v : H^1_s(\mathbb{Q}_v, T) \xrightarrow{\simeq} H^1_f(\mathbb{Q}_v, A)^\vee, \tag{2.56}$$

where for a $\mathbb{Z}_p$-module M we let

$$M^\vee := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p) \tag{2.57}$$

be the Pontryagin dual of M.

2.19.2 The case of modular motives

In the case of motives of modular forms, there is an isomorphism $V_p^*(1) \simeq V_p$ under which $T_p^*(1) \simeq T_p$, so the results above apply with $V = V_p$, $T = T_p$, $A = A_p$.

Lemma 2.63 *There is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v\in S} H^1_f(\mathbb{Q}_v, T_p) & \longrightarrow & \bigoplus_{v\in S} H^1(\mathbb{Q}_v, T_p) & \longrightarrow & \bigoplus_{v\in S} H^1_s(\mathbb{Q}_v, T_p) & \longrightarrow & 0 \\
 & & \simeq \downarrow \varphi_S & & \downarrow & & \downarrow & & \\
 & & \bigoplus_{v\in S} H^1_s(\mathbb{Q}_v, A_p)^\vee & \longrightarrow & H^1(G_S, A_p)^\vee & \longrightarrow & H^1_f(\mathbb{Q}, A_p)^\vee & \longrightarrow & 0
\end{array}$$

with exact rows.

Proof The top row is a direct consequence of the definitions of the groups involved, while the bottom row is obtained by taking Pontryagin duals of the exact sequence

$$0 \longrightarrow H^1_f(\mathbb{Q}, A_p) \longrightarrow H^1(G_S, A_p) \longrightarrow \bigoplus_{v\in S} H^1_s(\mathbb{Q}_v, A_p)$$

induced by (2.52) with $M = A_p$. The vertical isomorphism on the left is defined by setting $\varphi_S := \bigoplus_{v\in S}\varphi_v$, with φ_v as in (2.55). On the other hand, the middle vertical arrow is the composition of the map $\bigoplus_{v\in S} H^1(\mathbb{Q}_v, T_p) \to \bigoplus_{v\in S} H^1(\mathbb{Q}_v, A_p)^\vee$ induced by (2.54) and the dual of the map $H^1(G_S, A_p) \to \bigoplus_{s\in S} H^1(\mathbb{Q}_v, A_p)$ given by restriction in cohomology. Finally, the right vertical map is the composition of the map $\oplus_{v\in S}\psi_v$, where ψ_v is as in (2.56), and the dual of the map $H^1_f(\mathbb{Q}, A_p) \to \bigoplus_{s\in S} H^1_f(\mathbb{Q}_v, A_p)$ defined by restriction in cohomology. The commutativity of the diagram is immediate by construction. □

For a prime $\mathfrak{p}$ of F above the prime number p, let $A_\mathfrak{p}[\mathfrak{p}] = T_\mathfrak{p}/\mathfrak{p}T_\mathfrak{p}$ be the mod $\mathfrak{p}$ residual representation on the $\mathfrak{p}$-torsion of $A_\mathfrak{p}$ (*cf.* Sect. 3.1.2). In the rest of this article, we work under the following

Assumption 2.64 (1) $p \nmid N$;
(2) $A_\mathfrak{p}[\mathfrak{p}]$ is ramified at the primes dividing N for each $\mathfrak{p} \mid p$;
(3) ℓ prime, $\ell^2 \mid N \Rightarrow V_p^{I_\ell} = 0$.

In part (3), $I_\ell \subset G_{\mathbb{Q}_\ell}$ is the inertia subgroup at ℓ. As a consequence of Assumption 2.64, the (finite) set of places of $\mathbb{Q}$ from Sect. 2.13 is explicitly given by

$$S := \{\ell \text{ prime} \mid \ell \text{ divides } Np\} \cup \{\infty\}. \tag{2.58}$$

Lemma 2.65 *If $\ell \neq p$ is a prime number, then $H^0_f(\mathbb{Q}_\ell, V_p) = H^1_f(\mathbb{Q}_\ell, V_p) = 0$.*

Proof Let S be as in (2.58) and let ℓ be a prime number. First assume that $\ell \notin S$. Since V_p is unramified at ℓ, we have $V_p^{I_\ell} = V_p$. It follows from (2.32) and [134, Lemma 1.3.2, (i)] that there is an isomorphism

$$H^1_f(\mathbb{Q}_\ell, V_p) \simeq V_p/(\mathrm{Frob}_\ell - 1)V_p. \tag{2.59}$$

In this case, Frob_ℓ acts with eigenvalues α, β such that $|\alpha| = |\beta| = \ell^{k/2+1}$; we deduce that $\mathrm{Frob}_\ell - 1 : V_p \to V_p$ is an isomorphism, and the result follows from (2.59).

Assume now that $\ell \mid N$. If $\ell^2 \mid N$, then by part (3) of Assumption 2.64 we have $V_p^{I_\ell} = 0$, so again the result follows from (2.32). Finally, assume that $\ell \| N$. The restriction of V_p to $G_{\mathbb{Q}_\ell}$ is isomorphic to $\left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & c \\ 0 & 1 \end{smallmatrix}\right)$, where χ_{cyc} is the p-adic cyclotomic character and $c : G_{\mathbb{Q}_\ell} \to V_p$ is a 1-cocycle. Moreover, I_ℓ acts via the map $g \mapsto \left(\begin{smallmatrix} 1 & c(g) \\ 0 & 1 \end{smallmatrix}\right)$, so, since V_p is ramified at ℓ, we have $c \neq 0$ and $V_p^{I_\ell} \simeq F_p(1)$. In particular, $\mathrm{Frob}_\ell - 1$ is an isomorphism on $V_p^{I_\ell}$, and by (2.32) the lemma is proved. □

Lemma 2.66 $H^0(\mathbb{Q}_p, V_p) = 0$.

Proof By part (1) of Assumption 2.64, V_p is a crystalline, hence de Rham, representation, so $\mathbf{D}_{\mathrm{dR}}(V_p) = \mathbf{D}_{\mathrm{cris}}(V_p)$. By [15, Theorem 4.1, (ii)], the Bloch–Kato exponential map gives an isomorphism

$$\exp_{\mathrm{BK}} : t(V_p) \xrightarrow{\simeq} H^1_f(\mathbb{Q}_p, V_p),$$

and then [15, Corollary 3.8.4] implies that $H^0(\mathbb{Q}_p, V_p) = 0$. □

2.20 On the cohomology of T_p, V_p, A_p

Let K be a number field. Recall that we assume throughout that Conjecture 2.12 is true, *i.e.*, $H^1_{\mathrm{mot}}(K, \mathcal{M})$ has finite dimension, denoted by $r_{\mathrm{alg}}(\mathcal{M}/K)$, over F. For notational convenience, set $r := r_{\mathrm{alg}}(\mathcal{M}/K)$. Furthermore, assume also that

- the p-part of Conjecture 2.50 over K holds true.

This condition, for suitable choices of K, will essentially be in force until the end of the paper (namely, in Sect. 2.20, in Sect. 2.23 and in the main theorems of Section 5). Therefore, the p-adic regulator map from (2.41) is an isomorphism

$$\mathrm{reg}_{K,p} : H^1_{\mathrm{mot}}(K, \mathcal{M})_p \xrightarrow{\simeq} H^1_f(K, V_p)$$

of F_p-modules. As a consequence, $H^1_f(K, V_p)$ is free of rank r over F_p. Since $F_p = \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$, it follows from (2.53) with $M = V$ that the $F_{\mathfrak{p}}$-vector space $H^1_f(L, V_{\mathfrak{p}})$ has dimension r for all primes $\mathfrak{p}$ of F above p. In the following lines, let $\star \in \{p\} \cup \{\mathfrak{p} \mid p\}$.

2.20.1 Cohomology and divisible submodules

Set

$$\underline{H}^1_f(K, T_\star) := \mathrm{im}\Big(H^1_f(K, T_\star) \longrightarrow H^1_f(K, V_\star)\Big).$$

Since $H^1_f(K, T_{\mathfrak{p}})$ is finitely generated over $\mathcal{O}_{\mathfrak{p}}$ and there is a canonical isomorphism

$$H^1_f(L, T_{\mathfrak{p}}) \otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \xrightarrow{\simeq} H^1_f(L, V_{\mathfrak{p}})$$

of $F_{\mathfrak{p}}$-vector spaces (*cf.* [134, Proposition B.2.4] and [154, Proposition 2.3]), we conclude that $\underline{H}^1_f(K, T_{\mathfrak{p}})$ is a free $\mathcal{O}_{\mathfrak{p}}$-submodule of $H^1_f(K, V_{\mathfrak{p}})$ of rank r; in other words, $\underline{H}^1_f(K, T_{\mathfrak{p}})$ is an $\mathcal{O}_{\mathfrak{p}}$-lattice inside $H^1_f(K, V_{\mathfrak{p}})$. Again by (2.53), $\underline{H}^1_f(K, T_p) = \bigoplus_{\mathfrak{p}|p} \underline{H}^1_f(K, T_{\mathfrak{p}})$, so $\underline{H}^1_f(K, T_p)$ is a free $\mathcal{O}_p$-submodule of $H^1_f(K, V_p)$ of rank r.

It can be checked that there is an exact sequence

$$H^1_f(K, T_\star) \longrightarrow H^1_f(K, V_\star) \longrightarrow H^1_f(K, A_\star), \tag{2.60}$$

which induces an exact sequence

$$0 \longrightarrow \underline{H}^1_f(K, T_\star) \longrightarrow H^1_f(K, V_\star) \longrightarrow H^1_f(K, A_\star). \tag{2.61}$$

The group $H^1_f(K, V_\star)$ is a vector space over a field of characteristic 0, so it is divisible, and then the rightmost map in (2.60) gives a map

$$\Upsilon_\star : H^1_f(K, V_\star) \longrightarrow H^1_f(K, A_\star)_{\mathrm{div}}. \tag{2.62}$$

Clearly, $\Upsilon_p = \bigoplus_{\mathfrak{p}|p} \Upsilon_{\mathfrak{p}}$.

Proposition 2.67 *The map $\Upsilon_\star$ is surjective.*

Proof The surjectivity of Υ_p is equivalent to that of $\Upsilon_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$, which is well known (*cf.* [40], [117, §2.1.3]). □

2.20.2 On Pontryagin duals

Let $M \in \{T, V, A\}$. As is pointed out in Remark B.1, there is an identification

$$\mathrm{Hom}_{\mathrm{cont}}\big(H^1_f(L, M_{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p\big) = \mathrm{Hom}_{\mathrm{cont}}\big(H^1_f(L, M_{\mathfrak{p}}), F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}\big), \tag{2.63}$$

which provides an alternative description of the Pontryagin dual $H^1_f(L, M_{\mathfrak{p}})^\vee$ of $H^1_f(L, M_{\mathfrak{p}})$. Let us also define

$$H^1_f(K, M_p)^\vee := \mathrm{Hom}_{\mathrm{cont}}\big(H^1_f(K, M_p), F_p/\mathcal{O}_p\big)$$

and call this $\mathcal{O}_p$-module the *Pontryagin dual* of $H^1_f(K, M_p)$. It follows from (2.63) and the splitting $F_p/\mathcal{O}_p = \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$ that

$$H^1_f(K, M_p)^\vee = \bigoplus_{\mathfrak{p}|p} H^1_f(K, M_{\mathfrak{p}})^\vee. \tag{2.64}$$

Let us write $\mathrm{corank}_{\mathcal{O}_\star} H^1_f(K, A_\star)$ for the corank of $H^1_f(K, A_\star)$ over $\mathcal{O}_\star$, *i.e.*, the rank as an $\mathcal{O}_\star$-module of its Pontryagin dual $H^1_f(L, A_\star)^\vee$. The corollary below will be used on several occasions when proving results on $r_{\mathrm{alg}}(\mathcal{M})$ (see, *e.g.*, Theorem 5.22, Sects. 7.2, 8.2, 9.2).

Corollary 2.68 $\mathrm{corank}_{\mathcal{O}_\star} H^1_f(K, A_\star) = r$.

Proof To begin with, $\mathrm{corank}_{\mathcal{O}_\star} H^1_f(K, A_\star) = \mathrm{corank}_{\mathcal{O}_\star} H^1_f(K, A_\star)_{\mathrm{div}}$. On the other hand, it follows from (2.61) and Proposition 2.67 that there is an isomorphism of $\mathcal{O}_\star$-modules

$$H^1_f(K, A_\star)_{\mathrm{div}} \simeq H^1_f(K, V_\star)\big/\underline{H}^1_f(K, T_\star) \simeq (F_\star/\mathcal{O}_\star)^r,$$

whence the desired equality. □

2.20.3 Dual bases

In light of Proposition 2.67, taking Pontryagin duals of the map $\Upsilon_\star$ in (2.62) gives an injection of $\mathcal{O}_\star$-modules

$$\Upsilon_\star^\vee : H^1_f(K, A_\star)^\vee_{\mathrm{div}} \hookrightarrow H^1_f(K, V_\star)^\vee. \tag{2.65}$$

Again, $\Upsilon_p^\vee = \bigoplus_{\mathfrak{p}|p} \Upsilon_{\mathfrak{p}}^\vee$.

Now pick an F_p-basis $\tilde{\mathscr{B}} = \{x_1, \ldots, x_r\}$ of $H^1_f(K, V_p)$; it generates a rank r free $\mathcal{O}_p$-submodule $\Lambda_{\tilde{\mathscr{B}}}$ of $H^1_f(K, V_p)$. Fix an isomorphism

$$\varphi_{\tilde{\mathscr{B}}} : \Lambda_{\tilde{\mathscr{B}}} \xrightarrow{\simeq} \underline{H}^1_f(K, T_p)$$

of $\mathcal{O}_p$-modules and set $\xi_i := \varphi_{\tilde{\mathscr{B}}}(x_i)$ for $i = 1, \ldots, r$. Thus, $\{\xi_1, \ldots, \xi_r\}$ is an $\mathcal{O}_p$-basis of $\underline{H}^1_f(K, T_p)$; it is also an F_p-basis of $H^1_f(K, V_p)$. The elements $\xi_1, \ldots, \xi_n$ give rise to the dual basis $\{\xi_1^*, \ldots, \xi_n^*\}$ of the $\mathcal{O}_p$-linear dual of $\underline{H}^1_f(K, T_p)$ by the recipe $\xi_i^*(\xi_j) := \delta_{ij}$, where δ_{ij} is the Kronecker delta; this gives also a basis, which will be denoted in the same way, of the F_p-linear dual of $H^1_f(K, V_p)$. Composing the F_p-linear maps ξ_i^* with the canonical projection $F_p \twoheadrightarrow F_p/\mathcal{O}_p$, we obtain elements $\xi_i^\vee$ of the Pontryagin dual $H^1_f(K, V_p)^\vee$.

Lemma 2.69 *The elements $\xi_1^\vee, \ldots, \xi_r^\vee$ are linearly independent over $\mathcal{O}_p$.*

Proof In light of splitting (2.64) for $M = V$, we may

- assume that $\xi_1^\vee, \dots, \xi_r^\vee$ belong to $H^1_f(K, V_\mathfrak{p})^\vee$ for a prime $\mathfrak{p}$ of F above p,
- replace $\mathcal{O}_p$ with $\mathcal{O}_\mathfrak{p}$.

Now the lemma follows from Lemma B.8 with, in the notation of Sect. B.3, $\mathscr{K} = F_\mathfrak{p}$, $\mathscr{O} = \mathcal{O}_\mathfrak{p}$, $v_i = \xi_i$, $V = H^1_f(K, V_\mathfrak{p})$, $T_\mathscr{B} = \underline{H}^1_f(K, T_\mathfrak{p})$. □

Denote by $\Xi_{\tilde{\mathscr{B}}}$ the $\mathcal{O}_p$-submodule of $H^1_f(K, V_p)^\vee$ spanned by $\xi_1^\vee, \dots, \xi_r^\vee$; by Lemma 2.69, the $\mathcal{O}_p$-rank of $\Xi_{\tilde{\mathscr{B}}}$ is r.

Lemma 2.70 *If $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{B}}'$ are F_p-bases of $H^1_f(K, V_p)$, then $\Xi_{\tilde{\mathscr{B}}} = \Xi_{\tilde{\mathscr{B}}'}$.*

Proof With self-explaining notation, $\{\xi_1^*, \dots, \xi_r^*\}$ and $\{(\xi_1')^*, \dots, (\xi_r')^*\}$ are bases of the $\mathcal{O}_p$-linear dual of $\underline{H}^1_f(K, T_p)$. In particular, for every $i \in \{1, \dots, r\}$ there are $a_{i,1}, \dots, a_{i,r} \in \mathcal{O}_p$ such that $(\xi_i')^* = a_{i,1}\xi_1^* + \cdots + a_{i,r}\xi_r^*$. On the other hand, the $\mathcal{O}_p$-linearity of the quotient projection $F_p \twoheadrightarrow F_p/\mathcal{O}_p$ ensures that

$$(\xi_i')^\vee = a_{i,1}\xi_1^\vee + \cdots + a_{i,r}\xi_r^\vee,$$

so $\Xi_{\tilde{\mathscr{B}}'} \subset \Xi_{\tilde{\mathscr{B}}}$. Analogously, the inclusion $\Xi_{\tilde{\mathscr{B}}} \subset \Xi_{\tilde{\mathscr{B}}'}$ holds as well. □

In light of Lemma 2.70, from here on we set

$$\Xi_p := \Xi_{\tilde{\mathscr{B}}} \subset H^1_f(K, V_p)^\vee$$

for any F_p-basis $\tilde{\mathscr{B}}$ of $H^1_f(K, V_p)$. Now recall the injection of $\mathcal{O}_p$-modules $\Upsilon_p^\vee$ from (2.65).

Proposition 2.71 *The image of $\Upsilon_p^\vee$ is Ξ_p.*

Proof Since $\Upsilon_p^\vee = \bigoplus_{\mathfrak{p}|p} \Upsilon_\mathfrak{p}^\vee$, as in the proof of Lemma 2.69 we may work with a fixed prime $\mathfrak{p} \mid p$ in place of p. Taking Pontryagin duals in the short exact sequence

$$0 \longrightarrow \underline{H}^1_f(K, T_\mathfrak{p}) \xrightarrow{\iota_\mathfrak{p}} H^1_f(K, V_\mathfrak{p}) \xrightarrow{\Upsilon_\mathfrak{p}} H^1_f(K, A_\mathfrak{p})_{\mathrm{div}} \longrightarrow 0,$$

where $\iota_\mathfrak{p}$ is simply inclusion, gives a short exact sequence of $\mathcal{O}_\mathfrak{p}$-modules

$$0 \longrightarrow H^1_f(K, A_\mathfrak{p})^\vee_{\mathrm{div}} \xrightarrow{\Upsilon_\mathfrak{p}^\vee} H^1_f(K, V_\mathfrak{p})^\vee \xrightarrow{\iota_\mathfrak{p}^\vee} \underline{H}^1_f(K, T_\mathfrak{p})^\vee \longrightarrow 0.$$

Therefore, if $\Xi_\mathfrak{p}$ denotes the analogue of Ξ_p at $\mathfrak{p}$, then we need to show that $\ker(\iota_\mathfrak{p}^\vee) = \Xi_\mathfrak{p}$. Let $\varphi \in \ker(\iota_\mathfrak{p}^\vee)$. This means that $\varphi \in H^1_f(K, V_\mathfrak{p})^\vee$ and $\varphi|_{\underline{H}^1_f(K, T_\mathfrak{p})} = 0$, which is equivalent to $\varphi \in \Xi_\mathfrak{p}$ by Proposition B.10 with, in the notation of Sect. B.3, $\mathscr{K} = F_\mathfrak{p}$, $\mathscr{O} = \mathcal{O}_\mathfrak{p}$, $V = H^1_f(K, V_\mathfrak{p})$, $T_\mathscr{B} = \underline{H}^1_f(K, T_\mathfrak{p})$, $\Xi_\mathscr{B} = \Xi_\mathfrak{p}$. □

The next corollary will be used in the proof of Theorem 2.83.

Corollary 2.72 *The set $\{\xi_1^\vee, \dots, \xi_r^\vee\}$ is an $\mathcal{O}_p$-basis of the image of $\Upsilon_p^\vee$.*

Proof By definition of Ξ_p, this follows from Proposition 2.71. □

In light of the fact that $\Upsilon_p^\vee$ is the dual of the map induced by the canonical projection $V_p \twoheadrightarrow A_p$, from now on we shall usually identify $H^1_f(K, A_p)^\vee_{\mathrm{div}}$ with its image under $\Upsilon_p^\vee$; in particular, we shall regard $\{\xi_1^\vee, \dots, \xi_r^\vee\}$ as an $\mathcal{O}_p$-basis of $H^1_f(K, A_p)^\vee_{\mathrm{div}}$.

2.21 Tamagawa ideals of $\mathcal{M}$

For each prime $\mathfrak{p}$ of F above p, the completion $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}_F$ at $\mathfrak{p}$ is a PID (actually, a DVR), so we can apply the definition of determinants in Sect. A.3 to the case where $R = \mathcal{O}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}$. In particular, with notation that already made an appearance in Sect. 1.1.3, if T is a finite $\mathcal{O}_p$-module, then $\mathcal{I}_{\mathcal{O}_p}(T)$ stands for $\mathrm{Det}_{\mathcal{O}_p}(T)$ and $\mathcal{I}_{\mathcal{O}_p}^{-1}(T)$ stands for $\mathrm{Det}_{\mathcal{O}_p}^{-1}(T)$; it follows that $\mathcal{I}_{\mathcal{O}_p}(T)$ and $\mathcal{I}_{\mathcal{O}_p}^{-1}(T)$ are fractional $\mathcal{O}_p$-ideals. More explicitly, with T as above, let $T = \bigoplus_{\mathfrak{p}|p} T_\mathfrak{p}$ be the splitting of T as a product of $\mathcal{O}_\mathfrak{p}$-modules; then

$$\mathcal{I}_{\mathcal{O}_p}(T) = \prod_{\mathfrak{p}|p} \mathcal{I}_{\mathcal{O}_\mathfrak{p}}(T_\mathfrak{p}) \subset F_p.$$

Since we shall essentially work with $\mathcal{O}_p$-modules only, from now on we simply set

$$\mathcal{I}(T) := \mathcal{I}_{\mathcal{O}_p}(T), \quad \mathcal{I}^{-1}(T) := \mathcal{I}_{\mathcal{O}_p}^{-1}(T) \tag{2.66}$$

for every finite $\mathcal{O}_p$-module T, unless confusion may arise.

2.21.1 Finite primes $\ell \neq p$

Let $\ell \neq p$ be a prime number and for any $G_{\mathbb{Q}_\ell}$-module M set

$$H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, M) := H^1\big(\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{unr}}/\mathbb{Q}_\ell), M^{I_\ell}\big) = \ker\big(H^1(\mathbb{Q}_\ell, M) \longrightarrow H^1(I_\ell, M)\big),$$

where the equality on the right is a consequence of the inflation-restriction exact sequence (see, *e.g.*, [134, Proposition B.2.5]). In particular, observe that $H^1_f(\mathbb{Q}_\ell, A_p) \subset H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, A_p)$ and the inclusion has finite index. By definition, $H^1_f(\mathbb{Q}_\ell, V_p) = H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, V_p)$ and there is an exact sequence

$$0 \longrightarrow H^0_f(\mathbb{Q}_\ell, V_p) \longrightarrow V_p^{I_\ell} \xrightarrow{\mathrm{Frob}_\ell - 1} V_p^{I_\ell} \longrightarrow H^1_f(\mathbb{Q}_\ell, V_p) \longrightarrow 0.$$

Thus, we obtain a chain of two isomorphisms

$$\vartheta_\ell : \mathrm{Det}_{F_p}\big(H^0_f(\mathbb{Q}_\ell, V_p)\big) \cdot \mathrm{Det}^{-1}_{F_p}\big(H^1_f(\mathbb{Q}_\ell, V_p)\big) \xrightarrow{\simeq} \mathrm{Det}^{-1}_{F_p}\big(H^1_f(\mathbb{Q}_\ell, V_p)\big) \xrightarrow{\simeq} (F_p, 0),$$

the former being a consequence of Lemma 2.65.

Definition 2.73 The *p-part of the Tamagawa ideal of $\mathcal{M}$ at ℓ* is

$$\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) := \vartheta_\ell\Big(\mathrm{Det}^{-1}_{\mathcal{O}_p}\big(H^1_f(\mathbb{Q}_\ell, T_p)\big)\Big). \tag{2.67}$$

Notice that, in light of the definition given in (A.6), there is a splitting

$$\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) = \prod_{\mathfrak{p}|p} \mathrm{Tam}_\ell^{(\mathfrak{p})}(\mathcal{M}),$$

where the $\mathfrak{p}$ varies over all primes of F above p and $\mathrm{Tam}_\ell^{(\mathfrak{p})}(\mathcal{M})$, the *$\mathfrak{p}$-part of the Tamagawa ideal of $\mathcal{M}$ at ℓ*, is defined as in (2.67) by replacing p with $\mathfrak{p}$.

To lighten our notation, set $\mathscr{G}_\ell := \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{unr}}/\mathbb{Q}_\ell) \simeq \hat{\mathbb{Z}}$. The auxiliary result below will be used in the proof of Proposition 2.76.

Lemma 2.74 $H^2\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) = H^2\big(\mathscr{G}_\ell, V_p^{I_\ell}\big) = 0$.

Proof The $\mathscr{G}_\ell$-module $V_p^{I_\ell}$, which is a vector space over a field of characteristic 0, is divisible, so $H^2\big(\mathscr{G}_\ell, V_p^{I_\ell}\big) = 0$ by [144, Ch. XIII, Proposition 2]. As for the other vanishing, note that $T_p^{I_\ell} = \varprojlim_n (T_p/p^n T_p)^{I_\ell}$ and $T_p/p^n T_p$ is finite for all $n \in \mathbb{N}$. On the other hand, [134, Lemma 1.3.2, (i)] gives

$$H^1\big(\mathscr{G}_\ell, (T_p/p^n T_p)^{I_\ell}\big) \simeq (T_p/p^n T_p)^{I_\ell} \big/ (\mathrm{Frob}_\ell - 1)(T_p/p^n T_p)^{I_\ell},$$

so $H^1\big(\mathscr{G}_\ell, (T_p/p^n T_p)^{I_\ell}\big)$ is finite for all $n \in \mathbb{N}$. By [154, Corollary 2.2], it follows that there is an isomorphism

$$H^2\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) \simeq \varprojlim_n H^2\big(\mathscr{G}_\ell, (T_p/p^n T_p)^{I_\ell}\big). \tag{2.68}$$

Finally, since $(T_p/p^n T_p)^{I_\ell}$ is finite, hence torsion, [144, Ch. XIII, Proposition 2] ensures that

$$H^2\big(\mathscr{G}_\ell, (T_p/p^n T_p)^{I_\ell}\big) = 0 \tag{2.69}$$

for all $n \in \mathbb{N}$. Combining (2.68) and (2.69), we conclude that $H^2\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) = 0$. □

Recall that $H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_p) = H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big)$ and that Assumption 2.64 is in force.

Proposition 2.75 $\#H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_p) < \infty$.

Proof The kernel of the natural map $H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) \to H^1\big(\mathscr{G}_\ell, V_p^{I_\ell}\big)$ is isomorphic to a quotient of $H^0\big(\mathscr{G}_\ell, V_p^{I_\ell}/T_p^{I_\ell}\big)$, so it is torsion over $\mathcal{O}_p$. By Lemma 2.65, $H^1\big(\mathscr{G}_\ell, V_p^{I_\ell}\big) = 0$, and then $H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big)$ is $\mathcal{O}_p$-torsion. Since $H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) = \prod_{\mathfrak{p}\mid p} H^1\big(\mathscr{G}_\ell, T_{\mathfrak{p}}^{I_\ell}\big)$ as $\mathcal{O}_p$-modules, we deduce that $H^1\big(\mathscr{G}_\ell, T_{\mathfrak{p}}^{I_\ell}\big)$ is torsion over $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$. Furthermore, since $T_{\mathfrak{p}}$ is finitely generated over $\mathcal{O}_{\mathfrak{p}}$, it follows from [134, Proposition B.2.7] that $H^1(G_{\mathbb{Q}_\ell}, T_{\mathfrak{p}})$ is finitely generated over $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$, and then the same is true of its $\mathcal{O}_{\mathfrak{p}}$-submodule $H^1\big(\mathscr{G}_\ell, T_{\mathfrak{p}}^{I_\ell}\big)$. We conclude that $H^1\big(\mathscr{G}_\ell, T_{\mathfrak{p}}^{I_\ell}\big)$ is finite for all $\mathfrak{p} \mid p$, which implies that $H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_p) = H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big)$ is finite. □

Let S be the finite set of places of $\mathbb{Q}$ that was fixed in (2.58) and recall the notational convention from (2.66). Set $\mathcal{W}_p := A_p^{I_\ell} \big/ (A_p^{I_\ell})_{\mathrm{div}}$. The next result is a slight refinement, in our modular context, of [43, Ch. I, Proposition 4.2.2].

Proposition 2.76 *(1) If $\ell \notin S$, then* $\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) \simeq \mathcal{O}_p$.
(2) $\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) \simeq \mathcal{I}\big(\mathcal{W}_p^{\mathrm{Frob}_\ell = 1}\big)$ *for every ℓ.*

Proof Part (1) is [43, Ch. I, Proposition 4.2.2, i)]. As for part (2), the inflation-restriction exact sequence yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1\big(\mathscr{G}_\ell, T_p^{I_\ell}\big) & \longrightarrow & H^1(G_{\mathbb{Q}_\ell}, T_p) & \longrightarrow & H^1(I_\ell, T_p)^{G_{\mathbb{Q}_\ell}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1\big(\mathscr{G}_\ell, V_p^{I_\ell}\big) & \longrightarrow & H^1(G_{\mathbb{Q}_\ell}, V_p) & \longrightarrow & H^1(I_\ell, V_p)^{G_{\mathbb{Q}_\ell}} & \longrightarrow & 0
\end{array} \tag{2.70}$$

in which the surjectivity of the right non-trivial arrows on both rows follows from Lemma 2.74. As a consequence of Lemma 2.65, $H^1_f(\mathbb{Q}_\ell, T_p)$ is the kernel of the middle vertical map in diagram (2.70). By [154, Proposition 2.3], the kernels of the middle and the right vertical arrows in (2.70) are $H^1(G_{\mathbb{Q}_\ell}, T_p)_{\mathrm{tors}}$ and $H^1(I_\ell, T_p)^{G_{\mathbb{Q}_\ell}}_{\mathrm{tors}}$, respectively. By applying the snake lemma to (2.70), we get a short exact sequence

$$0 \longrightarrow H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_p) \longrightarrow H^1_f(\mathbb{Q}_\ell, T_p) \longrightarrow H^1(I_\ell, T_p)^{G_{\mathbb{Q}_\ell}}_{\mathrm{tors}} \longrightarrow 0. \tag{2.71}$$

By [43, Ch. I, Proposition 4.2.2, ii)], we know that

$$\mathrm{Tam}^{(p)}_\ell(\mathcal{M}) = \mathrm{Det}_{\mathcal{O}_p}\Big(H^1(I_\ell, T_p)^{G_{\mathbb{Q}_\ell}}_{\mathrm{tors}}\Big). \tag{2.72}$$

On the other hand, by [134, Lemma 1.3.5, (iii)], there is an isomorphism of $\mathcal{O}_p$-modules

$$H^1_f(\mathbb{Q}_\ell, T_p)\big/H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_p) \simeq \mathcal{W}_p^{\mathrm{Frob}_\ell=1}. \tag{2.73}$$

The desired result follows by combining (2.71), (2.72) and (2.73). □

2.21.2 The prime p

Now we consider the case $\ell = p$. There is an exact sequence of F_p-modules

$$0 \longrightarrow H^0_f(\mathbb{Q}_p, V_p) \longrightarrow \mathbf{D}_{\mathrm{cris}}(V_p) \xrightarrow{(\varphi-1,\mathrm{pr})} \mathbf{D}_{\mathrm{cris}}(V_p)\oplus t(\mathcal{M})_p \longrightarrow H^1_f(\mathbb{Q}_p, V_p) \longrightarrow 0. \tag{2.74}$$

By [15, Corollary 3.8.4], $H^0_f(\mathbb{Q}_p, V_p) = H^0(\mathbb{Q}_p, V_p)$, so $H^0_f(\mathbb{Q}_p, V_p) = 0$ by Lemma 2.66. Taking determinants in (2.74), we obtain an isomorphism

$$\vartheta_p : \mathrm{Det}^{-1}_{F_p}\big(H^1_f(\mathbb{Q}_p, V_p)\big) \xrightarrow{\simeq} \mathrm{Det}^{-1}_{F_p}\big(t_p(V_p)\big).$$

Define the $\mathcal{O}_p$-submodule

$$\Lambda_p := \vartheta_p\Big(\mathrm{Det}^{-1}_{\mathcal{O}_p}\big(H^1_f(\mathbb{Q}_p, T_p)\big)\Big)$$

of $\mathrm{Det}^{-1}_{F_p}\big(t_p(V_p)\big)$. Recall that $t_p(V_p)$ is a free F_p-module of rank 1 and fix an F_p-generator ω of $t(\mathcal{M})_p$. Then ω is a generator of the free F_p-module $\mathrm{Det}_{F_p}\big(t(\mathcal{M})_p\big)$ of rank 1; moreover, $\mathrm{Det}^{-1}_{F_p}\big(t(\mathcal{M})_p\big)$ is free of rank 1 over F_p, and we let ω^{-1} denote the generator of $\mathrm{Det}^{-1}_{F_p}\big(t(\mathcal{M})_p\big)$ corresponding to ω, which is characterized by $\omega^{-1}(\omega) = 1$. Now Λ_p is an $\mathcal{O}_p$-submodule of free F_p-module $\mathrm{Det}^{-1}_{F_p}\big(t(\mathcal{M})_p\big) = F_p \cdot \omega^{-1}$, so there exists an $\mathcal{O}_p$-ideal $\mathrm{Tam}_{p,\omega}(A_p)$ such that

$$\Lambda_p = \mathrm{Tam}_{p,\omega}(A_p) \cdot \omega^{-1}. \tag{2.75}$$

With T_{B} as in (2.5) and ϕ_∞ the involution from Sect. 2.4.2, we define $T^+_{\mathrm{B}} := T_{\mathrm{B}}^{\phi_\infty=1}$ and $T^+_p := H^0(\mathbb{R}, T_p)$. There are comparison isomorphisms

$$\mathrm{Comp}_{\mathrm{B,\acute{e}t}} : T^+_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} T^+_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \mathrm{Comp}_{\mathrm{B,dR}} : V^+_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} t(V_p). \tag{2.76}$$

We deduce from (2.44) that there is an induced isomorphism of $\mathcal{O}_p$-modules

$$\mathrm{Comp}_{\mathrm{B,\acute{e}t}} : T_{\mathrm{B}}^+ \otimes_{\mathcal{O}_F} \mathcal{O}_p \xrightarrow{\simeq} T_p^+.$$

Choose $\delta_f \in T_{\mathrm{B}}^+ \smallsetminus \{0\}$ and set $\omega_{\delta_f} := \mathrm{Comp}_{\mathrm{B,dR}}(\delta_f) \in t(V_p)$.

Definition 2.77 The *p-part of the Tamagawa ideal of $\mathcal{M}$ at p* is

$$\mathrm{Tam}_p^{(p)}(\mathcal{M}) := \mathrm{Tam}_{p,\omega_{\delta_f}}(A_p),$$

where $\mathrm{Tam}_{p,\omega_{\delta_f}}(A_p)$ is defined as in (2.75).

Now assume that $\mathrm{Comp}_{\mathrm{B,\acute{e}t}}(\delta_f)$ is an $\mathcal{O}_p$-generator of T_p^+ (later on, this condition will be satisfied for a suitable choice of p). Then, since V_p is crystalline, we actually have

$$\mathrm{Tam}_p^{(p)}(\mathcal{M}) = \mathcal{O}_p. \tag{2.77}$$

This follows from the proof in [5, Theorem 4.2.2] of Conjecture $C_{EP,\mathbb{Q}_p}(V)$ from [43, §4.5.4] (*cf.* also [6, Proposition II.2], and [125, Proposition 4.2.5] in the ordinary case, which will be the setting of interest for us in subsequent sections).

2.21.3 The archimedean prime

In the archimedean case, we introduce (the p-part of) the Tamagawa ideal of $\mathcal{M}$ as follows.

Definition 2.78 The *p-part of the Tamagawa ideal of $\mathcal{M}$ at ∞* is

$$\mathrm{Tam}_\infty^{(p)}(\mathcal{M}) := \mathrm{Det}_{\mathcal{O}_p}^{-1}\big(H_f^1(\mathbb{R}, T_p)\big).$$

This definition completes the list of the p-parts of Tamagawa ideals of $\mathcal{M}$. When p is odd, it can be checked that

$$\mathrm{Tam}_\infty^{(p)}(\mathcal{M}) = \mathcal{O}_p \tag{2.78}$$

(see, *e.g.*, [36, p. 708]).

2.22 Compact cohomology

Our goal is to calculate $\mathrm{Det}_{\mathcal{O}_p}\big(\mathbf{R}\Gamma_c(G_S, T_p)\big)$ as defined in (A.8), where $\mathbf{R}\Gamma_c(G_S, T_p)$ is the compact complex from Sect. 2.13.2.

2.22.1 A vanishing lemma

We begin with a basic vanishing result for cohomology with compact support.

Lemma 2.79 $H_c^0(G_S, T_p) = 0$.

Proof By (2.35), the group $H_c^0(G_S, T_p)$ consists of the elements of $H_f^0(\mathbb{Q}, T_p)$ whose image in $H_f^0(\mathbb{Q}_v, T_p)$ is zero for all $v \in S$. Now $H_f^0(\mathbb{Q}, T_p)$ injects into $H_f^0(\mathbb{Q}_p, T_p)$ and $H_f^0(\mathbb{Q}_p, T_p)$ injects into $H_f^0(\mathbb{Q}_p, V_p)$; since $p \in S$ and $H_f^0(\mathbb{Q}_p, V_p) = 0$ by Lemma 2.66, we conclude that $H_f^0(\mathbb{Q}, T_p) = 0$, as was completing the proof. □

2.22.2 *Computing* $\mathrm{Det}_{\mathcal{O}_p}\left(\mathbf{R}\Gamma_c(G_S, T_p)\right)$

The next result computes $\mathrm{Det}_{\mathcal{O}_p}\left(\mathbf{R}\Gamma_c(G_S, T_p)\right)$.

Proposition 2.80 *There is a canonical isomorphism*

$$\mathrm{Det}_{\mathcal{O}_p}\left(\mathbf{R}\Gamma_c(G_S, T_p)\right) \simeq \mathrm{Det}^{-1}_{\mathcal{O}_p}\left(H^1_f(\mathbb{Q}, T_p)\right) \cdot \mathrm{Det}_{\mathcal{O}_p}\left(H^1_f(\mathbb{Q}, A_p)^\vee\right) \cdot \mathrm{Det}^{-1}_{\mathcal{O}_p}\left(H^0(G_S, A_p)^\vee\right) \cdot \prod_{v\in S} \mathrm{Det}_{\mathcal{O}_p}\left(H^1_f(\mathbb{Q}_v, T_p)\right) \cdot \mathrm{Det}^{-1}_{\mathcal{O}_p}(T_p^+).$$

Proof Fix an integer $n \geq 1$ and write $A_p[p^n]$ for the p^n-torsion subgroup of A_p. Since $\mathcal{M}$ is self-dual, there is a canonical isomorphism $A_p[p^n] \simeq \mathrm{Hom}_{\mathbb{Z}_p}\left(A_p[p^n], (\mathbb{Q}_p/\mathbb{Z}_p)(1)\right)$, so the Poitou–Tate exact sequence (see, *e.g.*, [114, §5.1.6]) gives a long exact sequence

$$H^1\left(G_S, A_p[p^n]\right) \longrightarrow \bigoplus_{v\in S} H^1\left(\mathbb{Q}_v, A_p[p^n]\right) \longrightarrow H^1\left(G_S, A_p[p^n]\right)^\vee \longrightarrow H^2\left(G_S, A_p[p^n]\right) \longrightarrow \bigoplus_{v\in S} H^2\left(\mathbb{Q}_v, A_p[p^n]\right) \longrightarrow H^0\left(G_S, A_p[p^n]\right)^\vee \longrightarrow 0.$$

Since these groups are finite, passing to inverse limits on n produces an exact sequence

$$H^1(G_S, T_p) \longrightarrow \bigoplus_{v\in S} H^1(\mathbb{Q}_v, T_p) \longrightarrow H^1(G_S, A_p)^\vee \longrightarrow H^2(G_S, T_p) \longrightarrow \bigoplus_{v\in S} H^2(\mathbb{Q}_v, T_p) \longrightarrow H^0(G_S, A_p)^\vee \longrightarrow 0.$$

Using Lemma 2.63, we get an exact sequence

$$0 \longrightarrow H^1_f(\mathbb{Q}, T_p) \longrightarrow H^1(G_S, T_p) \longrightarrow \bigoplus_{v\in S} H^1_s(\mathbb{Q}_v, T_p) \longrightarrow H^1_f(\mathbb{Q}, A_p)^\vee \longrightarrow H^2(G_S, T_p) \longrightarrow \bigoplus_{v\in S} H^2(\mathbb{Q}_v, T_p) \longrightarrow H^0(G_S, A_p)^\vee \longrightarrow 0. \tag{2.79}$$

On the other hand, using Lemmas 2.65, 2.66 and 2.79, the definition of cohomology with compact support (*cf.* Sect. 2.13.2) yields an exact sequence

$$0 \longrightarrow H^0(\mathbb{R}, T_p) \longrightarrow H^1_c(G_S, T_p) \longrightarrow H^1(G_S, T_p) \longrightarrow \bigoplus_{s\in S} H^1(\mathbb{Q}_v, T_p) \longrightarrow H^2_c(G_S, T_p) \longrightarrow H^2(G_S, T_p) \longrightarrow \bigoplus_{s\in S} H^2(\mathbb{Q}_v, T_p) \longrightarrow H^3_c(G_S, T_p) \longrightarrow 0. \tag{2.80}$$

Taking $\mathrm{Det}_{\mathcal{O}_p}$ of (2.79) and (2.80), and using the multiplicativity of determinants, gives the desired result. □

2.23 A reformulation of *p*-TNC

We want to reformulate the p-part of Conjecture 2.56 in a convenient way. We assume the p-part of Conjecture 2.50 over $\mathbb{Q}$.

2.23.1 p-torsion of $\mathcal{M}$

Let $H^1(\mathbb{Q}, T_p)_{\mathrm{tors}}$ be the torsion submodule of $H^1(\mathbb{Q}, T_p)$.

Definition 2.81 The *p-torsion part of* $\mathcal{M}$ is

$$\mathrm{Tors}_p(\mathcal{M}) := \mathcal{I}^{-1}\big(H^0(G_S, A_p)^\vee\big) \cdot \mathcal{I}^{-1}\big(H^1(\mathbb{Q}, T_p)_{\mathrm{tors}}\big).$$

The $\mathcal{O}_p$-module $\mathrm{Tors}_p(\mathcal{M})$ will play a role in Theorem 2.83.

2.23.2 Some linear algebra of lattices

Let $\mathscr{B} = \{t_1, \dots, t_r\}$ be a basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$ as an F-vector space; this is also a basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_p$ over F_p. Recall that reg_p is a shorthand for $\mathrm{reg}_{\mathbb{Q},p}$ and for each $i \in \{1, \dots, r\}$ put $x_i := \mathrm{reg}_p(t_i)$. We are assuming the p-part of Conjecture 2.50 over $\mathbb{Q}$, so $\tilde{\mathscr{B}} := \{x_1, \dots, x_r\}$ is a basis of $H^1_f(\mathbb{Q}, V_p)$ as an F_p-module. As in Sect. 2.20, write $\Lambda_{\tilde{\mathscr{B}}}$ for the free $\mathcal{O}_p$-submodule of $H^1_f(\mathbb{Q}, V_p)$ of rank r generated by $\tilde{\mathscr{B}}$. Let $\{\xi_1, \dots, \xi_r\}$ be an $\mathcal{O}_p$-basis of $\underline{H}^1_f(\mathbb{Q}, T_p)$ and let $\mathrm{A}_{\tilde{\mathscr{B}}} \in \mathrm{GL}_r(F_p)$ be the transition matrix from the F_p-basis $\tilde{\mathscr{B}}$ to the F_p-basis $\{\xi_1, \dots, \xi_r\}$ of $H^1_f(\mathbb{Q}, V_p)$. Let $\xi_1^\vee, \dots, \xi_r^\vee \in H^1_f(\mathbb{Q}, V_p)^\vee$ be the dual elements from Sect. 2.20 (*cf.* also Sect. B.3) and define $x_1^\vee, \dots, x_r^\vee$ in an analogous way.

Remark 2.82 If $\Lambda_{\tilde{\mathscr{B}}} = \underline{H}^1_f(\mathbb{Q}, T_p)$, then $\mathrm{A}_{\tilde{\mathscr{B}}} \in \mathrm{GL}_r(\mathcal{O}_p)$. In general, $\det(\mathrm{A}_{\tilde{\mathscr{B}}})$ depends on the choice of an $\mathcal{O}_p$-basis of $\underline{H}^1_f(\mathbb{Q}, T_p)$ only up to multiplication by elements of $\mathcal{O}_p^\times$, which shows that the principal fractional $\mathcal{O}_p$-ideal $\big(\det(\mathrm{A}_{\tilde{\mathscr{B}}})\big)$ is independent of the choice of a basis of $\underline{H}^1_f(\mathbb{Q}, T_p)$ over $\mathcal{O}_p$. On the other hand, keeping $\{\xi_1, \dots, \xi_r\}$ fixed, if $\mathscr{B}'$ is another F-basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$, then $\det(\mathrm{A}_{\tilde{\mathscr{B}}})$ and $\det(\mathrm{A}_{\tilde{\mathscr{B}}'})$ differ by multiplication by the determinant of the transition matrix from $\mathscr{B}$ to $\mathscr{B}'$.

2.23.3 Reformulation of p-TNC

Recall the comparison isomorphism $\mathrm{Comp}_{\mathrm{B,ét}}$ from (2.76). In particular, $\mathrm{Comp}^{-1}_{\mathrm{B,ét}}(T_p^+)$ is an $\mathcal{O}_p$-submodule of the free F_p-module $V_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p$ of rank 1. As in Sect. 2.21.2, choose $\gamma_f \in T_{\mathrm{B}}^+ \smallsetminus \{0\}$ and let $\Omega_\infty := \Omega_\infty^{(\gamma_f)} \in F_\infty^\times$ be the period from Sect. 2.5.2. Let us consider the $\mathcal{O}_p$-submodule Λ_{γ_f} of T_p^+ generated by $\mathrm{Comp}_{\mathrm{B,ét}}(\gamma_f)$ and set $\mathcal{I}_p(\gamma_f) := \mathcal{I}\big(T_p^+/\Lambda_{\gamma_f}\big)$. Let us define the period

$$\Omega_{\mathcal{M}} := \frac{\Omega_\infty}{(2\pi i)^{k/2}} \in (F \otimes_{\mathbb{Q}} \mathbb{C})^\times. \tag{2.81}$$

From here on, in order to simplify some formulas, for two invertible $\mathcal{O}_p$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ we shall write $\frac{\mathfrak{a}}{\mathfrak{b}}$ in place of $\mathfrak{a} \cdot \mathfrak{b}^{-1}$. With all our current notation in force (in particular, recall the Shafarevich–Tate group $Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M})$ from part (1) of Definition 2.60 and the matrix $\mathrm{A}_{\tilde{\mathscr{B}}} \in \mathrm{GL}_r(F_p)$ in Sect. 2.23.2), we can state the reformulation of the p-part of Conjecture 2.56 we are interested in.

Theorem 2.83 *Assume that*

(1) Conjecture 2.18 *holds true;*
(2) Conjecture 2.40 *(or, equivalently,* Conjecture 2.41*) holds true;*
(3) *the p-part of* Conjecture 2.50 *for* $K = \mathbb{Q}$ *holds true.*

The p-part of Conjecture 2.56 *is equivalent to the equality*

$$\left(\frac{\Lambda^*(\mathcal{M},0)}{\Omega_{\mathcal{M}}\cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}\right) = \frac{\mathcal{I}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot \mathcal{I}_p(\gamma_f)\cdot \prod_{v\in S}\mathrm{Tam}_v^{(p)}(\mathcal{M})}{\big(\det(\mathrm{A}_{\tilde{\mathscr{B}}})\big)^2\cdot \mathrm{Tors}_p(\mathcal{M})} \qquad (p\text{-TNC}_{\mathscr{B}})$$

of fractional $\mathcal{O}_p$-ideals.

Observe that assumption (3) in the statement of the theorem ensures that Conjecture 2.12 for $K=\mathbb{Q}$ holds true (*cf.* Remark 2.52). In order to make sense of equality (p-TNC$_{\mathscr{B}}$), in particular of how the left-hand side is seen as a principal fractional $\mathcal{O}_p$-ideal, the reader is referred to Remark 2.8. With notation as in (2.22), it is also useful to bear in mind that $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M}) = \big(\mathrm{Reg}^{\sigma}_{\mathscr{B}}(\mathcal{M})\big)_{\sigma\in\Sigma}$.

Remark 2.84 Assumption (1) in Theorem 2.83 is imposed only to force $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})$ to be non-zero: if we know that $\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})\neq 0$, we can remove this non-degeneracy requirement.

Remark 2.85 Suppose that $\mathscr{B}$ and $\mathscr{B}'$ are bases of $H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})$ over F. Combining Remarks 2.22 and 2.82, one sees that (p-TNC$_{\mathscr{B}}$) holds if and only if (p-TNC$_{\mathscr{B}'}$) holds.

Remark 2.86 By Remark 2.82, the $\mathcal{O}_p$-ideal $\big(\det(\mathrm{A}_{\tilde{\mathscr{B}}})\big)$ on the right-hand side of (p-TNC$_{\mathscr{B}}$) is independent of the choice of an $\mathcal{O}_p$-basis of $\underline{H}^1_f(\mathbb{Q},T_p)$. Moreover, if $\Lambda_{\tilde{\mathscr{B}}} = \underline{H}^1_f(\mathbb{Q},T_p)$, then $\big(\det(\mathrm{A}_{\tilde{\mathscr{B}}})\big)=\mathcal{O}_p$; in accord with this fact, one can check that the left-hand side of (p-TNC$_{\mathscr{B}}$) does not depend on $\mathscr{B}$ if $\mathscr{B}$ varies over all F-bases of $H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})$ such that $\Lambda_{\tilde{\mathscr{B}}}=\underline{H}^1_f(\mathbb{Q},T_p)$.

Proof of Theorem 2.83 If γ_p is a generator of T_p^+, then $\mathrm{Det}_{\mathcal{O}_p}(T_p^+)=\mathcal{I}_p(\gamma_f)\cdot\gamma_p$. As before, given a ring R, an R-module M and $m_1,\dots,m_r\in M$, set $\underline{m}:=m_1\wedge\cdots\wedge m_r$. Keeping Remark/Notation A.2 in mind, there is an isomorphism of $\mathcal{O}_p$-modules

$$\mathrm{Det}^{-1}_{\mathcal{O}_p}\big(H^1_f(\mathbb{Q},T_p)\big)\simeq \mathcal{I}^{-1}\big(H^1_f(\mathbb{Q},T_p)_{\mathrm{tors}}\big)\cdot\underline{\xi}^{-1} = \det(\mathrm{A}_{\tilde{\mathscr{B}}})\cdot\mathcal{I}^{-1}\big(H^1_f(\mathbb{Q},T_p)_{\mathrm{tors}}\big)\cdot\underline{x}^{-1}.$$

By Corollary 2.72 and the convention introduced at the end of Sect. 2.20, $\{\xi_1^\vee,\dots,\xi_r^\vee\}$ is an $\mathcal{O}_p$-basis of $H^1_f(\mathbb{Q},A_p)^\vee_{\mathrm{div}}$. Therefore, setting $\underline{\xi}^\vee:=\xi_1^\vee\wedge\cdots\wedge\xi_r^\vee$ and $\underline{x}^\vee:=x_1^\vee\wedge\cdots\wedge x_r^\vee$, there is an isomorphism of $\mathcal{O}_p$-modules

$$\mathrm{Det}_{\mathcal{O}_p}\big(H^1_f(\mathbb{Q},A_p)^\vee\big)\simeq \mathcal{I}^{-1}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot\underline{\xi}^\vee = \det(\mathrm{A}_{\tilde{\mathscr{B}}})\cdot\mathcal{I}^{-1}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot\underline{x}^\vee.$$

Recall the isomorphism of F_p-modules $\theta_{p,S}$ from (2.47). Combining Proposition 2.76 with Proposition 2.80 and Definitions 2.77, 2.78 and 2.81, we get an isomorphism of $\mathcal{O}_p$-modules

$$\mathrm{Det}_{\mathcal{O}_p}\big(\mathbf{R}\Gamma_c(G_S,T_p)\big)\simeq \frac{\det(\mathrm{A}_{\tilde{\mathscr{B}}})^2\cdot\mathrm{Tors}_p(\mathcal{M})}{\mathcal{I}_p(\gamma_f)\cdot\mathcal{I}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot\prod_{v\in S}\mathrm{Tam}_v^{(p)}(\mathcal{M})}\cdot\theta_{p,S}(\beta_f), \quad (2.82)$$

where, as above, $\beta_f:=\underline{x}^{-1}\otimes\underline{x}^\vee\otimes\gamma_f^{-1}\otimes\omega_f^\vee\in\Delta(\mathcal{M})$. Let $\zeta_f\in\Delta(\mathcal{M})$ be as in Conjecture 2.40 and write $\zeta_f=a\beta_f$ for some $a\in F^\times$. By Proposition 2.43 and its proof, there is an equality

$$\frac{1}{a}=\frac{L^*(\mathcal{M},0)}{\Omega_\infty\cdot\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}, \quad (2.83)$$

which should be understood as in Remark 2.8. By (2.82) there is an isomorphism

$$\frac{\theta_{p,S}(\zeta_f)}{a}\cdot\mathcal{O}_p \simeq \mathrm{Det}_{\mathcal{O}_p}\big(\mathbf{R}\Gamma_c(G_S,T_p)\big)\cdot\det(\mathrm{A}_{\mathscr{B}})^{-2}\cdot\mathcal{I}_p(\gamma_f)\cdot\mathrm{Tors}_p(\mathcal{M})^{-1}\cdot\mathcal{I}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot\prod_{v\in S}\mathrm{Tam}_v^{(p)}(\mathcal{M}). \tag{2.84}$$

Combining (2.83), (2.84), the relation $\zeta_f^* = \zeta_f/(k/2-1)!$, formula (2.25) and the definition of $\Omega_{\mathcal{M}}$ given in (2.81) shows that the equality $\theta_{p,S}(\zeta_f^*)\cdot\mathcal{O}_p = \mathrm{Det}_{\mathcal{O}_p}\big(\mathbf{R}\Gamma_c(G_S,T_p)\big)$ predicted by the p-part of Conjecture 2.56 for $\mathcal{O}=\mathcal{O}_F$ (*cf.* (2.51)) is equivalent to equality (p-TNC$_{\mathscr{B}}$), as claimed. □

Remark 2.87 Equality (p-TNC$_{\mathscr{B}}$) is equivalent to the equality

$$\left(\frac{(k/2-1)!\cdot L^*(\mathcal{M},0)}{\Omega_\infty\cdot\mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}\right) = \frac{\mathcal{I}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big)\cdot\mathcal{I}_p(\gamma_f)\cdot\prod_{v\in S}\mathrm{Tam}_v^{(p)}(\mathcal{M})}{\big(\det(\mathrm{A}_{\mathscr{B}})\big)^2\cdot\mathrm{Tors}_p(\mathcal{M})} \tag{p-TNC$_{\mathscr{B}}$-bis}$$

of fractional $\mathcal{O}_p$-ideals: this follows immediately from (2.25).

To the best of our knowledge, Theorem 2.83 offers the first reformulation of this form of (the p-part of) the TNC for $\mathcal{M}$ in arbitrary analytic rank; the reader is referred to [38] for a similar interpretation in analytic rank 0.

3 Preliminaries on modular forms

Let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. As in Section 2, let $f\in S_k(\Gamma_0(N))$ be a newform of weight $k\geq 4$ and level N, with q-expansion $f(q)=\sum_{n\geq 1}a_n(f)q^n$. From here on we assume, as in the introduction, that

- f has no complex multiplication in the sense of [130, p. 34, Definition].

As before, let F be the Hecke field of f; by construction, it is naturally a subfield of $\bar{\mathbb{Q}}$. We write $\mathcal{O}_F$ (respectively, D_F) for the ring of integers (respectively, the discriminant) of F. Finally, let $\mathcal{O}_f := \mathbb{Z}\big[a_n(f)\mid n\geq 1\big]$ be the order of $\mathcal{O}_F$ generated over $\mathbb{Z}$ by the Fourier coefficients $a_n(f)$ and let $c_f := [\mathcal{O}_F:\mathcal{O}_f]$ be the index of $\mathcal{O}_f$ in $\mathcal{O}_F$.

Remark 3.1 A sufficient condition for f not to have complex multiplication is that N be square-free (*cf.* [130, p. 34]), which will be assumed in due course.

3.1 Big image assumptions

We collect two basic results on the Galois representations attached to f, which will allow us to introduce our assumption on these representations.

3.1.1 Big image

Let p be a prime number. With notation as in Sect. 2.4.3, denote by

$$\rho_p : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{O}_p}(T_{f,p}) \simeq \mathrm{GL}_2(\mathcal{O}_p)$$

the p-adic Galois representation attached to f and p. We say that ρ_p has *big image* if there is an inclusion

$$\left\{g \in \mathrm{GL}_2(\mathcal{O}_p) \mid \det(g) \in (\mathbb{Z}_p^\times)^{k-1}\right\} \subset \mathrm{im}(\rho_p).$$

Theorem 3.2 (Ribet) *The representation ρ_p has big image for all but finitely many p.*

Proof Since f is not CM, the theorem follows from [133, Theorem 3.1]. □

See [88] for an open image result for the adelic Galois representation attached to f.

3.1.2 Residual irreducibility

With notation as above, if $\mathfrak{p}$ is a prime of F above p, then we denote by

$$\rho_{\mathfrak{p}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{O}_{\mathfrak{p}}}(T_{f,\mathfrak{p}}) \simeq \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$$

the Galois representation associated with $T_{f,\mathfrak{p}}$. Reducing modulo the maximal ideal of $\mathcal{O}_{\mathfrak{p}}$, we obtain a residual representation

$$\bar{\rho}_{\mathfrak{p}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{F}_{\mathfrak{p}}}(T_{f,\mathfrak{p}}/\mathfrak{p}T_{f,\mathfrak{p}}) \simeq \mathrm{GL}_2(\mathbb{F}_{\mathfrak{p}}),$$

where $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ is the residue field of $F_{\mathfrak{p}}$. Moreover, write $\bar{\rho}_{\mathfrak{p}}^{\dagger}$ for the self-dual twist of $\bar{\rho}_{\mathfrak{p}}$, *i.e.*, the representation attached to $T_{f,\mathfrak{p}}^{\dagger} \simeq T_{\mathfrak{p}}$.

Proposition 3.3 *If p is a prime number such that ρ_p has big image, then the image of $\bar{\rho}_{\mathfrak{p}}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ for every prime $\mathfrak{p}$ of F above p. In particular, $\bar{\rho}_{\mathfrak{p}}$ and $\bar{\rho}_{\mathfrak{p}}^{\dagger}$ are irreducible for every $\mathfrak{p}$ above p.*

Proof Let p be a prime number such that ρ_p has big image and let $\mathfrak{p}$ be a prime of F above p. Then the image of $\rho_{\mathfrak{p}}$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$, which implies that the image of $\bar{\rho}_{\mathfrak{p}}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ and $\bar{\rho}_{\mathfrak{p}}$ is irreducible. Finally, the irreducibility of a representation is preserved by tensorization with 1-dimensional representations (see, *e.g.*, [84, Exercise 2.2.14, (2)]), so the irreducibility of $\bar{\rho}_{\mathfrak{p}}^{\dagger}$ follows from that of $\bar{\rho}_{\mathfrak{p}}$. □

Combining Theorem 3.2 and Proposition 3.3, we get that, for all but finitely many prime numbers p, the representation $\bar{\rho}_{\mathfrak{p}}^{\dagger}$ is irreducible for each prime $\mathfrak{p}$ of F above p (*cf.* also [133, Theorem 2.1, (a)]).

Remark 3.4 By a standard group-theoretical result (see, *e.g.*, [146, Ch. IV, §3.4, Lemma 3]), if the image of $\bar{\rho}_{\mathfrak{p}}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, then the image of $\rho_{\mathfrak{p}}$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$. Therefore, the converse to Proposition 3.3 holds as well. It is worth pointing out that the big image condition that we will impose in Assumptions 3.6, 4.5 and 5.38 will always be used in the guise of Proposition 3.3 (see, *e.g.*, Sect. 4.3.2, the proofs of Propositions 4.3 and 4.8, the proof of Theorem 5.29).

Remark 3.5 It is a natural problem to study the finitely many "exceptional" primes p for which Theorem 3.2 fails. Effective versions of the classical "open image" theorem of Serre for Galois representations attached to elliptic curves without complex multiplication [143] have been amply investigated in various directions (see, *e.g.*, [30,85,89,98,167]);

in particular, a result of Mazur says that the mod ℓ Galois representation associated with a semistable elliptic curve over $\mathbb{Q}$ is surjective for every prime number $\ell \geq 11$ ([100, Theorem 4]). On the contrary, much less seems to be known for Galois representations attached to higher (even) weight modular forms: the reader is referred to [11] for a description of the primes p for which Theorem 3.2 (or, rather, its consequence for mod $\mathfrak{p}$ representations, *cf.* Remark 3.4) does not hold. Here we would like to point out that, later in this paper (see, *e.g.*, Assumption 4.5), we will need to assume that $k \equiv 2 \pmod{2(p-1)}$, so that $p < k$. Therefore, as apparent from [11, Theorem 3.2], the "big image" property is a constraint we will need to impose on the pair (f, p): in particular, we are not allowed to simply take p to be "large enough" with respect to k.

3.1.3 Assumptions on Galois representations

To state (in Section 4) Kolyvagin's conjecture for higher (even) weight newforms, we will work under

Assumption 3.6 The prime number p satisfies the following conditions:

(1) $p \nmid 6ND_F c_f$;
(2) ρ_p has big image.

By Theorem 3.2, all but finitely many primes p satisfy Assumption 3.6 (we remark that the set of "good" primes p with this property is clearly not uniform in f).

3.2 p-isolation

Let us write $\mathscr{H}_k(\Gamma_0(N)) \subset \mathrm{End}_{\bar{\mathbb{Q}}}\big(S_k(\Gamma_0(N); \bar{\mathbb{Q}})\big)$ for the *full* Hecke algebra of weight k and level $\Gamma_0(N)$, which is generated over $\mathbb{Z}$ by the Hecke operators T_ℓ for all primes $\ell \nmid N$ and the Hecke operators U_ℓ for primes $\ell \mid N$. Moreover, for any $\mathbb{Z}$-algebra A we set $\mathscr{H}_k(\Gamma_0(N))_A := \mathscr{H}_k(\Gamma_0(N)) \otimes_{\mathbb{Z}} A$.

Proposition 3.7 *Put* $d_k := \dim_{\mathbb{C}} S_k(\Gamma_0(N)) \in \mathbb{N}$. *For any* $\mathbb{Z}$*-algebra* A*, the* A*-module* $\mathscr{H}_k(\Gamma_0(N))_A$ *is free of rank* d_k*; in particular,* $\mathscr{H}_k(\Gamma_0(N))_A$ *is flat over* A.

Proof It is well known that $S_k(\Gamma_0(N); \mathbb{Z})$ is free of rank d_k over $\mathbb{Z}$ (see, *e.g.*, [147, Theorem 3.52]) and that there is an isomorphism $\mathscr{H}_k(\Gamma_0(N)) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(S_k(\Gamma_0(N); \mathbb{Z}), \mathbb{Z}\big)$ induced by the natural duality between Hecke algebras and spaces of modular forms (see, *e.g.*, [131, §2]). This proves the proposition for $A = \mathbb{Z}$, and the statement for a general $\mathbb{Z}$-algebra A follows by extending scalars from $\mathbb{Z}$ to A. □

3.2.1 p-isolation of f

Let $g(q) = \sum_{n \geq 1} a_n(g) q^n \in S_k(\Gamma_0(N))$ be a normalized eigenform for $\mathscr{H}_k(\Gamma_0(N))$ and let $L := F\big(a_n(g) \mid n \geq 1\big)$ be the composite of F and the Hecke field of g. Let p be a prime number and pick a prime $\mathfrak{p}$ of F above p. The form f is said to be *congruent to* g *modulo* $\mathfrak{p}$ if

$$a_n(f) \equiv a_n(g) \pmod{\mathfrak{P}}$$

for some prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and for all $n \geq 1$. In this case, we write $f \equiv g \pmod{\mathfrak{p}}$ and say that $\mathfrak{p}$ (and $\mathfrak{P}$) is a *congruence prime for* f.

Definition 3.8 The form f is *p-isolated* if there is no normalized eigenform $g \in S_k(\Gamma_0(N))$ other than f such that $f \equiv g \pmod{\mathfrak{p}}$ for some prime $\mathfrak{p}$ of F above p.

The next result tells us that, for a given f, non-trivial congruences modulo primes of F above p can occur only for finitely many p.

Theorem 3.9 (Ribet) *The modular form f is p-isolated for all but finitely many p.*

Proof This follows from [131, Theorem 1.4] (*cf.* also [131, (1.5)]). □

We briefly elaborate on Theorem 3.9. Following Hida ([56, (6.7_b)]), one can introduce the quantity

$$C(f) := u(f)^{-1} \cdot 2^{(k-2)\cdot[F:\mathbb{Q}]} \cdot \prod_{\sigma\in\Sigma} (f^\sigma, f^\sigma)_{\Gamma_1(N)},$$

where $u(f) \in \mathbb{R}_+$ is the transcendental factor associated with f that is defined in [56, (6.6_c)] and $(\cdot,\cdot)_{\Gamma_1(N)}$ denotes the Petersson inner product. By [56, Theorem 6.2], $C(f) \in \mathbb{Z}$; notice that the integer $C(f)$ is (at least in principle) explicitly computable. By [131, (1.5)], the newform f is p-isolated for all p such that $p \geq k$ and $p \nmid 6 \cdot C(f)$. Furthermore, a necessary and sufficient condition for p-isolation in terms of the index of suitable lattices in parabolic cohomology groups with real coefficients is provided by [131, Theorem 1.4].

Remark 3.10 As was already observed, later on (see, *e.g.*, Assumption 4.5) we will assume that $k \equiv 2 \pmod{2(p-1)}$, so that $p < k$. Therefore, much like the "big image" property introduced in *cf.* Sect. 3.1.1 (*cf.* also Remark 3.5), the p-isolation of f is a constraint we will need to impose on the pair (f, p). From our point of view, a characterization of the p-isolation of f of the sort described above is useful because it shows that this property is, in general, compatible with other desirable features, the only price to pay being to throw away finitely many primes p, which can be explicitly detected.

3.2.2 Congruence module of f

Let L be the composite of the Hecke fields of all normalized eigenforms in $S_k(\Gamma_0(N))$, which is a number field whose ring of integers will be denoted by $\mathcal{O}_L$. Write $\theta_{f,\mathcal{O}_L} : \mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_L} \twoheadrightarrow \mathcal{O}_L$ for the (surjective) $\mathcal{O}_L$-linear extension of the morphism $\theta_f : \mathscr{H}_k(\Gamma_0(N)) \to \mathcal{O}_L$ attached to f, which is a homomorphism of $\mathcal{O}_L$-algebras. As a shorthand, set $\mathscr{H} := \mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_L}$; denote by $Q(\mathscr{H})$ the total quotient ring of $\mathscr{H}$. By Proposition 3.7, $\mathscr{H}$ is flat over $\mathcal{O}_L$.

As remarked, *e.g.*, in [62, p. 276], there is a unique splitting of L-algebras of the form $Q(\mathscr{H}) = L \times \mathcal{Z}$ for some $\mathscr{H}$-algebra $\mathcal{Z}$. Set $\mathfrak{a} := \ker(\mathscr{H} \to \mathcal{Z})$ for the kernel of the obvious map induced by the splitting above. Following Hida (and Ribet), we define the *congruence module of f* to be

$$C_0(f;\mathcal{O}_L) := (\mathscr{H}/\mathfrak{a}) \otimes_{\mathscr{H}} \mathcal{O}_L,$$

where the tensor product is taken via $\theta_{f,\mathcal{O}_L}$; this is an $\mathscr{H}$-algebra. As explained in [62, p. 277], there is an isomorphism $C_0(f;\mathcal{O}_L) \simeq \mathscr{H}\big/\big(\mathfrak{a} + \ker(\theta_{f,\mathcal{O}_L})\big)$ of $\mathscr{H}$-algebras, where the sum of ideals is, in fact, direct. In particular, the annihilator of $C_0(f;\mathcal{O}_L)$ as an $\mathscr{H}$-module is

$$\mathrm{Ann}_{\mathscr{H}}\left(C_0(f;\mathcal{O}_L)\right) = \mathfrak{a} + \ker(\theta_{f,\mathcal{O}_L}) = \mathrm{Ann}_{\mathscr{H}}\left(\ker(\theta_{f,\mathcal{O}_L})\right) + \ker(\theta_{f,\mathcal{O}_L}),$$

the second equality being a consequence of the identification $\mathfrak{a} = \mathrm{Ann}_{\mathscr{H}}\left(\ker(\theta_{f,\mathcal{O}_L})\right)$, which can be directly checked. We point out that the $\mathscr{H}$-algebra $C_0(f;\mathcal{O}_L)$ owes its name to the fact that it detects congruence primes for f in the sense of Sect. 3.2.1: the congruence primes for f are exactly the prime ideals of $\mathcal{O}_L$ belonging to the support of $C_0(f;\mathcal{O}_L)$ as an $\mathcal{O}_L$-module.

Remark 3.11 Since $C_0(f;\mathcal{O}_L)$ is finitely generated as an $\mathcal{O}_L$-module, $\mathrm{Supp}_{\mathcal{O}_L}\left(C_0(f;\mathcal{O}_L)\right)$ consists precisely of the prime ideals of $\mathcal{O}_L$ containing the annihilator of $C_0(f;\mathcal{O}_L)$ over $\mathcal{O}_L$ (see, *e.g.*, [99, p. 26]).

The reader is referred, *e.g.*, to [60, §6] and [62, §5.3.3] for further details (*cf.* also [47, §4] and [132]).

3.2.3 Congruence ideal of f

The notation from Sect. 3.2.2 is kept in force; in particular, $\mathscr{H}$ stands for $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_L}$. The *congruence ideal of* f is the ideal of $\mathcal{O}_L$ given by

$$\eta_{f,\mathcal{O}_L} := \theta_{f,\mathcal{O}_L}\Big(\mathrm{Ann}_{\mathscr{H}}\big(\ker(\theta_{f,\mathcal{O}_L})\big)\Big). \tag{3.1}$$

In order to lighten our notation, let us define $\mathscr{A} := \mathrm{Ann}_{\mathscr{H}}\left(\ker(\theta_{f,\mathcal{O}_L})\right)$, so that $\eta_{f,\mathcal{O}_L} = \theta_{f,\mathcal{O}_L}(\mathscr{A})$. In general, we write $\mathrm{Max}(\star)$ for the set of the maximal ideals of a ring $\star$. It is elementary to check that the surjective ring homomorphism $\theta_{f,\mathcal{O}_L}$ induces a bijection

$$\left\{\mathbf{m} \in \mathrm{Max}(\mathscr{H}) \;\middle|\; \ker(\theta_{f,\mathcal{O}_L}) \subset \mathbf{m}\right\} \xleftrightarrow{\sim} \mathrm{Max}(\mathcal{O}_L) \tag{3.2}$$

by the recipes $\mathbf{m} \mapsto \theta_{f,\mathcal{O}_L}(\mathbf{m})$ and $\mathfrak{m} \mapsto \theta^{-1}_{f,\mathcal{O}_L}(\mathfrak{m})$. Furthermore, bijection (3.2) restricts to a bijection

$$\left\{\mathbf{m} \in \mathrm{Max}(\mathscr{H}) \;\middle|\; \mathscr{A} + \ker(\theta_{f,\mathcal{O}_L}) \subset \mathbf{m}\right\} \xleftrightarrow{\sim} \left\{\mathfrak{M} \in \mathrm{Max}(\mathcal{O}_L) \;\middle|\; \eta_{f,\mathcal{O}_L} \subset \mathfrak{M}\right\}. \tag{3.3}$$

For later use, we record an easy lemma. Let R be a ring containing $\mathcal{O}_F$ as a subring and let $\theta_{f,R} : \mathscr{H}_k(\Gamma_0(N))_R \twoheadrightarrow R$ be the (surjective) homomorphism of R-algebras obtained by extending θ_f by R-linearity.

Lemma 3.12 *The R-module $\ker(\theta_{f,R})$ is finitely generated.*

Proof The (tautological) short exact sequence of R-modules

$$0 \longrightarrow \ker(\theta_{f,R}) \longrightarrow \mathscr{H}_k(\Gamma_0(N))_R \xrightarrow{\theta_{f,R}} R \longrightarrow 0$$

splits because the rightmost non-zero term is free. Since $\mathscr{H}_k(\Gamma_0(N))_R$ is finitely generated as an R-module, the lemma follows. □

Together with the correspondence between maximal ideals described above, Lemma 3.12 will be used in the proof of Proposition 3.13.

3.2.4 Congruence ideal of f at $\mathfrak{p}$

Let $\theta_{f,\mathfrak{p}} : \mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{p}} \twoheadrightarrow \mathcal{O}_\mathfrak{p}$ be the (surjective) $\mathcal{O}_\mathfrak{p}$-linear extension of θ_f and let $\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{p}}}\big(\ker(\theta_{f,\mathfrak{p}})\big)$ be the annihilator ideal of $\ker(\theta_{f,\mathfrak{p}})$ in $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{p}}$ (observe that, with notation as in Sect. 3.2.3, $\theta_{f,\mathfrak{p}}$ is a shorthand for $\theta_{f,\mathcal{O}_\mathfrak{p}}$). By analogy with (3.1), the *congruence ideal of f at $\mathfrak{p}$* is the ideal of $\mathcal{O}_\mathfrak{p}$ given by

$$\eta_{f,\mathfrak{p}} := \theta_{f,\mathfrak{p}}\Big(\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{p}}}\big(\ker(\theta_{f,\mathfrak{p}})\big)\Big).$$

Let us set $\eta_{f,p} := \prod_{\mathfrak{p}|p} \eta_{f,\mathfrak{p}} \subset \mathcal{O}_p$.

The result we are about to state is presumably well known to the experts; however, for want of an appropriate reference we provide a proof of it.

Proposition 3.13 *Let p be a prime number. If f is p-isolated, then $\eta_{f,p} = \mathcal{O}_p$.*

Proof Let p be a prime number such that f is p-isolated. Of course, we need equivalently to show that $\eta_{f,\mathfrak{p}} = \mathcal{O}_\mathfrak{p}$ for each prime $\mathfrak{p}$ of F above p. Let $\mathfrak{p}$ be such a prime. Recall the congruence ideal $\eta_{f,\mathcal{O}_L} \subset \mathcal{O}_L$ of f from (3.1). Take a prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and write $\mathcal{O}_\mathfrak{P}$ for the completion of $\mathcal{O}_L$ at $\mathfrak{P}$. With self-explaining notation, the congruence ideal of f at $\mathfrak{P}$ is

$$\eta_{f,\mathfrak{P}} := \theta_{f,\mathfrak{P}}\Big(\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{P}}}\big(\ker(\theta_{f,\mathfrak{P}})\big)\Big).$$

The flatness of $\mathcal{O}_\mathfrak{P}$ over $\mathcal{O}_L$ and the fact that, by Lemma 3.12, the $\mathcal{O}_L$-module $\ker(\theta_{f,\mathcal{O}_L})$ is finitely generated imply that, with notation as in Sect. 3.2.3, there is a canonical identification

$$\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_\mathfrak{P}}}\big(\ker(\theta_{f,\mathfrak{P}})\big) = \mathscr{A} \otimes_{\mathcal{O}_L} \mathcal{O}_\mathfrak{P}$$

(see, *e.g.*, [18, Ch. I, §2, Corollary 2 to Proposition 12]). This shows that there is also a canonical identification

$$\eta_{f,\mathfrak{P}} = \eta_{f,\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{O}_\mathfrak{P}; \tag{3.4}$$

in other words, $\eta_{f,\mathfrak{P}}$ is the $\mathfrak{P}$-adic completion of $\eta_{f,\mathcal{O}_L}$. Denote by $i_{k,L} : \mathcal{O}_L \hookrightarrow \mathscr{H}$ the natural (injective) map, then set $\mathbf{m} := \theta_{f,\mathcal{O}_L}^{-1}(\mathfrak{P}) \in \mathrm{Max}(\mathscr{H})$ and $\mathfrak{M} := i_{k,L}^{-1}(\mathbf{m}) \in \mathrm{Max}(\mathcal{O}_L)$. Notice that the prime $\mathfrak{M}$ of L lies above p. Now recall the congruence module $C_0(f;\mathcal{O}_L)$ of f from Sect. 3.2.2. In order to lighten our notation, from here until the end of this proof we put $\mathscr{C} := C_0(f;\mathcal{O}_L)$ and $\mathscr{O} := \mathcal{O}_L$. The map $i_{k,L}$ gives an injection of $\mathscr{O}$-modules $\mathscr{O}_{(\mathfrak{M})} \hookrightarrow \mathscr{H}_{(\mathbf{m})}$ between localizations. Since f is p-isolated, $\mathfrak{M} \notin \mathrm{Supp}_\mathscr{O}(\mathscr{C})$ (see, *e.g.*, [62, p. 277]), *i.e.*, $\mathscr{C} \otimes_\mathscr{O} \mathscr{O}_{(\mathfrak{M})} = 0$. Since $\mathscr{O}$ injects into $\mathscr{H}$ as a subring, there is a surjection of $\mathscr{O}$-modules $\mathscr{C} \otimes_\mathscr{O} \mathscr{H}_{(\mathbf{m})} \twoheadrightarrow \mathscr{C} \otimes_\mathscr{H} \mathscr{H}_{(\mathbf{m})}$. On the other hand, $\mathscr{C} \otimes_\mathscr{O} \mathscr{H}_{(\mathbf{m})} = (\mathscr{C} \otimes_\mathscr{O} \mathscr{O}_{(\mathfrak{M})}) \otimes_{\mathscr{O}_{(\mathfrak{M})}} \mathscr{H}_{(\mathbf{m})} = 0$, so $\mathscr{C} \otimes_\mathscr{H} \mathscr{H}_{(\mathbf{m})} = 0$, *i.e.*, $\mathbf{m} \notin \mathrm{Supp}_\mathscr{H}(\mathscr{C})$. Now, the $\mathscr{H}$-module $\mathscr{C}$ being finitely generated, $\mathrm{Supp}_\mathscr{H}(\mathscr{C})$ consists exactly of the prime ideals of $\mathscr{H}$ containing the annihilator of $\mathscr{C}$ over $\mathscr{H}$ (see, *e.g.*, [99, p. 26]). As observed in Sect. 3.2.2, this annihilator is equal to $\mathscr{A} + \ker(\theta_{f,\mathcal{O}_L})$, where, as in Sect. 3.2.3, $\mathscr{A}$ is the annihilator of $\ker(\theta_{f,\mathcal{O}_L})$ over $\mathscr{H}$. Therefore, bijection (3.3) tells us that $\mathfrak{P}$ does not contain $\eta_{f,\mathcal{O}_L}$, and then

$$\eta_{f,\mathfrak{P}} = \mathcal{O}_\mathfrak{P} \tag{3.5}$$

by equality (3.4).

To prove that $\eta_{f,\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$, one can proceed as follows. Formally identical arguments to those used above lead to a canonical identification

$$\eta_{f,\mathfrak{P}} = \eta_{f,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{P}} \tag{3.6}$$

that is induced by the inclusion $\eta_{f,\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}}$. The natural injection $\mathcal{O}_{\mathfrak{p}} \hookrightarrow \mathcal{O}_{\mathfrak{P}}$ is a flat local homomorphism of local rings, hence a faithfully flat ring homomorphism (see, *e.g.*, [18, Ch. I, §3, Proposition 9]). We consider the short exact sequence of $\mathcal{O}_{\mathfrak{p}}$-modules

$$0 \longrightarrow \eta_{f,\mathfrak{p}} \longrightarrow \mathcal{O}_{\mathfrak{p}} \longrightarrow \mathcal{O}_{\mathfrak{p}}/\eta_{f,\mathfrak{p}} \longrightarrow 0. \tag{3.7}$$

Upon tensoring (3.7) by $\mathcal{O}_{\mathfrak{P}}$, the flatness of $\mathcal{O}_{\mathfrak{P}}$ over $\mathcal{O}_{\mathfrak{p}}$ and equalities (3.5) and (3.6) show that

$$(\mathcal{O}_{\mathfrak{p}}/\eta_{f,\mathfrak{p}}) \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{P}} = 0.$$

Finally, by the faithful flatness of $\mathcal{O}_{\mathfrak{P}}$ over $\mathcal{O}_{\mathfrak{p}}$, we conclude that $\mathcal{O}_{\mathfrak{p}}/\eta_{f,\mathfrak{p}} = 0$, as was to be shown. □

Remark 3.14 In the course of proving Proposition 3.13, we showed that if a prime $\mathfrak{P}$ of L divides the congruence ideal $\eta_{f,\mathcal{O}_L}$ of f, then $\mathfrak{P}$ is a congruence prime for f.

Until the end of this subsection, we assume that f is p-isolated. For each $\mathfrak{p} \mid p$, there is a (tautological) short exact sequence of $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}$-modules

$$0 \longrightarrow \ker(\theta_{f,\mathfrak{p}}) \longrightarrow \mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}} \xrightarrow{\theta_{f,\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}} \longrightarrow 0. \tag{3.8}$$

Using the non-triviality of $\eta_{f,\mathfrak{p}}$ and the flatness (ensured by Proposition 3.7) of the $\mathcal{O}_{\mathfrak{p}}$-algebra $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}$, one can check that

$$\ker(\theta_{f,\mathfrak{p}}) \cap \mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}}\big(\ker(\theta_{f,\mathfrak{p}})\big) = \{0\}$$

(see, *e.g.*, [87, p. 250]). Thus, since $\eta_{f,\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ by Proposition 3.13, the map $\theta_{f,\mathfrak{p}}$ restricts to an isomorphism of $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}$-modules

$$\theta_{f,\mathfrak{p}} : \mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}}\big(\ker(\theta_{f,\mathfrak{p}})\big) \xrightarrow{\simeq} \mathcal{O}_{\mathfrak{p}}.$$

This shows that the short exact sequence (3.8) has a unique splitting with values in the annihilator of $\ker(\theta_{f,\mathfrak{p}})$. From now on, we view this distinguished splitting as canonical and consider (with a slight abuse of notation) the equality of $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}$-modules

$$\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}} = \ker(\theta_{f,\mathfrak{p}}) \oplus \mathcal{O}_{\mathfrak{p}}. \tag{3.9}$$

See, *e.g.*, [87] for more details.

Remark 3.15 The p-isolation of f will play an important role in Sect. 5.9.3, where we compare the periods of f and of a suitable imaginary quadratic twist of f; in particular, the formula in Sect. 5.9.3, which is due to Hida, will be used in the proof of Theorem 5.42. While we certainly expect our main result on the p-part of the Tamagawa number conjecture to hold without p-isolation in force, at the moment it is not clear to us how to modify some of our arguments in the absence of this condition.

3.3 On p-adic Abel–Jacobi maps

Following [111, Ch. II, (6.5)], with T_p and $T_{\mathfrak{p}}$ as in Sect. 2.4.3, let us define

$$J_p := \Pi_B \Pi_\epsilon \cdot H^1_{\text{ét}}\big(\bar{X}, \mathcal{O}_p(k/2)\big),$$

which we view as a subgroup of $H^1_{\text{ét}}\big(\bar{X}, F_p(k/2)\big)$; then $T_p = J_p[\theta_f]$. There is a splitting $J_p = \prod_{\mathfrak{p}|p} J_{\mathfrak{p}}$, where

$$J_{\mathfrak{p}} := \Pi_B \Pi_\epsilon \cdot H^1_{\text{ét}}\big(\bar{X}, \mathcal{O}_{\mathfrak{p}}(k/2)\big) \tag{3.10}$$

is regarded as a subgroup of $H^1_{\text{ét}}\big(\bar{X}, F_{\mathfrak{p}}(k/2)\big)$. Notice that

$$T_{\mathfrak{p}} = J_{\mathfrak{p}}[\theta_{f,\mathfrak{p}}] = J_{\mathfrak{p}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \mathcal{O}_{\mathfrak{p}}.$$

As is pointed out in [110, §3], there is a surjection $\varpi_p : J_p \twoheadrightarrow T_p$ whose restriction to T_p is the multiplication-by-p^m map for some $m \in \mathbb{N}$. Notation being as in (2.17), for any number field L the Abel–Jacobi map in (2.39) yields a map

$$\mathrm{AJ}_{L,p} : \mathrm{CH}_0^{k/2}(X/L)_{\mathcal{O}_p} \longrightarrow H^1_f(L, J_p) \xrightarrow{\varpi_{p,*}} H^1_f(L, T_p), \tag{3.11}$$

where $\varpi_{p,*}$ is induced by ϖ_p functorially. Furthermore, both the source and the target of (3.11) split over the primes $\mathfrak{p}$ of F above p, and we let

$$\mathrm{AJ}_{L,\mathfrak{p}} : \mathrm{CH}_0^{k/2}(X/L)_{\mathcal{O}_{\mathfrak{p}}} \longrightarrow H^1_f(L, T_{\mathfrak{p}}) \tag{3.12}$$

be the $\mathfrak{p}$-component of (3.11). Finally, set

$$\Lambda_{\mathfrak{p}}(L) := \mathrm{im}(\mathrm{AJ}_{L,\mathfrak{p}}) \subset H^1_f(L, T_{\mathfrak{p}}). \tag{3.13}$$

The $\mathcal{O}_{\mathfrak{p}}$-module $H^1_f(L, T_{\mathfrak{p}})$ is finitely generated, so $\Lambda_{\mathfrak{p}}(L)$ is finitely generated over $\mathcal{O}_{\mathfrak{p}}$.

Remark 3.16 By construction, $\mathrm{AJ}_{L,p}$ and $\mathrm{AJ}_{L,\mathfrak{p}}$ factor through $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/L)_{\mathcal{O}_p}$ and $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/L)_{\mathcal{O}_{\mathfrak{p}}}$, respectively (*cf.* [110, p. 105]). In particular, $\mathrm{AJ}_{L,p}$ and $\mathrm{AJ}_{L,\mathfrak{p}}$ induce Abel–Jacobi maps on $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_p}$ and $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_p}$, respectively, to be denoted by the same symbols.

3.4 p-integral motivic cohomology

As at the end of Sect. 3.2.4, from now until the end of this subsection we work under the assumption that f is p-isolated. By Theorem 3.9, this condition rules out only finitely many primes p, which can be characterized as explained in Remark 3.10. We use p-isolation to split $J_{\mathfrak{p}}$ over the Hecke algebra.

3.4.1 Splitting $J_{\mathfrak{p}}$

In the present situation, one can take ϖ_p so that its restriction to T_p is the identity (in other words, one can take $m = 0$ in Sect. 3.3). More precisely, if $J_{\mathfrak{p}}$ is the $\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}$-module from (3.10), then the splitting in (3.9) produces a splitting

$$J_{\mathfrak{p}} = \Big(J_{\mathfrak{p}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \mathcal{O}_{\mathfrak{p}}\Big) \bigoplus \Big(J_{\mathfrak{p}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \ker(\theta_{f,\mathfrak{p}})\Big). \tag{3.14}$$

On the other hand, we already observed that $T_{\mathfrak{p}} = J_{\mathfrak{p}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \mathcal{O}_{\mathfrak{p}}$, so from (3.14) we get a (projection) map $\varpi_{\mathfrak{p}} : J_{\mathfrak{p}} \twoheadrightarrow T_{\mathfrak{p}}$ for each $\mathfrak{p} \mid p$. Taking sums over all $\mathfrak{p} \mid p$ gives the desired surjection $\varpi_p : J_p \twoheadrightarrow T_p$.

3.4.2 $\mathfrak{p}$-integral motivic cohomology of $\mathcal{M}$

Let L be a number field. With notation as in (2.17) and using again (3.9), we can consider the splitting

$$\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_{\mathfrak{p}}} = \Big(\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_{\mathfrak{p}}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \mathcal{O}_{\mathfrak{p}}\Big) \bigoplus \Big(\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_{\mathfrak{p}}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \ker(\theta_{f,\mathfrak{p}})\Big). \tag{3.15}$$

For each $\mathfrak{p} \mid p$, we define the *first $\mathfrak{p}$-integral motivic cohomology group of $\mathcal{M}$ over L* to be

$$\begin{aligned} H^1_{\mathrm{mot}}(L, \mathcal{M})_{\mathfrak{p}\text{-int}} &:= \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L) \otimes_{\mathscr{H}_k(\Gamma_0(N))} \mathcal{O}_{\mathfrak{p}} \\ &= \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_{\mathfrak{p}}} \otimes_{\mathscr{H}_k(\Gamma_0(N))_{\mathcal{O}_{\mathfrak{p}}}} \mathcal{O}_{\mathfrak{p}}, \end{aligned} \tag{3.16}$$

where the $\mathscr{H}_k(\Gamma_0(N))$-algebra structure on $\mathcal{O}_{\mathfrak{p}}$ is induced by composing θ_f with the natural injection $\mathcal{O}_F \hookrightarrow \mathcal{O}_{\mathfrak{p}}$ and the bottom identification is a standard canonical isomorphism.

3.4.3 p-integral motivic cohomology of $\mathcal{M}$

We define the *first p-integral motivic cohomology group of $\mathcal{M}$ over L* as

$$H^1_{\mathrm{mot}}(L, \mathcal{M})_{p\text{-int}} := \bigoplus_{\mathfrak{p} \mid p} H^1_{\mathrm{mot}}(L, \mathcal{M})_{\mathfrak{p}\text{-int}}.$$

It follows that the p-adic étale regulator map from (2.41) yields maps

$$\mathrm{reg}_{L,\mathfrak{p}} : H^1_{\mathrm{mot}}(L, \mathcal{M})_{\mathfrak{p}\text{-int}} \longrightarrow H^1_f(L, T_{\mathfrak{p}}) \tag{3.17}$$

for each $\mathfrak{p} \mid p$ and

$$\mathrm{reg}_{L,p} : H^1_{\mathrm{mot}}(L, \mathcal{M})_{p\text{-int}} \longrightarrow H^1_f(L, T_p) \tag{3.18}$$

that satisfy $\mathrm{reg}_{L,p} = \bigoplus_{\mathfrak{p} \mid p} \mathrm{reg}_{L,\mathfrak{p}}$. Note that, since $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_p} = \bigoplus_{\mathfrak{p} \mid p} \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_{\mathfrak{p}}}$, for $\star \in \{p\} \cup \{\mathfrak{p} \mid p\}$ there is a commutative triangle

$$\begin{array}{ccc} \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{O}_\star} & \xrightarrow{\mathrm{AJ}_{L,\star}} & H^1_f(L, T_\star) \\ & {\scriptstyle \Pi_{\mathcal{M},L,\star}} \searrow \quad \nearrow {\scriptstyle \mathrm{reg}_{L,\star}} & \\ & H^1_{\mathrm{mot}}(L, \mathcal{M})_{\star\text{-int}} & \end{array} \tag{3.19}$$

in which, bearing (3.16) in mind, $\Pi_{\mathcal{M},L,\star}$ is the projection induced by (3.15) if $\star = \mathfrak{p}$ or the direct sum of such projections over all $\mathfrak{p} \mid p$ if $\star = p$, whereas $\mathrm{AJ}_{L,\star}$ is (the restriction of) the map in (3.11) if $\star = p$ or in (3.12) if $\star = \mathfrak{p}$.

Remark 3.17 Let $\star \in \{p\} \cup \{\mathfrak{p} \mid p\}$. Of course, extending $\mathrm{reg}_{L,\star}$ in (3.17) and (3.18) $F_\star$-linearly we recover the p-adic regulator map in (2.41) if $\star = p$ or the $\mathfrak{p}$-adic regulator map in (2.42) if $\star = \mathfrak{p}$, which justifies the slight abuse of notation.

3.4.4 A conjecture on $\mathtt{reg}_{L,p}$

The following conjecture on the p-adic regulator map $\mathrm{reg}_{L,p}$ in (3.18) is a stronger, integral version of Conjecture 2.50. It predicts that if f is p-isolated, then $\mathrm{reg}_{L,p}$ is not only injective (as implied by Conjecture 2.50), but also surjective.

Conjecture 3.18 *Let L be a number field. The map $\mathrm{reg}_{L,p}$ in (3.18) is an isomorphism of $\mathcal{O}_p$-modules for all but finitely many primes p at which f is p-isolated.*

In particular, we conjecture $\mathrm{reg}_{L,p}$ to be an isomorphism of $\mathcal{O}_p$-modules for all but finitely many p (*cf.* Theorem 3.9). At a certain point, we will assume that Conjecture 3.18 is true for a specific choice of L (namely, $L = \mathbb{Q}$ or L a suitable imaginary quadratic field).

Remark 3.19 Conjecture 3.18 implies Conjecture 2.50 for all primes p at which f is p-isolated (*cf.* Remark 3.17).

Remark 3.20 Our main motivation for proposing Conjecture 3.18 is the following. Let $\mathcal{T}$ be an abelian tensor category and let $\mathcal{H} = \big(H^\bullet(\cdot,\star), H_\bullet(\cdot,\star)\big)$ be a $\mathcal{T}$-valued twisted Poincaré duality theory with weights (see, *e.g.*, [67, §6]). Furthermore, let X be a smooth proper variety of dimension d over a field. As explained, *e.g.*, in [67, §9.1], there is an Abel–Jacobi map

$$r_{\mathcal{H}} : Z^j(X)_0 \longrightarrow \mathrm{R}^1\Gamma H^{2j-1}(X,j)$$

for all integers $0 \le j \le d$, where $Z^j(X)_0$ is the group of cycles of codimension j on X that are homologically equivalent to 0 and

$$\mathrm{R}^1\Gamma H^{2j-1}(X,j) = \mathrm{Ext}\big(1, H^{2j-1}(X,j)\big).$$

Now let $\mathcal{H}_{\mathrm{B}}$ be the Betti cohomology theory with coefficients in $\bar{\mathbb{Q}}$. The image of $r_{\mathcal{H}_{\mathrm{B}}}$ and integral Betti cohomology induce integral structures on $\mathrm{Ext}\big(1, H^{2j-1}(X,j)\big)$ that we expect to coincide after localization at a prime number p for all but finitely many p. Therefore, the comparison isomorphism between Betti and étale cohomology suggests that the p-adic Abel–Jacobi map is surjective when regulator maps can be defined (using an assumption of p-isolation on f), and this led us to Conjecture 3.18. Admittedly, at present we are at a loss to provide a more convincing and less vague motivation for this conjecture.

4 Kolyvagin's conjecture for modular forms

Our goal in this section is to state and prove Kolyvagin's conjecture for a large class of higher (even) weight modular forms.

4.1 Heegner cycles

We recall the definition of (classical) Heegner cycles in the sense of Nekovář [110,111].

4.1.1 Heegner hypothesis

Let K be an imaginary quadratic field of discriminant D_K such that

- all the prime factors of N split in K.

In other words, K satisfies the *Heegner hypothesis* relative to N. By virtue of this condition, if $\mathcal{O}_K$ is the ring of integers of K, then we can fix an N-cyclic ideal of $\mathcal{O}_K$, *i.e.*, an ideal $\mathcal{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$. Let us choose once and for all an embedding $K \hookrightarrow \mathbb{C}$. For every integer $n \geq 1$ prime to NpD_K let $\mathcal{O}_n := \mathbb{Z} + n\mathcal{O}_K$ be the order of K of conductor n. The isogeny $\mathbb{C}/\mathcal{O}_n \to \mathbb{C}/(\mathcal{O}_n \cap \mathcal{N})^{-1}$ of complex tori defines a Heegner point $x_n \in X_0(N)$ that, by the theory of complex multiplication, is rational over the ring class field K_n of K of conductor n (in particular, K_1 is the Hilbert class field of K).

4.1.2 Heegner cycles

Write $\pi_N : X(N) \to X_0(N)$ for the map induced by the inclusion $\Gamma(N) \subset \Gamma_0(N)$ and choose $\tilde{x}_n \in \pi_N^{-1}(x_n)$. The elliptic curve E_n corresponding to $\tilde{x}_n$ has complex multiplication by $\mathcal{O}_n$. Fix the unique square root $\xi_n = \sqrt{-n^2 D_K}$ of the discriminant of $\mathcal{O}_n$ with positive imaginary part under the chosen embedding $K \hookrightarrow \mathbb{C}$. For any $a \in \mathcal{O}_n$ let $\Gamma_{n,a} \subset E_n \times E_n$ denote the graph of a and let $i_{\tilde{x}_n} : \tilde{\pi}_{k-2}^{-1}(\tilde{x}_n) = E_n^{k-2} \hookrightarrow X$ be the canonical inclusion (recall that $X = \tilde{\mathcal{E}}_N^{k-2}$). We will frequently write the same symbol Z for a cycle Z and the class $[Z]$ of Z in the Chow group. Put

$$Z_k(\tilde{x}_n) := \Gamma_{n,\xi_n} \smallsetminus \big[(E_n \times \{0\}) \cup (\{0\} \times E_n)\big] \tag{4.1}$$

and, with notation as in (2.17), define

$$\tilde{\Gamma}_n := \Pi_B \Pi_\epsilon \cdot (i_{\tilde{x}_n})_* \big(Z_k(\tilde{x}_n)^{(k-2)/2}\big) \in \Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n), \tag{4.2}$$

where $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)$ is to be viewed as a subgroup of $\mathrm{CH}^{k/2}(X/K_n)_{\mathbb{Q}}$. As explained in [110, p. 105], there is an equality

$$\Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)_{\mathbb{Z}_p} = \Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathbb{Z}_p}$$

inside $\mathrm{CH}_0^{k/2}(X/K_n)_{\mathbb{Q}_p}$. We call the image

$$\Gamma_{n,p} \in \Pi_B \Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathcal{O}_p} \subset \mathrm{CH}_0^{k/2}(X/K_n)_{F_p} \tag{4.3}$$

of $\tilde{\Gamma}_n$ the *geometric Heegner cycle* (*at p*) *of conductor n*. Moreover, for each $\mathfrak{p} \mid p$ we write

$$\Gamma_{n,\mathfrak{p}} \in \Pi_B \Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathcal{O}_\mathfrak{p}} \subset \mathrm{CH}_0^{k/2}(X/K_n)_{F_\mathfrak{p}} \tag{4.4}$$

for the image of $\Gamma_{n,p}$ (here we are implicitly using the splitting $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathcal{O}_p} = \bigoplus_{\mathfrak{p}|p} \Pi_B \Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathcal{O}_\mathfrak{p}}$). The *arithmetic Heegner cycle* (*at p*) *of conductor n* is then the image

$$y_{n,p} := \mathrm{AJ}_{K_n,p}(\Gamma_{n,p}) \in H_f^1(K_n, T_p)$$

of the cycle in (4.3) via the p-adic Abel–Jacobi map from (3.11), which factors through $\Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_n)_{\mathcal{O}_p}$ (*cf.* Remark 3.16). With notation as in (3.13), for each $\mathfrak{p} \mid p$ we also set

$$y_{n,\mathfrak{p}} := \mathrm{AJ}_{K_n,\mathfrak{p}}(\Gamma_{n,\mathfrak{p}}) \in \Lambda_{\mathfrak{p}}(K_n),$$

where $\mathrm{AJ}_{K_n,\mathfrak{p}}$ is the $\mathfrak{p}$-adic Abel–Jacobi map in (3.12) and, as in (3.13), $\Lambda_{\mathfrak{p}}(K_n)$ is its image. In other words, $y_{n,\mathfrak{p}}$ is the natural image of $y_{n,p}$ in $H^1_f(K_n, T_{\mathfrak{p}})$. It turns out that $y_{n,p}$ and $y_{n,\mathfrak{p}}$ are independent of the choice of $\tilde{x}_n$ ([110, p. 107]). In the rest of this paper, the expression "Heegner cycle of conductor n" will always refer to $y_{n,\mathfrak{p}}$ for a fixed $\mathfrak{p}$ as above.

Finally, a crucial role in our arguments will be played by the cycle

$$y_{K,\mathfrak{p}} := \mathrm{cores}_{K_1/K}(y_{1,\mathfrak{p}}) \in \Lambda_{\mathfrak{p}}(K); \tag{4.5}$$

here we exploit the Galois-equivariance of the maps $\mathrm{AJ}_{\star,\mathfrak{p}}$, which implies that the square

$$\begin{array}{ccc} \Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K_1)_{\mathcal{O}_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{K_1,\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(K_1) \\ \downarrow{\scriptstyle \mathrm{tr}_{K_1/K}} & & \downarrow{\scriptstyle \mathrm{cores}_{K_1/K}} \\ \Pi_B\Pi_\epsilon \cdot \mathrm{CH}_0^{k/2}(X/K)_{\mathcal{O}_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{K,\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(K) \end{array} \tag{4.6}$$

commutes (as the notation suggests, $\mathrm{tr}_{K_1/K}$ is the Galois trace map on Chow groups).

4.2 Kolyvagin integers

Recall that, by Assumption 3.6, the prime p is unramified in F, hence p is a local uniformizer for F at $\mathfrak{p}$. Let $v_{\mathfrak{p}}$ be the valuation of $F_{\mathfrak{p}}$ normalized so that $v_{\mathfrak{p}}(p) = 1$.

4.2.1 Kolyvagin primes

A prime number ℓ is a *Kolyvagin prime* for the data $(f, \mathfrak{p}, K)$ if

(1) $\ell \nmid Np$;
(2) ℓ is inert in K;
(3) $M(\ell) := \min\{v_{\mathfrak{p}}(\ell+1), v_{\mathfrak{p}}(a_\ell(f))\} > 0$.

Denote by $\mathcal{P}_{\mathrm{Kol}}(f)$ the set of Kolyvagin primes for $(f, \mathfrak{p}, K)$.

4.2.2 Kolyvagin integers

Let us write

$$\Lambda_{\mathrm{Kol}}(f) := \{\text{square-free products of primes in } \mathcal{P}_{\mathrm{Kol}}(f)\}$$

for the set of *Kolyvagin integers* for $(f, \mathfrak{p}, K)$. If we need to specify the data $(\mathfrak{p}, K)$ we also write $\mathcal{P}_{\mathrm{Kol}}(f, \mathfrak{p}, K)$ and $\Lambda_{\mathrm{Kol}}(f, \mathfrak{p}, K)$ for $\mathcal{P}_{\mathrm{Kol}}(f)$ and $\Lambda_{\mathrm{Kol}}(f)$, respectively. Notice that $1 \in \Lambda_{\mathrm{Kol}}(f)$. Finally, for every $n \in \Lambda_{\mathrm{Kol}}(f)$ define

$$M(n) := \begin{cases} \min\{M(\ell) \mid \ell \mid n\} & \text{if } n \geq 2, \\ \infty & \text{if } n = 1. \end{cases} \tag{4.7}$$

We call $M(n)$ the *Kolyvagin index* of n.

4.3 Kolyvagin classes and Kolyvagin's conjecture

We attach to our newform f a systematic supply of Galois cohomology classes, which we call *Kolyvagin classes*, that are indexed by Kolyvagin integers and take values in quotients of $T_{\mathfrak{p}}$ (or, equivalently, in torsion submodules of $A_{\mathfrak{p}}$, *cf.* Sect. 2.17.2). Our strategy for producing these classes, which are defined in terms of the Heegner cycles of Sect. 4.1, follows the recipe proposed by Kolyvagin for modular abelian varieties (see, *e.g.*, [53, §4] and [166, §3.7]). In order to fix notation that we will use in the rest of the paper, and for the convenience of the reader, we describe the construction of Kolyvagin classes in our higher weight setting.

4.3.1 Kolyvagin derivatives

For all $n \in \Lambda_{\mathrm{Kol}}(f)$, let us set $G_n := \mathrm{Gal}(K_n/K_1)$ and $\mathcal{G}_n := \mathrm{Gal}(K_n/K) \simeq \mathrm{Pic}(\mathcal{O}_n)$, so that, by class field theory, $G_n = \prod_{\ell|n} G_\ell$ with G_ℓ cyclic of order $\ell+1$, where ℓ varies over the prime divisors of n. For all $\ell \in \mathcal{P}_{\mathrm{Kol}}$ choose a generator σ_ℓ of G_ℓ; define Kolyvagin derivative operators as

$$D_\ell := \sum_{i=1}^{\ell} i\sigma_\ell^i \in \mathbb{Z}[G_\ell], \quad D_n := \prod_{\ell|n} D_\ell \in \mathbb{Z}[G_n].$$

In particular, D_1 is the identity operator. Fix $n \in \Lambda_{\mathrm{Kol}}(f)$, let $\mathcal{G}$ be a system of representatives for $\mathcal{G}_n/G_n$ and set

$$z_{n,\mathfrak{p}} := \sum_{\sigma\in\mathcal{G}} \sigma\big(D_n(y_{n,\mathfrak{p}})\big) \in \Lambda_{\mathfrak{p}}(K_n).$$

Remark 4.1 Since G_1 is trivial and D_1 is the identity operator, $z_{1,\mathfrak{p}} = \sum_{\sigma\in\mathcal{G}_1} \sigma(y_{1,\mathfrak{p}}) \in \Lambda_{\mathfrak{p}}(K_1)$. A direct computation shows that

$$\mathrm{res}_{K_1/K}(y_{K,\mathfrak{p}}) = z_{1,\mathfrak{p}}, \tag{4.8}$$

where $y_{K,\mathfrak{p}} \in \Lambda_{\mathfrak{p}}(K)$ is the cycle defined in (4.5).

4.3.2 Kolyvagin classes

As a consequence of [93, Corollary 2.7, (3)] and [93, Proposition 2.8], for any number field L and every integer $M \geq 1$ there is a natural Galois-equivariant injection

$$\iota_{L,M} : \Lambda_{\mathfrak{p}}(L)\big/p^M\Lambda_{\mathfrak{p}}(L) \hookrightarrow H^1\big(L, A_{\mathfrak{p}}[p^M]\big). \tag{4.9}$$

This map should the thought of as a higher weight avatar of the usual Kummer map in the Galois cohomology of abelian varieties over number fields. The extension $K_n/\mathbb{Q}$, which is generalized dihedral, is solvable, so [93, Lemma 3.10, (2)] ensures that $H^0\big(K_n, A_{\mathfrak{p}}[p^M]\big) = 0$. It follows that restriction induces an isomorphism

$$\mathrm{res}_{K_n/K} : H^1\big(K, A_{\mathfrak{p}}[p^M]\big) \xrightarrow{\simeq} H^1\big(K_n, A_{\mathfrak{p}}[p^M]\big)^{\mathcal{G}_n}. \tag{4.10}$$

Moreover, one can easily check that if $[z_{n,\mathfrak{p}}]_M$ denotes the class of $z_{n,\mathfrak{p}}$ modulo p^M (analogous notation will be used, below, for $y_{K,\mathfrak{p}}$) and $M(n)$ is the Kolyvagin index of n from (4.7), then

$$M \leq M(n) \implies [z_{n,\mathfrak{p}}]_M \in \big(\Lambda_{\mathfrak{p}}(K_n)/p^M\Lambda_{\mathfrak{p}}(K_n)\big)^{\mathcal{G}_n},$$

whence

$$M \leq M(n) \implies \iota_{K_n,M}\big([z_{n,\mathfrak{p}}]_M\big) \in H^1\big(K_n, A_{\mathfrak{p}}[p^M]\big)^{\mathcal{G}_n}.$$

For notational convenience, and also for later reference, let us define

$$d_M(f,n) := \iota_{K_n,M}\big([z_{n,\mathfrak{p}}]_M\big) \in H^1\big(K_n, A_{\mathfrak{p}}[p^M]\big)^{\mathcal{G}_n}.$$

Keeping isomorphism (4.10) in mind, for all $M \leq M(n)$ we can define the *Kolyvagin class* $c_M(f,n)$ as

$$c_M(f,n) := \mathrm{res}^{-1}_{K_n/K}\big(d_M(f,n)\big) \in H^1\big(K, A_{\mathfrak{p}}[p^M]\big).$$

In particular, one has

$$c_M(f,n) = 0 \iff d_M(f,n) = 0. \tag{4.11}$$

The *Kolyvagin set* associated with $(f,\mathfrak{p},K)$ is

$$\kappa^{(K)}_{f,\mathfrak{p},\infty} := \big\{c_M(f,n) \mid n \in \Lambda_{\mathrm{Kol}}(f),\ 1 \leq M \leq M(n)\big\}. \tag{4.12}$$

If $\mathfrak{p}$ and K are clear from the context (which will usually be the case), then we shall write $\kappa_{f,\infty}$ in place of $\kappa^{(K)}_{f,\mathfrak{p},\infty}$.

4.3.3 Kolyvagin's conjecture in higher weight

The following conjecture was first proposed, with a slightly different formalism, in [97, Conjecture A].

Conjecture 4.2 (Kolyvagin's conjecture, higher weight) $\kappa_{f,\infty} \neq \{0\}$.

This is a higher (even) weight counterpart of a conjecture for rational elliptic curves due to Kolyvagin ([82, Conjecture A]).

It is convenient to introduce some more terminology. The *strict Kolyvagin set* attached to $(f,\mathfrak{p},K)$ is

$$\kappa^{(K),\mathrm{st}}_{f,\mathfrak{p},\infty} := \big\{c_1(f,n) \mid n \in \Lambda_{\mathrm{Kol}}(f)\big\}. \tag{4.13}$$

As above, we shall write $\kappa^{\mathrm{st}}_{f,\infty}$ in place of $\kappa^{(K),\mathrm{st}}_{f,\mathfrak{p},\infty}$ if no confusion is likely to arise. Clearly, there is an inclusion $\kappa^{\mathrm{st}}_{f,\infty} \subset \kappa_{f,\infty}$, so the non-triviality of $\kappa^{\mathrm{st}}_{f,\infty}$ implies Conjecture 4.2. In fact, we will verify that, under our assumptions, $\kappa^{\mathrm{st}}_{f,\infty} \neq \{0\}$, thus proving Conjecture 4.2 in our setting.

4.4 The Kolyvagin classes $c_M(f, 1)$

Of special interest will be the classes $c_M(f, 1)$ for $M \geq 1$; in the next result, we describe them more explicitly.

Proposition 4.3 $c_M(f, 1) = \iota_{K,M}\big([y_{K,\mathfrak{p}}]_M\big)$.

Proof As is explained, *e.g.*, in [16, §A.9.17], there is a natural base change map

$$\mathrm{CH}_0^{k/2}(X/K) \longrightarrow \mathrm{CH}_0^{k/2}(X/K_1)^{\mathcal{G}_1}. \tag{4.14}$$

By composing (4.14) with the (restriction of the) Abel–Jacobi map Φ_{K_1}, we get a map

$$\psi_{K,K_1} : \mathrm{CH}_0^{k/2}(X/K) \longrightarrow H^1(K_1, T_{\mathfrak{p}})^{\mathcal{G}_1}.$$

On the other hand, since the extension $K_1/\mathbb{Q}$ is solvable, by [93, Lemma 3.10, (2)] one has $H^0(K_1, A_{\mathfrak{p}}[p^m]) = 0$ for all $m \geq 1$, so $H^0(K_1, T_{\mathfrak{p}}) = \varprojlim_m H^0\big(K_1, A_{\mathfrak{p}}[p^m]\big) = 0$. It follows that restriction induces an isomorphism

$$\mathrm{res}_{K_1/K} : H^1(K, T_{\mathfrak{p}}) \xrightarrow{\simeq} H^1(K_1, T_{\mathfrak{p}})^{\mathcal{G}_1}.$$

Now it turns out (see, *e.g.*, the proof of [28, Proposition 6]) that the composition

$$\mathrm{res}_{K_1/K}^{-1} \circ \psi_{K,K_1} : \mathrm{CH}_0^{k/2}(X/K) \longrightarrow H^1(K, T_{\mathfrak{p}})$$

coincides with the Abel–Jacobi map. Since $\Lambda_{\mathfrak{p}}(K) = \mathrm{im}(r_{K,\mathfrak{p}})$ and $\Lambda_{\mathfrak{p}}(K_1) = \mathrm{im}(r_{K_1,\mathfrak{p}})$, we obtain a natural injection $\Lambda_{\mathfrak{p}}(K) \hookrightarrow \Lambda_{\mathfrak{p}}(K_1)^{\mathcal{G}_1}$ given by $x \mapsto \mathrm{res}_{K_1/K}(x)$. This map, in turn, induces a map η_{K,K_1} that is given by the composition

$$\Lambda_{\mathfrak{p}}(K)/p^M\Lambda_{\mathfrak{p}}(K) \longrightarrow \Lambda_{\mathfrak{p}}(K_1)^{\mathcal{G}_1}/p^M\Lambda_{\mathfrak{p}}(K_1)^{\mathcal{G}_1} \hookrightarrow \big(\Lambda_{\mathfrak{p}}(K_1)/p^M\Lambda_{\mathfrak{p}}(K_1)\big)^{\mathcal{G}_1}$$

and fits into the commutative square

$$\begin{array}{ccc} \Lambda_{\mathfrak{p}}(K)/p^M\Lambda_{\mathfrak{p}}(K) & \xhookrightarrow{\iota_{K,M}} & H^1\big(K, A_{\mathfrak{p}}[p^M]\big) \\ \downarrow{\scriptstyle \eta_{K,K_1}} & & \downarrow{\scriptstyle \simeq\ \mathrm{res}_{K_1/K}} \\ \big(\Lambda_{\mathfrak{p}}(K_1)/p^M\Lambda_{\mathfrak{p}}(K_1)\big)^{\mathcal{G}_1} & \xhookrightarrow{\iota_{K_1,M}} & H^1\big(K_1, A_{\mathfrak{p}}[p^M]\big)^{\mathcal{G}_1}. \end{array}$$

Finally, one has $\eta_{K,K_1}\big([y_{K,\mathfrak{p}}]_M\big) = \big[\mathrm{res}_{K_1/K}(y_{K,\mathfrak{p}})\big] = [z_{1,\mathfrak{p}}]_M$, where the second equality is a consequence of formula (4.8), so

$$\iota_{K,M}\big([y_{K,\mathfrak{p}}]_M\big) = \mathrm{res}_{K_1/K}^{-1}\Big(\iota_{K_1,M}\big(\eta_{K_1/K}([y_{K,\mathfrak{p}}]_M)\big)\Big) = \mathrm{res}_{K_1/K}^{-1}\big(\iota_{K_1,M}([z_{1,\mathfrak{p}}]_M)\big) = c_M(f, 1),$$

as desired. □

4.5 Toward a proof of Kolyvagin's conjecture: assumptions

Our goal in the next sections is to prove Conjecture 4.2 for a large class of modular forms. As hinted at in the introduction, our strategy is based on a deformation-theoretic approach via Hida theory.

4.5.1 $\mathfrak{p}$-ordinariness of f

Recall that f is $\mathfrak{p}$-*ordinary* if $a_p(f) \in \mathcal{O}_\mathfrak{p}^\times$. Furthermore, the semisimplification $\bar{\rho}_\mathfrak{p}^{ss}$ of $\bar{\rho}_\mathfrak{p}$ is *p-distinguished* if its restriction to $G_{\mathbb{Q}_p}$ can be put in the shape $\bar{\rho}_\mathfrak{p}^{ss}|_{G_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$ for characters $\varepsilon_1 \neq \varepsilon_2$ (see, *e.g.*, [48, §2]). Moreover, by [133, Theorem 2.1, (a)], if p is sufficiently large (*i.e.*, if p lies outside a suitable finite set of prime numbers), then $\bar{\rho}_{f,\mathfrak{p}}$ is irreducible for all $\mathfrak{p}$ as above.

Proposition 4.4 *If f is $\mathfrak{p}$-ordinary, then $\bar{\rho}_\mathfrak{p}^{ss}$ is p-distinguished.*

Proof This is a consequence of [163, Theorem 2.1.4]: see, *e.g.*, [159, Lemma 4.12] for details (*cf.* also [95, §2.3] for related computations). □

4.5.2 Assumptions

We work under the following assumption, which includes Assumption 3.6 stated before. We explicitly remark that no p-isolation condition on f is required.

Assumption 4.5 The pair $(f, \mathfrak{p})$ satisfies the following conditions:

(1) $N \geq 3$ is square-free;
(2) $p \nmid 6ND_F c_f$;
(3) $k \equiv 2 \pmod{2(p-1)}$;
(4) $a_p(f) \in \mathcal{O}_\mathfrak{p}^\times$;
(5) $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$;
(6) ρ_p has big image;
(7) $\bar{\rho}_\mathfrak{p}$ is ramified at all primes dividing N.

See Definition 3.8 for the notion of p-isolated form. By Proposition 4.4, if f and $\mathfrak{p}$ satisfy Assumption 4.5, then $\bar{\rho}_\mathfrak{p}^{ss}$ is p-distinguished.

Remark 4.6 In light of results of Serre on eigenvalues of Hecke operators ([145, §7.2]), it seems reasonable to expect that condition (4), which is an ordinariness property for f at $\mathfrak{p}$, holds for infinitely many p. In fact, questions of this sort appear to lie in the circle of ideas of the Lang–Trotter conjectures on the distribution of traces of Frobenius acting on elliptic curves [86] and of their extensions to higher weight modular forms (see, *e.g.*, [108], [109]).

Remark 4.7 It can be checked that f is p-distinguished also as a consequence of the fact that, by condition (3) in Assumption 4.5, k is not congruent to 1 modulo $p-1$: see, *e.g.*, [78, Remark 7.2.7] and the reference therein.

We record the following consequence of Assumption 4.5.

Proposition 4.8 *Let L be a number field such that the extension $L/\mathbb{Q}$ is solvable.*

(1) *The $\mathcal{O}_\mathfrak{p}$-module $H^1(L, T_\mathfrak{p})$ is torsion-free.*
(2) *The $\mathcal{O}_\mathfrak{p}$-modules $H^1_f(L, T_\mathfrak{p})$ and $\Lambda_\mathfrak{p}(L)$ are free of finite rank.*

Proof As p is a uniformizer for $\mathcal{O}_\mathfrak{p}$, in order to show that $H^1(L, T_\mathfrak{p})$ is torsion-free over $\mathcal{O}_\mathfrak{p}$ it is enough to check that the p-torsion of $H^1(L, T_\mathfrak{p})$ is trivial. Since $T_\mathfrak{p}$ is free (hence

torsion-free) over $\mathcal{O}_\mathfrak{p}$, we can consider the short exact sequence of Galois modules

$$0 \longrightarrow T_\mathfrak{p} \xrightarrow{p\cdot} T_\mathfrak{p} \longrightarrow T_\mathfrak{p}/pT_\mathfrak{p} = A_\mathfrak{p}[p] \longrightarrow 0,$$

where the first non-trivial arrow is the multiplication-by-p map and the equality denotes a canonical identification. Passing to cohomology, we see that the p-torsion subgroup of $H^1(L, T_\mathfrak{p})$ is a quotient of $H^0\big(L, A_\mathfrak{p}[p]\big)$. On the other hand, $H^0\big(L, A_\mathfrak{p}[p]\big)$ is trivial by [93, Lemma 3.10, (2)], and part (1) is proved. Finally, the $\mathcal{O}_\mathfrak{p}$-submodules $H^1_f(L, T_\mathfrak{p})$ and $\Lambda_\mathfrak{p}(L)$ of $H^1(L, T_\mathfrak{p})$ are finitely generated, so part (2) follows from part (1). □

4.6 Hida families of modular forms

We sketch the basics of Hida's theory of families of modular forms; see, *e.g.*, [58], [59], [61, Ch. 7] for details and proofs.

4.6.1 *p-stabilization of f*

Let $f \in S_k(\Gamma_0(N))$ be the newform fixed above. Let us write $f^\sharp(q) = \sum_{n\geq 1} a_n(f^\sharp)q^n \in S_k(\Gamma_0(Np))$ for the ordinary p-stabilization of f (see, *e.g.*, [52, p. 410] or [159, §2.4]). The cusp form $f^\sharp$ can be characterized as the unique (normalized) p-ordinary eigenform of weight k and level divisible by p with the property that $a_n(f^\sharp) = a_n(f)$ except for those n divisible by p ([57, Lemma 3.3]).

4.6.2 *Arithmetic primes*

Set $\Gamma := 1 + p\mathbb{Z}_p$, choose a finite extension L of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_L$ and form the Iwasawa algebra $\Lambda_L := \mathcal{O}_L[\![\Gamma]\!]$ of Γ with coefficients in $\mathcal{O}_L$; in the following, we will take $L = F_\mathfrak{p}$. Let A be a finitely generated commutative Λ-algebra. As in [65, Definition 2.1], an $\mathcal{O}_L$-algebra homomorphism $\kappa : A \to \bar{\mathbb{Q}}_p$ is said to be *arithmetic* if the composition

$$\Gamma \longrightarrow A^\times \xrightarrow{\kappa} \bar{\mathbb{Q}}_p^\times$$

with the canonical map $\Gamma \to A^\times$ has the form $\gamma \mapsto \psi(\gamma)\gamma^{k-2}$ for some integer $k \geq 2$ and some finite order character ψ of Γ. A prime ideal of A that is the kernel of an arithmetic homomorphism is an *arithmetic prime* of A; we write $\mathcal{X}^{\mathrm{arith}}(A)$ for the set of such primes. If $\wp$ is an arithmetic prime of A and $A_\wp$ is the localization of A at $\wp$, then the residue field $L_\wp := A_\wp/\wp A_\wp$ is a finite extension of L. The composition $\Gamma \to A^\times \to L_\wp^\times$ has the form $\gamma \mapsto \psi_\wp(\gamma)\gamma^{r_\wp - 2}$ for a finite order character $\psi_\wp : \Gamma \to L_\wp^\times$ and an integer $r_\wp \geq 2$; we call $\psi_\wp$ and $r_\wp$ the *wild character* and the *weight* of $\wp$, respectively.

4.6.3 *p-adic Hida family through f*

Let $\mathcal{R}$ denote the branch of the p-adic Hida family $\boldsymbol{f}$ passing through f (or, rather, through $f^\sharp$; *cf.* [65, §2.1]); we briefly explain the terminology, referring to, *e.g.*, [65] for details. The ring $\mathcal{R}$ is a complete local noetherian domain that is a finite flat Λ_L-algebra; we write $\mathfrak{m}_\mathcal{R}$ for the maximal ideal of $\mathcal{R}$, $\mathbb{F}_\mathcal{R} := \mathcal{R}/\mathfrak{m}_\mathcal{R}$ for its residue field, which is finite of characteristic p, and $\mathcal{F} := \mathrm{frac}(\mathcal{R})$ for its quotient field. Without loss of generality, we may assume that $\mathbb{F}_\mathcal{R}$ is equal to the residue field $\mathbb{F}_L := \mathcal{O}_L/\pi_L\mathcal{O}_L$ of L, where $\mathcal{O}_L$ is the valuation ring of L and $\pi_L \in \mathcal{O}_L$ is a uniformizer. As above, for every $\wp \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ let $L_\wp := \mathcal{R}_\wp/\wp\mathcal{R}_\wp$; moreover, set $s_\wp := \max\big\{1, \mathrm{cond}(\psi_\wp)\big\}$, where $\mathrm{cond}(\psi_\wp)$ is the

conductor of $\psi_{\wp}$. There exists a formal power series

$$\boldsymbol{f} = \sum_{n\geq 1} a_n(\boldsymbol{f})q^n \in \mathcal{R}[\![q]\!]$$

such that

- for every $\wp = \ker(\kappa) \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$, the power series

$$f_{\wp}(q) := \sum_{n\geq 1} \kappa\big(a_n(\boldsymbol{f})\big)q^n \in L_{\wp}[\![q]\!]$$

  is the q-expansion of an ordinary cuspidal eigenform of weight $r_{\wp}$, level $\Gamma_1(Np^{s_{\wp}})$ and character $\psi_{\wp}\omega^{2-r_{\wp}}$, where ω is the Teichmüller character;
- there is $\wp_{f^\sharp} \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ such that $f_{\wp_{f^\sharp}} = f^\sharp$.

Finally, denote by $\mathfrak{h}_N^{\mathrm{ord}}$ Hida's p-ordinary Hecke algebra of tame level N. The homomorphism $\mathfrak{h}_N^{\mathrm{ord}} \to \mathcal{R}$ associated with $\boldsymbol{f}$ corresponds to a minimal prime ideal $\mathfrak{a}$ of $\mathfrak{h}_N^{\mathrm{ord}}$, and then $\mathcal{R}$ is the integral closure of $\mathfrak{h}_N^{\mathrm{ord}}/\mathfrak{a}$ in its quotient field. In particular, $\mathcal{R}$ is a module over $\mathfrak{h}_N^{\mathrm{ord}}$.

4.7 Big Galois representations

As in Sect. 2.3, let $\bar{\mathbb{Z}}$ be the ring of integers of $\bar{\mathbb{Q}}$ and choose a prime ideal $\mathfrak{P}$ of $\bar{\mathbb{Z}}$ such that $\mathfrak{P}\cap\mathcal{O}_F = \mathfrak{p}$. Let us denote by F_g the Hecke field of a given eigenform g. Furthermore, notation being as in Sect. 2.3, we write V_g for the representation of $G_{\mathbb{Q}}$ attached to g and the prime $\mathfrak{P}\cap F_g$, then let $V_g^\dagger = V_g(k/2)$ be the self-dual twist of V_g, where k is the weight of g.

4.7.1 *The representation* $\mathbb{T}$

Let $\mathbb{T}$ denote the representation of $G_{\mathbb{Q}}$ attached to the Hida family $\boldsymbol{f}$ from Sect. 4.6; this "big" Galois representation was constructed by Hida in [58] (*cf.* also [65, Proposition 2.1.2]). Namely, for all $s \geq 1$ let $J_1(Np^s)$ be the Jacobian variety of the (compact) modular curve $X_1(Np^s)$. By Albanese (*i.e.*, covariant) functoriality, the degeneracy maps $X_1(Np^{s+1}) \to X_1(Np^s)$ yield maps $J_1(Np^{s+1}) \to J_1(Np^s)$, which in turn give maps $\mathrm{Ta}_p\big(J_1(Np^{s+1})\big) \to \mathrm{Ta}_p\big(J_1(Np^s)\big)$ between p-adic Tate modules. As in 4.6, let $\mathfrak{h}_N^{\mathrm{ord}}$ be Hida's p-ordinary Hecke algebra of tame level N, then define

$$\mathbb{T} := \left(\varprojlim_s \Big(\mathrm{Ta}_p\big(J_1(Np^s)\big) \otimes_{\mathbb{Z}_p} \mathcal{O}_L\Big)^{\mathrm{ord}}\right) \otimes_{\mathfrak{h}_N^{\mathrm{ord}}} \mathcal{R}, \tag{4.15}$$

where the superscript "ord" indicates ordinary parts, which are cut out by Hida's ordinary projector (see, *e.g.*, [58, p. 551] and [61, §7.2, Lemma 1]). The $\mathcal{R}$-module $\mathbb{T}$ is equipped with a natural action of $G_{\mathbb{Q}}$.

4.7.2 *Basic properties of* $\mathbb{T}$

Under standard assumptions on residual representations (*cf.* Sect. 4.8), $\mathbb{T}$ is a free $\mathcal{R}$-module of rank 2. It satisfies the following crucial property: for every arithmetic prime $\wp$ of $\mathcal{R}$, the *specialization* $\mathbb{T}_{\wp}/\wp\mathbb{T}_{\wp}$ *of* $\mathbb{T}$ *at* $\wp$ is equivalent over $\bar{\mathbb{Q}}_p$ (*i.e.*, after a finite base

change) to the dual (*i.e.*, contragradient) representation $V_{f_\wp}^*$ of the p-adic representation $V_{f_\wp}$ of $G_{\mathbb{Q}}$ attached to $f_\wp$ (see, *e.g.*, [117, (1.5.5)]). The representation

$$\rho_{\boldsymbol{f}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(\mathbb{T}) \simeq \mathrm{GL}_2(\mathcal{R})$$

is unramified outside Np and

$$\mathrm{tr}\big(\rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell)\big) = a_\ell(\boldsymbol{f})$$

for all prime numbers $\ell \nmid Np$, where Frob_ℓ denotes the conjugacy class in $G_{\mathbb{Q}}$ of an arithmetic Frobenius at ℓ (see, *e.g.*, [27, Theorem 4.3]).

Now let v be a place of $\bar{\mathbb{Q}}$ above p, let $G_v \subset G_{\mathbb{Q}}$ be the decomposition group at v and let $I_v \subset G_v$ be the inertia subgroup; by [65, Proposition 2.4.1], there is a short exact sequence of (left) $\mathcal{R}[G_v]$-modules

$$0 \longrightarrow F_v^+(\mathbb{T}) \longrightarrow \mathbb{T} \longrightarrow F_v^-(\mathbb{T}) \longrightarrow 0$$

in which both $F_v^+(\mathbb{T})$ and $F_v^-(\mathbb{T})$ are free of rank 1 over $\mathcal{R}$. The group G_v acts on $F_v^-(\mathbb{T})$ via the unamified character $\eta_v : G_{F_v}/I_v \to \mathcal{R}^\times$ taking the arithmetic Frobenius to U_p, while it acts on $F_v^+(\mathbb{T})$ via $\eta_v^{-1}\varepsilon_{\mathrm{cyc}}[\varepsilon_{\mathrm{cyc}}]$, where $\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $z \mapsto [z]$ denotes the inclusion $\mathbb{Z}_p^\times \hookrightarrow \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]^\times$ of group-like elements (here recall that $\mathbb{Z}_p^\times \simeq \boldsymbol{\mu}_{p-1} \times \Gamma$, where $\boldsymbol{\mu}_{p-1} \subset \bar{\mathbb{Q}}_p^\times$ is the group of $(p-1)$-st roots of unity).

4.8 Critical twist and residual representations

Rather than in $\mathbb{T}$ itself, we will be interested in a suitable twist $\mathbb{T}^\dagger$ of $\mathbb{T}$.

4.8.1 Critical character and critical twist

Let us fix a *critical character* $\Theta : G_{\mathbb{Q}} \to \Lambda^\times$ as in [65, Definition 2.1.3]; with notation and terminology as in [65], in our case $k \equiv 2 \pmod{2(p-1)}$ and $j = 0$, so only the "wild" part of Θ plays a role. We remark that the choice of Θ amounts to the choice of a square root of ω^{k-2}, *i.e.*, an integer a modulo $p-1$ such that $2a = k-2$; let us fix such a choice once and for all (*cf.* [65, Remark 2.1.4]). Let $\mathcal{R}^\dagger$ denote $\mathcal{R}$ viewed as a module over itself but with $G_{\mathbb{Q}}$ acting via Θ^{-1}, then define th *critical twist* of $\mathbb{T}$ to be

$$\mathbb{T}^\dagger := \mathbb{T} \otimes_{\mathcal{R}} \mathcal{R}^\dagger.$$

The twist $\mathbb{T}^\dagger$ has the property that for every arithmetic prime $\wp$ of $\mathcal{R}$ of weight $k_\wp \equiv 2 \pmod{2(p-1)}$ and trivial character the specialization $\mathbb{T}^\dagger_\wp/\wp\mathbb{T}^\dagger_\wp$ of $\mathbb{T}^\dagger$ at $\wp$ is equivalent to $V_{f_\wp}^\dagger$ after a finite base change (see, *e.g.*, [117, (3.2.4)]). As a consequence, there is a specialization map

$$\mathbb{T}^\dagger \longrightarrow \mathbb{T}^\dagger_\wp/\wp\mathbb{T}^\dagger_\wp \simeq V_{f_\wp}^\dagger,$$

which in turn induces specialization maps in cohomology. Summing up, $\mathbb{T}$ and $\mathbb{T}^\dagger$ enjoy the following interpolation properties, up to a finite base change:

- the specialization of $\mathbb{T}$ at an arithmetic prime $\wp$ of $\mathcal{R}$ is equivalent to $V_{f_\wp}^*$;
- if $k_\wp \equiv 2 \pmod{2(p-1)}$, then the specialization of $\mathbb{T}^\dagger$ at $\wp$ is equivalent to $V_{f_\wp}^\dagger$.

4.8.2 Residual representations

Define

$$\bar{\mathbb{T}} := \mathbb{T}/\mathfrak{m}_{\mathcal{R}}\mathbb{T} = \mathbb{T} \otimes_{\mathcal{R}} \mathbb{F}_{\mathcal{R}},$$

which is a 2-dimensional representation of $G_{\mathbb{Q}}$ over $\mathbb{F}_{\mathcal{R}}$. As above, let $\wp$ be an arithmetic prime of weight $k_\wp \equiv 2 \pmod{2(p-1)}$ and trivial character; as in Sect. 2.4, let $T_{f_\wp}$ be the lattice realizing the $\mathfrak{P}$-adic representation attached to $f_\wp$ and let $T_{f_\wp}^\dagger$ be the self-dual twist of $T_{f_\wp}$. As explained, *e.g.*, in [159, §2.2], we have reduced representations

$$\bar{\rho}_{f_\wp} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(\bar{T}_{f_\wp}), \quad \bar{\rho}_{f_\wp}^\dagger : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}\big(\bar{T}_{f_\wp}^\dagger\big)$$

and their semi-simplifications

$$\bar{\rho}_{f_\wp}^{ss} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(\bar{T}_{f_\wp}^{ss}), \quad \bar{\rho}_{f_\wp}^{\dagger,ss} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}\big(\bar{T}_{f_\wp}^{\dagger,ss}\big).$$

It turns out that if $\wp$ and $\wp'$ are two arithmetic primes, then

$$\bar{\rho}_{f_\wp}^{ss} \simeq \bar{\rho}_{f_{\wp'}}^{ss} \tag{4.16}$$

after a finite base change ([59, p. 251]). If $\bar{\rho}_{f_\wp}$ (equivalently, $\bar{\rho}_{f_\wp}^{ss}$) is irreducible and p-distinguished for one (hence for every) arithmetic prime $\wp$, then

- $\mathbb{T}$ is free of rank 2 over $\mathcal{R}$ ([103, Théorème 7]);
- $\bar{\mathbb{T}} \simeq \bar{\rho}_{f_\wp}$ after a finite base change for all such $\wp$ (see, *e.g.*, [91, Proposition 5.4]).

Notice that property (4.16) no longer holds unconditionally once $\bar{\rho}_{f_\wp}^{ss}$ and $\bar{\rho}_{f_{\wp'}}^{ss}$ have been replaced by $\bar{\rho}_{f_\wp}^{\dagger,ss}$ and $\bar{\rho}_{f_{\wp'}}^{\dagger,ss}$, respectively. However, it is true that if $\wp$ and $\wp'$ are arithmetic primes such that $k_\wp \equiv k_{\wp'} \equiv 2 \pmod{2(p-1)}$, then $\bar{\rho}_{f_\wp}^{\dagger,ss} \simeq \bar{\rho}_{f_{\wp'}}^{\dagger,ss}$ after a finite base change (*cf.* [159, Remark 2.7]).

Finally, set $\bar{\mathbb{T}}^\dagger := \mathbb{T}^\dagger/\mathfrak{m}_{\mathcal{R}}\mathbb{T}^\dagger$. An easy computation shows that Θ is trivial modulo $\mathfrak{m}_{\mathcal{R}}$, so there is a canonical identification $\bar{\mathbb{T}} = \bar{\mathbb{T}}^\dagger$ of representations of $G_{\mathbb{Q}}$ over $\mathbb{F}_{\mathcal{R}}$. We write $\pi_{\mathcal{R}} : \mathbb{T}^\dagger \twoheadrightarrow \bar{\mathbb{T}}$ for the surjection determined by the identification between $\bar{\mathbb{T}}$ and $\bar{\mathbb{T}}^\dagger$, and

$$\pi_{\mathcal{R},L} : H^1\big(L, \mathbb{T}^\dagger\big) \longrightarrow H^1(L, \bar{\mathbb{T}}) \tag{4.17}$$

for the map in cohomology induced functorially by $\pi_{\mathcal{R}}$, where L is a given number field.

4.9 Abelian varieties and Kummer maps in weight 2

Let f_2 be the specialization of $\boldsymbol{f}$ of weight 2 and trivial character; the cusp form f_2 is a p-ordinary, p-stabilized newform (in the sense of [52, Definition 2.5]) of level Np and conductor either N or Np. If f_2 has conductor N, then f_2 is the p-stabilization of a newform g of weight 2 and level N, otherwise we set $g := f_2$. In both cases, denote by N_g the level (*i.e.*, conductor) of the newform g and write $\sum_{n\geq 1} a_n(g)q^n$ for the q-expansion of $g \in S_2(\Gamma_0(N_g))$.

4.9.1 The abelian variety A_g

As in Sect. 4.7, let $F_g = \mathbb{Q}\big(a_n(g) \mid n \geq 1\big)$ be the Hecke field of g; denote by A_g the abelian variety over $\mathbb{Q}$ of conductor N_g attached to g via the Eichler–Shimura construction. Then A_g, which arises as a quotient of the Jacobian $J_0(N_g)$ of the modular curve $X_0(N_g)$, has dimension equal to the degree of F_g and is of GL_2-type. Furthermore, the endomorphism ring of A_g is (isomorphic to) the ring of integers $\mathcal{O}_g$ of F_g and all endomorphisms of A_g are defined over $\mathbb{Q}$.

4.9.2 Galois representations attached to g

Let $\mathfrak{P}$ be the prime ideal of $\bar{\mathbb{Z}}$ from Sect. 4.7. Set $\boldsymbol{p} := \mathfrak{P} \cap \mathcal{O}_g$, which is a prime of F_g above p, and let $F_{g,\boldsymbol{p}}$ be the completion of F_g at $\boldsymbol{p}$, whose valuation ring will be denoted by $\mathcal{O}_{g,\boldsymbol{p}}$. Let $\mathrm{Ta}_{\boldsymbol{p}}(A_g)$ be the $\boldsymbol{p}$-adic Tate module of A_g and let $V_{\boldsymbol{p}}(A_g) := \mathrm{Ta}_{\boldsymbol{p}}(A_g) \otimes \mathbb{Q}$ be the associated $F_{g,\boldsymbol{p}}$-linear representation of $G_{\mathbb{Q}}$. If $V_{g,\boldsymbol{p}}$ denotes the $F_{g,\boldsymbol{p}}$-linear Galois representation attached to g then there is an identification

$$V_{g,\boldsymbol{p}} = H^1_{\text{ét}}(A_g, F_{g,\boldsymbol{p}}) \simeq V_{\boldsymbol{p}}(A_g)^*, \tag{4.18}$$

where $V_{\boldsymbol{p}}(A_g)^* := \mathrm{Hom}_{F_{g,\boldsymbol{p}}}\big(V_{\boldsymbol{p}}(A_g), F_{g,\boldsymbol{p}}\big)$ is the $F_{g,\boldsymbol{p}}$-linear dual of $V_{\boldsymbol{p}}(A_g)$ equipped with its contragradient $G_{\mathbb{Q}}$-action (the isomorphism in (4.18) follows by combining [106, Theorem 15.1, (a)] with the $G_{\mathbb{Q}}$-equivariant splitting $\mathrm{Ta}_p(A_g) = \bigoplus_{\pi \mid p} \mathrm{Ta}_\pi(A_g)$, with π varying over all the primes of F_g above p). By taking the self-dual twist, we obtain

$$V^\dagger_{g,\boldsymbol{p}} = V_{g,\boldsymbol{p}}(1) \simeq V_{\boldsymbol{p}}(A_g)^*(1) \simeq V_{\boldsymbol{p}}(A_g), \tag{4.19}$$

where the rightmost isomorphism, which we fix once and for all, is a consequence of the Weil pairing. Let us choose a $G_{\mathbb{Q}}$-stable $\mathcal{O}_{g,\boldsymbol{p}}$-lattice $T_{g,\boldsymbol{p}} \subset V_{g,\boldsymbol{p}}$ whose Tate twist

$$T^\dagger_{g,\boldsymbol{p}} := T_{g,\boldsymbol{p}} \otimes_{\mathcal{O}_{g,\boldsymbol{p}}} \mathcal{O}_{g,\boldsymbol{p}}(1) \subset V^\dagger_{g,\boldsymbol{p}}$$

corresponds to $\mathrm{Ta}_{\boldsymbol{p}}(A_g)$ under isomorphism (4.19). For any number field L, let

$$\xi_{g,L} : H^1\big(L, T^\dagger_{g,\boldsymbol{p}}\big) \xrightarrow{\simeq} H^1\big(L, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big) \tag{4.20}$$

be the isomorphism induced by (4.19) functorially.

Notation 4.9 From here on, we drop dependence on $\boldsymbol{p}$ from our notation and simply write $T^\dagger_g$ (respectively, $V^\dagger_g$) in place of $T^\dagger_{g,\boldsymbol{p}}$ (respectively, $V^\dagger_{g,\boldsymbol{p}}$).

4.9.3 Kummer maps

Now let $A_g[\boldsymbol{p}]$ be the $\boldsymbol{p}$-torsion $\mathcal{O}_g$-submodule of $A_g(\bar{\mathbb{Q}})$. For any number field L and every integer $M \geq 1$ let

$$\pi_{g,L,M} : H^1\big(L, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big) \longrightarrow H^1\big(L, A_g[\boldsymbol{p}^M]\big) \tag{4.21}$$

be the map induced by the surjection $\mathrm{Ta}_{\boldsymbol{p}}(A_g) \twoheadrightarrow A_g\big[\boldsymbol{p}^M\big]$ and let

$$\delta_{g,L} : A_g(L) \longrightarrow H^1\big(L, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big), \quad \pi_{g,L,M} \circ \delta_{g,L} : A_g(K) \longrightarrow H^1\big(L, A_g[\boldsymbol{p}^M]\big) \tag{4.22}$$

be the Kummer maps in Galois cohomology (see, *e.g.*, [54, Appendix A.1]). In turn, the second map gives an injection

$$\bar{\delta}_{g,L,M} : A_g(L)\big/\boldsymbol{p}^M A_g(L) \hookrightarrow H^1\big(L, A_g[\boldsymbol{p}^M]\big) \tag{4.23}$$

that, in our context, is the weight 2 counterpart of the map $\iota_{L,M}$ introduced in (4.9). From now on, we shall simply set $\pi_{g,L} := \pi_{g,L,1}$ and $\bar{\delta}_{g,L} := \bar{\delta}_{g,L,1}$.

4.10 Kolyvagin classes in weight 2

Let $g = \sum_{n\geq 1} a_n(g)q^n \in S_2(\Gamma_0(N_g))$ be the newform from Sect. 4.9. Assume that p splits in K, so that the imaginary quadratic field K satisfies the Heegner hypothesis with respect to both N and Np.

4.10.1 Kolyvagin integers

Let $\nu_{\boldsymbol{p}}$ be the valuation of $F_{g,\boldsymbol{p}}$ normalized by declaring that $\nu_{\boldsymbol{p}}$ takes the value 1 at a uniformizer of $\mathcal{O}_{g,\boldsymbol{p}}$. By analogy with the definition given in Sect. 4.2, using $\nu_{\boldsymbol{p}}$ in place of $\nu_{\mathfrak{p}}$, one can introduce the sets $\mathcal{P}_{\mathrm{Kol}}(g)$ of Kolyvagin primes and $\Lambda_{\mathrm{Kol}}(g)$ of Kolyvagin integers for the data $(g, \boldsymbol{p}, K)$. Moreover, complex multiplication allows one to define Heegner points $\alpha_c \in A_g(K_c)$ indexed by integers $c \geq 1$ coprime to N. These points arise by modularity from the Heegner points $x_c \in X_0(N)(K_c)$ appearing in Sect. 4.1. For details, the reader is referred to [53,81,83].

4.10.2 Weight 2 Kolyvagin classes

Using the points α_n instead of the cycles $y_{n,\mathfrak{p}}$, and the maps $\bar{\delta}_{g,K_n,M}$ from (4.23) in place of the maps $\iota_{K_n,M}$, the recipe in Sect. 4.3 yields Kolyvagin cohomology classes

$$c_M(g,n) \in H^1\big(K, A_g[\boldsymbol{p}^M]\big), \quad d_M(g,n) \in H^1\big(K_n, A_g[\boldsymbol{p}^M]\big)^{\mathcal{G}_n}$$

for $n \in \Lambda_{\mathrm{Kol}}(g)$ and $M \leq M(g,n)$, and then a Kolyvagin set

$$\kappa_{g,\infty} := \big\{c_M(g,n) \mid n \in \Lambda_{\mathrm{Kol}}(g),\ 1 \leq M \leq M(g,n)\big\}$$

attached to $(g, \boldsymbol{p}, K)$. In particular, one has

$$c_M(g,n) = 0 \iff d_M(g,n) = 0. \tag{4.24}$$

The set $\kappa_{g,\infty}$ is the supply of Galois cohomology classes whose non-triviality was predicted (at least when A_g is an elliptic curve) by Kolyvagin in [82] and then confirmed (in a stronger form, under some assumptions) by Zhang in [166] and by Skinner–Zhang in [152]. For more details on the construction of $\kappa_{g,\infty}$, see, *e.g.*, [166, §3.7].

Remark 4.10 By analogy with what was done in Sect. 4.3, we should write $\kappa^{(K)}_{g,\boldsymbol{p},\infty}$ instead of $\kappa_{g,\infty}$. However, since this family of Kolyvagin classes will play only an auxiliary role, we prefer to keep our notation as light as possible.

Remark 4.11 The results of Zhang on Kolyvagin's conjecture for weight 2 newforms have been extended by Sweeting in [153], where some of the technical assumptions in [166] are relaxed. Observe, however, that Sweeting's results are proved, like Zhang's, exclusively for

modular forms of weight 2, so they do not overlap with ours. For results on Kolyvagin's conjecture for a rational elliptic curve E that cover, in particular, many cases where p is an Eisenstein prime for E, thus complementing Zhang's work, see also the recent paper [23] by Burungale–Castella–Grossi–Skinner.

For later use, we prove

Lemma 4.12 $\Lambda_{\mathrm{Kol}}(g) = \Lambda_{\mathrm{Kol}}(f)$.

Proof Let ℓ be a prime number. Clearly, $v_{\mathfrak{p}}(\ell+1) > 0$ if and only if $v_{\boldsymbol{p}}(\ell+1) > 0$. On the other hand, there is a congruence

$$a_\ell(f) \equiv a_\ell(g) \pmod{\mathfrak{P}} \tag{4.25}$$

(*cf.* [59, p. 251]), which immediately implies that $v_{\mathfrak{p}}\big(a_\ell(f)\big) > 0$ if and only if $v_{\boldsymbol{p}}\big(a_\ell(g)\big) > 0$. This shows that $\mathcal{P}_{\mathrm{Kol}}(f) = \mathcal{P}_{\mathrm{Kol}}(g)$, and the lemma is proved. □

4.11 Distinguished specialization maps

Let $\wp_f$ be the arithmetic prime of $\mathcal{R}$ such that $f_{\wp_f} = f^\sharp$. Similarly, with notation as in Sect. 4.9, let $\wp_g$ be the arithmetic prime of $\mathcal{R}$ such that $f_{\wp_g} = g$. Let $\star \in \{f, g\}$.

4.11.1 Specializations and reductions

As was already remarked in Sect. 4.8.1, the representation $T^\dagger_{\wp_\star}\big/\wp_\star\mathbb{T}^\dagger_{\wp_\star}$ of $G_{\mathbb{Q}}$ is equivalent (after a finite base change) to $V^\dagger_\star$; for later use in the statement of the specialization theorem for big Heegner points (Theorem 4.13) and in our proof of Kolyvagin's conjecture in higher weight (Theorem 4.14), here we fix once and for all an isomorphism between these two representations (*cf.* [122, Remark 1.3, (4)]). Following Ota, let

$$\mathrm{sp}_0^\star : \mathbb{T}^\dagger \longrightarrow T^\dagger_\star \tag{4.26}$$

be the $G_{\mathbb{Q}}$-equivariant specialization map that is described in [122, §2.6]. This map factors through the surjection $\mathbb{T}^\dagger \twoheadrightarrow \mathbb{T}^\dagger/\wp_\star\mathbb{T}^\dagger$ and induces, possibly after a finite base change, an isomorphism

$$\mathrm{sp}_0^\star : \mathbb{T}^\dagger_{\wp_\star}\big/\wp_\star\mathbb{T}^\dagger_{\wp_\star} \xrightarrow{\simeq} V^\dagger_\star \tag{4.27}$$

of representations of $G_{\mathbb{Q}}$. Note that $\mathbb{T}^\dagger/\wp_\star\mathbb{T}^\dagger$ sits as an $\mathcal{R}/\wp_\star$-lattice inside $\mathbb{T}^\dagger_{\wp_\star}\big/\wp_\star\mathbb{T}^\dagger_{\wp_\star}$. As explained, *e.g.*, in [159, §5.1], isomorphism (4.27) determines, up to finite base change (*i.e.*, over $\bar{\mathbb{F}}_p$), an isomorphism

$$\overline{\mathrm{sp}}_0^\star : \bar{\mathbb{T}} \xrightarrow{\simeq} \bar{T}^\dagger_\star \tag{4.28}$$

of representations of $G_{\mathbb{Q}}$ that makes the square

$$\begin{array}{ccc} \mathbb{T}^\dagger & \xrightarrow{\mathrm{sp}_0^\star} & T^\dagger_\star \\ \downarrow{\scriptstyle \pi_{\mathcal{R}}} & & \downarrow \\ \bar{\mathbb{T}} & \xrightarrow[\simeq]{\overline{\mathrm{sp}}_0^\star} & \bar{T}^\dagger_\star \end{array} \tag{4.29}$$

commute (*cf.* also [78, §7.3]); here $\pi_{\mathcal{R}}$ is, as before, given by reduction modulo $\mathfrak{m}_{\mathcal{R}}$ and the right vertical arrow is the canonical surjection.

4.11.2 Specializations and cohomology

By functoriality, for any number field L the maps in (4.26) and (4.28) determine maps

$$\mathrm{sp}^{\star}_{0,L} : H^1(L, \mathbb{T}^{\dagger}) \longrightarrow H^1\big(L, T^{\dagger}_{\star}\big), \quad \overline{\mathrm{sp}}^{\star}_{0,L} : H^1(L, \bar{\mathbb{T}}) \xrightarrow{\simeq} H^1\big(L, \bar{T}^{\dagger}_{\star}\big), \tag{4.30}$$

and then the square in (4.29) gives a commutative square

$$\begin{array}{ccc} H^1(L, \mathbb{T}^{\dagger}) & \xrightarrow{\mathrm{sp}^{\star}_{0,L}} & H^1\big(L, T^{\dagger}_{\star}\big) \\ \downarrow{\scriptstyle \pi_{\mathcal{R},L}} & & \downarrow{\scriptstyle \varpi_{\star,L}} \\ H^1(L, \bar{\mathbb{T}}) & \xrightarrow[\simeq]{\overline{\mathrm{sp}}^{\star}_{0,L}} & H^1\big(L, \bar{T}^{\dagger}_{\star}\big) \end{array} \tag{4.31}$$

in Galois cohomology, where $\pi_{\mathcal{R},L}$ is the map in (4.17) and $\varpi_{\star,L}$ is induced functorially by the surjection $T^{\dagger}_{\star} \twoheadrightarrow \bar{T}^{\dagger}_{\star}$.

4.12 Big Heegner points in Hida families

Let K be the imaginary quadratic field from Sect. 4.10; the ring class fields of K will be denoted as in Sect. 4.1.

4.12.1 Heegner points in towers of modular curves

Following a recipe described by Castella in [27], which refines a construction originally proposed by Howard in [65], we consider the tower of modular curves $\tilde{X}_s := X_1(Np^s)$ for integers $s \geq 1$. Let $n \geq 1$ be an integer coprime to Np. As in [27, §4.2], one can define Heegner points $P_{n,s} \in \tilde{X}_s\big(\tilde{L}_{np^s}(\mu\!\!\mu_{p^s})\big)$, where $\mu\!\!\mu_{p^s}$ is the set of p^s-th roots of unity (in a fixed algebraic closure of K) and $\tilde{L}_{np^s}$ is the compositum of K_{np^s} and the ray class field of K of conductor $\mathcal{N}$; here $\mathcal{N}$ is the ideal of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$ appearing in the construction of those Heegner points (*cf.* [27, §2.5]): we fix it as in Sect. 4.1.

4.12.2 Big Heegner points

As in Sect. 4.6, let $\mathfrak{h}^{\mathrm{ord}}_N$ be Hida's ordinary Hecke algebra of tame level N. Let $\mathrm{Ta}^{\mathrm{ord}}_p\big(\tilde{J}_s\big)$ be the ordinary part of the p-adic Tate module of the Jacobian variety $\tilde{J}_s$ of $\tilde{X}_s$ and define $\mathbb{T}^{\dagger}_s := \mathrm{Ta}^{\mathrm{ord}}_p(\tilde{J}_s) \otimes_{\mathfrak{h}^{\mathrm{ord}}_N} \mathcal{R}^{\dagger}$. For any number field L denote by $\mathfrak{G}_L$ the Galois group over L of the maximal extension of L unramified outside the primes above Np. Applying the twisted Kummer map defined in [65, p. 101] to $P_{n,s}$ gives rise to a cohomology class in $H^1\big(\mathfrak{G}_{\tilde{L}_{np^s}(\mu\!\!\mu_{p^s})}, \mathbb{T}^{\dagger}_s\big)$; inflating and then corestricting from $\tilde{L}_{np^s}(\mu\!\!\mu_{p^s})$ to K_n, we get a class $\mathcal{P}_{n,s} \in H^1\big(K_n, \mathbb{T}^{\dagger}_s\big)$. These classes satisfy the compatibility relation

$$\alpha_{s,*}(\mathcal{P}_{n,s}) = U_p \cdot \mathcal{P}_{n,s-1}$$

for all $s \geq 2$, where $\alpha_s : \tilde{X}_s \to \tilde{X}_{s-1}$ is the natural covering map and the map $\alpha_{s,*}$ is induced functorially by α_s in cohomology. The *big Heegner point of conductor n* is then

$$\mathfrak{X}_n := \varprojlim_s U_p^{-s} \cdot \mathcal{P}_{n,s} \in H^1(K_n, \mathbb{T}^{\dagger}); \tag{4.32}$$

observe that this expression does indeed make sense because the Hecke operator U_p acts invertibly on ordinary submodules.

4.13 Specializations of big Heegner points

Let $c \geq 1$ be an integer coprime to $ND_K p$. Let α_c and y_c be the Heegner point and the Heegner cycle of conductor c introduced in Sects. 4.10 and 4.1, respectively. Finally, let $\mathfrak{X}_c$ be the big Heegner point of conductor c from (4.32). We identify (big) Heegner points and Heegner cycles with their images via the natural group homomorphisms

$$H^1(L, M) \longrightarrow H^1\big(L, M \otimes \bar{\mathbb{Z}}_p\big), \quad H^1(L, M') \longrightarrow H^1\big(L, M' \otimes \bar{\mathbb{F}}_p\big)$$

where $M \in \big\{\mathrm{Ta}_{\boldsymbol{p}}(A_g), T_g^\dagger, \mathbb{T}^\dagger, T_f^\dagger\big\}$, $M' \in \big\{A_g[\boldsymbol{p}], \bar{T}_g^\dagger, \bar{\mathbb{T}}, \bar{T}_f^\dagger\big\}$ and L is either K or a ring class field of K. Recall the maps

$$\mathrm{sp}^f_{0,K_c} : H^1(K_c, \mathbb{T}^\dagger) \to H^1\big(K_c, T_\star^\dagger\big), \quad \xi_{g,K_c} \circ \mathrm{sp}^g_{0,K_c} : H^1(K_c, \mathbb{T}^\dagger) \to H^1\big(K_c, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big)$$

from Sect. 4.9.2 and Sect. 4.11.1. It is natural to consider $\mathrm{sp}^f_{0,K_c}(\mathfrak{X}_c)$ and $\xi_{g,K_c}\big(\mathrm{sp}^g_{0,K_c}(\mathfrak{X}_c)\big)$ and compare them to Heegner cycles and to Heegner points, respectively. The results we are interested in are due to Howard (in weight 2, [64]), Castella [27] and Ota [122].

4.13.1 Weight 2 specialization: the case $N_g = N$

This is the case covered by the results of Castella [27] and of Ota [122]: see part (2) of Theorem 4.13.

4.13.2 Weight 2 specialization: the case $N_g = Np$

In this case, the comparison between $\xi_{g,K_c}\big(\mathrm{sp}^g_{0,K_c}(\mathfrak{X}_c)\big)$ and Heegner points is easier; it is carried out, albeit somewhat in disguise, by Howard in [64, Section 3]. To explain this result, including the fact that the big Heegner points considered in [27] differ slightly from those originally defined in [65], we proceed as follows (the actual relation between Howard's classes and Castella's is immaterial for our goals, so we shall be quite brief and simply outline the arguments).

Notation from Sect. 4.9 is in force. For all $s \geq 1$ set $X_{0,1}(N, p^s) := X\big(\Gamma_0(N) \cap \Gamma_1(p^s)\big)$ and write $J_{0,1}(N, p^s)$ for the Jacobian variety of $X_{0,1}(N, p^s)$; there are obvious degeneracy maps $X_{0,1}(N, p^{s+1}) \to X_{0,1}(N, p^s)$ that yield maps between p-adic Tate modules of Jacobians. Set

$$\widetilde{\mathbb{T}} := \left(\varprojlim_s \Big(\mathrm{Ta}_p\big(J_{0,1}(N, p^s)\big) \otimes_{\mathbb{Z}_p} \mathcal{O}_L\Big)^{\mathrm{ord}}\right) \otimes_{\mathfrak{h}_N^{\mathrm{ord}}} \mathcal{R}.$$

This is the big Galois representation considered by Howard in [65]; its critical twist $\widetilde{\mathbb{T}}^\dagger$ is defined as in Sect. 4.8. The canonical degeneracy maps $X_1(Np^s) \to X_{0,1}(N, p^s)$ induce maps between p-adic Tate modules of Jacobians and, since the tame character of our Hida family is trivial, a natural isomorphism $\mathbb{T}^\dagger \xrightarrow{\simeq} \widetilde{\mathbb{T}}^\dagger$ of representations of $G_{\mathbb{Q}}$, where $\mathbb{T}$ is defined in (4.15). By functoriality, for any number field $\mathcal{K}$ we get an isomorphism

$$\Xi_{\mathcal{K}} : H^1(\mathcal{K}, \mathbb{T}^\dagger) \xrightarrow{\simeq} H^1\big(\mathcal{K}, \widetilde{\mathbb{T}}^\dagger\big). \tag{4.33}$$

Let

$$\tilde{\xi}_{g,\mathcal{K}} \circ \widetilde{\mathrm{sp}}^{g}_{0,\mathcal{K}} : H^1\big(\mathcal{K}, \widetilde{\mathbb{T}}^{\dagger}\big) \longrightarrow H^1\big(\mathcal{K}, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big)$$

be the analogue of $\xi_{g,\mathcal{K}} \circ \mathrm{sp}^{g}_{0,\mathcal{K}}$ for the representation $\widetilde{\mathbb{T}}^{\dagger}$; then

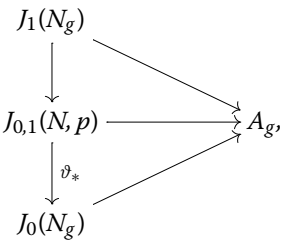
$$\xi_{g,\mathcal{K}} \circ \mathrm{sp}^{g}_{0,\mathcal{K}} = \tilde{\xi}_{g,\mathcal{K}} \circ \widetilde{\mathrm{sp}}^{g}_{0,\mathcal{K}} \circ \Xi_{\mathcal{K}}. \tag{4.34}$$

Given $c \geq 1$ coprime to Np, denote by $\mathfrak{X}_c^{\mathrm{How}} \in H^1\big(K_c, \widetilde{\mathbb{T}}^{\dagger}\big)$ the image under inflation of Howard's original big Heegner point of conductor c ([65, Definition 2.2.3]). As remarked in the proof of [27, Proposition 4.4], the Heegner points used in [27], which live on the tower of modular curves $X_1(Np^s)$, project to those from [65], which live on the tower of modular curves $X_{0,1}(N, p^s)$. On the other hand, the constructions in [27] and [65] are compatible with projections, and then one can check that

$$\Xi_{K_c}(\mathfrak{X}_c) = \mathfrak{X}_c^{\mathrm{How}}, \tag{4.35}$$

where Ξ_{K_c} is the map in (4.33) with $\mathcal{K} = K_c$.

Now, for all $s \geq 1$ embed $X_0(N_g)$ into its Jacobian $J_0(N_g)$ by sending the cusp ∞ to 0. Since g has trivial character, the degeneracy maps $X_1(N_g) \to X_{0,1}(N_g) \xrightarrow{\vartheta} X_0(N_g)$ induce by covariant functoriality a commutative diagram



where the horizontal arrow is defined as the composition of the maps in the lower triangle. Denote by $\tilde{x}_{cp} \in X_{0,1}(N,p)\big(K_{cp}(\mu_p)\big)$ the Heegner point that appears in [65, p. 98] in the construction of $\mathfrak{X}_c^{\mathrm{How}}$. If $x_{cp} \in X_0(N_g)(K_{cp})$ is the Heegner point of conductor cp from Sect. 4.1 and kolyvaginspsclasssps2spssubsec, then it is not restrictive to assume that $x_{cp} = \vartheta(\tilde{x}_{cp})$. It turns out that, since g has trivial character, $\tilde{\xi}_{g,K_c}\big(\widetilde{\mathrm{sp}}^{g}_{0,K_c}(\mathfrak{X}_c^{\mathrm{How}})\big)$ is the image under the Kummer map of the trace

$$\sum_{\sigma \in \mathrm{Gal}(K_{cp}(\mu_p)/K_c)} \big([\tilde{x}_{cp}] - [\infty]\big)^{\sigma} \tag{4.36}$$

(*cf.* [65, eq. (7)]). On the other hand, x_{cp} is rational over K_{cp}, so the image in $J_0(N_g)$ via ϑ_* of the divisor class in (4.36) is

$$(p-1) \cdot \sum_{\sigma \in \mathrm{Gal}(K_{cp}/K_c)} \big([x_{cp}] - [\infty]\big)^{\sigma}. \tag{4.37}$$

As explained, *e.g.*, in [8, §2.4], there is an equality

$$\sum_{\sigma \in \mathrm{Gal}(K_{cp}/K_c)} \big([x_{cp}] - [\infty]\big)^{\sigma} = (U_p - 1)\big([x_c] - [\infty]\big),$$

where $x_c \in X_0(N_g)(K_c)$ is the Heegner point of conductor c introduced in Sect. 4.1 and U_p is the usual Hecke operator. It follows that the image of (4.37) in A_g is

$$\big(a_p(g) - 1\big) \cdot (p-1) \cdot \alpha_c.$$

By part (5) of Assumption 4.5, $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$, so $a_p(g) \not\equiv 1 \pmod{\boldsymbol{p}}$, by congruence (4.25). Finally, combining all these facts with (4.34) for $\mathcal{K} = K_c$ and (4.35), a computation shows that

$$\xi_{g,K_c}\big(\mathrm{sp}^g_{0,K_c}(\mathfrak{X}_c)\big) = d \cdot \delta_{g,K_c}(\alpha_c) \tag{4.38}$$

for a suitable $d = d(c) \in \bar{\mathbb{Z}}_p^\times$, where δ_{g,K_c} is the Kummer map from (4.22).

4.13.3 The specialization theorem

We keep notation from (4.22) in force. The next result will play a crucial role in our proof of Kolyvagin's conjecture for f.

Theorem 4.13 (Castella, Howard, Ota) *There exist $d = d(c) \in \bar{\mathbb{Z}}_p^\times$ and $e = e(c) \in \bar{\mathbb{Z}}_p^\times$ such that*

(1) $\xi_{g,K_c}\big(\mathrm{sp}^g_{0,K_c}(\mathfrak{X}_c)\big) = d \cdot \delta_{g,K_c}(\alpha_c)$;
(2) $\mathrm{sp}^f_{0,K_c}(\mathfrak{X}_c) = e \cdot y_{c,\mathfrak{p}}$.

Proof Part (2) is [27, Theorem 6.5], while part (1) follows by the same arguments if $N_g = N$ (*cf.* [27, Remark 6.6]) and by the arguments sketched in Sect. 4.13.2 if $N_g = Np$ (*cf.* equality (4.38)). Note that, in our setting, the constant appearing in [27, Theorem 6.5] is a p-adic unit. See also [122, Theorem 1.2] for a refinement of the main result of [27] that works under our assumptions, which are slightly weaker than those in [27]. □

4.14 Big Kolyvagin classes

For every integer $c \geq 1$ coprime to N let $\mathfrak{X}_c \in H^1(K_c, \mathbb{T}^\dagger)$ be the big Heegner point of conductor c from (4.32). With notation as in Sect. 4.3, for every $n \in \Lambda_{\mathrm{Kol}}(f)$ define the *big Kolyvagin class* of conductor n as

$$d(\boldsymbol{f}, n) := \sum_{\sigma \in \mathcal{G}} \sigma\big(D_n(\mathfrak{X}_n)\big) \in H^1(K_n, \mathbb{T}^\dagger). \tag{4.39}$$

As we shall see in Sect. 4.15, the classes $d(\boldsymbol{f}, n)$ will be crucially used to prove, under suitable assumptions, Conjecture 4.2.

4.15 Proof of Kolyvagin's conjecture

Recall the strict Kolyvagin set $\kappa^{\mathrm{st}}_{f,\infty}$ that was attached to f, $\mathfrak{p}$, K in (4.13). We can now establish (under our running assumptions, *cf.* Sect. 4.5.2) the non-triviality of $\kappa^{\mathrm{st}}_{f,\infty}$; as a consequence, we deduce the validity of Conjecture 4.2 for f, thus proving Theorem C.

Theorem 4.14 *There exists $n_0 \in \Lambda_{\mathrm{Kol}}(f)$ such that $c_1(f, n_0) \neq 0$. In particular, $\kappa^{\mathrm{st}}_{f,\infty} \neq \{0\}$.*

Proof For any number field L, square (4.31) gives a commutative diagram

$$\begin{array}{ccccccc}
H^1\big(L, \mathrm{Ta}_{\boldsymbol{p}}(A_g)\big) & \xleftarrow[\simeq]{\xi_{g,L}} & H^1\big(L, T_g^\dagger\big) & \xleftarrow{\mathrm{sp}^g_{0,L}} & H^1(L, \mathbb{T}^\dagger) & \xrightarrow{\mathrm{sp}^f_{0,L}} & H^1\big(L, T_f^\dagger\big) \\
\downarrow{\scriptstyle \pi_{g,L}} & & \downarrow{\scriptstyle \varpi_{g,L}} & & \downarrow{\scriptstyle \pi_{\mathcal{R},L}} & & \downarrow{\scriptstyle \varpi_{f,L}} \\
H^1\big(L, A_g[\boldsymbol{p}]\big) & \xleftarrow[\simeq]{\bar{\xi}_{g,L}} & H^1\big(L, \bar{T}_g^\dagger\big) & \xleftarrow[\simeq]{\overline{\mathrm{sp}}^g_{0,L}} & H^1(L, \bar{\mathbb{T}}) & \xleftarrow[\simeq]{\left(\overline{\mathrm{sp}}^f_{0,L}\right)^{-1}} & H^1\big(L, \bar{T}_f^\dagger\big)
\end{array} \tag{4.40}$$

in which $\xi_{g,L}$ is as in (4.20), while the central and right horizontal maps are those appearing in (4.30) and the vertical ones are as in (4.21), (4.17) and Sect. 4.11. We remark that, although our notation does not reflect this, all characteristic 0 (respectively, residual) representations in (4.40) are base changed to $\bar{\mathbb{Z}}_p$ (respectively, $\bar{\mathbb{F}}_p$).

Recall from Lemma 4.12 that $\Lambda_{\mathrm{Kol}}(f) = \Lambda_{\mathrm{Kol}}(g)$. Let $n \in \Lambda_{\mathrm{Kol}}(f)$ and let $d(\boldsymbol{f}, n) \in H^1(K_n, \mathbb{T}^\dagger)$ be the big Kolyvagin class of conductor n introduced in (4.39). By part (2) of Theorem 4.13, $\mathrm{sp}^f_{0,K_n}(\mathfrak{X}_n) = e \cdot y_{n,\mathfrak{p}}$ for some $e \in \bar{\mathbb{Z}}_p^\times$. Since

- there is a canonical identification $\bar{T}_f^\dagger = A_f^\dagger[p]$,
- the square

$$\begin{array}{ccc}
\Lambda_{\mathfrak{p}}(K_n) & \hookrightarrow & H^1\big(K_n, T_f^\dagger\big) \\
\downarrow & & \downarrow{\scriptstyle \varpi_{f,K_n}} \\
\Lambda_{\mathfrak{p}}(K_n)\big/p\Lambda_{\mathfrak{p}}(K_n) & \xhookrightarrow{\iota_{K_n,1}} & H^1\big(K_n, A_f^\dagger[p]\big),
\end{array}$$

  where the top horizontal arrow is the set-theoretic inclusion, is commutative,
- all the maps in (4.40) are Galois-equivariant,

it follows that

$$\mathrm{sp}^f_{0,K_n}\big(d(\boldsymbol{f}, n)\big) = \sum_{\sigma\in\mathcal{G}} \sigma\Big(D_n\big(\mathrm{sp}^f_{0,K_n}(\mathfrak{X}_n)\big)\Big) = e \cdot \sum_{\sigma\in\mathcal{G}} \sigma\big(D_n(y_{n,\mathfrak{p}})\big) = e \cdot z_{n,\mathfrak{p}},$$

whence

$$\varpi_{f,K_n}\Big(\mathrm{sp}^f_{0,K_n}\big(d(\boldsymbol{f}, n)\big)\Big) = \bar{e} \cdot \iota_{K_n,1}\big([z_{n,\mathfrak{p}}]_1\big) = \bar{e} \cdot d_1(f, n) \tag{4.41}$$

with $\bar{e} \in \bar{\mathbb{F}}_p^\times$. On the other hand, $\xi_{g,K_n}\big(\mathrm{sp}^g_{0,K_n}(\mathfrak{X}_n)\big) = d \cdot \delta_{g,K_n}(\alpha_n)$ for some $d \in \bar{\mathbb{Z}}_p^\times$, by part (1) of Theorem 4.13. Thus, if $[\alpha_n]_1$ denotes the image of α_n in $A_g(K_n)/\boldsymbol{p}A_g(K_n)$, then

$$\begin{aligned}
\pi_{g,K_n}\Big(\xi_{g,K_n}\big(\mathrm{sp}^g_{0,K_n}(\mathfrak{X}_n)\big)\Big) &= \bar{d} \cdot \pi_{g,K_n}\big(\delta_{g,K_n}(\alpha_n)\big) \\
&= \bar{d} \cdot \bar{\delta}_{g,K_n}\big([\alpha_n]_1\big)
\end{aligned}$$

with $\bar{d} \in \bar{\mathbb{F}}_p^\times$. By definition of $d_1(g, n)$, this immediately implies that

$$\pi_{g,K_n}\Big(\xi_{g,K_n}\big(\mathrm{sp}^g_{0,K_n}\big(d(\boldsymbol{f}, n)\big)\big)\Big) = \bar{d} \cdot d_1(g, n). \tag{4.42}$$

For simplicity, set

$$\psi_n := \tilde{\xi}_{g,K_n} \circ \overline{\mathrm{sp}}^{g}_{0,K_n} \circ \left(\overline{\mathrm{sp}}^{f}_{0,K_n}\right)^{-1} : H^1\big(K_n, \bar{T}_f^{\dagger}\big) \xrightarrow{\simeq} H^1\big(K_n, A_g[\boldsymbol{p}]\big).$$

In light of (4.41) and (4.42), the commutativity of (4.40) with $L = K_n$ ensures that

$$\psi_n\big(d_1(f,n)\big) = \bar{e}^{-1} \cdot \psi_n\left(\varpi_{f,K_n}\left(\mathrm{sp}^{f}_{0,K_n}\big(d(\boldsymbol{f},n)\big)\right)\right) = \bar{e}^{-1}\bar{d} \cdot d_1(g,n), \tag{4.43}$$

with $\bar{e}^{-1}\bar{d} \in \bar{\mathbb{F}}_p^{\times}$. Now observe that, by [166, Theorem 1.1] (respectively, [152, Theorem 1.3]) if $N_g = N$ (respectively, $N_g = Np$), there is $n_0 \in \Lambda_{\mathrm{Kol}}(g) = \Lambda_{\mathrm{Kol}}(f)$ such that $c_1(g,n_0) \neq 0$. By (4.24), it follows that $d_1(g,n_0) \neq 0$, and then $d_1(f,n_0) \neq 0$ by (4.43). Finally, by (4.11) we conclude that $c_1(f,n_0) \neq 0$, so $\kappa^{\mathrm{st}}_{f,\infty} \neq \{0\}$. □

Remark 4.15 We expect that a similar deformation-theoretic strategy can be adopted to prove an analogue of Theorem 4.14 for newforms of finite slope at p, replacing the results due to Castella and Ota on specializations of big Heegner points in Hida families with those by Büyükboduk–Lei [25] and Jetchev–Loeffler–Zerbes [68] on the interpolation of Heegner points and Heegner cycles in Coleman families (*cf.* [124] for applications of the main results of [25] to the study of algebraic ranks, analytic ranks and Shafarevich–Tate groups when the modular forms they are attached to vary in a Coleman family).

Remark 4.16 In our recent article [90] with Pati, we give an alternative proof (under mild technical assumptions) of Conjecture 4.2 at ordinary primes. More precisely, our strategy in [90], which is inspired by the work of W. Zhang in weight 2, is based on a variant for modular forms of the congruence method originally introduced by Bertolini–Darmon to prove one divisibility in the anticyclotomic Iwasawa main conjecture for rational elliptic curves [9]. Namely, in [90] we adapt to higher (even) weight modular forms this approach via congruences, building crucially on results of Wang on the indivisibility of Heegner cycles over Shimura curves [161]. Since in [90] we need $p > k+1$, our main theorem and its applications in [90] can be viewed as complementary to results in the present paper, where $p < k$ because the congruence $k \equiv 2 \pmod{2(p-1)}$ is imposed.

5 The *p*-part of the TNC for $\mathcal{M}$

Our goal is to prove, under suitable assumptions on f and p, the p-part of the TNC for the motive $\mathcal{M}$ when the analytic rank of $\mathcal{M}$ is 1; this will be done in Sect. 5.9.4.

5.1 Heegner modules

We review the construction of Heegner cycles in [165] and compare it with that from [110]. Recall that we use the same symbol for an algebraic cycle and for its class in the corresponding Chow group. We point out that the assumption that the level N of f be square-free is not needed in Sects. 5.1, 5.2, pspsprimaryspsShaspssubsec. The freedom to work with newforms whose level is not necessarily square-free will be important in Sect. 9.1, when collecting some of the arithmetic consequences of our main results.

5.1.1 Zhang's cycles

Fix an imaginary quadratic field K in which all the primes dividing Np split. Let $x_n \in X_0(N)$ be a Heegner point of conductor n and let $\tilde{x}_n \in X(N)$ be the lift of x_n that was chosen in

Sect. 4.1. Recall the cycle $Z_k(\tilde{x}_n)$ from (4.1). To begin with, Zhang considers in [165, §2.4] the $k/2$-codimensional cycle

$$W_k(\tilde{x}_n) := \sum_{g \in S_{k-2}} \mathrm{sgn}(g) \cdot g^* \Big(Z_k(\tilde{x}_n)^{(k-2)/2} \Big) \in \mathrm{CH}^{k/2}(X/K_n) \tag{5.1}$$

(recall that $X = \tilde{\mathcal{E}}_N^{k-2}$), where $\mathrm{sgn}(g)$ is the sign of the permutation g. Recall that the geometric Heegner cycle $\tilde{\Gamma}_n$ defined in (4.2) belongs to $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)$.

Lemma 5.1 *The equalities*

(1) $\Pi_\epsilon \cdot \Big(Z_k(\tilde{x}_n)^{(k-2)/2} \Big) = \dfrac{W_k(\tilde{x}_n)}{(k-2)!}$,

(2) $\tilde{\Gamma}_n = \dfrac{\Pi_B \cdot W_k(\tilde{x}_n)}{(k-2)!}$

hold in $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)$.

Proof Part (2) is an immediate consequence of part (1) and the definitions, so we only need to check part (1). Recall from Sect. 2.2.2 that the restriction of ϵ to the subgroup $\mathfrak{S}_{k-2}$ of the group Γ_{k-2} in (2.3) is the sign character, so the projector associated with this restriction is $\Pi'_\epsilon := \frac{1}{(k-2)!} \sum_{g \in S_{k-2}} \mathrm{sgn}(g) g^*$. On the other hand, as in the proof of [165, Lemma 2.4.3], $\big((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}\big)^{k-2}$ acts on $Z_k(\tilde{x}_n)^{(k-2)/2}$ via the restriction of ϵ to $\big((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}\big)^{k-2}$. Now ϵ is trivial on $(\mathbb{Z}/N\mathbb{Z})^{2(k-2)}$ and the product map on $(\mathbb{Z}/2\mathbb{Z})^{k-2}$. It follows that the projector Π''_ϵ associated with the restriction of ϵ to $\big((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}\big)^{k-2}$ acts trivially on $W_k(\tilde{x}_n)$. Finally, $\Pi_\epsilon = \Pi'_\epsilon \cdot \Pi''_\epsilon$, so there are equalities

$$\Pi_\epsilon \cdot Z_k(\tilde{x}_n)^{(k-2)/2} = \Pi'_\epsilon \cdot Z_k(\tilde{x}_n)^{(k-2)/2} = \frac{W_k(\tilde{x}_n)}{(k-2)!},$$

as desired. □

5.1.2 Base change maps and trace maps

Let L_2/L_1 be a Galois extension of number fields and set $\mathscr{G} := \mathrm{Gal}(L_2/L_1)$. From here up to Sect. 5.1.3, let us adopt the convention that $\mathrm{CH}^n_\emptyset$ stands for CH^n. Let $\star \in \{\emptyset, 0, \mathrm{arith}\}$. Denote by

$$\iota_{L_1 \to L_2} : \mathrm{CH}^{k/2}_\star(X/L_1) \longrightarrow \mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}} \tag{5.2}$$

the base change map from L_1 to L_2, a special case of which was introduced in the proof of Proposition 4.3. Galois trace induces a map

$$\mathrm{tr}_{L_2/L_1} : \mathrm{CH}^{k/2}_\star(X/L_2) \longrightarrow \mathrm{CH}^{k/2}_\star(X/L_1); \tag{5.3}$$

the composition $\mathrm{tr}_{L_2/L_1} \circ \iota_{L_1 \to L_2}$ is equal to the multiplication-by-$[L_2 : L_1]$ map, so the kernel of $\iota_{L_1 \to L_2}$ is torsion, annihilated by $[L_2 : L_1]$. Extending scalars in (5.2) and (5.3) to any ring R in which $[L_2 : L_1]$ is invertible, we get maps

$$\mathrm{CH}^{k/2}_\star(X/L_1)_R \underset{\iota_{L_1 \to L_2}}{\overset{\mathrm{tr}_{L_2/L_1}}{\leftrightarrows}} \mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}}_R \tag{5.4}$$

such that $\mathrm{tr}_{L_2/L_1} \circ \iota_{L_1 \to L_2}$ and $\iota_{L_1 \to L_2} \circ \mathrm{tr}_{L_2/L_1}$ are isomorphisms. Therefore, the extension of scalars to such an R of the map in (5.2) is an isomorphism. In particular, when $R = F$ we obtain a map

$$\mathrm{CH}^{k/2}_\star(X/L_1) \longrightarrow \mathrm{CH}^{k/2}_\star(X/L_1)_F \xrightarrow[\simeq]{\iota_{L_1 \to L_2}} \mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}}_F \tag{5.5}$$

whose kernel is the torsion subgroup of $\mathrm{CH}^{k/2}_\star(X/L_1)$.

Notation/Convention 5.2 We identify a non-torsion element of $\mathrm{CH}^{k/2}_\star(X/L_1)$ with its image under the map in (5.5) and use the same symbol for both cycle classes. Conversely, we identify an element of $\mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}}_F$ with its image in $\mathrm{CH}^{k/2}_\star(X/L_1)_F$ under the trace map tr_{L_2/L_1} in (5.4). In particular, since the kernel of the obvious map

$$\mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}} \longrightarrow \mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}}_F$$

is the torsion subgroup of the left-hand side term, we shall not distinguish between a non-torsion element of $\mathrm{CH}^{k/2}_\star(X/L_2)^{\mathscr{G}}$ and its image in $\mathrm{CH}^{k/2}_\star(X/L_1)_F$ via tr_{L_2/L_1}. In a similar fashion, with our usual notation in force, a non-torsion subgroup of $\mathrm{CH}^{k/2}_\star(X/L_2)$ injects into $\mathrm{CH}^{k/2}_\star(X/L_2)_{\mathcal{O}_\mathfrak{p}}$, so that we shall freely identify it with its image in this $\mathcal{O}_\mathfrak{p}$-module.

Remark 5.3 The fact that $\ker(\iota_{L_1 \to L_2})$ is torsion is a special case of an analogous result for smooth projective varieties and arbitrary field extensions: see, *e.g.*, [14, Lemma (1A.3)], [128, p. 238] for details (the statement in [14] is given only for the Chow group CH^2 of a surface, but the arguments in the proof work for all groups CH^i of a variety).

5.1.3 The Heegner module of level N

As in Sect. 4.1.2, let $\pi_N : X(N) \to X_0(N)$ be the canonical degeneracy map; as in Sect. 2.2.2, put $t_N := \#\big(\Gamma_0(N)/\Gamma(N)\big)$, then write $\pi_N^*(x_n) = \sum_{i=1}^{t_N} \tilde{x}_{n,i}$. Set

$$W_k(x_n) := \Pi_B \cdot W_k(\tilde{x}_n) = \frac{1}{t_N} \cdot \sum_{i=1}^{t_N} W_k(\tilde{x}_{n,i}). \tag{5.6}$$

As in Sect. 4.3.1, let $\mathcal{G}_1 := \mathrm{Gal}(K_1/K)$. The *Heegner module* $\mathrm{Heeg}_{K,N}$ *of level* N is the subgroup of $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)$ generated by $T_m \cdot W_k(x_1^\sigma)$ for all $\sigma \in \mathcal{G}_1$ and all Hecke operators T_m with $(m, N) = 1$ (the cycles $T_m \cdot W_k(x_1^\sigma)$ do indeed lie in $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)$, *cf.* [165, §3.1]): we view $\mathrm{Heeg}_{K,N}$ as a subgroup of $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}_0(X/K_1)$, which in turn should be thought of as a subgroup of $\mathrm{CH}^{k/2}_0(X/K_1)_{\mathbb{Q}}$ (*cf.* Sect. 2.6.1).

Fix a prime number p such that $p \nmid 2N$. As was summarized in Sect. 5.1.2, if $\mathfrak{p}$ is a prime of F above p, then there are natural maps

$$\mathrm{Heeg}_{K,N}^{\mathcal{G}_1} \hookrightarrow \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathcal{G}_1} \longrightarrow \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathcal{G}_1}_{\mathcal{O}_\mathfrak{p}} \xrightarrow{\mathrm{tr}_{K_1/K}} \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_{\mathcal{O}_\mathfrak{p}}, \tag{5.7}$$

where the leftmost arrow is the set-theoretic inclusion and the middle one is extension of scalars. Thus, by composition with (5.7), the $\mathfrak{p}$-adic Abel–Jacobi map $\mathrm{AJ}_{K,\mathfrak{p}}$ yields a map

$$\mathrm{AJ}_{K,\mathfrak{p}} : \mathrm{Heeg}_{K,N}^{\mathcal{G}_1} \longrightarrow \Lambda_\mathfrak{p}(K), \tag{5.8}$$

which will be denoted by the same symbol (*cf.* Remark 3.16). Let us define the cycle

$$\mathscr{X}_K := \sum_{\sigma\in\mathcal{G}_1} W_k(x_1^\sigma) = \mathrm{tr}_{K_1/K}\big(W_k(x_1)\big) \in \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K), \tag{5.9}$$

where the second equality can be checked by unfolding the definition of the Galois action on Heegner cycles.

Remark 5.4 At some point in this article, we will need to consider also the counterpart of $\mathrm{AJ}_{K,\mathfrak{p}}$ with f replaced by the quadratic twist f^K. We will denote this map by $\mathrm{AJ}_{f^K,K',\mathfrak{p}}$, where K' is a suitable imaginary quadratic field. In this case, we will use the symbol $\mathcal{G}_1'$ for the analogue of the Galois group $\mathcal{G}_1$.

In the statement below, $y_{K,\mathfrak{p}} \in \Lambda_{\mathfrak{p}}(K)$ is the cycle introduced in (4.5).

Lemma 5.5 $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{X}_K) = (k-2)!\cdot y_{K,\mathfrak{p}}$.

Proof Recall the cycle $\Gamma_{1,\mathfrak{p}}$ in (4.4) and set

$$\Gamma_{K,\mathfrak{p}} := \mathrm{tr}_{K_1/K}(\Gamma_{1,\mathfrak{p}}) \in \Pi_B\Pi_\epsilon\cdot \mathrm{CH}^{k/2}_0(X/K)_{\mathcal{O}_{\mathfrak{p}}}.$$

By part (2) of Lemma 5.1, there is an equality $W_k(x_1) = (k-2)!\cdot\Gamma_{1,\mathfrak{p}}$, which implies that

$$\mathscr{X}_K = (k-2)!\cdot\Gamma_{K,\mathfrak{p}} \tag{5.10}$$

in $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_{\mathcal{O}_{\mathfrak{p}}}$. Combining equality (5.10), the fact that $y_{1,\mathfrak{p}} = \mathrm{AJ}_{K_1,\mathfrak{p}}(\Gamma_{1,\mathfrak{p}})$ and the commutativity of (4.6), we obtain

$$\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{X}_K) = (k-2)!\cdot\mathrm{cores}_{K_1/K}(y_{1,\mathfrak{p}}) = (k-2)!\cdot y_{K,\mathfrak{p}},$$

as claimed. □

5.2 Zhang's formula of Gross–Zagier type

We review the main result of [165], which is a counterpart of the Gross–Zagier formula ([55, Theorem 6.3]) for higher (even) weight modular forms.

5.2.1 *Zhang's cycles with coefficients in* $\mathbb{R}$

With $W_k(\tilde{x}_n)$ as in (5.1), let us consider the $k/2$-codimensional cycle with real coefficients

$$S_k(\tilde{x}_n) := c_n\cdot W_k(\tilde{x}_n) \tag{5.11}$$

on X, where $c_n\in\mathbb{R}$ is a positive constant such that the self-intersection of $S_k(\tilde{x}_n)$ on each fiber is equal to $(-1)^{(k-2)/2}$ (*cf.* [165, §2.4]).

We thank Congling Qiu for providing us with a proof of the next result, which will be crucial in our subsequent arguments (recall that we write D_K for the discriminant of K).

Lemma 5.6 *The self-intersection of* $W_k(\tilde{x}_n)$ *is* $(k/2-1)!^2\cdot(k-2)!\cdot(-2n^2D_K)^{k/2-1}$.

Proof The Heegner point $x_n \in X_0(N)$ corresponds to an elliptic curve E_n with complex multiplication by the order $\mathcal{O}_n$ of conductor n of K, equipped with a cyclic subgroup of order N; the elliptic curve E_n is defined over K_n. Now set $V := H^1_{\mathrm{dR}}(E/K_n) \otimes_{K_n} \mathbb{C}$, the tensor product being taken with respect to a fixed embedding $K_n \hookrightarrow \mathbb{C}$. Choose a basis $\{\omega, \eta\}$ of V such that $\langle \omega, \eta \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the intersection pairing. Consider the pairing $[\cdot, \cdot]$ in $V \otimes_{\mathbb{C}} V$ induced by $\langle \cdot, \cdot \rangle$, which has antidiagonal matrix $(1, -1, -1, 1)$ in the basis $\{\omega \otimes \omega, \omega \otimes \eta, \eta \otimes \omega, \eta \otimes \eta\}$. The self-intersection of $Z_k(\tilde{x}_n)$ is $-2n^2 D_K$ (*cf.* the proof of [110, Proposition 5.1]), so in $V \otimes_{\mathbb{C}} V$ we have

$$\big[Z_k(\tilde{x}_n), Z_k(\tilde{x}_n)\big] = -2n^2 D_K. \tag{5.12}$$

The induced symmetric pairing on $\mathrm{Sym}^{k-2}(V)$, denoted for simplicity by the same symbol, is given by the formula

$$\big[x_1 \otimes \cdots \otimes x_{k-2}, y_1 \otimes \cdots \otimes y_{k-2}\big] = \frac{1}{(k-2)!} \cdot \sum_{\sigma \in S_{k-2}} \prod_{i=1}^{k-2} \langle x_i, y_{\sigma(i)} \rangle. \tag{5.13}$$

Therefore, we may now compute

$$\begin{aligned} \big[W_k(\tilde{x}_n), W_k(\tilde{x}_n)\big] &= (k-2)!^2 \cdot \big[Z_k(\tilde{x}_n)^{k/2-1}, Z_k(\tilde{x}_n)^{k/2-1}\big] \\ &= (k-2)!^2 \cdot \big[Z_k(\tilde{x}_n), Z_k(\tilde{x}_n)\big]^{k/2-1} \cdot \frac{(k/2-1)!^2}{(k-2)!} \\ &= (k/2-1)!^2 \cdot (k-2)! \cdot (-2n^2 D_K)^{k/2-1}, \end{aligned} \tag{5.14}$$

where the first equality in (5.14) follows from Lemma 5.1, the second from (5.13) and the third from (5.12). □

It follows from Lemma 5.6 that

$$c_n = \frac{1}{(k/2-1)! \cdot \sqrt{(k-2)!} \cdot \big(\sqrt{-2D_K}\big)^{k/2-1}}. \tag{5.15}$$

With notation from Sect. 5.1, as in [165, §4.1] we define

$$S_k(x_n) := \frac{1}{\sqrt{\deg(\pi_N)}} \cdot \sum_{i=1}^{t_N} S_k(\tilde{x}_{n,i}) \in \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_n)_{\mathbb{R}}, \tag{5.16}$$

where the cycles $S_k(\tilde{x}_{n,i})$ are defined as in (5.11). As above, the fact that $S_k(x_n)$ belongs to $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_n)_{\mathbb{R}}$ follows from [165, §3.1].

Lemma 5.7 *The equality*

$$W_k(x_n) = \frac{S_k(x_n)}{c_n \cdot \sqrt{\deg(\pi_N)}}$$

holds in $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)_{\mathbb{R}}$.

Proof Let $\{\gamma_1, \ldots, \gamma_{t_N}\}$ be a set of representatives of $\Gamma_0(N)/\Gamma(N)$ and for all $i = 1, \ldots, t_N$ put $\tilde{x}_{n,i} := \gamma_i^*(x_n)$. By definition, there are equalities

$$W_k(x_n) = \Pi_B W_k(\tilde{x}_n) = \frac{1}{\deg(\pi_N)} \cdot \sum_{i=1}^{t_N} \gamma_i^* W_k(\tilde{x}_n) = \frac{1}{\deg(\pi_N)} \cdot \sum_{i=1}^{t_N} W_k(\tilde{x}_{n,i}).$$

Since each $W_k(\tilde{x}_{n,i})$ has self-intersection c_n, the result follows. □

Now we can prove

Proposition 5.8 *The equality*

$$S_k(\tilde{x}_n) = \sqrt{\frac{\deg(\pi_N) \cdot \binom{k-2}{k/2-1}}{(-2n^2 D_K)^{k/2-1}}} \cdot \tilde{\Gamma}_n$$

holds in $\Pi_B \Pi_\epsilon \cdot \mathrm{CH}^{k/2}(X/K_n)_{\mathbb{R}}$.

Proof Immediate by combining Lemmas 5.1 and 5.7 with the expression for c in (5.15). □

Let us consider the $\mathbb{R}$-linear extension

$$\langle \cdot, \cdot \rangle_{\mathrm{GS}} : \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)_{\mathbb{R}} \times \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

of the Gillet–Soulé height pairing from Sect. 2.7. Moreover, let $S_k(x_n)$ be the cycle from (5.16). Following [165, §0.1], define V to be the $\mathbb{R}$-subspace of $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)_{\mathbb{R}}$ that is generated by $T_m \cdot S_k(x_1^\sigma)$ for all $\sigma \in \mathcal{G}_1$ and all Hecke operators T_m with $(m, N) = 1$. It follows that $V = \mathrm{Heeg}_{K,N} \otimes_{\mathbb{Z}} \mathbb{R}$ and $S_k(\tilde{x}_1) \in V$. Let V' be the quotient of V by the null subspace for $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$. Then V' is a subquotient of the direct sum $S_k(\Gamma_0(N))^{h_K}$, where $h_K := \#\mathcal{G}_1$ is the class number of K ([165, Theorem 0.3.1]). Clearly, $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$ yields a height pairing, to be denoted in the same way, on V'. Choose an orthonormal basis $\{f = f_1, \ldots, f_t\}$ of V' with respect to the Petersson inner product $(\cdot, \cdot)_{\Gamma_0(N)}$, so that V' splits into f_j-eigencomponents V'_{f_j}. Now define $s'_{k,f}(x_1^\sigma)$ to be the image of $S_k(x_1^\sigma)$ in V'_f (where, as above, $\sigma \in \mathcal{G}_1$) and put

$$s'_f := \sum_{\sigma \in \mathcal{G}_1} s'_{k,f}(x_1^\sigma) \in V'_f,$$

so that s'_f is the image of $\sum_{\sigma \in \mathcal{G}_1} S_k(x_1^\sigma)$ in V'_f. Notice that, in fact, $s'_f \in (V'_f)^{\mathcal{G}_1}$.

Remark 5.9 If $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$ is non-degenerate on V, then $V' = V$ and

$$s'_f \in \left(\mathrm{Heeg}_{K,N}^{\mathcal{G}_1} \otimes_{\mathbb{Z}} \mathbb{R}\right)[\theta_f] \subset \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_{\mathbb{R}}[\theta_f].$$

This fact will play a role in the proof of Proposition 5.11.

5.2.2 Zhang's formula

In the statement below, $u_K := \#\mathcal{O}_K^\times/2$. There is a splitting

$$L(f/K,s) = L(f,s)\cdot L(f^K,s), \tag{5.17}$$

which implies that $r_{\mathrm{an}}(f/K) \geq 1$ (see, *e.g.*, [20, p. 543]).

Theorem 5.10 (S.-W. Zhang) $L'(f/K,k/2) = \dfrac{2^{2k-1}\cdot\pi^k\cdot(f,f)_{\Gamma_0(N)}}{(k-2)!\cdot u_K^2\cdot\sqrt{|D_K|}}\cdot\langle s_f', s_f'\rangle_{\mathrm{GS}}$.

Proof With notation as in [165], this follows immediately from [165, Corollary 0.3.2] upon taking χ to be the trivial character. □

Recall the map $\mathrm{AJ}_{K,\mathfrak{p}}$ from (5.8); by a slight abuse of terminology, it will be interpreted as the restriction to $\mathrm{Heeg}_{K,N}^{\mathcal{G}_1}$ of the map in (3.12). The following result is implicit in [165]; since we shall use it later, we provide a complete proof of it.

Proposition 5.11 (1) *Assume that* $\mathrm{AJ}_{K,\mathfrak{p}}$ *is injective on* $\mathrm{Heeg}_{K,N}^{\mathcal{G}_1}$. *If* $r_{\mathrm{an}}(f/K) = 1$, *then* $y_{K,\mathfrak{p}}$ *is not* $\mathcal{O}_{\mathfrak{p}}$*-torsion.*

(2) *Assume that* $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ *is non-degenerate on* $\mathrm{Heeg}_{K,N}\otimes_{\mathbb{Z}}\mathbb{R}$. *If* $r_{\mathrm{an}}(f/K) > 1$, *then* $y_{K,\mathfrak{p}}$ *is* $\mathcal{O}_{\mathfrak{p}}$*-torsion.*

Proof For simplicity, set $\mathrm{CH} := \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)$. We first show (1). By Theorem 5.10, s_f' is non-zero because $r_{\mathrm{an}}(f/K) = 1$, and so $\sum_{\sigma\in\mathcal{G}_1} S_n(x_1^\sigma)$ is non-zero as well. Thanks to Proposition 5.8, $\sum_{\sigma\in\mathcal{G}_1}\tilde{\Gamma}_1^\sigma$ is non-zero in $\Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\mathbb{R})$, hence in $\Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}R)$. Let $\mathscr{X}_K$ be as in (5.9). Part (2) of Lemma 5.1 implies that $\mathscr{X}_K = (k-2)!\sum_{\sigma\in\mathcal{G}_1}\tilde{\Gamma}_1^\sigma$, so $\mathscr{X}_K$ is non-torsion in $\mathrm{Heeg}_{K,N}^{\mathcal{G}_1}$. Finally, part (1) follows from Lemma 5.5 and the injectivity of $\mathrm{AJ}_{K,\mathfrak{p}}$ on $\mathrm{Heeg}_{K,N}^{\mathcal{G}_1}$, which we are assuming.

Now we prove (2), which is more delicate. In light of Lemma 5.5, we need equivalently to show that $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{X}_K)$ is $\mathcal{O}_{\mathfrak{p}}$-torsion in $\Lambda_{\mathfrak{p}}(K)$. If $\mathcal{F}\in\{\mathbb{R},\mathbb{R}\cap\bar{\mathbb{Q}},\bar{\mathbb{Q}}_p\}$, then there is a decomposition

$$\Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\mathcal{F}) = \bigoplus_g \Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\mathcal{F})[\theta_g], \tag{5.18}$$

where g varies over all normalized newforms in $S_k(\Gamma_0(N))$ and, with notation as in (2.4), $\Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\mathcal{F})[\theta_g]$ denotes the g-isotypic submodule of $\Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\mathcal{F})$; see, *e.g.*, [111, pp. 656–657] for details (strictly speaking, [111] deals solely with $\mathcal{F}=\bar{\mathbb{Q}}_p$, but what one only needs is that the field $\mathcal{F}$ contains all Hecke eigenvalues of all normalized newforms; when $\mathcal{F}=\bar{\mathbb{Q}}_p$, we are implicitly using an embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$, which allows us to view complex algebraic numbers as elements of $\bar{\mathbb{Q}}_p$). It is straightforward to check that

$$\Pi_B\Pi_\epsilon\cdot\mathcal{CH} = (\Pi_B\Pi_\epsilon\cdot\mathcal{CH})\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathbb{Q}}_p \tag{5.19}$$

with either $\mathcal{CH} = \mathrm{CH}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}}_p$ or $\mathcal{CH} = (\mathrm{CH}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}}_p)[\theta_g]$ for some g as above. Then, using the fact that the Abel–Jacobi map $\mathrm{AJ}_{K,\mathfrak{p}}$ is Hecke-equivariant and $\Lambda_{\mathfrak{p}}(K)$ is f-isotypic, it follows that $\mathrm{AJ}_{K,\mathfrak{p}}\otimes\mathrm{id}_{\bar{\mathbb{Q}}_p}$ factors through the f-isotypic component as

$$\mathrm{AJ}_{K,\mathfrak{p}}\otimes\mathrm{id}_{\bar{\mathbb{Q}}_p} : \Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}}_p) \twoheadrightarrow \Pi_B\Pi_\epsilon\cdot(\mathrm{CH}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}}_p)[\theta_f] \longrightarrow \Lambda_{\mathfrak{p}}(K)\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathbb{Q}}_p,$$

where the first map is the projection induced by (5.18) with $\mathcal{F} = \bar{\mathbb{Q}}_p$. There is a commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\mathrm{AJ}_{K,\mathfrak{p}} \otimes \mathrm{id}_{\bar{\mathbb{Q}}_p}}{\curvearrowright} & & \\
\Pi_B\Pi_\epsilon \cdot \left(\mathrm{CH} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_p\right) & \xrightarrow{\pi_{1,f}} & \Pi_B\Pi_\epsilon \cdot \left(\mathrm{CH} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_p\right)[\theta_f] & \longrightarrow & \Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} \bar{\mathbb{Q}}_p \\
\uparrow{\scriptstyle \iota_1} & & \uparrow & & \\
\Pi_B\Pi_\epsilon \cdot \left(\mathrm{CH} \otimes_{\mathbb{Z}} (\mathbb{R} \cap \bar{\mathbb{Q}})\right) & \xrightarrow{\pi_{2,f}} & \Pi_B\Pi_\epsilon \cdot \left(\mathrm{CH} \otimes_{\mathbb{Z}} (\mathbb{R} \cap \bar{\mathbb{Q}})\right)[\theta_f] & & \\
\downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & \\
\Pi_B\Pi_\epsilon \cdot (\mathrm{CH} \otimes_{\mathbb{Z}} \mathbb{R}) & \xrightarrow{\pi_{3,f}} & \Pi_B\Pi_\epsilon \cdot (\mathrm{CH} \otimes_{\mathbb{Z}} \mathbb{R})[\theta_f] & &
\end{array} \tag{5.20}$$

in which the horizontal surjections are the projections induced by (5.18) and the vertical injections are given by extension of scalars. Adopting the same symbol for $\mathscr{Z}_K$ and for its natural image in $\Pi_B\Pi_\epsilon \cdot \left(\mathrm{CH} \otimes_{\mathbb{Z}} (\mathbb{R} \cap \bar{\mathbb{Q}})\right)$, we want to show that

$$\left((\mathrm{AJ}_{K,\mathfrak{p}} \otimes \mathrm{id}_{\bar{\mathbb{Q}}_p}) \circ \iota_1\right)(\mathscr{Z}_K) = 0. \tag{5.21}$$

Since $r_{\mathrm{an}}(f/K) > 1$, Theorem 5.10 and the non-degeneracy of $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$ imply that $s'_f = 0$ in $\Pi_B\Pi_\epsilon \cdot (\mathrm{CH} \otimes_{\mathbb{Z}} \mathbb{R})[\theta_f]$ (*cf.* Remark 5.9). To lighten our notation, set $c := c_1$. Comparing (5.6), (5.9) and (5.11) gives the equality

$$\iota_2(\mathscr{Z}_K) = \frac{\Pi_B}{c} \cdot \sum_{\sigma \in \mathcal{G}_1} S_k(x_1^\sigma)$$

in $\Pi_B\Pi_\epsilon \cdot (\mathrm{CH} \otimes_{\mathbb{Z}} \mathbb{R})$, and then $\pi_{3,f}\left(\iota_2(\mathscr{Z}_K)\right) = (\Pi_B/c) \cdot s'_f = 0$. Now the commutativity of the lower square in (5.20) and the injectivity of ι_3 yield $\pi_{2,f}(\mathscr{Z}_K) = 0$, while the commutativity of the upper square in (5.20) shows that $\pi_{1,f}\left(\iota_1(\mathscr{Z}_K)\right) = 0$. This clearly implies (5.21).

Keeping equality (5.19) in mind (with $\mathcal{CH} = \mathrm{CH} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_p$), the vanishing in (5.21) shows that the image of $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{Z}_K) \in \Lambda_{\mathfrak{p}}(K)$ in $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} \bar{\mathbb{Q}}_p$ is trivial. Finally, the natural map $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \to \Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} \bar{\mathbb{Q}}_p$ is injective, so the image of $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{Z}_K)$ in $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}$ is trivial. This means that $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{Z}_K)$ is $\mathcal{O}_{\mathfrak{p}}$-torsion in $\Lambda_{\mathfrak{p}}(K)$, as desired. □

Remark 5.12 Unfortunately, while it is natural to impose a non-degeneracy condition like that in part (2) of Proposition 5.11 when studying the arithmetic of Heegner cycles (see, *e.g.*, [164, Assumption 4.1]), we are not aware of any result in this direction (except for weight 2 modular forms, which are not considered in this paper).

5.3 Periods of modular forms

We begin by connecting the periods $\Omega_\infty^{(\gamma)}$ from Sect. 2.5 to those appearing in the work of Vatsal [156] and of Skinner–Urban [151]. To do this, we clarify the relation between these periods and modular symbols. In what follows, we set

$$\epsilon := (-1)^{\frac{k-2}{2}}. \tag{5.22}$$

5.3.1 Modular symbols

Let R be a commutative ring, let $k \geq 2$ be an integer, set $n := k-2$ and let $L_n(R) := \mathrm{Sym}^n(R)$ be the R-module of homogeneous polynomials of degree n in the variables X ad Y with coefficients in R. We write $L_n(R)$ also for the corresponding locally constant sheaf on the open modular curve Y_Γ of level Γ, where $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$.

Let

$$\mathrm{Symb}_\Gamma\big(L_n(R)\big) := \mathrm{Hom}_\Gamma\big(\mathcal{D}_0, L_n(R)\big)$$

be the group of Γ-invariant $L_n(R)$-valued *modular symbols*, where $\mathcal{D}_0$ is the group of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$ equipped with its left action of $R[\Sigma]$ ([52, Definition 4.6]). We also let

$$\mathrm{Bound}_\Gamma\big(L_n(R)\big) := \mathrm{Hom}_\Gamma\big(\mathcal{D}, L_n(R)\big)$$

be the group of Γ-invariant $L_n(R)$-valued *boundary symbols*, where $\mathcal{D}$ is the group of divisors on $\mathbb{P}^1(\mathbb{Q})$ equipped with the natural left action of $R[\Sigma]$. There is an exact sequence

$$0 \longrightarrow H^0\Big(\Gamma, L_n(R)\Big) \longrightarrow \mathrm{Bound}_\Gamma\Big(L_n(R)\Big) \longrightarrow \mathrm{Symb}_\Gamma\Big(L_n(R)\Big) \longrightarrow H^1_{\mathrm{par}}\Big(\Gamma, L_n(R)\Big) \longrightarrow 0.$$

As before, $H^1_{\mathrm{par}}\big(\Gamma, L_n(R)\big)$ denotes parabolic cohomology of the open modular curve $Y(\Gamma)$ of level Γ with coefficients in the locally constant sheaf associated with $L_n(R)$ (*i.e.*, the image of the compact cohomology group $H^1_{\mathrm{cpt}}\big(\Gamma, L_n(R)\big) = H^1_{\mathrm{cpt}}\big(Y(\Gamma), L_n(R)\big)$ in $H^1\big(\Gamma, L_n(R)\big) = H^1\big(Y(\Gamma), L_n(R)\big)$). Using notation from [52, §4], the map $\mathrm{Symb}_\Gamma\big(L_n(R)\big) \to H^1_{\mathrm{par}}\big(\Gamma, L_n(R)\big)$ sends Φ to the cohomology class represented by the 1-cocycle $\gamma \mapsto \Phi\big(\{\gamma(c)\} - \{c\}\big)$.

Write $\mathfrak{H}_k(\Gamma)$ for the Hecke algebra acting on modular forms of level Γ and weight k (in particular, for $\Gamma = \Gamma(N)$ we recover the algebra $\mathfrak{H}_k(\Gamma(N))$ from Sect. 2.2). By [52, Theorem 4.2], there is an $\mathfrak{H}_k(\Gamma)$-equivariant isomorphism

$$\mathrm{Symb}_\Gamma\big(L_n(R)\big) \simeq H^1_{\mathrm{cpt}}\big(\Gamma, L_n(R)\big) \tag{5.23}$$

such that the action of complex conjugation on $H^1_{\mathrm{cpt}}\big(\Gamma, L_n(R)\big)$ corresponds to the action of the matrix $\iota := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathrm{Symb}_\Gamma\big(L_n(R)\big)$. The map

$$\Theta : H^1\big(\Gamma, L_n(R)\big) \longrightarrow \mathrm{Symb}_\Gamma\big(L_n(R)\big) \tag{5.24}$$

defined by $\omega \mapsto \eta_q \cdot \Phi$, where Φ is any lift of ω, T_q is the Hecke operator at a prime number $q \equiv 1 \pmod{Np}$ and

$$\eta_q := T_q - (q+1), \tag{5.25}$$

is independent of the choice of Φ because η_q kills $\mathrm{Bound}_\Gamma\big(L_n(R)\big)$, *i.e.*, $\eta_q \cdot x = 0$ for all $x \in \mathrm{Bound}_\Gamma\big(L_n(R)\big)$ ([156, §1.6]). Furthermore, the map Θ is $\mathfrak{H}_k(\Gamma)$-equivariant and also equivariant for the action of complex conjugation on $H^1\big(\Gamma, L_n(R)\big)$ and for the action of ι on $\mathrm{Symb}_\Gamma\big(L_n(R)\big)$. We also write

$$\Theta : H^1\big(\Gamma, L_n(R)\big) \longrightarrow H^1_{\mathrm{cpt}}\big(\Gamma, L_n(R)\big) \tag{5.26}$$

for the composition of (5.23) and (5.24).

If $R = \mathbb{C}$, then the modular symbol $\Phi_{f,\Gamma}$ associated with f is given by

$$\Phi_{f,\Gamma}\big(\{a\} - \{b\}\big) := 2\pi i \cdot \int_b^a f(z) \cdot (zX + Y)^{k/2-1} dz,$$

where the integral is computed along a geodesic path (with respect to the Poincaré metric) from b to a in the complex upper half-plane. Split the $\mathbb{C}$-vector space $\mathrm{Symb}_\Gamma\big(L_n(\mathbb{C})\big)$ into $\pm$-eigenspaces $\mathrm{Symb}_\Gamma^\pm\big(L_n(\mathbb{C})\big)$ for complex conjugation, then denote by $\Phi_{f,\Gamma}^\pm$ the projections of $\Phi_{f,\Gamma}$ to the respective eigenspaces. Observe, in particular, that there is an isomorphism

$$V_{\mathrm{B}}(-k/2) \otimes_F \mathbb{C} \simeq H^1\big(\Gamma, L_n(\mathbb{C})\big)[\theta_f] = \mathbb{C} \cdot \Phi_{f,\Gamma}^+ \oplus \mathbb{C} \cdot \Phi_{f,\Gamma}^- \tag{5.27}$$

satisfying the following property: an element of $V_{\mathrm{B}}(-k/2) \otimes_F \mathbb{C}$ lies in the $+$-eigenspace for complex conjugation if and only if its image under (5.27) lies in $\mathbb{C} \cdot \Phi_{f,\Gamma}^\epsilon$, with $\epsilon \in \{\pm 1\}$ as in (5.22). The comparison isomorphism between Betti and de Rham realizations gives rise to an isomorphism

$$\mathrm{Comp}_{\mathrm{B,dR}} : \mathrm{Symb}_\Gamma\big(L_n(\mathbb{C})\big) \xrightarrow{\simeq} V_{\mathrm{dR}} \otimes_F \mathbb{C}$$

that is equivariant with respect to the action of complex conjugation. Define

$$\varphi_{f,\Gamma} := \mathrm{Comp}_{\mathrm{B,dR}}(\Phi_{f,\Gamma}). \tag{5.28}$$

Note that the generator ω_f of $\mathrm{Fil}^{k/2-1}(V_{\mathrm{dR}})$ from (2.14) is sent to $(2\pi i)^{-1} \cdot \varphi_{f,\Gamma_0(N)}$ by the comparison isomorphism.

5.3.2 The periods $\Omega_{f,\Gamma}^\pm$

We further elaborate on the definition of periods given in Sect. 2.5; we find it more convenient to work integrally throughout. For this, define

$$\mathbb{T}_{\mathrm{par},\Gamma} := \mathrm{im}\Big(H^1\big(\Gamma, L_n(\mathcal{O}_F)\big) \xrightarrow{j} H^1\big(\Gamma, L_n(F)\big)\Big)$$

where j is the natural map. With the usual $\pm$-notation, pick $\delta_{f,\Gamma}^\pm \in \mathbb{T}_{\mathrm{par},\Gamma}^\pm[\theta_f] \smallsetminus \{0\}$ and set

$$\eta_{f,\Gamma}^\pm := (\mathrm{Comp}_{\mathrm{B,dR}} \circ \Theta)\big(\delta_{f,\Gamma}^\pm\big);$$

here Θ is the map in (5.24) for $R = F$, which we view Θ as taking values in $\mathrm{Symb}_\Gamma\big(L_n(\mathbb{C})\big)$ by means of the distinguished embedding $\iota_F : F \hookrightarrow \mathbb{R}$. Define periods $\Omega_{f,\Gamma}^\pm \in \mathbb{C}$ via the equality

$$\varphi_{f,\Gamma} = \Omega_{f,\Gamma}^+ \cdot \eta_{f,\Gamma}^+ + \Omega_{f,\Gamma}^- \cdot \eta_{f,\Gamma}^-,$$

where $\varphi_{f,\Gamma}$ was introduced in (5.28). Consider the module index

$$\mathfrak{a}_{f,\Gamma}^\pm := \big[\mathbb{T}_{\mathrm{par},\Gamma}^\pm : \delta_{f,\Gamma}^\pm \cdot \mathcal{O}_F\big]$$

as defined, *e.g.*, in [45, p. 10]; by [45, §3, Proposition 1, (ii)], $\mathfrak{a}_{f,\Gamma}^\pm$ is a non-zero (integral) ideal of $\mathcal{O}_F$.

Remark 5.13 The periods $\Omega_{f,\Gamma}^\pm$ depend on our choice of $\delta_{f,\Gamma}^\pm$, but the products $\Omega_{f,\Gamma}^\pm \cdot \mathfrak{a}_{f,\Gamma}^\pm$, where we see $\mathfrak{a}_{f,\Gamma}^\pm$ as an $\mathcal{O}_F$-submodule of $\mathbb{R}$ via the (set-theoretic) inclusion ι_F, do not.

5.3.3 Comparison of periods

Now we relate the period $\Omega_\infty^{(\gamma_f)}$ to the periods introduced in Sect. 5.3.2. For any $\sigma \in \Sigma$, replacing f with f^σ and ι_F with σ in Sect. 5.3.2, we obtain periods $\Omega^{\pm}_{f^\sigma,\Gamma}$ and $\mathcal{O}_F$-submodules $\mathfrak{a}^{\pm}_{f^\sigma,\Gamma}$ of $\mathbb{R}$ ($\mathfrak{a}^{\pm}_{f^\sigma,\Gamma}$ is endowed with an $\mathcal{O}_F$-module structure via σ). Briefly, choose $\delta^{\pm}_{f,\Gamma}$ as in Sect. 5.3.2, fix $\sigma \in \Sigma$ and define $\eta^{\pm}_{f^\sigma,\Gamma} := (\mathrm{Comp}_{\mathrm{B,dR}} \circ \Theta_\sigma)(\delta^{\pm}_{f,\Gamma})$, where Θ_σ is the map obtained by composing Θ with σ. Define periods $\Omega^{\pm}_{f^\sigma,\Gamma} \in \mathbb{C}^\times$ via the equality $\varphi_{f^\sigma,\Gamma} = \Omega^{+}_{f^\sigma,\Gamma}\eta^{+}_{f^\sigma,\Gamma} + \Omega^{-}_{f^\sigma,\Gamma}\eta^{-}_{f^\sigma,\Gamma}$. With a slight abuse of notation, for $\epsilon = (-1)^{k/2-1}$ as in (5.22), in the following lines we do not distinguish between ϵ and its sign; set

- $\delta_{f,\Gamma} := \delta^{\epsilon}_{f,\Gamma}$;
- $\Omega_{f^\sigma,\Gamma} := \Omega^{\epsilon}_{f^\sigma,\Gamma}$ for each $\sigma \in \Sigma$;
- $\mathfrak{a}_{f,\Gamma} := \mathfrak{a}^{\epsilon}_{f,\Gamma}$.

The proposition below provides the comparison result we need.

Proposition 5.14 *Set* $\gamma_f := (2\pi i)^{k/2} \cdot (\Pi_B \Pi_\epsilon) \cdot \delta_{f,\Gamma_0(N)} \in T^{+}_{\mathrm{B}} \smallsetminus \{0\}$. *The equality*

$$\Omega_\infty^{(\gamma_f)} = C \cdot \left((2\pi i)^{\frac{k-2}{2}} \cdot \Omega_{f^\sigma,\Gamma_0(N)}\right)_{\sigma\in\Sigma} \tag{5.29}$$

holds in $F_\infty^\times$ *for some* $C \in F^\times$ *satisfying* $\mathrm{ord}_\lambda(C) = 0$ *for all primes* λ *of* F *such that* $\lambda \nmid N \cdot \mathfrak{a}_{f,\Gamma_0(N)}$.

Proof We use the argument in [38, Lemma 4.1], which follows a duality argument from [34, §1.7]. Fix $\sigma \in \Sigma$ and denote by $\Omega^{(\gamma_f)}_{\infty,\sigma}$ the determinant of the comparison isomorphism

$$V^{\epsilon}_{\mathrm{B}} \otimes_{F,\sigma} \mathbb{C} \xrightarrow{\simeq} \left(V_{\mathrm{dR}}/\operatorname{Fil}^0(V_{\mathrm{dR}})\right) \otimes_{F,\sigma} \mathbb{C}$$

computed with respect to the lattice $\mathcal{O}_F \cdot \gamma_f$ and the image of T_{dR}, which is generated by ω_{f^σ}; thus, by definition, $\Omega_\infty^{(\gamma_f)} = \left(\Omega^{(\gamma_f)}_{\infty,\sigma}\right)_{\sigma\in\Sigma}$. In the notation of [38], we have $\Omega^{(\gamma_f)}_{\infty,\sigma} = \mathrm{vol}_\infty$ (observe that in [38] the term vol_∞ is the determinant of the isomorphism

$$V^{\epsilon}_{\mathrm{B}}(-k/2) \otimes_{F,\sigma} \mathbb{C} \xrightarrow{\simeq} \left(V_{\mathrm{dR}}/\operatorname{Fil}^0(V_{\mathrm{dR}})\right) \otimes_{F,\sigma} \mathbb{C}$$

computed with respect to the lattice $\mathcal{O}_F \cdot \delta_{f,\Gamma_0(N)}$ and the image of T_{dR} multiplied by $(2\pi i)^{k/2}$, and the last factor $(2\pi i)^{k/2}$ is taken into account directly by the twist isomorphism between $V^{\pm}_{\mathrm{B}}$ and $V^{\pm}_{\mathrm{B}}(-k/2)$). We view $\mathfrak{a}_{f^\sigma,\Gamma_0(N)} = \sigma(\mathfrak{a}_{f,\Gamma_0(N)})$ as an $\mathcal{O}_F$-submodule of $\mathbb{R}$ via σ. By [38, Lemma 4.1], there is an equality of sets

$$\Omega^{(\gamma_f)}_{\infty,\sigma} \cdot \mathfrak{a}^{-1}_{f^\sigma,\Gamma_0(N)} = \sigma(C_0) \cdot (2\pi i)^{k/2-1} \cdot \Omega_{f^\sigma,\Gamma_0(N)} \cdot \mathfrak{a}_{f^\sigma,\Gamma_0(N)}$$

for a suitable $C_0 \in F^\times$, independent of σ, with $\mathrm{ord}_\lambda(C_0) = 0$ for all primes λ of F such that $\lambda \nmid N$ (note that the term $\Omega_\pm$ in [38] is defined by comparison with $(2\pi i)^{-1}\varphi^{\pm}_{f^\sigma}$, which explains the power of $2\pi i$ appearing in (5.29)). Therefore, we get an equality

$$\Omega^{(\gamma_f)}_{\infty,\sigma} = \sigma(C) \cdot (2\pi i)^{k/2-1} \cdot \Omega_{f^\sigma,\Gamma_0(N)}$$

for some $C \in F^\times$ with the property that $\mathrm{ord}_\lambda(C) = 0$ for all primes λ of F such that $\lambda \nmid N \cdot \mathfrak{a}_{f,\Gamma_0(N)}$, as desired. □

5.3.4 Special values and their algebraic parts

For each $\sigma \in \Sigma$, set $\Omega_{f^\sigma} := \Omega^{\epsilon}_{f^\sigma,\Gamma_0(N)}$ and $\Omega_f := \Omega_{f^{\iota_F}}$ (as before, ϵ is as in (5.22)). The special value of $L(f,s)$ at $s = k/2$ is described (see, *e.g.*, [102, Ch. I, §7]; *cf.* also [38, §5]) by the formula

$$L(f,k/2) = \frac{(2\pi)^{k/2}}{(k/2-1)!} \cdot \int_0^\infty f(it) t^{\frac{k-2}{2}} dt. \tag{5.30}$$

Let us define the *algebraic part* of $L(f,k/2)$ as

$$L^{\mathrm{alg}}(f,k/2) := \frac{L(f,k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_f}; \tag{5.31}$$

it is well known that $L^{\mathrm{alg}}(f,k/2)$ belongs to F. The period Ω_f depends on $\delta^{\epsilon}_{f,\Gamma_0(N)}$, so the same is true of $L^{\mathrm{alg}}(f,k/2)$.

5.3.5 Algebraic parts of special values and real embeddings

The algebraic part $L^{\mathrm{alg}}(f,k/2)$ in (5.31) belongs to F, so it makes sense to consider $\sigma\left(L^{\mathrm{alg}}(f,k/2)\right)$ for $\sigma \in \Sigma$.

Proposition 5.15 *For each $\sigma \in \Sigma$, there is an equality*

$$\sigma\left(L^{\mathrm{alg}}(f,k/2)\right) = L^{\mathrm{alg}}(f^\sigma,k/2).$$

Proof For $P \in L_n(R)$ we write $P(X,Y) = \sum_{j=0}^{k-2} r_j(P) X^j Y^{k-2-j}$. From (5.30) we get

$$L^{\mathrm{alg}}(f^\sigma,k/2) = \frac{(-i)^{k/2}}{(2\pi i)\cdot(k/2-1)!\cdot\Omega_{f^\sigma}} \cdot \int_0^\infty f^\sigma(it) t^{\frac{k-2}{2}} dt.$$

Since f^σ has real Fourier coefficients, the integral belongs to the ϵ-eigenspace for complex conjugation. Thus, taking the definition of modular symbols into account, we obtain an equality

$$L^{\mathrm{alg}}(f^\sigma,k/2) = \frac{r_{\frac{k-2}{2}}\left(\varphi_{f^\sigma}\left(\{i\infty\}-\{0\}\right)\right)}{(k/2-1)!\cdot\Omega_{f^\sigma}}.$$

Let η_q be the Hecke element introduced in (5.25). Since $\sigma\left(\theta_f(\eta_q)\right) = \theta_{f^\sigma}(\eta_q)$ by definition, it suffices to show that $\sigma\left(\Theta(\delta_f)\right) = \Theta(\delta_{f^\sigma})$. Thus, we are reduced to showing that the square

$$\begin{array}{ccc} H^1_{\mathrm{cpt}}\left(\Gamma_0(N), L_n(F)\right) & \xleftarrow{\Theta} & H^1_{\mathrm{B}}\left(\Gamma_0(N), L_n(F)\right) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ H^1_{\mathrm{cpt}}\left(\Gamma_0(N), L_n(\mathbb{C})\right) & \xleftarrow{\Theta} & H^1_{\mathrm{B}}\left(\Gamma_0(N), L_n(\mathbb{C})\right) \end{array}$$

is commutative (here Θ stands for the map in (5.26)). This is an immediate consequence of the definition of Θ and the fact that σ commutes with Hecke operators. □

5.3.6 A comparison of periods

Now we make our choice of $\delta_{f,\Gamma}^{\pm}$ more precise. Specifically, we choose elements $\delta_{f,\Gamma}^{\pm} \in T_{\mathrm{B},\Gamma}^{\pm} \smallsetminus \{0\}$ such that their natural images in $T_{\mathrm{B},\Gamma}^{\pm} \otimes_{\mathbb{Z}} \mathcal{O}_p$ generate this free $\mathcal{O}_p$-module of rank 1. At the cost of discarding finitely many primes p, one can proceed as follows. Take $\delta_{f,\Gamma}^{\pm} \in T_{\mathrm{B},\Gamma}^{\pm} \smallsetminus \{0\}$ and recall the module index $\mathfrak{a}_{f,\Gamma}^{\pm}$ from Sect. 5.3.2, which is an ideal of $\mathcal{O}_F$ defined in terms of $\delta_{f,\Gamma}^{\pm}$. Let $\mathrm{N}\big(\mathfrak{a}_{f,\Gamma}^{\pm}\big) := \#\big(\mathcal{O}_F/\mathfrak{a}_{f,\Gamma}^{\pm}\big)$ be the norm of $\mathfrak{a}_{f,\Gamma}^{\pm}$. Basic properties of the module index (see, *e.g.*, [45, §3]) allow one to check that if $p \nmid \mathrm{N}\big(\mathfrak{a}_{f,\Gamma}^{\pm}\big)$, then the image of $\delta_{f,\Gamma}^{\pm}$ in $T_{\mathrm{B},\Gamma}^{\pm} \otimes_{\mathbb{Z}} \mathcal{O}_p$ generates $T_{\mathrm{B},\Gamma}^{\pm} \otimes_{\mathbb{Z}} \mathcal{O}_p$.

Thus, we assume that $p \nmid \mathrm{N}\big(\mathfrak{a}_{f,\Gamma_0(N)}^{\pm}\big)$; we want to compare the periods $\Omega_{f,\Gamma_1(N)}$ and $\Omega_{f,\Gamma_0(N)}$. Before doing this, we need to fix some more notation: for each prime $\mathfrak{p}$ of F above p, denote by $\mathcal{O}_{(\mathfrak{p})}$ the localization of $\mathcal{O}_F$ at $\mathfrak{p}$, then set $\mathcal{O}_{(p)} := \prod_{\mathfrak{p}|p} \mathcal{O}_{(\mathfrak{p})}$.

Proposition 5.16 *The periods $\Omega_{f,\Gamma_1(N)}$ and $\Omega_{f,\Gamma_0(N)}$ differ by multiplication by elements of $\mathcal{O}_{(p)}^{\times}$.*

Proof Recall that the $\mathbb{C}$-vector space $S_k(\Gamma_0(N))$ is isomorphic to the $\mathfrak{H}_k(\Gamma_1(N))$-submodule of $S_k(\Gamma_1(N))$ consisting of those forms on which $\Gamma_0(N)/\Gamma_1(N)$ acts via the trivial character. Therefore, there is a canonical map $\mathfrak{H}_k(\Gamma_1(N)) \to \mathfrak{H}_k(\Gamma_0(N))$, so any $\mathfrak{H}_k(\Gamma_0(N))$-module is also equipped with a structure of $\mathfrak{H}_k(\Gamma_1(N))$-module by means of this map. There is a commutative diagram of $\mathfrak{H}_k(\Gamma_1(N))$-modules with exact rows

$$\begin{array}{ccccccc}
\mathrm{Bound}_{\Gamma_0(N)}\big(L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & \mathrm{Symb}_{\Gamma_0(N)}\big(L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & H^1_{\mathrm{B}}\big(\Gamma_0(N), L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Bound}_{\Gamma_1(N)}\big(L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & \mathrm{Symb}_{\Gamma_1(N)}\big(L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & H^1_{\mathrm{B}}\big(\Gamma_1(N), L_n(\mathcal{O}_{(p)})\big) & \longrightarrow & 0
\end{array}$$

in which the vertical arrows are induced by restriction in cohomology.

Let $\mathfrak{p}$ be a prime of F above p, denote by $\mathbb{F}_{\mathfrak{p}}$ the residue field of F at $\mathfrak{p}$ and let $\Gamma \in \big\{\Gamma_1(N), \Gamma_0(N)\big\}$; there is a canonical map $\mathfrak{H}_k(\Gamma) \to \mathbb{F}_{\mathfrak{p}}$, whose kernel will be denoted by $\mathfrak{m}_\Gamma$. If M is an $\mathfrak{H}_k(\Gamma)$-module, then we write $M_{\mathfrak{m}_\Gamma}$ for the localization of M at $\mathfrak{m}_\Gamma$. Then

$$\mathrm{Bound}_{\Gamma}\big(L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_\Gamma} = 0$$

because the action of $\mathfrak{H}_k(\Gamma)$ on boundary symbols is Eisenstein, so we get a commutative square of $\mathfrak{H}_k(\Gamma)$-modules

$$\begin{array}{ccc}
\mathrm{Symb}_{\Gamma_0(N)}\big(L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_0(N)}} & \xrightarrow{\simeq} & H^1_{\mathrm{B}}\big(\Gamma_0(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_0(N)}} \\
\downarrow & & \downarrow \\
\mathrm{Symb}_{\Gamma_1(N)}\big(L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}} & \xrightarrow{\simeq} & H^1_{\mathrm{B}}\big(\Gamma_1(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}
\end{array} \tag{5.32}$$

in which the horizontal maps are isomorphisms. Now we prove that the right vertical arrow is an isomorphism; to do this, we show that the map of free $\mathcal{O}_{(\mathfrak{p})}$-modules

$$\mathrm{Hom}_{\Gamma_0(N)}\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_0(N)}} \longrightarrow \mathrm{Hom}_{\Gamma_1(N)}\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}} \tag{5.33}$$

is an isomorphism (note that $\mathrm{Hom}_\Gamma\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_\Gamma}$ is a finitely generated torsion-free, and hence free, $\mathcal{O}_{(\mathfrak{p})}$-module). The map $\mathrm{Hom}_{\Gamma_0(N)}\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big) \to \mathrm{Hom}_{\Gamma_1(N)}\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big)$ is injective, so (5.33) is injective, as localization is a flat operation. By Nakayama's lemma, it suffices to show that the map

$$\mathrm{Hom}_{\Gamma_0(N)}\big(\mathcal{D}_0, L_n(\mathbb{F}_\mathfrak{p})\big)[\mathfrak{m}_{\Gamma_0(N)}] \longrightarrow \mathrm{Hom}_{\Gamma_1(N)}\big(\mathcal{D}_0, L_n(\mathbb{F}_\mathfrak{p})\big)[\mathfrak{m}_{\Gamma_1(N)}] \tag{5.34}$$

is surjective. If this map is not surjective, then there is $\phi \in \mathrm{Hom}_{\Gamma_1(N)}\big(\mathcal{D}_0, L_n(\mathbb{F}_\mathfrak{p})\big)[\mathfrak{m}_{\Gamma_1(N)}]$ on which $\Gamma_0(N)/\Gamma_1(N)$ acts via a non-trivial character ε; by Nakayama's lemma, there is a non-zero $\Phi \in \mathrm{Hom}_{\Gamma_1(N)}\big(\mathcal{D}_0, L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}[\varepsilon]$ mapping to ϕ under the canonical surjection $\mathcal{O}_{(\mathfrak{p})} \twoheadrightarrow \mathbb{F}_\mathfrak{p}$. Therefore, we obtain two modular symbols $\Phi_{f,\Gamma_1(N)}$ and Φ in $\mathrm{Symb}_{\Gamma_1(N)}\big(L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}$ that are distinct (because ε is non-trivial) and share the eigenvalues for the action of $\mathfrak{H}_k(\Gamma_1(N))$. The images of $\Phi_{f,\Gamma_1(N)}$ and Φ in $H^1_\mathrm{B}\big(\Gamma_1(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}$ coincide because f is a newform and then, in light of (5.32), we deduce that $\Phi = \Phi_{f,\Gamma_1(N)}$: this contradiction proves the surjectivity of (5.34), whence the surjectivity of (5.33). Using (5.32), we conclude that there is an isomorphism of $\mathcal{O}_{(\mathfrak{p})}$-modules

$$H^1_\mathrm{B}\big(\Gamma_0(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_0(N)}} \simeq H^1_\mathrm{B}\big(\Gamma_1(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}.$$

Upon taking $\pm$-eigenspaces for complex conjugation and $\mathfrak{m}_{\Gamma_0(N)}$- and $\mathfrak{m}_{\Gamma_1(N)}$-torsion submodules, respectively, we get an isomorphism

$$H^1_\mathrm{B}\big(\Gamma_0(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_0(N)}}\big[\theta_{f,\Gamma_0(N)}, \pm\big] \simeq H^1_\mathrm{B}\big(\Gamma_1(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)_{\mathfrak{m}_{\Gamma_1(N)}}\big[\theta_{f,\Gamma_1(N)}, \pm\big]$$

of free $\mathcal{O}_{(\mathfrak{p})}$-modules of rank 1, where $\theta_{f,\Gamma_0(N)}$ and $\theta_{f,\Gamma_1(N)}$ denote the two ring homomorphisms associated with f. Therefore, $\gamma^\pm_{f,\Gamma_1(N)}$ and the image of $\gamma^\pm_{f,\Gamma_0(N)}$ in $H^1_\mathrm{B}\big(\Gamma_1(N), L_n(\mathcal{O}_{(\mathfrak{p})})\big)$ are both generators of these free $\mathcal{O}_{(\mathfrak{p})}$-modules, which implies that the periods $\Omega_{f,\Gamma_0(N)}$ and $\Omega_{f,\Gamma_1(N)}$ differ by a unit of $\mathcal{O}_{(\mathfrak{p})}$, as was to be shown. □

Remark 5.17 When the modular form f has weight 2 (a case that we have excluded from the outset), Proposition 5.16 is proved in [152, Lemma 9.4] by different arguments. More precisely, the proof of [152, Lemma 9.4] uses in a crucial way Eisenstein properties of the Shimura subgroup, which in our higher weight context are replaced by Eisenstein properties of modular symbols.

From here on, as in Sect. 5.3.3, we set $\delta_{f,\Gamma_0(N)} := \delta^\epsilon_{f,\Gamma_0(N)}$; moreover, put $\mathfrak{a}_{f,\Gamma_0(N)} := \mathfrak{a}^\epsilon_{f,\Gamma_0(N)}$ and, as in Proposition 5.14, define

$$\gamma_f := (2\pi i)^{k/2} \cdot (\Pi_B \Pi_\epsilon) \cdot \delta_{f,\Gamma_0(N)} \in T^+_\mathrm{B} \smallsetminus \{0\}.$$

By what we noticed previously, if $p \nmid \mathrm{N}(\mathfrak{a}_{f,\Gamma_0(N)})$, then $\mathrm{Comp}_{\mathrm{B},\text{ét}}(\gamma_f)$ generates T^+_p. In other words, notation being as in Sect. 2.23.3, we know that

$$p \nmid \mathrm{N}(\mathfrak{a}_{f,\Gamma_0(N)}) \implies \mathcal{I}_p(\gamma_f) = \mathcal{O}_p. \tag{5.35}$$

This implication will be used in the proof of our main results.

Remark 5.18 Although T_{B} and $\gamma_f \in T_{\mathrm{B}}^{+}$ are defined in terms of the congruence subgroup $\Gamma(N)$, Proposition 5.16 ensures that, for our goals and arguments, we can equivalently work with the period $\Omega_{f,\Gamma_0(N)}$.

5.4 Choice of auxiliary imaginary quadratic fields

In the remainder of the paper, we will need to fix auxiliary imaginary quadratic fields in a judicious way. For our purposes, we may restrict ourselves to $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$.

5.4.1 The $r_{\mathrm{an}}(\mathcal{M}) = 0$ case

Assume that $r_{\mathrm{an}}(\mathcal{M}) = 0$. Let us consider the imaginary quadratic fields K satisfying the following two conditions:

- the primes dividing Np split in K;
- $r_{\mathrm{an}}(f^K) = 1$.

Denote by $\mathscr{I}_0(f, p)$ the set of all such fields. By Lemma 2.34, $r_{\mathrm{an}}(f) = 0$, so $\varepsilon(f) = +1$, and then it follows from [20, p. 543, Theorem, (i)] (*cf.* also [107]) that $\mathscr{I}_0(f, p) \neq \emptyset$. Note that $r_{\mathrm{an}}(f/K) = 1$ for all $K \in \mathscr{I}_0(f, p)$.

5.4.2 The $r_{\mathrm{an}}(\mathcal{M}) = 1$ case

Assume that $r_{\mathrm{an}}(\mathcal{M}) = 1$. Let us consider the imaginary quadratic fields K satisfying the following two conditions:

- the primes dividing Np split in K;
- $r_{\mathrm{an}}(f^K) = 0$.

Denote by $\mathscr{I}_1(f, p)$ the set of all such fields. By Lemma 2.34, $r_{\mathrm{an}}(f) = 1$, so $\varepsilon(f) = -1$, and then $\mathscr{I}_1(f, p) \neq \emptyset$ by [20, p. 543, Theorem, (ii)] (*cf.* also [160]). Again, observe that $r_{\mathrm{an}}(f/K) = 1$ for all $K \in \mathscr{I}_1(f, p)$

5.5 Rationality conjecture for $\mathcal{M}$ in analytic rank 0

Here we prove (under suitable assumptions) the rationality conjecture (Conjecture 2.40) when $r_{\mathrm{an}}(\mathcal{M}) = 0$.

5.5.1 Nekovář's theorem

Let L be a number field. Following [110], we define the Shafarevich–Tate group of $\mathcal{M}$ over L at $\mathfrak{p}$ *à la* Nekovář via the short exact sequence of $\mathcal{O}_{\mathfrak{p}}$-modules

$$0 \longrightarrow \Lambda_{\mathfrak{p}}(L) \otimes_{\mathcal{O}_{\mathfrak{p}}} (F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}) \longrightarrow H^1_f(L, A_{\mathfrak{p}}) \longrightarrow Ш^{\mathrm{Nek}}_{\mathfrak{p}}(L, \mathcal{M}) \longrightarrow 0 \tag{5.36}$$

(see, *e.g.*, [93, §2.4] for details on the leftmost non-trivial map, which will be tacitly regarded as a set-theoretic inclusion). Let us also define the Shafarevich–Tate group of $\mathcal{M}$ over L at p *à la* Nekovář by setting

$$Ш^{\mathrm{Nek}}_{p}(L, \mathcal{M}) := \bigoplus_{\mathfrak{p}|p} Ш^{\mathrm{Nek}}_{\mathfrak{p}}(L, \mathcal{M}), \tag{5.37}$$

where the direct sum ranges over all primes of F above p.

Remark 5.19 The group $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(L, \mathcal{M})$ should be thought of as a higher weight counterpart of the classical Shafarevich–Tate group of an abelian variety, which is given by the recipe "Selmer group modulo rational points".

As before, let K be an imaginary quadratic field in which all the prime divisors of Np split and let $\mathfrak{p}$ be a prime of F above p. The theorem below is a higher weight analogue of a well-known result of Kolyvagin for Mordell–Weil and Shafarevich–Tate groups of elliptic curves (see, *e.g.*, [53, Theorem 1.3]).

Theorem 5.20 (Nekovář) *If $y_{K,\mathfrak{p}}$ is not torsion, then*

(1) $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = F_{\mathfrak{p}} \cdot y_{K,\mathfrak{p}}$;
(2) $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K, \mathcal{M})$ *is finite;*
(3) $\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} H^1_f(K, A_{\mathfrak{p}}) = 1$.

Proof This is [110, Theorem 13.1] (*cf.* also [111, Ch. II, (6.5)]). □

As an immediate consequence of Theorem 5.20 and (5.37), if $y_{K,\mathfrak{p}}$ is not torsion for each $\mathfrak{p} \mid p$, then $Ш^{\mathrm{Nek}}_{p}(K, \mathcal{M})$ is finite.

5.5.2 Comparing Shafarevich–Tate groups

Given a number field L, there is an inclusion $\Lambda_{\mathfrak{p}}(L) \otimes_{\mathcal{O}_{\mathfrak{p}}} (F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}) \subset H^1_f(L, A_{\mathfrak{p}})_{\mathrm{div}}$, which induces a surjection $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(L, \mathcal{M}) \twoheadrightarrow Ш^{\mathrm{BK}}_{\mathfrak{p}}(L, \mathcal{M})$ of $\mathcal{O}_{\mathfrak{p}}$-modules. To our knowledge, no finer, general comparison between $Ш^{\mathrm{BK}}_{\mathfrak{p}}(L, \mathcal{M})$ and $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(L, \mathcal{M})$ is available in the literature.

Take an imaginary quadratic field K as above. The next result, which is a by-product of Theorem 5.20, offers an alternative description of $Ш^{\mathrm{BK}}_{\mathfrak{p}}(K, \mathcal{M})$ in an important special case.

Proposition 5.21 *[Nekovář] If $y_{K,\mathfrak{p}}$ is not torsion, then $Ш^{\mathrm{BK}}_{\mathfrak{p}}(K, \mathcal{M}) = Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K, \mathcal{M})$.*

Proof If $y_{K,\mathfrak{p}}$ is not torsion, then, by Theorem 5.20, both $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} (F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})$ and $H^1_f(K, A_{\mathfrak{p}})$ have corank 1 over $\mathcal{O}_{\mathfrak{p}}$, so $\Lambda_{\mathfrak{p}}(K) \otimes_{\mathcal{O}_{\mathfrak{p}}} (F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})$ is the maximal p-divisible submodule of $H^1_f(K, A_{\mathfrak{p}})$, whence the claim of the proposition. □

5.5.3 On Conjecture 2.30, $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$

The next result, which is basically a consequence of Theorems 5.10 and 5.20, establishes (conditionally on certain assumptions on p-adic Abel–Jacobi maps and regulators) the "algebraic rank = analytic rank" conjecture (Conjecture 2.30) over $\mathbb{Q}$ when $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$. We use notation from Sect. 5.4.

Theorem 5.22 (Nekovář, S.-W. Zhang) *Assume that $r_{\mathrm{an}}(\mathcal{M}) \in \{0, 1\}$ and that*

(1) *there exists $K \in \mathscr{I}_{r_{\mathrm{an}}(\mathcal{M})}(f, p)$ such that $\mathrm{AJ}_{K,\mathfrak{p}}$ is injective on $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ for some $\mathfrak{p} \mid p$;*
(2) *the p-part of Conjecture 2.50 over $\mathbb{Q}$ holds true.*

Then $r_{\mathrm{alg}}(\mathcal{M}) = r_{\mathrm{an}}(\mathcal{M})$.

Proof Assume that $r_{\mathrm{an}}(\mathcal{M}) = 1$. Fix $K \in \mathscr{I}_1(f, p)$ and a prime $\mathfrak{p}$ of F above p for which condition (1) holds. It follows from splitting (5.17) that $r_{\mathrm{an}}(f/K) = 1$, hence, by part (1) of Proposition 5.11, $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion. As explained in the proof of [159, Theorem 5.26],

an analysis of the action of complex conjugation on $y_{K,\mathfrak{p}}$ combined with Theorem 5.20 shows that $\Lambda_{\mathfrak{p}}(K)\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}$ and $\Lambda_{\mathfrak{p}}(\mathbb{Q})\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}$ are both 1-dimensional over $F_{\mathfrak{p}}$ and that $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K,\mathcal{M})$ and $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(\mathbb{Q},\mathcal{M})$ are both finite (*cf.* also [159, Proposition 5.25]). It follows from (5.36) that $\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} H^1_f(\mathbb{Q},A_{\mathfrak{p}})=1$, and then, since we are assuming condition (2), $r_{\mathrm{alg}}(\mathcal{M})=1$ by Corollary 2.68.

Now assume that $r_{\mathrm{an}}(\mathcal{M})=0$. Fix $K\in\mathscr{I}_0(f,p)$ and a prime $\mathfrak{p}$ of F above p for which condition (1) holds. As in the previous case, it follows from (5.17) that $r_{\mathrm{an}}(f/K)=1$, hence, by part (1) of Proposition 5.11, $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion. As in the proof of [159, Theorem 7.4], an analysis of complex conjugation acting on $y_{K,\mathfrak{p}}$ allows one to show that $\Lambda_{\mathfrak{p}}(\mathbb{Q})\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}$ is trivial. On the other hand, it follows from Theorem 5.20 and [159, Proposition 5.25] that $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(\mathbb{Q},\mathcal{M})$ is finite. Therefore, $\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}} H^1_f(\mathbb{Q},A_{\mathfrak{p}})=0$, and then, since we are assuming condition (2), $r_{\mathrm{alg}}(\mathcal{M})=0$ by Corollary 2.68. □

Remark 5.23 In the statement of Theorem 5.22 and elsewhere in this paper, we impose injectivity assumptions on p-adic Abel–Jacobi maps; assumptions of this sort are natural to ask for if one wants to get (by combining Theorems 5.10 and 5.20) results on algebraic ranks from information on analytic ranks. While it is a "folklore" conjecture that such maps are always injective, it is worth emphasizing that, in the present article, conditions of this kind are only exploited, in the guise of part (1) of Proposition 5.11, to use Theorem 5.20 and prove Conjecture 2.30 and the finiteness of $Ш^{\mathrm{Nek}}_{p}(\mathbb{Q},\mathcal{M})$ in the special cases we are interested in. In particular, if one is willing to assume the validity of Conjecture 2.30 and the finiteness of $Ш^{\mathrm{Nek}}_{p}(\mathbb{Q},\mathcal{M})$ in low rank situations, then the aforementioned conditions can be dispensed with (specifically, we are referring to Theorems 5.24, 5.29, 5.37, 5.42).

5.5.4 Proof of Conjecture 2.40, $r_{\mathrm{an}}(\mathcal{M})=0$

Now we can prove the main result of this subsection.

Theorem 5.24 *Assume that* $r_{\mathrm{an}}(\mathcal{M})=0$ *and that*

(1) *there exists* $K\in\mathscr{I}_0(f,p)$ *such that* $\mathrm{AJ}_{K,\mathfrak{p}}$ *is injective on* $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ *for some* $\mathfrak{p}\mid p$*;*
(2) *the p-part of Conjecture 2.50 over* $\mathbb{Q}$ *holds true.*

Then Conjecture 2.40 is true.

Proof Assume that $r_{\mathrm{an}}(\mathcal{M})=0$; this implies, by Lemma 2.34, that $r_{\mathrm{an}}(f^\sigma)=0$ for all $\sigma\in\Sigma$. Thanks to Theorem 5.22, $r_{\mathrm{alg}}(\mathcal{M})=0$, so $\mathrm{Reg}(\mathcal{M})=1$. Let Ω_∞ be the period in Sect. 2.23.3; by Proposition 2.43, to prove the theorem we can equivalently show that $L^*(\mathcal{M},0)\big/\Omega_\infty\in F^\times$. Namely (*cf.* Remark 2.8), we need to show that there exists $\mathscr{L}_{\mathcal{M}}\in F^\times$ such that

$$\iota_\Sigma(\mathscr{L}_{\mathcal{M}})=\frac{L^*(\mathcal{M},0)}{\Omega_\infty}. \tag{5.38}$$

By definition of ι_Σ, and in light of (5.29), equality (5.38) means that $\sigma(\mathscr{L}_{\mathcal{M}})=L^{\mathrm{alg}}(f^\sigma,k/2)$ for all $\sigma\in\Sigma$. By Proposition 5.15, the element $\mathscr{L}_{\mathcal{M}}:=L^{\mathrm{alg}}(f,k/2)\in F^\times$ does the job, and we are done. □

As explained in Sect. 2.11.2, Theorem 5.24 shows that, under the specified assumptions, the analytic rank 0 case of Conjecture 2.41 holds true as well.

5.6 *p*-TNC for $\mathcal{M}$ in analytic rank 0

As before, D_F is the discriminant of F, while the ideal $\mathfrak{a}_{f,\Gamma_0(N)}$ of $\mathcal{O}_F$ was introduced at the end of Sect. 5.3.6 and $c_f = [\mathcal{O}_F : \mathcal{O}_f]$. Furthermore, recall from Sect. 3.1.1 that the Galois representation ρ_p is said to have "big image" if the image of ρ_p contains all $g \in \mathrm{GL}_2(\mathcal{O}_p)$ such that $\det(g) \in (\mathbb{Z}_p^\times)^{k-1}$.

We work under the following list of conditions on the pair (f, p).

Assumption 5.25 (1) $p \nmid 6ND_F\mathrm{N}(\mathfrak{a}_{f,\Gamma_0(N)})c_f$;
(2) $k \equiv 2 \pmod{2(p-1)}$;
(3) $a_p(f) \in \mathcal{O}_\mathfrak{p}^\times$;
(4) $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$ for each prime $\mathfrak{p} \mid p$;
(5) ρ_p has big image;
(6) $N \geq 3$ and there exists a prime ℓ dividing N exactly such that $\bar{\rho}_\mathfrak{p}$ is ramified at ℓ for each prime $\mathfrak{p} \mid p$.

Our purpose is to prove the p-part of the TNC for $\mathcal{M}$ when $r_{\mathrm{an}}(\mathcal{M}) = 0$. Unless otherwise stated, from here on we make our choice of periods as in Sect. 5.3.6.

5.6.1 *p*-adic *L*-functions

Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, define

$$\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$$

and write $\Lambda_\mathfrak{p} := \mathcal{O}_\mathfrak{p}[\![\Gamma]\!]$ for the Iwasawa algebra of Γ with coefficients in $\mathcal{O}_\mathfrak{p}$. For every $n \in \mathbb{N}$ let $\mathbb{Q}_n$ be the subfield of $\mathbb{Q}_\infty$ such that $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$; in particular, $\mathbb{Q}_0 = \mathbb{Q}$. For any prime v of $\mathbb{Q}_n$ denote by $\mathbb{Q}_{n,v}$ the completion of $\mathbb{Q}_n$ at v and write $I_{n,v} = I_{\mathbb{Q}_{n,v}}$ for the corresponding inertia group. Finally, let $\chi_{\mathrm{cyc}} : \Gamma \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character.

Let $\Omega_f^\pm := \Omega_{f,\Gamma_0(N)}^\pm \in \mathbb{C}^\times$ be the periods from Sect. 5.3.2. In [151], the periods $\Omega_{f,\Gamma_1(N)}^\pm$ are considered instead, but the choice of $\Omega_{f,\Gamma_0(N)}^\pm$ is equivalent for us since, by Proposition 5.16, these complex numbers differ by a p-adic unit. Let $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ be an embedding corresponding to $\mathfrak{p}$, which allows us to consider $L^{\mathrm{alg}}(f, k/2) \in F$ as a p-adic number in $\bar{\mathbb{Q}}_p$; here recall that $L^{\mathrm{alg}}(f, k/2) = L(f, k/2)\big/(2\pi i)^{k/2-1} \cdot \Omega_f$, where $\Omega_f = \Omega_f^\epsilon$ and ϵ is the sign of $(-1)^{k/2-1}$.

Let $\mathcal{L}_{f,\mathfrak{p}} \in \Lambda$ be the cyclotomic p-adic L-function of f and $\mathfrak{p}$ constructed in [74, Theorem 16.2] and [151, §3.4.4] (*cf.* also [102]). Adopting the conventions in [74, Theorem 16.2], there is an interpolation formula

$$\mathcal{L}_{f,\mathfrak{p}}(\chi_{\mathrm{cyc}}^{k/2}) = \left(1 - \frac{p^{\frac{k-2}{2}}}{\alpha}\right)^2 \cdot \left(\frac{k-2}{2}\right)! \cdot L^{\mathrm{alg}}(f, k/2). \tag{5.39}$$

Since $k > 2$, the multiplicative factor $\left(1 - \frac{p^{\frac{k-2}{2}}}{\alpha}\right)^2$ is a unit of $\mathcal{O}_{(\mathfrak{p})}$. Therefore, there is an equality of $\mathcal{O}_\mathfrak{p}$-ideals

$$\left(\mathcal{L}_{f,\mathfrak{p}}(\chi_{\mathrm{cyc}}^{k/2})\right) = \left((k/2-1)! \cdot L^{\mathrm{alg}}(f, k/2)\right). \tag{5.40}$$

5.6.2 Comparing the periods of f and f^K

As in Remark 2.3, let f^K be the twist of f by the Dirichlet character ϵ_K attached to K. Of course, the Hecke field of f^K is F.

Lemma 5.26 (1) *For any choice of $\delta_f^{\pm}$ as in Sect. 5.3.2, the equality $\Omega_{f^K}^{\mp} = \sqrt{D_K} \cdot \Omega_f^{\pm}$ holds up to elements of $F^\times$.*

(2) *For any choice of $\delta_f^{\pm}$ and $\delta_{f^K}^{\pm}$ as in Sect. 5.3.6, the equality $\Omega_f^{\pm} = \Omega_{f^K}^{\mp}$ holds up to p-adic units.*

Proof For part (1), we adapt (with minor changes) the proof of [152, Lemma 9.6]. For any ring R as before, let us define a homomorphism $H^1\big(\Gamma_0(N), L_n(R)\big) \to H^1\big(\Gamma_0(ND_K^2), L_n(R)\big)$ of R-modules by

$$\varphi \longmapsto \left(\gamma \mapsto \sum_{a \in (\mathbb{Z}/D_K\mathbb{Z})^\times} \epsilon_K(a) \cdot \varphi\left(\left(\begin{smallmatrix} 1 & -a/D_K \\ 0 & 1 \end{smallmatrix}\right) \cdot \gamma \cdot \left(\begin{smallmatrix} 1 & a/D_K \\ 0 & 1 \end{smallmatrix}\right)\right)\right).$$

For $R = \mathbb{C}$, $\omega_f^{\pm}$ is mapped to $\tau(\epsilon_K) \cdot \omega_{f^K}^{\mp}$, where $\tau(\epsilon_K)$ is the Gauss sum of ϵ_K. For $R = F$, $\delta_f^{\pm}$ is mapped to an $F^\times$-multiple of $\delta_{f^K}^{\mp}$. Therefore, $\Omega_{f^K}^{\pm}$ is an $F^\times$-multiple of $\Omega_f^{\mp}/\tau(\epsilon_K)$, and the conclusion follows because $\tau(\epsilon_K)^2 = D_K$.

Part (2) is a consequence of the interpolation formulas satisfied by the p-adic L-functions associated with f and f^K. With notation as in [151, §3.4.4], $\mathcal{L}_{f,\epsilon_K}(\mathbf{1})$ and $\mathcal{L}_{f^K,\mathbf{1}}(\mathbf{1})$ differ by a p-adic unit, so comparing with the interpolation formulas in [151, §3.4.4] we see that $\Omega_{f^K}^{\pm}$ is an $\mathcal{O}_p^\times$-multiple of $\Omega_f^{\mp}/\tau(\epsilon_K)$, and the conclusion follows because $\tau(\epsilon_K)^2 = D_K$ and p splits in K. □

Remark 5.27 In the case of elliptic curves, the analogue of part (2) of Lemma 5.26 is proved, *e.g.*, in [152, Lemma 9.6] with a different method.

5.6.3 Greenberg's Selmer group

Let $\eta_p : G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \bar{\mathbb{Q}}_p^\times$ denote the unramified character of $G_{\mathbb{Q}_p}$ taking arithmetic Frobenius to $a_p(f)$. Since f is ordinary at $\mathfrak{p}$ and $V_{\mathfrak{p}}$ is (equivalent to) the self-dual twist of the $\mathfrak{p}$-adic representation attached to f, there is a short exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \longrightarrow V_{\mathfrak{p}}^+ \longrightarrow V_{\mathfrak{p}} \longrightarrow V_{\mathfrak{p}}^- \longrightarrow 0$$

such that $V_{\mathfrak{p}}^+$ and $V_{\mathfrak{p}}^-$ are 1-dimensional $F_{\mathfrak{p}}$-vector spaces, and $G_{\mathbb{Q}_p}$ acts on $V_{\mathfrak{p}}^+$ and $V_{\mathfrak{p}}^-$ through $\eta_p^{-1}\chi_{\mathrm{cyc}}^{k/2}$ and $\eta_p\chi_{\mathrm{cyc}}^{-k/2+1}$, respectively. Define $T_{\mathfrak{p}}^+ := T_{\mathfrak{p}} \cap V_{\mathfrak{p}}^+$, $A_{\mathfrak{p}}^+ := V_{\mathfrak{p}}^+/T_{\mathfrak{p}}^+$, $A_{\mathfrak{p}}^- := A_{\mathfrak{p}}/A_{\mathfrak{p}}^+$ and set

$$H^1_{\mathrm{ord}}(\mathbb{Q}_{n,v}, A_{\mathfrak{p}}) := \ker\Big(H^1(\mathbb{Q}_{n,v}, A_{\mathfrak{p}}) \longrightarrow H^1(I_{\mathbb{Q}_{n,v}}, A_{\mathfrak{p}}^-)\Big).$$

For every $n \in \mathbb{N}$, define the $\mathfrak{p}$-*primary Greenberg Selmer group of f over $\mathbb{Q}_n$* to be

$$\mathrm{Sel}_{\mathfrak{p}}(f/\mathbb{Q}_n) := \ker\left(H^1(\mathbb{Q}_n, A_{\mathfrak{p}}) \longrightarrow \prod_{v \nmid p} \frac{H^1(\mathbb{Q}_{n,v}, A_{\mathfrak{p}})}{H^1_{\mathrm{ur}}(\mathbb{Q}_{n,v}, A_{\mathfrak{p}})} \times \prod_{v|p} \frac{H^1(\mathbb{Q}_{n,v}, A_{\mathfrak{p}})}{H^1_{\mathrm{ord}}(\mathbb{Q}_{n,v}, A_{\mathfrak{p}})}\right),$$

where v denotes a prime of $\mathbb{Q}_n$. By [121, Proposition 4.2], for all n there is an injection $H^1_f(\mathbb{Q}_n, A_\mathfrak{p}) \hookrightarrow \mathrm{Sel}_\mathfrak{p}(f/\mathbb{Q}_n)$ with finite cokernel whose order is bounded independently of n. As in [151], let us consider the $\mathfrak{p}$-*primary Greenberg Selmer group of* f *over* $\mathbb{Q}_\infty$ given by

$$S_\mathfrak{p} = \mathrm{Sel}_\mathfrak{p}(f/\mathbb{Q}_\infty) := \varinjlim_n \mathrm{Sel}_\mathfrak{p}(f/\mathbb{Q}_n),$$

where the direct limit is taken with respect to restriction maps. By [74, Theorem 17.4, (1)] (*cf.* also [151, Theorem 3.15]), the $\Lambda_\mathfrak{p}$-module $S_\mathfrak{p}$ is cotorsion, *i.e.*, the Pontryagin dual $X_\mathfrak{p} := \mathrm{Hom}(S_\mathfrak{p}, F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})$ of $S_\mathfrak{p}$ is torsion over $\Lambda_\mathfrak{p}$; we write $(\mathcal{F}_{f,\mathfrak{p}}) \subset \Lambda_\mathfrak{p}$ for the characteristic ideal of $X_\mathfrak{p}$.

5.6.4 Proof of p-TNC, $r_{\mathrm{an}}(\mathcal{M}) = 0$

The following theorem is a special case of a result for Selmer groups over finite abelian extensions of $\mathbb{Q}$.

Theorem 5.28 (Kato) *If* $r_{\mathrm{an}}(f) = 0$, *then* $H^1_f(\mathbb{Q}, A_\mathfrak{p})$ *is finite.*

Proof This is [74, Theorem 14.2, (2)] for $K = \mathbb{Q}$. □

Now we prove the p-part of the Tamagawa number conjecture for $\mathcal{M}$ when $r_{\mathrm{an}}(\mathcal{M}) = 0$; in fact, we prove, more generally, a rank 0 counterpart of Theorem B. As will be clear, our proof builds crucially on work of Kato and of Skinner–Urban. Recall that Assumption 5.25 is in force.

Theorem 5.29 *Assume that* $r_{\mathrm{an}}(\mathcal{M}) = 0$ *and that*

(1) *there exists* $K \in \mathscr{I}_0(f, p)$ *such that* $\mathrm{AJ}_{K,\mathfrak{p}}$ *is injective on* $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ *for some* $\mathfrak{p} \mid p$;
(2) *the* p*-part of Conjecture 2.50 over* $\mathbb{Q}$ *holds true.*

Then the following results hold:

(a) $r_{\mathrm{alg}}(\mathcal{M}) = 0$;
(b) $Ш^{\mathrm{BK}}_\mathfrak{p}(\mathbb{Q}, \mathcal{M}) = Ш^{\mathrm{Nek}}_\mathfrak{p}(\mathbb{Q}, \mathcal{M})$ *for each* $\mathfrak{p} \mid p$ *satisfying (1);*
(c) *the* p*-part of Conjecture 2.56 is true.*

Furthermore, if condition (1) holds for each $\mathfrak{p} \mid p$, *then* $Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M}) = Ш^{\mathrm{Nek}}_p(\mathbb{Q}, \mathcal{M})$.

Proof Part (a) was already proved in Theorem 5.22. Let $\mathfrak{p} \mid p$ be a prime of F satisfying condition (1). As explained in the proof of Theorem 5.22, $\Lambda_\mathfrak{p}(\mathbb{Q}) \otimes_{\mathcal{O}_\mathfrak{p}} F_\mathfrak{p}$ is trivial, and then $\Lambda_\mathfrak{p}(\mathbb{Q}) \otimes_{\mathcal{O}_\mathfrak{p}} (F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})$ is trivial as well. Thus, $Ш^{\mathrm{Nek}}_\mathfrak{p}(\mathbb{Q}, \mathcal{M}) = H^1_f(\mathbb{Q}, A_\mathfrak{p})$. On the other hand, $H^1_f(\mathbb{Q}, A_\mathfrak{p}) = Ш^{\mathrm{BK}}_\mathfrak{p}(\mathbb{Q}, \mathcal{M})$ because, by Theorem 5.28, $H^1_f(\mathbb{Q}, A_\mathfrak{p})$ is finite: this proves part (b) and (by taking direct sums over all $\mathfrak{p} \mid p$) the last statement.

We prove part (c) by proving equality (p-TNC$_{\mathscr{B}}$) in Theorem 2.83. To start with, observe that Conjecture 2.40 is true in this case, by Theorem 5.24. Thus, all the assumptions in Theorem 2.83 are verified (*cf.* Remark 2.84).

By Lemma 2.34, $r_{\mathrm{an}}(f) = 0$. Let $\mathbf{1}$ denote the trivial character. We first show that

$$\big(\mathcal{F}_{f,\mathfrak{p}}(\mathbf{1})\big) = \big(\mathcal{L}_{f,\mathfrak{p}}(\chi_{\mathrm{cyc}}^{k/2})\big) \tag{5.41}$$

as ideals of $\mathcal{O}_\mathfrak{p}$. Define the $\Lambda_\mathfrak{p}$-torsion module

$$X_\mathfrak{p}(k/2) := \mathrm{Hom}\big(S_\mathfrak{p}(-k/2), F_\mathfrak{p}/\mathcal{O}_\mathfrak{p}\big)$$

and write $\mathcal{C} = \mathrm{Char}\big(X_\mathfrak{p}(k/2)\big)$ for its characteristic power series, which does not depend on k (*cf.* [74, Proposition 17.2]). By [74, Theorem 17.4] and [151, Theorem 3.29], there is an equality $(\mathcal{C}) = (\mathcal{L}_{f,\mathfrak{p}})$, so $\mathcal{C}\big(\chi_{\mathrm{cyc}}^{k/2}\big)$ and $\mathcal{L}_{f,\mathfrak{p}}\big(\chi_{\mathrm{cyc}}^{k/2}\big)$ generate the same ideal of $\mathcal{O}_\mathfrak{p}$ (here, for an element $x \in \Lambda$ and a character $\chi : \Gamma \to \bar{\mathbb{Q}}_p^\times$, we set $x(\chi) := \chi(x)$). In order to show (5.41), it therefore remains to note that $\mathcal{C}\big(\chi_{\mathrm{cyc}}^{k/2}\big)$ and $\mathcal{F}_{f,\mathfrak{p}}(\mathbf{1})$ generate the same ideal of $\mathcal{O}_\mathfrak{p}$, which follows easily by taking into consideration the Tate twist in the relevant definitions (see, *e.g.*, [134, Lemma 1.2]).

Combining (5.39) and (5.41), we obtain the equivalences

$$L(f, k/2) \neq 0 \iff \mathcal{L}_{f,\mathfrak{p}}\big(\chi_{\mathrm{cyc}}^{k/2}\big) \neq 0 \iff \mathcal{F}_{f,\mathfrak{p}}(\mathbf{1}) \neq 0.$$

Since $r_{\mathrm{an}}(f) = 0$, we deduce from Theorem 5.28 that $H^1_f(\mathbb{Q}, A_\mathfrak{p})$ is finite. A generalization to our setting of the arguments in [51, Section 4] (see [95]) shows that

$$\mathcal{I}_{\mathcal{O}_\mathfrak{p}}\big(\mathcal{O}_\mathfrak{p}/\mathcal{F}_{f,\mathfrak{p}}(\mathbf{1})\cdot\mathcal{O}_\mathfrak{p}\big) = \mathcal{I}_{\mathcal{O}_\mathfrak{p}}\big(S_\mathfrak{p}^\Gamma/(\#S_\mathfrak{p})_\Gamma\big) = \mathcal{I}_{\mathcal{O}_\mathfrak{p}}\big(H^1_f(\mathbb{Q}, A_\mathfrak{p})\big)\cdot\prod_{\ell\mid N}\mathrm{Tam}_\ell^{(\mathfrak{p})}(\mathcal{M}). \tag{5.42}$$

Combining (5.40), (5.41) and (5.42), we see that for each $\mathfrak{p}\mid p$ there is an equality

$$\Big((k/2-1)!\cdot L^{\mathrm{alg}}(f, k/2)\Big) = \mathcal{I}_{\mathcal{O}_\mathfrak{p}}\big(H^1_f(\mathbb{Q}, A_\mathfrak{p})\big)\cdot\prod_{\ell\mid N}\mathrm{Tam}_\ell^{(\mathfrak{p})}(\mathcal{M}) \tag{5.43}$$

of (fractional) $\mathcal{O}_\mathfrak{p}$-ideals. On the other hand, by Proposition 5.15 (*cf.* the proof of Theorem 5.24), $\iota_\Sigma\big((k/2-1)!\cdot L^{\mathrm{alg}}(f, k/2)\big) = (k/2-1)!\cdot L^*(\mathcal{M}, 0)\big/\Omega_\infty^{(\gamma_f)}$. Thus, with the convention from Remark 2.8, one can replace $(k/2-1)!\cdot L^{\mathrm{alg}}(f, k/2)$ with $(k/2-1)!\cdot L^*(\mathcal{M}, 0)\big/\Omega_\infty^{(\gamma_f)}$ in (5.43). Since $H^1_f(\mathbb{Q}, A_p) = Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M})$ and (5.43) holds for each $\mathfrak{p}\mid p$, we obtain an equality

$$\left(\frac{(k/2-1)!\cdot L^*(\mathcal{M}, 0)}{\Omega_\infty}\right) = \mathcal{I}_{\mathcal{O}_p}\big(Ш^{\mathrm{BK}}_p(\mathbb{Q}, \mathcal{M})\big)\cdot\prod_{\ell\mid N}\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) \tag{5.44}$$

of (fractional) $\mathcal{O}_p$-ideals. Now notice that, by part (1) of Proposition 4.8, $H^1(\mathbb{Q}, T_\mathfrak{p})_{\mathrm{tors}}$ is trivial for each $\mathfrak{p}\mid p$, so $H^1(\mathbb{Q}, T_p)_{\mathrm{tors}}$ is trivial. Furthermore, it follows from Proposition 3.3 that $H^0(G_S, A_p)$ is trivial (*cf.* also the proof of [94, Lemma 2.4]). This shows that $\mathrm{Tors}_p(\mathcal{M}) = \mathcal{O}_p$ (*cf.* Sect. A.3). By (2.77), $\mathrm{Tam}_p^{(p)}(\mathcal{M}) = \mathcal{O}_p$, while $\mathrm{Tam}_\infty^{(p)}(\mathcal{M}) = \mathcal{O}_p$ by (2.78) and $\mathcal{I}_p(\gamma_f) = \mathcal{O}_p$ by (5.35). Finally, by Theorem 5.22, $r_{\mathrm{alg}}(\mathcal{M}) = 0$, so $\mathrm{Reg}(\mathcal{M}) = 1$. Therefore, equality (5.44) coincides with equality (p-TNC$_{\mathscr{B}}$) (or, better, with its equivalent form (p-TNC$_{\mathscr{B}}$-bis) in Remark 2.87) in our setting, which completes the proof. □

Remark 5.30 We want to explain how to relax an assumption in [95] so that the results in [95] can be safely applied in our current setting. In [95], one requires ([95, Assumption 6(b)]) that $H^0(I_v, A^-) = 0$, which is too restrictive for the applications described above. This condition is used in the proof of [95, Lemma 5.4] to show that

$H^1\big(G_v^{\mathrm{un}}, H^1(I_v, T^-)_{\mathrm{tors}}\big) = 0$. In that argument, we identify $H^1(I_v, T^-)_{\mathrm{tors}}$ with the largest cotorsion quotient of $H^0(I_v, A^-)$ and then conclude using $H^0(I_v, A^-) = 0$. It turns out that we can relax the assumption $H^0(I_v, A^-) = 0$ and still show that $H^1\big(G_v^{\mathrm{un}}, H^1(I_v, T^-)_{\mathrm{tors}}\big)$ is trivial, arguing as follows. As observed above, the group G_v^{un} acts on $H^0(I_v, A^-)$ via the unramified character taking the geometric Frobenius to the unit root α of the Hecke polynomial. In the case of weight 2, we have $H^0(I_v, A^-) = A^-$, hence the largest cotorsion quotient of $H^0(I_v, A^-)$ is trivial (as A^- is divisible) and we conclude that $H^1\big(G_v^{\mathrm{un}}, H^1(I_v, T^-)_{\mathrm{tors}}\big)$ is trivial using the argument in [95]. If the weight is bigger than 2, then $H^0(I_v, A^-) = A^-[p^n]$ for some integer n, due to the fact that inertia acts on A^- via the $(1-k/2)$-th power of the cyclotomic character. Since $a_p(f) \not\equiv 1 \pmod{\mathfrak{p}}$, we also have $\alpha \not\equiv 1 \pmod{\mathfrak{p}}$. Thus, $H^1\big(G_v^{\mathrm{un}}, H^1(I_v, T^-)_{\mathrm{tors}}\big)$ is isomorphic to $A^-[p^n]\big/(\alpha - 1)\cdot A^-[p^n]$. Now $\alpha - 1$ is invertible modulo $\mathfrak{p}$, so $(\alpha-1)\cdot A^-[p^n] = A^-[p^n]$ and $H^1\big(G_v^{\mathrm{un}}, H^1(I_v, T^-)_{\mathrm{tors}}\big)$ is trivial.

5.7 Kolyvagin's conjecture and Shafarevich–Tate groups

As usual, let $\mathfrak{p}$ be a prime of F above p. We gather some results on $Ш_{\mathfrak{p}}^{\mathrm{BK}}(K, \mathcal{M})$, where K is, as usual, an imaginary quadratic field in which all the prime factors of Np split.

5.7.1 *A structure theorem for* $Ш_{\mathfrak{p}}^{\mathrm{BK}}(K, \mathcal{M})$

Following [97, §4.2], for every integer $M \geq 1$ we define $\tilde{S}_1(M)$ to be the set of prime numbers ℓ such that

- $\ell \nmid Np$;
- ℓ is inert in K;
- $\mathfrak{p}^M \mid \ell + 1$.

Denote by $\tilde{S}_n(M)$ the set of square-free products of n primes in $\tilde{S}_1(M)$ (here $\tilde{S}_0(M) := \{1\}$) and define

$$\tilde{S}(M) := \bigcup_{n\in\mathbb{N}} \tilde{S}_n(M).$$

For every integer $m \geq 1$, let $\mathscr{M}(m)$ be the set of integers $M \geq 1$ such that $m \in \tilde{S}(M)$.

As in [97, §7.1], we also consider the set $S_1(M)$ of prime numbers ℓ such that

- $\ell \nmid Np$;
- ℓ is inert in K;
- $\mathfrak{p}^M \mid a_\ell(f)$, $\mathfrak{p}^M \mid \ell + 1$;
- $\mathfrak{p}^{M+1} \nmid \ell + 1 \pm a_\ell(f)$.

Write $S_n(M)$ for the set of square-free products of n primes in $S_1(M)$ (with $S_0(M) := \{1\}$) and define

$$S(M) := \bigcup_{n\in\mathbb{N}} S_n(M).$$

With notation as in Sect. 4.3, for every integer $n \geq 1$ set

$$\omega(n) := \max\big\{M \in \mathscr{M}(n) \mid [z_{n,\mathfrak{p}}]_M = 0\big\},$$

where we put $\omega(n) := \infty$ if this maximum does not exist. Finally, for every $r \in \mathbb{N}$ set

$$M_r := \min\{\omega(n) \mid n \in S_r(\omega(n)+1)\},$$

with $M_r := \infty$ if $\omega(n) = \infty$.

Lemma 5.31 *M_0 is finite and $M_r \geq M_{r+1} \geq 0$ for all $r \in \mathbb{N}$.*

Proof This is [97, Lemma 7.4]. □

Set $M_\infty := \inf\{M_r \mid r \in \mathbb{N}\}$ and note that Conjecture 4.2 is equivalent to the statement that $M_\infty < \infty$. Let $Ш_{\mathfrak{p}}^{\mathrm{BK},\pm}(f/K)$ be the $\pm$-eigenspaces of $\mathrm{Gal}(K/\mathbb{Q})$ acting on $Ш_{\mathfrak{p}}^{\mathrm{BK}}(K,\mathcal{M})$ and let $\varepsilon \in \{\pm\}$ be the sign of the root number of f (*cf.* Sect. 2.9).

In the next theorem, which is a higher weight analogue of the main result of [105, §5], we let $N_i := M_{i-1} - M_i$ for all $i \geq 1$. Observe that, by Lemma 5.31, $N_i \geq 0$ for all i.

Theorem 5.32 (Nekovář, Masoero) *If $y_{K,\mathfrak{p}}$ is not torsion, then there are isomorphisms*

$$Ш_{\mathfrak{p}}^{\mathrm{BK},-\varepsilon}(f/K) \simeq \left(\mathcal{O}_{\mathfrak{p}}/p^{N_1}\mathcal{O}_{\mathfrak{p}}\right)^2 \oplus \left(\mathcal{O}_{\mathfrak{p}}/p^{N_3}\mathcal{O}_{\mathfrak{p}}\right)^2 \oplus \dots$$

and

$$Ш_{\mathfrak{p}}^{\mathrm{BK},\varepsilon}(f/K) \simeq \left(\mathcal{O}_{\mathfrak{p}}/p^{N_2}\mathcal{O}_{\mathfrak{p}}\right)^2 \oplus \left(\mathcal{O}_{\mathfrak{p}}/p^{N_4}\mathcal{O}_{\mathfrak{p}}\right)^2 \oplus \dots$$

of $\mathcal{O}_{\mathfrak{p}}$-modules.

Proof By Proposition 5.21, $Ш_{\mathfrak{p}}^{\mathrm{BK}}(K,\mathcal{M}) = Ш_{\mathfrak{p}}^{\mathrm{Nek}}(K,\mathcal{M})$. Thus, the theorem is essentially [97, Theorem 7.3], the only difference being that in [97] one is interested in the structure of $Ш_{\mathfrak{p}}^{\mathrm{Nek}}(K,\mathcal{M})$ as an abelian group, whereas here we are looking at it as an $\mathcal{O}_{\mathfrak{p}}$-module. □

5.7.2 Some consequences on $Ш_{\mathfrak{p}}^{\mathrm{BK}}(K,\mathcal{M})$ and $y_{K,\mathfrak{p}}$

The next theorem is a consequence of Theorem 4.14, which asserts the validity of Conjecture 4.2, and Theorem 5.32; it will play a key role in the proof of our results on the p-part of the TNC for $\mathcal{M}$ in analytic rank 1.

Theorem 5.33 *If $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion, then* $\mathrm{length}_{\mathcal{O}_{\mathfrak{p}}}\left(Ш_{\mathfrak{p}}^{\mathrm{BK}}(K,\mathcal{M})\right) = 2M_0$.

Proof As in the proof of [97, Corollary 7.11], it follows from Theorem 5.32 that

$$\mathrm{length}_{\mathcal{O}_{\mathfrak{p}}}\left(Ш_{\mathfrak{p}}^{\mathrm{BK}}(K,\mathcal{M})\right) = 2(M_0 - M_\infty). \tag{5.45}$$

On the other hand, Theorem 4.14 says that $c_1(f,n) \neq 0$ for a suitable $n \in \Lambda_{\mathrm{Kol}}(f)$, which implies that $M_\infty = 0$. The theorem is then a consequence of equality (5.45). □

For later reference, we prove

Proposition 5.34 *If $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion, then* $M_0 = \mathrm{length}_{\mathcal{O}_{\mathfrak{p}}}\left(\Lambda_{\mathfrak{p}}(K)/y_{K,\mathfrak{p}} \cdot \mathcal{O}_{\mathfrak{p}}\right)$.

Proof By definition, M_0 is the largest integer $M \geq 1$ such that $[z_{1,\mathfrak{p}}]_M = 0$. Equivalently, M_0 is the largest integer $M \geq 1$ such that $z_{1,\mathfrak{p}} \in \mathfrak{p}^M \Lambda_{\mathfrak{p}}(K_1)$. Now denote by M_0' the length of $\Lambda_{\mathfrak{p}}(K)/y_{K,\mathfrak{p}} \cdot \mathcal{O}_{\mathfrak{p}}$ as an $\mathcal{O}_{\mathfrak{p}}$-module, which can be described as the largest integer $M \geq 1$ such that $y_{K,\mathfrak{p}} \in \mathfrak{p}^M \Lambda_{\mathfrak{p}}(K)$. We want to show that $M_0 = M_0'$.

As explained in the proof of Proposition 4.3, restriction gives an $\mathcal{O}_{\mathfrak{p}}$-linear injection

$$\mathrm{res}_{K_1/K} : \Lambda_{\mathfrak{p}}(K) \hookrightarrow \Lambda_{\mathfrak{p}}(K_1)^{\mathcal{G}_1} \subset \Lambda_{\mathfrak{p}}(K_1) \tag{5.46}$$

such that, by (4.8), $\mathrm{res}_{K_1/K}(y_{K,\mathfrak{p}}) = z_{1,\mathfrak{p}}$. It follows immediately that $M_0' \leq M_0$. On the other hand, part (1) of Proposition 4.8 ensures that $\Lambda_{\mathfrak{p}}(K_1)$ is torsion-free, so (5.46) induces for every integer $M \geq 1$ an $\mathcal{O}_{\mathfrak{p}}$-linear injection

$$\Lambda_{\mathfrak{p}}(K)\big/\mathfrak{p}^M\Lambda_{\mathfrak{p}}(K) \hookrightarrow \Lambda_{\mathfrak{p}}(K_1)\big/\mathfrak{p}^M\Lambda_{\mathfrak{p}}(K_1).$$

This shows that $M_0 \leq M_0'$, and the proof is complete. □

5.8 Rationality conjecture for $\mathcal{M}$ in analytic rank 1

Our goal here is to prove the rationality conjecture (Conjecture 2.40) when $r_{\mathrm{an}}(\mathcal{M}) = 1$. Notation from Remark 5.4 and Sect. 5.4 is in force.

5.8.1 Splitting $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F$ over the Hecke algebra

The arguments that follow are very similar to those in Sect. 3.2.4. We assume that f is p-isolated. Let $\theta_{f,F} : \mathscr{H}_k(\Gamma_0(N))_F \twoheadrightarrow F$ be the (surjective) F-linear extension of θ_f and denote by $\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_F}\big(\ker(\theta_{f,F})\big)$ the annihilator ideal of $\ker(\theta_{f,F})$ in $\mathscr{H}_k(\Gamma_0(N))_F$. The *congruence ideal of* $\theta_{f,F}$ is the ideal of F given by

$$\eta_{f,F} := \theta_{f,F}\Big(\mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_F}\big(\ker(\theta_{f,F})\big)\Big).$$

As explained, *e.g.*, in [87, p. 250], $\ker(\theta_{f,F}) \cap \mathrm{Ann}_{\mathscr{H}_k(\Gamma_0(N))_F}\big(\ker(\theta_{f,F})\big) = \{0\}$: as before, this follows from the fact that $\mathscr{H}_k(\Gamma_0(N))_F$ is (trivially) flat over F and f is p-isolated. In particular, $\eta_{f,F} = F$ and the (tautological) short exact sequence of $\mathscr{H}_k(\Gamma_0(N))_F$-modules

$$0 \longrightarrow \ker(\theta_{f,F}) \longrightarrow \mathscr{H}_k(\Gamma_0(N))_F \xrightarrow{\theta_{f,F}} F \longrightarrow 0$$

has a unique splitting with values in the annihilator of $\ker(\theta_{f,F})$ (actually, in [87] it is assumed that the counterpart of $\theta_{f,F}$ takes values in a complete DVR, but this property is not necessary for the conclusion above to hold). As in Sect. 3.2.4, we shall regard this distinguished splitting as canonical, so that there is an identification (or, rather, a canonical isomorphism) of $\mathscr{H}_k(\Gamma_0(N))_F$-modules

$$\mathscr{H}_k(\Gamma_0(N))_F = \ker(\theta_{f,F}) \oplus F. \tag{5.47}$$

Now let L be a number field. The splitting in (5.47) yields a splitting

$$\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F = \Big(\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F \otimes_{\mathscr{H}_k(\Gamma_0(N))_F} F\Big) \oplus \Big(\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F \otimes_{\mathscr{H}_k(\Gamma_0(N))_F} \ker(\theta_{f,F})\Big). \tag{5.48}$$

On the other hand, there is an identification

$$H^1_{\mathrm{mot}}(L,\mathcal{M}) = \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F \otimes_{\mathscr{H}_k(\Gamma_0(N))_F} F. \tag{5.49}$$

Combining (5.48) and (5.49), we get a canonical surjection

$$\Pi_{\mathcal{M},L} : \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_F \twoheadrightarrow H^1_{\mathrm{mot}}(L,\mathcal{M}) \tag{5.50}$$

of $\mathscr{H}_k(\Gamma_0(N))_F$-modules. In order not to make our notation heavier than necessary, if $\mathcal{R}$ is any subring of F, then we shall use the same symbol for the obvious natural map induced on $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/L)_{\mathcal{R}}$ by $\Pi_{\mathcal{M},L}$.

5.8.2 A distinguished $\mathbb{Q}$-rational cycle

Recall the cycle $\mathscr{X}_K \in \mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ introduced in (5.9) and the map

$$\Pi_{\mathcal{M},\mathbb{Q}} : \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_F \longrightarrow H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})$$

from (5.50) with $L = \mathbb{Q}$. It turns out that, in the setting we are concerned with, $\mathscr{X}_K$ is $\mathbb{Q}$-rational, as we prove in the following proposition. We use notation from Sect. 5.4.2.

Proposition 5.35 *Assume that $r_{\mathrm{an}}(\mathcal{M}) = 1$ and that $K \in \mathscr{I}_1(f,p)$ has the property that $\mathrm{AJ}_{K,\mathfrak{p}}$ is injective on $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ for some $\mathfrak{p} \mid p$. Then*

(1) $\mathscr{X}_K \in \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_F$;
(2) $\Pi_{\mathcal{M},\mathbb{Q}}(\mathscr{X}_K) \neq 0$.

Proof Let $\mathfrak{p}$ be a prime of F above p such that $\mathrm{AJ}_{K,\mathfrak{p}}$ is injective on $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$. Since $K \in \mathscr{I}_1(f,p)$, we know that $r_{\mathrm{an}}(f/K) = 1$. Part (1) of Proposition 5.11 tells us that $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion. Furthermore, an analysis of the action of complex conjugation on $y_{K,\mathfrak{p}}$ shows that $y_{K,\mathfrak{p}} \in \Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})}$ (see, *e.g.*, the proof of [159, Theorem 5.27]). On the other hand, $\mathrm{AJ}_{K,\mathfrak{p}}(\mathscr{X}_K) = (k-2)!\cdot y_{K,\mathfrak{p}}$ by Lemma 5.5, so $\mathscr{X}_K$ is not torsion. In addition, the map $\mathrm{AJ}_{K,\mathfrak{p}}$ is $\mathrm{Gal}(K/\mathbb{Q})$-equivariant, so the injectivity of $\mathrm{AJ}_{K,\mathfrak{p}}$ on $\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}$ (which we are assuming) allows us to conclude that

$$\begin{aligned}\mathscr{X}_K \in \left(\mathrm{Heeg}^{\mathcal{G}_1}_{K,N}\right)^{\mathrm{Gal}(K/\mathbb{Q})} &\subset \left(\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathcal{G}_1}\right)^{\mathrm{Gal}(K/\mathbb{Q})} \\ &= \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathrm{Gal}(K_1/\mathbb{Q})}.\end{aligned}$$

Let $\mathcal{K} \in \{F, F_{\mathfrak{p}}\}$. By what we explained in Sect. 5.1.2 (*cf.* Notation/Convention 5.2), the cyclic subgroup of $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathrm{Gal}(K_1/\mathbb{Q})}$ generated by $\mathscr{X}_K$ injects into $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K_1)^{\mathrm{Gal}(K_1/\mathbb{Q})}_{\mathcal{K}}$: we will not distinguish between $\mathscr{X}_K$ and its image in this vector space; moreover, we identify this element (hence $\mathscr{X}_K$ as well) with its image in $\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_{\mathcal{K}}$. With this convention in force, this shows, in particular, that $\mathscr{X}_K \in \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_F$. There is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_F & \longrightarrow & \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_{F_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(\mathbb{Q})\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \\
\downarrow{\scriptstyle \iota_{\mathbb{Q}\to K}} & & \downarrow{\scriptstyle \iota_{\mathbb{Q}\to K}} & & \downarrow{\scriptstyle \mathrm{res}_{K/\mathbb{Q}}} \\
\mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_F & \longrightarrow & \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/K)_{F_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{K,\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(K)\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}
\end{array} \tag{5.51}$$

in which, as before, the left and middle vertical arrows are the base change maps from (5.2) and the unlabelled maps are extensions of scalars. By a slight abuse of notation, we write $\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}$ (respectively, $\mathrm{AJ}_{K,\mathfrak{p}}$) also for the composition of the two upper (respectively, lower) horizontal maps. Since $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion, diagram (5.51) shows that $\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}(\mathscr{X}_K) \neq 0$ in $H^1_f(\mathbb{Q}, V_{\mathfrak{p}})$. Finally, triangle (3.19) with $L = \mathbb{Q}$ and $\star = \mathfrak{p}$ induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_F & \longrightarrow & \mathrm{CH}^{k/2}_{\mathrm{arith}}(X/\mathbb{Q})_{F_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}} & H^1_f(\mathbb{Q}, V_{\mathfrak{p}}) \\ & \searrow^{\Pi_{\mathcal{M},\mathbb{Q}}} & & \searrow^{\Pi_{\mathcal{M},\mathbb{Q},\mathfrak{p}}} & \uparrow_{\mathrm{reg}_{\mathfrak{p}}} \\ & & H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) & \longrightarrow & H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_{\mathfrak{p}} \end{array} \tag{5.52}$$

in which $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_{\mathfrak{p}}$ is the $F_{\mathfrak{p}}$-vector space from (2.38) and the unlabelled horizontal maps are extensions of scalars. Since $\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}(\mathscr{X}_K) \neq 0$ in $H^1_f(\mathbb{Q}, V_{\mathfrak{p}})$, diagram (5.52) gives $\Pi_{\mathcal{M},\mathbb{Q}}(\mathscr{X}_K) \neq 0$. □

From here on, set

$$\mathscr{Y}_{\mathcal{M}} := \frac{1}{(k-2)!} \cdot \Pi_{\mathcal{M},\mathbb{Q}}(\mathscr{X}_K) \in H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) \smallsetminus \{0\}. \tag{5.53}$$

Set $\mathscr{Y}_{\mathcal{M},K} := \iota_{\mathbb{Q}\to K}(\mathscr{Y}_{\mathcal{M}}) \in H^1_{\mathrm{mot}}(K, \mathcal{M}) \smallsetminus \{0\}$. As a consequence of triangle (3.19) with $L = K$ and Lemma 5.5, there is an equality

$$\mathrm{reg}_{K,\mathfrak{p}}(\mathscr{Y}_{\mathcal{M},K}) = y_{K,\mathfrak{p}} \tag{5.54}$$

for each prime $\mathfrak{p}$ of F above p.

The next result establishes, for each $\mathfrak{p} \mid p$, an isomorphism between $\Lambda_{\mathfrak{p}}(\mathbb{Q})$ and $\Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})}$.

Proposition 5.36 *For each prime $\mathfrak{p}$ of F above p, restriction induces an isomorphism*

$$\mathrm{res}_{K/\mathbb{Q}} : \Lambda_{\mathfrak{p}}(\mathbb{Q}) \xrightarrow{\simeq} \Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})}$$

of $\mathcal{O}_{\mathfrak{p}}$-modules.

Proof Let $\mathfrak{p}$ be a prime of F above p. Since $[K : \mathbb{Q}] = 2$ and p is odd, base change gives an isomorphism

$$\iota_{\mathbb{Q}\to K} : \mathrm{CH}^{k/2}_0(X/\mathbb{Q})_{\mathcal{O}_{\mathfrak{p}}} \xrightarrow{\simeq} \mathrm{CH}^{k/2}_0(X/K)^{\mathrm{Gal}(K/\mathbb{Q})}_{\mathcal{O}_{\mathfrak{p}}}$$

of $\mathcal{O}_{\mathfrak{p}}$-modules (*cf.* Sect. 5.1.2). On the other hand, there is a commutative square

$$\begin{array}{ccc} \mathrm{CH}^{k/2}_0(X/\mathbb{Q})_{\mathcal{O}_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{\mathbb{Q},\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(\mathbb{Q}) \\ \simeq \downarrow \iota_{\mathbb{Q}\to K} & & \downarrow \mathrm{res}_{K/\mathbb{Q}} \\ \mathrm{CH}^{k/2}_0(X/K)^{\mathrm{Gal}(K/\mathbb{Q})}_{\mathcal{O}_{\mathfrak{p}}} & \xrightarrow{\mathrm{AJ}_{K,\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})} \end{array}$$

in which the bottom horizontal arrow is surjective by [159, Lemma 5.8] and the injectivity of $\mathrm{res}_{K/\mathbb{Q}}$ follows as in the proof of Proposition 4.3. We conclude that $\mathrm{res}_{K/\mathbb{Q}}$ is necessarily surjective, hence an isomorphism. □

Under the assumptions of Proposition 5.35, we know that $y_{K,\mathfrak{p}} \in \Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})}$. Thus, in light of Proposition 5.36, we define

$$y_{\mathbb{Q},\mathfrak{p}} := \mathrm{res}_{K/\mathbb{Q}}^{-1}(y_{K,\mathfrak{p}}) \in \Lambda_{\mathfrak{p}}(\mathbb{Q}). \tag{5.55}$$

In particular, $y_{\mathbb{Q},\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion. Observe that the square

$$\begin{array}{ccc} H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})_{\mathfrak{p}} & \xrightarrow{\mathrm{reg}_{\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(\mathbb{Q})\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \\ \simeq \downarrow \iota_{\mathbb{Q}\to K} & & \simeq \downarrow \mathrm{res}_{K/\mathbb{Q}} \\ H^1_{\mathrm{mot}}(K,\mathcal{M})_{\mathfrak{p}}^{\mathrm{Gal}(K/\mathbb{Q})} & \xrightarrow{\mathrm{reg}_{K,\mathfrak{p}}} & \Lambda_{\mathfrak{p}}(K)^{\mathrm{Gal}(K/\mathbb{Q})}\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \end{array}$$

is commutative, so equality (5.54) guarantees that

$$\mathrm{reg}_{\mathfrak{p}}(\mathscr{Y}_{\mathcal{M}}) = y_{\mathbb{Q},\mathfrak{p}} \tag{5.56}$$

for each prime $\mathfrak{p}$ of F above p.

5.8.3 Proof of Conjecture 2.40, $r_{\mathrm{an}}(\mathcal{M}) = 1$

We are in a position to prove the main result of this section.

Theorem 5.37 *Assume that $r_{\mathrm{an}}(\mathcal{M}) = 1$ and that there exists $K \in \mathscr{I}_1(f,p)$ such that*

(1) *there is $K' \in \mathscr{I}_0(f^K,p)$ with $\mathrm{AJ}_{f^K,K',\mathfrak{p}}$ injective on $\mathrm{Heeg}_{K',N}^{\mathcal{G}_1'}$ for some $\mathfrak{p}\mid p$;*
(2) *the p-part of Conjecture 2.50 for $\mathcal{M}(f^K)$ over $\mathbb{Q}$ holds true.*

Then Conjecture 2.40 is true.

Proof Assume that $r_{\mathrm{an}}(\mathcal{M}) = 1$; this implies, by Lemma 2.34, that $r_{\mathrm{an}}(f^\sigma) = 1$ for all $\sigma \in \Sigma$. Let $\mathscr{B}$ be a basis of $H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})$ over F. By definition of the embedding ι_Σ in Remark 2.8, and bearing (2.22) and (5.29) in mind, we need to show that there exists $\mathscr{L}' \in F^\times$ such that

$$\sigma(\mathscr{L}') = \frac{L'(f^\sigma,k/2)}{(2\pi i)^{k/2-1}\cdot\Omega_{f^\sigma}\cdot \mathrm{Reg}^\sigma_{\mathscr{B}}(\mathcal{M})} \tag{5.57}$$

for all $\sigma \in \Sigma$.

Choose an imaginary quadatic field $K \in \mathscr{I}_1(f,p)$ satisfying conditions (1) and (2); then $r_{\mathrm{an}}(f/K) = 1$. Set $g := f^K$; again by Lemma 2.34, $r_{\mathrm{an}}(g^\sigma) = 0$ for all $\sigma \in \Sigma$. Theorem 5.10 ensures that for each $\sigma \in \Sigma$ there is an equality

$$\frac{L'(f^\sigma/K,k/2)\cdot(k-2)!\cdot u_K^2\cdot\sqrt{|D_K|}}{2^{2k-1}\cdot\pi^k\cdot(f^\sigma,f^\sigma)_{\Gamma_0(N)}\cdot\langle s'_{f^\sigma},s'_{f^\sigma}\rangle_{\mathrm{GS}}} = 1.$$

Thanks to the factorization in (5.17) for $L(f^\sigma/K,s)$ and the fact that $g^\sigma = (f^\sigma)^K$, we can write

$$\frac{L'(f^\sigma,k/2)\cdot(k-2)!\cdot u_K^2\cdot\sqrt{|D_K|}\cdot i^{k/2-1}\cdot\Omega_{g^\sigma}}{2^{3k/2}\cdot\pi^{k/2+1}\cdot(f^\sigma,f^\sigma)_{\Gamma_0(N)}\cdot\langle s'_{f^\sigma},s'_{f^\sigma}\rangle_{\mathrm{GS}}}=\left(L^{\mathrm{alg}}(g^\sigma,k/2)\right)^{-1}.$$

Properties (1) and (2) ensure that we can apply Theorem 5.24 to g. Therefore, there exists $\widetilde{\mathscr{L}'}\in F^\times$ such that

$$\sigma\left(\widetilde{\mathscr{L}'}\right)=\frac{L'(f^\sigma,k/2)\cdot\sqrt{|D_K|}\cdot i^{k/2-1}\cdot\Omega_{g^\sigma}}{\pi^{k/2+1}\cdot(f^\sigma,f^\sigma)_{\Gamma_0(N)}\cdot\langle s'_{f^\sigma},s'_{f^\sigma}\rangle_{\mathrm{GS}}}\tag{5.58}$$

for all $\sigma\in\Sigma$. Comparing (5.57) and (5.58), we see that it suffices to show that there exists $\mathscr{A}\in F^\times$ such that

$$\sigma(\mathscr{A})=\frac{\Omega_{f^\sigma}\cdot\Omega_{g^\sigma}\cdot\sqrt{|D_K|}}{\pi^2\cdot(f^\sigma,f^\sigma)_{\Gamma_0(N)}}\cdot\frac{\mathrm{Reg}^\sigma_{\mathscr{B}}(\mathcal{M})}{\langle s'_{f^\sigma},s'_{f^\sigma}\rangle_{\mathrm{GS}}}\tag{5.59}$$

for all $\sigma\in\Sigma$, for then (5.57) follows with $\mathscr{L}':=\widetilde{\mathscr{L}'}/\mathscr{A}$. To prove (5.59), we deal with the two factors on the right-hand side separately. By Lemma 5.26, $\Omega_f\Omega_g\sqrt{|D_K|}=i\Omega_f^+\Omega_f^-$ up to elements of $F^\times$; furthermore, $i\Omega_f^+\Omega_f^-/\pi^2(f,f)_{\Gamma_0(N)}$ belongs to $F^\times$ and satisfies the equality $\sigma\left(i\Omega_f^+\Omega_f^-/\pi^2(f,f)_{\Gamma_0(N)}\right)=i\Omega_{f^\sigma}^+\Omega_{f^\sigma}^-/\pi^2(f^\sigma,f^\sigma)_{\Gamma_0(N)}$ for all $\sigma\in\Sigma$ (see, *e.g.*, [66, §1.4]). This ensures that $\Omega_f\Omega_g\sqrt{|D_K|}/\pi^2(f,f)_{\Gamma_0(N)}$ belongs to $F^\times$ and $\sigma\left(\Omega_f\Omega_g\sqrt{|D_K|}/\pi^2(f,f)_{\Gamma_0(N)}\right)=\Omega_{f^\sigma}\Omega_{g^\sigma}\sqrt{|D_K|}/\pi^2(f^\sigma,f^\sigma)_{\Gamma_0(N)}$ for all $\sigma\in\Sigma$.

As for the other term, up to replacing the chosen basis $\mathscr{B}$ (consisting of a single element) with a multiple in $F^\times$, we can take $\mathscr{B}=\{\mathscr{Y}_{\mathcal{M}}\}$, where $\mathscr{Y}_{\mathcal{M}}\in H^1_{\mathrm{mot}}(\mathbb{Q},\mathcal{M})$ was defined in (5.53). Then Proposition 5.8 implies that $\mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M})/\langle s'_f,s'_f\rangle_{\mathrm{GS}}$ also belongs to $F^\times$ and satisfies $\sigma\left(\mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M})/\langle s'_f,s'_f\rangle_{\mathrm{GS}}\right)=\mathrm{Reg}^{\sigma}_{\mathscr{B}}(\mathcal{M})/\langle s'_{f^\sigma},s'_{f^\sigma}\rangle_{\mathrm{GS}}$ for all $\sigma\in\Sigma$. Summing up, the element

$$\mathscr{A}:=\frac{\Omega_f\cdot\Omega_g\cdot\sqrt{|D_K|}}{\pi^2\cdot(f,f)_{\Gamma_0(N)}}\cdot\frac{\mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M})}{\langle s'_f,s'_f\rangle_{\mathrm{GS}}}\in F^\times$$

satisfies (5.59), which concludes the proof. □

As observed in Sect. 2.11.2, Theorem 5.24 also shows that, under the assumptions appearing in its statement, the analytic rank 1 case of Conjecture 2.41 holds true.

5.9 *p*-TNC for $\mathcal{M}$ in analytic rank 1

We prove the main result of this paper, which says that, under the assumptions in Sect. 5.9.1 and those directly described in the statement of Theorem 5.42, the p-part of the TNC for $\mathcal{M}$ is true when $r_{\mathrm{an}}(\mathcal{M})=1$.

5.9.1 Assumptions

We work under the following assumption on the form f and the prime p; we freely use notation from previous sections.

Assumption 5.38 (1) Assumptions 4.5 and 5.25 are satisfied by $(f,\mathfrak{p})$ for all $\mathfrak{p}\mid p$.
(2) f is p-isolated.

(3) Conjecture 3.18 holds true for $L = \mathbb{Q}$, *i.e.*, the p-adic regulator

$$\mathrm{reg}_p : H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_{p\text{-int}} \longrightarrow H^1_f(\mathbb{Q}, T_p)$$

from (3.18) is an isomorphism of $\mathcal{O}_p$-modules.

These hypotheses could certainly be relaxed, but we prefer to state here assumptions that sit well in the general framework of Section 2. Notice that condition (3) in Assumption 5.38 implies the validity of Conjecture 2.50 for the number fields specified above.

Remark 5.39 Since N is assumed to be square-free, f has no complex multiplication.

5.9.2 Tamagawa ideals of $\mathcal{M}(f^K)$

Recall that $\mathcal{M}(f^K)$ denotes the motive of the twist f^K of f, which is a newform of level ND_K^2, as $(N, D_K) = 1$. In the statement that follows, ℓ is a prime number.

Proposition 5.40 (1) *If $\ell \mid N$, then* $\mathrm{Tam}_\ell^{(p)}\big(\mathcal{M}(f^K)\big) = \mathrm{Tam}_\ell^{(p)}(\mathcal{M}) = \mathcal{O}_p$.
(2) *If $\ell \mid D_K$, then* $\mathrm{Tam}_\ell^{(p)}\big(\mathcal{M}(f^K)\big) = \mathcal{O}_p$.

Proof Write A_p^K for the analogue of A_p relative to f^K; there is an isomorphism $A_p^K \simeq A_p$ of G_K-modules. If $\ell \mid N$ is a prime number, then ℓ splits in K, so the representations A_p and A_p^K are isomorphic as $G_{\mathbb{Q}_\ell}$-modules and the first equality in part (1) follows. To conclude the proof of (1), recall from Lemma 5.40 that $\mathrm{Tam}_\ell^{(p)}(\mathcal{M}) \simeq \mathcal{I}\big(\mathcal{W}_p^{\mathrm{Frob}_\ell = 1}\big)$ where $\mathcal{W}_p := A_p^{I_\ell}/(A_p^{I_\ell})_{\mathrm{div}}$ is a finite group. Now, $V_p^{I_\ell}$ is 1-dimensional (see, *e.g.*, [114, §12.4.4.2 and Lemma 12.4.5, (ii)]), so the divisible subgroup of $A_p^{I_\ell}$ contains a subgroup isomorphic to the divisible group $F_p/\mathcal{O}_p$. If $\mathcal{W}_p$ were not trivial, then $A_{\mathfrak{p}}[\mathfrak{p}]$ would be unramified for some $\mathfrak{p} \mid p$, which contradicts part (1) of Assumption 5.38. Therefore, $\mathcal{W}_p = 0$ and the second equality in (1) follows. Now we turn to part (2) (*cf.* also [152, Corollary 9.2]). Take a prime number $\ell \mid D_K$, let λ be the unique prime of K above ℓ and let I_ℓ (respectively, I_λ) be the inertia subgroup of $G_{\mathbb{Q}_\ell}$ (respectively, G_{K_λ}); note that the residue field of K at λ is $\mathbb{F}_\ell$. The prime ℓ ramifies in K, so $I_\ell = I_\lambda$. Since $p \neq 2$, we may split $H^1_{\mathrm{unr}}(K_\lambda, A_p) = H^1(\mathbb{F}_\ell, A_p^{I_\ell})$ into the direct sum of its eigenspaces for the action of $\mathcal{G} := \mathrm{Gal}(K_\lambda/\mathbb{Q}_\ell)$; the eigenspace on which $\mathcal{G}$ acts trivially (respectively, as -1) is $H^1_{\mathrm{unr}}(K_\lambda, A_p)^{\mathcal{G}}$ (respectively, is isomorphic to $H^1_{\mathrm{unr}}(K_\lambda, A_p^K)^{\mathcal{G}}$). Therefore, the canonical map

$$H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, A_p) \oplus H^1_{\mathrm{unr}}\big(\mathbb{Q}_\ell, A_p^K\big) \longrightarrow H^1_{\mathrm{unr}}(K_\lambda, A_p)$$

is an isomorphism. Keeping Proposition 2.76 in mind, since

$$\mathcal{I}\big(H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, A_p^K)\big) = \mathcal{I}\big(H^1_{\mathrm{unr}}(K_\lambda, A_p)\big) = \mathcal{O}_p$$

because $\ell \nmid N$, we deduce that

$$\mathrm{Tam}_\ell^{(p)}\big(\mathcal{M}(f^K)\big) = \mathcal{I}\big(H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, A_p^K)\big) = \mathcal{O}_p,$$

concluding the proof. □

5.9.3 Comparison of periods

Since f is p-isolated (*cf.* Assumption 5.38), by [60, Theorem 0.1] and Lemma 5.26 there is an equality

$$\left(\frac{\pi^2 \cdot (f,f)_{\Gamma_0(N)}}{\Omega_f \cdot \Omega_{f^K}}\right) = \mathcal{O}_p \tag{5.60}$$

of (fractional) $\mathcal{O}_p$-ideals. See also [66, §1.4] for details.

5.9.4 Proof of Theorem B

Retaining the assumptions of Proposition 5.35, now we state a technical result that will be used in the proof of our main theorem. We fix a prime $\mathfrak{p}$ of F above p.

Lemma 5.41 $\mathrm{length}_{\mathcal{O}_\mathfrak{p}}\left(\Lambda_\mathfrak{p}(\mathbb{Q})/y_{\mathbb{Q},\mathfrak{p}} \cdot \mathcal{O}_\mathfrak{p}\right) = \mathrm{length}_{\mathcal{O}_\mathfrak{p}}\left(\Lambda_\mathfrak{p}(K)/y_{K,\mathfrak{p}} \cdot \mathcal{O}_\mathfrak{p}\right)$.

Proof Since $\mathrm{res}_{K/\mathbb{Q}}(y_{\mathbb{Q},\mathfrak{p}}) = y_{K,\mathfrak{p}}$ by (5.55), one can proceed *mutatis mutandis* as in the proof of Proposition 5.34. □

We restate Theorem B in a precise form; as before, we employ notation from Remark 5.4 and Sect. 5.4. Recall that Assumption 5.38 is in force. Moreover, recall the sets of imaginary quadratic fields $\mathscr{I}_0(f,p)$ and $\mathscr{I}_1(f,p)$ introduced in Sects. 5.4.1 and 5.4.2, respectively.

Theorem 5.42 *Assume that* $r_{\mathrm{an}}(\mathcal{M}) = 1$ *and that there exists* $K \in \mathscr{I}_1(f,p)$ *such that*

(1) $\mathrm{AJ}_{K,\mathfrak{p}}$ *is injective on* $\mathrm{Heeg}_{K,N}^{\mathcal{G}_1}$ *for all* $\mathfrak{p} \mid p$;
(2) *there is* $K' \in \mathscr{I}_0(f^K,p)$ *with* $\mathrm{AJ}_{f^K,K',\mathfrak{p}}$ *injective on* $\mathrm{Heeg}_{K',N}^{\mathcal{G}_1'}$ *for some* $\mathfrak{p} \mid p$;
(3) *the p-part of Conjecture 2.50 for* $\mathcal{M}(f^K)$ *over* $\mathbb{Q}$ *holds true.*

Moreover, assume that

(4) *the p-part of Conjecture 2.50 over* $\mathbb{Q}$ *holds true.*

Then the following results hold:

(a) $r_{\mathrm{alg}}(\mathcal{M}) = 1$;
(b) $Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M}) = Ш_p^{\mathrm{Nek}}(\mathbb{Q},\mathcal{M})$;
(c) *the p-part of Conjecture 2.56 is true.*

Proof As was already observed, parts (a) and (b) are due to Nekovář. More precisely, part (a) was proved in Theorem 5.22. On the other hand, the arguments used in the proof of Proposition 5.21 show that there is an equality $Ш_\mathfrak{p}^{\mathrm{BK}}(\mathbb{Q},\mathcal{M}) = Ш_\mathfrak{p}^{\mathrm{Nek}}(\mathbb{Q},\mathcal{M})$ for each $\mathfrak{p} \mid p$ (*cf.* also the proof of Theorem 5.22), and then part (b) follows upon taking direct sums over all such $\mathfrak{p}$.

As we did in rank 0 in Theorem 5.29, we prove part (c) by checking equality (p-TNC$_{\mathscr{B}}$) in Theorem 2.83. First of all, the existence of $K \in \mathscr{I}_1(f,p)$ satisfying properties (2) and (3) guarantees, by Theorem 5.37, that Conjecture 2.40 is true in this case. Thus, keeping property (4) in mind and noting that Theorem 5.10 implies that the regulator we will be working with is non-zero (*cf.*), all the assumptions in Theorem 2.83 are verified (*cf.* Remark 2.84). As a further preliminary observation (implicit in the proofs of parts (a) and (b)), notice that if we fix $K \in \mathscr{I}_1(f,p)$ with the properties in the statement of the theorem

and let $\mathfrak{p}$ be a prime of F above p, then $r_{\mathrm{an}}(f/K) = 1$ and, thanks to property (1) and part (1) of Proposition 5.11, $y_{K,\mathfrak{p}}$ is not torsion.

By Theorem 5.22, $r_{\mathrm{alg}}(\mathcal{M}) = 1$. Let $\mathscr{Y}_{\mathcal{M}} \in H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M}) \smallsetminus \{0\}$ be as in (5.53). Thus, $\mathscr{B} := \{\mathscr{Y}_{\mathcal{M}}\}$ is an F-basis of $H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})$. Let $y_{\mathbb{Q},\mathfrak{p}} \in \Lambda_{\mathfrak{p}}(\mathbb{Q})$ be as in (5.55) and recall from (5.56) that $\mathrm{reg}_{\mathfrak{p}}(\mathscr{Y}_{\mathcal{M}}) = y_{\mathbb{Q},\mathfrak{p}}$. Recall the $\mathfrak{p}$-adic regulators

$$\mathrm{reg}_{\mathfrak{p}} : H^1_{\mathrm{mot}}(\mathbb{Q}, \mathcal{M})_\star \longrightarrow H^1_f(\mathbb{Q}, M_{\mathfrak{p}})$$

from (2.42) if $(\star, M) = (\mathfrak{p}, V)$ or from (3.17) if $(\star, M) = (\mathfrak{p}\text{-int}, T)$. The surjectivity of $\Pi_{\mathcal{M},\mathbb{Q},\mathfrak{p}}$ implies that $\mathrm{im}(\mathrm{reg}_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}(\mathbb{Q})$. It follows that condition (3) in Assumption 5.38 yields an equality $\Lambda_{\mathfrak{p}}(\mathbb{Q}) = H^1_f(\mathbb{Q}, T_{\mathfrak{p}})$ of free $\mathcal{O}_{\mathfrak{p}}$-modules of rank 1; once we view them in $H^1_f(\mathbb{Q}, V_{\mathfrak{p}})$, these two $\mathcal{O}_{\mathfrak{p}}$-lattices coincide with $\underline{H}^1_f(\mathbb{Q}, T_{\mathfrak{p}})$.

For each $\mathfrak{p} \mid p$, pick $\varpi_{\mathfrak{p}} \in \mathfrak{p} \smallsetminus (\mathfrak{p}^2 \cup \bigcup_{\mathfrak{p}' \mid p,\, \mathfrak{p}' \neq \mathfrak{p}} \mathfrak{p}')$; in particular, $\varpi_{\mathfrak{p}}$ is a uniformizer at $\mathfrak{p}$. Let us write

$$H^1_f(\mathbb{Q}, T_{\mathfrak{p}})/y_{\mathbb{Q},\mathfrak{p}} \cdot \mathcal{O}_{\mathfrak{p}} \simeq \mathcal{O}_{\mathfrak{p}}/(\mathfrak{p}\mathcal{O}_{\mathfrak{p}})^{f_{\mathfrak{p}}} = \mathcal{O}_{\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{f_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}) \tag{5.61}$$

for some $f_{\mathfrak{p}} \in \mathbb{N}$, then set $\mathbf{\Pi}_p := \prod_{\mathfrak{p} \mid p} \varpi_{\mathfrak{p}}^{f_{\mathfrak{p}}} \in \mathcal{O}_F \smallsetminus \{0\}$. Now define

$$y_{\mathbb{Q},p} := (y_{\mathbb{Q},\mathfrak{p}})_{\mathfrak{p} \mid p} \in \bigoplus_{\mathfrak{p} \mid p} H^1_f(\mathbb{Q}, T_{\mathfrak{p}}) = H^1_f(\mathbb{Q}, T_p)$$

and $\hat{y}_{\mathbb{Q},p} := \mathbf{\Pi}_p^{-1} \cdot y_{\mathbb{Q},p} \in H^1_f(\mathbb{Q}, T_p)$. Observe that, by construction, $\{\hat{y}_{\mathbb{Q},p}\}$ is an $\mathcal{O}_p$-basis of $H^1_f(\mathbb{Q}, T_p) = \underline{H}^1_f(\mathbb{Q}, T_p)$. Furthermore, with notation as in Sect. 2.23.2, $\mathrm{reg}_p(\mathscr{Y}_{\mathcal{M}}) = y_{\mathbb{Q},p}$, so that $\tilde{\mathscr{B}} = \{y_{\mathbb{Q},p}\}$. Therefore, there is an equality

$$\mathrm{A}_{\tilde{\mathscr{B}}} = (\mathbf{\Pi}_p). \tag{5.62}$$

Recall from Sect. 5.8.2 the element $\mathscr{Y}_{\mathcal{M},K} = \iota_{\mathbb{Q}\to K}(\mathscr{Y}_{\mathcal{M}}) \in H^1_{\mathrm{mot}}(K, \mathcal{M}) \smallsetminus \{0\}$. By Proposition 5.8 and our choice of p, there are equalities

$$\begin{aligned}\langle s'_f, s'_f \rangle_{\mathrm{GS}} &= \frac{\deg(\pi_N) \cdot \binom{k-2}{k/2-1}}{(-2D_K)^{k/2-1}} \cdot \langle \mathscr{Y}_{\mathcal{M},K}, \mathscr{Y}_{\mathcal{M},K} \rangle^K_{\mathrm{GS},\iota_F} \\ &= \frac{\deg(\pi_N) \cdot \binom{k-2}{k/2-1}}{(-2D_K)^{k/2-1}} \cdot \langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^K_{\mathrm{GS},\iota_F},\end{aligned}$$

where the superscripts indicate that the pairings are taken with respect to the ground field K. On the other hand, $\langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^K_{\mathrm{GS},\iota_F} = 2\langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^{\mathbb{Q}}_{\mathrm{GS},\iota_F}$, where the pairing on the right is taken relative to the ground field $\mathbb{Q}$ (see, *e.g.*, [17, §3.1.4]). Thus, since $p \nmid 2D_K \deg(\pi_N)$, we can replace the term $\langle s'_f, s'_f \rangle_{\mathrm{GS}}$ in Theorem 5.10 with $\binom{k-2}{k/2-1} \cdot \langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^{\mathbb{Q}}_{\mathrm{GS},\iota_F}$. Observe that Theorem 5.10 ensures that $\langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^{\mathbb{Q}}_{\mathrm{GS},\iota_F} \neq 0$ because $r_{\mathrm{an}}(f/K) = 1$; since $\mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M}) = \langle \mathscr{Y}_{\mathcal{M}}, \mathscr{Y}_{\mathcal{M}} \rangle^{\mathbb{Q}}_{\mathrm{GS},\iota_F}$, we have $\mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M}) \neq 0$.

Combining Theorem 5.10 with the period comparison formula (5.60) yields, for each prime $\mathfrak{p}$ of F above p, an equality

$$\left(\frac{\big((k/2-1)!\big)^2 \cdot L'(f/K, k/2)}{\pi^{k-2} \cdot \Omega_f \cdot \Omega_{fK} \cdot \mathrm{Reg}^{\iota_F}_{\mathscr{B}}(\mathcal{M})} \right) = \mathcal{O}_{\mathfrak{p}} \tag{5.63}$$

of fractional $\mathcal{O}_{\mathfrak{p}}$-ideals. Since equality (5.63) holds for each $\mathfrak{p} \mid p$, we get an equality

$$\begin{aligned}\left(\frac{\big((k/2-1)!\big)^2 \cdot L'(f/K, k/2)}{\pi^{k-2} \cdot \Omega_f \cdot \Omega_{f^K} \cdot \mathrm{Reg}_{\mathscr{B}}^{\iota_F}(\mathcal{M})}\right) &= (\mathbf{\Pi}_p^{-2}) \cdot (\mathbf{\Pi}_p^2) \cdot \mathcal{O}_p \\ &= \big(\det(\mathrm{A}_{\mathscr{B}})\big)^{-2} \cdot \Big(\mathcal{I}_{\mathcal{O}_p}\big(H_f^1(\mathbb{Q}, T_p)/y_{\mathbb{Q},p} \cdot \mathcal{O}_p\big)\Big)^2 \\ &= \big(\det(\mathrm{A}_{\mathscr{B}})\big)^{-2} \cdot \mathcal{I}_{\mathcal{O}_p}\big(Ш_p^{\mathrm{BK}}(K, \mathcal{M})\big)\end{aligned} \tag{5.64}$$

of fractional $\mathcal{O}_p$-ideals, where the second equality follows from (5.61) and (5.62), while the third, in light of Definition 2.60 and Lemma 5.41, is a consequence of Theorem 5.33 and Proposition 5.34.

The p-adic Galois representation attached to f^K is the twist of ρ_p by ϵ_K, so Assumption 4.5 holds for f^K. Thus, by Theorem 5.29 and Proposition 5.40, there is an equality

$$\left(\frac{(k/2-1)! \cdot L(f^K, k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_{f^K}}\right) = \mathcal{I}_{\mathcal{O}_p}\Big(Ш_p^{\mathrm{BK}}\big(\mathbb{Q}, \mathcal{M}(f^K)\big)\Big) \tag{5.65}$$

of fractional $\mathcal{O}_p$-ideals. Combining the factorization

$$L'(f/K, k/2) = L'(f, k/2) \cdot L(f^K, k/2)$$

and formula (5.60) with equalities (5.64) and (5.65), we obtain an equality

$$\begin{aligned}\left(\frac{(k/2-1)! \cdot L'(f, k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_f \cdot \mathrm{Reg}_{\mathscr{B}}^{\iota_F}(\mathcal{M})}\right) = \big(\det(\mathrm{A}_{\mathscr{B}})\big)^{-2} \cdot \mathcal{I}_{\mathcal{O}_p}\big(Ш_p^{\mathrm{BK}}(K, \mathcal{M})\big) \\ \cdot \mathcal{I}_{\mathcal{O}_p}^{-1}\Big(Ш_p^{\mathrm{BK}}\big(\mathbb{Q}, \mathcal{M}(f^K)\big)\Big).\end{aligned}$$

Furthermore, the splitting $Ш_p^{\mathrm{BK}}(K, \mathcal{M}) = Ш_p^{\mathrm{BK}}(\mathbb{Q}, \mathcal{M}) \oplus Ш_p^{\mathrm{BK}}\big(\mathbb{Q}, \mathcal{M}(f^K)\big)$ of Shafarevich–Tate groups, which is a consequence of an analogous decomposition of Selmer groups (see, *e.g.*, [92, Proposition 6.2]), induces an equality

$$\left(\frac{(k/2-1)! \cdot L'(f, k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_f \cdot \mathrm{Reg}_{\mathscr{B}}^{\iota_F}(\mathcal{M})}\right) = \big(\det(\mathrm{A}_{\mathscr{B}})\big)^{-2} \cdot \mathcal{I}_{\mathcal{O}_p}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q}, \mathcal{M})\big). \tag{5.66}$$

of fractional $\mathcal{O}_p$-ideals.

Now we compare (5.66) with equality (p-TNC$_{\mathscr{B}}$) in Theorem 2.83. Thanks to Theorem 5.37, we already know that

$$\frac{L^*(\mathcal{M}, 0)}{\Omega_\infty \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})} = \left(\frac{L'(f^\sigma, k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_{f^\sigma} \cdot \mathrm{Reg}_{\mathscr{B}}^{\sigma}(\mathcal{M})}\right)_{\sigma \in \Sigma}$$

belongs to $F^\times$ in the sense of Remark 2.8; more explicitly, there is an equality

$$\iota_\Sigma\left(\frac{L'(f, k/2)}{(2\pi i)^{k/2-1} \cdot \Omega_f \cdot \mathrm{Reg}_{\mathscr{B}}^{\iota_F}(\mathcal{M})}\right) = \frac{L^*(\mathcal{M}, 0)}{\Omega_\infty \cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}. \tag{5.67}$$

Combining (5.66) and (5.67), we get an equality

$$\left(\frac{(k/2-1)!\cdot L^*(\mathcal{M},0)}{\Omega_\infty\cdot \mathrm{Reg}_{\mathscr{B}}(\mathcal{M})}\right)=\big(\det(\mathrm{A}_{\mathscr{B}})\big)^{-2}\cdot \mathcal{I}_{\mathcal{O}_p}\big(Ш_p^{\mathrm{BK}}(\mathbb{Q},\mathcal{M})\big) \tag{5.68}$$

of fractional $\mathcal{O}_p$-ideals. As we explained in the proof of Theorem 5.29, $\mathrm{Tors}_p(\mathcal{M})=\mathcal{O}_p$. In addition, if $\ell\,|\,N$, then $\mathrm{Tam}_\ell^{(p)}(\mathcal{M})=\mathcal{O}_p$ by part (1) of Proposition 5.40, while $\mathrm{Tam}_p^{(p)}(\mathcal{M})=\mathcal{O}_p$ by (2.77) and $\mathrm{Tam}_\infty^{(p)}(\mathcal{M})=\mathcal{O}_p$ by (2.78). Finally, $\mathcal{I}_p(\gamma_f)=\mathcal{O}_p$ by (5.35). Therefore, equality (5.68) coincides with (p-TNC$_{\mathscr{B}}$) (or, rather, with its equivalent form (p-TNC$_{\mathscr{B}}$-bis) in Remark 2.87) in our setting, and the proof is complete. □

6 On the structure of Selmer groups

As an application of Theorem 4.14, we deduce results on the structure of Selmer groups of modular forms. As will be apparent, these results basically follow from [97].

Fix a newform f with Hecke field F and a prime $\mathfrak{p}$ of F above p that satisfy Assumption 4.5. For any number field L, let

$$r_{\mathfrak{p}}(f/L):=\mathrm{corank}_{\mathcal{O}_{\mathfrak{p}}}H^1_f(L,A_{\mathfrak{p}}) \tag{6.1}$$

be the corank of $H^1_f(L,A_{\mathfrak{p}})$ over $\mathcal{O}_{\mathfrak{p}}$. Set also $r_{\mathfrak{p}}(f):=r_{\mathfrak{p}}(f/\mathbb{Q})$.

Throughout this section, K is an imaginary quadratic field in which all the prime factors of Np split.

6.1 Vanishing order of $\kappa_{f,\infty}$

Let $\kappa_{f,\infty}$ be the Kolyvagin set attached to $f,\mathfrak{p},K$ from (4.12). By Theorem 4.14, $\kappa_{f,\infty}\neq\{0\}$. For every $n\in\Lambda_{\mathrm{Kol}}(f)=\Lambda_{\mathrm{Kol}}(f,\mathfrak{p},K)$, denote by $\nu(n)$ the number of prime factors of n and let $M(n)$ be the Kolyvagin index of n that was introduced in (4.7).

Definition 6.1 The *vanishing order* of $\kappa_{f,\infty}$ is

$$\nu_\infty:=\min\big\{\nu(n)\mid n\in\Lambda_{\mathrm{Kol}}(f)\text{ and }c_M(f,n)\neq 0\text{ for some }M\le M(n)\big\}\in\mathbb{N}.$$

The following result will be used in the proof of Theorem 9.2.

Proposition 6.2 *If $y_{K,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p}}$-torsion, then $\nu_\infty\ge 1$.*

Proof By part (2) of Proposition 4.8, the $\mathcal{O}_{\mathfrak{p}}$-module $\Lambda_{\mathfrak{p}}(K)$ is free, so $y_{K,\mathfrak{p}}=0$. It follows from Proposition 4.3 that $c_M(f,1)=0$ for all M, whence $\nu_\infty\ge 1$. □

As before, let $\varepsilon(f)\in\{\pm1\}$ be the root number of f. It is convenient to consider the sign

$$\varepsilon_\infty:=\mathrm{sign}\big(\varepsilon(f)\cdot(-1)^{\nu_\infty+1}\big)\in\{\pm\}, \tag{6.2}$$

which will appear in Theorem 6.4.

6.2 A structure theorem for $H^1_f(K, A_\mathfrak{p})$

From now on, let $H^1_f(K, A_\mathfrak{p})^\pm$ denote the ± 1-eigenspaces of complex conjugation acting on $H^1_f(K, A_\mathfrak{p})$ and write

$$r_\mathfrak{p}^\pm(f/K) := \mathrm{corank}_{\mathcal{O}_\mathfrak{p}} H^1_f(K, A_\mathfrak{p})^\pm$$

for the corresponding coranks over $\mathcal{O}_\mathfrak{p}$. Observe that

$$r_\mathfrak{p}(f/K) = r_\mathfrak{p}^+(f/K) + r_\mathfrak{p}^-(f/K). \tag{6.3}$$

6.2.1 A lemma on p^m-torsion

The next auxiliary result will be used in the proof of the structure theorem for $H^1_f(K, A_\mathfrak{p})$ (Theorem 6.4).

Lemma 6.3 *For all $m \in \mathbb{N}$ there is a Galois-equivariant identification*

$$H^1_f\big(K, A_\mathfrak{p}[p^m]\big) = H^1_f(K, A_\mathfrak{p})[p^m].$$

Proof By [94, Lemma 2.4, (1)], $H^0(K, A_\mathfrak{p}) = 0$, so the inclusion $A_\mathfrak{p}[p^m] \hookrightarrow A_\mathfrak{p}$ induces an identification

$$H^1(K, A_\mathfrak{p})[p^m] = H^1\big(K, A_\mathfrak{p}[p^m]\big). \tag{6.4}$$

By definition, $H^1_f\big(K, A_\mathfrak{p}[p^m]\big)$ consists of the elements of $H^1\big(K, A_\mathfrak{p}[p^m]\big)$ whose image in $H^1(K, A_\mathfrak{p})$ lies in $H^1_f(K, A_\mathfrak{p})$, and then the lemma follows from (6.4). □

6.2.2 Structure theorem

Recall that p, which is unramified in F, is a uniformizer for $\mathcal{O}_\mathfrak{p}$. As in Sect. 5.7, let $\varepsilon \in \{\pm\}$ be the sign of the root number of f. Write

$$H^1_f(K, A_\mathfrak{p})^\pm \simeq (F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})^{r_\mathfrak{p}^\pm(f/K)} \oplus \mathcal{X}_\mathfrak{p}^\pm$$

where $\mathcal{X}_\mathfrak{p}^\pm$ is a finite $\mathcal{O}_\mathfrak{p}$-module, then introduce splittings

$$\mathcal{X}_\mathfrak{p}^{-\varepsilon} \simeq (\mathcal{O}_\mathfrak{p}/p^{n_1}\mathcal{O}_\mathfrak{p})^2 \oplus (\mathcal{O}_\mathfrak{p}/p^{n_3}\mathcal{O}_\mathfrak{p})^2 \oplus \dots$$

and

$$\mathcal{X}_\mathfrak{p}^{\varepsilon} \simeq (\mathcal{O}_\mathfrak{p}/p^{n_2}\mathcal{O}_\mathfrak{p})^2 \oplus (\mathcal{O}_\mathfrak{p}/p^{n_4}\mathcal{O}_\mathfrak{p})^2 \oplus \dots$$

of $\mathcal{O}_\mathfrak{p}$-modules. Finally, let the integers $N_i \in \mathbb{N}$ be defined as in Sect. 5.7 (*cf.* Theorem 5.32) and let $\varepsilon_\infty \in \{\pm\}$ be the sign from (6.2).

Theorem 6.4 (1) $r_\mathfrak{p}^{\varepsilon_\infty}(f/K) = \nu_\infty + 1$ *and* $r_\mathfrak{p}^{-\varepsilon_\infty}(f/K) \le \nu_\infty$.
(2) $\nu_\infty = \max\big\{r_\mathfrak{p}^+(f/K), r_\mathfrak{p}^-(f/K)\big\} - 1$.
(3) $n_i = N_i$ *for all* $i > \nu_\infty + 1$.
(4) $0 \le \nu_\infty - r_\mathfrak{p}^{-\varepsilon_\infty}(f/K) \equiv 0 \pmod 2$.

Proof Parts (1) and (3) are consequences of the techniques exploited in the proof of [97, Theorem 8.4], using the decomposition as $\mathcal{O}_{\mathfrak{p}}$-modules rather than as groups; details are left to the reader. Part (2) is a restatement of (1), since $\max\{r_{\mathfrak{p}}^{+}(f/K), r_{\mathfrak{p}}^{-}(f/K)\} = r_{\mathfrak{p}}^{\varepsilon_\infty}(f/K)$, by (1). By the second statement in (1), it remains to show the congruence in (4). We write $r = r_{\mathfrak{p}}^{-\varepsilon_\infty}(f/K)$ to simplify the notation. As above, consider the decomposition

$$H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big) = H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{+} \oplus H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{-}$$

under the action of $\mathrm{Gal}(K/\mathbb{Q})$. By Lemma 6.3, the invariants of $H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{-\varepsilon_\infty}$ are those of $\mathcal{X}_{f,\mathfrak{p}}^{-\varepsilon_\infty}$ shifted by r terms all equal to M coming from the divisible subgroup $(F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})^r$ of $\mathrm{Sel}_{\mathfrak{p}}(f/K)^{-\varepsilon_\infty}$. Thus, by (3), the last $\nu_\infty+1+r$ invariants of $H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{-\varepsilon_\infty}$ are in even number. The existence of a Flach–Cassels pairing on $H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{-\varepsilon_\infty}$ that is alternating and non-degenerate ([40], [97, Section 6]) ensures that the total number of the invariants of $H_f^1\big(K, A_{\mathfrak{p}}[p^M]\big)^{-\varepsilon_\infty}$ is even, and therefore $\nu_\infty + r$ is even. □

7 Parity results

We prove a p-parity result for modular forms (Sect. 7.1) and then deduce from it part (1) of Theorem D (Sect. 7.2). We fix throughout a newform f with Hecke field F and a prime $\mathfrak{p}$ of F above p satisfying Assumption 4.5.

7.1 A $\mathfrak{p}$-parity result

Let $r_{\mathfrak{p}}(f)$ be defined as in (6.1) and, as in Sect. 2.9, let $\varepsilon(f)$ be the root number of f. The following is a $\mathfrak{p}$-parity result for f.

Theorem 7.1 $(-1)^{r_{\mathfrak{p}}(f)} = \varepsilon(f)$.

Proof Choose an imaginary quadratic field K in which all the prime factors of Np split. Combining the inflation-restriction exact sequence

$$\begin{aligned} 0 \longrightarrow H^1\big(\mathrm{Gal}(K/\mathbb{Q}), A_{\mathfrak{p}}(K)\big) &\longrightarrow H^1(\mathbb{Q}, A_{\mathfrak{p}}) \\ &\longrightarrow H^1(K, A_{\mathfrak{p}})^{\mathrm{Gal}(K/\mathbb{Q})} \longrightarrow H^2\big(\mathrm{Gal}(K/\mathbb{Q}), A_{\mathfrak{p}}(K)\big) \end{aligned}$$

and the triviality of $A_{\mathfrak{p}}(K)$ from [94, Lemma 2.4, (1)] gives an identification

$$H^1(\mathbb{Q}, A_{\mathfrak{p}}) = H^1(K, A_{\mathfrak{p}})^{+}. \tag{7.1}$$

By keeping track of local conditions, one can then check that (7.1) induces an identification

$$H_f^1(\mathbb{Q}, A_{\mathfrak{p}}) = H_f^1(K, A_{\mathfrak{p}})^{+}. \tag{7.2}$$

The desired equality follows from Theorem 6.4 by an easy combinatorial argument. □

Remark 7.2 The analogue of Theorem 7.1 for a large class of elliptic curves and, more generally, Hilbert modular forms of parallel weight has been proved by Nekovář [113, 115, 116]. It is worth pointing out that a crucial ingredient in Nekovář's arguments is the fact that the truth of the parity conjecture is constant in Hida families: Theorem 7.1 can be seen as an incarnation of this general principle.

7.2 Proof of part (1) of Theorem D

Recall the set Σ of real (equivalently, complex) embeddings of F. For all $\sigma \in \Sigma$, the representation of $G_{\mathbb{Q}}$ attached to f and $\mathfrak{p}$ is equivalent (over $\bar{\mathbb{Q}}_p$) to the representation of $G_{\mathbb{Q}}$ attached to f^σ and the prime $\sigma(\mathfrak{p})$ of the Hecke field $\sigma(F)$ of f^σ. In particular, $r_{\mathfrak{p}}(f) = r_{\sigma(\mathfrak{p})}(f^\sigma)$ for all $\sigma \in \Sigma$. Then, by Theorem 7.1, $\varepsilon(f^\sigma)$ is constant as σ varies in Σ, which means, by (2.26), that the parity of $r_{\mathrm{an}}(f^\sigma)$ is constant as σ varies in Σ. In light of equality (2.27) and Theorem 7.1, we get the congruence

$$r_{\mathfrak{p}}(f) \equiv r_{\mathrm{an}}(\mathcal{M}) \pmod 2.$$

Now, by Corollary 2.68, $r_{\mathfrak{p}}(f) = r_{\mathrm{alg}}(\mathcal{M})$, and the proof is complete. □

8 Converse theorems

We prove p-converse theorems for modular forms (Sect. 8.1) and then deduce from them part (2) of Theorem D (Sect. 8.2). We fix throughout a newform f with Hecke field F and a prime $\mathfrak{p}$ of F above p satisfying Assumption 4.5.

8.1 $\mathfrak{p}$-converse theorems

For any number field L, define the $F_{\mathfrak{p}}$-vector space

$$X_{\mathfrak{p}}(L) := \Lambda_{\mathfrak{p}}(L) \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_{\mathfrak{p}}(L) \otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}}.$$

8.1.1 Results over K

The following result is a higher weight analogue of the part of [166, Theorem 1.3] that deals with Mordell–Weil ranks and Shafarevich–Tate groups.

Theorem 8.1 *Let K be an imaginary quadratic field in which all the prime factors of Np split. If $r_{\mathfrak{p}}(f/K) = 1$, then*

(1) *$y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion;*
(2) *$\dim_{F_{\mathfrak{p}}}\big(X_{\mathfrak{p}}(K)\big) = 1$;*
(3) *$Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K, \mathcal{M})$ is finite.*

Proof If (1) holds, then (2) and (3) follow from Theorem 5.20, so we need to show only (1). If $r_{\mathfrak{p}}(f/K) = 1$, then equality (6.3) implies that

$$\max\big\{r^+_{\mathfrak{p}}(f/K), r^-_{\mathfrak{p}}(f/K)\big\} = 1.$$

By part (2) of Theorem 6.4, this is equivalent to $\nu_\infty = 0$, *i.e.*, $c_M(f, 1) \neq 0$ for some $M \geq 1$. On the other hand, Proposition 4.3 says that $c_M(f, 1) = \iota_{K,M}\big([y_{K,\mathfrak{p}}]_M\big)$, so we surmise that $y_{K,\mathfrak{p}} \neq 0$. By part (2) of Proposition 4.8, the $\mathcal{O}_{\mathfrak{p}}$-module $\Lambda_{\mathfrak{p}}(K)$ is free of finite rank, hence $y_{K,\mathfrak{p}}$ is not torsion, as was to be shown. □

For another higher weight converse to the Kolyvagin–Gross–Zagier theorem, the reader is referred to [161, Theorem 2].

Recall that if K is a field as in Theorem 8.1, then $r_{\mathrm{an}}(f/K) \geq 1$. The following is a $\mathfrak{p}$-converse result over K.

Corollary 8.2 *Let K be an imaginary quadratic field in which all the prime factors of Np split. Assume that $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ is non-degenerate on $\mathrm{Heeg}_{K,N}\otimes_{\mathbb{Z}}\mathbb{R}$. If $r_{\mathfrak{p}}(f/K)=1$, then $r_{\mathrm{an}}(f/K)=1$.*

Proof By part (1) of Theorem 8.1, $y_{K,\mathfrak{p}}$ is not $\mathcal{O}_{\mathfrak{p}}$-torsion. Since we are assuming that $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ is non-degenerate on $\mathrm{Heeg}_{K,N}\otimes_{\mathbb{Z}}\mathbb{R}$, the claim follows from part (2) of Proposition 5.11. □

Focusing now on the case where the base field is $\mathbb{Q}$, we can prove an analogue in higher weight of the algebraic part of [166, Theorem 1.4, (i)].

Theorem 8.3 *If $r_{\mathfrak{p}}(f)=1$, then*

(1) $\dim_{F_{\mathfrak{p}}}\big(X_{\mathfrak{p}}(\mathbb{Q})\big)=1$;
(2) $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(\mathbb{Q},\mathcal{M})$ *is finite.*

Proof Since $r_{\mathfrak{p}}(f)=1$, it follows from Theorem 7.1 that $\varepsilon(f)=-1$. Choose an imaginary quadratic field K such that

- all the prime factors of Np split in K;
- $r_{\mathrm{an}}(f^K)=0$.

The existence of such a K is guaranteed by [20, p. 543, Theorem, (ii)]. Let $A^K_{\mathfrak{p}}$ be the analogue for f^K of the $\mathcal{O}_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$ associated with f. By Theorem 5.28 with f^K in place of f, the Selmer group $H^1_f\big(\mathbb{Q},A^K_{\mathfrak{p}}\big)$ is finite, so $r_{\mathfrak{p}}(f^K)=0$. It can be checked (see, *e.g.*, the proof of [92, Proposition 6.2]) that there is a canonical identification

$$H^1_f\big(\mathbb{Q},A^K_{\mathfrak{p}}\big)=H^1_f(K,A_{\mathfrak{p}})^-. \tag{8.1}$$

Combining (6.3), (7.2) and (8.1), we obtain

$$r_{\mathfrak{p}}(f/K)=r_{\mathfrak{p}}(f)+r_{\mathfrak{p}}(f^K)=1, \tag{8.2}$$

and then Theorem 8.1 ensures that $\dim_{F_{\mathfrak{p}}}\big(X_{f,\mathfrak{p}}(K)\big)=1$ and $Ш^{\mathrm{Nek}}_{\mathfrak{p}}(K,\mathcal{M})$ is finite. Finally, part (1) and part (2) of the theorem follow from [159, Theorem 5.26] and [159, Proposition 5.25], respectively. □

8.1.2 Assumption (GS) *and results over* $\mathbb{Q}$

To complete the picture, we prove a $\mathfrak{p}$-converse result over $\mathbb{Q}$ that can be regarded as a higher weight counterpart of [150, Theorem A], [157, Theorem A] and [166, Theorem 1.4, (i)]. To do this, with notation as in Sect. 5.4.2, we need to introduce hypotheses concerning, in particular, the non-degeneracy of the Gillet–Soulé height pairings:

(GS) there is $K\in\mathscr{S}_1(f,p)$ such that $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ is non-degenerate on $\mathrm{Heeg}_{K,N}\otimes_{\mathbb{Z}}\mathbb{R}$.

Now we can prove

Theorem 8.4 *If $r_{\mathfrak{p}}(f)=1$ and* (GS) *holds, then $r_{\mathrm{an}}(f)=1$.*

Proof On the one hand, Theorem 5.28 gives $r_{\mathfrak{p}}(f^K)=0$, and then $r_{\mathfrak{p}}(f/K)=1$ by the first equality in (8.2). Therefore, we can apply Corollary 8.2 to deduce that $r_{\mathrm{an}}(f/K)=1$.

On the other hand, factorization (5.17) yields the equality

$$r_{\mathrm{an}}(f/K) = r_{\mathrm{an}}(f) + r_{\mathrm{an}}(f^K),$$

whence $r_{\mathrm{an}}(f) = 1$. □

Remark 8.5 If we knew that Gillet–Soulé height pairings are non-degenerate (at least on the $\mathbb{R}$-vector space $\mathrm{Heeg}_{K,N} \otimes_{\mathbb{Z}} \mathbb{R}$, or in full generality, as predicted by the conjectures in [4,12,50]), then Corollary 8.2 and Theorem 8.4 would become unconditional. Unfortunately, non-degeneracy results of this kind appear to lie well beyond the scope of currently available techniques.

Remark 8.6 Recently, Burungale and Tian proved a p-converse result for CM elliptic curves over $\mathbb{Q}$ at good ordinary primes p ([24, Theorem 1.2]). It would be desirable to obtain $\mathfrak{p}$-converse theorems (possibly conditional, like Theorem 8.4, on the non-degeneracy of suitable height pairings) for higher weight CM newforms.

8.2 Proof of part (2) of Theorem D

Recall that we are assuming that

- Conjecture 2.50 holds true for p and $\mathbb{Q}$;
- condition (GS) from Sect. 8.1.2 is satisfied.

With notation as above, if $r_{\mathrm{alg}}(\mathcal{M}) = 1$, then $r_{\mathfrak{p}}(f) = 1$ by Corollary 2.68 (with $\star = \mathfrak{p}$), so Theorem 8.4 gives $r_{\mathrm{an}}(f) = 1$. By Lemma 2.34, $r_{\mathrm{an}}(\mathcal{M}) = 1$, as desired. □

9 Higher rank results

In this final section, we collect higher rank results for f and its motive $\mathcal{M}$. As before, we require throughout that f and the prime $\mathfrak{p}$ of F above p satisfy Assumption 4.5.

9.1 Higher rank results for f

We begin with results on f, in particular on the invariant $r_{\mathfrak{p}}(f)$ from (6.1). We use notation from Sect. 5.4.1.

9.1.1 Assumption (reg)

According to the sign of the root number of f, we will need to assume one of two different sets of hypotheses. The first is (GS) from Sect. 8.1.2, whereas the second takes care, in addition, of the injectivity (at least on Heegner modules) of $\mathfrak{p}$-adic regulators over imaginary quadratic fields:

(reg) there is $K \in \mathscr{I}_0(f,p)$, of discriminant D_K, such that

- $\langle \cdot,\cdot \rangle_{\mathrm{GS}}$ is non-degenerate on $\mathrm{Heeg}_{K,N} \otimes_{\mathbb{Z}} \mathbb{R}$,
- $\mathrm{reg}_{f^K,K_1',\mathfrak{p}}$ is injective on $\mathrm{Heeg}_{K',ND_K^2}$ for every imaginary quadratic field K' in which all the prime factors of ND_Kp split.

Here $\mathrm{reg}_{f^K,K_1',\mathfrak{p}}$ is the counterpart of the $\mathfrak{p}$-adic regulator $\mathrm{reg}_{K_1',\mathfrak{p}}$ relative to the motive $\mathcal{M}(f^K)$. As before (*cf.* Remark 8.5), note that the results in [20] guarantee that if $\varepsilon(f) = -1$, then there is always an imaginary quadratic field satisfying the first two conditions in (reg).

Remark 9.1 Since $(N, D_K) = 1$, the level of f^K is ND_K^2: this is the reason why in (reg) we consider Heegner modules of level ND_K^2.

9.1.2 Higher rank results for f

The following result is a higher weight analogue of [166, Theorem 1.4, (ii)]; in Sect. 9.2 we shall deduce from it analogous results for $\mathcal{M}$.

Theorem 9.2 *Assume that either*

- $\varepsilon(f) = -1$ *and* (GS) *holds*

or

- $\varepsilon(f) = +1$ *and* (reg) *holds.*

If $r_{\mathrm{an}}(f) > 1$, *then*

$$r_{\mathfrak{p}}(f) \in \left\{ 2n + \frac{1-\varepsilon(f)}{2} \;\middle|\; n \in \mathbb{Z}_{\geq 1} \right\}.$$

In particular, $r_{\mathfrak{p}}(f) \geq \dfrac{5-\varepsilon(f)}{2}$.

Proof We need to show that

- $r_{\mathfrak{p}}(f) \geq 3$ if $\varepsilon(f) = -1$,
- $r_{\mathfrak{p}}(f) \geq 2$ if $\varepsilon(f) = +1$.

Assume first that $\varepsilon(f) = -1$ and (GS) holds. By Theorem 7.1, $r_{\mathfrak{p}}(f)$ is odd. If $r_{\mathfrak{p}}(f) = 1$, then $r_{\mathrm{an}}(f) = 1$ by Theorem 8.4, so $r_{\mathfrak{p}}(f) \in \{2n+1 \mid n \geq 1\}$.

Assume now that $\varepsilon(f) = +1$ and (reg) holds; recall the vanishing order ν_∞ of $\kappa_{f,\infty}$ introduced in Definition 6.1. By Theorem 7.1, $r_{\mathfrak{p}}(f)$ is even. Choose an imaginary quadratic field K satisfying (reg). Then $r_{\mathrm{an}}(f/K) > 1$, so part (2) of Proposition 5.11 guarantees that $y_{K,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p}}$-torsion. It follows from Proposition 6.2 that $\nu_\infty \geq 1$; equivalently, by part (2) of Theorem 6.4, we obtain

$$\max\left\{r_{\mathfrak{p}}^+(f/K), r_{\mathfrak{p}}^-(f/K)\right\} \geq 2. \tag{9.1}$$

On the other hand, since $r_{\mathrm{an}}(f^K) = 1$, by [20, p. 543, Theorem, (ii)] there exists an imaginary quadratic field K' (which we fix) such that

- all the prime factors of $ND_K p$ split in K';
- $r_{\mathrm{an}}(f^K/K') = 1$.

Using the injectivity of $\mathrm{reg}_{f^K,K_1',\mathfrak{p}}$ on Heeg_{K,ND_K^2}, we apply Theorem 5.20 to f^K and obtain, in particular, $r_{\mathfrak{p}}(f^K/K') = 1$. Finally, reasoning as, *e.g.*, in the proof of [159, Theorem 5.27], we get $r_{\mathfrak{p}}(f^K) = 1$, and then the equality $r_{\mathfrak{p}}^-(f/K) = r_{\mathfrak{p}}(f^K)$ combined with (9.1) gives $r_{\mathfrak{p}}(f) = r_{\mathfrak{p}}^+(f/K) \in \{2n \mid n \geq 1\}$. □

Remark 9.3 We sketch an alternative approach to the $\varepsilon(f) = +1$ part of Theorem 9.2 that does not use the injectivity on Heeg_{K,ND_K^2} of the $\mathfrak{p}$-adic regulator, but relies instead on certain conjectural (non-)vanishing properties of p-adic L-functions of modular forms.

Namely, let g be a weight $k \geq 4$ newform on $\Gamma_0(M)$ with $p \nmid M$ and let $L_p(g,s)$ be the p-adic L-function of g in the sense of Mazur–Tate–Teitelbaum ([102, Ch. I, §13]; for simplicity, we suppress dependence of $L_p(g,s)$ on the "allowable p-root for g", *cf.* [102, Ch. I, §12]). We normalize $L_p(g,s)$ as in [52, p. 430]. It is conjectured in [102, Ch. I, §16] that, in our non-exceptional setting (*cf.* [102, Ch. I, §15]), $\mathrm{ord}_{s=k/2} L_p(g,s) = r_{\mathrm{an}}(g)$. Here we just assume the following implication:

$$r_{\mathrm{an}}(g) = 1 \implies \mathrm{ord}_{s=\frac{k}{2}} L_p(g,s) = 1. \tag{9.2}$$

With notation as in the proof of Theorem 9.2, rather than using the last condition in (reg), we combine the equality $r_{\mathrm{an}}(f^K) = 1$ and (9.2) to get $\mathrm{ord}_{s=k/2} L_p(f^K, s) = 1$. Then, keeping Corollary 2.68 in mind, [111, Theorem C, (2)] gives $r_{\mathfrak{p}}(f^K) = 1$, and by (9.1) we conclude that $r_{\mathfrak{p}}(f) \in \{2n \mid n \geq 1\}$.

9.2 Proof of parts (3) and (4) of Theorem D

For the sake of clarity, we restate the result we want to prove on the motive $\mathcal{M}$.

Theorem 9.4 *Assume that either*

- $r_{\mathrm{an}}(\mathcal{M})$ *is odd and* (GS) *holds*

or

- $r_{\mathrm{an}}(\mathcal{M})$ *is even and* (reg) *holds.*

Furthermore, assume that Conjecture 2.50 holds true for p and $\mathbb{Q}$.

If $r_{\mathrm{an}}(\mathcal{M}) > 1$, then

$$r_{\mathrm{alg}}(\mathcal{M}) \in \left\{ 2n + \frac{1-(-1)^{r_{\mathrm{an}}(\mathcal{M})}}{2} \;\middle|\; n \in \mathbb{Z}_{\geq 1} \right\}.$$

In particular, $r_{\mathrm{alg}}(\mathcal{M}) \geq \dfrac{5-(-1)^{r_{\mathrm{an}}(\mathcal{M})}}{2}$.

Proof To begin with, note that, by Lemma 2.34, $r_{\mathrm{an}}(\mathcal{M}) > 1$ if and only if $r_{\mathrm{an}}(f) > 1$. Moreover, recall from Sect. 7.2 that $r_{\mathrm{an}}(\mathcal{M})$ and $r_{\mathrm{an}}(f)$ have the same parity; in other words, $(-1)^{r_{\mathrm{an}}(\mathcal{M})} = \varepsilon(f)$. Finally, since we are assuming Conjecture 2.50 for p and $\mathbb{Q}$, Corollary 2.68 gives $r_{\mathrm{alg}}(\mathcal{M}) = r_{\mathfrak{p}}(f)$, and then the desired result follows from Theorem 9.2. □

Appendix A Determinants of projective modules

We sketch the basic elements of the theory of determinants of projective modules over commutative rings, putting a focus on modules over (products of finitely many) principal ideal domains. In doing so, we follow [73, §2.1] and [77, Lecture 1, §5] quite closely.

A.1 Determinants over commutative rings

Let R be a commutative ring. Denote by $\mathcal{L}_R$ the category of isomorphism classes of graded invertible R-modules: the objects of $\mathcal{L}_R$ are pairs (L, r) consisting of an invertible (*i.e.*, projective, rank 1) R-module L and a function $r : \mathrm{Spec}(R) \to \mathbb{Z}$, which is to be thought of as a grading, that is locally constant for the Zariski topology, while morphisms between

two objects (L, r) and (M, s) are trivial if $r \neq s$ and isomorphisms of R-modules $L \to M$ otherwise. One defines a product in $\mathcal{L}_R$ by

$$(L, r) \cdot (M, s) := (L \otimes_R M, r + s), \tag{A.1}$$

and then $(R, 0)$ is the unit object for this product. The inverse of a pair (L, r) is $(L, r)^{-1} := (L^*, -r)$, where $L^* := \mathrm{Hom}_R(L, R)$ is the R-linear dual of L; more precisely, there is a canonical isomorphism

$$(L, r) \cdot (L^*, -r) \simeq (R, 0)$$

induced by the usual evaluation map $L \otimes_R \mathrm{Hom}_R(L, R) \to R$. The monoidal category $\mathcal{L}_R$ is equipped with a modified commutativity constraint involving a sign that depends on the grading (see, *e.g.*, [21, §2.5], [77, Definition 1.27]). If $f : R \to R'$ is a ring homomorphism, then one defines a functor

$$\mathcal{L}_R \longrightarrow \mathcal{L}_{R'}$$

by the recipe $(L, r) \mapsto (L \otimes_R R', r \circ f^*)$ on objects and in the obvious way (*i.e.*, by extension of scalars) on morphisms, where $f^* : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is the map induced by f by (contravariant) functoriality.

Let us write $\mathrm{Proj}_R^{\mathrm{fg}}$ for the category of finitely generated, projective R-modules. For every object M of $\mathrm{Proj}_R^{\mathrm{fg}}$ let

$$\mathrm{rk}_R(M) : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}$$

be the rank function attached to M, which maps $\mathfrak{p}$ to $\mathrm{rk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ and is locally constant with respect to the Zariski topology on $\mathrm{Spec}(R)$ (see, *e.g.*, [162, Ch. I, Corollary 2.2.2]); notice that $M_\mathfrak{p}$ is projective, hence free, of finite rank over the local ring $R_\mathfrak{p}$. If $\mathrm{rk}_R(M)$ is constant, then $\wedge_R^{\mathrm{rk}_R(M)} M$ denotes the usual $\mathrm{rk}_R(M)$-th exterior power of M over R. If $\mathrm{rk}_R(M)$ is not constant, then the definition of $\wedge_R^{\mathrm{rk}_R(M)} M$ is more delicate: see, *e.g.*, [127, Definition 1.3.1] or [162, p. 21]. In both cases, $\wedge_R^{\mathrm{rk}_R(M)} M$ is an invertible R-module, called the *determinant of* M; see, *e.g.*, [41, Part 3] for motivation for retaining the rank information. There is a (covariant) functor $\mathrm{Proj}_R^{\mathrm{fg}} \to \mathcal{L}_R$ defined by

$$M \longmapsto \mathrm{Det}_R(M) := \left(\bigwedge_R^{\mathrm{rk}_R(M)} M,\ \mathrm{rk}_R(M) \right)$$

on objects and in the obvious fashion on morphisms. Notice that $\mathrm{Det}_R(0) = (R, 0)$. For notational convenience, for every object M of $\mathrm{Proj}_R^{\mathrm{fg}}$ we set $\mathrm{Det}_R^{-1}(M) := \mathrm{Det}_R(M)^{-1}$.

Caveat A.1 In this paper, we frequently use expressions like "N is an R-submodule of $\mathrm{Det}_R(M)$", meaning that N is a submodule of the R-module underlying $\mathrm{Det}_R(M)$. A similar interpretation must be given to statements like "The set $\star$ is a basis of $\mathrm{Det}_R(M)$ over R".

It turns out that Det_R is multiplicative on short exact sequences: if we are given an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow C \longrightarrow 0$$

in $\mathrm{Proj}_R^{\mathrm{fg}}$, then

$$\mathrm{Det}_R(P) \simeq \mathrm{Det}_R(K) \cdot \mathrm{Det}_R(C), \tag{A.2}$$

where the product on the right is defined as in (A.1). Moreover, if there is a short exact sequence of R-modules

$$0 \longrightarrow P \longrightarrow Q \longrightarrow T \longrightarrow 0 \tag{A.3}$$

with P, Q objects of $\mathrm{Proj}_R^{\mathrm{fg}}$, then we define $\mathrm{Det}_R(T) := \mathrm{Det}_R(Q) \cdot \mathrm{Det}_R^{-1}(P)$. One can check that $\mathrm{Det}_R(T)$ is independent of the choice of an exact sequence as in (A.3).

Remark/Notation A.2 Suppose that M is free of finite rank r over R. If $\{m_1, \dots, m_r\}$ is an R-basis of M, then $\{m_1 \wedge \cdots \wedge m_r\}$ is an R-basis of $\mathrm{Det}_R(M)$. In this case, we denote by $(m_1 \wedge \cdots \wedge m_r)^{-1}$ the dual element of $m_1 \wedge \cdots \wedge m_r$, so that $\big\{(m_1 \wedge \cdots \wedge m_r)^{-1}\big\}$ is an R-basis of $\mathrm{Det}_R^{-1}(M)$. Furthermore, for every $n \geq 1$, the natural pairing

$$\bigwedge^n M \times \bigwedge^n M^* \longrightarrow R, \quad (m_1 \wedge \cdots \wedge m_n, \ell_1 \wedge \cdots \wedge \ell_n) \longmapsto \det\big(\ell_i(m_j)\big)$$

is perfect, so it induces a canonical isomorphism

$$\bigwedge^n M^* \simeq \left(\bigwedge^n M\right)^* \tag{A.4}$$

of R-modules, which can be regarded as an identification. With standard notation, the basis $\big\{(m_1 \wedge \cdots \wedge m_r)^{-1}\big\}$ of $\mathrm{Det}_R^{-1}(M)$ corresponds, under the identification (A.4), to the basis $\{m_1^* \wedge \cdots \wedge m_r^*\}$ of $\mathrm{Det}_R(M^*)$. Finally, there is obviously an equality $\mathrm{rk}_R(M) = \mathrm{rk}_R(M^*)$ of rank functions, so it follows that $\mathrm{Det}_R^{-1}(M)$ and $\mathrm{Det}_R(M^*)$ have the same underlying R-module but opposite rank functions: this is the reason why we use different symbols for the bases of $\mathrm{Det}_R^{-1}(M)$ and of $\mathrm{Det}_R(M^*)$ that are built out of a given basis of M over R. It is useful to keep this observation in mind when computing with determinants of modules.

Remark A.3 Assume that R is noetherian (which is always the case in the main body of the article). If M is an object of $\mathrm{Proj}_R^{\mathrm{fg}}$, then M^* is an object of $\mathrm{Proj}_R^{\mathrm{fg}}$. Since R is noetherian, M is finitely presented over R, so for every $\mathfrak{p} \in \mathrm{Spec}(R)$ there is a canonical isomorphism $(M^*)_{\mathfrak{p}} \simeq M_{\mathfrak{p}}^*$ of $R_{\mathfrak{p}}$-modules, where the right-hand term is the $R_{\mathfrak{p}}$-linear dual of $M_{\mathfrak{p}}$ (see, *e.g.*, [99, p. 52, Corollary]). Thus, there is an equality $\mathrm{rk}_R(M) = \mathrm{rk}_R(M^*)$ of rank functions.

A.2 Determinants over PID's

Let R be a principal ideal domain, write $\mathrm{frac}(R)$ for its fraction field and let T be a finitely generated torsion R-module. Choose a resolution

$$0 \longrightarrow P \xrightarrow{\phi} Q \longrightarrow T \longrightarrow 0$$

of T with objects P, Q of $\mathrm{Proj}_R^{\mathrm{fg}}$. Since R is a PID, P and Q are free over R; furthermore, T being torsion forces the ranks of P and Q to be equal. It turns out that

$$\mathrm{Det}_R(T) = \det(\phi)^{-1} \cdot R \subset \mathrm{frac}(R),$$

where $\det(\phi)$ is computed with respect to fixed bases of P and Q. Equivalently, $\mathrm{Det}_R(T)$ is the inverse of the ideal that is generated by the product of the elementary divisors of the R-module T. In this setting, we usually write $\mathcal{I}_R(T)$ in place of $\mathrm{Det}_R(T)$ to stress the fact that $\mathrm{Det}_R(T)$ is a fractional ideal of R; we also set $\mathcal{I}_R^{-1}(T) := \mathrm{Det}_R^{-1}(T)$. In particular, $\mathcal{I}_R\big((0)\big) = R$. See, *e.g.*, [77, Example 1.31] for details in the $R = \mathbb{Z}_p$ case.

Remark/Notation A.4 Let r be the rank of P and Q and let $\{p_1, \dots, p_r\}$ (respectively, $\{q_1, \dots, q_r\}$) be a basis of P (respectively, Q) over R. Keeping Remark/Notation A.2 in mind, there is an equality

$$\mathrm{Det}_R(Q) \cdot \mathrm{Det}_R^{-1}(P) = R \cdot (q_1 \wedge \cdots \wedge q_r) \cdot (p_1 \wedge \cdots \wedge p_r)^{-1}, \tag{A.5}$$

where $(q_1 \wedge \cdots \wedge q_r) \cdot (p_1 \wedge \cdots \wedge p_r)^{-1}$ is just a shorthand for $(q_1 \wedge \cdots \wedge q_r) \otimes (p_1 \wedge \cdots \wedge p_r)^{-1}$. Finally, combining (A.2) and (A.5) yields a natural isomorphism

$$\mathrm{Det}_R(T) \simeq R \cdot (q_1 \wedge \cdots \wedge q_r) \cdot (p_1 \wedge \cdots \wedge p_r)^{-1},$$

which will often be viewed as an identification.

A.3 Determinants over products of PID's

Let $R_1, \dots, R_n$ be PID's and consider their product $R := \prod_{i=1}^n R_i$. For $i = 1, \dots, n$ let $e_i \in R$ be the idempotent corresponding to R_i, so that there is an identification $R_i = e_i R$. Let T be an R-module. For $i = 1, \dots, n$ set $T_i := e_i T$; equivalently, $T_i = T \otimes_R R_i$. The R-submodule T_i of T is naturally an R_i-module and there is a canonical identification

$$T = \bigoplus_{i=1}^n T_i$$

of R-modules. Assume now that T is finitely generated and torsion over R; of course, this is tantamount to T_i being finitely generated and torsion over R_i for every $i = 1, \dots, n$. Let $Q(R)$ be the total quotient ring of R and set

$$\mathrm{Det}_R(T) := \prod_{i=1}^n \mathrm{Det}_{R_i}(T_i) \subset \prod_{i=1}^n \mathrm{frac}(R_i) = Q(R), \tag{A.6}$$

where $\mathrm{Det}_{R_i}(T_i)$ is defined as in Sect. A.2. As in the $n = 1$ case treated above, we also set $\mathcal{I}_R(T) := \mathrm{Det}_R(T)$ and $\mathcal{I}_R^{-1}(T) := \mathrm{Det}_R^{-1}(T)$. In particular, $\mathcal{I}_R\big((0)\big) = R$. Finally, a *fractional R-ideal* will be, by definition, a product of the form $I = I_1 \times \cdots \times I_n$ where I_j is a fractional R_j-ideal for $j = 1, \dots, n$; in this case, $\mathcal{I}_R(I) = \prod_{j=1}^n \mathcal{I}_{R_j}(I_j)$.

Remark A.5 In the applications we have in mind, the R_i will be discrete valuation rings (namely, they will be the completions of $\mathcal{O}_F$ at prime ideals above a fixed prime number p), so R will be regular.

A.4 Determinants and base change

If $R \to S$ is a ring homomorphism and P is an object of $\mathrm{Proj}_R^{\mathrm{fg}}$, then $P \otimes_R S$ is an object of $\mathrm{Proj}_S^{\mathrm{fg}}$ and there is a base change isomorphism

$$\mathrm{Det}_R(P) \otimes_R S \simeq \mathrm{Det}_S(P \otimes_R S) \tag{A.7}$$

of S-modules.

A.5 Determinants of complexes

Let $\mathrm{Proj}_R^\bullet$ denote the category of complexes of R-modules that are quasi-isomorphic to a bounded complex of R-modules in $\mathrm{Proj}_R^{\mathrm{fg}}$. For an object $C^\bullet$ in $\mathrm{Proj}_R^\bullet$, fix a quasi-isomorphism $\tilde{C}^\bullet \to C^\bullet$ with a bounded complex $\tilde{C}^\bullet$ of modules in $\mathrm{Proj}_R^{\mathrm{fg}}$ and define

$$\mathrm{Det}_R(C^\bullet) := \prod_{i \in \mathbb{Z}} \mathrm{Det}_R^{-1}\big(\tilde{C}^i\big) \tag{A.8}$$

([77, Definition 1.29]). If an object $C^\bullet$ of $\mathrm{Proj}_R^\bullet$ has the property that $H^j(C^\bullet)$ is an object of $\mathrm{Proj}_R^{\mathrm{fg}}$ for all j, then there is an isomorphism

$$\mathrm{Det}_R(C^\bullet) \simeq \prod_{j \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^j}\big(H^j(C^\bullet)\big)$$

of R-modules.

Appendix B Remarks on Pontryagin duals

We gather some facts about Pontryagin duals that are used in the main body of the paper. These results are consequences of well-known properties of duals of local fields; however, for several of them we could not find a convenient reference in the literature, so we decided to collect them here for the reader's benefit.

B.1 Generalities on Pontryagin duals

Let $\mathbf{T} := \{z \in \mathbb{C} \mid |z| = 1\} \simeq \mathbb{R}/\mathbb{Z}$ be the unit circle in the complex plane, viewed as a subgroup of $\mathbb{C}^\times$. The *Pontryagin dual* of a locally compact, Hausdorff topological abelian group G is the group

$$G^\wedge := \mathrm{Hom}_{\mathrm{cont}}(G, \mathbf{T}) \tag{B.1}$$

of characters of G, *i.e.*, continuous homomorphisms from G to $\mathbf{T}$, where $\mathbf{T}$ is equipped with its natural complex topology (equivalently, the quotient topology of $\mathbb{R}/\mathbb{Z}$). In turn, $G^\wedge$ can be endowed with the compact-open topology. This definition of Pontryagin dual does not coincide, in general, with the one that is given in Sect. 2.19 for $\mathbb{Z}_p$-modules; however, it is well known that if G is profinite, then the image of any continuous homomorphism $G \to \mathbf{T}$ is finite (take, *e.g.*, $n = 1$ in [62, Proposition 2.2]), so $G^\wedge = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Q}/\mathbb{Z})$. Of course, this equality is true also if G is a torsion abelian group. Thus, if G is either

- a $\mathbb{Z}_p$-module that is a torsion abelian group

or

- a profinite $\mathbb{Z}_p$-module,

then $G^\wedge = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Q}_p/\mathbb{Z}_p)$. In other words, with notation as in (2.57), $G^\wedge = G^\vee$.

Remark B.1 If G is a profinite $\mathbb{Z}_p$-module, then every element of $\mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Q}_p/\mathbb{Z}_p)$ is $\mathbb{Z}_p$-linear. Conversely, if G is a finitely generated $\mathbb{Z}_p$-module, then a $\mathbb{Z}_p$-linear homomorphism $G \to \mathbb{Q}_p/\mathbb{Z}_p$ is always continuous, so $G^\wedge = G^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(G, \mathbb{Q}_p/\mathbb{Z}_p)$ in this case. More generally, suppose that $\mathscr{K}$ is a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathscr{O}$. As explained, *e.g.*, in [114, §2.9.1, §2.9.2], if G is a (co)finitely generated $\mathscr{O}$-module, then $G^\wedge = G^\vee \simeq \mathrm{Hom}_{\mathscr{O}}(G, \mathscr{K}/\mathscr{O})$.

B.2 Pontryagin duals of finite extensions of $\mathbb{Q}_p$

Let $\mathscr{K}$ be a finite extension of $\mathbb{Q}_p$. Recall the definition of $\mathscr{K}^\wedge$ from (B.1). Fix a compatible system $(\zeta_{p^n})_{n\geq 1}$ of p-power roots of unity: $\zeta_{p^n} \in \bar{\mathbb{Q}}$ is a primitive p^n-root of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$; the standard choice is $\zeta_{p^n} := e^{2\pi i/p^n}$ for all $n \geq 1$. Let us define the (non-trivial) standard character $\chi_0 \in \mathbb{Q}_p^\wedge$ by

$$\chi_0\left(\sum_{k=-n}^{+\infty} a_k p^k\right) := \prod_{k=-n}^{-1} \zeta_{p^{|k|}}^{a_k}; \tag{B.2}$$

here $n \in \mathbb{N}$ and $a_k \in \mathbb{Z}$ for all $k \geq -n$. More succinctly, with the choice of roots of unity specified above, χ_0 is the composition

$$\chi_0 = \left(\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i(\cdot)}} \mathbf{T}\right).$$

Now fix a non-trivial $\varphi \in \mathscr{K}^\wedge$. It turns out that for every $\psi \in \mathscr{K}^\wedge$ there exists a unique $a_\varphi(\psi) \in \mathscr{K}$ such that $\psi(x) = \varphi\big(a_\varphi(\psi) \cdot x\big)$ for all $x \in \mathscr{K}$ and the map

$$\mathscr{K}^\wedge \xrightarrow{\simeq} \mathscr{K}, \quad \psi \longmapsto a_\varphi(\psi) \tag{B.3}$$

is an isomorphism of topological groups. See, *e.g.*, [61, §8.3, Proposition 1] and [129, Ch. 7, Exercise 1] for details.

Remark B.2 The (non-trivial) standard character of $\mathscr{K}$ is $\chi_0 \circ \mathrm{tr}_{\mathscr{K}/\mathbb{Q}_p}$, with χ_0 as in (B.2).

Let $\mathscr{K}^\vee$ be defined as in (2.57).

Proposition B.3 *There is an isomorphism $\mathscr{K}^\vee \simeq \mathscr{K}^\wedge$ of topological groups.*

Proof Observe that $\mathbb{Z}_p$ is contained in the kernel of the character χ_0 from (B.2), so there is an induced map

$$\bar{\chi}_0 : \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbf{T}, \quad [x] \longmapsto \chi_0(x), \tag{B.4}$$

where $[x]$ is the image of $x \in \mathbb{Q}_p$ in $\mathbb{Q}_p/\mathbb{Z}_p$. One can show that $\bar{\chi}_0$ yields an isomorphism of groups between $\mathbb{Q}_p/\mathbb{Z}_p$ and the subgroup $\mu_{p^\infty} \subset \mathbf{T}$ of p-power roots of unity. In light of this fact, from here on we shall view the map $\bar{\chi}_0$ from (B.4) as an isomorphism

$$\bar{\chi}_0 : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \mu_{p^\infty}. \tag{B.5}$$

Given $\varphi \in \mathscr{K}^\vee$, set $\varphi^\wedge := \bar{\chi}_0 \circ \varphi \in \mathscr{K}^\wedge$. On the other hand, given $\chi \in \mathscr{K}^\wedge$, note that $\mathrm{im}(\chi) \subset \mu_{p^\infty}$ (the proof of [61, §8.3, Proposition 1], which treats the $\mathscr{K} = \mathbb{Q}_p$ case, carries over *verbatim* to our setting), so we can define $\chi^\vee := \bar{\chi}_0^{-1} \circ \chi \in \mathscr{K}^\vee$. Clearly, $\varphi \mapsto \varphi^\wedge$ and $\chi \mapsto \chi^\vee$ are topological group homomorphisms $\mathscr{K}^\vee \to \mathscr{K}^\wedge$ and $\mathscr{K}^\wedge \to \mathscr{K}^\vee$ that are inverse to each other. □

Since we regard the character $\chi_0 \in \mathbb{Q}_p^\wedge$ described in (B.2) as canonical, we shall tacitly view the isomorphism $\mathscr{K}^\vee \simeq \mathscr{K}^\wedge$ as an identification. In light of (B.3), it follows that for every non-trivial $\varphi \in \mathscr{K}^\wedge$ there is an isomorphism of topological groups

$$a_\varphi : \mathscr{K}^\vee \xrightarrow{\simeq} \mathscr{K}. \tag{B.6}$$

Namely, for every $\psi \in \mathscr{K}^\vee$ there is a unique $a_\varphi(\psi) \in \mathscr{K}$ such that $\psi^\wedge(x) = \varphi\big(a_\varphi(\psi) \cdot x\big)$ for all $x \in \mathscr{K}$; equivalently, $\psi(x) = \varphi^\vee\big(a_\varphi(\psi) \cdot x\big)$ for all $\psi \in \mathscr{K}^\vee$ and $x \in \mathscr{K}$, where $\varphi^\vee \in \mathscr{K}^\vee$ corresponds to φ under the isomorphism of Proposition B.3.

In the special case $\mathscr{K} = \mathbb{Q}_p$ we shall take $\varphi = \chi_0$ and set $a := a_{\chi_0}$; here notice that $\chi_0^\vee$ is given by

$$\chi_0^\vee\left(\sum_{k=-n}^{+\infty} a_k p^k\right) = \bar{\chi}_0^{-1}\left(\prod_{k=-n}^{-1} \zeta_{p^{|k|}}^{a_k}\right) = \left[\sum_{k=-n}^{-1} a_k p^k\right],$$

where, as above, $[x]$ is the class of $x \in \mathbb{Q}_p$ in $\mathbb{Q}_p/\mathbb{Z}_p$ and $\bar{\chi}_0$ is the isomorphism from (B.5). In other words, $\chi_0^\vee$ is just the canonical projection $\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p$.

It is convenient to introduce the following notation: given $\psi \in \mathscr{K}^\wedge$ and $c \in \mathscr{K}$, we define $c \cdot \psi \in \mathscr{K}^\wedge$ by $(c \cdot \psi)(x) := \psi(cx)$ for all $x \in \mathscr{K}$. This endows $\mathscr{K}^\wedge$ with a $\mathscr{K}$-vector space structure, and an analogous definition can be given for $\mathscr{K}^\vee$. In particular, the image of $\psi \in \mathscr{K}^\vee$ under the isomorphism a_φ in (B.6) can be described by requiring $a_\varphi(\psi) \in \mathscr{K}$ to satisfy $\psi = a_\varphi(\psi) \cdot \varphi^\vee$.

Proposition B.4 *There is an isomorphism $\mathscr{K}^\vee \simeq \mathrm{Hom}_{\mathscr{O}}(\mathscr{K}, \mathscr{K}/\mathscr{O})$ of topological groups.*

Proof Let ϖ be a uniformizer for $\mathscr{O}$, so that $\mathscr{K} = \varinjlim_{n \in \mathbb{Z}} \varpi^n \mathscr{O}$. For every $n \in \mathbb{Z}$, the $\mathscr{O}$-module $\varpi^n \mathscr{O}$ is (topologically) free of rank 1, so Remark B.1 ensures that there is an isomorphism $(\varpi^n \mathscr{O})^\vee \simeq \mathrm{Hom}_{\mathscr{O}}(\varpi^n \mathscr{O}, \mathscr{K}/\mathscr{O})$ of topological groups. It follows that there are isomorphisms of topological groups

$$\mathscr{K}^\vee = \varprojlim_{n \in \mathbb{Z}} (\varpi^n \mathscr{O})^\vee \simeq \varprojlim_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}}(\varpi^n \mathscr{O}, \mathscr{K}/\mathscr{O}) = \mathrm{Hom}_{\mathscr{O}}(\mathscr{K}, \mathscr{K}/\mathscr{O}),$$

as was to be shown. □

From here on, we view the isomorphism provided by Proposition B.4 as an identification $\mathscr{K}^\vee = \mathrm{Hom}_{\mathscr{O}}(\mathscr{K}, \mathscr{K}/\mathscr{O})$.

Remark B.5 The isomorphisms in (B.3) and Proposition B.3 are $\mathscr{K}$-linear, so $\mathscr{K}^\wedge$ and $\mathscr{K}^\vee$ are $\mathscr{K}$-vector spaces of dimension 1.

B.3 On Pontryagin duals of $\mathscr{K}$-vector spaces

Let $\chi_{\mathscr{K}} : \mathscr{K} \twoheadrightarrow \mathscr{K}/\mathscr{O}$ be the canonical projection and let $\boldsymbol{X}_{\mathscr{K}} := \mathscr{O}\chi_{\mathscr{K}}$ be the $\mathscr{O}$-submodule of $\mathscr{K}^\vee$ generated by $\chi_{\mathscr{K}}$.

Lemma B.6 *An element of $\mathscr{K}^\vee$ is trivial on $\mathscr{O}$ if and only if it belongs to $\boldsymbol{X}_{\mathscr{K}}$.*

Proof It is obvious that $\chi_{\mathscr{K}}$ is trivial on $\mathscr{O}$, so every element of $\boldsymbol{X}_{\mathscr{K}}$ is trivial on $\mathscr{O}$. Now pick $\psi \in \mathscr{K}^\vee$. Thanks to isomorphism (B.3) and Proposition B.3, there exists a unique $a_{\mathscr{K}}(\psi) \in \mathscr{K}$ such that $\psi = a_{\mathscr{K}}(\psi) \cdot \chi_{\mathscr{K}}$. On the other hand, the square

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow[\simeq]{a_{\mathscr{K}}^{-1}} & \mathscr{K}^\vee \\ \downarrow & & \downarrow \\ \mathscr{K}/\mathscr{O} & \xrightarrow{\simeq} & \mathscr{O}^\vee \end{array} \tag{B.7}$$

is commutative. If ψ is trivial on $\mathscr{O}$, then ψ has trivial image in $\mathscr{O}^\vee$, and the commutativity of (B.7) ensures that the image of $a_{\mathscr{K}}(\psi)$ in $\mathscr{K}/\mathscr{O}$ is trivial as well, *i.e.*, $a_{\mathscr{K}}(\psi) \in \mathscr{O}$. This shows that ψ belongs to $\boldsymbol{X}_{\mathscr{K}}$. □

Now let V be a $\mathscr{K}$-vector space of finite dimension, say r. Let $\mathscr{B} = \{v_1, \dots, v_r\}$ be a basis of V over $\mathscr{K}$; it induces topological isomorphisms $V \simeq \mathscr{K}^r$ and

$$V^\vee \simeq (\mathscr{K}^\vee)^r. \tag{B.8}$$

Analogously to what was done in Sect. B.2 for $\mathscr{K}^\wedge$ and $\mathscr{K}^\vee$, one can endow $V^\vee$ with a natural $\mathscr{K}$-vector space structure, and then $V^\vee$ is r-dimensional over $\mathscr{K}$ (*cf.* Remark B.5).

Remark B.7 There is an isomorphism $V^\vee \simeq V^\wedge$ of topological groups, which we can view as a canonical identification.

For each $i \in \{1, \dots, r\}$, write $v_i^\vee$ for the element of $V^\vee$ corresponding under isomorphism (B.8) to the element of $(\mathscr{K}^\vee)^r$ with all components equal to 0 except the i-th that is equal to $\chi_{\mathscr{K}}$. Equivalently, the elements $v_1, \dots, v_r$ give rise to the dual basis $\{v_1^*, \dots, v_r^*\}$ of the $\mathscr{K}$-linear dual of V by the recipe $v_i^*(v_j) := \delta_{ij}$, where δ_{ij} is the Kronecker delta: composing the $\mathscr{K}$-linear maps v_i^* with $\chi_{\mathscr{K}}$, we obtain the elements $v_i^\vee$ of the Pontryagin dual $V^\vee$.

Lemma B.8 *The elements $v_1^\vee, \dots, v_r^\vee$ are linearly independent over $\mathscr{O}$.*

Proof Suppose there is an equality

$$a_1 v_1^\vee + \cdots + a_r v_r^\vee = 0 \tag{B.9}$$

with $a_1, \dots, a_r \in \mathscr{O}$ and there is j such that $a_j \neq 0$; let $\nu_j \in \mathbb{N}$ be the valuation of a_j. Let $\varpi \in \mathscr{O}$ be a uniformizer and notice that $\chi_{\mathscr{K}}$ is $\mathscr{O}$-linear. Then

$$\left(a_1 v_1^\vee + \cdots + a_r v_r^\vee\right)\left(\varpi^{-(\nu_j+1)} v_j\right) = \chi_{\mathscr{K}}\left(a_j\, \varpi^{-(\nu_j+1)}\right) \neq 0,$$

where the inequality on the right is a consequence of the fact that $a_j\, \varpi^{-(\nu_j+1)} \notin \mathscr{O}$. This contradicts (B.9). □

Since $\mathscr{K}$ is the quotient field of $\mathscr{O}$, it follows from Lemma B.8 that $v_1^\vee, \dots, v_r^\vee$ are linearly independent over $\mathscr{K}$ as well. Keeping in mind that $V^\vee$ is an r-dimensional $\mathscr{K}$-vector space, it makes sense to give

Definition B.9 The *Pontryagin dual basis* of $\mathscr{B}$ is the basis $\mathscr{B}^\vee := \{v_1^\vee, \dots, v_r^\vee\}$ of $V^\vee$ as a $\mathscr{K}$-vector space.

Now let $T_{\mathscr{B}}$ be the $\mathscr{O}$-lattice in V spanned by $\mathscr{B}$ and write $\Xi_{\mathscr{B}}$ for the $\mathscr{O}$-lattice in $V^\vee$ spanned by $\mathscr{B}^\vee$.

Proposition B.10 *An element of $V^\vee$ belongs to $\Xi_{\mathscr{B}}$ if and only if it is trivial on $T_{\mathscr{B}}$.*

Proof Immediate from Lemma B.6. □

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Declarations

Conflict of interest
The authors declare no conflict of interest.

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