

# Symplectic quantization: numerical results for the Feynman propagator on a 1+1 lattice and the theoretical relation with quantum field theory

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**ABSTRACT:** We present here the first lattice simulation of symplectic quantization, a new functional approach to quantum field theory which allows to define an algorithm to numerically sample the quantum fluctuations of fields directly in Minkowski space-time, at variance with all other present approaches. Symplectic quantization is characterized by a Hamiltonian deterministic dynamics evolving with respect to an additional time parameter  $\tau$  analogous to the fictitious time of Parisi-Wu stochastic quantization. The difference between stochastic quantization and the present approach is that the former is well defined only for Euclidean field theories, while the latter allows to sample the causal structure of space-time. In this work we present the numerical study of a real scalar field theory on a 1+1 space-time lattice with a  $\lambda\phi^4$  interaction. We find that for  $\lambda \ll 1$  the two-point correlation function obtained numerically reproduces qualitatively well the shape of the free Feynman propagator. Within symplectic quantization the expectation values over quantum fluctuations are computed as dynamical averages along the dynamics in  $\tau$ , in force of a natural ergodic hypothesis connecting Hamiltonian dynamics with a generalized microcanonical ensemble. Analytically, we prove that this *microcanonical* ensemble, in the continuum limit, is equivalent to a *canonical-like* one where the probability density of field configurations is  $P[\phi] \propto \exp(zS[\phi]/\hbar)$ . The results from our simulations correspond to the value  $z = 1$  of the parameter in the canonical weight, which in this case is a well-defined probability density for field configurations in causal space-time, provided that a lower bounded interaction potential is considered. The form proposed for  $P[\phi]$  suggests that our theory can be connected to ordinary quantum field theory by analytic continuation in the complex- $z$  plane.

**KEYWORDS:** Algorithms and Theoretical Developments, Correlation Functions, Lattice Quantum Field Theory

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## 1 Introduction

Since its invention by Kenneth Wilson [1], lattice field theory had an enormous development [2, 3] as a method to handle non-perturbative problems in quantum field theory, in particular concerning the theory of strong interactions with respect to problems such as the estimate of hadronic masses [4] or heavy ions collisions [5]. Nevertheless, despite the great achievements, any numerical approach to quantum field theory on the lattice has retained so far a major limitation: all importance-sampling protocols are well defined only for Euclidean field theory. Since the latter is obtained by Wick-rotating real into imaginary time, the causal structure of correlation functions in quantum field theory cannot be usually sampled numerically. The convenience/necessity to analytically continue real to imaginary time is to transform the Feynman path integral, characterized by the oscillating factor  $\exp(iS[\phi]/\hbar)$  —  $S[\phi]$  is the relativistic action and  $\phi$  a generic quantum field — into a normalizable probability density  $\exp(-S_E[\phi]/\hbar)$ , with  $\hbar$  playing the same role of temperature in the Boltzmann weight of statistical mechanics and where  $S_E[\phi]$  is the positive-definite Euclidean action. The mapping to imaginary time has been so far the unavoidable condition to set up any importance sampling numerical protocol to study the quantum fluctuations of fields. The goal of the present work is to present a conceptual framework and a method which goes beyond this limitation, allowing for a straightforward procedure to numerically sample quantum fluctuations of fields in Minkowskian spacetime. The use of imaginary time and Euclidean field theory forbids the representation on the lattice of any process or phenomenon intrinsically related to the causal structure of space-time, in particular all processes on the light cone. By

definition the probability density  $\exp(-S_E[\phi]/\hbar)$  works as an effective “equilibrium” measure for quantum fluctuations. Any importance-sampling protocol built from the Euclidean weight projects, for large lattice sizes, on the ground states of the corresponding Minkowskian theory. In fact, Monte Carlo simulations built from Euclidean field theory allow us to reproduce with extreme precision the physics of *stable/equilibrium* bound states of the strong interactions [4], whereas it has been so far much more problematic to reproduce metastable resonances with short lifetimes, like for instance tetraquark or pentaquark states [6, 7], or the dynamics of scattering processes with a strong relativistic character, namely processes with a different number of degrees of freedom in the asymptotic initial and final states. It is for this reason that we believe it is of crucial interest the possibility to test numerically any new proposal for a quantum field theory formulation which allows first to define and then to study the dynamics of quantum fields fluctuations directly in Minkowski space-time.

An interesting idea in this direction, namely the proposal of a functional approach to field theory which is well defined from the probabilistic point of view already in Lorentzian space-time, has been recently put forward by one of us and goes under the name of “*symplectic quantization*” [8–10]. The first ingredient of this approach is to assume, for a given quantum field  $\phi(x)$ , with  $x = (ct, \mathbf{x})$  a point in four-dimensional space-time, a dependence on an additional time parameter  $\tau$ :

$$\phi(x) \rightarrow \phi(x, \tau), \tag{1.1}$$

which controls the continuous sequence of quantum fluctuations in each point of space-time. Theories with such an additional time parameter are not a novelty, the whole Parisi-Wu stochastic quantization approach being based on this idea [11, 12].

Within the stochastic quantization approach, Euclidean multipoint correlation functions are obtained as a time average over the fictitious time parameter,

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \phi(x_1, s) \dots \phi(x_n, s) = \langle \phi(x_1) \dots \phi(x_n) \rangle_E, \tag{1.2}$$

where on the right hand term of the above equation  $\langle \rangle_E$  denotes expectation with respect to the Euclidean weight and on the left hand term each  $\phi(x_i, s)$  represents the solution at point  $x_i$  of an auxiliary Langevin dynamics of the kind

$$\frac{d\phi}{d\tau} = -\frac{\delta S_E[\phi]}{\delta \phi(x, \tau)} + \eta(x, \tau), \tag{1.3}$$

where  $\eta(x, \tau)$  represents is a zero mean white noise. It is a standard result in the theory of stochastic processes [13] that the stochastic dynamics of eq. (1.3) allows to sample asymptotically the equilibrium distribution  $\exp(-S_E[\phi]/\hbar)$ . Shortly after the seminal idea of stochastic quantization, people realized that, on the basis of statistical ensemble equivalence, the Euclidean probability density of quantum fields can be sampled also by following the solutions of the Hamilton equations generated by a generalized Hamiltonian functional  $\mathbb{H}_E[\pi(x), \phi(x)]$  of the kind

$$\mathbb{H}_E[\pi(x), \phi(x)] = \mathbb{K}_E[\pi(x)] + S_E[\phi(x)], \tag{1.4}$$

where  $\pi(x, \tau)$  is a generalized momentum conjugated to the field  $\phi(x, \tau)$  with respect to the flowing of the fictitious time  $\tau$ . The idea of [14, 15], which goes under the name of “*Microcanonical approach to quantum field theory*”, is to achieve a sampling of the Euclidean probability of the fields by studying the following deterministic equations:

$$\begin{aligned}\dot{\phi}(x, \tau) &= \frac{\delta}{\delta\pi(x)} \mathbb{H}_E[\pi, \phi] \\ \dot{\pi}(x, \tau) &= -\frac{\delta}{\delta\phi(x)} \mathbb{H}_E[\pi, \phi].\end{aligned}\tag{1.5}$$

The replacement of the stochastic dynamics in eq. (1.3) with the deterministic one in eq. (1.5) was solely motivated by its major computational efficiency in certain specific situations, for instance in the case of non-local bosonic actions obtained from the integration of fermionic variables [3], where the deterministic equations are more suited to parallel updates of the variables. As such, the role of the microcanonical ensemble built on the conservation of the generalized Hamiltonian  $\mathcal{H}_E[\pi(x), \phi(x)]$  was merely that of an alternative technique to sample  $\exp(-S_E[\phi]/\hbar)$ , where no physical interpretation was given neither to the additional time  $\tau$  nor to the conjugated momenta  $\pi(x, \tau)$ . On the contrary, the key idea of symplectic quantization is to claim that the microcanonical approach to quantum field theory is something more fundamental and general, valid independently from its formal equivalence to the Euclidean field theory: this, as we are going to show, allows to sample quantum fluctuations directly in Minkowskian space-time. The logic of the following exposition will be therefore precisely the opposite of the one used to introduce the microcanonical approach to Euclidean quantum field theory, justified solely on its formal equivalence with the latter. First, without knowing which is the corresponding canonical ensemble, we claim the existence of a microcanonical ensemble built on a generalized Hamiltonian of the kind

$$\mathbb{H}[\pi(x), \phi(x)] = \mathbb{K}[\pi(x)] - S[\phi(x)],\tag{1.6}$$

where again we have a generalized kinetic energy term  $\mathbb{K}[\pi(x)]$  and where this time the generalized potential is  $-S[\phi(x)]$ , with  $S[\phi(x)]$  the original Minkowskian action. We will show that the two point correlation function obtained by generating the quantum fluctuations of a weakly non-linear  $\lambda\phi^4$  theory from the dynamics generated by  $\mathbb{H}[\pi(x), \phi(x)]$  looks precisely like the standard Feynman propagator. It will be only after having checked that the deterministic dynamics generated by  $\mathbb{H}[\pi(x), \phi(x)]$  allows to reproduce results in qualitative agreement to ordinary quantum field theory that we will find out which is the canonical weight corresponding to our deterministic dynamics, obtaining it as the main result of the section 7. In particular, in section 7 we will show that in the thermodynamic limit the sampling of quantum fluctuation with our symplectic dynamics is equivalent to the sampling according to a canonical weight of the kind

$$P_z[\phi] \propto \exp(zS[\phi]/\hbar),\tag{1.7}$$

where the results of our simulations correspond to fixing  $z = 1$  in eq. (1.7). Clearly, a probability density like the one in eq. (1.7) is not equivalent to the complex amplitude which enters the Feynman path integral, but can be related to it by analytic continuation in the

complex  $z$  plane. Investigations in this direction are in progress and will be presented in a forthcoming paper explaining how the symplectic quantization approach works for a quantum particle in a harmonic potential, with particular emphasis on how the analytic continuation to complex  $z$  plane of the above density, eq. (1.7), must be handled [16].

In synthesis, main idea of the symplectic quantization approach is the possibility to sample quantum fluctuation directly in Lorentzian space-time by means of a generalized microcanonical ensemble, built by adding the conjugated momenta with respect to the intrinsic time  $\tau$  *directly* to the relativistic action with Minkowski signature, with no need of considering any sort of analytic continuation from real to imaginary time and therefore preserving the causal structure of space-time. This intuition has been strongly inspired from the evidence that in statistical mechanics there are physically important situations where only the microcanonical ensemble is well defined [17, 18], namely physical phenomena which cannot be described within the canonical ensemble.

## 2 Symplectic quantization: from dynamics to ensemble averages

Let us summarize here the main steps for the derivation of the symplectic quantization dynamics. First of all, inspired by the stochastic quantization approach [11, 12], we assume that quantum fields  $\phi(x, \tau)$  depend on an additional time variable  $\tau$  which parametrizes the dynamics of quantum fluctuations in a given point of Minkowski space-time. Since for a relativistic quantum field theory the ambient space includes observer's time, necessarily the intrinsic time  $\tau$  must be a different variable, as thoroughly discussed in [8, 9]. The symplectic quantization approach to field theory assumes, consistently with the existence of an intrinsic time  $\tau$ , the existence of conjugated momenta of the kind

$$\pi(x, \tau) \propto \dot{\phi}(x, \tau), \tag{2.1}$$

which can be obtained as follows. First, we introduce a generalized Lagrangian of the kind

$$\mathbb{L}[\phi, \dot{\phi}] = \int d^d x \left[ \frac{1}{2c_s^2} \dot{\phi}^2(x) + S[\phi] \right], \tag{2.2}$$

where  $c_s$ , in natural units, is a dimensionless parameter, and  $S[\phi]$  is the standard action for a quantum field, e.g.,

$$\begin{aligned} S[\phi] &= \int d^d x \left( \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - V[\phi(x)] \right) \\ &= \int d^d x \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x^0} \right)^2 - \frac{1}{2} \sum_{i=1}^d \left( \frac{\partial \phi}{\partial x^i} \right)^2 - V[\phi(x)] \right] \end{aligned} \tag{2.3}$$

where the potential is, for instance

$$V[\phi] = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \tag{2.4}$$

By means of a Legendre transform one then passes to the Hamiltonian:

$$\begin{aligned}
 \mathbb{H}[\phi, \pi] &= \frac{1}{2} \int d^d x \, c_s^2 \pi^2(x) - S[\phi] \\
 &= \int d^d x \left[ \frac{c_s^2}{2} \pi^2(x) - \frac{1}{2} \left( \frac{\partial \phi}{\partial x^0} \right)^2 + \frac{1}{2} \sum_{i=1}^d \left( \frac{\partial \phi}{\partial x^i} \right)^2 + V[\phi] \right] \\
 &= \int d^d x \left[ \frac{c_s^2}{2} \pi^2(x) + \frac{1}{2} \phi \partial_0^2 \phi - \sum_{i=1}^d \phi \partial_i^2 \phi + V[\phi] \right]
 \end{aligned} \tag{2.5}$$

For both the ease of notation and for conceptual simplicity we will assume hereafter  $c_s = 1$ . Let us remark that there is no a priori reason forcing the new constant  $c_s$  to be precisely equal to the speed of velocity. Yet, the fact that in natural units  $c_s$  is dimensionless suggests that no new physical constants need to be introduced. From eq. (2.5) we have that, within the symplectic quantization approach, the dynamics of quantum fluctuations is the one governed by the following Hamilton equations:

$$\begin{aligned}
 \dot{\phi}(x) &= \frac{\delta \mathbb{H}[\phi, \pi]}{\delta \pi(x)} \\
 \dot{\pi}(x) &= -\frac{\delta \mathbb{H}[\phi, \pi]}{\delta \phi(x)} = -\frac{\delta \mathbb{V}[\phi]}{\delta \phi(x)},
 \end{aligned} \tag{2.6}$$

from which one gets

$$\ddot{\phi}(x, \tau) = -\partial_0^2 \phi(x, \tau) + \sum_{i=1}^d \partial_i^2 \phi(x, \tau) - \frac{\delta V[\phi]}{\delta \phi(x, \tau)}. \tag{2.7}$$

At this stage one can legitimately wonder how a classical deterministic theory can account for quantum fluctuations. Let us notice that in the expression of the generalized Hamiltonian  $\mathbb{H}[\phi, \pi]$  we can recognize a “*generalized potential energy*”  $\mathbb{V}[\phi]$ , corresponding to the original relativistic action, and a “*generalized kinetic energy*”  $\mathbb{K}[\pi]$ , namely the quadratic part related to the new conjugated momenta:

$$\mathbb{H}[\phi, \pi] = \mathbb{K}[\pi] + \mathbb{V}[\phi] \tag{2.8}$$

with:

$$\begin{aligned}
 \mathbb{V}[\phi] &= -S[\phi] \\
 \mathbb{K}[\pi] &= \frac{1}{2} \int d^d x \, \pi^2(x).
 \end{aligned} \tag{2.9}$$

The classical field  $\phi_{cl}$  solution corresponds to a minimum of the new generalized potential  $\mathbb{V}[\phi]$ :

$$\left. -\frac{\delta \mathbb{V}[\phi]}{\delta \phi(x)} \right|_{\phi_{cl}} = 0. \tag{2.10}$$

On the contrary, quantum fluctuations are naturally sampled by the generalized Hamiltonian dynamics, along which the functional derivative of  $\mathbb{V}[\phi]$  is not zero but equal to the rate of change of the conjugated momenta, see eq. (2.6). The generalized energy  $\mathbb{H}[\phi, \pi]$  is constant

along the dynamics: quantum fluctuations are encoded in the fluctuations of the generalized potential energy,  $S[\phi]$ .

Having defined the above deterministic dynamics for the quantum fluctuations of the field  $\phi(x, \tau)$ , eq. (2.7), one can then legitimately wonder how this functional formalism connects to the standard one, for instance to the standard Feynman path-integral formulation of quantum field theory. In first instance, in order to find out the connection between the dynamic approach of symplectic quantization and any functional formulation of field theory where the additional time  $\tau$  is absent, one needs to find out which probability density  $\rho[\phi(x)]$  corresponds to the dynamics. There are then two possibilities:  $\rho[\phi(x)]$  might have either an integral or a punctual correspondence with quantum field theory. The integral correspondence is when the path-integral is recovered by means of an integral transformation, as it will be discussed below here, and the *punctual* correspondence is when, in a certain limit, one can directly show that  $\rho[\phi(x)] \approx e^{iS[\phi]/\hbar}$ . This last procedure is more subtle, because it requires an analytic continuation of the action and of its degrees of freedom in the complex plane, and will be discussed thoroughly in [16, 19]. But let us now go back to the relation between symplectic dynamics and probability densities, for which we make an ergodic hypothesis for the Hamiltonian dynamics in eq. (2.6): if we assume that this dynamics samples at long time  $\tau$  the constant generalized energy hypersurface with uniform probability [8, 9], then we can associate to the dynamics of eq. (2.6) the following measure:

$$\rho_{\text{micro}}[\phi(x)] = \frac{1}{\Omega[\mathcal{A}]} \delta(\mathcal{A} - \mathbb{H}[\phi, \pi]), \tag{2.11}$$

where  $\Omega[\mathcal{A}]$  is a sort of *microcanonical* partition function

$$\Omega[\mathcal{A}] = \int \mathcal{D}\phi \mathcal{D}\pi \delta(\mathcal{A} - \mathbb{H}[\phi, \pi]), \tag{2.12}$$

with  $\mathcal{D}\phi = \prod_x d\phi(x)$  and  $\mathcal{D}\pi = \prod_x d\pi(x)$  the standard notation for functional integration. From the above partition function we can define the microcanonical adimensional entropy of symplectic quantization:

$$\Sigma_{\text{sym}}[\mathcal{A}] = \ln \Omega[\mathcal{A}] \tag{2.13}$$

The *ergodicity assumption* for the symplectic quantization dynamics amounts to say that, considering  $\mathcal{O}[\phi(x)]$  a generic observable of the quantum fields, symplectic quantization can be related to a stationary probability measure free of the additional parameter  $\tau$  by claiming that for generic initial conditions the following equivalence between averages holds:

$$\lim_{\Delta\tau \rightarrow \infty} \frac{1}{\Delta\tau} \int_{\tau_0}^{\tau_0 + \Delta\tau} d\tau \mathcal{O}[\phi(x, \tau)] = \int \mathcal{D}\phi \mathcal{D}\pi \rho_{\text{micro}}[\phi(x)] \mathcal{O}[\phi(x)], \tag{2.14}$$

where  $\tau_0$  is a large enough time for the system to have reached stationarity and “lost memory” of initial conditions. How to relate then the microcanonical partition function in eq. (2.12) to the path integral? It is quite intuitive to understand that the two expression must be related by some sort of statistical ensemble change. The crucial point of this change of ensemble, as stressed already in [8], is that the microcanonical partition function  $\Omega[\mathcal{A}]$  is built on the conservation of a non-positive quantity, the generalized Hamiltonian  $\mathbb{H}[\phi, \pi]$ . The latter,

from the point of view of physical dimensions, is a relativistic action: it therefore takes both arbitrarily large positive and arbitrarily large negative values due to the negative sign in front of the coordinate-time derivative term in the second line of eq. (2.5). The absence of positive definiteness for the generalized Hamiltonian  $\mathbb{H}[\phi, \pi]$ , which is the true relativistic signature of the theory, is what forbids a standard change of ensemble with a Laplace transform, that is customary in statistical mechanics when passing from microcanonical to canonical. The only integral transform which allows us to map *formally* the ensemble where  $\mathbb{H}[\phi, \pi]$  is constrained to the one where it is free to fluctuate is the Fourier transform. It is by Fourier transforming the microcanonical partition function  $\Omega[\mathcal{A}]$  that one obtains straightforwardly the Feynman path integral:

$$\mathcal{Z}[u] = \int_{-\infty}^{\infty} dA e^{-iuA} \Omega[\mathcal{A}] = \int \mathcal{D}\phi \mathcal{D}\pi e^{-\frac{i}{2}u \int d^d x \pi^2(x) + iuS[\phi]} = \mathcal{N}(u) \int \mathcal{D}\phi e^{iuS[\phi]}, \quad (2.15)$$

where  $u$  is a variable conjugated to the action and in the second line of eq. (2.15) we have integrated out momenta thanks to the quadratic dependence on them, contributing the infinite normalization constant  $\mathcal{N}(u)$ , which is typical of path integrals. Finally, if we fix  $u = \hbar^{-1}$  into the last line of eq. (2.15) we have the Feynman path integral:

$$\mathcal{Z}[\hbar] = \int_{-\infty}^{\infty} d\mathcal{A} e^{-i\mathcal{A}/\hbar} \Omega[\mathcal{A}] \propto \int \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}. \quad (2.16)$$

The one above is to our knowledge the first derivation from first principles of the Feynman path-integral formula in the context of a more extended framework. We could say that this larger framework is the statistical mechanics of action-preserving systems, opposed to the statistical mechanics of energy-preserving systems, which is the standard one.

At this stage one can therefore legitimately wonder which is the relation between the probability density of the extended framework just introduced, i.e., the  $\rho_{\text{micro}}[\phi(x)]$  of eq. (2.11), and standard quantum field theory probability amplitudes. The key to this, at the present stage of development of this new approach, is precisely the Fourier transform of eq. (2.16): what we expect from the microcanonical symplectic quantization ensemble is the possibility to sample disconnected correlation functions at fixed generalized action  $\mathcal{A}$  and then, by Fourier transforming with respect to  $\mathcal{A}$ , obtain the original disconnected correlation functions of quantum field theory. We will see in section 6 that remarkably good results for the two point correlation function can be obtained already in the fixed generalized action ensemble! We speak about disconnected correlation functions because, at the present stage of understanding, we are able to write down an explicit relation only between the generating functionals of disconnected correlations, respectively  $\Omega[\mathcal{A}]$  for symplectic quantization and  $\mathcal{Z}[\hbar]$  for quantum field theory. Clearly the new approach needs to be improved, for at least three main reasons. First, to recover disconnected correlation functions of quantum field theory by doing many deterministic simulations at different fixed values of action  $\mathcal{A}$  and then try to Fourier transform the result looks a quite impractical protocol. We are presently working on an attempt to improve the correspondence between symplectic quantization and quantum field theory, but is still in progress [16, 19]. Second, the physical information of a quantum field theory is contained in the connected correlators, not the disconnected ones, so to rebuild the whole theory from this path seems a very long way. Last, but not

least, as we find in the result of simulations in section 5 and we will derive analytically in section 7, the microcanonical density of eq. (2.11) is ill-defined for a free theory, which looks quite problematic when compared to the importance and the success of perturbative expansions in quantum field theory.

Let us now come back for a moment on the reason why, differently from what is customary in the theory of statistical ensembles, symplectic quantization is related to quantum field theory by Fourier rather than Laplace transforming. This comes from the fact that the microcanonical statistical ensemble of symplectic quantization is built on the conservation of a non positive-defined quantity, landmark of the relativistic nature of the theory, that implies the necessity of Fourier transforming and determines the fact that *locally* we can only access complex probability amplitudes and not real probabilities. From the perspective of symplectic quantization the replacement at the local level of probabilities with probability amplitudes is therefore a direct consequence of special relativity and a wise use of statistical ensembles. To better understand this statement let us consider an unrealistic situation where the symplectic action (generalized Hamiltonian)  $\mathbb{H}[\phi, \pi]$  was positive definite. In this case one could change ensemble with Laplace rather than Fourier transform,

$$\mathcal{Z}[\mu] = \int_0^\infty d\mathcal{A} e^{-\mu\mathcal{A}} \Omega[\mathcal{A}] \propto \int \mathcal{D}\phi e^{\mu S[\phi]}, \tag{2.17}$$

leading to a theory which is perfectly equivalent to standard statistical mechanics in the canonical ensemble: locally there is a probability density for the field configuration,  $\rho(\phi) \propto e^{\mu S[\phi]}$ . We notice that the factor  $\rho(\phi)$  is intuitively well defined as a local probability density because for typical configuration of the field, far from those corresponding to ultrarelativistic particles, the relativistic action is usually negative  $S[\phi] < 0$ .

We have just shown how the standard path-integral formulation can be recovered, on the basis of an ergodicity assumption, from the symplectic quantization dynamics approach and which is the role played by  $\hbar$  within this, let us say, *change of ensemble*. At the same time it is not only legitimate but also necessary to wonder if and how there is a *quantization constraint* involving  $\hbar$  which can be imposed directly on the microcanonical ensemble of symplectic quantization. The indication coming from the stochastic quantization framework is that  $\hbar$  must play a role analogous to that of temperature. Therefore, as suggested in [10], we believe that the most natural assumption for the role of  $\hbar$  in the symplectic quantization formalism is to be analogous to the microcanonical temperature:

$$\frac{1}{\hbar} = \frac{d\Sigma_{\text{sym}}[\mathcal{A}]}{d\mathcal{A}} \tag{2.18}$$

Although satisfactory conceptually and formally consistent, a definition of  $\hbar$  as in eq. (2.18) is very difficult to implement in practice. For this reason we will resort in this paper to another more trivial but effective way to impose the quantization constraint in the symplectic quantization dynamics, the one analogous to the way which is customarily used to assign the temperature in the context a microcanonical molecular dynamics. Usually, if we have  $M$  degrees of freedom and we wish the system to be on the fixed energy hypersurface such that  $T^{-1} = \partial S(E)/\partial E$ , we simply assign initial conditions such that the total energy is  $E = Mk_B T$ : here we follow the same strategy. In particular, counting as “degrees of freedom” the number

of components in reciprocal space of the Fourier transform of the fields, i.e.,  $\pi(k)$  and  $\phi(k)$ , in order to set at  $\hbar$  the typical scale of generalized energy for each mode we can choose initial conditions in the ensemble characterized at stationarity by the following condition:

$$\langle \pi(x)\pi(y) \rangle = \frac{\hbar}{2} \delta^{(d)}(x - y), \tag{2.19}$$

where the angular brackets indicates intrinsic time average along the symplectic quantization dynamics:

$$\langle \pi(x)\pi(y) \rangle = \lim_{\Delta\tau \rightarrow \infty} \frac{1}{\Delta\tau} \int_{\tau_0}^{\tau_0 + \Delta\tau} d\tau \pi(x, \tau)\pi(y, \tau) \tag{2.20}$$

Eq. (2.19) for the expectation value of momenta can be rewritten for a discretized  $d$ -dimensional space-time lattice with lattice spacing  $a$ , as is the case for the numerical simulations discussed below, as follows:

$$\langle \pi(x_i)\pi(x_j) \rangle = \frac{\hbar}{2} \frac{\delta_{ij}}{a^d}, \tag{2.21}$$

where  $\delta_{ij}$  is the Kronecker delta. By Fourier transforming eq. (2.19) it is then straightforward to get

$$\langle \pi^*(k)\pi(k) \rangle = \frac{\hbar}{2}, \tag{2.22}$$

so that in Fourier space the “kinetic” contribution coming from each degree of freedom to the total action amounts to  $\hbar/2$ . The relation in eq. (2.22) can be also applied to the discretized momenta usually considered for a numerical simulation on the lattice:

$$\langle \pi^*(k_i)\pi(k_i) \rangle = \frac{\hbar}{2} \quad \forall i. \tag{2.23}$$

This will be the sort of quantization constraint which will be applied to all our numerical simulations, choosing initial conditions which are compatible with that. Since we have chosen to work with natural units we will replace  $\hbar = 1$  everywhere in the above formulas. Suitable initial conditions to expect something such as eq. (2.23) at stationarity is for instance the following:

$$|\pi(k_i; \tau = 0)|^2 = \hbar \quad \forall i, \tag{2.24}$$

which will be used for all simulations presented in this work.

Before moving to the presentation of simulation details and the exposition of numerical results let us make a final remark on the assumption that dynamical averages along the intrinsic time Hamiltonian dynamics, like the one of eq. (2.20), yields indeed a reliable sampling of the microcanonical probability density in eq. (2.11). The general attitude in order to claim the equivalence between dynamical and ensemble sampling of observables expectation values is to assume a certain degree of ergodicity/chaoticity of the microscopic dynamics, which in general has to be considered case by case. To this respect, beside recalling that the case of a interacting scalar field theory with non-linear quartic potential studied here is generally

considered the one of a system where the dynamics has good mixing properties, let us remark that in the present approach we do not really regard ergodicity as an issue. More precisely, we follow the perspective outlined in a recent review from one of us on the foundations of statistical mechanics [20] where it is explained how the use of statistical ensembles to draw predictions on the expectation values of generic observables requires ergodicity only in a weak sense, i.e., the mixing properties of the microscopic dynamics are not really the point. As for “generic” observables one has to think about all the observables which depend on large number of the system’s degrees of freedom. Just to make an example/analogy, we expect the behaviour of the two-point correlation function in real space for a weakly interacting field theory to “thermalize well” under the Hamiltonian dynamics in the same way that a good thermalization is achieved for individual particles in a classical harmonic chain, even if the Fourier modes of the harmonic chain does not thermalize at all. For further speculations on this point see, e.g., [21].

### 3 Simulation details

The deterministic dynamics of symplectic quantization can be defined for both Euclidean and Minkowski metric: to validate the new approach we have tested both scenarios. In order to do that we have discretized the Hamiltonian equations of motion, writing them in a general form where the nature of the metric is specified by the variable  $s = \{0, 1\}$ . All equations are written in natural units  $\hbar = c = 1$ .

In the present work we have considered a 1+1 lattice with either Euclidean or Minkowski metric, which we denote as  $\Gamma$ :

$$\Gamma : \left\{ x : x_\mu = an_\mu, \quad n_\mu = -\frac{M_\mu}{2}, \dots, \frac{M_\mu}{2} \quad \mu = 0, 1 \right\}. \tag{3.1}$$

Due to the finite size of the simulation grid momenta are also discretized:

$$p_\mu = \frac{2\pi}{a} \frac{k_\mu}{M_\mu} \quad |p_\mu| < \frac{\pi}{a}, \tag{3.2}$$

where  $\mu = 0, 1$ ,  $k_\mu \in [-M_\mu/2, M_\mu/2]$ ,  $L = M_\mu a$  is the lattice side and  $a$  is the lattice spacing. In the discretized theory the total number of degrees of freedom is identified with the number of points in the lattice:

$$M = \prod_{\mu=0}^{d-1} M_\mu \tag{3.3}$$

The discretized Hamiltonian of symplectic quantization reads then as

$$\mathbb{H}[\phi, \pi] = \frac{1}{2} \sum_{x \in \Gamma} \left[ \pi(x)^2 - (-1)^s \frac{1}{a^2} \phi(x) \Delta^{(0)} \phi(x) - \frac{1}{a^2} \phi(x) \Delta^{(1)} \phi(x) + m^2 \phi^2(x) + \frac{\lambda}{4} \phi^4(x) \right], \tag{3.4}$$

where the symbol  $\Delta^{(\mu)} \phi(x)$  denotes the discrete one-dimensional Laplacian along the  $\mu$ -th coordinate axis:

$$\Delta^{(\mu)} \phi(x) = \phi(x + a^\mu) + \phi(x - a^\mu) - 2\phi(x). \tag{3.5}$$

We have used a general expression in eq. (3.4), which, depending on the value chosen for the integer index  $s = \{0, 1\}$ , describes a theory with Euclidean,  $s = 0$ , or Minkowskian,  $s = 1$ , metric. From the expression of the Hamiltonian in eq. (3.4) we have that the force acting on the field on a two-dimensional lattice is:

$$F[\phi(x)] = -\frac{\delta\mathbb{H}[\phi, \pi]}{\delta\phi(x)} = \frac{(-1)^s}{a^2}\Delta^{(0)}\phi(x) + \frac{1}{a^2}\Delta^{(1)}\phi(x) - m^2\phi(x) - \lambda\phi^3(x), \quad (3.6)$$

so that the equation of motion for the field itself is:

$$\frac{d\phi(x, \tau)}{d\tau^2} = F[\phi(x, \tau)]. \quad (3.7)$$

Equations (3.6), (3.7) define the Hamiltonian dynamics which we have studied numerically using the leap-frog algorithm, a symplectic algorithm described in appendix A, which guarantees the conservation of (generalized) energy at the order  $\mathcal{O}(\tau^2)$ .

An important point for the study of this paper is the definition of boundary conditions. We used two different kinds of boundary conditions for the simulations. For all results on Euclidean lattice and for the study of dynamics stability with or without non-linear interaction on Minkowski lattice, discussed respectively in section 4 and in section 5, we have used standard periodic boundary conditions on the lattice. Differently, in section 6, aimed at studying the free propagation of physical signals across the lattice we used *fringe* boundary conditions [22], introduced with the purpose of mimicking the existence of an infinite lattice outside the simulation grid. Fringe boundary conditions are realized considering a larger lattice, which we denote as  $\Gamma_f$ , where the subscript “ $f$ ” is for fringe, which is composed by the original lattice  $\Gamma$  plus several additional layer of points which we denote as  $\Gamma_{\text{ext}}$ , in such a way that the *fringe* lattice is  $\Gamma_f = \Gamma + \Gamma_{\text{ext}}$ . For the fringe lattice one also considers periodic boundary conditions, but the generalized Hamiltonian for points belonging to  $\Gamma$  and to  $\Gamma_{\text{ext}}$  is different. Namely, the fringe lattice is characterized by the Hamiltonian:

$$\mathbb{H}_f[\pi, \phi] = \mathbb{H}_{\text{ext}}[\pi, \phi] + \mathbb{H}[\pi, \phi], \quad (3.8)$$

where  $\mathbb{H}[\pi, \phi]$  is the original discretized Hamiltonian of the system, see eq. (3.4), while  $\mathbb{H}_{\text{ext}}[\pi, \phi]$  reads as

$$\mathbb{H}_{\text{ext}}[\pi, \phi] = \frac{1}{2} \sum_{x \in \Gamma} \left[ \pi(x)^2 + m^2\phi^2(x) + \frac{\lambda}{4}\phi^4(x) + \epsilon \left( \frac{1}{a^2}\phi(x)\Delta^{(0)}\phi(x) - \frac{1}{a^2}\phi(x)\Delta^{(1)}\phi(x) \right) \right], \quad (3.9)$$

where the coefficient  $\epsilon$  is very small,  $\epsilon \ll 1$ . This choice of boundary conditions allows us to have a free propagation of signals across the boundary layer of  $\Gamma$ , our true simulation lattice, but the signal is then strongly damped when going across  $\Gamma_{\text{ext}}$ , the “external” boundary layer before making sort of interference at the periodic boundaries at the border of  $\Gamma_{\text{ext}}$ . This choice of boundary conditions allows us not only to deal with an overall system which is still Hamiltonian (apart from small corrections scaling as  $1/L$ ), but also to have quite satisfactory results for the study of the Feynman propagator, as shown in section 6.

We have done all simulations for a lattice with side  $M_0 = M_1 = 128$ , lattice spacing  $a = 1.0$  and using an integration time-step  $\delta\tau = 0.001$ . According to the discussion in the

previous section, we have fixed the energy scale by choosing initial conditions such that each degree of freedom in Fourier space carries a “*quantum*” of energy  $\hbar = 1$ . We have therefore assigned an initial total energy equal to  $M_0 \cdot M_1 = 16384$  for all simulations. Since we have studied both linear and non-linear interactions, in order to set precisely the initial value of the energy, we started all simulations with:

$$\begin{aligned} |\phi(k; 0)|^2 &= 0 & \forall k \\ |\pi(k; 0)|^2 &= 1 & \forall k. \end{aligned} \tag{3.10}$$

#### 4 Euclidean propagator

Our first test of the symplectic quantization approach consists in the study of its deterministic dynamics in the case of a two-dimensional Euclidean lattice, showing that it provides the correct two-point correlation function, also consistently with the results of stochastic quantization. For the simulation on the Euclidean lattice we have used simple periodic boundary conditions, since all correlation functions decay exponentially with the distance and there should be no signals propagating underdamped across the system.

Let us then recall here how the expectation values over quantum fluctuations of fields are computed within the symplectic quantization approach dynamics. If we indicate with  $\phi_{\mathbb{H}}(x, \tau)$  the solutions of the Hamiltonian equations of motion written in eq. (3.7), we have that the expectation value of a generic  $n$ -point correlation function can be computed as follows:

$$\langle \phi(x_1), \dots, \phi(x_n) \rangle = \lim_{\Delta\tau \rightarrow \infty} \frac{1}{\Delta\tau} \int_{\tau_0}^{\tau_0 + \Delta\tau} d\tau \phi_{\mathbb{H}}(x_1, \tau) \dots \phi_{\mathbb{H}}(x_n, \tau), \tag{4.1}$$

where  $\tau_0$  is a large enough time, for which the system has reached equilibrium and forgot any detail on the initial conditions of the dynamics. For a free field theory the propagator on a two-dimensional lattice take the simple form:

$$\tilde{G}(p; a) = \left[ \frac{4}{a^2} \sin^2 \left( \frac{ak_0}{2} \right) + \frac{4}{a^2} \sin^2 \left( \frac{ak_1}{2} \right) + m^2 \right]^{-1} \tag{4.2}$$

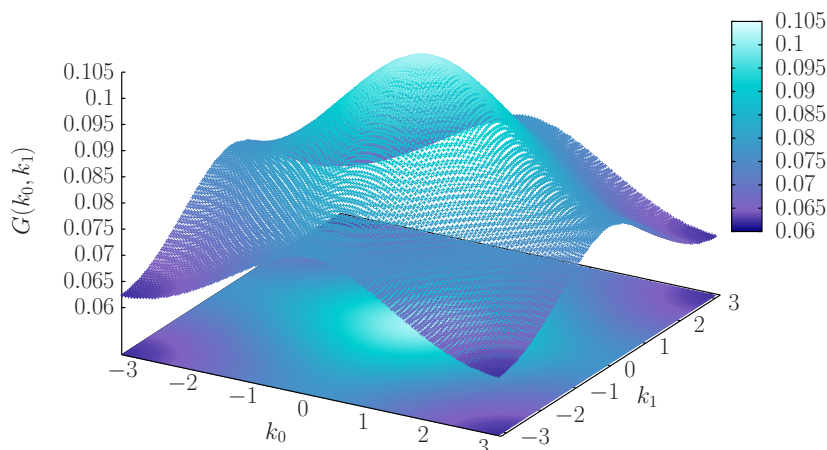
If we define the Fourier component of the field as

$$\hat{\phi}(k, \tau) = \frac{a^2}{2\pi} \sum_{x \in \Gamma} e^{-i(k_0 x_0 + k_1 x_1)} \phi(x, \tau), \tag{4.3}$$

we can then write the Fourier spectrum of the two-point correlation function, according to the discretized-time version of eq. (4.1), as the following dynamical average:

$$G(k) = \langle \hat{\phi}^*(k) \hat{\phi}(k) \rangle = \frac{1}{\Delta\tau} \sum_{i=0}^M \phi^*(k, \tau_0 + \delta\tau_i) \phi(k, \tau_0 + \delta\tau_i), \tag{4.4}$$

where  $\delta\tau_i = i \cdot \delta\tau$ . For the Euclidean lattice we have studied the lattice dynamics with the parameters and initial conditions given at the end of section 3, considering, in addition, value of mass  $m = 3.0$  and nonlinearity coefficient  $\lambda = 0.001$ : the numerical value of the propagator in Fourier space perfectly reproduces the expected dumbbell shape, as shown in figure 1. We have also checked that the two-point correlation function exhibits in real space the typical exponential decay  $C(x) \sim e^{-mx}$ .



**Figure 1.** Real part of the two-point correlation function Fourier spectrum (Euclidean propagator) for a  $\lambda\phi^4$  theory in  $d = 2$  euclidean dimensions. Numerical value from the interacting theory with nonlinearity  $\lambda = 0.001$ , lattice spacing  $a = 1.0$ , lattice side  $M_\mu = 128$ , mass  $m = 3.0$ .

### 5 Minkowski lattice: linear and non-linear theory

The numerical and analytical study of the free field theory in 1 + 1 Minkowski space-time presents a new problem with respect to the Euclidean space: the dynamics of quantum fluctuations for the *linear* non-interacting theory in the symplectic quantization approach turns out to be *unstable*. This can be recognized immediately from the free field equations in the continuum.

In the case of a purely quadratic potential  $V[\phi] = \frac{1}{2}m^2\phi^2$ , the explicit solution of eq. (2.7) can be obtained by exploiting the translational symmetry of space-time, which allow to Fourier transform the equations:

$$\ddot{\phi}(k, \tau) + \omega_k^2 \phi(k, \tau) = 0, \tag{5.1}$$

with

$$\omega_k^2 = |\mathbf{k}|^2 + m^2 - k_0^2. \tag{5.2}$$

The general solution of eq. (5.1) can be then written in terms of the initial conditions as

$$\begin{aligned} \phi(k, \tau) &= \phi(k, 0) \cos(\omega_k \tau) + \frac{\dot{\phi}(k, 0)}{\omega_k} \sin(\omega_k \tau) \quad \forall \omega_k^2 > 0 \\ \phi(k, \tau) &= \phi(k, 0) \cosh(z_k \tau) + \frac{\dot{\phi}(k, 0)}{z_k} \sinh(z_k \tau) \quad \forall \omega_k^2 < 0, \end{aligned} \tag{5.3}$$

where

$$iz_k = \sqrt{\omega_k^2}. \tag{5.4}$$

Without any loss of generality and consistently with what we have done numerically on the lattice, one can consider the following initial conditions:

$$\begin{aligned} \phi(k, 0) &= 0 \\ \dot{\phi}(k, 0) &= 1, \end{aligned} \tag{5.5}$$

so that the general time-dependent solution reads as

$$\begin{aligned} \omega_k^2 > 0 &\implies \phi(k, \tau) = \frac{\sin(\omega_k \tau)}{\omega_k} \\ \omega_k^2 < 0 &\implies \phi(k, \tau) = \frac{\sinh(z_k \tau)}{z_k}. \end{aligned} \tag{5.6}$$

Rewriting the generalized Hamiltonian in Fourier space we have

$$\mathbb{H}[\phi, \pi] = \frac{1}{2} \int d^d k \left( |\pi(k)|^2 + \omega_k^2 |\phi(k)|^2 \right), \tag{5.7}$$

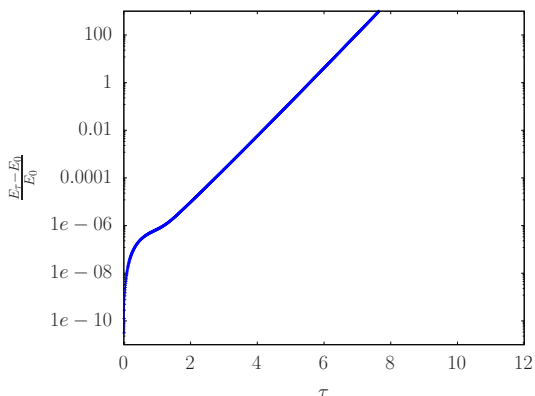
so that, by plugging into it the time-dependent solutions we have:

$$\begin{aligned} \omega_k^2 > 0 &\implies \mathbb{H}[\phi(\tau), \pi(\tau)] = \frac{1}{2} \int d^d k \left[ \cos^2(\omega_k \tau) + \sin^2(\omega_k \tau) \right] \\ \omega_k^2 < 0 &\implies \mathbb{H}[\phi(\tau), \pi(\tau)] = \frac{1}{2} \int d^d k \left[ \cosh^2(z_k \tau) - \sinh^2(z_k \tau) \right]. \end{aligned} \tag{5.8}$$

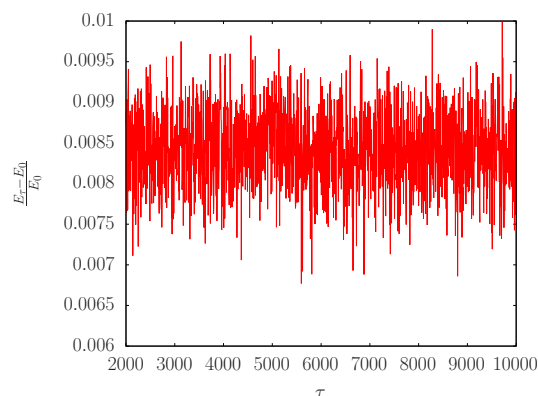
Considering the expressions in eq. (5.8) we realize that, despite the conservation of the symplectic quantization Hamiltonian, it exists an infinite set of momenta, namely all  $k$ 's with  $\omega_k^2 < 0$ , such that the “potential” and “kinetic” part of the generalized energy in eq. (5.7), namely  $\mathbb{K}[\phi, \pi]$  and  $\mathbb{V}[\phi, \pi]$ , both diverge exponentially with  $\tau$ . This fact presents two problems, one conceptual and the second numerical. The conceptual problem is represented by the fact that, irrespectively to the behaviour of moments  $\pi(k, \tau)$ , which *might* also be regarded as unphysical auxiliary variables, we have that also the contribution to generalized potential energy  $\mathbb{V}[\phi, \pi]$  (corresponding in practice to the relativistic action) of an infinite amount of field modes  $\phi(k, \tau)$  diverges exponentially with  $\tau$ . This divergence of the field amplitude is clearly unphysical: interpreting in fact as “particles” the modes of the free field, this would correspond to infinite growth of the action of an isolated particle, which is clearly not observed in the real world. At the same time, attempting to study numerically the free-field dynamics on Minkowski lattice, the leap-frog algorithm, which proceeds alternating the update of kinetic and potential energy, cannot handle the situation where the overall energy is conserved but the two contributions diverge. Eventually, due to the accumulation of numerical errors, total energy starts to diverge exponentially as well with elapsing time, see figure 2(a) and the discussion below.

Since in the present section we are just interested in the stability of the theory, irrespectively of a realistic study of signals propagation across the lattice, we have considered for simplicity periodic boundary conditions. Let us remark that these conditions would not be appropriate for a more realistic study of two-point correlation functions with Minkowski metric, since in this case we would like to probe the causal structure of space-time: a signal escaping from the lattice at  $+ct$  cannot appear back at  $-ct$ . For a similar reason even fixed boundary conditions would not be appropriate.

Using periodic boundary conditions we have checked numerically that the symplectic quantization dynamics of a free scalar field suffers from the pathology which can be conjectured already from the exact solution: after a certain time the whole energy starts to grow exponentially with  $\tau$ . In figure 2(a) we present the results of simulations of the free-field with Minkowski metric, all the parameters declared at the end of section 3 and  $m = 1.0$ ,



(a) Behaviour of the normalized energy  $(E(\tau) - E_0)/E_0$  vs  $\tau$  for a scalar free theory ( $\lambda = 0$ ) with  $m = 1.0$ ,  $a = 1.0$ ,  $M_\mu = 128$ , with initial conditions  $\pi(k; 0) = 1$  and  $\phi(k; 0) = 0$  for all  $k$ 's. Notice the exponential growth with  $\tau$  which sets in after a short transient.



(b) Behaviour of the normalized energy  $(E(\tau) - E_0)/E_0$  vs  $\tau$  for a scalar theory with a small self-interaction term,  $\lambda = 0.001$ , and with  $m = 1.0$ ,  $a = 1.0$ ,  $M_\mu = 128$ , with initial conditions  $\pi(k; 0) = 1$  and  $\phi(k; 0) = 0$  for all  $k$ 's. Oscillations are of order  $\delta t$ .

**Figure 2.** Comparison of normalized energy behaviour for scalar free theory and scalar theory with self-interaction.

showing a clear evidence of the exponential divergence with  $\tau$ . What seemed a good solution to both the conceptual and numerical shortcomings of the free theory has been to consider that the physically relevant theory is only the interacting one: physical fields are always in interactions and the “free-field theory” is just an approximation, with some internal inconsistencies which are revealed by the symplectic quantization approach. Let us for instance consider a potential of the kind

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{5.9}$$

for which the equations of motions in the continuum read as

$$\ddot{\phi}(x, \tau) = -\partial_0^2\phi(x, \tau) + \sum_{i=1}^d \partial_i^2\phi(x, \tau) - m\phi(x, \tau) - \lambda\phi^3(x, \tau). \tag{5.10}$$

Clearly, due to the non-linear term in eq. (5.10), it is not possible anymore to diagonalize the equations in Fourier space, so that both the sin/cos and the sinh/cosh solutions cannot be taken into account as a reference. Yet to be proven mathematically, the stability of eq. (5.10) is a quite delicate problem, since in general for many Fourier components the equations are linearly unstable. The intuition suggests that for each point of space-time  $x$  the cubic force acts as a restoring term which prevents the amplitude  $\phi(x, \tau)$  to grow without bounds. This intuition has been confirmed, up to the accuracy of our analysis, from our numerical results. By using periodic boundary conditions, the parameters and initial conditions declared at the end of section 3, setting the non-linearity coefficient  $\lambda = 0.001$  we find that the energy is no more divergent. The system relaxes to a stationary state with oscillations of order  $|E(t) - E_0|/E_0 = \mathcal{O}(\delta\tau)$ , as is shown in figure 4. Let us remark, anticipating some

of the upcoming results, that the instability of the deterministic dynamics of eq. (2.6) in the case of a free theory is perfectly consistent with the shape of the canonical probability density  $P[\phi] \sim \exp[S[\phi]/\hbar]$  for which we proved equivalence with the microcanonical weight  $\rho_{\text{micro}}[\phi(x)] \sim \delta(\mathcal{A} - \mathbb{H}[\phi, \pi])$ , as will be shown in section 7:  $P[\phi]$  turns out to be ill-defined for the action of a free scalar field.

Having assessed the stability of the symplectic quantization dynamics in the presence of non-linear interactions and periodic boundary conditions, it is now time to consider, keeping the non-linearity switched on, the more physical case of fringe boundary conditions [22]. This procedure will allow us to sample numerically the Feynman propagator for small non-linearity, as will be discussed in the next section.

## 6 Feynman propagator: numerical results

In the previous section we have shown how the presence of non-linear interactions solves the instability problem of the linear theory, still keeping periodic boundary conditions. But periodic boundary conditions are clearly unphysical, because one of the directions of our lattice corresponds to  $ct$ , so that periodicity of the boundaries is clearly meaningless. We need to devise a strategy to mimic the free propagation of any kind of signal across the boundaries as if outside there was an infinitely large lattice. This strategy is provided by the use of fringe boundary conditions, introduced in section 3.

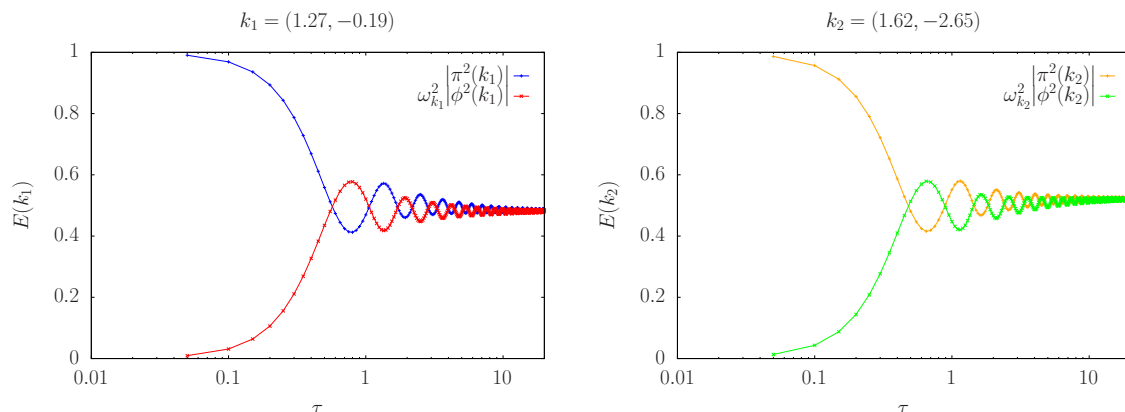
In this part of the paper we will therefore provide the numerical evidence that for perturbative values of the non-linearity coefficient  $\lambda$  we recover qualitatively the correct shape of the free Feynman propagator.

The strategy is very simple: having set the coefficient of the non-linear interaction  $\lambda$  to a small but finite value,  $\lambda = 0.001$ , we have run the symplectic dynamics with fringe boundary conditions until stationarity is reached at a certain time, which we call  $\tau_{\text{eq}}$ . According to the premises of section 2, where we assumed that at long enough times the symplectic quantization dynamics allows us to sample an equilibrium ensemble, we have checked that equipartition between positional and kinetic degrees of freedom is in fact reached. In figure 3 is shown how, for two given choices of  $k = \{k_0, k_1\}$  (corresponding respectively to small and large scales), we have that  $\overline{E}_{\text{harm}}(k, \tau)$  and  $\overline{E}_{\text{kin}}(k, \tau)$  reach asymptotically a value close to  $1/2$ , starting respectively from  $E_{\text{harm}}(k, 0) = 0$  and  $E_{\text{kin}}(k, 0) = 1$ , where the two energies are defined respectively as

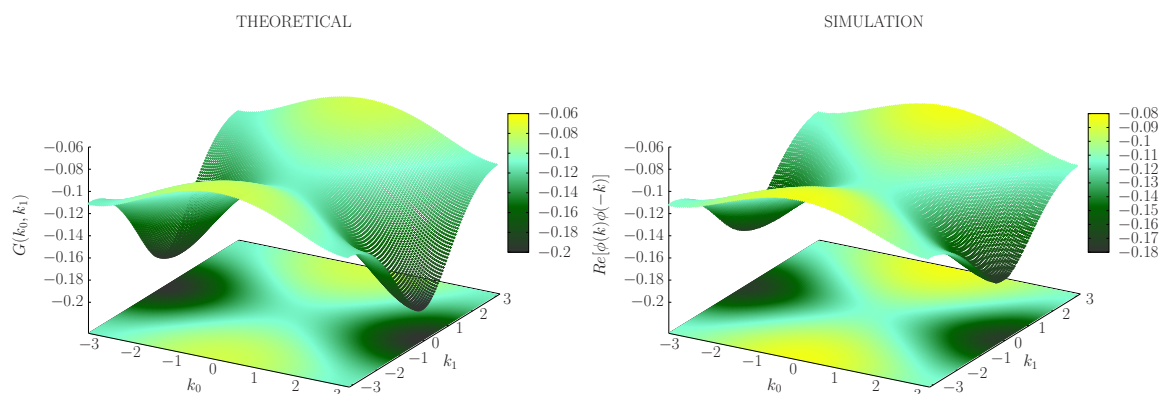
$$\begin{aligned} \overline{E}_{\text{harm}}(k, \tau) &= \frac{1}{\tau} \int_0^\tau ds \frac{1}{2} \omega_k^2 |\phi(k, s)|^2 \\ \overline{E}_{\text{kin}}(k, \tau) &= \frac{1}{\tau} \int_0^\tau ds \frac{1}{2} |\pi(k, s)|^2. \end{aligned} \tag{6.1}$$

We have found that this standard equipartition condition is fulfilled well when all  $k$ 's in the lattice are such that  $\omega_k^2 > 0$ , while the stationary state reached when a finite fraction of the modes is such that  $\omega_k^2 < 0$  has less trivial properties, which will be analysed in further details elsewhere.

Having thus assessed that the system reaches some equilibrium/stationary state within some time  $\tau_{\text{eq}}$ , we have computed for all times  $\tau > \tau_{\text{eq}}$  the Fourier spectrum of the two-point



**Figure 3.** Behaviour of the time averaged harmonic  $\overline{E}_{\text{harm}}(k, \tau)$  and kinetic  $\overline{E}_{\text{kin}}(k, \tau)$  energies for two different choices of  $k$ , corresponding respectively to small (left panel) and large (right panel) scales. Non-linearity coefficient is  $\lambda = 0.001$  and lattice parameters are with  $m = 3.0$ ,  $a = 1.0$ ,  $M_\mu = 128$ , with initial conditions  $\pi(k; 0) = 1$  and  $\phi(k; 0) = 0$  for all  $k$ 's. For this choice of parameters there are no unstable modes, i.e. for all  $k$ 's we have  $\omega_k^2 > 0$ .



**Figure 4.** Real part of the two-point correlation function Fourier spectrum  $G(k_0, k_1) = \langle \phi^*(k_0, k_1) \phi(k_0, k_1) \rangle$  (Feynman propagator) for a  $\lambda \phi^4$  theory in  $1 + 1$  space-time dimensions. *Left:* theoretical value of the free propagator with lattice spacing  $a = 1.0$ , lattice side  $M_\mu = 128$ , mass  $m = 3.0$ ; *Right:* numerical value from the interacting theory with the same parameters and nonlinearity  $\lambda = 0.001$ . Initial conditions are set to  $\phi(k; 0) = 0$  and  $\pi(k; 0) = 1$  for all  $k$ 's. For this choice of parameters there are no unstable modes, i.e. for all  $k$ 's we have  $\omega_k^2 > 0$ .

correlation function  $G(k) = \langle \phi^*(k) \phi(k) \rangle$  by averaging (quantum) fluctuations over intrinsic time. That is, we have defined an interval  $\Delta\tau$  large enough and we have computed

$$\langle \phi^*(k) \phi(k) \rangle = \frac{1}{\Delta\tau} \sum_{i=0}^M \phi^*(k, \tau_{\text{eq}} + \tau_i) \phi(k, \tau_{\text{eq}} + \tau_i), \quad (6.2)$$

where  $\tau_i = i \cdot \delta\tau$  and  $\Delta\tau = M\delta\tau$ .

In figure 4 we show (bottom panel) the result for the Fourier spectrum of the two-point correlation function obtained by setting all the parameters of the simulation and the initial

conditions as declared at the end of section 3, apart from the value of the mass that is set here at  $m = 3.0$  in order to better appreciate the shape of the propagator, and taking the value  $\lambda = 0.001$  for the non-linearity parameter. In order to compare our numerical data at small non-linearity with the theory, we have also reported in the top panel of figure 4 the theoretical shape of the free Feynman propagator  $G_{\text{th}}(k_0, k_1)$  on a discretized space-time grid in  $1 + 1$  dimensions, using for the lattice the same parameters of the simulation, i.e.,  $a = 1.0$ ,  $m = 1.0$ , and  $M_\mu = 128$ , where  $G_{\text{th}}(k_0, k_1)$  reads as

$$G_{\text{th}}(k_0, k_1) = \left[ \frac{4}{a^2} \sin^2 \left( \frac{ak_0}{2} \right) - \frac{4}{a^2} \sin^2 \left( \frac{ak_1}{2} \right) - m^2 \right]^{-1}. \quad (6.3)$$

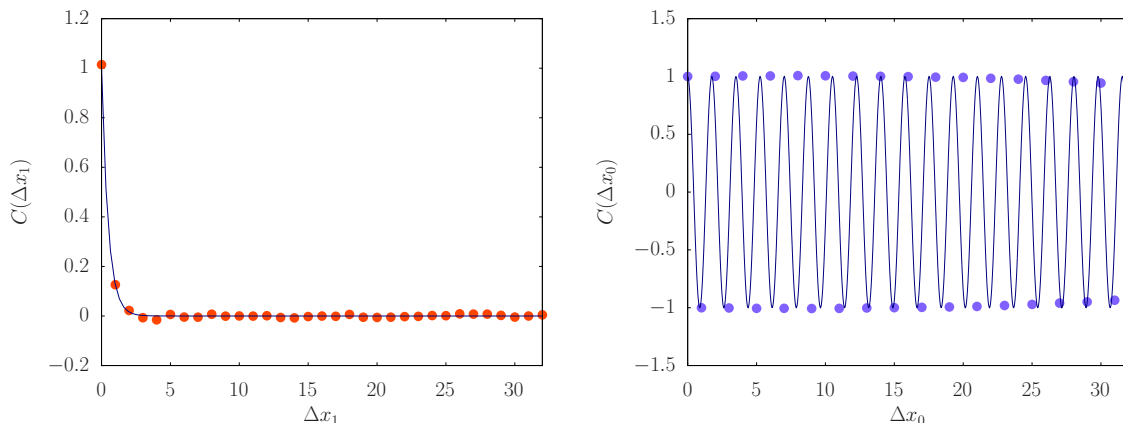
Let us stress the beautiful qualitative agreement between the theoretical prediction of the free propagator and the numerical results: at variance with the Euclidean propagator, which is a function decreasing monotonically in all directions moving away from the origin (see figure 1 above), we find that the Feynman propagator sampled numerically here has the characteristic shape of a saddle, denoting a different behaviour between *time-like* directions and *space-like* directions. This is the first and incontrovertible strong evidence that the symplectic quantization approach opens up new possibilities so far out of reach within the Euclidean formulation of lattice field theory. Even more clear is the signature of the causal structure of space-time probed by means of the new approach if we look at the two-point correlation function in real space. According to the theoretical predictions for the free theory in the continuum one would expect undamped oscillations along the purely *time-like* directions and an exponential decay along the purely *space-like* directions for the Feynman propagator  $\Delta_F(x - y)$ :

$$\Delta_F(x - y) = \frac{1}{(2\pi)^2} \int d^2k \frac{e^{ik(x-y)}}{k^2 - m^2}, \quad (6.4)$$

with

$$\begin{aligned} \Delta_F(x - y) &\sim e^{im|x-y|} && \text{for } x - y \parallel x_0 \\ \Delta_F(x - y) &\sim e^{-m|x-y|} && \text{for } x - y \parallel x_1. \end{aligned} \quad (6.5)$$

Clearly, when the same correlation function is sampled on a finite and discrete grid there will be finite-size effects at play so that, for instance, also the oscillations along the time-like direction will be slightly modulated by a tiny exponential decay: this is precisely what we find in numerical simulations. In figure 5 are shown, respectively in top and bottom panels, the exponential decay along the purely space-like direction and the oscillations along the purely time-like direction, obtained for the following choice of parameters:  $a = 1.0$ ,  $M_\mu = 128$ , mass  $m = 1.0$  and nonlinearity  $\lambda = 0.001$ . Let us notice that the value of the mass which can be obtained from either the fit of the exponential decay as  $C(\Delta x_1) \sim e^{-m\Delta x_1}$  or the oscillating part as  $C(\Delta x_0) \sim e^{im\Delta x_0}$  is  $m \sim 2.06 \pm 0.04$ , i.e., quite different from the value  $m = 1$  put in the Lagrangian. This effect, which we do not find for the deterministic dynamics in Euclidean space-time, is most probably a finite-size effect related to the propagation of signals across fringe boundary conditions. We made some attempts, discussed in appendix B, to investigate a possible interplay between the measured mass discrepancy with the way fringe boundary



**Figure 5.** Real space two-point correlation function for a  $\lambda\phi^4$  theory in 1 + 1 space-time dimensions with fringe boundary conditions, lattice spacing  $a = 1.0$ , lattice side  $M_\mu = 128$ , mass  $m = 1.0$  and nonlinearity  $\lambda = 0.001$ . *Left:* exponential decay along the direction parallel to the  $x_1$  axis; *Right:* oscillations along the direction parallel to the  $x_0 = ct$  axis.

conditions are imposed, but we didn't find any clear indication on the possible origin of the effect. A stronger effort is for sure necessary to put under control the finite-size effects related to fringe boundary conditions: we plan to devote another paper to this problem.

### 7 Canonical form of Minkowskian statistical mechanics

In section 6 we have shown that the Hamiltonian dynamics of a quantum field theory with an additional time parameter  $\tau$  and corresponding conjugated momenta allows to recover qualitatively well the shape of the free Feynman propagator for a small value of the interaction constant  $\lambda$ . It is therefore legitimate to wonder which is the precise relation between the correlation functions obtained in this generalized microcanonical ensemble and the one generated by the Feynman path integral and/or the corresponding Euclidean Field Theory. As a first step in this direction we propose an explicit calculation of the microcanonical partition function in the large- $M$  limit, where  $M$  is the number of degrees of freedom. The calculation shows that also for this peculiar system, where the microcanonical ensemble is built on the conservation of an action rather than an energy functional, the sampling of fluctuations in this ensemble is formally equivalent to the sampling of a *canonical* one at a *temperature*  $\hbar$ , namely fields fluctuations are sampled with probability  $\exp(S/\hbar)$ .

The explicit computation of the microcanonical partition function in the large- $M$  limit proceeds then as follows. As it is customary for the purpose of computing correlation functions, we assume the presence of an external source  $J(x)$  linearly coupled to the field:

$$\Omega[\mathcal{A}, J] = \int \mathcal{D}\phi \mathcal{D}\pi \delta \left( \mathcal{A} - \mathbb{H}[\phi, \pi] + \int d^d x J(x)\phi(x) \right). \tag{7.1}$$

Since we are using the lattice a regularizer for the theory, the field can be conveniently expanded in an orthonormal basis as follows [23]:

$$\phi(x) = \sum_{n=1}^M \phi_n(x) c_n, \tag{7.2}$$

where

$$\int d^d x \phi_n(x) \phi_m(x) = \delta_{mn}. \tag{7.3}$$

We can then define a *finite* measure over the field configuration, reading as:

$$\int \mathcal{D}_M \phi \equiv \prod_{n=1}^M \int_{-\infty}^{\infty} dc_n. \tag{7.4}$$

Contrary to the usual convention, there is no  $\hbar$  in this measure. In a  $d$ -dimensional box of volume  $L^d$  with lattice spacing  $a$ , the number of basis functions is [23]:

$$M = \frac{L^d}{a^d} = \frac{1}{\pi^d} L^d \Lambda^d, \tag{7.5}$$

where  $\Lambda = \pi/a$  is the momentum cutoff. Let us specify that  $M$  is not the number of classical dynamical degrees of freedom, which grows on the contrary simply as  $(L\Lambda)^3$ . From eq. (7.5) we see that the *field* limit  $M \rightarrow \infty$  can be obtained either as the continuum limit,  $\Lambda \rightarrow \infty$ , or as the thermodynamic limit,  $L \rightarrow \infty$ . Nevertheless, we will make a crucial step in the calculation of the microcanonical partition function in the large- $M$  which is well justified *only* in the *continuum* limit and not in the thermodynamic one, so that from here on we will refer to the limit  $M \rightarrow \infty$  as the continuum limit.

By lightening the notation according to the following conventions

$$\begin{aligned} \pi^2 &\equiv \int d^d x \pi^2(x) \\ J \cdot \phi &\equiv \int d^d x J(x) \phi(x). \end{aligned} \tag{7.6}$$

we can then rewrite the partition function on the lattice as:

$$\Omega[\mathcal{A}, J] = \int \mathcal{D}\phi_M \mathcal{D}\pi_M \delta \left( \mathcal{A} - \frac{\pi^2}{2} + S[\phi] + J\phi \right). \tag{7.7}$$

The functional integration over  $\pi(x)$  can be then done by taking advantage of the following formula, valid for  $R^2 > 0$ :

$$I_M(R) = \int_{-\infty}^{\infty} dx_1 \dots dx_M \delta \left( \frac{1}{2} \sum_{i=1}^M x_i^2 - R^2 \right) = \frac{(2\pi)^{\frac{M}{2}}}{\Gamma\left(\frac{M}{2}\right)} R^{M-2}, \tag{7.8}$$

from which, putting  $R = (\mathcal{A} + S[\phi] + J\phi)^{\frac{1}{2}}$ , we get:

$$\Omega[\mathcal{A}, J] = \frac{(2\pi)^{\frac{M}{2}}}{\Gamma\left(\frac{M}{2}\right)} \int \mathcal{D}\phi_M (\mathcal{A} + S[\phi] + J\phi)^{\frac{M}{2}-1}. \tag{7.9}$$

The positivity of  $R^2 = \mathcal{A} + S[\phi] + J\phi$ , which is crucial for the whole calculation, is ensured by construction of the microcanonical ensemble, since the kinetic energy term related to conjugate momenta is positive definite. In order to consider a large- $M$  limit in the computation is

convenient at this stage to rewrite the partition function in eq. (7.9) in the following form, which puts in evidence the dependence on  $M$ :

$$\Omega[\mathcal{A}, J] = \kappa_M \int \mathcal{D}\phi_M \exp \left\{ \left( \frac{M}{2} - 1 \right) \ln (\mathcal{A} + S[\phi] + J\phi) \right\}, \quad (7.10)$$

where  $\kappa_M = (2\pi)^{\frac{M}{2}} / \Gamma(M/2)$ . In order to now fulfill the same quantization constraint used for the numerical simulation discussed in the previous section we assign  $\hbar$  to every degree of freedom, equally sharing this amount among “positional” and “kinetic” components.

Since momenta have been integrated out, to make the integral finite, in expression eq. (7.10) we need to fix  $\mathcal{A}$  to half of the total value, since we need to account only for “momenta” degrees of freedom, namely we write

$$\mathcal{A}_z = \frac{\hbar M}{2z}, \quad (7.11)$$

while we consider the counterterms for the “positional” degrees of freedom to be already in the action. We have introduced at this point the dimensionless parameter  $z$  in order to be able to tune the value of the average *quantum of action* per degree of freedom in the final expression that we will derive for  $\Omega[\mathcal{A}, J]$  and also with the purpose to highlight how the present theory connects to ordinary Feynman path integral by analytic continuation in  $z$ . We now proceed to expand the partition function  $\Omega[\mathcal{A}_z, J]$  in powers of  $J$  so that we can write explicitly the generating functional in terms of correlators. By doing this, ignoring the subleading  $O(1)$  in  $M$  term in the exponent, we get:

$$\begin{aligned} \Omega[\mathcal{A}_z, J] &= \kappa_M \left( \frac{\hbar M}{2z} \right)^{\frac{M}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{\hbar} \right)^n \left( \frac{2}{M} \right)^n \frac{\Gamma\left(\frac{M}{2} + 1\right)}{\Gamma\left(\frac{M}{2} + 1 - n\right)} \\ &\int d^d x_1 \dots d^d x_n J(x_1) \dots J(x_n) \int \mathcal{D}_M \phi \phi(x_1) \dots \phi(x_n) \left( 1 + \frac{2}{M} \frac{z}{\hbar} S[\phi] \right)^{\frac{M}{2} - n}. \end{aligned} \quad (7.12)$$

Now, we proceed to expand  $\left( 1 + \frac{2}{M} \frac{z}{\hbar} S[\phi] \right)^{\frac{M}{2} - n}$  in powers of  $1/M$ . A tedious but straightforward calculation yields:

$$\left( 1 + \frac{2}{M} \frac{z}{\hbar} S[\phi] \right)^{\frac{M}{2} - n} = e^{\frac{z}{\hbar} S[\phi]} \exp \left( \sum_{j=1}^{\infty} (-1)^j \frac{2(2j-2)!!}{(j+1)!} \left( \frac{z}{\hbar} \frac{S[\phi]}{M} \right)^j [jS[\phi] + (j+1)n] \right). \quad (7.13)$$

from which we have

$$\begin{aligned} \Omega[\mathcal{A}_z, J] &= \kappa_M \left( \frac{\hbar M}{2z} \right)^{\frac{M}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{\hbar} \right)^n \left( \frac{2}{M} \right)^n \frac{\Gamma\left(\frac{M}{2} + 1\right)}{\Gamma\left(\frac{M}{2} + 1 - n\right)} \\ &\int d^d x_1 \dots d^d x_n J(x_1) \dots J(x_n) \left\langle \phi(x_1) \dots \phi(x_n) e^{\sum_{j=1}^{\infty} (-1)^j \frac{2(2j-2)!!}{(j+1)!} \left( \frac{z}{\hbar} \frac{S[\phi]}{M} \right)^j [jS[\phi] + (j+1)n]} \right\rangle, \end{aligned} \quad (7.14)$$

where the expectation value, denoted  $\langle \cdot \rangle$ , is taken with respect to the weight  $\exp(zS[\phi]/\hbar)$  with  $S[\phi]$  being the renormalized action. Consider now the correlators:

$$\begin{aligned} & \left\langle \phi(x_1) \dots \phi(x_n) e^{\sum_{j=1}^{\infty} (-1)^j \frac{2(2j-2)!!}{(j+1)!} \left(\frac{z}{\hbar} \frac{S[\phi]}{M}\right)^j [jS[\phi] + (j+1)n]} \right\rangle = \\ & = \langle \phi(x_1) \dots \phi(x_n) \rangle + \sum_{j=1}^{\infty} \frac{c_j(M, n)}{M^j} \langle \phi_1(x_1) \dots \phi_n(x_n) S^j[\phi] \rangle, \end{aligned} \quad (7.15)$$

where in the second line we performed a large- $M$  expansion of the exponential. It turns out that the coefficients  $c_j(M, n)$  are polynomials in  $M$  with an asymptotic behavior of the kind

$$\frac{c_j(M, n)}{M^j} = o\left(\frac{1}{M}\right). \quad (7.16)$$

We need now to ascertain whether  $\Omega[\mathcal{A}_z, J]$  has a sensible field limit,  $M \rightarrow \infty$ . We begin by noticing that

$$\lim_{M \rightarrow \infty} \left(\frac{2}{M}\right)^n \frac{\Gamma\left(\frac{M}{2} + 1\right)}{\Gamma\left(\frac{M}{2} + 1 - n\right)} \rightarrow 1, \quad (7.17)$$

which tells us that the coefficient of each term in the sum goes to unity in the large- $M$  limit. We make now the (very reasonable) assumption that all the insertions of powers of the renormalized action  $c_j(M, n)S[\phi]^j/M^j$  in the correlators appearing in eq. (7.15) go smoothly to zero in the continuum limit: this is in fact equivalent to assume that in the continuum limit the renormalized action remains finite in a finite volume, i.e.,  $S[\phi]/M \rightarrow 0$  when  $M \rightarrow \infty$ . Clearly the assumption that  $S[\phi]$  remains finite in the limit  $M \rightarrow \infty$  would not equally apply to thermodynamic limit, where we expect the renormalized action to be extensive,  $S[\phi] \sim M$ . We therefore have that for each term of the series the following holds:

$$\lim_{M \rightarrow \infty} \left\langle \phi_1(x_1) \dots, \phi_n(x_n) \frac{c_j(M, n)}{M^j} S^j[\phi] \right\rangle = 0 \quad \forall j, \quad (7.18)$$

which finally leads to:

$$\begin{aligned} \Omega[\mathcal{A}_z, J] &= \\ &= \kappa_M \left(\frac{\hbar M}{2z}\right)^{\frac{M}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{\hbar}\right)^n \int d^d x_1 \dots d^d x_n J(x_1) \dots J(x_n) \langle \phi(x_1) \dots \phi(x_n) \rangle \\ &= \kappa_M \left(\frac{\hbar M}{2z}\right)^{\frac{M}{2}} \int \mathcal{D}\phi_M e^{\frac{z}{\hbar} S[\phi]} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{\hbar}\right)^n \int d^d x_1 \dots d^d x_n J(x_1) \dots J(x_n) \phi(x_1) \dots \phi(x_n) \\ &= \kappa_M \left(\frac{\hbar M}{2z}\right)^{\frac{M}{2}} \int \mathcal{D}\phi_M e^{\frac{z}{\hbar} S[\phi] + \frac{z}{\hbar} \int d^d x J(x)\phi(x)}. \end{aligned} \quad (7.19)$$

From eq. (7.19) we can therefore conclude that, up to irrelevant multiplicative constants, the symplectic quantization microcanonical generating functional in then *continuum* limit takes the form:

$$\Omega[\hbar/z, J] = \int \mathcal{D}\phi \exp\left(\frac{z}{\hbar} S[\phi] + \frac{z}{\hbar} J\phi\right). \quad (7.20)$$

The choice of  $z$  corresponding to the simulations presented in the first part of this work is  $z = 1$ : in this case the expression of  $\Omega[\hbar, J]$  obtained in eq. (7.20) tells us that the correlation functions measured the Hamiltonian dynamics of symplectic quantization are identical, to the leading order in  $M$  and provided that ergodicity holds, to those obtained from a canonical probability distribution of the kind

$$P[\phi] = \frac{e^{S[\phi]/\hbar}}{\Omega[\hbar]}. \tag{7.21}$$

It comes quite natural at this point a short remark on how the continuum limit can be possibly be considered in our numerical setup, in order to gain full consistency between simulation results and the present analytical derivation. Very simply we assume that in the symplectic quantization framework the continuum limit can be taken exactly as in ordinary lattice quantum field theory. One has to consider the limit of a vanishing lattice spacing,  $a \rightarrow 0$ , while tuning the bare parameters so as to hold a chosen renormalization group (RG) invariant observable fixed. In practice, the prescription to consider the continuum limit can be realized as follows: after having identified a given RG-invariant quantity  $X$ , one has to perform simulations at several values of the lattice spacings  $a$ , then adjusting the coupling(s) and the mass(es) so that the RG-invariant observable  $X$  remains constant, thus ensuring a correct RG flow. The results of simulations can be then used to extrapolate smoothly the values of physical observables at  $a = 0$ , i.e., in the continuum limit. Let us also notice that this procedure, while being necessary for any future use of the Symplectic Quantization approach to extract physical information for realistic theories, it is not particularly interesting for an asymptotically trivial theory such as  $\lambda\phi^4$ , where the RG-flow is to a Gaussian fixed point, which makes the continuum limit trivial.

Let us conclude with two main remarks about the results in eqs. (7.20), (7.21). First of all we have shown that the microcanonical sampling is equivalent to the sampling from a probability distribution  $P[\phi]$  which is well defined for an interacting theory with a potential bounded from below, since for configuration of the field with large values and smooth variations we have approximatively

$$e^{S[\phi]/\hbar} \sim e^{-V[\phi]/\hbar}. \tag{7.22}$$

This is completely in agreement with the results of numerical simulations in the micro-canonical ensemble, where the Hamiltonian dynamics of the free theory develops run-away solutions. Second, the result of our derivation in eq. (7.20) shows us that this new “canonical Minkowskian measure” can be connected to the standard Feynman path integral by means of analytic continuation in the dimensionless parameter  $z$ . More investigations in this direction are actually in progress.

## 8 Conclusions and perspectives

In this work we have presented the first numerical test of symplectic quantization, a new functional approach to quantum field theory [8, 9] which allows for an importance sampling procedure directly in Minkowski space-time. The whole idea, which parallels the one of

stochastic quantization, is based on the assumption that fields have a dependence of an additional time parameter, the intrinsic time  $\tau$ , with respect to which conjugated momenta  $\pi(x)$  are defined. Quantum fluctuations of the fields are sampled by means of a deterministic dynamics flowing along the new time  $\tau$ , which controls the internal dynamics of the system and is distinguished from the coordinate time of observers and clocks. Such a dynamics is generated by a generalized Hamiltonian where the original relativistic action plays the role of a potential energy part and therefore fluctuates naturally along the flow of  $\tau$ . This whole construction does not need any sort of rotation from real to imaginary time to be consistent and to efficiently allow the numerical sampling of field fluctuations. Furthermore, under the hypothesis of ergodicity, symplectic quantization allows to define a generalized microcanonical ensemble which represents a probabilistically well defined functional approach to quantum field theory. In section 7 we have shown that the microcanonical partition function corresponding to the symplectic quantization dynamics is simply connected by means of an integral transformation to the Feynman path integral, thus implying that also all the disconnected correlation functions measured from the symplectic quantization approach are connected by means of an integral transformation to quantum field theoretic correlations. This said, there is also another possible way to connect the microcanonical functionals studied in this work to the standard Feynman path integral. We have shown that it is possible to explicitly compute  $\Omega[\hbar/z, J]$  in the continuum limit, where it turns out to be equivalent to a Minkowskian statistical mechanics theory with canonical weight  $P[\phi] \propto \exp(zS[\phi]/\hbar)$ . First of all, consistently with the results of our simulations, it must be noticed that this canonical probability is well defined for a Minkowskian theory, provided that the potential is bounded from below, it is therefore a very promising tool to study non-perturbative problems in causal space-time. Second, the above canonical expression suggests that a punctual correspondence with standard quantum field theory can be drawn by analytically continuing along a suitable integration path the above weight in the complex  $z$  plane [24, 25]. This last path looks very promising and is currently under investigation.

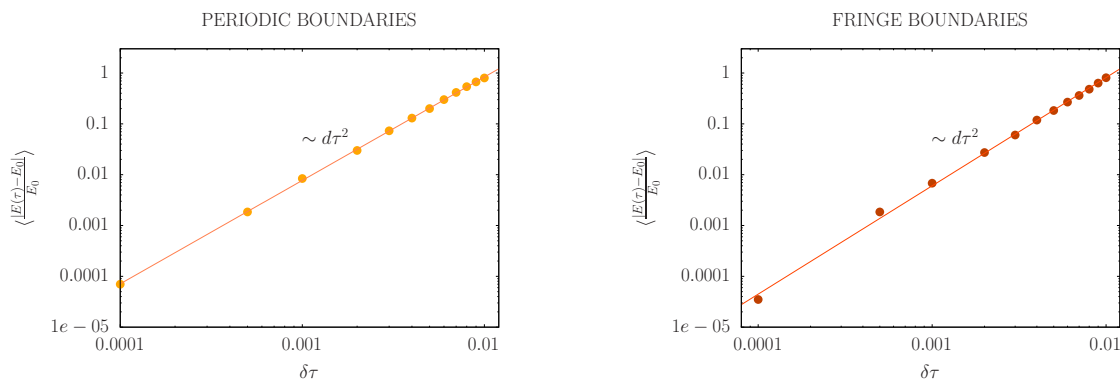
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## A Numerical algorithm

All numerical calculations in this paper have been performed using a splitting algorithm of second order, which takes advantage of the generalized Hamiltonian separability. Using the notation of section 2, the algorithm can be characterized as a map

$$\Psi_{\delta\tau} : \phi(x, \tau), \pi(x, \tau) \longrightarrow \phi(x, \tau + \delta\tau), \pi(x, \tau + \delta\tau), \quad (\text{A.1})$$



(a) Energy fluctuations  $\delta E(\delta\tau)$  as a function of the timestep  $\delta\tau$  of the numerical algorithm in the case of Minkowski metric and *periodic* boundary conditions. Energy conservation at the algorithmic precision, i.e.,  $\delta E(\delta\tau) \sim \delta\tau^2$ , is fulfilled.

(b) Energy fluctuations  $\delta E(\delta\tau)$  as a function of the timestep  $\delta\tau$  of the numerical algorithm in the case of Minkowski metric and *fringe* boundary conditions. Energy conservation at the algorithmic precision, i.e.,  $\delta E(\delta\tau) \sim \delta\tau^2$ , is fulfilled.

**Figure 6.** Energy fluctuations  $\delta E(\delta\tau)$  for different boundary conditions in the Minkowski metric.

with the following structure

$$\Psi_{\delta\tau} = \Phi_{\mathbb{K}}^{\delta\tau/2} \circ \Phi_{\mathbb{V}}^{\delta\tau} \circ \Phi_{\mathbb{K}}^{\delta\tau/2}, \tag{A.2}$$

where  $\Phi_{\mathbb{K}}^{\delta\tau/2}$  denotes the Hamiltonian flow of  $\mathbb{K}[\pi]$ , i.e., the flow of generalized momenta, while  $\Phi_{\mathbb{V}}^{\delta\tau}$  denotes the Hamiltonian flow of  $\mathbb{V}[\phi]$ , i.e., the flow of generalized coordinates (in this case, the field). In formulae, each time step of the algorithm is represented by the following sequence of operations, to be realized for each point of  $x$  of the lattice:

$$\begin{aligned} \pi(x, \tau + \delta\tau/2) &= \pi(x, \tau) + \frac{\delta\tau}{2} \cdot F[\phi(x, \tau)] & \forall x \\ \phi(x, \tau + \delta\tau) &= \phi(x, \tau) + \delta\tau \cdot \pi(x, \tau + \delta\tau/2) & \forall x \\ \pi(x, \tau + \delta\tau) &= \pi(x, \tau + \delta\tau/2) + \frac{\delta\tau}{2} \cdot F[\phi(x, \tau + \delta\tau)] & \forall x \end{aligned} \tag{A.3}$$

The splitting algorithm which we have just described is usually known as the leapfrog algorithm, the name coming from the fact the updated of generalized positions and velocities takes place at interleaved time points. Given  $E_0 = \mathbb{H}[\phi(x, 0), \pi(x, 0)]$  and  $E(\tau) = \mathbb{H}[\phi(x, \tau), \pi(x, \tau)]$ , where  $\phi(x, \tau)$  and  $\pi(x, \tau)$  are the numerical solutions computed at  $\tau$ , the leapfrog dynamics has the following algorithmic bound on energy fluctuations

$$\delta E(\delta\tau) = \langle |E(\tau)/E_0 - 1| \rangle \propto \delta\tau^2 \tag{A.4}$$

We have verified that the bound in eq. (A.4) is fulfilled by the fluctuations of both the Hamiltonian  $E(\tau) = \mathbb{H}[\phi(x, \tau), \pi(x, \tau)]$  in the case of Minkowski metric with periodic boundary conditions and the total Hamiltonian (system + boundary layers)  $\mathbb{H}_f[\phi(x, \tau), \pi(x, \tau)]$  in the case of *fringe* boundary conditions (see eq. (3.8) and the following discussion for the definition of  $\mathbb{H}_f[\phi, \pi]$ ). In figure 6(a) and figure 6(b) is shown the behavior of  $\delta E(\delta\tau)$  as a function of  $\delta\tau$  respectively for the case of periodic and fringe boundary conditions.

Test Type	Parameters	Measured Mass
Constant $\varepsilon$ ( $L = 128, L_{\text{fringe}} = L$ )	$\varepsilon = 10^{-3}$	$2.09 \pm 0.04$
	$\varepsilon = 10^{-5}$	$2.06 \pm 0.03$
	$\varepsilon = 10^{-10}$	$2.06 \pm 0.05$
	$\varepsilon = 10^{-15}$	$1.97 \pm 0.04$
Varying $\varepsilon(x)$ form ( $L = 128, L_{\text{fringe}} = L$ )	Linear Decay	$2.31 \pm 0.06$
	Exponential Decay, $c = 0.05$	$2.22 \pm 0.07$
	Exponential Decay, $c = 0.5$	$2.13 \pm 0.06$
	Exponential Decay, $c = 1.0$	$1.99 \pm 0.04$
Varying Lattice Size ( $L_{\text{fringe}} = 128$ )	$L = 128, L_{\text{fringe}} = L, \varepsilon = 10^{-10}$	$2.06 \pm 0.05$
	$L = 256, L_{\text{fringe}} = L/2, \varepsilon = 10^{-10}$	$1.91 \pm 0.02$
	$L = 128, L_{\text{fringe}} = L, \varepsilon = 10^{-15}$	$1.97 \pm 0.04$
	$L = 256, L_{\text{fringe}} = L/2, \varepsilon = 10^{-15}$	$1.91 \pm 0.02$

**Table 1.** Investigation of the dependence of the measured mass on fringe boundary condition parameters. The baseline from the paper is  $L = 128, L_{\text{fringe}} = L$ , constant  $\varepsilon = 10^{-10}$ , yielding a mass of  $2.06 \pm 0.05$ .

## B Fringe boundaries

In this section we present results of a preliminary investigation into the sensitivity of the measured mass to the parameters governing the fringe boundary conditions. These numerical tests were conducted with the same parameters used for figure 5 in the main text, namely  $m = 1, \lambda = 0.001, a = 1$  and  $L = 128$ , unless otherwise specified. Our findings are summarized in table 1 and discussed below.

In the first place, we investigated the behavior of the measured mass upon varying the damping parameter  $\varepsilon$ , while keeping it constant across space. The top section of table 1 shows that even by varying  $\varepsilon$  over ten orders of magnitude (from  $10^{-5}$  down to  $10^{-15}$ ) we do not find evidence of any trend in the change of the measured mass, which therefore seems not affected by variations of  $\varepsilon$ . We also found that simply increasing the fringe region thickness  $L_{\text{fringe}}$  from  $L$  to  $2L$  also makes apparently no difference.

We then investigated whether a spatially varying profile  $\varepsilon(x)$  may influence the finite-size effects on the measured mass. We considered both a linear and an exponential decay for  $\varepsilon(x)$  across the fringe layer. As shown in the middle section of table 1, a linear decay profile seems to have no effect on the measured mass. On the contrary, by exploiting an exponential decay of the form  $\varepsilon(x) = \exp(-cx)$  we find an encouraging signal: the measured mass value seems to have a trend towards the expected value as long as the decay becomes sharper (i.e., for larger  $c$ ).

Finally, we made a very preliminary investigation on how the measured mass value depends on the ratio  $L_{\text{fringe}}/L$ , upon increasing  $L$  at fixed  $L_{\text{fringe}}$ . From the bottom section of table 1 we have some preliminary indication that decreasing the ratio  $L_{\text{fringe}}/L$  there is a trend towards the expected value of the mass, but a more systematic investigation is clearly in order.

**Data Availability Statement.** This article has no associated data or the data will not be deposited.

**Code Availability Statement.** This article has no associated code or the code will not be deposited.

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