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Arithmetic jet spaces

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ABSTRACT

One of the primary objectives of this paper is to establish compatibility between two different jet space functors in the most general context. We first show an adjunction between the jet and the Witt functor on algebras. Following Borger's approach, we then construct the algebraic jet space functor in the general setting where the base is an arbitrary prolongation sequence. We then show that this functor is representable in the category of schemes and that Buium's jet space can be recovered by the π-adic completion of this representable scheme. As an application, this allows us to strengthen a result of Buium on the relation between Greenberg's transform and the special fiber of jet spaces, including the ramified case.
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1. Introduction

The theory of arithmetic jet spaces, developed by Buium, is inspired from differential algebra where the notion of derivation is replaced by π-derivations δ. Recall that to endow a ring R with a derivation is equivalent to giving a ring homomorphism from R

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to $D_1(R)$ where $D_n(R) := R[\epsilon]/(\epsilon^{n+1})$. In other words, the linearity and the Leibniz rule of a derivative operator are equivalent to the ring structure of the dual numbers $D_1(R)$. Based on this point of view, given a scheme X defined over a function field K with a fixed derivation ∂ on it, for each n, one canonically constructs the jet space scheme J^nX which also represents the D_n-valued points of the scheme X. As an example, when the fixed derivation ∂ on K is trivial, then J^1X is isomorphic to the tangent bundle of X. However, in general, J^1X is a torsor under the tangent bundle.

The ring of functions $\mathcal{O}(J^nX)$ can be thought of as the ring of n-th order differential functions on X. In particular, if X is an abelian scheme, then the non-triviality of the group of additive characters (also known as Manin characters) is a consequence of the Picard-Fuchs equation. Explicitly, if X is given by the Legendre equation $y^2 = x(x-1)(x-t)$ over $R = \mathbb{C}(t)$ with derivation $\partial = \frac{d}{dt}$, then

$$\Theta(x, y, x', y', x'', y'') = \frac{y}{2(x-t)^2} - \frac{d}{dt}\left[2t(t-1)\frac{x'}{y}\right] + 2t(t-1)x'\frac{y'}{y^2},$$

where x, y, x', y', x'', y'' are the induced coordinates of the jet space J^2X. Such characters lead to a natural generalization for abelian varieties. Manin gave a proof of the Lang-Mordell conjecture for abelian varieties over function fields [25] using such characters. Buium gave an alternative proof using them [10], although his method differed from that of Manin.

The theory of arithmetic jet spaces proceeds similarly. In the case when R is a number ring, the notion of derivation is replaced by a π-derivation δ. Just as the notion of derivation comes from the ring structure of dual numbers, analogously, the notion of a π-derivation δ comes from Witt vectors. In particular, when $R = \mathbb{Z}$ and for a fixed prime $\pi = p$ we have

$$\delta(x) = \frac{x - x^p}{p}$$

for all $x \in \mathbb{Z}$. Note that the above expression is integral due to Fermat's little theorem. Moreover, δ satisfies the following:

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

for all $x, y \in R$. Such a δ should be thought of as a "derivative operator" with respect to a prime π. In general, for any such δ, one can associate a ring homomorphism that is a lift of Frobenius $\phi : R \to R$ given by $\phi(x) = x^p + p\delta(x)$ for all $x \in R$. In other words, the theory of such π-derivations δ is related to the theory of lifts of Frobenius ϕ.

For any scheme X over Spec$R$ where on R we fix a π-derivation, one similarly constructs arithmetic jet spaces $\{J^nX\}_{n=0}^{\infty}$ which form a prolongation sequence. The geometry of these spaces has led to interesting applications in diophantine geometry, such as the proof of the effective Manin-Mumford conjecture [12] and the proof of finite intersection of the set of Heegner points with finite rank subgroups on a modular curve [16].

The jet space construction on the Hodge bundle over a modular curve leads to the theory of differential modular forms [13], [14], [15]. In [8], [9], [26] a canonical isocrystal was attached using the theory of arithmetic jet spaces. In [27], for an elliptic curve E defined over $\mathbb{Z}_p$, it was shown that this canonical isocrystal is isomorphic to the first crystalline cohomology in the category of filtered isocrystals when E does not have a canonical lift. Otherwise, the isocrystal is a subobject of the crystalline cohomology. This result can be regarded as a character theoretic interpretation of the crystalline cohomology. Very recently, the theory of δ-rings (rings endowed with a π-derivation δ) has led to the development of prismatic cohomology by Bhatt and Scholze [5].

However, in the foundations of the theory of arithmetic jet spaces, there was a requirement for a reconciliation of two different approaches of the jet functor. One is the approach developed by Buium, where jet spaces are defined in the category of π-adic formal schemes as a prolongation sequence that satisfies a universal property. On the other hand, Borger in [7] developed the algebraic jet schemes over Spec$\mathbb{Z}$ from the functor of points viewpoint. In this article, we construct the algebraic jet space functor in the general setting where the base is an arbitrary prolongation sequence. We then show that this functor is representable in the category of schemes and that Buium's jet space can be recovered by the π-adic completion of this representable scheme. This compatibility allows us to use the interplay between two different viewpoints of the jet functor as, for example, in [8] and [9]. As another application, we strengthen a result of Buium on the relation between Greenberg's transform and the special fiber of jet spaces, including the ramified case.

Now we will explain our results in greater detail. Let $\mathcal{O}$ be a Dedekind domain with K as its field of fractions and fix a non-zero prime ideal $\mathfrak{p} \subset \mathcal{O}$. Let us also fix π as a generator of the prime ideal $\mathfrak{p}$. Let $k = \mathcal{O}/\mathfrak{p}$ be the residue field and also assume it to be a finite field of characteristic p and cardinality $q = p^h$. The identity map on $\mathcal{O}$ is also tacitly fixed as the lift of the q-th power Frobenius, i.e., the identity on k, and let $\delta\colon \mathcal{O} \to \mathcal{O}$ denote the map $x \mapsto \pi^{-1}(x - x^q)$. The Witt vectors and jet spaces considered in this paper are regarding lifts of Frobenius compatible with respect to this choice of the identity map on $\mathcal{O}$.

The Witt vectors W_n described in [6] are always considered with respect to the above data, which we will denote by the pair $(\mathcal{O}, \mathfrak{p})$. In particular the most commonly used p-typical Witt vectors come from considering $\mathcal{O} = \mathbb{Z}$ and $\mathfrak{p} = (p)$, for some prime $p \in \mathbb{Z}$. It is noteworthy to mention that the Witt vector functor W_n depends on the choice of the *base ring* $\mathcal{O}$ and the maximal ideal $\mathfrak{p}$.

Given $\mathcal{O}$-algebras A and B, we define a *prolongation* $A \xrightarrow{(u,\delta)} B$ to be the data of two maps u, δ, where $u\colon A \to B$ is a morphism of $\mathcal{O}$-algebras and $\delta\colon A \to B$ is a set-theoretic map (also known as a π*-derivation relative to* u) that satisfies $\delta(1) = 0$ and

$$\delta(x+y) = \delta(x) + \delta(y) + C_\pi(u(x), u(y)), \tag{1.1}$$

$$\delta(xy) = u(x)^q\delta(y) + u(y)^q\delta(x) + \pi\delta(x)\delta(y), \tag{1.2}$$

for all $x, y \in A$, where $C_\pi(X,Y) = \frac{X^q+Y^q-(X+Y)^q}{\pi} \in \mathcal{O}[X,Y]$. A sequence of prolongations of $\mathcal{O}$-algebras $\{C_0 \xrightarrow{(u_0,\delta_0)} C_1 \xrightarrow{(u_1,\delta_1)} C_2 \xrightarrow{(u_2,\delta_2)} \dots\}$ is called a *prolongation sequence* if for each $n \geq 0$,

$$u_{n+1} \circ \delta_n = \delta_{n+1} \circ u_n;$$

we will denote it by $C_* := \{C_n\}_{n=0}^\infty$. As an example, consider the prolongation sequence given by $C_i = \mathcal{O}, u_i = \mathrm{id}, \delta_i(x) = \delta(x) = \pi^{-1}(x - x^q)$. We will denote this prolongation sequence by $\mathcal{O}_*$. Prolongation sequences naturally form a category; see Section 2.1.

Let R_* be a fixed prolongation sequence over $\mathcal{O}_*$, i.e., with a fixed morphism $\mathcal{O}_* \to R_*$, and let $\mathcal{C}_{R_*}$ denote the category of prolongation sequences over R_*. Following [13] we consider the functor

$$J_* : \mathrm{Alg}_{R_0} \to \mathcal{C}_{R_*}, \quad A \mapsto J_*(A) = \{J_n(A)\}_{n=0}^\infty$$

which associates to any R_0-algebra the canonical prolongation sequence of its π-jet algebras where J_nA are R_n-algebras for all n. Then, as in Proposition 1.1 in [13] J_*A, satisfies the following universal property:

$$\mathrm{Hom}_{R_0}(A, C_0) \simeq \mathrm{Hom}_{\mathcal{C}_{R_*}}(J_*(A), C_*)$$

for any prolongation sequence $C_* \in \mathcal{C}_{R_*}$. Our first result is the following adjunction between the functors J_n and W_n:

Theorem 1.3. *The J_n functor from the category of R_0-algebras to the R_n-algebras is left adjoint to the Witt functor W_n, that is, for all R_0-algebra A and R_n-algebra B we have*

$$\mathrm{Hom}_{R_n}(J_nA, B) \simeq \mathrm{Hom}_{R_0}(A, W_n(B)).$$

The R_0-algebra structure of $W_n(B)$ will be explained later (see Corollary 2.12). Note that we adopt Borger's indexing convention on Witt vectors. In other words if $X = \mathrm{Spec}A$ is an affine scheme over $\mathrm{Spec}R_0$ then the above theorem says that $(\mathrm{Spec}J_nA)(B) \simeq X(W_n(B))$. To prove the above result, we use an explicit expression (2.16), which is an inductive formula that provides a correspondence between the Witt and the Buium-Joyal delta coordinates.

Following along the lines in [7], given an arbitrary scheme X (not necessary affine), for each n, we define the sheaf J^nX in the category of sheaves of sets on affine schemes over $\mathrm{Spec}R_n$ as

$$J^nX(B) := X(W_n(B))$$

for every R_n-algebra B. Then we show the following:

Theorem 1.4. *If X is an scheme over* $\mathrm{Spec}(R_0)$ *then J^nX is a scheme over* $\mathrm{Spec}(R_n)$ *for all $n \geq 0$.*

Given a scheme X along with a covering by affine open subschemes $\{U_i\}$, Buium defines the n-th jet space $\widehat{\mathfrak{J}^n X}$ by glueing the affine π-adic formal schemes $\widehat{\mathfrak{J}^n U_i}$ over the π-adic completion of X. We show that Buium's jet space is indeed the π-adic completion of J^nX.

Theorem 1.5. *Let X be a scheme over* $\mathrm{Spec}(R_0)$. *Then the π-adic formal completion of J^nX is isomorphic to $\widehat{\mathfrak{J}^n X}$.*

In the last part of the paper we compare Greenberg's realization of a scheme X defined over a discrete valuation ring with the special fiber of the algebraic jet space associated to X; see Theorem 4.7. In particular, we show that the special fiber of the p-jet space of a smooth scheme X defined over a complete unramified extension of $\mathbb{Z}_p$ coincides with the Greenberg realization of X; see Corollary 4.9. This generalizes [12, Theorem 2.10] removing a condition on the residue field.

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2. Prolongation sequence algebras

2.1. π-derivations

We look more closely at π-derivations and prolongation sequences. Let A, B be $\mathcal{O}$-algebras, $\rho_A \colon \mathcal{O} \to A, \rho_B \colon \mathcal{O} \to B$ the structure morphisms and $u \colon A \to B$ a morphism of $\mathcal{O}$-algebras. Recall that a *π-derivation relative to u* is a map $\delta_A = \delta \colon A \to B$ that satisfies conditions (1.1) and (1.2). In the following we will sometimes write δx in place of $\delta(x)$ to relax the notation.

The ring $\mathcal{O}$ is endowed with a unique π-derivation $\delta(x) = \pi^{-1}(x - x^q)$ relative to the identity on $\mathcal{O}$. We will say that A is an *$\mathcal{O}$-algebra with π-derivation* if A is endowed with a π-derivation δ_A relative to $\mathrm{id}_A \colon A \to A$ such that $\delta_A \circ \rho_A = \rho_A \circ \delta$. Therefore, if A is an $\mathcal{O}$-algebra such that $\pi A = 0$, i.e., a k-algebra, A admits no structure of $\mathcal{O}$-algebra with π-derivation. On the other hand, if A has no π-torsion, any $\mathcal{O}$-algebra endomorphism $\phi \colon A \to A$ which lifts the q-Frobenius modulo πA produces a π-derivation $\delta_A(x) =$

$\pi^{-1}(\phi(x) - x^q)$ so that A has a structure of $\mathcal{O}$-algebra with π-derivation. Morphisms between $\mathcal{O}$-algebras with π-derivation are defined in the obvious way.

Given $\mathcal{O}$-algebras A and B, we define a *prolongation* to be the data (u, δ) of two maps

$$A \underset{\delta}{\overset{u}{\rightrightarrows}} B$$

where $u\colon A \to B$ is a morphism of $\mathcal{O}$-algebras and $\delta\colon A \to B$ is a π-derivation relative to u. For brevity, we will denote the above datum by

$$A \xrightarrow{(u,\delta)} B.$$

Prolongations form a category where morphisms are given by pairs of homomorphisms (f, g) of $\mathcal{O}$-algebras making the obvious squares

$$\begin{array}{ccc} A & \underset{\delta}{\overset{u}{\rightrightarrows}} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A' & \underset{\delta}{\overset{v}{\rightrightarrows}} & B' \end{array}$$

commute, i.e., $g \circ u = v \circ f$, $g \circ \delta = \delta \circ f$. Note that for $B = A, B' = A', f = g$, u and v the identity homomorphisms, the above diagram denotes a morphism of $\mathcal{O}$-algebras with π-derivations.

Let $\{C_n, u_n, \delta_n\}_{n=0}^{\infty}$ or simply $C_* := \{C_n\}_{n=0}^{\infty}$ denote a *prolongation sequence* of $\mathcal{O}$-algebras, that is, a sequence of prolongations of $\mathcal{O}$-algebras

$$C_0 \xrightarrow{(u_0,\delta_0)} C_1 \xrightarrow{(u_1,\delta_1)} C_2 \longrightarrow \dots$$

such that for each $n \geq 0$,

$$u_{n+1} \circ \delta_n = \delta_{n+1} \circ u_n. \tag{2.1}$$

We will denote by $\mathcal{O}_*$ the prolongation sequence given by $C_i = \mathcal{O}, u_i = \mathrm{id}, \delta_i(x) = \delta(x) = \pi^{-1}(x - x^q)$.

A morphism $f_*\colon C_* \to D_*$ between two prolongation sequences is a sequence of $\mathcal{O}$-algebra homomorphisms $f_n\colon C_n \to D_n$ which induces homomorphisms of prolongations

$$\begin{array}{ccc} C_n & \underset{\delta_n}{\overset{u_n}{\rightrightarrows}} & C_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ D_n & \underset{\delta_n}{\overset{v_n}{\rightrightarrows}} & D_{n+1} \end{array}$$

If f_* exists, we will say that D_* is over C_*. It is immediate to check that prolongation sequences form a category.

Remark 2.2. *Note that in general $\mathcal{O}$ has liftings of Frobenius different from the identity morphism. Further, in the definition of prolongation sequences C_*, we do not ask the π-derivation $C_i \to C_{i+1}$ to be compatible with the one fixed on $\mathcal{O}$. We then force this compatibility working with prolongation sequences over $\mathcal{O}_*$.*

We denote by $\mathcal{C}_{R_*}$ the category of prolongation sequences over the fixed prolongation sequence R_*.

In this paper, we will only consider prolongation sequences C_* over $\mathcal{O}_*$.

Now we introduce polynomials Q^δ whose role will be more clear after the introduction of π-jet algebras; see Remark 2.15. In fact Q^δ will be the image of Q via the universal π-derivation. Following Buium's notation, we will work with indeterminates $T = T^{(0)}, T' = T^{(1)}, T'' = T^{(2)}, \dots$

Proposition 2.3. *Consider the polynomial ring $A = \mathcal{O}[T, T', T'', \dots]$ and the $\mathcal{O}$-linear endomorphism $\phi_A \colon A \to A$ that maps $T^{(i)}$ to $(T^{(i)})^q + \pi T^{(i+1)}$. It lifts the q-Frobenius on $k[T, T', \dots]$. Let δ_A be the corresponding π-derivation and let Q^δ denote $\delta_A(Q)$ for any polynomial $Q \in \mathcal{O}[T, T', \dots]$.*

i) *If $Q = T^{(i)}$ then $Q^\delta = T^{(i+1)}$.*
ii) *If $Q \in \mathcal{O}[T, \dots, T^{(n)}]$ then $Q^\delta \in \mathcal{O}[T, \dots, T^{(n+1)}]$.*
iii) *For any $\mathcal{O}$-algebra with π-derivation (B, δ) and any $b \in B$ we have*

$$\delta\left(Q(b, \delta b, \dots, \delta^n b)\right) = Q^\delta(b, \delta b, \dots, \delta^{n+1} b) \in B$$

where $\delta^n \colon B \to B$ denotes the n-fold composition of δ.

Proof. Assertions i) and ii) are immediate since $\delta_A(Q) = \pi^{-1}(\phi_A(Q) - Q^q)$. For iii), let $\phi\colon B \to B$ be the ring endomorphism lifting the q-Frobenius on $B/\pi B$ and associated to δ, i.e., $\phi(b) = b^q + \pi\delta(b)$ for any $b \in B$. For a fixed $b \in B$, let $f_b \colon A \to B$ denote the morphism of $\mathcal{O}$-algebras mapping $T^{(i)}$ to $\delta^i(b)$. Since

$$\phi(f_b(T^{(i)})) = \delta^i(b)^q + \pi\delta^{i+1}(b) = f_b\left((T^{(i)})^q + \pi T^{(i+1)}\right) = f_b(\phi_A(T^{(i)})),$$

f_b is a morphism of $\mathcal{O}$-algebras with π-derivation, i.e., $\delta \circ f_b = f_b \circ \delta_A$. Hence

$$\delta\left(Q(b, \delta b, \dots, \delta^n b)\right) = \delta(f_b(Q)) = f_b(\delta_A(Q)) = Q^\delta(b, \delta b, \dots, \delta^{n+1} b),$$

as desired. □

2.2. π-typical Witt vectors

With the fixed $\mathcal{O}$, π and q as above, we define the π-typical Witt vector functors $W_n = W_{\pi,q,n}$ (where we omit the indices π, q if clear from the context) in terms of the Witt polynomials; see [21,28] for $\mathcal{O}$ a discrete valuation ring, [6] for $\mathcal{O}$ a Dedekind domain. See also [1, §1.1] for the ramification ring structure $(\mathcal{O}, \pi, q)$.

For each $n \geq 0$ let us define the n-th *ghost ring* $\prod_{i=0}^n B$ to be the $(n+1)$-product $\mathcal{O}$-algebra $B \times \cdots \times B$ where $\lambda(b_0, \dots, b_n) = (\lambda b_0, \dots, \lambda b_n)$ for any $\lambda \in \mathcal{O}$, and similarly for the infinite product $\prod_{i=0}^\infty B := B \times B \times \cdots$. Then for all $n \geq 1$ there exists a *restriction*, or *truncation*, map

$$T_w \colon \prod_{i=0}^{n} B \to \prod_{i=0}^{n-1} B, \qquad (b_0, \cdots, b_n) \mapsto (b_0, \cdots, b_{n-1}),$$

which is a morphism of $\mathcal{O}$-algebras.

Now define $W_n(B) = B^{n+1}$ as sets, and define the map of sets

$$w \colon W_n(B) \to \prod_{i=0}^{n} B, \qquad b_. = (b_0, \dots, b_n) \mapsto (w_0(b_.), \dots, w_n(b_.))$$

where

$$w_i = x_0^{q^i} + \pi x_1^{q^{i-1}} + \cdots + \pi^i x_i$$

are the so-called *Witt polynomials* or *ghost polynomials*. The map w is known as the *ghost* map. Note that we adopt Borger's indexing notation for Witt vectors.

The ghost map w is injective when B has no π-torsion, e.g., if B is a ring of polynomials over $\mathcal{O}$ with possibly infinitely many indeterminates. This implies that we can endow $W_n(B)$ with a unique $\mathcal{O}$-algebra structure such that w becomes a morphism of $\mathcal{O}$-algebras. On the other hand, for general B, we can induce such $\mathcal{O}$-algebra structure by representing B as quotient of $\mathcal{O}[x_i]$ for a suitable family of indeterminates x_i; cf. [18, Ch 1. §1.1], [28, Proposition 1.1.8], [6, §2.4]. This construction is functorial in B.

There are *restriction*, or *truncation*, maps

$$T \colon W_n(B) \to W_{n-1}(B), \qquad (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{n-1}),$$

which satisfy $T_w \circ w_i = w_i \circ T$ for $i \geq 1$. We define the ring of *π-typical Witt vectors* $W_{q,\pi}(B) = W(B) = \varprojlim W_n(B)$ with restrictions T as transition maps, where subscripts q, π are usually omitted since clear from the context.

Remark 2.4. *Given $\mathcal{O}$-algebras A and B any prolongation (u, δ) of A to B determines a morphism of rings $A \to W_1(B), a \mapsto (u(a), \delta(a))$, and conversely ([11, 1.2]).*

2.3. Operations on Witt vectors

Now we recall some important functorial operators on the Witt vectors. See [20,28]. They are constructed from operators on the ghost rings defined above.

First consider the left shift *Frobenius* operator on the ghost rings

$$F_w\colon \prod_{i=0}^{n} B \to \prod_{i=0}^{n-1} B, \qquad (b_0, \dots, b_n) \mapsto (b_1, \dots, b_n),$$

and passing to the limit

$$F_w\colon \prod_{i=0}^{\infty} B \to \prod_{i=0}^{\infty} B, \qquad (b_0, b_1, \dots) \mapsto (b_1, b_2, \dots).$$

They are morphisms of $\mathcal{O}$-algebras. The *Frobenius* morphism $F\colon W_n(B) \to W_{n-1}(B)$, is the unique map which is functorial in B and makes the following diagram

$$\begin{array}{ccc} W_n(B) & \xrightarrow{w} & \prod_{i=0}^{n} B \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_w} \\ W_{n-1}(B) & \xrightarrow[w]{} & \prod_{i=0}^{n-1} B \end{array}$$

commute. It is a morphism of $\mathcal{O}$-algebras. Similarly, one defines the Frobenius morphism $F\colon W(B) \to W(B)$.

Next we consider the $\mathcal{O}$-linear map

$$V_w\colon \prod_{i=0}^{n-1} B \to \prod_{i=0}^{n} B, \qquad V_w(b_0, \dots, b_{n-1}) \mapsto (0, \pi b_0, \dots, \pi b_{n-1}).$$

The *Verschiebung* $V\colon W_{n-1}(B) \to W_n(B)$ is the map given by

$$V(x_0, \dots, x_{n-1}) = (0, x_0, \dots, x_{n-1});$$

it makes the following diagram commute:

$$\begin{array}{ccc} W_{n-1}(B) & \xrightarrow{w} & \prod_{i=0}^{n-1} B \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V_w} \\ W_n(B) & \xrightarrow[w]{} & \prod_{i=0}^{n} B \end{array} .$$

Similarly, one defines the Verschiebung $V\colon W(B) \to W(B)$. Frobenius and Verschiebung satisfy the identities

$$FV(x) = \pi x \quad \forall x \in W(B); \tag{2.5}$$

$$x \cdot V(y) = V(F(x) \cdot y) \quad \forall x, y \in W(B). \tag{2.6}$$

The Verschiebung is not a ring homomorphism, but it is $\mathcal{O}$-linear. Indeed, this is clear when B has no π-torsion since V_w and w are morphisms of $\mathcal{O}$-algebras; the general case follows considering any surjective homomorphism $\mathcal{O}[x_i, i \in I] \to B$.

Finally, we have the multiplicative *Teichmüller map* $[\ ]\colon B \to W(B)$ given by $x \mapsto [x] = (x, 0, 0, \dots)$.

Remark 2.7. *The Frobenius on $W(B)$ lifts the q-Frobenius on $W(B)/\pi W(B)$, and hence it induces a π-derivation Δ on $W(B)$ relative to the identity. Although this is a well-known fact, we demonstrate by using the previous identities. Let $x = (x_0, x_1, \dots) \in W(B)$ and write $x = [x_0] + V(y)$ with V the Verschiebung map and $y = (x_1, x_2, \dots)$. Then by (2.5)*

$$F(x) = F[x_0] + FV(y) \equiv [x_0^q] \pmod{\pi W(B)}.$$

On the other hand,

$$x^q = ([x_0] + V(y))^q = [x_0^q] + V(y)^q + p(\dots) \equiv [x_0^q] + V(y)^q \equiv [x_0^q] \pmod{\pi W(B)},$$

where the first equality is due to the fact that p divides $\binom{q}{j}$ for $0 < j < q$, the equivalence in the middle follows from $p \in \pi\mathcal{O}$ and the latter equivalence follows from (2.5) and (2.6) writing $V(y)^2 = V(FV(y) \cdot y) = V(\pi y^2) = \pi V(y^2)$ by $\mathcal{O}$-linearity of V. One concludes then that $F(x) - x^q \in \pi W(B)$.

Now we recall the universal property of π-typical Witt vectors:

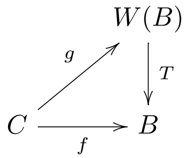
Theorem 2.8. *The functor W is the right adjoint of the forgetful functor $\mathrm{Alg}_\delta \to \mathrm{Alg}$ from the category of $\mathcal{O}$-algebras with π-derivation to the category of $\mathcal{O}$-algebras, i.e., there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Alg}_\delta}(C, W(B)) \simeq \mathrm{Hom}_{\mathrm{Alg}}(C, B),$$

for any $\mathcal{O}$-algebra B and any $\mathcal{O}$-algebras with π-derivation C.

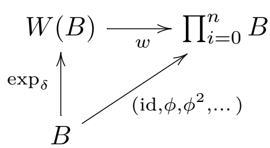
This fact is explained in [22] for the case $\pi = p$ and in [6, §1] for the general case. Here we review the theory as follows:

The ring $W(B)$ of π-typical Witt vectors is the unique (up to unique isomorphism) $\mathcal{O}$-algebra $W(B)$ with a π-derivation Δ on $W(B)$ relative to the identity (see Remark 2.7) and an $\mathcal{O}$-algebra restriction homomorphism $T\colon W(B) \to B$ such that, given any $\mathcal{O}$-algebra C with a π-derivation δ relative to id_C and an $\mathcal{O}$-algebra morphism $f\colon C \to B$, there exists a unique $\mathcal{O}$-algebra morphism $g\colon C \to W(B)$ such that the diagram



commutes and $g \circ \delta = \Delta \circ g$.

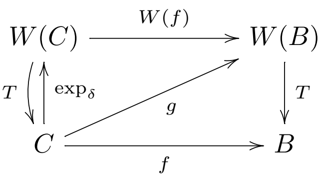
Note that if B has a π-derivation δ relative to the identity, the above theorem asserts the existence of a canonical section $\exp_\delta \colon B \to W(B)$ of T; one can check that $\exp_\delta$ makes the diagram



(2.9)

commute, where $\phi \colon B \to B$ maps b to $b^q + \pi\delta(b)$; cf. [28, Prop. 1.1.23]. In particular, $\exp_\delta(b) = (b, \delta(b), \dots)$ and the other entries are polynomials in $\delta^i(b) = \delta(\delta(\dots\delta(b)))$ with coefficients in B.

In the general case, one defines g as the composition of the natural section $\exp_\delta$ of $T \colon W(C) \to C$ with $W(f)$:



Thanks to Proposition 2.3 we can make more explicit the description of $\exp_\delta$ in (2.9):

Proposition 2.10. *There exist unique polynomials $P_n \in \mathcal{O}[T, T' \dots T^{(n)}]$ such that for any $\mathcal{O}$-algebra with π-derivation B and any $b \in B$*

$$\exp_\delta(b) = (P_0(b), P_1(b), \dots),$$

where $P_n(b) = P_n(b, \delta b, \dots, \delta^n b)$. Further $P_0(T) = T, P_1(T, T') = T'$ and for every $n \geq 1$

$$P_n = P_{n-1}^\delta + \sum_{i=0}^{n-2} \left[\sum_{j=1}^{q^{n-1-i}} \pi^{i+j-n} \binom{q^{n-1-i}}{j} P_i^{q(q^{n-1-i}-j)} (P_i^\delta)^j \right],$$

with $P_i^\delta(b) = \delta(P_i(b, \delta b, \dots, \delta^i b))$.

Proof. From diagram (2.9) we get $P_0(b) = b, P_1(b) = \delta(b)$, and more generally

$$\begin{aligned}
\sum_{i=0}^{n} \pi^i P_i(b)^{q^{n-i}} &= \phi^n(b) \\
&= \phi(\phi^{n-1}(b)) \\
&= \phi\left(\sum_{i=0}^{n-1} \pi^i P_i(b)^{q^{n-1-i}}\right) \\
&= \sum_{i=0}^{n-1} \pi^i (\phi(P_i(b))^{q^{n_i}} \\
&= \sum_{i=0}^{n-1} \pi^i \left(P_i(b)^q + \pi\delta(P_i(b))\right)^{q^{n_i}} \\
&= \sum_{i=0}^{n-1} \pi^i \left[P_i(b)^{q^{n-i}} + \sum_{j=1}^{q^{n_i}} \binom{q^{n_i}}{j} P_i(b)^{(q^{n_i}-j)q} \pi^j (\delta P_i(b))^j\right] \\
&= \sum_{i=0}^{n-1} \pi^i P_i(b)^{q^{n-i}} + \sum_{i=0}^{n-1} \left[\sum_{j=1}^{q^{n_i}} \pi^{i+j} \binom{q^{n_i}}{j} P_i(b)^{(q^{n_i}-j)q} (\delta P_i(b))^j\right]
\end{aligned}$$

with $n_i = n - 1 - i$. Hence, cancelling the common terms on both sides of the above equality, we get

$$\pi^n P_n(b) = \pi^n \delta P_{n-1}(b) + \sum_{i=0}^{n-2} \left[\sum_{j=1}^{q^{n-1-i}} \pi^{i+j} \binom{q^{n-1-i}}{j} P_i(b)^{q(q^{n-1-i}-j)} (\delta P_i(b))^j\right].$$

It is then clear that the polynomial

$$P_n = P_{n-1}^\delta + \sum_{i=0}^{n-2} \left[\sum_{j=1}^{q^{n-1-i}} c_{i,j} P_i^{q(q^{n-1-i}-j)} (P_i^\delta)^j\right] \quad \text{with} \quad c_{i,j} = \pi^{i+j-n} \binom{q^{n-1-i}}{j}$$

does the job since $c_{i,j} \in \mathcal{O}$. We would like to add a few words explaining why $c_{i,j}$ lies in $\mathcal{O}$ (although it is expected due to functorial reasons). If $i + j - n \geq 0$ the fact is clear. Now assuming $n - i > j$, we will show that the p-adic valuation of $c_{i,j}$ is non-negative. Let $q = p^h$ and let e denote the ramification of p in $\mathcal{O}$. Since $j > 0$, $i + j - n$ is negative and $n - 1 - i$ is positive, we have

$$v_p(c_{i,j}) = \frac{i+j-n}{e} + h(n-1-i) - v_p(j) \geq (i+j-n) + (n-1-i) - v_p(j) = j - 1 - v_p(j) \geq 0,$$

where we have used the formula

$$v_p \binom{p^m}{j} = m - v_p(j).$$

Uniqueness of the polynomials P_n is immediate. □

In the case when $\pi = p, q = p$ the map $\exp_\delta$ in (2.9) is the one in [22, Proposition 1] and hence the polynomials P^δ describe Joyal's δ_n-operation.

From now until the end of the section let R_* denote a fixed prolongation sequence over $\mathcal{O}_*$.

Proposition 2.11. *Given any prolongation sequence $C_* \in \mathcal{C}_{R_*}$, for each n there exists a canonical morphisms of $\mathcal{O}$-algebras*

$$C_0 \to W_n(C_n).$$

Proof. Consider the prolongation sequence C_* written as

$$C_0 \overset{(u_0,\delta_0)}{\longrightarrow} C_1 \overset{(u_1,\delta_1)}{\longrightarrow} \cdots,$$

and consider the $\mathcal{O}$-algebra obtained by taking the direct limit with respect to u_n, denoted $C_\infty = \varinjlim C_n$. Thanks to the maps in (2.1) the δ_n's induce a π-derivation δ on C_∞ relative to the identity. Then by the universal property of Witt vectors in Theorem 2.8 we have a canonical morphism of $\mathcal{O}$-algebras

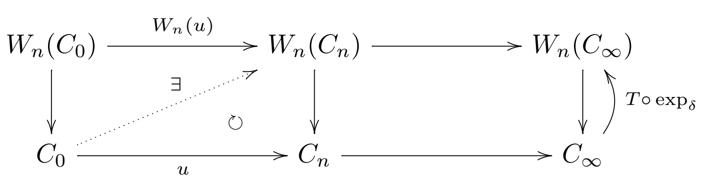
$$\exp_\delta \colon C_\infty \to W(C_\infty)$$

where $\exp_\delta(c) = (P_0(c), P_1(c), \dots)$ with $P_n(c) := P_n(c, \delta c, \dots, \delta^n c)$ and P_n is the polynomial introduced in Proposition 2.10. Therefore, by composition we obtain

$$C_0 \overset{u}{\longrightarrow} C_\infty \overset{\exp_\delta}{\longrightarrow} W(C_\infty) \overset{T}{\to} W_n(C_\infty)$$

where T is the restriction map of Witt vectors. Now note that the map $C_0 \to W_n(C_\infty)$ is given by $c \mapsto (P_0(c), \dots, P_n(c))$ for all $c \in C_0$. Since $P_i(c)$, for each $i \leq n$, is a polynomial in $c, \dots, \delta^i c$, with coefficients in $\mathcal{O}$, the map $T \circ \exp_\delta \circ u$ factors through $W_n(C_n)$ and we are done. □

The above results say that there are, in general, no sections of the reduction map $W_n(C_0) \to C_0$. However, one section exists by replacing C_0 with C_∞ and the latter section induces a lifting of $C_0 \to C_n$ to $W_n(C_n)$ as depicted below



Corollary 2.12. *Let B be an R_n-algebra. Then $W_n(B)$ is an R_0-algebra.*

Proof. Let $f\colon R_n \to B$ be the structure map. Now by Proposition 2.11, we have the $\mathcal{O}$-algebra map $R_0 \to W_n(R_n)$ and hence composition with $W_n(f)\colon W_n(R_n) \to W_n(B)$ gives us the claim. □

2.4. π-jet algebras

We continue with the notations of the previous subsection and let R_* denote a fixed prolongation sequence of $\mathcal{O}$-algebras. We now consider the functor

$$J_*\colon \mathrm{Alg}_{R_0} \to \mathcal{C}_{R_*}, \quad A \mapsto J_*A = \{J_nA\}_{n=0}^{\infty}$$

as defined by Buium in [13]. Assume A has presentation $A = R_0[\mathbf{x}]/(\mathbf{f})$ where $\mathbf{x}$ is a collection (possibly infinite) of variables $x_\gamma, \gamma \in \Gamma$, and $\mathbf{f}$ is the system of defining polynomials $\mathbf{f} = (f_j)_{j\in I}$ where $f_j \in R[\mathbf{x}]$ and I is an indexing set. One has a prolongation sequence

$$R_0[\mathbf{x}] \overset{(u_0,\delta_0)}{\longrightarrow} R_1[\mathbf{x}, \mathbf{x}^{(1)}] \overset{(u_1,\delta_1)}{\longrightarrow} R_2[\mathbf{x}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}] \dots \tag{2.13}$$

where $\mathbf{x}^{(i)}$ is a set of indeterminates $x_\gamma^{(i)}, \gamma \in \Gamma$, u_i is the obvious extension of $R_i \to R_{i+1}$ mapping $x_\gamma^{(r)}$, $0 \le r \le i$, to itself and δ_i is a π-derivation that extends the one on R_i and maps $x_\gamma^{(r)}$ to $x_\gamma^{(r+1)}$ for $0 \le r < i$. Then as in [13] we set for each n

$$J_nA = R_n[\mathbf{x}, \dots \mathbf{x}^{(n)}]/(\mathbf{f}, \dots, \delta^n \mathbf{f}), \tag{2.14}$$

where $\delta^i \mathbf{f} = (\delta^i f_j)_{j\in I}$; we obtain in this way a prolongation sequence

$$A = J_0A \overset{(u_0,\delta_0)}{\longrightarrow} J_1A \overset{(u_1,\delta_1)}{\longrightarrow} J_2A \dots$$

where u_i, δ_i still denote the maps induced by the u_i and δ_i in (2.13). We further have ring homomorphisms

$$\phi = \phi_n\colon J_nA \to J_{n+1}A, \quad a \mapsto u_n(a)^q + \pi\delta_n(a),$$

and with $\phi^n\colon A \to J_nA$ we will denote the composition $\phi_{n-1} \circ \phi_{n-2} \circ \dots \circ \phi_0$.

For any R_0-algebra A, $J_*A = \{J_nA\}_{n=0}^{\infty}$ is called the *canonical prolongation sequence* of π-jet algebras over R_* associated to A. Then for each n, J_nA is an R_n-algebra and we can consider the direct limit $J_\infty A = \varinjlim J_nA$, with transition maps u_n, which is endowed with a canonical π-derivation $\delta = \varinjlim \delta_n$ relative to the identity which is $\varinjlim u_n$. Moreover, any J_nA is an A-algebra via the u_i; for any $a \in A$, we let the same letter denote its image in J_nA.

Remark 2.15. *For $A = \mathcal{O}[T]$, then $J_n A = \mathcal{O}[T, T', \dots, T^{(n)}]$ with u_n the obvious inclusion and δ_n the map sending a polynomial Q to Q^δ, as defined in Proposition 2.3.*

Consider the canonical morphism $\exp_{\delta,n}\colon A \to W_n(J_n A)$ as constructed in Proposition 2.11; it is given by

$$\exp_{\delta,n}(a) = (P_0(a), \dots P_n(a)) \tag{2.16}$$

where $P_i(a)$ are polynomials in $a, \delta(a), \dots, \delta^i(a)$ with coefficients in $\mathcal{O}$ as introduced in Proposition 2.10. In particular, we have the following recursive formula

$$P_n(a) = P_{n-1}^\delta(a) + \sum_{i=0}^{n-2}\left[\sum_{j=1}^{q^{n-1-i}} \pi^{i+j-n}\binom{q^{n-1-i}}{j} P_i(a)^{q(q^{n-1-i}-j)}(P_i^\delta(a))^j\right], \tag{2.17}$$

for any $n \geq 1$ and $a \in A$, where $P_i^\delta(a) = \delta(P_i(a, \delta(a), \dots, \delta^i(a)))$.

Note that for $A = R_0[\mathbf{x}] = R_0[x_\gamma, \gamma \in \Gamma]$, inductively applying (2.17), we have

$$P_n(x_\gamma) = x_\gamma^{(n)} + S_{n-1}(x_\gamma) \tag{2.18}$$

with $S_{n-1}(x_\gamma) \in R_n[x_\gamma, \dots, x_\gamma^{(n-1)}]$, for $n \geq 1$, and $S_0(x_\gamma) = 0$. In particular, $J_n(R_0[\mathbf{x}]) = R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}] = R_n[\mathbf{x}, P_1(\mathbf{x}), \dots, P_n(\mathbf{x})]$, where $P_i(\mathbf{x})$ means the collection of the polynomials $P_i(x_\gamma), \gamma \in \Gamma$. More generally, for any polynomial $f \in R_0[\mathbf{x}]$ we have

$$P_n(f) = \delta^n f + S_{n-1}(f) \quad \text{with} \quad S_{n-1}(f) \in R_n[f, \delta f, \dots, \delta^{n-1} f], \quad n \geq 1. \tag{2.19}$$

Formulas (2.14), (2.19), and (2.18) yield:

Lemma 2.20. *For any R_0-algebra $A = R_0[\mathbf{x}]/(\mathbf{f})$ we have*

$$J_n A \simeq \frac{R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}]}{(P_0(\mathbf{f}), \dots, P_n(\mathbf{f}))} \simeq \frac{R_n[P_0(\mathbf{x}), \dots, P_n(\mathbf{x})]}{(P_0(\mathbf{f}), P_1(\mathbf{f}), \dots, P_n(\mathbf{f}))}.$$

We can now prove that Buium's π-jet algebras functor is left adjoint to the functor of π-typical Witt vectors which is Theorem 1.3 in our introduction.

Theorem 2.21. *The J_n functor from the category of R_0-algebras to the category R_n-algebras is left adjoint to the Witt functor W_n, that is, for all R_0-algebra A and R_n-algebra B we have*

$$\mathrm{Hom}_{R_n}(J_n A, B) \simeq \mathrm{Hom}_{R_0}(A, W_n(B)).$$

Proof. Let B be an R_n-algebra and view $W_n(B)$ as an R_0-algebra as in Corollary 2.12. We first consider the case $A = R_0[\mathbf{x}] = R_0[x_\gamma, \gamma \in \Gamma]$. For any homomorphism of R_0-algebras

$$\mathfrak{g}\colon R_0[\mathbf{x}] \longrightarrow W_n(B), \quad x_\gamma \mapsto (b_{\gamma,0}, \dots, b_{\gamma,n})$$

let $\Phi(\mathfrak{g})$ denote the homomorphism of R_n-algebras

$$\Phi(\mathfrak{g})\colon R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}] = R_n[P_0(\mathbf{x}), \dots, P_n(\mathbf{x})] \longrightarrow B, \quad P_i(x_\gamma) \mapsto b_{\gamma,i}.$$

Then we have a natural bijection

$$\operatorname{Hom}_{R_0}(R_0[\mathbf{x}], W_n(B)) \simeq \operatorname{Hom}_{R_n}(R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}], B), \quad \mathfrak{g} \mapsto \Phi(\mathfrak{g}).$$

By construction, the following diagram

$$\begin{array}{ccccc} R_0[\mathbf{x}] & \xrightarrow{\exp_{\delta,n}} & W_n(R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}]) & \xrightarrow{\mathrm{pr}_i} & R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}] \\ & \searrow^{\mathfrak{g}} & \downarrow{\scriptstyle W_n(\Phi(\mathfrak{g}))} & & \downarrow{\scriptstyle \Phi(\mathfrak{g})} \\ & & W_n(B) & \xrightarrow{\mathrm{pr}_i} & B \end{array} \tag{2.22}$$

commutes for all $0 \le i \le n$, where the map pr_i is the set-theoretic projection onto the $i$the component, i.e., $\mathrm{pr}_i(b_0, \dots, b_n) = b_i$. Note that the commutativity of the square on the right is trivial, while the commutativity of the triangle on the left depends on $\exp_{\delta,n}(x_\gamma) = (P_0(x_\gamma), \dots, P_n(x_\gamma))$ (2.16), and the definition $\Phi(\mathfrak{g})(P_i(x_\gamma)) = b_{\gamma,i}$.

Applying again (2.16), one checks that $\mathfrak{g}(f) = 0$ if and only if $\phi(\mathfrak{g})(P_i(f)) = 0$ for any $0 \le i \le n$. Together with Lemma 2.20 this fact says that $\mathfrak{g}$ in diagram (2.22) factors through $R_0[\mathbf{x}]/(\mathbf{f})$ if and only if $\Phi(\mathfrak{g})$ factors through

$$R_n[\mathbf{x}, \dots, \mathbf{x}^{(n)}]/(P_0(\mathbf{f}), \dots, P_0(\mathbf{f})) = R_n[P_0(\mathbf{x}), \dots, P_n(\mathbf{x})]/(P_0(\mathbf{f}), \dots, P_0(\mathbf{f})).$$

Hence, for any R_0-algebra A and any R_n-algebra B there is a natural bijection

$$\operatorname{Hom}_{R_0}(A, W_n(B)) \simeq \operatorname{Hom}_{R_n}(J_n A, B), \quad \mathfrak{g} \mapsto \Phi(\mathfrak{g}), \tag{2.23}$$

where W_n stands for $W_{\pi,q,n}$, such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\exp_{\delta,n}} & W_n(J_n A) & \xrightarrow{\mathrm{pr}_i} & J_n A \\ & \searrow^{\mathfrak{g}} & \downarrow{\scriptstyle W_n(\Phi(\mathfrak{g}))} & & \downarrow{\scriptstyle \Phi(\mathfrak{g})} \\ & & W_n(B) & \xrightarrow{\mathrm{pr}_i} & B \end{array}$$

commutes, and we are done. □

3. Jet spaces

In this section, we wish to reconcile two apparently different approaches to π-jet spaces, one due to Buium and the other to Borger. When working with affine schemes, we generalize both constructions to our context showing that they coincide. It is easy to see that Borger's construction extends naturally to non-affine schemes, while Buium's construction requires additional assumptions, e.g., nilpotency of p, and one is led to work with formal schemes. After introducing both constructions, we compare them in Corollaries 3.4 & 3.12. Our result was known to Borger in classical cases (cf. [7, 12.8]).

3.1. Buium's π-jet spaces

In [12] Buium develops the theory of π-jet spaces in the category of π-adic formal schemes. However, we want to talk about it in two distinct steps: first for the affine case and then for the non-affine case where we will be in the context of π-adic formal schemes.

Note that from (2.14), one can naturally define the *Buium jet space* of an affine scheme $X = \mathrm{Spec} A$ setting

$$\mathcal{J}^n X := \mathrm{Spec} J_n A.$$

Now we will pass to the category of π-adic formal schemes to follow Buium's original construction in [12]. Let $\hat{\mathcal{O}}$ be the π-adic completion of $\mathcal{O}_{\mathfrak{p}}$ and assume that our fixed prolongation sequence R_* consists of complete π-adic rings over $\hat{\mathcal{O}}$. For an affine π-adic formal scheme $Y = \mathrm{Spf} A$, set $\widehat{\mathcal{J}^n Y} = \mathrm{Spf} \widehat{J_n A}$ where $\widehat{J_n A}$ is the π-adic completion of $J_n A$. Now given a π-adic formal scheme X over R_0 along with an open affine cover

$$X = \bigcup_i V_i,$$

Buium in [11, p.315] defines the n-th jet space $\widehat{\mathcal{J}^n X}$ as

$$\widehat{\mathcal{J}^n X} = \bigcup_i \widehat{\mathcal{J}^n V_i}, \tag{3.1}$$

where the glueing data of the spaces $\widehat{\mathcal{J}^n V_i}$ naturally come from that of the open subschemes V_i. The fundamental reason why such a glueing is possible is the fact that for π-adically complete algebras A over R_0 one has the following canonical isomorphisms of localized R_n-algebras

$$\widehat{J_n(A_s)} = \widehat{(J_n A)_s}, \text{ for all } s \in A. \tag{3.2}$$

Here, one may ask the question of following a similar procedure as above for constructing a n-th jet space $\widehat{\mathcal{J}^n X}$ for a general scheme X over R_0. For any algebra A (not necessarily π-adically complete), we have

$$J_n(A_s) = J_n(A)_{s\phi(s)\cdots\phi^n(s)}, \text{ for all } s \in A. \tag{3.3}$$

If A is a π-adically complete algebra, then (3.3) leads to (3.2). However, this is not true in general for an A that is not π-adically complete and hence this approach will not directly apply to the scheme context.

3.2. The algebraic jet spaces

In this subsection, we will develop some of the functorial theory of jet spaces over a fixed prolongation sequence R_*. Much of the notation is borrowed from [7]. One of the goals of this section is to prove Theorem 3.8. This follows naturally from [7, Theorem 12.1]. However, since our base prolongation sequence is not $\mathcal{O}_*$ in general, it is worthwhile to give some details.

For a fixed affine scheme $S = \mathrm{Spec}R$, let $\mathbf{Aff}_S$ denote the category of affine schemes over S equipped with the étale topology and let $\mathbf{Sp}_S$ denote the category of sheaves of sets on $\mathbf{Aff}_S$. Then we can consider the fully faithful embedding of the category of S-schemes, denoted by $\mathbf{Sch}_S$, inside $\mathbf{Sp}_S$. If $f\colon S' \to S$ is a morphism of affine schemes and X a sheaf in $\mathbf{Sp}_S$, we write $S' \times_S X$ for the pullback of X to $\mathbf{Sp}_{S'}$. When R is a ring decorated with subscripts and superscripts, we will write $\mathbf{Aff}_R$ in place of $\mathbf{Aff}_S$ and $\mathbf{Sp}_R$ in place of $\mathbf{Sp}_S$.

Let $R_* = (R_0 \to R_1 \to \dots)$ be a fixed prolongation sequence and set $S^{(n)} = \mathrm{Spec}R_n$ and $S^{(n)}_m = S^{(n)} \times_{\mathrm{Spec}\mathcal{O}} \mathrm{Spec}\mathcal{O}/(\pi^m)$ its reduction modulo π^m; note that, following Borger's notation, subscripts on rings become superscripts on schemes.

For any $n \geq 0$ and any sheaf X in $\mathbf{Sp}_{R_0}$ we generalize Borger's construction introducing a sheaf J^nX in $\mathbf{Sp}_{R_n}$ given by

$$J^nX(B) := X(W_n(B))$$

for every R_n-algebra B, where $W_n(B)$ is here considered with its R_0-algebra structure as in Corollary 2.12. When $R_* = \mathcal{O}_*$, this sheaf is the one denoted by $W_{n*}(X)$ in [7, §10.3].

Theorem 2.21 immediately implies the following

Corollary 3.4. *Let X be an affine scheme over R_0. Then there is a natural isomorphism of sheaves*

$$J^nX \simeq \mathfrak{J}^nX.$$

In particular $J^n(X)$ is representable by an affine R_n-scheme.

Following [7, §10.10] we say that a map $f\colon X \to Y$ in $\mathbf{Sp}_{R_0}$ is *formally étale* if for all nilpotent closed immersions $\bar{T} \to T$ of affine schemes the induced map

$$X(T) \to X(\bar{T}) \times_{Y(\bar{T})} Y(T)$$

is a bijection. Further, f is *locally of finite presentation* if for any cofiltered system $(T_i)_{i\in I}$ of affine schemes over Y, the induced map $\mathrm{colim}_i \mathrm{Hom}_Y(T_i, X) \to \mathrm{Hom}_Y(\lim_i T_i, X)$ is bijective. Finally, f is said to be étale if it is locally of finite presentation and formally étale. For f a morphism of schemes, these definitions are the usual ones.

Proposition 3.5. *Let $f\colon X \to Y$ be a formally étale (respectively locally of finite presentation, respectively étale) map in $\mathbf{Sp}_{R_0}$. Then the map $J^n f\colon J^n X \to J^n Y$ in $\mathbf{Sp}_{R_n}$ has the same property.*

Proof. The proof in [7, 11.1] works in this context also. We recall it briefly. Assume f is formally étale. Let $\bar{T} = \mathrm{Spec}(B/I) \to T = \mathrm{Spec}(B)$ be a closed immersion of affine R_n-schemes. Assume $I^m = 0$ and note that $W_n(I)$ is a nilpotent ideal in $W_n(B)$ [6, Corollary 6.4]. Then

$$J^n X(T) = X(W_n(B)) = X(W_n(B)/W_n(I)) \times_{Y(W_n(B)/W_n(I))} Y(W_n(B)) = \\ X(W_n(B/I)) \times_{Y(W_n(B/I))} Y(W_n(B)) = J^n X(\bar{T}) \times_{J^n Y(\bar{T})} J^n Y(T).$$

Hence $J^n f$ is formally étale.

Assume now f is locally of finite presentation. We can repeat word by word the argument in [7, 11.1(a)] setting $W_n^*(T) := \mathrm{Spec} W_n(B)$ for any affine scheme $T = \mathrm{Spec} B$ over Y. The étale case follows then by definition of étaleness. □

Corollary 3.6. *Let $f\colon X \to Y$ be an open immersion of R_0-schemes. Then $J^n f\colon J^n X \to J^n Y$ is an open immersion of sheaves in $\mathbf{Sp}_{R_n}$, i.e., it is an étale monomorphism.*

Proof. The functor J^n preserves monomorphisms since it is right adjoint to the functor $\mathrm{Spec}(W_n(-))$ on affine R_n-schemes. By the previous proposition it preserves étaleness. □

Lemma 3.7. *Let $f\colon X \to Y$ be a closed immersion of R_0-schemes. Then $J^n f\colon J^n X \to J^n Y$ is a closed immersion of sheaves in $\mathbf{Sp}_{R_n}$, i.e., its base change along any map $T \to J^n Y$ with T an R_n-scheme is a closed immersion of schemes.*

Proof. As in the scheme theoretic case, it is sufficient to consider the case where T is affine. Let $T = \mathrm{Spec} C$ and consider a map $g\colon T \to J^n Y$, i.e., $g \in J^n Y(C)$ and let $g'\colon \mathrm{Spec} W_n(C) \to Y$ be the corresponding map, i.e., the corresponding section in $Y(W_n(C))$. The map of functors g factors as $T \to J^n(\mathrm{Spec} W_n(C)) \stackrel{J^n g'}{\to} J^n Y$ where the first map is the one associated to the identity on $\mathrm{Spec} W_n(C)$ due to the adjunction in Theorem 2.21. Hence it suffices to prove that the projection $J^n X \times_{J^n Y} J^n(\mathrm{Spec} W_n(C)) \to J^n(\mathrm{Spec} W_n(C))$ is a closed immersion of schemes. Let

$X' = X \times_Y \mathrm{Spec} W_n(C)$; it is an affine closed subscheme of $\mathrm{Spec} W_n(C)$ by hypothesis. Hence $J^n X' \to J^n(\mathrm{Spec} W_n(C))$ is a closed immersion by Corollary 3.4 and the explicit description of Buium jet spaces. Since the functor J^n preserves fiber products, $J^n X' = J^n X \times_{J^n Y} J^n(\mathrm{Spec} W_n(C))$ and the proof is completed. □

3.3. Representability of algebraic jet spaces

We prove here the representability of the algebraic jet functor which was Theorem 1.4 in our introduction.

Theorem 3.8. *If X is an $S^{(0)}$-scheme then $J^n X$ is an $S^{(n)}$-scheme for all n.*

Proof. Let B be an R_n-algebra and $g : \mathrm{Spec} B \to S^{(n)}$ be the associated structure map of schemes. By Corollary 2.12, $W_n(B)$ naturally becomes an R_0-algebra and let $\tilde{g} : \mathrm{Spec} W_n(B) \to S^{(0)}$ be the induced structure map. Let W_{n*} be the absolute arithmetic jet space functor introduced by Borger as in [7]. Then the above gives a canonical map between functors $S^{(n)} \to W_{n*}(S^{(0)})$ by sending g to $\tilde{g}$, for any $g \in S^{(n)}(B)$.

Let $f \in J^n X(B)$. Then f, by definition, satisfies the following commutative diagram of morphism of schemes

$$\begin{array}{ccc} \mathrm{Spec} W_n(B) & \xrightarrow{f} & X \\ {\scriptstyle \tilde{g}}\downarrow & \swarrow & \\ S^{(0)}. & & \end{array}$$

Hence $f \in (W_{n*}(X) \times_{W_{n*}(S^{(0)})} S^{(n)})(B)$. And clearly any section of $W_{n*}(X) \times_{W_{n*}(S^{(0)})} S^{(n)}$ can be realized as a section of $J^n X$. Therefore we have the isomorphism

$$J^n X \simeq W_{n*}(X) \times_{W_{n*}(S^{(0)})} S^{(n)}$$

as functors. By [7], Theorem 12.1, $W_{n*}(X)$ and $W_{n*}(S^{(0)})$ are representable by schemes. Hence $J^n X$ is also representable by a scheme. □

The scheme $J^n X$ is called *algebraic* (π, q)*-jet space of level* n, or simply *jet space of level* n*, associated to* X.

Let us use the same notation for the restriction $J^n : \mathbf{Sch}_{R_0} \to \mathbf{Sch}_{R_n}$ from the category of R_0-schemes to the category of R_n-schemes. This latter functor has a left adjoint, denoted by W_n^*, which associates to any R_n-scheme Y the $W_n(R_n)$-scheme $W_n^*(Y)$ constructed by gluing $\mathrm{Spec} W_n(A)$ as $\mathrm{Spec} A$ varies among the open subschemes of Y. For the construction of $W_n^*(Y)$ see also [23], [7]. Due to Corollary 2.12 it is an R_0-scheme.

3.4. The π-fiber

For a sheaf X in $\mathbf{Sp}_{R_0}$ let $X^{(n)}$ denote its pullback $S^{(n)} \times_{S^{(0)}} X$ to $\mathbf{Sp}_{R_n}$. Note that there is a canonical morphism of sheaves in $\mathbf{Sp}_{R_n}$

$$J^n X \to X^{(n)} \tag{3.9}$$

which is given by the maps $X(W_n(B)) \to X(B)$ induced by the projection $W_n(B) \to B$ for every R_n-algebra B (cf. the co-ghost map κ_0 in [7, 10.6.9]).

Let $f\colon X \to Y$ be a map in $\mathbf{Sp}_{R_0}$. By (3.9) and functoriality of J^n there exists a morphism

$$J^n X \longrightarrow J^n Y \times_{Y^{(n)}} X^{(n)} \tag{3.10}$$

in $\mathbf{Sp}_{R_n}$. The following proposition states that this map becomes an isomorphism when working modulo powers of π.

Proposition 3.11. *Let $f\colon X \to Y$ be a map in $\mathbf{Sp}_{R_0}$ which is formally étale. Then the map*

$$S_m^{(n)} \times_{S^{(n)}} J^n X \longrightarrow S_m^{(n)} \times_{S^{(n)}} J^n Y \times_{Y^{(n)}} X^{(n)}$$

is an isomorphism of sheaves over $S_m^{(n)}$ for all n and m.

Proof. Let B be an $(R_n/\pi^m R_n)$-algebra. Then (3.10) gives the following canonical map

$$J^n X(B) = X(W_n(B)) \longrightarrow Y(W_n(B)) \times_{Y(B)} X(B) = (J^n Y \times_{Y^{(n)}} X^{(n)})(B)$$

on B-sections. Now we will construct an inverse to the above map. Given an element $g \in Y(W_n(B)) \times_{Y(B)} X(B)$ this is determined by a pair of sections (g', g'') making the following diagram commute

$$\begin{array}{ccc} \mathrm{Spec} B & \xrightarrow[g'']{} & X \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Spec} W_n(B) & \xrightarrow[g']{} & Y \end{array}$$

Note that the left vertical arrow is a nilpotent thickening of $\mathrm{Spec}B$ since π is nilpotent in B and hence the ideal $V(W_n(B))$ is nilpotent by (2.5) and (2.6). Since $f\colon X \to Y$ is formally étale, there is a unique lift $\tilde{g}\colon \mathrm{Spec}W_n(B) \to X$ of g'' and we are done. $\square$

As a consequence, one can get that jet functors respect open affine coverings of formal schemes.

Corollary 3.12. *Let X be an S_0-scheme. Then $S_m^{(n)} \times_{S^{(n)}} J^n X$ is an $S_m^{(n)}$-scheme for any n and m. Further if $\{U_i\}_{i\in I}$ is an affine open covering of the scheme X then $\{S_m^{(n)} \times_{S^{(n)}} J^n U_i\}_{i\in I}$ is an affine open covering of $S_m^{(n)} \times_{S^{(n)}} J^n X$.*

Hence the formal π-completion of $J^n X$ is isomorphic to $\widehat{\mathfrak{J}^n X}$ and the special fiber of $J^n X$ is covered by the special fibers of the open subschemes $J^n U_i$'s.

Note that the above corollary is Theorem 1.5 in our introduction.

Proof. Representability by a scheme is clear by Theorem 3.8. By Corollaries 3.6 and 3.4 we know that the canonical map $S_m^{(n)} \times_{S^{(n)}} J^n U_i \to S_m^{(n)} \times_{S^{(n)}} J^n X$ are open immersions with $S_m^{(n)} \times_{S^{(n)}} J^n U_i$ affine schemes. Furthermore, the open immersions $U_i \times_X U_j \to U_i$ induce corresponding open immersions on the n-th jet spaces restricted to $S_m^{(n)}$. We can then glue the affine schemes $S_m^{(n)} \times_{S^{(n)}} J^n U_i$ getting an $S_m^{(n)}$-scheme $\bar{J}$. It remains to check that the induced map $\bar{J} \to S_m^{(n)} \times_{S^{(n)}} J^n X$ is an isomorphism. Given any section $s \in J^n(X)(B) = X(W_n(B))$ with B an R_0-algebra, up to localization on $W_n(B)$ (or equivalently on B, since the corresponding affine spectra are homeomorphic) we may assume that s comes from a section $s_i\colon \operatorname{Spec}(W_n(B)) \to U_i$. Since $U_i(W_n(B)) = J^n(U_i)(B) \subset \bar{J}(B)$, the section s lies in $\bar{J}(B)$ and we are done. □

Let X be an R_0-scheme, let $\overline{J^n X}$ denote the special fiber of $J^n X$, i.e.,

$$\overline{J^n X} := \overline{J^n X} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec}\mathcal{O}/(\pi) = \overline{J^n X} \times_{S^{(n)}} S_1^{(n)}$$

and let $\overline{\mathfrak{J}^n X}$ denote the special fiber of the π-adic formal scheme $\widehat{\mathfrak{J}^n X}$.

Corollary 3.13. *Let X be an R_0-scheme. For all $n \geq 0$ we have $\overline{J^n X} \simeq \overline{\mathfrak{J}^n X}$.*

Proof. This follows from (3.1) and Corollaries 3.4 and 3.12. □

4. Jet spaces and Greenberg transform

In this section, we let $W(B)$ denote the ring of p-typical Witt vectors with coefficients in a ring B (i.e., $W_{\pi,q}(B)$ with $\pi = p, q = p$). In addition, let $\mathfrak{R}$ be a fixed local artinian ring with perfect residue field k' of characteristic p. We recall the definition of the Greenberg algebra and the Greenberg transform.

4.1. Greenberg algebra and Greenberg transform

The *Greenberg algebra* associated to $\mathfrak{R}$, is the affine ring scheme $\mathcal{R}$ over $\operatorname{Spec}(k')$ that represents the fpqc sheaf associated to the presheaf

$$(\text{affine } k'\text{-schemes}) \to (\mathfrak{R}\text{-algebras}), \qquad \operatorname{Spec}(B) \mapsto W(B) \otimes_{W(k')} \mathfrak{R};$$

see [19,24,3] for an explicit description. There exists a canonical surjective homomorphism $\mathcal{R}(B) \to W(B) \otimes_{W(k')} \mathfrak{R}$ which is an isomorphism if B is semiperfect, i.e., if the absolute Frobenius on B is surjective [24, Corollary A.2].

Example 4.1. If $\mathfrak{R} = W_m(k')$, then $\mathcal{R} = \mathbb{W}_{m,k'}$ the k'-ring scheme of p-typical Witt vectors of length $m+1$, [24, proof of Prop. A.1].

Let now $Z = \mathrm{Spec}(A)$ be an affine $\mathfrak{R}$-scheme and consider the functor

$$(\text{affine } k'\text{-schemes}) \to (\text{Sets}), \qquad \mathrm{Spec}(B) \mapsto Z(\mathcal{R}(B)) := \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Spec}(\mathcal{R}(B)), Z).$$

By [19] and [3, Proposition 6.4], the above functor is representable by an affine scheme $\mathrm{Gr}^{\mathfrak{R}}(Z) = \mathrm{Spec}(\mathrm{Gr}^{\mathfrak{R}}(A))$, called *the Greenberg transform* or *Greenberg realization of Z* and hence

$$\mathrm{Hom}_{\mathfrak{R}}\big(\mathrm{Spec}\mathcal{R}(B), \mathrm{Spec}(A)\big) \simeq \mathrm{Hom}_{k'}\big(\mathrm{Spec}(B), \mathrm{Spec}(\mathrm{Gr}^{\mathfrak{R}}(A))\big),$$

for any k'-algebra B. In particular, there is a natural bijection

$$\mathrm{Hom}_{\mathfrak{R}}\big(A, \mathcal{R}(B)\big) \simeq \mathrm{Hom}_{k'}\big(\mathrm{Gr}^{\mathfrak{R}}(A), B\big). \tag{4.2}$$

If $\mathfrak{R} = W_m(k')$ then the above adjunction gives for any k'-algebra B

$$\mathrm{Hom}_{W_m(k')}\big(A, W_m(B)\big) \simeq \mathrm{Hom}_{k'}\big(\mathrm{Gr}^{W_m(k')}(A), B\big). \tag{4.3}$$

More generally, for any k'-scheme Y one can glue $\mathrm{Spec}\mathcal{R}(B)$ as $\mathrm{Spec}B$ varies among the open subschemes of Y thus getting a scheme $h^{\mathfrak{R}}(Y)$ over $\mathfrak{R}$. The functor $h^{\mathfrak{R}}$ from the category of k'-schemes to the category of $\mathfrak{R}$-schemes admits a right adjoint $\mathrm{Gr}^{\mathfrak{R}}$ so that there are natural bijections

$$\mathrm{Hom}_{\mathfrak{R}}\big(h^{\mathfrak{R}}Y, X\big) \simeq \mathrm{Hom}_{k'}\big(Y, \mathrm{Gr}^{\mathfrak{R}}(X)\big),$$

for any k'-scheme Y and any $\mathfrak{R}$-scheme X; see [3, §6]. By construction, any affine open covering $\{U_i\}$ of X induces an affine open covering $\{\mathrm{Gr}^{\mathfrak{R}}(U_i)\}$ of $\mathrm{Gr}^{\mathfrak{R}}(X)$.

Example 4.4. If $\mathfrak{R} = W_m(k')$, then $h^{\mathcal{R}}(Y)$ coincides with the scheme $W_m^*(Y)$ introduced in §3.3.

4.2. Comparison between jet spaces and Greenberg transform

Let $(\mathcal{O}', \pi', q')$ be another triple with $\mathcal{O}'$ a Dedekind domain above $\mathcal{O}$, π' a generator of a fixed prime ideal $\mathfrak{p}' \subset \mathcal{O}'$ such that $\pi \in (\pi')^e\mathcal{O}'$, $q' = q^r$ the cardinality of the residue field $\mathcal{O}'/\mathfrak{p}'$. We write $W(B)$ for the ring of p-typical Witt vectors with coefficients in B, and $W_{\pi,q}(B)$ for the π-typical variant constructed in Section 2.2 from the data $(\mathcal{O}, \pi, q)$. Similarly, for the finite length rings $W_m(B)$ and $W_{\pi,q,m}(B)$. For any $\mathcal{O}'$-algebra B there exists a natural homomorphism of $\mathcal{O}$-algebras (cf. [17, Proposition 1.2])

$$u\colon W_{\pi,q}(B) \to W_{\pi',q'}(B)$$

determined by the conditions

$$u([b]) = [b], \qquad u(F^r(x)) = F(u(x)), \qquad u(V(x)) = \frac{\pi}{\pi'} V(u(F^{r-1}(x))),$$

for any $b \in B$ and $x \in W_{\pi,q}(B)$; here F denotes Frobenius in both rings of Witt vectors and similarly for the Verschiebung V. Now, for any $\mathbb{F}_q$-algebra B there are Drinfeld maps

$$u\colon W(B) \to W_{p,q}(B), \qquad u_m\colon W_m(B) \to W_{p,q,m}(B), \tag{4.5}$$

which are isomorphisms if B is perfect but not in general [18, 1.2.1]; in fact, if $q = p^r, r \neq 1$ and $0 \neq b \in B$ satisfies $b^p = 0$, then $u(V[b]) = V[b^{p^{r-1}}] = 0$. In the general case $e > 1$, u induces unique homomorphisms

$$u_{m-1}\colon W_{\pi,q,m-1}(B) \to W_{\pi',q',me-1}(B) \tag{4.6}$$

for any $\mathbb{F}_q$-algebra B and any $m \geq 1$.

4.3. Greenberg realization

We now show that the special fiber of a π-jet space is very close to a Greenberg realization, thus generalizing [12, Thm. 2.10].

Let $k = \mathbb{F}_q$ and let $\mathcal{O} = W(\mathbb{F}_q)[\pi]/(\pi^e + \dots)$ be a finite totally ramified extension of $W(\mathbb{F}_q)$ of degree e with maximal ideal $\pi\mathcal{O}$. Let $\tilde{k}$ be any (possibly infinite) perfect field containing $\mathbb{F}_q$ and let $\tilde{\mathcal{O}} = W_{\pi,q}(\tilde{k}) = W(\tilde{k})[\pi]/(\pi^e + \dots)$ be the totally ramified extension of $W(\tilde{k})$ of degree e with uniformizer π. Let $\mathfrak{R} = \tilde{\mathcal{O}}/\pi^{me}\tilde{\mathcal{O}}$ for $m \geq 1$ and write Gr_{me-1} for the Greenberg realization $\mathrm{Gr}^{\mathfrak{R}}$ (thus shifting indices with respect to the notation in [3, §7]).

Theorem 4.7. *Let X be an $\mathfrak{R}$-scheme, $\mathrm{Gr}_{me-1}(X)$ the Greenberg realization of X with respect to $\mathfrak{R}$ and $J^{me-1}X$ the arithmetic (π, q)-jet space of level $me - 1$ of X. There is a natural morphism*

$$v\colon \mathrm{Gr}_{me-1}(X) \to \overline{J^{me-1}X} := J^{me-1}X \times_{\tilde{\mathcal{O}}} \mathrm{Spec}(\tilde{k})$$

of $\tilde{k}$-schemes which induces an isomorphism on inverse perfections, i.e., for any perfect $\tilde{k}$-algebra B the map $\mathrm{Gr}_{me-1}(X)(B) \to J^{me-1}X(B)$ is an isomorphism. Further

i) *If* $\mathcal{O} = \mathbb{Z}_p$ *and* $\mathfrak{R} = W_{m-1}(\tilde{k})$, *then* v *is an isomorphism.*
ii) *If* $e = 1$ *and* $m = 1$ *then the morphism* v *may be identified with the identity on* $X \times_{\mathfrak{R}} \mathrm{Spec}\tilde{k}$.

Proof. Since the Greenberg realization of X is determined by the Greenberg realization of an affine open covering of X and the same holds for the special fiber of the jet space by Corollary 3.12, we may work locally on X and assume that $X = \mathrm{Spec}A$ is affine. Let $\mathrm{Gr}_i(A)$ denote the ring of global sections of $\mathrm{Gr}_i(X)$. Assume first that $\mathcal{O} = W(\mathbb{F}_q), \tilde{\mathcal{O}} = W(\tilde{k}), \pi = p, e = 1, \mathfrak{R} = W_{m-1}(\tilde{k})$. By Example 4.1 we have $\mathcal{R} = \mathbb{W}_{m-1,\tilde{k}}$, the ring scheme of p-typical Witt vectors of length m over $\tilde{k}$. Note that, since $\tilde{k}$ is perfect, Drinfeld's map (4.5) gives an isomorphism $W(\tilde{k}) \simeq W_{p,q}(\tilde{k})$. By (4.3) and (2.23) for $R_* = \tilde{\mathcal{O}}_*$ we get

$$\begin{aligned}\mathrm{Hom}_{\tilde{k}}\big(\mathrm{Gr}_{m-1}(A), B\big) &\simeq \mathrm{Hom}_{W_{m-1}(\tilde{k})}\big(A, W_{m-1}(B)\big) \simeq \\ &\mathrm{Hom}_{W(\tilde{k})}\big(A, W_{m-1}(B)\big) \to \mathrm{Hom}_{W(\tilde{k})}\big(A, W_{p,q,m-1}(B)\big) \simeq \\ &\qquad \mathrm{Hom}_{W(\tilde{k})}(J_{m-1}A, B) \simeq \mathrm{Hom}_{\tilde{k}}(J_{m-1}A \otimes_{W_{m-1}(\tilde{k})} \tilde{k}, B),\end{aligned}$$

for any $\tilde{k}$-algebra B, where the arrow in the middle is induced by Drinfeld's map in (4.5) $u_{m-1} \colon W_{m-1}(B) \to W_{p,q,m-1}(B)$. Note that u_0 is the identity on B and u_{m-1}, $m \geq 2$, is the identity if $q = p$. Hence the identity map on $\mathrm{Gr}_{m-1}(A)$ produces a natural homomorphism of $\tilde{k}$-algebras $J_{m-1}A \otimes_{W_{m-1}(\tilde{k})} \tilde{k} \to \mathrm{Gr}_{m-1}(A)$ which is an isomorphism if $q = p$ or $m = 1$. This proves i) and ii) for $\mathcal{O} = \mathbb{Z}_p$. Furthermore, since u_{m-1} is an isomorphism for B perfect, it induces a morphism of schemes $v \colon \mathrm{Gr}_{m-1}(X) \to J^{m-1}(X) \times_{\tilde{\mathcal{O}}} \mathrm{Spec}(\tilde{k})$ with the indicated property on inverse perfections.

We now prove the theorem for general $\mathcal{O}$ and $\mathfrak{R} = \tilde{\mathcal{O}}/\pi^{me}\tilde{\mathcal{O}}$. We first introduce a generalization of Drinfeld's map. Since $\mathfrak{R} = \tilde{\mathcal{O}}/\pi^{me}\tilde{\mathcal{O}} = \oplus_{i=0}^{e-1} W_{m-1}(\tilde{k})\pi^i$ [4, Remark 3.1b)], for any $\tilde{k}$-algebra B there is a natural homomorphism of $\tilde{\mathcal{O}}$-algebras

$$\begin{aligned}\tilde{u} \colon \mathcal{R}(B) = W_{m-1}(B) \otimes_{W_{m-1}(\tilde{k})} \mathfrak{R} = \oplus_{i=0}^{e-1} W_{m-1}(B)\pi^i &\to W_{\pi,q,me-1}(B) \\ \sum_{i=0}^{e-1} w_i \otimes \pi^i &\mapsto \sum_{i=0}^{e-1} u(w_i)\pi^i,\end{aligned}$$

where the first equality follows by [3, Lemma 4.3] applied to the extension $W(\tilde{k}) \to \tilde{\mathcal{O}}$, $w_i \in W_{m-1}(B)$ and $u \colon W_{m-1}(B) = W_{p,p,m-1}(B) \to W_{\pi,q,me-1}(B)$ is Drinfeld's map in (4.6). By (4.2) and (2.23) with $R_* = \tilde{\mathcal{O}}_*$, for any $\tilde{k}$-algebra B we have

$$\begin{aligned}\mathrm{Hom}_{\tilde{k}}\big(\mathrm{Gr}_{me-1}(A), B\big) &\simeq \mathrm{Hom}_{\mathfrak{R}}\big(A, \mathcal{R}(B)\big) \simeq \mathrm{Hom}_{\tilde{\mathcal{O}}}\big(A, \mathcal{R}(B)\big) \to \\ &\mathrm{Hom}_{\tilde{\mathcal{O}}}(A, W_{\pi,q,me-1}(B)) \simeq \mathrm{Hom}_{\tilde{\mathcal{O}}}(J_{me-1}(A), B) \simeq \mathrm{Hom}_{\tilde{k}}(J_{me-1}(A) \otimes_{\tilde{\mathcal{O}}} \tilde{k}, B),\end{aligned}$$

where the arrow in the middle is induced by $\tilde{u}$. Hence we have a natural homomorphism $J_{me-1}(A) \otimes_{\tilde{\mathcal{O}}} \tilde{k} \to \mathrm{Gr}_{me-1}(A)$, or in other words there is a natural morphism

$\mathrm{Gr}_{me-1}(X) \to \overline{J^{me-1}(X)}$ of $\tilde{k}$-schemes. Since $\tilde{u}$ maps $\mathcal{R}(B) \simeq W_{m-1}(B) \otimes_{W(\mathbb{F}_q)} \mathcal{O}/\pi^{me}\mathcal{O}$ isomorphically to $W_{\pi,q,me-1}(B)$, when B is perfect [18, 1.2.1], the morphism v is an isomorphism when passing to inverse perfection [4, §5]. Furthermore, since for $e = 1$ and $m = 1$ the map $\tilde{u} = u_0$ is the identity on B, assertion ii) is clear. □

Example 4.8. Assume $e = 1$ and let $X = \mathbb{A}^1_{\tilde{\mathcal{O}}}$. Then $\tilde{\mathcal{O}} = W(\tilde{k})$ and $\mathrm{Gr}_{m-1}(X)$ is $\mathbb{W}_{m-1,\tilde{k}}$, the scheme of Witt vectors of length m over $\mathrm{Spec}(\tilde{k})$ and $J^{m-1}(X)$ is $\mathbb{W}_{p,q,m-1}$, the $W(\tilde{k})$-scheme of truncated ramified Witt vectors. The morphism $v\colon \mathrm{Gr}_{m-1}(X) \to J^{m-1}(X) \times_{\tilde{\mathcal{O}}} \mathrm{Spec}(\tilde{k})$ in Theorem 4.7, evaluated on a $\tilde{k}$-algebra B, coincides with Drinfeld's map u_{m-1} in (4.5) and therefore is not an isomorphism in general.

The map $\tilde{u}$ in the proof of the previous theorem is studied in detail in [2] when showing that the perfection of the Greenberg algebra of $\tilde{\mathcal{O}}$ is the special fiber of the ring scheme of ramified Witt vectors.

Let X be an $\mathfrak{R}$-scheme of dimension r. As noted in [11, p.316], one cannot expect that, in the ramified case, the Greenberg transform and the special fiber of the jet space have the same dimension at all levels. However, Theorem 4.7 implies equality for a cofinal subset of the set of levels.

By Corollary 3.13 the special fiber of J^nX coincides with the special fiber of the formal jet space $\widehat{\mathfrak{J}^n X}$. The above theorem states that, for X a $W(\tilde{k})$-scheme, the Greenberg realization $\mathrm{Gr}_n(X)$ is isomorphic to the special fiber of $\widehat{\mathfrak{J}^n X} = \widehat{J^n(X)}$. Furthermore, the reduction morphism $\mathbb{W}_n \to \mathbb{W}_{n-1}$, induces a canonical morphism $J^n(X) \to J^{n-1}(X)$, and the change of level morphism $\mathrm{Gr}^{W_n(\tilde{k})}(X) \to \mathrm{Gr}^{W_{n-1}(\tilde{k})}(X)$. We then obtain the following

Corollary 4.9. *Let $\tilde{k}$ be any perfect field of characteristic p and let X be a $W(\tilde{k})$-scheme. Then the sequence $J^*(X) = (J^n(X))_n$ restricts to the sequence $(\mathrm{Gr}_n(X))_n$ on special fibers. In particular, if the p-jet space of infinite level $J^\infty(X) = \varprojlim J^n(X)$ exists (e.g., if X is affine or smooth over $W(\tilde{k})$) then its special fiber is isomorphic to the $\tilde{k}$-scheme $\mathrm{Gr}(X) = \varprojlim \mathrm{Gr}^{W_n(\tilde{k})}(X)$.*

Note that the above result generalizes the one in [12, Theorem 2.10] removing the condition that $\tilde{k}$ is algebraically closed.

Data availability

No data was used for the research described in the article.

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