

# A condition for scattered linearized polynomials involving Dickson matrices

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## Abstract

A linearized polynomial over  $\mathbb{F}_{q^n}$  is called scattered when for any  $t, x \in \mathbb{F}_{q^n}$ , the condition  $xf(t) - tf(x) = 0$  holds if and only if  $x$  and  $t$  are  $\mathbb{F}_q$ -linearly dependent. General conditions for linearized polynomials over  $\mathbb{F}_{q^n}$  to be scattered can be deduced from the recent results in [4, 7, 15, 19]. Some of them are based on the Dickson matrix associated with a linearized polynomial. Here a new condition involving Dickson matrices is stated. This condition is then applied to the Lunardon-Polverino binomial  $x^{q^s} + \delta x^{q^{n-s}}$ , allowing to prove that for any  $n$  and  $s$ , if  $N_{q^n/q}(\delta) = 1$ , then the binomial is not scattered. Also, a necessary and sufficient condition for  $x^{q^s} + bx^{q^{2s}}$  to be scattered is shown which is stated in terms of a special plane algebraic curve.

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## 1 Introduction

A point  $P$  of the projective space  $\text{PG}(d-1, q^n)$  is a one-dimensional subspace of the vector space  $\mathbb{F}_{q^n}^d$ ; that is,  $P = \langle v \rangle_{\mathbb{F}_{q^n}} = \{cv : c \in \mathbb{F}_{q^n}\}$  for some nonzero  $v \in \mathbb{F}_{q^n}^d$ .

Let  $U$  be an  $r$ -dimensional  $\mathbb{F}_q$ -subspace of  $\mathbb{F}_{q^n}^d$ . Then

$$L_U := \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in U, v \neq 0\}$$

is an  $\mathbb{F}_q$ -linear set (or just *linear set*) of rank  $r$  in  $\text{PG}(d-1, q^n)$ . Let  $u, v \in U$ . If  $u = cv$ ,  $c \in \mathbb{F}_q$ , then clearly  $\langle u \rangle_{\mathbb{F}_{q^n}} = \langle v \rangle_{\mathbb{F}_{q^n}}$ . If this is the only case in which two vectors of  $U$  determine the same point of  $\text{PG}(d-1, q)$ , that is,  $\langle v \rangle_{\mathbb{F}_{q^n}} = \langle u \rangle_{\mathbb{F}_{q^n}}$  if and only if  $\langle v \rangle_{\mathbb{F}_q} = \langle u \rangle_{\mathbb{F}_q}$ , then  $L_U$  is called a *scattered* linear set. Equivalently,  $L_U$  is scattered if and only if it has maximum size  $(q^r - 1)/(q - 1)$  with respect to  $r$ . The linear sets are related to combinatorial objects, such as blocking sets, two-intersection sets, finite semifields, rank-distance codes, and many others. The interested reader is referred to the survey by O. Polverino [18] and to [20], where J. Sheekey builds a bridge with the rank-distance codes.

Assume that in particular  $U$  is an  $\mathbb{F}_q$ -subspace of  $\mathbb{F}_{q^n}^2$ ,  $\dim_{\mathbb{F}_q} U = n$ . In this case  $L_U := \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in U, v \neq 0\} \subseteq \text{PG}(1, q^n)$ , is called a *maximum* linear set of  $\text{PG}(1, q^n)$ , since by the dimension formula any linear set of rank greater than  $n$  equals  $\text{PG}(1, q^n)$ . Up to projectivities of  $\text{PG}(1, q^n)$  it may be assumed that  $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}} \notin L_U$ . Hence

$$L_U = L_f = \{\langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$$

where  $f(x)$  is a suitable  $\mathbb{F}_q$ -linear map, that is a linearized polynomial:

$$f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}, \quad a_i \in \mathbb{F}_{q^n}, \quad i = 0, 1, \dots, n-1. \quad (1)$$

If  $L_f$  is scattered, then  $f(x)$  is called a *scattered* linearized polynomial, or *scattered  $q$ -polynomial* with respect to  $n$ . A property characterizing the scattered  $q$ -polynomials is that for any  $x, y \in \mathbb{F}_{q^n}^*$ ,  $f(x)/x = f(y)/y$  if and only if  $\langle x \rangle_{\mathbb{F}_q} = \langle y \rangle_{\mathbb{F}_q}$ .

A first example of scattered  $q$ -polynomial is  $f(x) = x^q$  [3], with respect to any  $n$ . Indeed, for any  $x, y \in \mathbb{F}_{q^n}^*$ ,  $f(x)/x = f(y)/y$ , is equivalent to  $x^{q-1} = y^{q-1}$ , hence to  $x/y \in \mathbb{F}_q^*$ . A derived example is  $f(x) = x^{q^s}$ ,  $\gcd(n, s) = 1$ . Indeed  $(x/y)^{q^s-1} = 1$  implies  $x/y \in \mathbb{F}_{q^s} \cap \mathbb{F}_{q^n}^* = \mathbb{F}_q^*$ . In both cases above,  $L_f = \{\langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\} = \{\langle (1, z) \rangle_{\mathbb{F}_{q^n}} : z \in \mathbb{F}_{q^n}, N_{q^n/q}(z) = 1\}$ , where  $N_{q^n/q}(z) = z^{(q^n-1)/(q-1)}$  denotes the norm over  $\mathbb{F}_q$  of  $z \in \mathbb{F}_{q^n}$ . The related linear set is called a *linear set of pseudoregulus type*.

The next example has been given by G. Lunardon and O. Polverino [12] and generalized in [11, 20]:

$$f(x) = x^{q^s} + \delta x^{q^{n-s}}, \quad n \geq 4, \quad \gcd(n, s) = 1, \quad N_{q^n/q}(\delta) \neq 1.$$

In particular cases, the condition  $N_{q^n/q}(\delta) \neq 1$  has been proved to be necessary for  $f(x)$  to be scattered [2, 10, 11, 22]. In section 3 it will be proved that actually it is necessary for any  $n$  and  $s$ . Further examples of scattered  $q$ -polynomials are given in [6, 5, 14, 22]. All of them are with respect to  $n = 6$  or  $n = 8$ . D. Bartoli, M. Giulietti, G. Marino, and O. Polverino [1] proved that if  $\hat{f}(x)$  is the adjoint of  $f(x)$  with respect to the bilinear form  $\langle x, y \rangle = \text{Tr}_{q^n/q}(xy)$  in  $\mathbb{F}_{q^n}^2$ , where  $\text{Tr}_{q^n/q}(z) = \sum_{i=0}^{n-1} z^{q^i}$  denotes the trace over  $\mathbb{F}_q$  of  $z \in \mathbb{F}_{q^n}$ , then  $L_f = L_{\hat{f}}$ . This implies that if the polynomial  $f(x)$  in (1) is scattered, then also  $\hat{f}(x) = \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}$  is. Up to the knowledge of the author of this paper, no more examples of scattered  $q$ -polynomials are known. So, it would seem that scattered  $q$ -polynomials are rare. D. Bartoli and Y. Zhou [2] formalized such an idea of scarcity by proving that the pseudoregulus and Lunardon-Polverino polynomials are, roughly speaking, the only  $q$ -polynomials of a certain type which are scattered for infinitely many  $n$ .

Recently, a great deal of effort has been put in finding conditions for  $q$ -polynomials to be scattered [4, 7, 15, 19]. Some of them are based on the *Dickson matrix* associated with the  $q$ -polynomial in (1), that is, the  $n \times n$  matrix

$$M_{q,f} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1}^q & a_0^q & a_1^q & \cdots & a_{n-2}^q \\ a_{n-2}^{q^2} & a_{n-1}^{q^2} & a_0^{q^2} & \cdots & a_{n-3}^{q^2} \\ \vdots & & & & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & a_3^{q^{n-1}} & \cdots & a_0^{q^{n-1}} \end{pmatrix}.$$

It is well-known that the rank of  $M_{q,f}$  equals the rank of  $f(x)$ , see for example [21, Proposition 4.4]. This rank can be computed by applying the following result by B. Csajbók:

**Theorem 1.1** ([4, Theorem 3.4]). *Let  $M_{q,f}$  be the Dickson matrix associated with the  $q$ -polynomial in (1). Denote by  $M_{q,f}^{(r)}$  the  $r \times r$  submatrix of  $M_{q,f}$  obtained by considering the last  $r$  columns and the first  $r$  rows of  $M_{q,f}$ . Then the rank of  $f(x)$  is  $t$  if and only if  $\det(M_{q,f}^{(n)}) = \det(M_{q,f}^{(n-1)}) = \cdots = \det(M_{q,f}^{(t+1)}) = 0$ , and  $\det(M_{q,f}^{(t)}) \neq 0$ .*

A  $q$ -polynomial  $f(x) \in \mathbb{F}_{q^n}[x]$  is scattered if and only if for any  $m \in \mathbb{F}_{q^n}$  the dimension of the kernel of  $f_m(x) = mx + f(x)$  is at most one. So, by Theorem 1.1 a necessary and sufficient condition for  $f(x)$  to be scattered is

that the system of two equations

$$\det(M_{q,f_m}^{(n)}) = \det(M_{q,f_m}^{(n-1)}) = 0$$

has no solution in the variable  $m \in \mathbb{F}_{q^n}$ .

In this paper a condition consisting of one equation (Proposition 2.2) is proved, and applied to two binomials. It would seem that one equation is better than two in order to prove that a given  $q$ -polynomial  $f(x)$  is not scattered, while two equations will usually be more helpful in the proof that  $f(x)$  is. As a matter of fact, here the condition  $N_{q^n/q}(\delta) \neq 1$  is proved to be necessary for the Lunardon-Polverino binomial to be scattered (cf. Theorem 3.4). Furthermore, two necessary and sufficient conditions for  $x^{q^s} + bx^{q^{2s}}$  (where  $\gcd(s, n) = 1$ ) to be scattered are stated in Propositions 3.5 and 3.10. This leads to the fact that the polynomial  $x^q + bx^{q^2}$ ,  $b \neq 0$ , is never scattered if  $n \geq 5$  (cf. Proposition 3.8 and Remark 3.11).

## 2 A condition for scattered linearized polynomials

In this paper  $s$ ,  $n$ ,  $q$  and  $\sigma$  will always denote natural numbers such that  $n \geq 3$ ,  $\gcd(s, n) = 1$ ,  $q$  is the power of a prime and  $\sigma = q^s$ . Any  $\mathbb{F}_q$ -linear endomorphism of  $\mathbb{F}_{q^n}$  can be represented in the form

$$f(x) = a_0x + a_1x^\sigma + a_2x^{\sigma^2} + \cdots + a_{n-1}x^{\sigma^{n-1}} \in \mathbb{F}_{q^n}[x]. \quad (2)$$

As a matter of fact, if  $\tau$  is the permutation  $i \mapsto is$  of  $\mathbb{Z}/(n)$ , then  $f(x)$  is the same function of  $\tilde{f}(x) = \sum_{i=0}^{n-1} a_{\tau^{-1}(i)}x^{q^i}$ . Generalizing the notion of Dickson matrix given in the previous section, the  $\sigma$ -matrix of Dickson associated with the linearized polynomial  $g(t) = \sum_{i=0}^{n-1} a_i t^{\sigma^i}$  is

$$M_{\sigma,g} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1}^\sigma & a_0^\sigma & a_1^\sigma & \cdots & a_{n-2}^\sigma \\ a_{n-2}^{\sigma^2} & a_{n-1}^{\sigma^2} & a_0^{\sigma^2} & \cdots & a_{n-3}^{\sigma^2} \\ \vdots & & & & \vdots \\ a_1^{\sigma^{n-1}} & a_2^{\sigma^{n-1}} & a_3^{\sigma^{n-1}} & \cdots & a_0^{\sigma^{n-1}} \end{pmatrix}.$$

This is just the Dickson matrix  $M_{q,\tilde{g}}$  associated with  $\tilde{g}(t)$  after a permutation of the row and columns. Indeed, the element in row  $r$  and column  $c$  of  $M_{q,\tilde{g}}$ ,

$r, c \in \{0, 1, \dots, n-1\}$ , is  $m_{rc} = a_{\tau^{-1}(c-r)}^{q^r} = a_{\tau^{-1}(c)-\tau^{-1}(r)}^{q^r}$ . By applying  $\tau$  to both the row and column index,  $m_{\tau(r)\tau(c)} = a_{c-r}^{\sigma^r}$  follows. Therefore, the rank of  $M_{\sigma,g}$  equals the rank of  $g(t)$ .

**Remark 2.1.** *Each row of an  $n$ -order  $\sigma$ -matrix of Dickson is obtained from the previous one (cyclically) by the map*

$$\phi : (X_0, X_1, \dots, X_{n-1}) \mapsto (X_{n-1}, X_0, \dots, X_{n-2})^\sigma$$

which is an invertible semilinear map of  $\mathbb{F}_{q^n}^n$  into itself.

The polynomial (2) is scattered if and only if  $f_1(x) = \sum_{i=1}^{n-1} a_i x^{\sigma^i}$  is. Hence in the following  $a_0$  will always be zero.

**Proposition 2.2.** *Let  $f(x) = \sum_{i=1}^{n-1} a_i x^{\sigma^i}$  be a linearized polynomial over  $\mathbb{F}_{q^n}$ , and*

$$g(t) = g_x(t) = -f(x)t + \sum_{i=1}^{n-1} a_i x^{\sigma^i} t^{\sigma^i} = -f(x)t + f(xt).$$

Then the following conditions are equivalent:

- (i) the polynomial  $f(x)$  is scattered;
- (ii) for any  $x \in \mathbb{F}_{q^n}^*$ , a nonsingular  $(n-1)$ -order minor of  $M_{\sigma,g}$  exists;
- (iii) for any  $x \in \mathbb{F}_{q^n}^*$ , all  $(n-1)$ -order minors of  $M_{\sigma,g}$  are nonsingular.

*Proof.* The polynomial  $f(x)$  is scattered if and only if for any  $x \in \mathbb{F}_{q^n}^*$  the rank of  $h(t) = xf(t) - tf(x)$  is  $n-1$ , that is, the rank of

$$M_{\sigma,h} = \begin{pmatrix} -f(x) & a_1 x & a_2 x & \dots & a_{n-1} x \\ a_{n-1}^\sigma x^\sigma & -f(x)^\sigma & a_1^\sigma x^\sigma & \dots & a_{n-2}^\sigma x^\sigma \\ a_{n-2}^{\sigma^2} x^{\sigma^2} & a_{n-1}^{\sigma^2} x^{\sigma^2} & -f(x)^{\sigma^2} & \dots & a_{n-3}^{\sigma^2} x^{\sigma^2} \\ \vdots & & & & \vdots \\ a_1^{\sigma^{n-1}} x^{\sigma^{n-1}} & a_2^{\sigma^{n-1}} x^{\sigma^{n-1}} & a_3^{\sigma^{n-1}} x^{\sigma^{n-1}} & \dots & -f(x)^{\sigma^{n-1}} \end{pmatrix}$$

is always  $n-1$ . By dividing the rows of  $M_{\sigma,h}$  by  $x, x^\sigma, x^{\sigma^2}, \dots, x^{\sigma^{n-1}}$ , respectively, and then multiplying the columns for that same elements, one

obtains

$$\begin{pmatrix} -f(x) & a_1 x^\sigma & a_2 x^{\sigma^2} & \dots & a_{n-1} x^{\sigma^{n-1}} \\ a_{n-1}^\sigma x & -f(x)^\sigma & a_1^\sigma x^{\sigma^2} & \dots & a_{n-2}^\sigma x^{\sigma^{n-1}} \\ a_{n-2}^{\sigma^2} x & a_{n-1}^{\sigma^2} x^\sigma & -f(x)^{\sigma^2} & \dots & a_{n-3}^{\sigma^2} x^{\sigma^{n-1}} \\ \vdots & & & & \vdots \\ a_1^{\sigma^{n-1}} x & a_2^{\sigma^{n-1}} x^\sigma & a_3^{\sigma^{n-1}} x^{\sigma^2} & \dots & -f(x)^{\sigma^{n-1}} \end{pmatrix},$$

that is, the matrix  $M_{\sigma,g}$ . By Remark 2.1, if a  $\sigma$ -matrix of Dickson is singular, then any row is a linear combination of the remaining ones. Hence the rank of  $M_{\sigma,g}$  equals the rank of any  $(n-1) \times n$  matrix obtained from it by deleting a row. Furthermore, since the sum of the columns of  $M_{\sigma,g}$  is zero, all  $(n-1)$ -order minors have the same rank of  $M_{\sigma,g}$ .  $\square$

### 3 Two linearized binomials

**Definition 3.1.** For any  $\delta \in \mathbb{F}_{q^n}$ ,

$$f_{\sigma,\delta}(x) = x^\sigma + \delta x^{\sigma^{n-1}}$$

is the Lunardon-Polverino binomial.

If  $N_{q^n/q}(\delta) \neq 1$ , then  $f_{\sigma,\delta}$  is scattered [11, 12, 13, 20].

**Proposition 3.2.** The polynomial  $f_{\sigma,\delta}(x)$  is scattered if only if there is no  $x \in \mathbb{F}_{q^n}^*$  such that

$$\sum_{i=0}^{n-1} z^{(\sigma^i - 1)/(\sigma - 1)} = 0, \quad (3)$$

where  $z = \delta x^{\sigma^{n-1} - \sigma}$ .

*Proof.* The  $(n-1)$ -th order North-West principal minor of the  $\sigma$ -matrix of Dickson associated with the polynomial

$$g(t) = -f_{\sigma,\delta}(x)t + \sum_{i=1}^{n-1} a_i x^{\sigma^i} t^{\sigma^i} = -f_{\sigma,\delta}(x)t + x^\sigma t^\sigma + \delta x^{\sigma^{n-1}} t^{\sigma^{n-1}},$$

further normalized row by row, is

$$B(z) = \begin{pmatrix} -(1+z) & 1 & 0 & 0 & \cdots & 0 & 0 \\ z^\sigma & -(1+z)^\sigma & 1 & 0 & \cdots & 0 & 0 \\ 0 & z^{\sigma^2} & -(1+z)^{\sigma^2} & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(1+z)^{\sigma^{n-3}} & 1 \\ 0 & 0 & 0 & 0 & \cdots & z^{\sigma^{n-2}} & -(1+z)^{\sigma^{n-2}} \end{pmatrix}. \quad (4)$$

By Laplace expansion along the last column and induction on  $n$ , the determinant of  $B(z)$  can be computed as  $(-1)^{n+1} \sum_{i=0}^{n-1} z^{(\sigma^i-1)/(\sigma-1)}$ .  $\square$

The following can be useful in understanding the role of  $\delta$ :

**Proposition 3.3.** *Let  $z \in \mathbb{F}_{q^n}$ . Then (3) holds if and only if there exists a  $y \in \mathbb{F}_{q^n}^*$  such that  $z = y^{\sigma-1}$  and  $\text{Tr}_{q^n/q}(y) = 0$ .*

*Proof.* Any solution of (3) is nonzero. Raising  $\sum_{i=0}^{n-1} z^{(\sigma^i-1)/(\sigma-1)}$  to the  $\sigma$ , multiplying by  $z$  and then subtracting to the original equation yields  $1 - N_{q^n/q}(z) = 0$ . So,  $z$  is a solution of (3) if and only if  $z = y^{\sigma-1}$  for some  $y \in \mathbb{F}_{q^n}^*$ , and  $\sum_{i=0}^{n-1} y^{\sigma^i-1} = 0$ . The latter equation is equivalent to  $\text{Tr}_{q^n/q}(y) = 0$ .  $\square$

Propositions 3.2 and 3.3 together show that, if  $f_{\sigma,\delta}$  is not scattered, then there is an  $x \in \mathbb{F}_{q^n}$  such that  $N_{q^n/q}(\delta x^{\sigma^{n-1}-\sigma}) = N_{q^n/q}(\delta) N_{q^n/q}(x^{\sigma^{n-1}-\sigma}) = 1$ . On the other hand,  $q-1$  divides  $\sigma^{n-1}-\sigma$ , hence  $N_{q^n/q}(x^{\sigma^{n-1}-\sigma}) = 1$ . Summarizing, if  $N_{q^n/q}(\delta) \neq 1$ , then the Lunardon-Polverino binomial is scattered, as is known.

**Theorem 3.4.** *If  $N_{q^n/q}(\delta) = 1$ , then the Lunardon-Polverino binomial  $f_{\sigma,\delta}(x)$  is not scattered.*

*Proof.* Case odd  $n$ . Since

$$x^{\sigma^{n-1}-\sigma} = (x^\sigma)^{\sigma^{n-2}-1}$$

and  $\gcd(s(n-2), n) = 1$ , the expression  $x^{\sigma^{n-1}-\sigma}$  takes all values in  $\mathbb{F}_{q^n}$  whose norm over  $\mathbb{F}_q$  is equal to one. This allows the substitution  $\delta x^{\sigma^{n-1}-\sigma} = w^{\sigma-1}$  into (3). So,  $f_{\sigma,\delta}(x)$  is not scattered if and only if  $\text{Tr}_{q^n/q}(w) = 0$  for some nonzero  $w$  and this is trivial.

Case even  $n$ . Since  $\gcd(\sigma^{n-2} - 1, \sigma^n - 1) = \sigma^2 - 1$ , the set of all powers of elements in  $\mathbb{F}_{q^n}$  with exponent  $\sigma^{n-2} - 1$  coincides with the set of all powers with exponent  $\sigma^2 - 1$ . Hence for any  $x \in \mathbb{F}_{q^n}$  there exists  $u \in \mathbb{F}_{q^n}$  such that  $x^{\sigma^{n-1}-\sigma} = u^{\sigma^2-1}$ , and conversely. This allows the substitution  $z = \delta u^{\sigma^2-1}$  in (3), meaning that if there is  $u$  such that

$$\sum_{i=0}^{n-1} \left( \delta u^{\sigma^2-1} \right)^{(\sigma^i-1)/(\sigma-1)} = 0, \quad (5)$$

then  $f_{\sigma,\delta}(x)$  is not scattered. So, taking  $\delta = d^{\sigma-1}$ , (5) is equivalent to  $\text{Tr}_{q^n/q}(du^{\sigma+1}) = 0, u \neq 0$ . This is a quadratic form in  $u$  in a vector space over  $\mathbb{F}_q$  of dimension greater than two which has at least one nontrivial zero.  $\square$

The theorem above has been proved in the particular cases  $n = 4$  in [10],  $s = 1$  in [2], both  $n$  and  $q$  odd in [11], and odd  $n$  in [22].

**Proposition 3.5.** *The polynomial  $f(x) = x^\sigma + bx^{\sigma^2}$  is scattered if only if there is no  $x \in \mathbb{F}_{q^n}^*$  such that*

$$\sum_{i=0}^{n-1} w^{(\sigma^i-1)/(\sigma-1)} = 0, \quad \text{where } w = -(1 + b^{-1}x^{\sigma-\sigma^2}). \quad (6)$$

*Proof.* The  $\sigma$ -matrix of Dickson associated with the polynomial

$$g(t) = -f(x)t + x^\sigma t^\sigma + bx^{\sigma^2} t^{\sigma^2},$$

further normalized by dividing the rows by  $bx^{\sigma^2}, b^\sigma x^{\sigma^3}, \dots$  is

$$A = \begin{pmatrix} w & -(1+w) & 1 & \cdots & 0 & 0 \\ 0 & w^\sigma & -(1+w)^\sigma & \cdots & 0 & 0 \\ 0 & 0 & w^{\sigma^2} & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 1 & 0 & 0 & \cdots & w^{\sigma^{n-2}} & -(1+w)^{\sigma^{n-2}} \\ -(1+w)^{\sigma^{n-1}} & 1 & 0 & \cdots & 0 & w^{\sigma^{n-1}} \end{pmatrix}.$$

The matrix obtained by deleting the last row and first column is  $B(w)$  (cf. (4)).  $\square$

**Corollary 3.6.** *Assume  $b_1, b_2 \in \mathbb{F}_{q^n}$  and  $N_{q^n/q}(b_1) = N_{q^n/q}(b_2)$ . Then the polynomials  $f_i(x) = x^\sigma + b_i x^{\sigma^2}$ ,  $i = 1, 2$ , are either both scattered, or both non-scattered.*



*Proof.* If the norm of  $b_1$  is zero then the statement is trivial, so assume that it is not. Define  $w_i(x) = -(1 + b_i^{-1}x^{\sigma-\sigma^2})$  for  $i = 1, 2$ , and note that  $w_1(x) = w_2(y)$  is equivalent to  $b_1/b_2 = ((x/y)^\sigma)^{\sigma-1}$ , that is,  $((x/y)^\sigma)^{\sigma-1} = c^{\sigma-1}$  for some  $c \in \mathbb{F}_{q^n}^*$ . This equation can be always solved in both  $x$  and  $y$ , whence  $w_1(x)$  and  $w_2(y)$  take the same set of values.  $\square$

**Remark 3.7.** *Corollary 3.6 allows to look at only  $q-1$  linearized polynomials, given  $s$ ,  $n$ , and  $q$ . This makes a computer search easier. Computations with GAP<sup>1</sup> show that there are no scattered linearized polynomials of the form  $l_b(x) = x^q + bx^{q^2}$ ,  $b \neq 0$ , for any  $q < 223$  if  $n = 5$ . In [17] it is proved that for  $n = 5$  and  $q \geq 223$  the linearized polynomial  $l_b(x)$  is not scattered for any  $b \neq 0$ . The next proposition summarizes this.*

**Proposition 3.8.** *If  $n = 5$  and  $b \in \mathbb{F}_{q^5}^*$ , then the  $q$ -polynomial  $l_b(x) = x^q + bx^{q^2} \in \mathbb{F}_{q^5}[x]$  is non-scattered.*

**Remark 3.9.** *For  $n = 4$  there are scattered polynomials of type  $l_b(x)$ ,  $b \neq 0$ . By the results in [9, 10], all the related linear sets are of Lunardon-Polverino type, up to collineations.*

**Proposition 3.10.** *Let  $b \in \mathbb{F}_{q^n}^*$ . The polynomial  $x^\sigma + bx^{\sigma^2} \in \mathbb{F}_{q^n}[x]$  is not scattered if and only if the algebraic curve  $b^{-1}X^{q-1} + Y^{\sigma-1} + 1 = 0$  in  $\text{AG}(2, q^n)$  has a point  $(x_0, y_0)$  with coordinates in  $\mathbb{F}_{q^n}^*$ , such that  $\text{Tr}_{q^n/q}(y_0) = 0$ .*

*Proof.* By Proposition 3.3, the first equation in (6) is equivalent to the existence of  $y \in \mathbb{F}_{q^n}^*$  such that  $w = y^{\sigma-1}$ ,  $\text{Tr}_{q^n/q}(y) = 0$ . The second equation  $y^{\sigma-1} + 1 + b^{-1}x^{q-q^2} = 0$  has solutions with  $x \neq 0$  if and only if  $b^{-1}x^{q-1} + y^{\sigma-1} + 1 = 0$  does.  $\square$

**Remark 3.11.** *Very recently, M. Montanucci [16] proved that if  $n > 5$ , then for any  $q$  the algebraic curve  $b^{-1}X^{q-1} + Y^{q-1} + 1 = 0$  has a point with the properties above. Together with Propositions 3.8 and 3.10 this implies that for  $n \geq 5$  no  $q$ -polynomial of type  $l_b(x) = x^q + bx^{q^2}$ ,  $b \neq 0$ , is scattered.*

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<sup>1</sup>Code: <https://pastebin.com/pgTXX76C>.

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