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ASYMPTOTIC BEHAVIOUR OF A CLASS OF DEGENERATE ELLIPTIC-PARABOLIC OPERATORS: A UNITARY APPROACH

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Abstract. We study the asymptotic behaviour of a sequence of strongly degenerate parabolic equations $\partial_t(r_h u) - \operatorname{div}(a_h \cdot Du)$ with $r_h(x,t) \geqslant 0$, $r_h \in L^\infty(\Omega \times (0,T))$. The main problem is the lack of compactness, by-passed via a regularity result. As particular cases, we obtain G-convergence for elliptic operators ($r_h \equiv 0$), G-convergence for parabolic operators ($r_h \equiv 1$), singular perturbations of an elliptic operator ($a_h \equiv a$ and $r_h \to r$, possibly $r \equiv 0$).

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1. Introduction

In the papers [5, 13, 14] De Giorgi and Spagnolo introduced G-convergence for a class of elliptic operators, precisely for a class of elliptic operators in divergence form defined by an elliptic and simmetric matrix with bounded coefficients. Tartar extended this convergence to the non-simmetric (and then non-linear) case (see, for instance, [16] and [17]).

Later in [3] Colombini and Spagnolo defined G-convergence for a class of parabolic operators in divergence form still defined by a simmetric matrix with bounded coefficients depending, in this case, also on time. Before introducing the aim of this paper we recall the definition of G-convergence in both cases, denoting the convergence by EG in the elliptic case and by PG in the parabolic one, as extended to non-simmetric operators by Tartar (for a book containing results about both EG and PG convergences we refer to [7]).

Consider $n \in \mathbf{N}$ fixed. Moreover, for $\lambda_0 \leqslant \Lambda_0$ and M positive real numbers, denote by $\mathcal{M}_U(\lambda_0, \Lambda_0, M)$, with U open set of $\mathbf{R}^k$, $k \in \mathbf{N}$, the class of $n \times n$ matrices defined as follows:

$$\begin{aligned} & a = [a_{ij}(y)]_{i,j=1}^n \in L^\infty(U) \qquad \text{such that} \\ & \lambda_0|\xi|^2 \leqslant \big(a(y)\cdot\xi,\xi\big) \leqslant \Lambda_0|\xi|^2 \\ & \big|\big(a(y)\cdot\xi,\eta\big)\big| \leqslant M\big(a(y)\cdot\xi,\xi\big)^{1/2}\big(a(y)\cdot\eta,\eta\big)^{1/2} \end{aligned} \tag{1}$$

for every $\xi, \eta \in \mathbf{R}^n$, for *a.e.* $y \in U$.

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Definition 1.1. Let Ω be a bounded open set of $\mathbf{R}^n$ and $T > 0$. Consider a sequence $(a_h)_h \subset \mathcal{M}_\Omega(\lambda_0, \Lambda_0, M)$, $a_h = a_h(x)$ (referring to (1), in this case $k = n$). Given $a = a(x) \in \mathcal{M}_\Omega(\lambda_0, \Lambda_0, M)$ we say that

$$a_h \stackrel{EG}{\longrightarrow} a \qquad \text{in } \Omega$$

if for every $f \in H^{-1}(\Omega)$ it results that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(\Omega) \\ a_h \cdot Du_h \to a \cdot Du & \text{in } L^2(\Omega)^n\text{-weak,} \end{array}$$

where u_h and u denote respectively the solutions (see Def. 2.5 with $r \equiv 0$) of

$$\begin{cases} -\operatorname{div}(a_h \cdot Dv) = f & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases} \qquad \begin{cases} -\operatorname{div}(a \cdot Dv) = f & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

For a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$, $a_h = a_h(x,t)$ (referring to (1), in this case $k = n+1$), and given $a = a(x,t) \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ we say that

$$a_h \stackrel{PG}{\longrightarrow} a \qquad \text{in } \Omega \times (0,T)$$

if for every $f \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi \in L^2(\Omega)$ it results that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,L^2(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n)\text{-weak,} \end{array}$$

where u_h and u denote respectively the solutions (see Def. 2.5 with $r \equiv 1$) of

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(a_h \cdot Dv) = f & \textit{in } \Omega \times (0,T) \\ v = 0 & \textit{in } \partial\Omega \times (0,T) \\ v = \varphi & \textit{in } \Omega \times \{0\} \end{cases} \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(a \cdot Dv) = f & \textit{in } \Omega \times (0,T) \\ v = 0 & \textit{in } \partial\Omega \times (0,T) \\ v = \varphi & \textit{in } \Omega \times \{0\}. \end{cases}$$

In [3] the authors studied the connection between EG and PG convergence: in particular they proved that if $(a_h)_h \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ satisfies

$$\lim_{\tau\to 0} \sup_h \int_I \int_\omega |a_h(x,t+\tau) - a_h(x,t)| \mathrm{d}x \; \mathrm{d}t = 0 \qquad \forall\; I \times \omega \subset\subset (0,T) \times \Omega, \tag{2}$$

then

$$a_h(\cdot,t) \stackrel{EG}{\longrightarrow} a(\cdot,t) \qquad \text{in } \Omega \text{ for } a.e.\ t \in (0,T) \quad \text{ iff } \quad a_h \stackrel{PG}{\longrightarrow} a \qquad \text{in } \Omega \times (0,T) \tag{3}$$

and showed with a counterexample that this is not always true.

In this paper we consider strongly degenerate parabolic, or elliptic-parabolic, operators like

$$Pu = \frac{\partial}{\partial t}(ru) - \operatorname{div}(a \cdot Du) \qquad \text{with } r = r(x,t) \geqslant 0 \tag{4}$$

and study the limit behaviour of the sequence of Cauchy-Dirichlet problems (for the existence result we refer to [8], but see also [11])

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = f & \text{in } \Omega_{h,+}(t) \ \times (0,T) \\ -\operatorname{div}(a_h \cdot Du) = f & \text{in } \Omega_{h,0}(t) \ \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases} \tag{5}$$

where (the initial condition on $\Omega_{h,+}(0)$ will be clarified at the end of Sect. 2) $\Omega_{h,+}(t) := \{x \in \Omega \mid r_h(x,t) > 0\}$ and $\Omega_{h,0}(t) := \{x \in \Omega \mid r_h(x,t) = 0\}$, $(a_h)_h$ is a sequence in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$, the class of matrices a satisfying (1) and

$$\big|a(x,t) - a(x,s)\big| \leqslant N|t-s|$$

for *a.e.* $x \in \Omega$ and every $s, t \in [0,T]$, r_h belonging to a suitable class $\mathcal{F}$ defined in (6). Arising from (3) the aim of the present paper is to give a more general definition of G-convergence, for problems (5) (see Def. 4.3), which is *independent* of the sequence $(r_h)_h$ and a compactness result with respect to it (see Th. 4.5). This in particular justifies (3) and includes other phenomena, as singular perturbations (in which the result is stronger, see Prop. 5.1), but we refer to the lastgli Studi di Lecce section for examples. We want to stress that, since r in (4) may be equal to zero on some region with positive measure, a difficulty in this situation is that the natural compactness result (see Th. 2.7) is not guaranteed. Only for the sequence of the solutions (the solutions u_h to the problems (5)), we are able to obtain the compactness via a regularity result (see Th. 2.8).

We recall that, in the general situation, a first study in this direction was already made, in the periodic case and with $r = r(x)$, in [9].

Elliptic-parabolic operators like those in (4) were already studied, as regards the existence of the solution, probably first by Showalter (see, for instance, [10] for one of the first papers and [11] for a recent book) and recently in [8] for a more general class of operators (nonlinear and possibly forward, backward and stationary).

The interest to study such problems lies on the fact that many diffusion problems lead to differential equations like

$$\frac{\partial}{\partial t}(r(x,t)u) - \operatorname{div}(a(x,t)\cdot Du) = f$$

which may be also of mixed type (see for example [1], Chap. 3, and the references therein), *i.e.* partially elliptic and partially parabolic. For some applications see also the examples in the last section.

The scheme of the paper is the following: Section 2 is dedicated to existence of the solution to an equation $Pu = f$ and to the position of the problem. In Section 3 there are some compactness results: since a "classical type" compactness result (see Th. 2.7, with $r_h \equiv 1$ for the classical case) does not hold in general, we pass through a regularity result (Th. 2.8) to obtain it. In Section 4 we define G-convergence and prove a compactness result in two steps: Theorem 4.1 furnishes, given a sequence of operators $P_h u = \frac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du)$ in a suitable class, the existence, up to a subsequence, of a limit operator $Pu = \frac{\partial}{\partial t}(ru) - \operatorname{div}(a_r \cdot Du)$, Theorem 4.5 states that a_r is *independent of* r. In the last section we give some examples, including also singular perturbations.

2. Elliptic-parabolic equations and statement of the problem

From now on T, λ_0, Λ_0, M will be fixed positive constants, N, C_1, C_2 non-negative constants, μ_0 a non-positive constant (indeed μ_0 could also be positive, but in that case it is sufficient to consider $\mu_0 = 0$) and Ω a bounded open set of $\mathbf{R}^n$ with Lipschitzian boundary. We will denote for brevity

$$\mathcal{V} := L^2(0,T;H^1_0(\Omega)), \quad \mathcal{H} := L^2(0,T;L^2(\Omega)), \quad \mathcal{V}' := L^2(0,T;H^{-1}(\Omega)).$$

We denote by $\mathcal{F}(C_1, C_2, \mu_0)$ the class of measurable functions r satisfying

$$
\begin{array}{ll}
\text{(i)} & r \in L^\infty(\Omega \times (0,T)), \quad r \geqslant 0,\\
\text{(ii)} & \|r\|_{L^\infty(\Omega\times(0,T))} \leqslant C_1,\\
\text{(iii)} & t \mapsto \displaystyle\int_\Omega u(x)v(x)r(x,t)\mathrm{d}x \quad \text{absolutely continuous on } [0,T],\\
\text{(iv)} & \left|\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega u(x)v(x)r(x,t)\mathrm{d}x\right| \leqslant C_2\ \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)} \quad \text{for } a.e.\ t\in[0,T],\\
\text{(v)} & \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega u^2(x)r(x,t)\mathrm{d}x \geqslant \mu_0\ \|u\|^2_{H^1_0(\Omega)} \qquad \text{for } a.e.\ t\in[0,T]
\end{array}
\tag{6}
$$

for every $u, v \in H^1_0(\Omega)$. For a $r \in \mathcal{F}(C_1, C_2, \mu_0)$ we define

$$
\Omega^r_+(t) := \big\{x\in\Omega \bigm| r(x,t) > 0\big\}, \qquad \Omega^r_0(t) := \big\{x\in\Omega \bigm| r(x,t) = 0\big\}. \tag{7}
$$

Remark 2.1. The class just defined is compact, *i.e.* if $(r_h)_h$ is a sequence in $\mathcal{F}(C_1, C_2, \mu_0)$, there is a subsequence $(r_{h_j})_j$ and a function $r \in \mathcal{F}(C_1, C_2, \mu_0)$ such that $r_{h_j} \to r$ in $L^\infty(\Omega\times(0,T))$-weak$*$. In fact, there is a subsequence $(r_{h_j})_j$ and a function r such that $r_{h_j} \to r$ in $L^\infty(\Omega\times(0,T))$-weak$*$. Now verify that r belongs to $\mathcal{F}(C_1, C_2, \mu_0)$. Clearly $r \geqslant 0$ and $\|r\|_\infty \leqslant C_1$. To verify that r satisfies also (iii), (iv), (v), consider a countable set Z, dense in $H^1_0(\Omega)$, and for every $u, v \in Z$ define the functions $F^{u,v}_h(t) = \int_\Omega u(x)v(x)r_h(x,t)\mathrm{d}x$. Since $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ the sequence $(F^{u,v}_h)_h$ turns out to be equicontinuous and equibounded. Then there is a function, denoted by $F^{u,v}$, and a subsequence $F^{u,v}_{h_j}$ such that $F^{u,v}_{h_j} \to F^{u,v}$ uniformly in $[0,T]$. Since Z is countable we can find a sequence h_{j_k} such that $F^{u,v}_{h_{j_k}} \to F^{u,v}$ uniformly in $[0,T]$ for every $u, v \in Z$ (and in fact for every $u, v \in H^1_0(\Omega)$). This in particular implies that

$$
\int_0^T F^{u,v}_{h_{j_k}}(t)\eta(t)\mathrm{d}t \to \int_0^T F^{u,v}(t)\eta(t)\mathrm{d}t
$$

for every $\eta \in L^1(\Omega)$. Since $r_{h_j} \to r$ in $L^\infty(\Omega\times(0,T))$-weak$*$

$$
\int_0^T F^{u,v}_{h_{j_k}}(t)\eta(t)\mathrm{d}t = \int_0^T\int_\Omega u(x)v(x)\eta(t)r_{h_{j_k}}(x,t)\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega u(x)v(x)\eta(t)r(x,t)\mathrm{d}x\,\mathrm{d}t
$$

and then $F^{u,v}(t) = \displaystyle\int_\Omega u(x)v(x)r(x,t)\mathrm{d}x$.

Notice that $F^{u,v} \in W^{1,\infty}$: then for every $\eta \in C^1_c(0,T)$ we have

$$
\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big[F^{u,v}_{h_{j_k}}(t)\big]\eta(t)\mathrm{d}t = -\int_0^T F^{u,v}_{h_{j_k}}(t)\eta'(t)\mathrm{d}t \to \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big[F^{u,v}(t)\big]\eta(t)\mathrm{d}t
$$

and then we derive

$$
-C_2\ \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}\int_0^T \eta(t)\mathrm{d}t \leqslant \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big[F^{u,v}(t)\big]\eta(t)\mathrm{d}t \leqslant C_2\ \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}\int_0^T \eta(t)\mathrm{d}t
$$

from which

$$
\left|\frac{\mathrm{d}}{\mathrm{d}t}\big[F^{u,v}(t)\big]\right| \leqslant C_2\ \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}.
$$

Analogously we derive that $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2(x)r(x,t)\mathrm{d}x \geqslant \mu_0\ \|u\|^2_{H^1_0(\Omega)}$.

Given $r \in \mathcal{F}(C_1, C_2, \mu_0)$ we introduce the families of operators

$$
\begin{aligned}
&R : [0,T] \longrightarrow \mathcal{L}(L^2(\Omega)), \qquad R(t)u := r(\cdot,t)u(\cdot) \\
&R' : [0,T] \longrightarrow \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)), \quad \langle R'(t)u, v\rangle := \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x)v(x)r(x,t)\mathrm{d}x \\
&\mathcal{R} : \mathcal{H} \longrightarrow \mathcal{H} \qquad \mathcal{R}u(t) := R(t)u(t) \\
&\mathcal{R}' : \mathcal{V} \longrightarrow \mathcal{V}' \qquad \langle \mathcal{R}'u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}\mathrm{d}t
\end{aligned}
\tag{8}
$$

and define the following Banach space

$$
\mathcal{W} = \big\{u \in \mathcal{V} \;\big|\; (\mathcal{R}u)' \in \mathcal{V}'\big\}, \qquad \|u\|_{\mathcal{W}} := \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} \tag{9}
$$

where $(\mathcal{R}u)'$ denotes the derivative in the distributional sense of $\mathcal{R}u$ with respect to the variable t.

An approximation result we will need later is the following.

Lemma 2.2. *Consider R defined in* (8)*. Then for every $u \in \mathcal{W}$ and $\sigma > 0$ there exist $v \in C^1([0,T]; H^1_0(\Omega))$ and $S \in C^2([0,T]; \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)))$ (defined analogously to R by a $s \in \mathcal{F}(C_1, C_2, \mu_0)$, $\frac{\partial s}{\partial t} \in \mathcal{F}(C_2, K, -K)$ where $K = K(C_1, \sigma)$) such that*

$$
\|u - v\|_{\mathcal{W}} < \sigma, \qquad \|(\mathcal{R}u)' - (\mathcal{S}v)'\|_{\mathcal{V}'} < \sigma.
$$

Moreover $\mathcal{S}$ can be chosen in such a way that $S'(0) = 0$.

Proof. Fix $u \in \mathcal{W} = \mathcal{W}_{\mathcal{R}}$ and $\sigma > 0$. From Proposition 2.4 in [8] we derive the existence of $v \in C^1([0,T]; H^1_0(\Omega))$ such that

$$
\|u - v\|_{\mathcal{W}} < \sigma/2 .
$$

Consider a family of mollifiers $(\rho_\epsilon)_{\epsilon>0}$ and, after defining

$$
\bar{R}(t) := \begin{cases} R(t) & \text{if } t \in [0,T] \\ 0 & \text{if } t \notin [0,T], \end{cases}
$$

consider

$$
R_\epsilon(t) = \int_{\mathbf{R}} \bar{R}(\tau)\rho_\epsilon(t-\tau)\mathrm{d}\tau
$$

and the corresponding $\mathcal{R}_\epsilon u(t) := R_\epsilon(t)u(t)$. Note that $R \in W^{1,\infty}(0,T; \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)))$ and $R_\epsilon \to R$ in $L^\infty(0,T; \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))) \cap W^{1,q}(0,T; \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)))$ for every $q < +\infty$.
Clearly $\langle R'_\epsilon(t)u, u\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \langle \int_{\mathbf{R}} \bar{R}'(\tau)\rho_\epsilon(t-\tau)\mathrm{d}\tau\, u, u\rangle \geqslant \mu_0\|u\|^2_{H^1_0(\Omega)}$.

Observe that, since $v \in C^1([0,T]; H^1_0(\Omega))$, $v \in \mathcal{W}_{\mathcal{R}}$ and $v \in \mathcal{W}_{\mathcal{R}_\epsilon}$ for every $\epsilon > 0$. Then, since

$$
\|(\mathcal{R}u)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'} \leqslant \|(\mathcal{R}u)' - (\mathcal{R}v)'\|_{\mathcal{V}'} + \|(\mathcal{R}v)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'},
$$

to prove the result it is sufficient to estimate $\|(\mathcal{R}v)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'}$. Since $R_\epsilon \to R$ in $L^\infty(0,T; \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)))$ we have that

$$
\mathcal{R}_\epsilon v' \to \mathcal{R}v' \qquad \text{in } \mathcal{H},
$$

and therefore it is sufficient to analyse the quantity $\mathcal{R}'_\epsilon v - \mathcal{R}'v$. We consider a function $\phi \in \mathcal{V}$ and by the generalized Hölder's inequality we have for every $p > 2$

$$\begin{aligned}\Big|\langle \mathcal{R}'_\epsilon v - \mathcal{R}' v, \phi\rangle_{\mathcal{V}'\times\mathcal{V}}\Big| &= \Big| \int_0^T \langle R'_\epsilon(t)v(t) - R'(t)v(t), \phi(t)\rangle_{H^{-1}\times H^1_0}\, \mathrm{d}t\Big| \\ &\leqslant \Big[\int_0^T \|R'_\epsilon(t) - R'(t)\|^{2p/(p-2)}_{\mathcal{L}(H^1_0,H^{-1})}\mathrm{d}t\Big]^{\frac{p-2}{2p}} \Big[\int_0^T \|v(t)\|^p_{H^1_0(\Omega)}\mathrm{d}t\Big]^{\frac{1}{p}} \Big[\int_0^T \|\phi(t)\|^2_{H^1_0(\Omega)}\mathrm{d}t\Big]^{\frac{1}{2}}.\end{aligned}$$

Taking the supremum with respect to ϕ, $\|\phi\|_{H^1_0(\Omega)} = 1$, we obtain that

$$\|\mathcal{R}'_\epsilon v - \mathcal{R}' v\|_{\mathcal{V}'} \leqslant \|R'_\epsilon - R'\|_{L^{2p/(p-2)}(0,T;\mathcal{L}(H^1_0(\Omega),H^{-1}(\Omega)))}\|v\|_{L^p(0,T;H^1_0(\Omega))}$$

for a $p > 2$ and since $R'_\epsilon \to R'$ in $L^q(0,T;\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)))$ for every $q < +\infty$ we conclude choosing $\mathcal{S} = \mathcal{R}_\epsilon$ for ϵ sufficiently small.

Moreover we can choose $\mathcal{S}$ in such a way $S'(0) = 0$. To do this it is sufficient to consider the function $\eta(t) = \delta^{-1}t$ in $[0,\delta]$ and $\eta(t) = 1$ in $[\delta, T]$, and choose

$$S(t) = \int_0^t \eta(\tau)R'_\epsilon(\tau)\mathrm{d}\tau + R_\epsilon(0).$$

It is sufficient to estimate $\|(\mathcal{R}_\epsilon v)' - (\mathcal{S}v)'\|_{\mathcal{V}'}$. First we estimate $\mathcal{R}_\epsilon v' - \mathcal{S}v'$ in $\mathcal{H}$:

$$\int_0^T \|R_\epsilon(t)v'(t) - S(t)v'(t)\|^2_{L^2(\Omega)}\mathrm{d}t \leqslant \|R'_\epsilon\|^2_{L^\infty(0,T;\mathcal{L}(H^1_0(\Omega),H^{-1}(\Omega)))}\|v'\|^2_{\mathcal{V}}\,\delta^2\,.$$

Similarly we can obtain $\|\mathcal{R}'_\epsilon v - \mathcal{S}'v\|_{\mathcal{V}'} \leqslant \|R'_\epsilon\|_{L^\infty(0,T;\mathcal{L}(H^1_0(\Omega),H^{-1}(\Omega)))}\|v\|_{\mathcal{V}}\sqrt{\delta}$. Since δ is arbitrary we are done. □

For the following result see Proposition 2.6 in [8].

Theorem 2.3. *For every $u, v \in \mathcal{W}$ the following holds:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}u(t), v(t))_{L^2(\Omega)} &= \langle \mathcal{R}'u(t), v(t)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} \\ &\quad + \langle \mathcal{R}u'(t), v(t)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} + \langle \mathcal{R}v'(t), u(t)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}.\end{aligned} \tag{10}$$

Moreover the function $t \mapsto (R(t)u(t), u(t))_{L^2(\Omega)}$ is continuous and there is a constant $c = c(T, \|\mathcal{R}\|)$ (depending only on $T, \|\mathcal{R}\| \leqslant C_1$) such that

$$\max_{[0,T]} |(R(t)u(t), u(t))_{L^2(\Omega)}| \leqslant c\ \|u\|^2_{\mathcal{W}}. \tag{11}$$

Remark 2.4. Observe that, if $R(t)v(x) = r(x,t)v(x)$ for every $v \in L^2(\Omega)$ and for a $r \in \mathcal{F}(C_1, C_2, \mu_0)$, by Theorem 2.3 we deduce that if $u \in \mathcal{W}$ then

$$u(t) \in L^2(\Omega^r_+(t), r(\cdot,t)) \qquad \text{for every } t \in [0,T]\,,$$

where $\Omega^r_+(t)$ is defined in (7) and $L^2(\Omega^r_+(t), r(\cdot,t))$ denotes the completion of $L^2(\Omega^r_+(t))$ with respect to the norm $\|v\|^2 := \int_\Omega v^2(x)r(x,t)\mathrm{d}x$. Observe that

$$L^2(\Omega) \subset L^2(\Omega^r_+(t), r(\cdot,t)) \quad \text{and} \quad \int_{\Omega^r_+(t)} v^2(x)r(x,t)\mathrm{d}x \leqslant C_1 \int_\Omega v^2(x)\mathrm{d}x$$

for every $v \in L^2(\Omega)$ and every $t \in [0,T]$.

We recall the definition of the class $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$, with $\lambda_0 \leqslant \Lambda_0$ and M positive real numbers, given in (1), characterised by

$$
\begin{aligned}
&a = [a_{ij}(x,t)]_{i,j=1}^{n} \qquad \text{such that}\\
&\lambda_0|\xi|^2 \leqslant \big(a(x,t)\cdot\xi,\xi\big) \leqslant \Lambda_0|\xi|^2\\
&\big|\big(a(x,t)\cdot\xi,\eta\big)\big| \leqslant M\big(a(x,t)\cdot\xi,\xi\big)^{1/2}\big(a(x,t)\cdot\eta,\eta\big)^{1/2}
\end{aligned}
\tag{12}
$$

for every $\xi,\eta\in\mathbf{R}^n$, for *a.e.* $(x,t)\in\Omega\times(0,T)$. By $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M,N)$, N non-negative constant, we denote the subclass of $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$ satisfying the further assumption

$$
\big|a(x,t)-a(x,s)\big| \leqslant N|t-s| \tag{13}
$$

for *a.e.* $x\in\Omega$ and every $s,t\in[0,T]$. For simplicity we define the family of operators

$$
\begin{aligned}
&A:[0,T]\to\mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega)) \quad \langle A(t)u,v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega \big(a(x,t)\cdot Du(x),Dv(x)\big)\mathrm{d}x\\
&\mathcal{A}:\mathcal{V}\to\mathcal{V}' \qquad \mathcal{A}u(t)=A(t)u(t).
\end{aligned}
\tag{14}
$$

Observe that under assumption (13) if we choose $a\in\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M,N)$ we can define $A':[0,T]\to\mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega))$

$$
\big\langle A'(t)u,v\big\rangle_{H^{-1}\times H_0^1} = \int_\Omega \big(a'(x,t)\cdot Du(x),Dv(x)\big)\mathrm{d}x \quad \text{where } a'_{ij}(x,t)=\frac{\partial a_{ij}}{\partial t}(x,t)
$$

which by (13) turns out to be bounded. We recall now an existence result contained in [8] (Th. 3.8). Before we give the definition of solution.

Definition 2.5. Given $a\in\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$, $r\in\mathcal{F}(C_1,C_2,\mu_0)$ with $\mu_0>-2\lambda_0$, $f\in\mathcal{V}'$, $\varphi\in L^2(\Omega_+^r(0),r(\cdot,0))$ (see Rem. 2.4 for the definition of this space) a function $u\in\mathcal{W}$ is a solution of

$$
\begin{cases}
\dfrac{\partial}{\partial t}(ru)-\operatorname{div}(a\cdot Du)=f & \text{on } \Omega\times(0,T)\\
u=0 & \text{in } \partial\Omega\times(0,T)\\
u(x,0)=\varphi & \text{in } \Omega_+^r(0)
\end{cases}
\tag{15}
$$

if

$$
\begin{aligned}
&\langle(\mathcal{R}u)'(t),v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}+\langle\mathcal{A}u(t),v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}=\langle f(t),v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\
&\int_\Omega \big(u(x,0)-\varphi(x)\big)^2 r(x,0)\,\mathrm{d}x=0
\end{aligned}
$$

for every $v\in H_0^1(\Omega)$ and for *a.e.* $t\in[0,T]$ and the initial datum makes sense in $L^2(\Omega_+^r(0),r(\cdot,0))$ thanks to Theorem 2.3. If $r\equiv 0$ the initial condition has no meaning and in this case a solution is a function $u\in\mathcal{V}$ such that

$$
\langle\mathcal{A}u(t),v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}=\langle f(t),v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}
$$

for every $v\in H_0^1(\Omega)$ and for *a.e.* $t\in[0,T]$.

Theorem 2.6. *Consider $a\in\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$, $r\in\mathcal{F}(C_1,C_2,\mu_0)$ with $\mu_0>-2\lambda_0$. For every $f\in\mathcal{V}'$ and $\varphi\in L^2(\Omega_+^r(0),r(\cdot,0))$ problem* (15) *has a unique solution $u\in\mathcal{W}$ and there exists a constant $c=c(\mu_0,\lambda_0,\Lambda_0,C_2)$*

(*depending only on* μ_0, λ_0, Λ_0, C_2) *such that*

$$\|u\|_{\mathcal{W}} \leqslant c \left[\|f\|_{\mathcal{V}'} + \|\varphi(\cdot) r^{1/2}(\cdot, 0)\|_{L^2(\Omega)}\right]. \tag{16}$$

Statement of the problem - Fix $f \in \mathcal{V}'$ and $\varphi \in L^2(\Omega)$ and consider

$$\begin{aligned} &(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N), \\ &(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0) \\ &\mu_0 > -2\lambda_0. \end{aligned} \tag{17}$$

Assumption $\mu_0 > -2\lambda_0$ is required to have existence to problems (18) (see Th. 2.6).

Then we consider the sequence of elliptic-parabolic problems

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u(x,0) = \varphi & \text{in } \Omega_{h,+}(0) \end{cases} \tag{18}$$

where $\Omega_{h,+}(t) := \Omega_+^{r_h}(t)$. We consider $\varphi \in L^2(\Omega)$ so that problem (18) makes sense for every $h \in \mathbf{N}$, since $L^2(\Omega)$ is dense in $L^2(\Omega_{h,+}(0), r_h(\cdot,0))$ for every $h \in \mathbf{N}$ (see Rem. 2.4).

We want to study the asymptotic behaviour of the solutions u_h when $h \to +\infty$ and characterise the limit problem.

The main difficulty is the lack of compactness of the solutions in $L^2(0,T;L^2(\Omega))$ which is natural in the classical case, *i.e.* when $r_h \equiv 1$.

In this framework the natural compactness result reads as follows in the theorem below (see Th. 2.14 and Th. 2.18 in [8]). Before we define

$$\mathcal{W}_h = \big\{u \in \mathcal{V} \;\big|\; (r_h u)' \in \mathcal{V}'\big\}. \tag{19}$$

Theorem 2.7. *Consider a sequence* $(u_h)_h$ *such that* $u_h \in \mathcal{W}_h$ *and* $\|u_h\|_{\mathcal{W}_h} \leqslant c$ *for a positive constant* c*. Then* $(u_h)_h$ *is relatively compact*

(i) *in* $L^2(0,T;L^2(\Omega))$ *if* $r_h \to r$ *in* $L^\infty(\Omega\times(0,T))$*-weakly* $*$ *and* $r > 0$ *almost everywhere;*

(ii) *in* $L^2(Q_+;r)$*, the completion of* $C_c(Q_+)$ *with respect to the norm* $\|ur^{1/2}\|_{L^2(Q_+)}$ *where* $Q_+ = \{(x,t) \in \Omega\times(0,T) \,|\, r(x,t) > 0\}$*, if* $r_h \to r$ *in* $L^\infty(\Omega\times(0,T))$ *(strongly).*

We by-pass the problem of the lack of compactness in $L^2(0,T;L^2(\Omega))$ *via* the following regularity result (see Th. 3.11 and Cor. 3.13 in [8]).

Theorem 2.8. *Consider the problem* (15)*. Assume that, besides to assumptions of Theorem* 2.6*,* $\partial r/\partial t \in \mathcal{F}(K_1,K_2)$*, i.e. satisfies* (i)$-$(iv) *of* (6) *with constants* K_1, K_2*, and a satisfies* (13)*. Suppose moreover that* $f \in H^1(0,T;H^{-1}(\Omega))$ *and that there exists* $u_0 \in H_0^1(\Omega)$ *such that* $u_0 = \varphi$ *in* $\Omega_+^r(0)$ *and* $f(0) + \operatorname{div}(a(x,0)Du_0(x)) - \frac{\partial r}{\partial t}(x,0) = r(x,0)v(x)$ *for some* $v \in L^2(\Omega_+^r(0))$*. Then the solution* u *satisfies*

$$u \in H^1(0,T;H_0^1(\Omega)) \qquad \textit{and} \qquad \|u\|_{H^1(0,T;H_0^1(\Omega))} \leqslant c$$

for a positive constant c *depending (only) on* $\mu_0, \lambda_0, \Lambda_0, C_2, N, K_2$, $\|f\|_{H^1(0,T;H^{-1}(\Omega))}$,
$\|r^{1/2}(\cdot,0)v\|_{L^2(\Omega)}$, $\|r^{1/2}(\cdot,0)u_0\|_{L^2(\Omega)}$.

3. Preliminary compactness results

In this section we will suppose more regularity on the sequence $(r_h)_h$ than we will require to state the main theorem (see Th. 4.5). Precisely in this section we require (see (12), (13) and (6))

$$\begin{aligned} &(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N), \\ &(r_h), \subset \mathcal{F}(C_1, C_2, \mu_0), \qquad \left(\frac{\partial r_h}{\partial t}\right)_h \subset \mathcal{F}(K_1, K_2, \nu_0) \end{aligned} \tag{20}$$

for some constants K_1, K_2, ν_0. Consider the problems

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = g & \text{on } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \psi_h & \text{in } \Omega_{h,+}(0)\times\{0\} \end{cases} \tag{21}$$

where $g \in H^1(0,T;H^{-1}(\Omega)) \subset C([0,T];H^{-1}(\Omega))$ and ψ_h is the solution to

$$\begin{cases} E_h w := -\operatorname{div}(a_h(x,0)\cdot Dw(x)) + \dfrac{\partial r_h}{\partial t}(x,0)w(x) = g(x,0) & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega, \end{cases} \tag{22}$$

where the linear, elliptic and bounded operator $E_h : H_0^1(\Omega) \to H^{-1}(\Omega)$ can be considered because a_h and r_h are continuous in the variable t.

Before stating the main result we recall the following lemma. Denote by c_P the constant appearing in the Poincaré's inequality

$$\int_\Omega u^2(x)\mathrm{d}x \leqslant c_P \int_\Omega |Du|^2(x)\mathrm{d}x, \qquad u \in H_0^1(\Omega). \tag{23}$$

Lemma 3.1. *Consider $a, a_1, a_2... \in \mathcal{M}_\Omega(\lambda_0, \Lambda_0, M)$ and suppose $a_h \xrightarrow{EG} a$ (see Def. 1.1). Consider a sequence of functions $(b_h)_h \subset L^\infty(\Omega)$, $b_h \geqslant \gamma$ for every $h \in \mathbf{N}$ where $-\lambda_0/c_P < \gamma \leqslant 0$ and suppose $b_h \to b$ in $L^\infty(\Omega)$-weak*. Then for every $f \in H^{-1}(\Omega)$ it results that*

$$\begin{aligned} &w_h \to w && \text{in } L^2(\Omega) \\ &a_h\cdot Dw_h \to a\cdot Dw && \text{in } L^2(\Omega)^n\text{-weak}, \end{aligned}$$

where w_h and w denote respectively the solutions

$$\begin{cases} -\operatorname{div}(a_h\cdot Dv) + b_h v = f & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases} \qquad \begin{cases} -\operatorname{div}(a\cdot Dv) + bv = f & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Proof. Since $-\lambda_0/c_P < \gamma \leqslant 0$ we have that

$$\int_\Omega b_h v^2 \mathrm{d}x \geqslant \int_\Omega \gamma v^2\mathrm{d}x \geqslant c_P\gamma \int_\Omega |Dv|^2\mathrm{d}x$$

and then the elliptic operators $v \mapsto -\operatorname{div}(a_h\cdot Dv) + b_h v$ are equicoercive. Since the solutions are compact in $L^2(\Omega)$ we have, up to a subsequence, that $-b_h w_h$ converges to $-bw$ weakly in $L^2(\Omega)$ and then strongly in $H^{-1}(\Omega)$. Then we obtain the thesis observing that w_h solve the problems

$$\begin{cases} -\operatorname{div}(a_h\cdot Dv) = f_h := f - b_h w_h & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

□

Remark 3.2. As a consequence we have that, if $a_h(\cdot,0) \xrightarrow{EG} a(\cdot,0)$ and $(r_h)_h \subset \mathcal{F}(C_1,C_2,\ \mu_0)$, $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(K_1,K_2,\nu_0)$ for some K_1,K_2,ν_0 and $|\frac{\partial r_h}{\partial t}|(x,0) \leqslant \alpha$ for every $h \in \mathbf{N}$ with $0 \leqslant \alpha < \lambda_0/c_P$ are such that $r_h \to r$ in $L^\infty(\Omega \times (0,T))$-weak$*$ (see also Rem. 2.1) equation (22) admits a unique solution since

$$\int_\Omega u^2(x)\frac{\partial r_h}{\partial t}(x,0)\mathrm{d}x \geqslant -\alpha \int_\Omega u^2(x)\mathrm{d}x > -\lambda_0 \int_\Omega |Du|^2(x)\mathrm{d}x.$$

The solutions ψ_h of (22) satisfy

$$\psi_h \to \psi \qquad \text{in } L^2(\Omega), \qquad a_h \cdot D\psi_h \to a \cdot D\psi \qquad \text{in } L^2(\Omega)^n\text{-weak},$$

where ψ is the solution of

$$\begin{cases} Ew := -\mathrm{div}(a(x,0)\cdot Dw(x)) + \dfrac{\partial r}{\partial t}(x,0)w(x) = g(x,0) & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega. \end{cases}$$

Now we state the first compactness result.

Lemma 3.3. *Consider the problems* (21) *with the data* $g \in H^1(0,T;H^{-1}(\Omega))$ *and* $\psi_h = E_h^{-1}g(0)$ *and denote by* $u_h \in \mathcal{W}_h$ *the corresponding solutions. Then the sequence*

$$(u_h)_h \qquad \textit{is bounded in } H^1(0,T;H_0^1(\Omega)).$$

As a consequence we obtain that

$$\begin{aligned} &(u_h)_h \quad \textit{is relatively compact in } C([0,T];L^2(\Omega)),\\ &\textit{the sequence} \quad t \mapsto \int_\Omega u_h^2(x,t)r_h(x,t)\mathrm{d}x \qquad \textit{is relatively compact in } C([0,T]). \end{aligned}$$

Proof. Since $g(0) + \mathrm{div}(a_h(\cdot,0)\cdot D\psi_h) - \dfrac{\partial r_h}{\partial t}(\cdot,0)\psi_h = 0$, hypotheses of Theorem 2.8 are satisfied. Then we have that the solutions satisfy the estimations

$$\|u_h\|_{H^1(0,T;H_0^1(\Omega))} \leqslant c, \qquad \|u_h\|_{C([0,T];H_0^1(\Omega))} \leqslant c.$$

We deduce that $(u_h)_h$ is equibounded in $C([0,T];H_0^1(\Omega))$ and therefore the sets $\{u_h(t) \mid h \in \mathbf{N}\}$ are relatively compact in $L^2(\Omega)$ for every $t \in [0,T]$. Moreover

$$u_h(t) - u_h(s) = \int_s^t u_h'(\tau)\mathrm{d}\tau$$

and then

$$\|u_h(t) - u_h(s)\|_{H_0^1(\Omega)} \leqslant \int_s^t \|u_h'(\tau)\|_{H_0^1(\Omega)}\mathrm{d}\tau \leqslant |t-s|^{1/2}\|u_h'\|_{\mathcal{V}},$$

so $(u_h)_h$ is also equicontinuous valued in $H_0^1(\Omega)$ (and in particular in $L^2(\Omega)$). Then by Lemma 1 in [12] we obtain $(u_h)_h$ relatively compact in $C([0,T];L^2(\Omega))$.

To prove the second statement denote for simplicity by $(u_h)_h$ a subsequence converging in $C([0,T];L^2(\Omega))$ and call u the limit in $C([0,T];L^2(\Omega))$. Consider the sequence $(r_h)_h$: by Remark 2.1 we have the existence of

$r \in \mathcal{F}(C_1, C_2, \mu_0)$ such that, up to a subsequence, $r_h \to r$ in $L^\infty(\Omega \times (0,T))$-weak$*$. Then, for $t, s \in [0,T]$,

$$\Big| \int_\Omega u_h^2(x,t) r_h(x,t) \mathrm{d}x - \int_\Omega u_h^2(x,s) r_h(x,s) \mathrm{d}x \Big|$$
$$\leqslant \Big| \int_\Omega u_h^2(x,t) r_h(x,t) \mathrm{d}x - \int_\Omega u_h^2(x,t) r_h(x,s) \mathrm{d}x \Big|$$
$$+ \Big| \int_\Omega u_h^2(x,t) r_h(x,s) \mathrm{d}x - \int_\Omega u_h^2(x,s) r_h(x,s) \mathrm{d}x \Big|.$$

By (6) and the continuity of $u_h : [0,T] \to L^2(\Omega)$ we infer that $t \mapsto \int_\Omega u_h^2(x,t) r_h(x,t) \mathrm{d}x$ is continuous. Moreover

$$\Big| \int_\Omega u_h^2(x,t) r_h(x,t) \mathrm{d}x - \int_\Omega u^2(x,t) r(x,t) \mathrm{d}x \Big|$$
$$\leqslant \Big| \int_\Omega u_h^2(x,t) r_h(x,t) \mathrm{d}x - \int_\Omega u^2(x,t) r_h(x,t) \mathrm{d}x \Big|$$
$$+ \Big| \int_\Omega u^2(x,t) r_h(x,t) \mathrm{d}x - \int_\Omega u^2(x,t) r(x,t) \mathrm{d}x \Big|$$
$$\leqslant \|r_h\|_\infty \int_\Omega |u_h^2(x,t) - u^2(x,t)| \mathrm{d}x + \Big| \int_\Omega u^2(x,t) \big(r_h(x,t) - r(x,t)\big) \mathrm{d}x \Big|$$

by which we obtain the thesis. □

Theorem 3.4. *Consider a sequence of functions $v_h \in \mathcal{W}_h$ $(h = 1, 2, ...)$, a function $v \in \mathcal{W}$, a constant c_1 such that*

$$\|v_h\|_{\mathcal{W}_h} \leqslant c_1 \quad \textit{for every } h, \quad v_h \to v \quad \textit{in } C([0,T]; L^2(\Omega)), \tag{24}$$

and a sequence of vectorial functions $m_h, m \in L^2(0,T;(L^2(\Omega))^n)$ $(h = 1, 2, ...)$, a constant c_2 such that

$$\|m_h\|_{L^2(0,T;(L^2(\Omega))^n)} \leqslant c_2 \qquad m_h \to m \quad \textit{in } L^2(0,T;(L^2(\Omega))^n)\textit{-weak}. \tag{25}$$

Moreover suppose that

$$(r_h v_h)' - \operatorname{div}\,(m_h) = f \in \mathcal{V}' \quad \textit{in } C^1_{\mathrm{c}}(\Omega \times (0,T)). \tag{26}$$

Then

$$\big(m_h, Dv_h\big) \to \big(m, Dv\big) \quad \textit{in } \mathcal{D}'(\Omega \times (0,T)).$$

Proof. Fix a function $\varphi \in C^\infty_{\mathrm{c}}(\Omega \times (0,T))$ and multiply equation (26) by $v_h\varphi$. We obtain

$$\int_0^T \int_\Omega (m_h, Dv_h)\varphi \,\mathrm{d}x\,\mathrm{d}t = \langle f - (r_h v_h)', v_h\varphi\rangle - \int_0^T \int_\Omega (m_h, D\varphi) v_h \,\mathrm{d}x\,\mathrm{d}t. \tag{27}$$

Clearly $\langle f, v_h\varphi\rangle \to \langle f, v\varphi\rangle$ as $h \to \infty$ and $\int_0^T \int_\Omega (m_h, D\varphi) v_h \mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega (m, D\varphi) v \mathrm{d}x\,\mathrm{d}t$. By (10) and since φ has compact support in $\Omega \times (0,T)$ we have that

$$-2\langle (r_h v_h)', v_h\varphi\rangle = \int_0^T \int_\Omega v_h^2(x,t) r_h(x,t) \frac{\partial\varphi}{\partial t}(x,t) \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega v_h^2(x,t) \varphi \frac{\partial r_h}{\partial t}(x,t) \mathrm{d}x\,\mathrm{d}t$$

which converges to

$$\int_0^T \int_\Omega v^2(x,t) r(x,t) \frac{\partial\varphi}{\partial t}(x,t) \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega v^2(x,t) \varphi \frac{\partial r}{\partial t}(x,t) \mathrm{d}x\,\mathrm{d}t = -2\langle (rv)', v\varphi\rangle.$$

Then

$$\int_0^T \int_\Omega (m_h, Dv_h)\varphi \,\mathrm{d}x\,\mathrm{d}t \to \langle f - (rv)', v\varphi\rangle - \int_0^T \int_\Omega (m, D\varphi)v \,\mathrm{d}x\,\mathrm{d}t$$

and since, multiplying (26) by $\phi \in C_c^\infty(\Omega \times (0,T))$ and taking the limit, we also have

$$(rv)' - \mathrm{div}\ (m) = f \qquad \text{in } \mathcal{V},$$

we obtain the thesis. □

4. The definition of G-convergence and the compactness result

In this section we give the main result, a compactness result with respect to G-convergence defined below (see Def. 4.3) for a sequence of operators (see (12) and (6))

$$\begin{aligned} &\mathcal{P}_h u = \frac{\partial}{\partial t}(r_h u) - \mathrm{div}(a_h \cdot Du), \\ &(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N), \qquad (r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0). \end{aligned} \tag{28}$$

The statement of this result is divided in two theorems. In the first one (Th. 4.1) we suppose the regularity required in the previous section, *i.e.* (20), and prove a partial result: the existence of a limit operator in divergence form. The second result (Th. 4.5) is a kind of uniqueness result: with less assumptions on the coefficients, *i.e.* satisfying the assumptions in (28), we will prove that the matrix defining the limit operator is independent of the sequence (r_h).

Theorem 4.1. *Consider a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ and $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ with $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(K_1, K_2, \nu_0)$ and $\frac{\partial r_h}{\partial t}(x, 0) = 0$ for every $h \in \mathbf{N}$. There exist a matrix $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\ M^2\Lambda_0, M\sqrt{\Lambda_0/\lambda_0})$ and a function $r \in \mathcal{F}(C_1, C_2, \mu_0)$ such that for every $f \in \mathcal{V}'$ and $\varphi \in L^2(\Omega)$ the solutions u_h of problems (18), $h \in \mathbf{N}$, satisfy, up to a subsequence,*

$$u_h \to u \quad \text{in } L^2(0,T;L^2(\Omega)) \qquad \text{and} \qquad a_h \cdot Du_h \to a \cdot Du \quad \text{in } L^2(0,T,L^2(\Omega)^n)\text{-weak}$$

where u is the solution of

$$\begin{cases} \dfrac{\partial(ru)}{\partial t} - \mathrm{div}(a \cdot Du) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_+^r(0) \times \{0\}. \end{cases} \tag{29}$$

Remark 4.2. The result is true also if in (18) we consider a sequence of data $(f_h)_h \subset \mathcal{V}'$, $(f_h)_h$ relatively compact in $\mathcal{V}'$, and $(\varphi_h)_h \subset L^2(\Omega)$, φ_h relatively compact in $L^2(\Omega)$.

Proof. First, by Remark 2.1, from $(r_h)_h$ we can extract a subsequence, still denoted by $(r_h)_h$, such that $r_h \to r$ in $L^\infty(\Omega \times (0,T)$-weak $*$ and $r \in \mathcal{F}(C_1, C_2, \mu_0)$. Denote by $\mathcal{R}$ the operator defined by

$$\mathcal{R} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega)), \qquad (\mathcal{R}u)(x,t) = r(x,t)u(x,t). \tag{30}$$

Analogously (to short) we denote by $\mathcal{R}_h$ the operators associated to r_h and by $\mathcal{A}_h$ the operators associated to a_h as defined in (14). Fix X a countable and dense subset of $H^1(0,T;H^{-1}(\Omega))$ and $Y = \{y \in H_0^1(\Omega) \mid y = E^{-1}g(0)$ for $g \in X\}$ where E is the operator defined in Remark 3.2 (the EG-limit, up to a subsequence, of $\{a_h(\cdot,0)\}_h$). Then consider $g \in X$ and $\psi = E^{-1}g(0)$ (and denote by $u_h(g,\psi)$ the solutions to the problems (21) with $\psi_h = E_h^{-1}\big(E\psi\big)$ where E_h are the operators in (22).

By Lemma 3.3 we have that the solutions $u_h(g,\psi)$ are compact in $C([0,T];L^2(\Omega))$, and in particular $\psi_h \to \psi$ in $L^2(\Omega)$. Denote by $\mathcal{B}_\mathcal{R}(g,\psi)$ the limit in $C([0,T];L^2(\Omega))$ of $u_h(g,\psi)$. In this way we have defined an operator in $Z = \{(g,\psi) \in X \times Y \,|\, \psi = E^{-1}g(0)\}$. $\mathcal{B}_\mathcal{R}$ is linear and continuous (see Th. 2.6). For every $h \in \mathbf{N}$ we have

$$\|u_h(g,\psi)\|_{\mathcal{W}_h} \leqslant c\,\left[\|g\|_{\mathcal{V}'} + \|\sqrt{r_h(\cdot,0)}\psi_h\|_{L^2(\Omega)}\right].$$

Since $(r_h u_h)' \to (ru)'$ weakly in $\mathcal{V}'$ and, by Lemma 3.3, $\sqrt{r_h(\cdot,0)}\psi_h \to \sqrt{r(\cdot,0)}\psi$ in $L^2(\Omega)$, by the lower semicontinuity of the norm we obtain

$$\|\mathcal{B}_\mathcal{R}(g,\psi)\|_{\mathcal{W}} \leqslant c\,\left[\|g\|_{\mathcal{V}'} + \|\sqrt{r(\cdot,0)}\psi\|_{L^2(\Omega)}\right]. \tag{31}$$

Since Z is dense in $\mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0))$, we can extend $\mathcal{B}_\mathcal{R}$ (and denote it in the same way)

$$\mathcal{B}_\mathcal{R} : \mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0)) \to \mathcal{W}.$$

Then we define

$$\begin{aligned}
&\mathcal{K}_\mathcal{R} : \mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0)) \to \mathcal{V}',\\
&\mathcal{K}_\mathcal{R} f = f - \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi)\big),\\
&\|\mathcal{K}_\mathcal{R}(f,\varphi)\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant c'\,\left[\|f\|_{\mathcal{V}'} + \|\sqrt{r(\cdot,0)}\varphi\|_{L^2(\Omega)}\right].
\end{aligned}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}_h u_h) + \mathcal{A}_h u_h = f \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi)) + \mathcal{K}_\mathcal{R}(f,\varphi) = f. \tag{32}$$

Multiplying the first in (32) by u_h we have (where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{V}'$ and $\mathcal{V}$)

$$\begin{aligned}
\frac{1}{2}\int_\Omega u_h^2(x,T)r_h(x,T)\mathrm{d}x - \frac{1}{2}\int_\Omega \varphi^2(x)r_h(x,0)\mathrm{d}x + \int_0^T\int_\Omega \frac{\partial r_h}{\partial t}\,u_h^2\,\mathrm{d}x\,\mathrm{d}t&\\
+\int_0^T\int_\Omega \big(a_h\cdot Du_h, Du_h\big)\mathrm{d}x\ \mathrm{d}t &= \langle f, u_h\rangle
\end{aligned}$$

and the second by $\mathcal{B}_\mathcal{R}(f,\varphi)$ we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega \mathcal{B}_\mathcal{R}(f,\varphi)^2(x,T)r(x,T)\mathrm{d}x - \frac{1}{2}\int_\Omega \varphi^2(x)r(x,0)\mathrm{d}x + \int_0^T\int_\Omega \frac{\partial r}{\partial t}\mathcal{B}_\mathcal{R}(f,\varphi)^2\mathrm{d}x\ \mathrm{d}t&\\
+\langle\mathcal{K}_\mathcal{R}(f,\varphi), \mathcal{B}_\mathcal{R}(f,\varphi)\rangle &= \langle f, \mathcal{B}_\mathcal{R}(f,\varphi)\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&\langle f, u_h\rangle \to \langle f, \mathcal{B}_\mathcal{R}(f,\varphi)\rangle\\
&\int_\Omega u_h^2(x,T)r_h(x,T)\mathrm{d}x \to \int_\Omega \mathcal{B}_\mathcal{R}(f,\varphi)^2(x,T)r(x,T)\mathrm{d}x\\
&\int_\Omega \varphi^2(x)r_h(x,0)\mathrm{d}x \to \int_\Omega \varphi^2(x)r(x,0)\mathrm{d}x\\
&\int_0^T\int_\Omega \frac{\partial r_h}{\partial t}u_h^2\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \frac{\partial r}{\partial t}\mathcal{B}_\mathcal{R}(f,\varphi)^2\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{33}$$

we deduce that

$$\int_0^T \int_\Omega \big(a_h \cdot Du_h, Du_h\big)\mathrm{d}x\,\mathrm{d}t = \langle \mathcal{A}_h u_h, u_h\rangle \to \langle \mathcal{K}_\mathcal{R}(f,\varphi), \mathcal{B}_\mathcal{R}(f,\varphi)\rangle. \tag{34}$$

The operator $\mathcal{B}_\mathcal{R}$ is injective: indeed if $\mathcal{B}_\mathcal{R}(f,\varphi)=0$, $\langle \mathcal{A}_h u_h, u_h\rangle \to 0$ and then $\|u_h\|_\mathcal{V} \to 0$. Since $(\mathcal{R}_h u_h)' \to (\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi))'$ weakly in $\mathcal{V}'$ we have that $\mathcal{A}_h u_h \to \mathcal{K}_\mathcal{R}(f,\varphi)$ weakly in $\mathcal{V}'$. Since $\mathcal{A}_h$ are equibounded and $u_h \to 0$ we conclude that $\mathcal{K}_\mathcal{R}(f,\varphi)=0$, *i.e.* $f - \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi)) = 0$. Than $f=0$. Clearly, since $\mathcal{B}_\mathcal{R}(f,\varphi)=0$ and, by Theorem 2.3, we also have that $\varphi = 0$ in $L^2(\Omega_+^r(0), r(\cdot,0))$.
Now we show that $\mathcal{B}_\mathcal{R}(\mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0)))$ is dense in $\mathcal{V}$: fix $g \in \mathcal{V}'$ such that

$$\langle g, \mathcal{B}_\mathcal{R}(f,\varphi)\rangle = 0 \qquad \text{for every } f \in \mathcal{V}, \varphi \in L^2(\Omega_+^r(0), r(\cdot,0)).$$

In particular $\langle g, \mathcal{B}_\mathcal{R}(g,0)\rangle = 0$ and then

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{R}\mathcal{B}_\mathcal{R}(g,0)\big) + \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,0)\right\rangle = 0$$

i.e. $\langle \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,0)\rangle \leqslant 0$. But by (34) we know that $\langle \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,0)\rangle \geqslant 0$, then it is zero. Therefore $u_h(g,0) \to 0$ and as above we deduce that $g=0$.

Thus we can define the inverse of $\mathcal{B}_\mathcal{R}$: if we define

$$\begin{array}{rccc} \mathcal{A}_\mathcal{R} : & \mathcal{B}_\mathcal{R}\big(\mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0))\big) & \longrightarrow & \mathcal{V}' \\ & \mathcal{B}_\mathcal{R}(f,\varphi) & \mapsto & \mathcal{K}_\mathcal{R}(f,\varphi), \end{array}$$

by density we can define an operator, still denoted by $\mathcal{A}_\mathcal{R}$,

$$\mathcal{A}_\mathcal{R} : \mathcal{V} \to \mathcal{V}'.$$

Now we need to define a last operator: for every $(g,\psi) \in (X,Y)$ the sequence $a_h \cdot Du_h(g,\psi)$ is bounded in $L^2(0,T;L^2(\Omega)^n)$ and, up to a subsequence, weakly converges in $L^2(0,T;L^2(\Omega)^n)$

$$a_h \cdot Du_h(g,\psi) \to \mu(g,\psi). \tag{35}$$

We have that

$$\begin{array}{rcl} |a_h(x,t)\cdot Du_h(g,\psi)(x,t)| & = & \sup_{|\eta|=1}\big(a_h(x,t)\cdot Du_h(x,t), \eta\big) \\ & \leqslant & M\Lambda_0^{1/2}\big(a_h(x,t)\cdot Du_h(x,t), Du_h(x,t)\big)^{1/2}. \end{array}$$

By Lemma 7.8 in [2] and Theorem 3.4 we have that

$$|\mu(g,\psi)| \leqslant M\Lambda_0^{1/2}\big(\mu(g,\psi), D\mathcal{B}_\mathcal{R}(g,\psi)\big)^{1/2}$$

from which

$$|\mu(g,\psi)| \leqslant M^2\Lambda_0 |D\mathcal{B}_\mathcal{R}(g,\psi)|. \tag{36}$$

Again by density, using (31), we can extend

$$\mu : \mathcal{V}' \times L^2(\Omega_+^r(0), r(\cdot,0)) \to L^2(0,T;L^2(\Omega)^n).$$

We define now $\mathcal{P}_\mathcal{R} u := (\mathcal{R}u)' + \mathcal{A}_\mathcal{R} u$ and

$$\begin{array}{rcccc} M_\mathcal{R} & : & \mathcal{W} & \longrightarrow & L^2(0,T;L^2(\Omega)^n) \\ & & u & \mapsto & \mu\big(\mathcal{P}_\mathcal{R} u, P_+(0)u(0)\big) \end{array} \tag{37}$$

where $P_+(0)w$ is the restriction to $\Omega_+^r(0)$ of a function w defined in Ω. Observe that by (36), (32) and the definition of $\mathcal{A}_\mathcal{R}$ and $\mathcal{P}_\mathcal{R}$ one has, for every $u \in \mathcal{W}$,

$$|M_\mathcal{R} u| \leqslant M^2 \Lambda_0 |Du|. \tag{38}$$

By definition we have that for every $v \in \mathcal{V}$

$$\langle \mathcal{A}_\mathcal{R} u, v\rangle = \int_0^T \int_\Omega \big(M_\mathcal{R} u, Dv\big) \mathrm{d}x \ \mathrm{d}t. \tag{39}$$

Now we want to construct a matrix a such that for every $u, v \in \mathcal{V}$

$$\int_0^T \int_\Omega \big(M_\mathcal{R} u, Dv\big) \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_\Omega \big(a \cdot Du, Dv\big) \mathrm{d}x \ \mathrm{d}t.$$

For this purpose fix $\omega \subset\subset \Omega$ and a function $\eta \in C^1(\overline{\Omega} \times [0,T])$ with $\eta(\cdot,t) \in C^1_c(\Omega)$ for every $t \in [0,T]$ such that $\eta = 1$ on $\omega \times [0,T]$. Define $\phi_\xi(x,t) = (\xi,x)\eta(x,t)$ ((ξ,x) denotes the scalar product in $\mathbf{R}^n$). Finally, if r is the function in (30), define

$$a_r(x,t) \cdot \xi = M_\mathcal{R}\big(\mathcal{P}_\mathcal{R}\phi_\xi, P_+(0)\phi_\xi(x,0)\big) \qquad \text{for } (x,t) \in \omega \times [0,T].$$

From (39) we obtain that

$$\langle \mathcal{A}_\mathcal{R} u, v\rangle = \int_0^T \int_\Omega \big(a_r Du, Dv\big) \mathrm{d}x \ \mathrm{d}t \tag{40}$$

and arguing as in [15], Theorem 3, we obtain

$$\lambda_0 |\xi|^2 \leqslant (a_r(x,t) \cdot \xi, \xi), \qquad (a_r(x,t) \cdot \xi, \eta) \leqslant M \Lambda_0^{1/2} (a_r(x,t) \cdot \xi, \xi)^{1/2} |\eta|^{1/2}$$

for *a.e.* $(x,t) \in \Omega \times (0,T)$ and for every $\xi, \eta \in \mathbf{R}^n$. Then also for the operator $\mathcal{A}_\mathcal{R}$ we have

$$\lambda_0 \|u\|_\mathcal{V} \leqslant \langle \mathcal{A}_\mathcal{R} u, u\rangle \quad \text{and} \quad |\langle \mathcal{A}_\mathcal{R} u, v\rangle| \leqslant M^2 \Lambda_0 \|u\|_\mathcal{V} \|v\|_\mathcal{V}. \qquad \square$$

Definition 4.3. Consider a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$. We say that

$$a_h \xrightarrow{G} a \qquad \text{in } \Omega \times (0,T)$$

if for every $f \in \mathcal{V}'$, for every $\varphi \in L^2(\Omega)$ and for every $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ and $r \in \mathcal{F}(C_1, C_2, \mu_0)$ with $\mu_0 > -2\lambda_0$ and

$$r_h \to r \qquad L^\infty(\Omega \times (0,T))\text{-weak}*$$

it results that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,L^2(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n)\text{-weak}, \end{array}$$

where u_h and u denote respectively the solutions of

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h v) - \operatorname{div}(a_h \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \\ v = \varphi & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial t}(r v) - \operatorname{div}(a \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \\ v = \varphi & \text{in } \Omega^r_+(0) \times \{0\}. \end{cases}$$

Before stating the main result we need a short and preliminary lemma regarding the following problems

$$\begin{cases} A_h u = -\mathrm{div}(a_h \cdot Du) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \end{cases} \tag{41}$$

where $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$.

Lemma 4.4. *Consider the problems* (41) *with* $f \in H^1(0,T; H^{-1}(\Omega))$ *and suppose the solutions* u_h *satisfy*

$$u_h \to u \quad \text{in } L^2(0,T;L^2(\Omega)) \qquad \text{and} \qquad a_h \cdot Du_h \to a \cdot Du \quad \text{in } L^2(0,T,L^2(\Omega)^n)\text{-weak}$$

where u *is the solution of*

$$\begin{cases} Au = -\mathrm{div}(a \cdot Du) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T). \end{cases}$$

Then $t \mapsto a(\cdot,t)$ *is continuous* ($\lim_{t\to s} |||a(\cdot,t) - a(\cdot,s)|||_{L^\infty(\Omega)} = 0$ *for every* $s \in [0,T]$) *and* $a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t)$ *for every* $t \in [0,T]$.

Proof. Let u_h be the solution of (41), where $A_h : \mathcal{V} \to \mathcal{V}'$. Observe that $u_h(t)$ solves $A_h(t)u_h(t) = f(t)$ for almost every $t \in [0,T]$. Then, by Theorem 2.8, $(u_h)_h$ is equibounded and equicontinuous in $[0,T]$ with respect to the $H^1_0(\Omega)$-norm and moreover $u_h \to u$ in $C([0,T];L^2(\Omega))$. Then we have for every $t \in [0,T]$

$$A_h^{-1}(t)f(t) \to A^{-1}(t)f(t) \quad \text{in } L^2(\Omega)\,.$$

Moreover we have that, for every $\Phi \in L^2(\Omega)^n$, the functions $t \mapsto \int_\Omega (a_h(x,t)Du_h(x,t), \Phi)\mathrm{d}x$ are equicontinuous and equibounded. Indeed

$$\Big| \int_\Omega (a_h(x,t)Du_h(x,t), \Phi)\mathrm{d}x - \int_\Omega (a_h(x,s)Du_h(x,s), \Phi)\mathrm{d}x\Big| \leqslant$$
$$\leqslant N|t-s| \int_\Omega \Big|(Du_h(x,t), \Phi)\Big|\mathrm{d}x + M\Lambda_0 \Big[\int_\Omega \Big|Du_h(x,t) - Du_h(x,s)\Big|^2 \mathrm{d}x \int_\Omega \big|\Phi\big|^2 \mathrm{d}x\Big]^{1/2}.$$

By equicontinuity of $(u_h)_h$ in $H^1_0(\Omega)$ we conclude that the sequence $t \mapsto \int_\Omega (a_h(x,t)Du_h(x,t),\, \Phi)\mathrm{d}x$ is relatively compact in $C([0,T])$, and consequently there is a subsequence $(a_{h_j})_j$ (since $H^1_0(\Omega)$ is separable) and a vectorial function V such that for every $\Phi \in L^2(\Omega)^n$ and $\eta \in C[0,T]$, one has

$$\int_0^T \eta(t)\Big[\int_\Omega (a_{h_j}(x,t)Du_{h_j}(x,t), \Phi(x))\mathrm{d}x\Big]\mathrm{d}t \to \int_0^T \eta(t)\Big[\int_\Omega (V(x,t), \Phi(x))\mathrm{d}x\Big]\mathrm{d}t.$$

By assumptions we conclude that $V = a \cdot Du$. Finally, since this can be derived for every subsequence of $(a_h \cdot Du_h)_h$ we conclude that the whole sequence $a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t)$ for every $t \in [0,T]$. The continuity follows by Theorem 2.4 in [3]. □

Theorem 4.5. *The class* $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ *of matrices satisfying* (2) *is relatively compact with respect to G-convergence.*

Proof. Consider $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$, $f \in \mathcal{V}'$ and problems (18) with $r_h \equiv 0$ for every $h \in \mathbf{N}$ (see Def. (2.5)) and denote by w_h the corresponding solutions. By Theorem 4.1 we have that there exists a subsequence $(a_{h_k})_k$ and a matrix

$$\bar{a} \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, M^2\Lambda_0, M\sqrt{\Lambda_0/\lambda_0}) \tag{42}$$

such that the sequence $(w_{h_k})_k$ satisfies

$$w_{h_k} \to w \quad \text{in } L^2(0,T;L^2(\Omega)) \qquad \text{and} \qquad a_{h_k} \cdot Dw_{h_k} \to \bar{a} \cdot Dw \quad \text{in } L^2(0,T;L^2(\Omega)^n)\text{-weak}$$

where w is the solution of

$$\begin{cases} -\mathrm{div}(\bar{a}(x,t) \cdot Du(x,t)) = f(x,t) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T)\,. \end{cases}$$

Now consider a sequence $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ with $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(K_1, K_2, \nu_0)$. Up to a subsequence

$$r_h \to r \quad \text{and} \quad \frac{\partial r_h}{\partial t} \to \frac{\partial r}{\partial t} \qquad \text{in } L^\infty(\Omega \times (0,T))\text{-weak}*$$

for a $r \in \mathcal{F}(C_1, C_2, \mu_0)$ with $\frac{\partial r}{\partial t} \in \mathcal{F}(K_1, K_2, \nu_0)$ (see Rem. 2.1). Consider this function r and the matrix $\bar{a}$ in (42) and define the operators in $\mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$

$$E_h(t)u := -\mathrm{div}(a_h(\cdot,t) \cdot Du) + \frac{\partial r_h}{\partial t}(\cdot,t)u\,, \qquad E(t)u := -\mathrm{div}(\bar{a}(\cdot,t) \cdot Du) + \frac{\partial r}{\partial t}(\cdot,t)u$$

for every $t \in [0,T]$ (this is possible thanks to Lem. 4.4). Now consider $f \in \mathcal{V}'$ and $\varphi \in L^2(\Omega)$, the following problems

$$\begin{cases} \frac{\partial}{\partial t}(r_h u) - \mathrm{div}(a_h \cdot Du) = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases}$$

and let u_h be the corresponding solutions. For every $\epsilon > 0$ we can find $\psi \in H_0^1(\Omega)$ and $g \in H^1(0,T;H^{-1}(\Omega))$ such that

$$E(0)\psi = g(0)\,, \qquad \|\varphi - \psi\|_{L^2(\Omega)} < \epsilon/2\,, \qquad \|f - g\|_{\mathcal{V}'} < \epsilon/2$$

and define $\psi_h := E_h(0)^{-1} g(0)$. We have that the sequence of the solutions $(v_h)_h$ of

$$\begin{cases} \frac{\partial}{\partial t}(r_h u) - \mathrm{div}(a_h \cdot Du) = g & \text{on } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \psi_h & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases}$$

satisfies, by Theorem 2.6,

$$\begin{aligned} &\|u_h - v_h\|_{L^2(0,T;L^2(\Omega))} \leqslant c(C_1)\epsilon \\ &\|a_h \cdot Du_h - a_h \cdot Dv_h\|_{L^2(0,T;L^2(\Omega)^n)} \leqslant c(M, \Lambda_0)\epsilon \end{aligned} \tag{43}$$

and moreover is relatively compact in $C([0,T];L^2(\Omega))$ by Lemma 3.3. Denote by $\mathcal{R}_h$ the operator in $\mathcal{L}(L^2(0,T;L^2(\Omega)))$ defined by $\mathcal{R}_h u = r_h u$ and $\mathcal{R}$ the operator in $\mathcal{L}(L^2(0,T;L^2(\Omega)))$ defined by $\mathcal{R}u = ru$. By Theorem 2.8 we have that, in particular, $(\mathcal{R}_h v_h)'$ is relatively compact in $C([0,T];H^{-1}(\Omega))$. Then, up to a subsequence,

$$(\mathcal{R}_h v_h)' \to (\mathcal{R}v)' \qquad \text{in } C([0,T];H^{-1}(\Omega)).$$

Observe then that v_h solves the following problem in $\mathcal{V}$

$$\begin{cases} -\operatorname{div}(a_h(x,t)\cdot Dv(x,t)) = g(x,t) - \dfrac{\partial}{\partial t}\big(r_h(x,t)v_h(x,t)\big) =: g_h(x,t) & \text{in } \Omega\times(0,T) \\ v = 0 & \text{in } \partial\Omega\times(0,T) \end{cases}$$

with, up to a subsequence, $g_h \to g - \frac{\partial(rv)}{\partial t}$ strongly in $\mathcal{V}'$ and moreover $v_h(x,0) = \psi_h(x)$ in $\Omega_{h,+}(0)$. By Remark 4.2 we have that if we consider the subsequence $(a_{h_k})_k$ (or if necessary extracting from this another subsequence because of $(r_h)_h$) we have that

$$\begin{aligned} &v_{h_k} \to v \quad \text{in } C([0,T];L^2(\Omega)) \qquad \text{and} \\ &a_{h_k}\cdot Dv_{h_k} \to \bar a\cdot Dv \quad \text{in } L^2(0,T;L^2(\Omega)^n)\text{-weak} \end{aligned} \tag{44}$$

where v is the solution of

$$\begin{cases} -\operatorname{div}(\bar a(x,t)\cdot Dv(x)) = f(x,t) - \dfrac{\partial}{\partial t}(r(x,t)v(x,t)) & \text{in } \Omega\times(0,T) \\ v = 0 & \text{in } \partial\Omega\times(0,T) \end{cases}$$

and moreover, by Lemma 3.3,

$$\big(r_{h_k}(x,0)\big)^{1/2} v_{h_k}(x,0) \to \big(r(x,0)\big)^{1/2} v(x,0) \qquad \text{in } L^2(\Omega).$$

Since $(r_{h_k}(x,0))^{1/2}v_{h_k}(x,0) = (r_{h_k}(x,0))^{1/2}\psi_{h_k}(x)$ and $\psi_{h_k} \to \psi$ in $L^2(\Omega)$ (by Lemma 3.1 and Remark 3.2) we conclude that v solves

$$\begin{cases} \dfrac{\partial}{\partial t}(rv) - \operatorname{div}(\bar a\cdot Dv) = g & \text{on } \Omega\times(0,T) \\ v = 0 & \text{in } \partial\Omega\times(0,T) \\ v = \psi & \text{in } \Omega^r_+(0)\times\{0\}. \end{cases}$$

Now consider the solution u to the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(ru) - \operatorname{div}(\bar a\cdot Du) = f & \text{on } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \varphi & \text{in } \Omega^r_+(0)\times\{0\}. \end{cases}$$

Then taking the limit in

$$\|u_{h_k} - u\|_{\mathcal{V}} \leqslant \|u_{h_k} - v_{h_k}\|_{\mathcal{V}} + \|v_{h_k} - v\|_{\mathcal{V}} + \|v - u\|_{\mathcal{V}}$$

we obtain thanks to (43) and (44) that $\lim_k \|u_{h_k} - u\|_{\mathcal{V}} \leqslant c\,\epsilon$. In the same way we obtain that

$$\Big|\lim_k \int_0^T \int_\Omega \big[\big(a_{h_k}\cdot Du_{h_k},\Phi\big) - \big(\bar a\cdot Du,\Phi\big)\big]\mathrm{d}x\ \mathrm{d}t\Big| \leqslant c\,\epsilon$$

for every $\Phi \in C^1_c(\Omega)^n$. This concludes the proof that the matrix $\bar a$ does not depend on the sequence $(r_h)_h$ and on the limit r.

Now we show that the hypothesis $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(K_1,K_2,\nu_0)$ can be dropped. Then suppose only $(r_h)_h \subset \mathcal{F}(C_1,C_2,\mu_0)$, let r be the limit of r_h as before and denote by u_h the solutions to problems (18). By Lemma 2.2 for every $h \in \mathbf{N}$ we can consider a sequence of functions $v_h \in \mathcal{V}$ and a sequence of functions s_h, $(s_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ and $(\frac{\partial s_h}{\partial t})_h \subset \mathcal{F}(C_2, K, -K)$, with K independent of h, such that (we denote by $\mathcal{S}_h$ the operator defined by $\mathcal{S}_h w = s_h w$ and recall the definition of $\mathcal{W}_h$ is given in (19))

$$\|u_h - v_h\|_{\mathcal{W}_h} < \frac{1}{h}, \qquad \|(\mathcal{R}_h u_h)' - (\mathcal{S}_h v_h)'\|_{\mathcal{V}'} < \frac{1}{h}, \qquad \text{and moreover } S_h'(0) = 0.$$

By the first of these estimates and by (11) we obtain that

$$\int_\Omega |u_h(x,0) - v_h(x,0)|^2 r_h(x,0)\mathrm{d}x < c\,\frac{1}{h^2}$$

and since $u_h(x,0) = \varphi(x)$ in $\Omega_{h,+}$ we have

$$\int_\Omega |\varphi(x) - v_h(x,0)|^2 r_h(x,0)\mathrm{d}x < c\,\frac{1}{h^2}\cdot$$

Observe that the function v_h solve the following problem

$$\begin{cases} (\mathcal{S}_h w)' + \mathcal{A}_h w = f_h := f + (\mathcal{S}_h v_h)' - (\mathcal{R}_h u_h)' + \mathcal{A}_h v_h - \mathcal{A}_h u_h \\ P_{+,h}(0) w(0) = \varphi_h := \varphi + \big(v_h(0) - u_h(0)\big). \end{cases}$$

By Lemma 2.2 we have that $f_h \to f$ in $\mathcal{V}'$ and finally, by Remark 4.2 since $\mathcal{S}_h$ satisfy hypotheses of Theorem 4.1, we obtain, taking the limit, that there exist a function $v \in \mathcal{V}$ and a function $s \in \mathcal{F}(C_1, C_2, \mu_0)$ such that, up to a subsequence,

$$\begin{aligned} & v_h \to v \quad \text{ in } L^2(0,T; L^2(\Omega)) \\ & a_h \cdot D v_h \to \bar{a} \cdot Dv \quad \text{ in } L^2(0,T; L^2(\Omega)^n)\text{-weak}, \end{aligned}$$

where v solves the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(sv) - \operatorname{div}(\bar{a} \cdot Dv) = f & \text{on } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \\ v = \varphi & \text{in } \Omega_+^s(0) \times \{0\}. \end{cases}$$

By estimations above we deduce that $v = u$ and that $s = r$ and that

$$\begin{aligned} & u_h \to u \quad \text{ in } L^2(0,T; L^2(\Omega)) \\ & a_h \cdot D u_h \to \bar{a} \cdot Du \quad \text{ in } L^2(0,T; L^2(\Omega)^n)\text{-weak} \end{aligned}$$

and then the result is completely proved. □

5. Examples

In this section we present first some possible choices in the class $\mathcal{F}(C_1, C_2, \mu_0)$, then some particular cases of G-convergence: in Subsection 5.2 the classical variational convergences, in 5.3 the result in homogenization, in 5.4 the singular perturbations, in which the convergence of the solutions is stronger (Lem. 5.1).

5.1. **Examples of admissible** r

1) $r \equiv r(x)$ – Besides $r \equiv 1$ and $r \equiv 0$, one can consider more general situations. Every non-negative $r \in L^\infty(\Omega)$, *i.e.* $r = r(x)$ depending only on x, belongs to the class $\mathcal{F}(C_1, C_2, \mu_0)$ with $C_1 = \|r\|_\infty$ and $C_2 = \mu_0 = 0$. Then also

$$r(x) = \begin{cases} 1 & \text{in a Cantor type set } \Omega_+ \\ 0 & \text{outside } \Omega_+, \end{cases}$$

Ω_+ with positive measure, is admitted.

2) $r = r(t)$ – Every regular function satisfying suitable bounds can be admitted. Since

$$t \mapsto \int_\Omega u(x)v(x)r(t)\mathrm{d}x$$

is required to be absolutely continuous we will require $r \in W^{1,\infty}(0,T)$. In this way

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x)v(x)r(t)\mathrm{d}x = r'(t)\int_\Omega u(x)v(x)\mathrm{d}x$$

and it is sufficient to require (remember $\mu_0 \leqslant 0$)

$$0 \leqslant r \leqslant C_1\,, \qquad \frac{\mu_0}{c_P} \leqslant r'(t) \leqslant C_2,$$

where c_P is the constant appearing in (23), to have $r \in \mathcal{F}(C_1, C_2, \mu_0)$.

As particular case we want to observe that if $r(0) = 0$ and $r(t) > 0$ for every $t > 0$ the initial condition is not needed. For example, if $r(t) = t$ the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(tv) - \operatorname{div}(a(x,t)\cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \end{cases}$$

has a unique solution (without any condition at time $t = 0$).

3) $r = r(x,t)$ - If $r, \frac{\partial r}{\partial t} \in L^\infty(\Omega \times (0,T))$ then $r \in \mathcal{F}(C_1, C_2, \mu_0)$ with $C_1 = \|r\|_\infty$ and $C_2 = \|\frac{\partial r}{\partial t}\|_\infty$. Precisely, if c_P is the constant appearing in (23), r can be choosen such that

$$0 \leqslant r \leqslant C_1\,, \qquad \frac{\mu_0}{c_P} \leqslant \frac{\partial r}{\partial t} \leqslant C_2$$

($C_2 \geqslant 0$, $\mu_0 \leqslant 0$).

Also functions r for which $\frac{\partial r}{\partial t} \notin L^\infty(\Omega \times (0,T))$ can be considered. For example if

$$r(x,t) = \chi_A(x,t), \qquad A \subset \Omega \times (0,T).$$

In this case, if we denote $\Omega_+(t) = \{x \in \Omega \mid r(\cdot,t) > 0\}$, we need

$$t \mapsto \int_{\Omega_+(t)} u(x)v(x)\mathrm{d}x \qquad \text{differentiable.} \tag{45}$$

We refer to [8] for more details and to [6] (Prop. 3, Sect. 3.4.4) for differentiability of (45).

5.2. Variational convergences

4) If $r_h \equiv 0$ for every h we have a result for a class of elliptic operators and we have that

$$a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t) \quad \text{in } \Omega \quad \text{for } a.e.\ t \in [0,T].$$

5) If $r_h \equiv 1$ for every h we have a result for a class of parabolic operators and we have that

$$a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h \xrightarrow{PG} a \quad \text{in } \Omega \times (0,T).$$

6) Suppose a_h to be symmetric matrices. Then, choosing $r_h \equiv 0$ and using the classical result (see for instance [4] for the definition of Γ-convergence) we obtain that

$$a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h(\cdot,t) \xrightarrow{\Gamma} a(\cdot,t) \quad \text{in } \Omega \quad \text{for } a.e.\ t \in [0,T].$$

5.3. Homogenization

7) If $r(x)$ is a Q-periodic function in the variable x and a_{ij}, the entries of a matrix $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$, are Q-periodic in the variable x, Q cube of $\mathbf{R}^n$, then the solution (see [9]) of

$$\begin{cases} r(hx)\dfrac{\partial u}{\partial t} - \operatorname{div}(a(hx,t)\cdot Du) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{h,+} \times \{0\} \end{cases} \tag{46}$$

converge in $L^2(0,T;L^2(\Omega))$ to the solution of the problem

$$\begin{cases} \left[\dfrac{1}{|Q|}\displaystyle\int_Q r\right]\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \widehat{a}_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega \times \{0\} \end{cases}$$

(for the definition of $\widehat{a}$ see for example [9]). Notice that if the mean value of r $|Q|^{-1}\int_Q r > 0$ the limit problem is given by a standard parabolic equation and in this case the initial condition is obtained in all Ω even if for every h the problems above are partially elliptic and partially parabolic. This happens also if r is positive only on a Cantor set of positive measure.

The only case in which the limit problem is elliptic is when $r \equiv 0$: in this case problems (46) are the sequence of elliptic problems

$$\begin{cases} -\operatorname{div}(a(hx,t)\cdot Du) = f(x,t) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \qquad \text{for } a.e.\ t \in (0,T)$$

and the initial conditions $u = \varphi$ in $\Omega_{h,+} \times \{0\}$ are meaningless.

5.4. Singular perturbations

8) Suppose to have a fixed elliptic operator $u \mapsto -\mathrm{div}(a(x,t)\cdot Du)$, with $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\, \Lambda_0, M)$. Then $a \xrightarrow{G} a$ in particular means that the solutions of

$$\begin{cases} \dfrac{1}{h}\dfrac{\partial u}{\partial t} - \mathrm{div}(a\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \varphi & \text{in } \Omega\times\{0\} \end{cases}$$

converge in $L^2(0,T;L^2(\Omega))$ to the solution of the problem

$$\begin{cases} -\mathrm{div}(a\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T). \end{cases}$$

Indeed $r_h = 1/h$ and $r \equiv 0$ belong to the class $\mathcal{F}(C_1, C_2, \mu_0)$.
But in fact we have more. The following convergence result holds.

Proposition 5.1. *Consider $\mathcal{A}, \mathcal{A}_h : L^2(0,T;H^1_0(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ the operators $\mathcal{A}u = -\mathrm{div}(a(x,t)\cdot Du)$, $\mathcal{A}_h u = -\mathrm{div}(a_h(x,t)\cdot Du)$ with $a, a_1, a_2, \ldots \in \mathcal{M}_{\Omega\times(0,T)}$ $(\lambda_0,\, \Lambda_0, M, N)$. Suppose $\mathcal{A}_h v \to \mathcal{A}v$ in $L^2(0,T;H^{-1}(\Omega))$ for every $v \in L^2(0,T;H^1_0(\Omega))$. Then*

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,H^1_0(\Omega)) \\ a_h\cdot Du_h \to a\cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n), \end{array}$$

where u_h and u are respectively the solutions to (18) *and* (29)*, and $a_h \xrightarrow{G} a$.*

Proof. Since $\mathcal{A}_h - \mathcal{A} = \mathcal{A}_h(\mathcal{A}^{-1} - \mathcal{A}_h^{-1})\mathcal{A}$, fix $f \in L^2(0,T;H^{-1}(\Omega))$ and choose $v = \mathcal{A}^{-1}f$. Then

$$\mathcal{A}^{-1}f - \mathcal{A}_h^{-1}f = \mathcal{A}_h^{-1}(\mathcal{A}_h v - \mathcal{A}v).$$

By our assumptions and (16) we conclude that $\|\mathcal{A}_h^{-1}f - \mathcal{A}^{-1}f\|_{L^2(0,T,H^1_0(\Omega))} \to 0$. In particular $\|\mathcal{A}_h^{-1}f - \mathcal{A}^{-1}f\|_{L^2(0,T,L^2(\Omega))} \to 0$ and, for every $\Phi \in L^2(0,T;L^2(\Omega)^n)$ by (12)

$$\begin{aligned} &\sup_{\|\Phi\|=1}\left|\int_0^T\int_\Omega (a_h\cdot D\mathcal{A}_h^{-1}f - a\cdot D\mathcal{A}^{-1}f, \Phi)\mathrm{d}x\ \mathrm{d}t\right| \\ &\qquad \leqslant c(M,\Lambda_0)\langle \mathcal{A}_h u - \mathcal{A}u, u\rangle_{L^2(0,T;H^{-1}(\Omega))\times L^2(0,T;H^1_0(\Omega))} \to 0. \end{aligned}$$

By Theorem 4.5 we derive that $a_h \xrightarrow{G} a$. □

Remark 5.2. Observe that assumptions of Proposition 5.1 are guaranteed if $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ and $a_h \to a$ in $L^1_{loc}(\Omega\times(0,T))$, but the converse is not true (see [14]).

We conclude that under assumptions of Proposition 5.1, for every $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ converging to $r \in \mathcal{F}(C_1, C_2, \mu_0)$ in $L^\infty(\Omega\times(0,T))$-weak$*$, we have that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,H^1_0(\Omega)) \\ a_h\cdot Du_h \to a\cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n), \end{array} \tag{47}$$

where u_h and u denote respectively the solutions of

$$\begin{cases} \partial_t(r_h v) - \operatorname{div}(a_h \cdot Dv) = f & \Omega \times (0,T) \\ v = 0 & \partial\Omega \times (0,T) \\ v = \varphi & \Omega_{h,+}(0) \times \{0\} \end{cases} \qquad \begin{cases} \partial_t(rv) - \operatorname{div}(a \cdot Dv) = f & \Omega \times (0,T) \\ v = 0 & \partial\Omega \times (0,T) \\ v = \varphi & \Omega^r_+(0) \times \{0\}. \end{cases}$$

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