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Maximal L^q-Regularity for Parabolic Hamilton–Jacobi Equations and Applications to Mean Field Games

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Abstract

In this paper we investigate maximal L^q-regularity for time-dependent viscous Hamilton–Jacobi equations with unbounded right-hand side and superlinear growth in the gradient. Our approach is based on the interplay between new integral and Hölder estimates, interpolation inequalities, and parabolic regularity for linear equations. These estimates are obtained via a duality method à la Evans. This sheds new light on the parabolic counterpart of a conjecture by P.-L. Lions on maximal regularity for Hamilton–Jacobi equations, recently addressed in the stationary framework by the authors. Finally, applications to the existence problem of classical solutions to Mean Field Games systems with unbounded local couplings are provided.

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1 Introduction

The purpose of this paper is two-fold. First, to establish maximal L^q-regularity for parabolic Hamilton–Jacobi equations with unbounded right-hand side of the form

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) + H(x, Du(x,t)) = f(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0, T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d \end{cases}$$

(HJ)

where H has superlinear growth in the gradient variable and $f \in L^q(Q_T)$ for some $q > 1$. Second, to apply these regularity results to prove the existence of classical solutions for a large class of second-order Mean Field Games systems with local coupling, i.e.

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = g(m(x,t)) & \text{in } Q_T \\ \partial_t m - \Delta m - \text{div}(D_p H(x, Du)m) = 0 & \text{in } Q_T \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{T}^d. \end{cases}$$

(MFG)

Maximal regularity for (HJ). The problem of maximal L^q-regularity for (HJ) amounts to show that bounds on the right-hand side f in $L^q(Q_T)$ imply bounds on individual terms $\partial_t u$, Δu and $H(x, Du)$ in $L^q(Q_T)$ (see Theorem 1.1 below for a more precise statement). This problem has been proposed for stationary Hamilton–Jacobi equations by P.-L. Lions in a series of seminars (see, e.g., [46,47]). Under the assumption that

$$H(x, p) = |p|^\gamma, \qquad \gamma > 1,$$

he conjectured that maximal regularity holds provided that q is above the threshold $d(\gamma - 1)/\gamma$. The conjecture has been proved recently in [21] via a refined Bernstein method, which unfortunately breaks down in the parabolic setting. Here, we are able to obtain parabolic maximal regularity via different methods, assuming that q is above a certain (parabolic) threshold, that is

$$q \geq (d+2)\frac{\gamma - 1}{\gamma} = \frac{d+2}{\gamma'}$$

in the regime of *subnatural growth*, that is, for $\gamma < 2$. For $\gamma \geq 2$, we obtain the larger threshold

$$q > (d+2)\frac{\gamma - 1}{2}.$$

Besides maximal regularity, we prove new results on the Hölder regularity of solutions when $\gamma > 2$.

Before stating our results, we briefly review some contributions on the existence and regularity of solutions to Hamilton–Jacobi equations with unbounded right-hand side.

We first discuss the case $\gamma \leq 2$, that is, when the nonlinear term $H(Du)$ plays a mild role, being the diffusion term (formally) dominating at small scales. When $f \in L^q$, $q > (d+2)/2$, the boundedness and Hölder continuity of weak solutions are classical [41, Chapter 5]. For such values of q, boundedness has actually been established for much more general quasi-linear equations, see, e.g., [58]. For these problems, it was shown [27,33] that the existence of weak (and possibly unbounded) solutions holds up to $q = (d+2)/2$. For $\gamma < 2$, namely, strictly below the natural growth, the existence assumption has been relaxed to $q \geq (d+2)/\gamma'$ in the recent paper [49]. Still, the solutions obtained are in a weak or renormalized sense, and although they need not be bounded, the question of further regularity beyond $H(Du) \in L^1$ in the existence regime $(d+2)/\gamma' \leq q < (d+2)/2$ has remained open so far. Beyond the natural growth, that is, in the super-quadratic case $\gamma > 2$, the existence and regularity are even less understood. Under the sign condition $f \geq 0$, it has been proven in [13,66] that viscosity solutions enjoy Hölder bounds depending on $\|f\|_q$, with $q > 1 + d/\gamma$. It is worth mentioning that these results only rely on the super-quadratic nature of H, and tolerate degenerate diffusions. Below this exponent, bounds in L^p spaces have been shown in [14] (see also [35, Section 3.4]). It is remarkable that the exponent $1 + d/\gamma$ decreases as γ grows in the super-quadratic regime, so estimates in L^∞ require milder assumptions on f than in the sub-quadratic regime. On the other hand, further regularity, especially involving Du, seems to be difficult to achieve. For general $\gamma > 1$, Lipschitz regularity has been recently investigated in [20], and proven under the (optimal) assumption $q > d + 2$ when $\gamma \leq 3$ and larger q when $\gamma > 3$. Note that whenever Lipschitz regularity is established, maximal regularity for (HJ) follows from maximal regularity for linear equations (e.g. [29,38,42,60,61]), being $H(Du)$ controlled in L^∞. We finally mention that within the context of L^p-viscosity solutions, Hamilton–Jacobi equations with unbounded right-hand side have been considered, see e.g. [26].

Maximal regularity typically requires some mild smoothness of coefficients in the equation (diffusions with coefficients merely in L^∞ are not allowed even in the linear framework), but provide the integrability of $D^2 u$, $\partial_t u$, thus allowing to recover most of the aforementioned properties of solutions by means of Sobolev embeddings, as q varies. To our knowledge, there are only a few instances of this type of results in the literature, and in the regime $\gamma \leq 2$ only. In [64] maximal regularity is stated for $f \in L^q$, $q \geq d + 1$ (see also [51,63]). For Hamilton–Jacobi equations driven by the Laplacian, some results can be found in [35], under the more general assumption $q > (d+2)/2$, but no results are available below this exponent, nor for $\gamma > 2$.

As we previously announced, the first main result of this paper consists in achieving maximal regularity in the full range $q > (d+2)/\gamma'$ in the sub-quadratic setting $\gamma \leq 2$, and for $q > (d+2)(\gamma-1)/2$ when $\gamma > 2$. Data will be periodic in the x-variable, namely, functions will be defined over the d-dimensional flat torus $\mathbb{T}^d$. This will be convenient for the applications to MFG systems. We suppose that $H \in C(\mathbb{T}^d \times \mathbb{R}^d)$ is convex in the second variable, and that there exist constants $\gamma > 1$ and $C_H > 0$ such that

$$C_H^{-1}|p|^\gamma - C_H \leq H(x, p) \leq C_H(|p|^\gamma + 1),\qquad (H)$$

for every $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$. Concerning the case $\gamma \geq 2$, we will further assume some additional regularity in the x-variable, i.e. for $\alpha \in (0, 1)$ to be specified,

$$H(x, p) - H(x + \xi, p) \leq C_H |\xi|^\alpha \left(|D_p H(x, p)|^{\gamma'} + 1 \right) \qquad (H_\alpha)$$

for all $x, \xi \in \mathbb{T}^d$ and $p \in \mathbb{R}^d$. A typical example of H satisfying (H) is

$$H(x, p) = h(x)|p|^\gamma + b(x) \cdot p, \qquad 0 < h_0 \leq h(x), \quad h, b \in C(\mathbb{T}^d).$$

If $h \in C^\alpha(\mathbb{T}^d)$, this Hamiltonian will satisfy also (H_α).

Hoping to help the reader to have a clearer picture, we sketch known and new regularity regimes as γ and q vary in Fig. 1.

Theorem 1.1 *Assume that (H) holds, and (H_α) also when $\gamma \geq 2$ (with α as in Theorem 1.2 below). Let $u \in W_q^{2,1}(Q_T)$ be a strong solution to (HJ) and assume that for some $K > 0$*

$$\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)} \leq K.$$

If

$$q > \begin{cases} (d+2)\frac{\gamma-1}{\gamma} & \text{if } 1 + \frac{2}{d+2} < \gamma < 2 \\ (d+2)\frac{\gamma-1}{2} & \text{if } \gamma \geq 2 \end{cases}$$

then, there exists a constant $C > 0$ depending on K, q, d, C_H, T such that

$$\|u\|_{W_q^{2,1}(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \leq C.$$

The strategy of the proof is based on the following procedure. By maximal regularity for linear equations, one has

$$\|D^2 u\|_{L^q} \lesssim \|H(Du)\|_{L^q} + \|f\|_{L^q} + \|u_0\|_{W^{2-2/q,q}} \lesssim \|Du\|_{L^{\gamma q}}^\gamma + \|f\|_{L^q} + \|u_0\|_{W^{2-2/q,q}}.$$

Then, one looks for a suitable norm $\|\|\cdot\|\|$ so that a Gagliardo-Nirenberg type interpolation inequality of the form

$$\|Du\|_{L^{\gamma q}}^\gamma \lesssim \|D^2 u\|_{L^q}^{\gamma\theta} \|\|u\|\|^{\gamma(1-\theta)}$$

with $\gamma\theta < 1$ holds. Combining the two inequalities, maximal regularity is achieved whenever it is possible to produce bounds on $\|\|u\|\|$. This way of producing estimates for nonlinear problems, using linear estimates and interpolation inequalities, goes back to the works of Amann and Crandall [2]. When $\gamma = 2$, a good choice for $\|\|\cdot\|\|$ is the L^∞ norm, which is easily controlled by $\|f\|_\infty$ (ABP type estimates allow to reach $\|f\|_{d+1}$, as in [64]). Here, we start with the observation that when $\gamma < 2$, the optimal choice is a suitable L^p norm, while for $\gamma > 2$, a Hölder bound on u is needed. A crucial step in

this work is the derivation of such estimates. These are obtained by duality arguments, inspired by [20,32]. The main idea behind duality is to shift the attention from (HJ) to the formal adjoint of its linearization, which is a Fokker–Planck equation. Crossed information on u and the solution ρ to the "dual" equation allow to retrieve estimates on ρ, that are then transferred back to u. A more detailed heuristic explanation of this method can be found in [20] (see also the references therein). Note that [20] is devoted to Lipschitz regularity of u, while here we investigate Hölder regularity, which requires a further inspection of the regularity of ρ at the level of Nikol'skii spaces. As already mentioned, the nonlinear adjoint method started with the works of L.C. Evans [31,32]. Further results for stationary equations have been obtained in [69], see also [11,54] for further applications. This method has been successfully applied to study the regularity of parabolic Hamilton–Jacobi equations and Mean Field Games systems in recent works [19,20,25,36], see also [22] for maximal $L_t^q - L_x^p$ regularity estimates in the long-time (quadratic) regime.

We underline that our results on Hölder regularity in the super-quadratic case are new in the following sense. Recall that Hölder bounds have been obtained in [13,66] when $q > 1 + d/\gamma$ for $\gamma > 2$, while here we assume the stronger requirement $q > (d + 2)/\gamma'$ (which is the natural one for maximal regularity). In [13,66] sign assumptions on f are in force, and explicit Hölder exponents are not provided. Here, we do not require any assumption on the sign of f, and produce explicit Hölder exponents. The statement is as follows.

Theorem 1.2 *Assume that (H) and (H_α) hold with $\gamma \geq 2$. Let u be a strong solution to (HJ) in $W_q^{2,1}(Q_T)$, $q > \frac{d+2}{\gamma'}$. Then, there exists a positive constant C (depending on $\|u_0\|_{C^\alpha(\mathbb{T}^d)}, \|f\|_{L^q(Q_T)}, H, q, d, T$) such that*

$$\sup_{t \in [0,T]} \|u(t)\|_{C^\alpha(\mathbb{T}^d)} \leq C,$$

where $\alpha = \gamma' - \frac{d+2}{q}$ if $q < \frac{d+2}{\gamma'-1}$, while $\alpha \in (0, 1)$ if $q \geq \frac{d+2}{\gamma'-1}$.

We mention that the proof of the Hölder bounds could be localized in time, thus assuming merely $u_0 \in C(\mathbb{T}^d)$. The constant C would then depend on $\|u\|_\infty$, see Remark 3.7.

In the sub-quadratic regime $\gamma < 2$, the existence and uniqueness of weak solutions can be obtained up to the critical integrability exponent $q = (d + 2)\frac{\gamma-1}{\gamma}$ [49]. From the viewpoint of maximal regularity, this endpoint situation is a bit more delicate than the one treated in Theorem 1.1, which concerns $q > (d+2)\frac{\gamma-1}{\gamma}$. Indeed, the heuristic procedure previously discussed yields

$$\|D^2 u\|_{L^q} \lesssim \|D^2 u\|_{L^q} \|u\|_{L^p}^{\gamma-1} + \|f\|_{L^q} + \|u_0\|_{W^{2-2/q,q}},$$

which is meaningful only if $\|u\|_{L^p}$ is small. We circumvent this issue by shifting the analysis from u to $u - u_k$, where u_k is the solution to a suitable regularized problem. Crucial stability estimates on $\|u - u_k\|_{L^p}$ then lead to the second main maximal regularity result of the paper.

Theorem 1.3 *Assume that ([H]) holds, and $1 + \frac{2}{d+2} < \gamma < 2$. Let $u \in W_q^{2,1}(Q_T)$ be a strong solution to ([HJ]), and*

$$q = (d+2)\frac{\gamma - 1}{\gamma}.$$

Then, there exists a constant $C > 0$ depending on f, $\|u_0\|_{W^{2-2/q,q}}$, q, d, C_H, T such that

$$\|u\|_{W_q^{2,1}(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \leq C.$$

We stress that in this limiting case the dependence of the constant C with respect to f is not just through its norm, as in the previous Theorem 1.1, but on subtler properties; see Remark 4.2 below for additional details.

Further comments concerning the thresholds for q in Theorems 1.1 and 1.3 are now in order. When $q < (d+2)/\gamma'$ we believe that maximal regularity results are false in general. This would be in line with known results for the associated stationary problem [21], for which maximal regularity does not hold when $q < d/\gamma'$. We also believe that the endpoint $q = \frac{d+2}{2}$ for quadratic problems $\gamma = 2$ could be treated by our methods, using more refined interpolation estimates in Orlicz spaces (or the analysis of a linear equation obtained via the Hopf–Cole transformation). Regarding the super-quadratic regime $\gamma > 2$, we do not know whether our assumption $q > (d+2)(\gamma - 1)/2$ is optimal or not.

It is finally worth noticing that our results, and in particular the Hölder estimates in Theorem 1.2, apply to equations with repulsive gradient term (e.g. Kardar–Parisi–Zhang type PDEs), i.e.

$$\partial_t v - \Delta v = G(x, Dv(x,t)) - f(x,t)$$

with G satisfying ([H]). In other words, the sign in front of H and f does not matter in ([HJ]), since it is sufficient to observe that $u(x,t) = -v(x,t)$ solves ([HJ]) with $H(x,p) = G(x, -p)$, which satisfies ([H]) too. We refer to [4,5] and the references therein for further discussions on these equations.

Mean Field Games. Armed with maximal regularity results for Hamilton–Jacobi equations, we describe our contributions on Mean Field Games (MFG) systems of the form ([MFG]). These arise in the MFG framework, which is a class of methods inspired by ideas in statistical physics to study differential games with a population of infinitely many indistinguishable players [43,44]. The PDE system ([MFG]) appears naturally when the running cost of a single player depends on the population's density in a pointwise manner. This results in the nonlinear coupling term $g(m(x,t))$ between the Hamilton–Jacobi and the Fokker–Planck equation in ([MFG]). When $g(\cdot)$ is an unbounded function, the regularity of solutions is still a challenging problem that has not been solved in full generality.

In the so-called *monotone* case, that is, when $g(\cdot)$ is increasing, there are no restrictions on the growth of f if one looks for solutions in some weak sense [14,59], or for classical solutions when the time horizon T is small [3,24]. Classical solutions
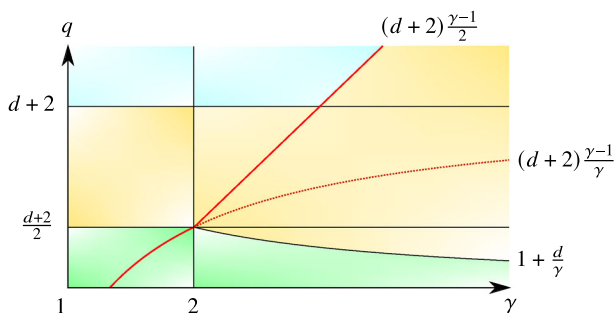
Fig. 1 A sketch of estimates that are known for solutions to (HJ), as γ and q vary: L^p (green region), Hölder (yellow region), Lipschitz (light blue region, [20]). Above the solid red line we prove here maximal regularity estimates (Theorems 1.1 and 1.3). Above the dashed red line we prove here new Hölder estimates (Theorem 1.2). When $\gamma > 2$ and $1 + \frac{d}{\gamma} < q < \frac{d+2}{\gamma'}$ (yellow zone between the dashed red line and the solid black line) Hölder estimates have been obtained in [13,66] under sign conditions on the right-hand side, while integrability estimates in the green region below the threshold $q = 1 + \frac{d}{\gamma}$ can be found in [14]

for arbitrary T can be obtained by requiring a mild behaviour of $g(m)$ as $m \to \infty$, or a mild behaviour of $H(p)$, i.e. $\gamma \leq 1 + 1/(d+1)$ (as suggested in [46]). We deal here with the first situation. For the model problem $g(m) = m^r$, an upper bound on r depending on γ and the space dimension d is usually required. Concerning the subquadratic case $\gamma < 2$, a main reference is [34] (see also [6]), while [36] explores the superquadratic case $2 < \gamma < 3$.

We will assume here that $g : \mathbb{T}^d \times [0, \infty) \to \mathbb{R}$ is of class C^1, and that there exist $r > 0$ and $C_g > 0$ such that

$$C_g^{-1} m^{r-1} \leq g'(m) \leq C_g(m^{r-1} + 1) \qquad \forall m \geq 0. \qquad (M^+)$$

This implies that $g(\cdot)$ is monotone increasing and bounded from above and below by power-like functions of type m^r. As for the Hamiltonian, we will further assume some additional smoothness and uniform convexity in the p-variable, having always in mind a power-like growth $|p|^\gamma$: for every $p \in \mathbb{R}^d$, $M \in \mathrm{Sym}_d$

$$
\begin{aligned}
\mathrm{Tr}(D_{pp}^2 H(x, p) M^2) &\geq C_H^{-1}(1 + |p|^2)^{\frac{\gamma-2}{2}} |M|^2 - C_H, \\
|D_{px}^2 H(x, p)| &\leq C_H(|p|^{\gamma-1} + 1), \\
|D_{xx}^2 H(x, p)| &\leq C_H(|p|^\gamma + 1).
\end{aligned}
\qquad (H2)
$$

Our main result on this class of problems reads as follows.

Theorem 1.4 *Under the assumptions (H), (H2) and (M^+), there exists a (unique) smooth solution to (MFG) if*

$$
r < \begin{cases} \dfrac{\gamma'}{d-2} \dfrac{d}{(d+2-\gamma')} & \text{if } 1 + \frac{1}{d+1} < \gamma \leq 2 \\[2mm] \dfrac{2}{d(\gamma-1)-2} & \text{if } \gamma \geq 2. \end{cases}
\qquad (1)
$$

It is understood here that there are no restrictions on r when $d = 1, 2$. To our knowledge, the results of this manuscript extend known classical regularity regimes in the sub-quadratic case. Note that

$$\frac{\gamma'}{d-2}\frac{d}{(d+2-\gamma')} \to +\infty \quad \text{as } \gamma \to 1 + \frac{1}{d+1},$$

$$\frac{\gamma'}{d-2}\frac{d}{(d+2-\gamma')} \to \frac{2}{d-2} \quad \text{as } \gamma \to 2.$$

Regarding the super-quadratic case, we actually obtain the same restriction $r < \frac{2}{d(\gamma-1)-2}$ as in [36], but we are able to cover the full interval $\gamma \in [2, \infty)$, thus unlocking some smoothness regimes for $\gamma > 3$. Still, one may conjecture that, for all $d \geq 1$, no restrictions on r should be required to get classical solutions (as for the purely quadratic case $H(p) = |p|^2$ investigated in [12, Theorem 4.1]), but this remains an open question.

In the *non-monotone* framework, the assumption of increasing monotonicity of $g(\cdot)$ is dropped. Conversely, one may even consider a $g(\cdot)$ which is strictly decreasing (focusing case), and model aggregation phenomena as in [15,17,23]. In this framework, a control on the growth of g is structurally needed. For stationary problems, it was shown in [16] that when $g(m) = -m^r$, no solutions on $\mathbb{R}^d$ exist when $r > \frac{\gamma'}{d-\gamma'}$, and even for $r \geq \frac{\gamma'}{d}$ some further assumptions might be needed for solutions to be obtained. On the other hand, when $r < \frac{\gamma'}{d}$, the existence of weak solutions is shown in [23], but their classical regularity is proven under much stronger hypotheses, that impose $\gamma < 2$.

Now, we suppose

$$-C_g(m^r + 1) \leq g(m) \leq C_g(m^r + 1). \tag{M^-}$$

With respect to the previous assumptions (M^+) in the monotone case, g need not be monotone increasing; in contrast, it can be strictly decreasing, for example, $g(m) = -m^r$. We have the following

Theorem 1.5 *Under the assumptions (H) and (M^-), there exists a smooth solution to* (MFG) *if*

$$r < \begin{cases} \frac{\gamma'}{d} & \text{if } 1 + \frac{1}{d+1} < \gamma \leq 2 \\ \frac{2}{(d+2)(\gamma-1)-2} & \text{if } \gamma \geq 2. \end{cases} \tag{2}$$

In other words, we prove that solutions in the focusing case obtained in [23] are always classical in the subquadratic regime. In the superquadratic regime, a new class of smooth solutions is obtained. We stress that the threshold $\frac{\gamma'}{d}$ is crucial, as one may have nonexistence of solutions for a large time horizon T when $r \geq \frac{\gamma'}{d}$ [18]. For a summary sketch of the existence regimes that we obtain here, see Fig. 2.
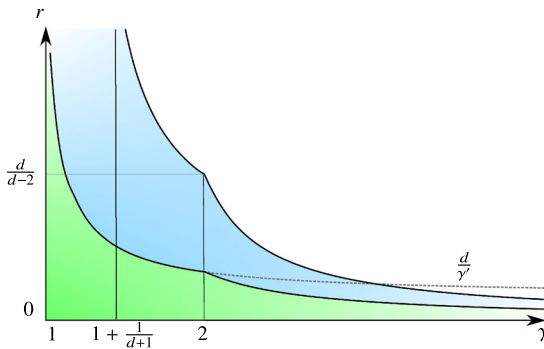
Fig. 2 A sketch of regions where existence of classical solutions to (MFG) are obtained, in the monotone case (light blue and green regions) and in the non-monotone case (green region only)

Theorems 1.4 and 1.5 are proven as follows. First, we use structural estimates to deduce some a priori bounds on $\|g(m)\|_{L^q}$. These are mainly second-order type estimates in the monotone framework, and first-order estimates in the non-monotone one (plus some additional interpolation procedures). The assumptions on the growth r of $g(\cdot)$ guarantee that q is large enough to apply our maximal regularity results for the HJ equation. Then, once the crucial a priori bounds on u in $W_q^{2,1}$ are established, a bootstrap procedure allows to get estimates up to second-order derivatives in Hölder spaces, and the existence follows via standard methods.

Outline. In Sections 2.1 and 2.2 we introduce the functional setting that is used throughout the paper, while Section 2.3 is devoted to Sobolev regularity results for Fokker–Planck equations. Section 3 comprehends the proof of a priori integral and Hölder estimates for Hamilton–Jacobi equations with unbounded ingredients. In Section 4, Theorem 1.1 on maximal regularity for (HJ) is proven, while in Section 5 the existence Theorems 1.4 and 1.5 for (MFG) are proven.

2 Preliminaries

2.1 Fractional Spaces of Periodic Functions

Let $\mu \in (0, 1)$ and $1 \leq p, q \leq \infty$. The Besov space $B_{pq}^{\mu}(\mathbb{T}^d)$ consists of all functions $u \in L^p(\mathbb{T}^d)$ such that the norm

$$\|u\|_{B_{pq}^{\mu}(\mathbb{T}^d)} := \|u\|_{L^p(\mathbb{T}^d)} + \left(\int_{\mathbb{T}^d} \frac{\|f(x+h) - f(x)\|_{L^p(\mathbb{T}^d)}^q}{|h|^{d+\mu q}} \, dh \right)^{\frac{1}{q}}$$

is finite. When $p = q = \infty$ and $\mu = \alpha \in (0, 1)$, the space $B_{\infty\infty}^{\mu}(\mathbb{T}^d) \simeq C^{\alpha}(\mathbb{T}^d)$ and it is endowed with the equivalent norm

$$\|u\|_{C^{\alpha}(\mathbb{T}^d)} := \|u\|_{C(\mathbb{T}^d)} + \sup_{x \neq y \in \mathbb{T}^d} \frac{|u(x) - u(y)|}{\text{dist}(x, y)^{\alpha}}$$

where $\text{dist}(x, y)$ is the geodesic distance on $\mathbb{T}^d$. When, instead, either $p = q \in (1, \infty)$ and μ is not an integer or μ is an integer and $p = q = 2$, it is immediate to recognize that $B_{pp}^{\mu}(\mathbb{T}^d) \simeq W^{\mu,p}(\mathbb{T}^d)$, where $W^{\mu,p}(\mathbb{T}^d)$ is the classical Sobolev–Slobodeckii scale in the periodic setting. When $q = \infty$, the space $B_{p\infty}^{\mu}(\mathbb{T}^d) \simeq N^{\mu,p}(\mathbb{T}^d)$ is known as Nikol'skii space [55] and the above norm is interpreted as usual as

$$\|u\|_{N^{\mu,p}(\mathbb{T}^d)} := \|u\|_{L^p(\mathbb{T}^d)} + \underbrace{\sup_h |h|^{-\mu}\|u(x+h) - u(x)\|_{L^p(\mathbb{T}^d)}}_{[u]_{N^{\mu,p}(\mathbb{T}^d)}} ,$$

see [45, Chapter 17] for analogous spaces defined on $\mathbb{R}^d$, and [67, p. 460], [62, Section 3.5.4] for details in the periodic case. For general $\mu > 0$, let $\mu = k + \sigma > 0$ with $k \in \mathbb{N} \cup \{0\}$, $\sigma \in (0, 1]$, $1 \le p < \infty$. Then, we define the Nikol'skii class

$$N^{\mu,p}(\mathbb{T}^d) := \{u \in W^{k,p}(\mathbb{T}^d) : [D^{\alpha}u]_{N^{\sigma,p}(\mathbb{T}^d)} < \infty , |\alpha| = k\}$$

We mention that in view of [70, Corollary 2 p. 143], Nikol'skii spaces $N^{\mu,p}(\mathbb{T}^d)$ can be endowed with equivalent norms of the form

$$\|u\|_{N^{\mu,p}(\mathbb{T}^d)} := \|u\|_{L^p(\mathbb{T}^d)} + \sum_{|\alpha|=k} \sup_{|h|>0} |h|^{k-\mu}\|\Delta_h^r u\|_{L^p(\mathbb{T}^d)}$$

where $k, r \in \mathbb{Z}$ are such that $0 \le k < \mu$ and $r > s - k$, where Δ_h^r are the increments of order r and step h, see [45]. Moreover, these spaces can be characterized via real interpolation: for $m \in \mathbb{N}$, $p, q \in [1, \infty]$ and $\theta \in (0, 1)$ we have[1]

$$(L^p(\mathbb{T}^d), W^{m,p}(\mathbb{T}^d))_{\theta,q} \simeq B_{p,q}^{\theta m}(\mathbb{T}^d) ,$$

with equivalence of the respective norms, see e.g. [48], [45, Theorem 17.24].

2.2 Space–time Anisotropic Spaces

For any time interval $(t_1, t_2) \subseteq \mathbb{R}$, let $Q_{t_1,t_2} := \mathbb{T}^d \times (t_1, t_2)$. We will also use the notation $Q_{t_2} := \mathbb{T}^d \times (0, t_2)$. For any $p \ge 1$ and $Q = Q_{t_1,t_2}$, we denote by $W_p^{2,1}(Q)$ the space of functions u such that $\partial_t^r D_x^{\beta} u \in L^p(Q)$ for all multi-indices β and r such that $|\beta| + 2r \le 2$, endowed with the norm

$$\|u\|_{W_p^{2,1}(Q)} = \left(\iint_Q \sum_{|\beta|+2r \le 2} |\partial_t^r D_x^{\beta} u|^p dx dt \right)^{\frac{1}{p}} .$$

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[1] We denote here with $(X_0, X_1)_{\theta,q}$ the standard real interpolation space between Banach spaces X_0, X_1.

The space $W_p^{1,0}(Q)$ is defined similarly, and is endowed with the norm

$$\|u\|_{W_p^{1,0}(Q)} := \|u\|_{L^p(Q)} + \sum_{|\beta|=1} \|D_x^\beta u\|_{L^p(Q)} .$$

We define the space $\mathcal{H}_p^1(Q)$ as the space of functions $u \in W_p^{1,0}(Q)$ with $\partial_t u \in (W_{p'}^{1,0}(Q))'$, equipped with the norm

$$\|u\|_{\mathcal{H}_p^1(Q)} := \|u\|_{W_p^{1,0}(Q)} + \|\partial_t u\|_{(W_{p'}^{1,0}(Q))'} .$$

Denoting by $C([t_1, t_2]; X)$, $C^\alpha([t_1, t_2]; X)$ and $L^q(t_1, t_2; X)$ the usual spaces of continuous, Hölder and Lebesgue functions respectively, with values in a Banach space X, we have the following isomorphisms: $W_2^{1,0}(Q) \simeq L^2(t_1, t_2; W^{1,2}(\mathbb{T}^d))$, and

$$\mathcal{H}_2^1(Q) \simeq \{u \in L^2(t_1, t_2; W^{1,2}(\mathbb{T}^d)), \ \partial_t u \in (L^2(t_1, t_2; W^{1,2}(\mathbb{T}^d)))'\}$$
$$\simeq \{u \in L^2(t_1, t_2; W^{1,2}(\mathbb{T}^d)), \ \partial_t u \in L^2(t_1, t_2; (W^{1,2}(\mathbb{T}^d))')\},$$

and the latter is known to be continuously embedded into $C([t_1, t_2]; L^2(\mathbb{T}^d))$ (see, e.g., [28, Theorem XVIII.2.1]). Sometimes, we will use the compact notation $C(X)$ and $L^q(X)$.

2.3 Sobolev Regularity of Solutions to Fokker–Planck Equations

In this section, we collect a few existence and regularity properties of solutions to

$$\begin{cases} -\partial_t \rho(x, t) - \Delta\rho(x, t) + \mathrm{div}(b(x, t)\rho(x, t)) = 0 & \text{in } Q_\tau , \\ \rho(x, \tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d \end{cases} \tag{3}$$

Here, $\tau \in (0, T]$, $Q_\tau := \mathbb{T}^d \times (0, \tau)$ and $Q_{s,\tau} := \mathbb{T}^d \times (s, \tau)$. We will assume that

$$\rho_\tau \in L^\infty(\mathbb{T}^d), \quad \rho_\tau \geq 0 \ \text{a.e.} \tag{4}$$

We first recall that such a transport equation with diffusion is well-posed in an energy space and has bounded solutions, when $|b| \in L^{\mathcal{P}}(Q_\tau)$ with $\mathcal{P}$ satisfying the so-called Aronson-Serrin condition $\mathcal{P} \geq d + 2$. For a proof of the following classical result, see e.g. [20, Proposition 2.3], or [7,8,41].

Proposition 2.1 *Let $|b| \in L^{\mathcal{P}}(Q_\tau)$, with $\mathcal{P} \geq d + 2$, and ρ_τ be as in (4). Then, there exists a unique weak solution $\rho \in \mathcal{H}_2^1(Q_\tau) \cap L^\infty(0, \tau; L^r(\mathbb{T}^d))$ for all $r \geq 1$ to (3), i.e.*

$$-\int_s^\tau \langle \partial_t \rho(t), \varphi(t) \rangle dt + \iint_{Q_{s,\tau}} D\rho \cdot D\varphi - b\rho \cdot D\varphi \, dx dt = 0 \tag{5}$$

for all $s > 0$ and $\varphi \in \mathcal{H}_2^1(Q_{s,\tau})$, and $\rho(\tau) = \rho_\tau$ in the L^2-sense. Moreover, ρ is a.e. nonnegative on Q_τ.

Our analysis will be based on the regularity properties of ρ in Sobolev spaces that depend on the integrability of $|b|$ against ρ itself. Similar results already appeared in [9,23,52,59].

Proposition 2.2 *Let ρ be the nonnegative weak solution to* (3) *and* $1 < \sigma' < d + 2$. *Then, there exists $C > 0$, depending on T, σ', d, such that*

$$\|\rho\|_{\mathcal{H}_{\sigma'}^1(Q_\tau)} \le C \left(\iint_{Q_\tau} |b(x,t)|^{m'} \rho \, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)} \right) \tag{6}$$

where $m' = 1 + \frac{d+2}{\sigma}$ and $p' = \frac{d\sigma}{\sigma(d+1)-(d+2)}$ if $\sigma' > \frac{d+2}{d+1}$, while $p' = 1$ if $\sigma' < \frac{d+2}{d+1}$.

As we will se below, if $\sigma' > \frac{d+2}{d+1}$ we obtain also

$$\|\rho(t)\|_{L^{p'}(\mathbb{T}^d)} \le C \left(\iint_{Q_\tau} |b(x,t)|^{m'} \rho \, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)} \right) \quad \text{for all } t \in [0, \tau]. \tag{7}$$

Proof The case $\sigma' < \frac{d+2}{d+1}$ is covered by [20, Proposition 2.5], which is based on duality combined with maximal regularity arguments (following [52]). We focus here on the case $\sigma' > \frac{d+2}{d+1}$, and prove the theorem via a (standard) method from weak solutions, that does not exploit parabolic Caldèron–Zygmund regularity. This is possible because $\frac{d+2}{d+1} < \sigma' < d + 2$ implies $2 < m' < d + 2$.

Set $\beta := \frac{m'-2}{d+2-m'}$. To simplify, we use $\varphi := \rho^\beta$ as a test function in the weak formulation of (3) integrating on $Q_{t,\tau} := \mathbb{T}^d \times (t, \tau)$, while the argument can be made rigorous by testing against $\varphi := (\rho + \epsilon)^\beta$, and then letting $\epsilon \to 0$. We have

$$\frac{1}{\beta+1} \int_{\mathbb{T}^d} \rho(x,t)^{\beta+1} \, dx + \beta \iint_{Q_{t,\tau}} \rho^{\beta-1} |D\rho|^2 \, dxdt$$

$$\le \iint_{Q_{t,\tau}} |b| \rho^\beta |D\rho| \, dxdt + \frac{1}{\beta+1} \int_{\mathbb{T}^d} |\rho_\tau|^{\beta+1} \, dx$$

$$\le C_\beta \iint_{Q_{t,\tau}} |b|^2 \rho^{\beta+1} \, dxdt + \frac{\beta}{4} \iint_{Q_{t,\tau}} \rho^{\beta-1} |D\rho|^2 \, dxdt + \frac{1}{\beta+1} \int_{\mathbb{T}^d} |\rho_\tau|^{\beta+1} \, dx$$

In particular, noting that $D\rho^{\frac{\beta+1}{2}} = \frac{\beta+1}{2} \rho^{\frac{\beta-1}{2}} D\rho$, we write

$$\beta \iint_{Q_{t,\tau}} \rho^{\beta-1} |D\rho|^2 \, dxdt = \frac{4\beta}{(\beta+1)^2} \iint_{Q_{s,\tau}} |D\rho^{\frac{\beta+1}{2}}|^2 \, dxdt$$

to get that

$$\frac{1}{\beta+1} \int_{\mathbb{T}^d} \rho(x,t)^{\beta+1}\, dx + \beta \iint_{Q_{t,\tau}} \rho^{\beta-1}|D\rho|^2\, dxdt$$

$$\geq c_\beta \left[\int_{\mathbb{T}^d} (|\rho(x,t)|^{\frac{\beta+1}{2}})^2\, dx + \iint_{Q_{t,\tau}} |D\rho^{\frac{\beta+1}{2}}|^2\, dxdt \right].$$

We first pass to the supremum over $t \in (0, \tau)$ and, by means of [30, Proposition I.3.1] we deduce

$$c_\beta \left[\operatorname{ess\,sup}_{t\in(0,\tau)} \int_{\mathbb{T}^d} (|\rho(x,t)|^{\frac{\beta+1}{2}})^2\, dx + \iint_{Q_\tau} |D\rho^{\frac{\beta+1}{2}}|^2\, dxdt \right]$$

$$\geq c_{\beta,d} \left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}}\, dxdt \right)^{1-\frac{2}{d+2}}.$$

We then have

$$c_{\beta,d} \left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}}\, dxdt \right)^{1-\frac{2}{d+2}} + \frac{\beta}{2} \iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt$$

$$\leq c_1 \left(\iint_{Q_\tau} |b|^{m'}\rho\, dxdt \right)^{\frac{2}{m'}} \left(\iint_{Q_\tau} \rho^{\beta\frac{m'}{m'-2}+1}\, dxdt \right)^{1-\frac{2}{m'}} + c_2 \|\rho_\tau\|_{L^{\beta+1}(\mathbb{T}^d)}^{\beta+1}.$$

We then note that $\sigma < d + 2$ implies $m' > 2$ and

$$(\beta+1)\frac{d+2}{d} = \beta\frac{m'}{m'-2} + 1 = \frac{d+2}{d+2-m'}.$$

We apply Young's inequality to the first term on the right-hand side to conclude

$$c_1 \left(\iint_{Q_\tau} |b|^{m'}\rho\, dxdt \right)^{\frac{2}{m'}} \left(\iint_{Q_\tau} \rho^{\beta\frac{m'}{m'-2}+1}\, dxdt \right)^{1-\frac{2}{m'}}$$

$$\leq c_3 \left(\iint_{Q_\tau} |b|^{m'}\rho\, dxdt \right)^{\frac{d}{d+2-m'}} + \frac{c_{\beta,d}}{2} \left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}}\, dxdt \right)^{\frac{d}{d+2}}.$$

Hence

$$c_{\beta,d} \left(\iint_{Q_\tau} \rho^{\frac{d+2}{d+2-m'}}\, dxdt \right)^{\frac{d}{d+2}} + \frac{\beta}{2} \iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt$$

$$\leq c_4 \left[\left(\iint_{Q_\tau} |b|^{m'}\rho\, dxdt \right)^{\frac{d}{d+2-m'}} + \|\rho_\tau\|_{L^{\frac{d}{d+2-m'}}(\mathbb{T}^d)}^{\frac{d}{d+2-m'}} \right]$$

$$\leq c_5 \left[\iint_{Q_\tau} |b|^{m'}\rho\, dxdt + \|\rho_\tau\|_{L^{\frac{d}{d+2-m'}}(\mathbb{T}^d)} \right]^{\frac{d}{d+2-m'}}.$$

We thus conclude the estimates

$$\|\rho\|_{L^{\frac{d+2}{d+2-m'}}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |b|^{m'} \rho \, dxdt + \|\rho_\tau\|_{L^{\frac{d}{d+2-m'}}(\mathbb{T}^d)} \right)$$

and

$$\iint_{Q_\tau} \rho^{\beta-1} |D\rho|^2 \, dxdt \leq \bar{C} \left[\iint_{Q_\tau} |b|^{m'} \rho \, dxdt + \|\rho_\tau\|_{L^{\frac{d}{d+2-m'}}(\mathbb{T}^d)} \right]^{\frac{d}{d+2-m'}}. \quad (8)$$

Finally, recalling that $\sigma' = \frac{d+2}{d+3-m'}$, we get the Sobolev estimate applying Hölder's inequality, using (8) and finally exploiting Young's inequality as

$$\|D\rho\|_{L^{\sigma'}(Q_\tau)} \leq \|\rho^{(\beta-1)/2} D\rho\|_{L^2(Q_\tau)} \|\rho^{(1-\beta)/2}\|_{L^{\frac{2(d+2)}{d+4-2m'}}(Q_\tau)}$$

$$= \left(\iint_{Q_\tau} \rho^{\beta-1} |D\rho|^2 \, dxdt \right)^{\frac{1}{2}} \left(\iint_{Q_\tau} \rho^{\frac{d+2}{d+2-m'}} \, dxdt \right)^{\frac{d+4-2m'}{2(d+2)}}$$

$$\leq c_6 \left(\iint_{Q_\tau} |b|^{m'} \rho \, dxdt + \|\rho_\tau\|_{L^{\frac{d}{d+2-m'}}(\mathbb{T}^d)} \right).$$

The estimate on the time derivative in $(W_\sigma^{1,0}(Q_\tau))'$ can be obtained by duality, as in [20, Proposition 2.4].   □

Corollary 2.3 *Let ρ be the nonnegative weak solution to (3). Then, there exists $C_1 > 0$, depending on T, q, d, such that if $q < \frac{d+2}{2}$*

$$\sup_{t\in[0,\tau]} \|\rho(t)\|_{L^{p'}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)} \leq C_1 \left(\iint_{Q_\tau} |b(x,t)|^{\frac{d+2}{q}} \rho \, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)} \right)$$

where $p = \frac{dq}{(d+2)-2q}$, while if $q > \frac{d+2}{2}$,

$$\sup_{t\in[0,\tau]} \|\rho(t)\|_{W^{\frac{d}{d+2}-\frac{2}{q'}, \frac{(d+2)q'}{d+2+q'}}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)}$$

$$\leq C_1 \left(\iint_{Q_\tau} |b(x,t)|^{\frac{d+2}{q}} \rho \, dxdt + \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \right).$$

Proof The first estimate follows by Proposition 2.2, applied with $m' = \frac{d+2}{q}$ (see also (7)), and the continuous embedding of $\mathcal{H}_{\sigma'}^1(Q_\tau)$ into $L^{q'}(Q_\tau)$. The second estimate follows analogously, using also the embedding of $\mathcal{H}_{\sigma'}^1(Q_\tau)$ into $C([0, \tau]; W^{\frac{d}{d+2}-\frac{2}{q'}, \frac{(d+2)q'}{d+2+q'}}(\mathbb{T}^d))$.   □

3 Bounds of Solutions to HJ Equations in Lebesgue and Hölder Spaces

We derive in this section some preliminary bounds on solutions to Hamilton–Jacobi equations via duality methods. Since we are in the setting of maximal regularity and $f \in L^q(Q_T)$, we assume that $u \in W_q^{2,1}(Q_T) \cap W_{\gamma q}^{1,0}(Q_T)$ is a strong solution to (HJ), i.e. it solves the Hamilton–Jacobi equation almost everywhere. The initial datum u_0 is achieved in $W^{2-2/q,q}(\mathbb{T}^d)$, as suggested by the Lions–Peetre trace method in interpolation theory (cf Lemma A.1 below). Note that by classical embedding properties for Sobolev–Slobodeckii spaces, we have the following inclusions

$$W^{2-\frac{2}{q},q}(\mathbb{T}^d) \hookrightarrow \begin{cases} C^{2-\frac{d+2}{q}}(\mathbb{T}^d) & \text{for } q > \frac{d+2}{2}, \\ L^p(\mathbb{T}^d) & \text{for } p \in [1,\infty) \text{ and } q = \frac{d+2}{2}, \\ L^{\frac{dq}{d+2-2q}}(\mathbb{T}^d) & \text{for } q < \frac{d+2}{2}. \end{cases} \qquad (9)$$

Since we will work under the assumptions

$$q \geq \frac{d+2}{\gamma'}, \qquad \gamma > 1 + \frac{2}{d+2},$$

embeddings of $W_q^{2,1}(Q_T)$ imply that u is bounded in $W_{\gamma q}^{1,0}(Q_T)$ (so we can drop $u \in W_{\gamma q}^{1,0}(Q_T)$ in the statements of our results), and $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d))$. Furthermore,

$$|Du|^{\gamma-1} \in L^{\mathcal{P}}(Q_T) \qquad \text{for some } \mathcal{P} \geq d+2,$$

so the dual equation (3) is well-posed in $\mathcal{H}_2^1$ whenever $b(x,t) = -D_p H(x, Du(x,t))$. Note finally that when $q > (d+2)/2$, any solution is automatically continuous and u solves (HJ) in the weak sense used in [20]. This always happens in the superquadratic regime $\gamma > 2$. On the other hand, in the subquadratic case $\gamma < 2$, u is not necessarily continuous when $(d+2)/\gamma' \leq q < (d+2)/2$.

Remark 3.1 The assumption

$$\gamma > 1 + \frac{2}{d+2}$$

guarantees that $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d))$, so u has finite energy and it can be safely used as a test function for the dual equation (3). One can drop this requirement in all following statements, relaxing to

$$\gamma > 1 + \frac{1}{d+1},$$

so that $q > \frac{d+2}{\gamma'} > 1$, and assuming *a priori* that $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d))$ (which is always true for example when u is a classical solution, as in Section 5 on MFG). One

could also invoke methods from renormalized solutions, to deal also with the case $\gamma \leq 1 + \frac{1}{d+1}$ but this is beyond the scope of this paper.

3.1 Bounds in Lebesgue Spaces

We obtain in this section the bounds on the L^p-norm of u, elaborating in particular the case $\frac{d+2}{\gamma'} \leq q < \frac{d+2}{2}$, that is when $\gamma < 2$.

$T_k(s) = \max\{-s, \min\{s, k\}\}$ below will denote the truncation operator at level $k > 0$, $u^+ = \max\{u, 0\}$ and $u^- = (-u)^+$ the positive and negative part of u respectively.

We start with some bounds on u^+, that are obtained with no restrictions on $q \geq 1$.

Lemma 3.2 *Assume that H is nonnegative. Let u be a strong solution to* (HJ) *in* $W_q^{2,1}(Q_T)$, $q \geq 1$. *There exists a positive constant C_0 (depending on q, d, T) such that*

$$\|u^+(\tau)\|_{L^p(\mathbb{T}^d)} \leq \|u_0^+\|_{L^p(\mathbb{T}^d)} + C_0 \|f^+\|_{L^q(Q_\tau)}, \tag{10}$$

where $p = \frac{dq}{(d+2)-2q}$ if $q < \frac{d+2}{2}$, while $p = \infty$ if $q > \frac{d+2}{2}$.

Note that $W_q^{2,1}(Q_T)$ is embedded into $C([0, T]; L^p(\mathbb{T}^d))$ (p as in the previous statement) so (10) has the form of an a priori bound.

Proof We detail the proof in the case $q < \frac{d+2}{2}$ only. The case $q > \frac{d+2}{2}$ can be treated in an analogous way (essentially following [20, Section 3]).

For $k > 0$, let $\mu = \mu_k$ be the weak nonnegative solution of the following backward problem

$$\begin{cases} -\partial_t \mu(x, t) - \Delta \mu(x, t) = 0 & \text{in } Q_\tau, \\ \mu(x, \tau) = \dfrac{\left[T_k\left(u^+(x, \tau)\right)\right]^{p-1}}{\|u^+(\tau)\|_p^{p-1}} & \text{in } \mathbb{T}^d \end{cases}$$

Note that $\|\mu_\tau\|_{L^{p'}(\mathbb{T}^d)} \leq 1$. Since μ solves an equation of the form (3) with $b \equiv 0$, by Corollary 2.3 we have

$$\|\mu\|_{L^{q'}(Q_\tau)} \leq C,$$

where C does not depend on k. Since u^+ is a weak subsolution of

$$\partial_t u^+(x, t) - \Delta u^+(x, t) \leq [f(x, t) - H(x, Du(x, t))]\chi_{\{u>0\}} \quad \text{in } Q_\tau,$$

testing against μ, and testing the equation for μ against u we get that

$$\int_{\mathbb{T}^d} u^+(\tau)\mu(\tau)\,dx \leq \int_{\mathbb{T}^d} u_0^+ \mu(0)\,dx + \iint_{Q_\tau \cap \{u>0\}} f\mu\,dxdt$$
$$- \iint_{Q_\tau \cap \{u>0\}} H(x, Du)\mu\,dxdt.$$

We apply Hölder's inequality to the second term of the right-hand side of the above inequality, the assumption $H \geq 0$ on the Hamiltonian, and the fact that the backward heat equation preserves the $L^{p'}$ norm, i.e. $\|\mu(t)\|_{L^{p'}(\mathbb{T}^d)} \leq 1$ for all $t \in [0, \tau]$, to get, after sending $k \to \infty$, the desired inequality.   □

We now proceed with some more delicate bounds on u^-, under the restriction $q > \frac{d+2}{\gamma'}$. We will use the following property of H: under the standing assumptions on H, the Lagrangian $L(x, v) = \sup_p\{v \cdot p - H(x, p)\}$ satisfies for some $C_L > 0$ (depending on C_H)

$$C_L^{-1}|v|^{\gamma'} - C_L \leq L(x, v) \leq C_L|v|^{\gamma'} + C_L. \tag{11}$$

Lemma 3.3 *Assume that (H) holds. Let u be a strong solution to (HJ) in $W_q^{2,1}(Q_T)$, $q > \frac{d+2}{\gamma'}$. There exists a positive constant C (depending on C_H, q, d, T) such that*

$$\|u^-(\tau)\|_{L^p(\mathbb{T}^d)} \leq C\left(\|u_0^-\|_{L^p(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)}\right)$$

$$+2CC_L\tau\left(\|u_0^-\|_{L^p(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)}\right)^{\frac{q\gamma'}{q\gamma'-(d+2)}} + C_L\tau. \tag{12}$$

where $p = \frac{dq}{(d+2)-2q}$ if $q < \frac{d+2}{2}$, while $p = \infty$ if $q > \frac{d+2}{2}$.

Proof As before, we detail the case $q < \frac{d+2}{2}$ only. For $k > 0$, let $\rho = \rho_k$ be the weak nonnegative solution of

$$\begin{cases} -\partial_t\rho(x,t) - \Delta\rho(x,t) + \text{div}\left(D_pH(x, Du(x,t))\chi_{\{u<0\}}\rho(x,t)\right) = 0 & \text{in } Q_\tau, \\ \rho(x,\tau) = \dfrac{\left[T_k\left(u^-(x,\tau)\right)\right]^{p-1}}{\|u^-(\tau)\|_p^{p-1}} & \text{in } \mathbb{T}^d \end{cases}$$

As before, $\|\mu_\tau\|_{L^{p'}(\mathbb{T}^d)} \leq 1$. By Corollary 2.3 we have

$$\|\rho(0)\|_{L^{p'}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)}$$

$$\leq C_1\left(\iint_{Q_\tau} |D_pH(x, Du)|^{\frac{d+2}{q}}\chi_{\{u<0\}}\rho\,dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)}\right), \tag{13}$$

where C does not depend on k. Since u^- is a weak subsolution of

$$\partial_t u^-(x,t) - \Delta u^-(x,t) \leq \left[-f(x,t) + H(x, Du(x,t))\right]\chi_{\{u<0\}} \quad \text{in } Q_\tau,$$

testing against ρ, and testing the equation for ρ against u^- we get that

$$\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\,dx + \iint_{Q_\tau} \left[-D_pH(x, Du) \cdot Du^- - H(x, Du)\right]\chi_{\{u<0\}}\rho\,dxdt$$

$$\leq \int_{\mathbb{T}^d} u_0^-\rho(0)\,dx - \iint_{Q_\tau \cap \{u<0\}} f\rho\,dxdt.$$

On the one hand, in view of (11) note that

$$
\begin{aligned}
\big[- D_p H(x, Du) \cdot Du^- - H(x, Du) \big] \chi_{\{u<0\}} &= L(D_p H(x, -Du^-)) \chi_{\{u<0\}} \\
&\geq \big[C_L^{-1} |D_p H(x, Du)|^{\gamma'} - C_L \big] \chi_{\{u<0\}},
\end{aligned}
$$

and on the other hand by Hölder's inequality

$$
\begin{aligned}
\int_{\mathbb{T}^d} u^-(\tau) \rho(\tau) \, dx &+ C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du)|^{\gamma'} \chi_{\{u<0\}} \rho \, dx dt \\
-C_L \iint_{Q_\tau} &\chi_{\{u<0\}} \rho \, dx dt \\
&\leq \|u_0^-\|_{L^p(\mathbb{T}^d)} \|\rho(0)\|_{L^{p'}(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)} \|\rho\|_{L^{q'}(Q_\tau)}.
\end{aligned} \tag{14}
$$

Then, plugging (13) into (14) we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^d} u^-(\tau) \rho(\tau) \, dx &+ C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du)|^{\gamma'} \chi_{\{u<0\}} \rho \, dx dt \\
\leq C_1 \left(\|u_0^-\|_{L^p(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)} \right) &\left(\iint_{Q_\tau} |D_p H(x, Du)|^{\frac{d+2}{q}} \chi_{\{u<0\}} \rho \, dx dt + 1 \right) \\
+ C_L \iint_{Q_\tau} \chi_{\{u<0\}} \rho \, dx dt.&
\end{aligned} \tag{15}
$$

Since $q > \frac{d+2}{\gamma'}$ one can use Young's inequality and the fact that $\int \rho(t) dx \leq \|\rho(t)\|_{L^{p'}(\mathbb{T}^d)} \leq 1$ to get the desired inequality (after passing to the limit $k \to \infty$). □

Combining (10) and (12), we get the following estimate in the case $q > \frac{d+2}{\gamma'}$.

Corollary 3.4 *Assume that (H) holds. Let u be a strong solution to (HJ) in $W_q^{2,1}(Q_T)$, $q > \frac{d+2}{\gamma'}$, and assume that*

$$
\|u_0\|_{L^p(\mathbb{T}^d)}, \ \|f\|_{L^q(Q_T)} \leq K.
$$

Then, there exists a positive constant C (depending on K, C_H, q, d, T) such that

$$
\sup_{t \in [0,T]} \|u(t)\|_{L^p(\mathbb{T}^d)} \leq C, \tag{16}
$$

where $p = \frac{dq}{(d+2)-2q}$ if $q < \frac{d+2}{2}$, while $p = \infty$ if $q > \frac{d+2}{2}$.

Note that the previous result does not cover the critical case $q = \frac{d+2}{\gamma'}$. The rest of the section is devoted to this endpoint situation, and more precise information on the stability of solutions in L^p will be obtained. This will be crucial in the subsequent analysis of maximal regularity. It is worth noting that the constants appearing in the estimates below will not depend just on the norms $\|f\|_{L^q}$, $\|u_0\|_{L^p}$, but on finer properties of f in L^q and u_0 in L^p; see Remark 3.6.

Let $\Gamma(x, t)$ be the fundamental solution of the heat equation on $\mathbb{T}^d \times (0, \infty)$. Consider, for $k > 0$, the solution u_k to the problem with truncated / regularized data, i.e.

$$\begin{cases} \partial_t u_k(x, t) - \Delta u_k(x, t) + H(x, Du_k(x, t)) = T_k\big(f(x, t)\big) & \text{in } Q_T = \mathbb{T}^d \times (0, T), \\ u_k(x, 0) = u_0(\cdot) \star \Gamma(\cdot, 1/k)\,(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{17}$$

where $\star$ stands for the standard convolution computed in space only, i.e. $f_1 \star f_2(x) = \int_{\mathbb{T}^d} f_1(x - y) f_2(y)\, dy$. In other words, u_k solves an Hamilton–Jacobi equation with L^∞ right-hand side and an initial datum in C^∞. Such an initial datum is actually $u_k(x, 0) = z(x, 1/k)$, where z is the solution to the heat equation

$$\begin{cases} \partial_t z(x, t) - \Delta z(x, t) = 0 & \text{in } Q_T = \mathbb{T}^d \times (0, T), \\ z(x, 0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases}$$

and converges to u_0 in $W^{2-2/q,q}(\mathbb{T}^d)$ as $k \to \infty$. The existence and uniqueness of a strong solution $u_k \in W_p^{2,1}(Q_T)$ for all $p \geq 1$ can be obtained using, for example, the results in [20]. We prove now the estimates on $u - u_k$.

Proposition 3.5 *Assume that ([H]) holds. Let u and u_k be strong solutions to ([HJ]) and ([17]) respectively, $\gamma < 2$ and $q = \frac{d+2}{\gamma'}$. Then, there exists a positive constant C (depending on f, u_0, C_H, q, d, T) such that*

$$\sup_{t \in [0,T]} \|u(t) - u_k(t)\|_{L^p(\mathbb{T}^d)} \leq C\big(\|f - T_k(f)\|_{L^q(Q_T)} + \|u_0 - u_0 \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)}\big),$$

$$\tag{18}$$

where $p = \frac{dq}{(d+2)-2q} = d\frac{\gamma-1}{2-\gamma}$.

Proof Let $w = u - u_k$. Note that w depends of course on k, but we will drop the subscript for simplicity.

As before, we argue by duality, and estimate w^+ first. Fix $\tau \in (0, T]$ and $\rho = \rho_k$ be the weak nonnegative solution of

$$\begin{cases} -\partial_t \rho(x, t) - \Delta \rho(x, t) + \mathrm{div}\big(D_p H(x, Du_k(x, t))\rho(x, t)\big) = 0 & \text{in } Q_\tau, \\ \rho(x, \tau) = \dfrac{\big[w^+(x, \tau)\big]^{p-1}}{\|w^+(\tau)\|_p^{p-1}} & \text{in } \mathbb{T}^d. \end{cases}$$

Note that as in previous lemmas, one should further truncate $\rho(\tau)$ to ensure the existence of ρ in an energy space (cf Proposition 2.1), and then pass to the limit, but we will omit this step for brevity.

Step 1: bounds on $\iint |D_p H(x, Du_k)|^{\gamma'} \rho$. Testing ([17]) against ρ, and testing the equation for ρ against u_k we get that

$$\iint_{Q_\tau} L\big(x, D_p H(x, Du_k)\big)\rho \, dxdt = -\iint_{Q_\tau} T_k(f)\rho \, dxdt - \int_{\mathbb{T}^d} u_k(0)\rho(0) \, dx$$

$$+ \int_{\mathbb{T}^d} u_k(\tau)\rho(\tau) \, dx. \qquad (19)$$

Recall that for all t, $\int_{\mathbb{T}^d} \rho(t)dx = \int_{\mathbb{T}^d} \rho(\tau)dx \le \|\rho(\tau)\|_{L^{p'}(\mathbb{T}^d)} = 1$. Then, for $h > 0$ that will be chosen below,

$$-\iint_{Q_\tau} T_k(f)\rho \, dxdt \le \iint_{Q_\tau} f^- \rho \, dxdt \le \iint_{Q_\tau \cap \{f^- \ge h\}} f^- \rho \, dxdt + h\tau$$

$$\le \|f^- \chi_{\{f^- \ge h\}}\|_{L^q(Q_\tau)} \|\rho\|_{L^{q'}(Q_\tau)} + hT.$$

Similarly, for any $h_0 > 0$, by Young's inequality for convolutions and Hölder inequality,

$$-\int_{\mathbb{T}^d} u_k(0)\rho(0) \, dx = -\int_{\mathbb{T}^d} u_0 \, \Gamma(1/k) \star \rho(0)$$

$$\le \int_{\mathbb{T}^d \cap \{u_0^- \ge h_0\}} u_0^- \, \Gamma(1/k) \star \rho(0) \, dx + h_0 \int_{\mathbb{T}^d} \Gamma(1/k) \star \rho(0) dx$$

$$\le \|u_0^- \chi_{\{u_0^- \ge h_0\}}\|_{L^p(\mathbb{T}^d)} \|\Gamma(1/k) \star \rho(0)\|_{L^{p'}(\mathbb{T}^d)} + h_0 \int_{\mathbb{T}^d} \Gamma(1/k) dx \int_{\mathbb{T}^d} \rho(0) dx$$

$$\le \|u_0^- \chi_{\{u_0^- \ge h_0\}}\|_{L^p(\mathbb{T}^d)} \|\rho(0)\|_{L^{p'}(\mathbb{T}^d)} + h_0.$$

Finally, applying Lemma 3.2 to u_k and noting that $(u_0 \star \Gamma(1/k))^+ \le u_0^+ \star \Gamma(1/k)$ by the comparison principle,

$$\int_{\mathbb{T}^d} u_k(\tau)\rho(\tau) \, dx \le \|u_k^+(\tau)\|_{L^p(\mathbb{T}^d)} \|\rho(\tau)\|_{L^{p'}(\mathbb{T}^d)}$$

$$\le \|(u_0 \star \Gamma(1/k))^+\|_{L^p(\mathbb{T}^d)} + C_0 \|T_k(f)^+\|_{L^q(Q_\tau)} \le \|u_0^+\|_{L^p(\mathbb{T}^d)} + C_0 \|f^+\|_{L^q(Q_\tau)}$$

Plugging the previous inequalities back in (19), and using the bounds from below on L, we then get

$$C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du_k)|^{\gamma'} \rho \, dxdt - C_L \iint_{Q_\tau} \rho \, dxdt$$

$$\le hT + h_0 + \|u_0^+\|_{L^p(\mathbb{T}^d)} + C_0 \|f^+\|_{L^q(Q_\tau)}$$

$$+ \big(\|u_0^- \chi_{\{u_0^- \ge h_0\}}\|_{L^p(\mathbb{T}^d)} + \|f^- \chi_{\{f^- \ge h\}}\|_{L^q(Q_\tau)}\big)\big(\|\rho(0)\|_{L^{p'}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)}\big)$$

Then Corollary 2.3 yields bounds on ρ, i.e.

$$C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du_k)|^{\gamma'} \rho \, dxdt \leq (h + C_L)T + h_0 + \|u_0^+\|_{L^p(\mathbb{T}^d)} + C_0 \|f^+\|_{L^q(Q_\tau)}$$

$$+ C_1 \big(\|u_0^- \chi_{\{u_0^- \geq h_0\}}\|_{L^p(\mathbb{T}^d)} + \|f^- \chi_{\{f^- \geq h\}}\|_{L^q(Q_\tau)} \big)$$

$$\left(\iint_{Q_\tau} |D_p H(x, Du_k)|^{\gamma'} \rho \, dxdt + 1 \right)$$

for some C_1 depending only on T, d, γ. Finally, h and h_0 are chosen large enough so that

$$\|u_0^- \chi_{\{u_0^- \geq h_0\}}\|_{L^p(\mathbb{T}^d)} + \|f^- \chi_{\{f^- \geq h\}}\|_{L^q(Q_\tau)} \leq \frac{1}{2C_1 C_L},$$

that gives

$$\iint_{Q_\tau} |D_p H(x, Du_k)|^{\gamma'} \rho \, dxdt \leq 2C_L[(h + C_L)T + h_0 + \|u_0^+\|_{L^p(\mathbb{T}^d)}$$

$$+ C_0 \|f^+\|_{L^q(Q_\tau)}] + 1 =: \overline{C}. \tag{20}$$

Step 2: bounds on w^+. Taking the difference between (17) and (HJ), by the convexity of $H(x, \cdot)$, w^+ is a weak subsolution of

$$\partial_t w^+(x, t) - \Delta w^+(x, t) + D_p H(x, Du_k(x, t)) \cdot Dw^+ \leq \big[f - T_k \big(f(x, t) \big) \big] \chi_{\{w > 0\}}$$

Testing against ρ, and testing the equation for ρ against w^+ gives

$$\|w^+(\tau)\|_{L^p(\mathbb{T}^d)} = \int_{\mathbb{T}^d} w^+(\tau)\rho(\tau) \, dx \leq \int_{\mathbb{T}^d} w^+(0)\rho(0) \, dx$$

$$+ \iint_{Q_\tau} \big[f - T_k \big(f(x, t) \big) \big] \chi_{\{w > 0\}} \rho \, dxdt.$$

Hence, using Corollary 2.3 and the estimate (20) we obtain

$$\|w^+\|_{L^p(\mathbb{T}^d)} \leq C_1 \big(\|w^+(0)\|_{L^p(\mathbb{T}^d)} + \|f - T_k(f)\|_{L^q(\mathbb{T}^d)} \big)$$

$$\left(\iint_{Q_\tau} |D_p H(x, Du_k)|^{\gamma'} \rho \, dxdt + 1 \right)$$

$$\leq C_1 (\overline{C} + 1) \big(\|u_0 - u_0 \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)} + \|f - T_k(f)\|_{L^q(\mathbb{T}^d)} \big),$$

which is "half" of the desired estimate.

To get an analogous estimate for $\|w^-\|_{L^p(\mathbb{T}^d)}$, which allows to conclude since $w = u - u_k$, one can proceed as in Step 1 and 2, noting that w^- satisfies

$$\partial_t w^-(x, t) - \Delta w^-(x, t) + D_p H(x, Du(x, t)) \cdot Dw^- \leq \big[T_k \big(f(x, t) \big) - f \big] \chi_{\{w < 0\}}.$$

To argue as before, it is sufficient to exploit properties of the dual problem

$$
\begin{cases}
-\partial_t \hat{\rho}(x,t) - \Delta \hat{\rho}(x,t) + \mathrm{div}\big(D_p H(x, Du(x,t))\hat{\rho}(x,t)\big) = 0 & \text{in } Q_\tau , \\
\hat{\rho}(x,\tau) = \dfrac{\big[w^-(x,\tau)\big]^{p-1}}{\|w^-(\tau)\|_p^{p-1}} & \text{in } \mathbb{T}^d.
\end{cases}
$$

$\square$

Remark 3.6 Note that (18) directly yields the estimate

$$
\sup_{t\in[0,T]} \|u(t)\|_{L^{d\frac{\gamma-1}{2-\gamma}}(\mathbb{T}^d)} \le C,
$$

thus extending (16) up to the critical case $q = (d+2)/\gamma'$. Let us focus on the the way C depends on f and u_0. When $q > (d+2)/\gamma'$, C depends on $\|f\|_{L^q}$ and $\|u_0\|_{L^p}$ only. At the endpoint $q = (d+2)/\gamma'$, the constant C above (and similarly the constant in (18)) is proportional to $\overline{C}$ defined in (20), that depends in turn on $\|u_0^+\|_{L^p(\mathbb{T}^d)}$, $\|f^+\|_{L^q(Q_\tau)}$, h, h_0, where h, h_0 are such that

$$
\|u_0^- \chi_{\{u_0^- \ge h_0\}}\|_{L^p(\mathbb{T}^d)} + \|f^- \chi_{\{f^- \ge h\}}\|_{L^q(Q_\tau)} \le \frac{1}{2C_1 C_L},
$$

and $C_1 = C_1(T, q, d)$ is as in Corollary 2.3. Thus, these constants *remain bounded when f and u_0 vary in bounded and equi-integrable sets in $L^q(Q_T)$ and $L^p(\mathbb{T}^d)$ respectively*. This is completely in line with the results in [49].

3.2 Bounds in Hölder Spaces

We now proceed with bounds on u in Hölder spaces, which will be obtained in particular when $\gamma \ge 2$, that is when q is necessarily greater than $\frac{d+2}{2}$ (and therefore u is continuous).

Proof of Theorem 1.2 **Step 1.** Since we have the representation $H(x, Du(x,t)) = \sup_{v\in\mathbb{R}^d}\{v \cdot Du(x,t) - L(x,v)\}$ for a.e. $(x,t) \in Q_T$, we get

$$
\int_0^\tau \langle \partial_t u(t), \varphi(t)\rangle dt + \iint_{Q_\tau} \partial_i u(x,t)\, \partial_i (\varphi(x,t))
$$
$$
+ [\Xi(x,t) \cdot Du(x,t) - L(x, \Xi(x,t))]\varphi \, dxdt
$$
$$
\le \iint_{Q_\tau} f(x,t)\varphi(x,t)\, dxdt \tag{21}
$$

for any measurable $\Xi : Q_\tau \to \mathbb{R}^d$ such that $L(\cdot, \Xi(\cdot, \cdot)) \in L^r(Q_\tau)$ and $\Xi \cdot Du \in L^r(Q_\tau)$, $r > 1$, and test function $\varphi \in \mathcal{H}_2^1(Q_\tau) \cap L^{r'}(Q_\tau)$. The previous inequality becomes an equality if $\Xi(x,t) = D_p H(x, Du(x,t))$ in Q_τ.

Fix now any $\tau \in [0, T]$, and let $\bar{x}, \bar{y} \in \mathbb{T}^d$ be such that

$$
u(\bar{y}) - u(\bar{x}) = |\bar{y} - \bar{x}|^\alpha \cdot [u(\cdot, \tau)]_{C^\alpha(\mathbb{T}^d)}.
$$

Let ρ_τ be any smooth nonnegative function satisfying $\int_{\mathbb{T}^d} \rho_\tau = 1$, and $\rho \in \mathcal{H}_2^1(Q_\tau) \cap L^{r'}(Q_\tau)$ (for all $r' > 1$) be the solution to

$$\begin{cases} -\partial_t \rho(x,t) - \Delta \rho(x,t) + \mathrm{div}\big(D_p H(x, Du(x,t))\rho(x,t)\big) = 0 & \text{in } Q_\tau, \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d. \end{cases}$$

Use now (21) with $\Xi(x,t) = D_p H(x, Du(x,t))$ and $\varphi = \rho$, and $u \in \mathcal{H}_2^1(Q_T)$ as a test function for the equation satisfied by ρ to get

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx = \int_{\mathbb{T}^d} u_0(x)\rho(x,0)dx + \iint_{Q_\tau} f(x,t)\rho(x,t)dxdt$$
$$+ \iint_{Q_\tau} L\big(x, D_p H(x, Du(x,t))\big)\rho(x,t)dxdt. \quad (22)$$

Setting $\xi = \bar{y} - \bar{x}$, one can easily check that $\hat{\rho}(x,t) := \rho(x - \xi, t)$ satisfies

$$\begin{cases} -\partial_t \hat{\rho}(x,t) - \Delta \hat{\rho}(x,t) + \mathrm{div}\big(D_p H(x - \xi, Du(x - \xi, t))\hat{\rho}(x,t)\big) = 0 & \text{in } Q_\tau, \\ \hat{\rho}(x,\tau) = \rho_\tau(x - \xi) & \text{in } \mathbb{T}^d. \end{cases}$$

As before, plugging $\Xi(x,t) = D_p H(x - \xi, Du(x - \xi, t))$ and $\varphi = \hat{\rho}$ into (21), and using $u \in \mathcal{H}_2^1(Q_T)$ as a test function for the equation satisfied by $\hat{\rho}$ yields

$$\int_{\mathbb{T}^d} u(x,\tau)\hat{\rho}_\tau(x)dx \leq \int_{\mathbb{T}^d} u_0(x)\hat{\rho}(x,0)dx$$
$$+ \iint_{Q_\tau} L(x, D_p H(x - \xi, Du(x - \xi, t)))\hat{\rho}\, dxdt + \iint_{Q_\tau} f\hat{\rho}\, dxdt$$

which, after the change of variables $x - \xi \mapsto x$, becomes

$$\int_{\mathbb{T}^d} u(x + \xi, \tau)\rho_\tau(x)dx \leq \int_{\mathbb{T}^d} u_0(x + \xi)\rho(x,0)dx$$
$$+ \iint_{Q_\tau} L(x + \xi, D_p H(x, Du(x,t)))\rho\, dxdt + \iint_{Q_\tau} f\hat{\rho}\, dxdt. \quad (23)$$

Taking the difference between (23) and (22) we obtain

$$\int_{\mathbb{T}^d} \big(u(x + \xi, \tau) - u(x, \tau)\big)\rho_\tau(x)dx \leq \int_{\mathbb{T}^d} \big(u_0(x + \xi) - u_0(x)\big)\rho(x,0)dx$$
$$+ \iint_{Q_\tau} \Big(L(x + \xi, D_p H(x, Du(x,t))) - L(x, D_p H(x, Du(x,t)))\Big)\rho(x,t)\, dxdt$$
$$+ \iint_{Q_\tau} f(x,t)\big(\rho(x - \xi, t) - \rho(x,t)\big)\, dxdt. \quad (24)$$

Step 2. To estimate the terms appearing in the right hand side of (24), we first derive bounds on ρ. We stress that constants $C, C_1, \ldots$ below are not going to depend on τ and ρ_τ. Rearranging (22) we have

$$\iint_{Q_\tau} L\big(x, D_p H(x, Du(x,t))\big)\rho(x,t)dxdt$$

$$= \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} u_0(x)\rho(x,0)dx - \iint_{Q_\tau} f(x,t)\rho(x,t)dxdt,$$

and by (11) and the bounds on $\|u\|_\infty$ of Proposition 3.4 we get

$$C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du(x,t))|^{\gamma'}\rho(x,t)dxdt \leq C + \|f\|_{L^q(Q_\tau)}\|\rho\|_{L^{q'}(Q_\tau)}.$$

Using Corollary 2.3,

$$C_L^{-1} \iint_{Q_\tau} |D_p H(x, Du(x,t))|^{\gamma'}\rho(x,t)dxdt \leq C$$

$$+ C_1\|f\|_{L^q(Q_\tau)} \left(\iint_{Q_\tau} |D_p H(x, Du(x,t))|^{\frac{d+2}{q}}\rho\, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)} \right).$$

This provides a control on $\iint_{Q_\tau} |D_p H(Du)|^{\gamma'}\rho$, and by means of Proposition 2.2,

$$\iint_{Q_\tau} |D_p H(x, Du(x,t))|^{\gamma'}\rho(x,t)dxdt + \|\rho\|_{\mathcal{H}^1_{\frac{d+2}{d+3-\gamma'}}(Q_\tau)} \leq C_2. \qquad (25)$$

Step 3. First, recall that $\int_{\mathbb{T}^d} \rho(0) = 1$,

$$\int_{\mathbb{T}^d} \big(u_0(x+\xi) - u_0(x)\big)\rho(x,0)dx \leq |\xi|^\alpha [u_0]_{C^\alpha(\mathbb{T}^d)}.$$

As for the second term in (24), $L(x,v) = \sup_{p \in \mathbb{R}^d}\{v \cdot p - H(x,p)\}$, and if $v = D_p H(x,p)$, then $L(x,v) = v \cdot p - H(x,p)$, hence

$$L(x+\xi, D_p H(x, Du(x,t))) - L(x, D_p H(x, Du(x,t)))$$
$$\leq H(x, Du(x,t)) - H(x+\xi, Du(x,t)).$$

Next, using (H$_\alpha$),

$$\iint_{Q_\tau} \Big(L(x+\xi, D_p H(x, Du(x,t))) - L(x, D_p H(x, Du(x,t))) \Big)\rho(x,t)\, dxdt$$

$$\leq \iint_{Q_\tau} \Big(H(x, Du(x,t)) - H(x+\xi, Du(x,t)) \Big)\rho(x,t)\, dxdt$$

$$\leq C_H |\xi|^\alpha \iint_{Q_\tau} \Big(|D_p H(x, Du(x, t))|^{\gamma'} + 1 \Big) \rho(x, t)\, dxdt.$$

Finally, we apply the embeddings of Propositions A.3 and A.2 (with $\delta = q'$, $p = \frac{d+2}{d+3-\gamma'}$ and hence $\alpha = \gamma' - \frac{d+2}{q}$) to get

$$\left| \iint_{Q_\tau} f(x, t)\big(\rho(x - \xi, t) - \rho(x, t)\big)\, dxdt \right|$$

$$\leq |\xi|^\alpha \iint_{Q_\tau} |f(x, t)| \frac{|\big(\rho(x - \xi, t) - \rho(x, t)\big)|}{|h|^\alpha}\, dxdt \leq |\xi|^\alpha \|f\|_{L^q(Q_\tau)} \|\rho\|_{L^{q'}(N^{\alpha, q'}(\mathbb{T}^d))}$$

$$\leq C|\xi|^\alpha \|f\|_{L^q(Q_\tau)} \|\rho\|_{L^{q'}(W^{\alpha, q'}(\mathbb{T}^d))}$$

$$\leq |\xi|^\alpha \|f\|_{L^q(Q_\tau)} \|\rho\|_{\mathcal{H}^1_{\frac{d+2}{d+3-\gamma'}}(Q_\tau)}.$$

Plugging now all the estimates in (24) and using (25) we obtain

$$\int_{\mathbb{T}^d} \big(u(x + \xi, \tau) - u(x, \tau)\big) \rho_\tau(x) dx \leq C_1 |\xi|^\alpha.$$

It is now sufficient to recall that ρ_τ can be any smooth nonnegative function satisfying $\int_{\mathbb{T}^d} \rho_\tau = 1$, so

$$|\bar{y} - \bar{x}|^\alpha \cdot [u(\cdot, \tau)]_{C^\alpha(\mathbb{T}^d)} = u(\bar{y}) - u(\bar{x}) \leq C_1 |\bar{y} - \bar{x}|^\alpha$$

and we have the assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.7 Adding an additional time localization term in the previous procedure, as in [20], it is possible to obtain Hölder bounds that are independent of the initial datum, but just depend on the sup-norm of the solution. This indicates that the equation regularizes at Hölder scales, and the weak solutions (in an appropriate sense) become instantaneously Hölder continuous at positive times.

4 Maximal L^q-Regularity

We start with a straightforward consequence of parabolic regularity results for linear equations.

Proposition 4.1 *Assume that (H) holds. Let u be a strong solution to (HJ) in $W_q^{2,1}(Q_T)$. Then,*

$$\|u\|_{W_q^{2,1}(Q_T)} \leq C(\|Du\|_{L^{\gamma q}(Q_T)}^\gamma + \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)} + 1) \qquad (26)$$

for some positive constant C depending on q, d, C_H.

Proof The proof is an easy consequence of well-known Caldèron-Zygmund type maximal regularity results for heat equations with potential

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = V(x,t) & \text{in } Q_\tau , \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases}$$

which satisfies the estimate (see [41,42], or [38] and the references therein)

$$\|u\|_{W_q^{2,1}(Q_T)} \le C(\|V\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}).$$

To get (26), it is now sufficient to choose $V = -H(x, Du) + f$, and use the assumption (H). $\qquad\square$

We now proceed with our main result on maximal regularity for (HJ) when $q > (d+2)/\gamma'$.

Proof of Theorem 1.1 To prove the assertion, we will combine the estimates derived in Section 3 with Gagliardo-Nirenberg type interpolation inequalities.

The subquadratic case $\gamma < 2$. We start from Proposition 3.4, which gives

$$\sup_{t \in [0,T]} \|u(t)\|_{L^s(\mathbb{T}^d)} \le C, \tag{27}$$

for any $s \le p = \frac{dq}{(d+2)-2q}$ if $q < \frac{d+2}{2}$, while $s \le \infty$ if $q > \frac{d+2}{2}$. Recall then the classical Gagliardo-Nirenberg inequality ([56, Lecture II Theorem p.125-126])

$$\|Du(t)\|_{L^{\gamma q}(\mathbb{T}^d)} \le C_1 \|u(t)\|_{W^{2,q}(\mathbb{T}^d)}^{\theta} \|u(t)\|_{L^s(\mathbb{T}^d)}^{1-\theta} \tag{28}$$

for $s \in [1, \infty]$ and $\theta \in [1/2, 1)$ satisfying

$$\frac{1}{\gamma q} = \frac{1}{d} + \theta \left(\frac{1}{q} - \frac{2}{d} \right) + (1-\theta)\frac{1}{s}.$$

Note that since $q > \frac{d+2}{\gamma'}$, we have $p > \frac{d(\gamma-1)}{2-\gamma}$, and therefore it is possible to choose s (close to $\frac{d(\gamma-1)}{2-\gamma}$) so that $\theta \in [1/2, 1/\gamma)$ and (27) and (28) holds. Then, raising (28) to γq and integrating on $(0, T)$ yields

$$\int_0^T \|Du(t)\|_{L^{\gamma q}(\mathbb{T}^d)}^{\gamma q} dt \le C_1^{\gamma q} \left(\sup_{t \in [0,T]} \|u(t)\|_{L^s(\mathbb{T}^d)}^{1-\theta} \right)^{\gamma q} \int_0^T \|u(t)\|_{W^{2,q}(\mathbb{T}^d)}^{\gamma \theta q} dt,$$

and since $\gamma\theta < 1$,

$$\|Du\|_{L^{\gamma q}(Q_T)} \le C_2 \|u\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))}^{\theta} \|u\|_{L^\infty(0,T;L^s(\mathbb{T}^d))}^{1-\theta}. \tag{29}$$

Plugging (27) and (29) into (26) we obtain

$$\|u\|_{W_q^{2,1}(Q_T)} \le C_3 (\|u\|_{W_q^{2,1}(Q_T)}^{\theta\gamma} + \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}),$$

and we conclude the assertion because $\theta\gamma < 1$.

The superquadratic case $\gamma \ge 2$. We start from Hölder bounds of Theorem 1.2, namely,

$$\sup_{t\in[0,T]} \|u(t)\|_{C^\alpha(\mathbb{T}^d)} \le C_1,$$

where $\alpha = \gamma' - \frac{d+2}{q}$ (or $\alpha \in (0,1)$ when $q \ge \frac{d+2}{\gamma'-1}$), and invoke the following Miranda-Nirenberg interpolation inequality (see [37,50,53,57])

$$\|Du(t)\|_{L^{\gamma q}(\mathbb{T}^d)} \le C\|u(t)\|_{W^{2,q}(\mathbb{T}^d)}^\theta \|u(t)\|_{C^\alpha(\mathbb{T}^d)}^{1-\theta},$$

where $\theta \in \left[\frac{1-\alpha}{2-\alpha}, 1\right)$ satisfies

$$\frac{1}{\gamma q} = \frac{1}{d} + \theta\left(\frac{1}{q} - \frac{2}{d}\right) - (1-\theta)\frac{\alpha}{d}.$$

Choosing $\theta = \frac{1-\alpha}{2-\alpha}$ (or α close enough to 1 when $q \ge \frac{d+2}{\gamma'-1}$), we have $\theta\gamma < 1$ if and only if

$$q > \frac{(d+2)(\gamma-1)}{2}.$$

Hence,

$$\|Du\|_{L^{\gamma q}(Q_T)} \le CC_1^{1-\theta}\|u\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))}^\theta.$$

Plugging this inequality into (26) and using the fact that $\gamma\theta < 1$, we conclude. $\quad\square$

We now consider the maximal regularity problem in the limiting case $q = (d+2)/\gamma'$. The scheme of the proof is similar to the one of Theorem 1.1, but requires an additional step involving solutions u_k to the regularized problem (17), that is; for $k > 0$,

$$\begin{cases} \partial_t u_k(x,t) - \Delta u_k(x,t) + H(x, Du_k(x,t)) = T_k\big(f(x,t)\big) \text{ in } Q_T = \mathbb{T}^d \times (0,T), \\ u_k(x,0) = u_0(\cdot)\star\Gamma(\cdot,1/k)(x) \qquad\qquad\qquad \text{in } \mathbb{T}^d. \end{cases}$$
$$(30)$$

Proof of Theorem 1.3 Let $w = u - u_k$, k to be chosen. From Proposition 3.5, we have the existence of C depending on f, u_0, C_H, q, d, T such that

$$\sup_{t\in[0,T]} \|w(t)\|_{L^p(\mathbb{T}^d)} \leq C\big(\|f - T_k(f)\|_{L^q(Q_T)} + \|u_0 - u_0\star\Gamma(1/k)\|_{L^p(\mathbb{T}^d)}\big),$$

$$p = d\frac{\gamma - 1}{2 - \gamma}.$$

The Gagliardo–Nirenberg inequality reads

$$\|Dw(t)\|_{L^{\gamma q}(\mathbb{T}^d)} \leq C_2 \|w(t)\|_{W^{2,q}(\mathbb{T}^d)}^{\frac{1}{\gamma}} \|w(t)\|_{L^p(\mathbb{T}^d)}^{1-\frac{1}{\gamma}},$$

where C_2 depends on d, p, q. Thus,

$$\|Dw\|_{L^{\gamma q}(Q_T)}^\gamma \leq C_2^\gamma C^{\gamma-1}\big(\|f - T_k(f)\|_{L^q(Q_T)} + \|u_0 - u_0\star\Gamma(1/k)\|_{L^p(\mathbb{T}^d)}\big)^{\gamma-1}$$
$$\|w\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))}. \tag{31}$$

Note now that w solves a.e. on Q_T

$$\partial_t w(x,t) - \Delta w(x,t) = H(x, Du_k(x,t)) - H(x, Du(x,t)) + f(x,t)$$
$$-T_k\big(f(x,t)\big), \tag{32}$$

and that, by assumptions on H, $|D_p H(x,p)| \leq C'_H(|p|^{\gamma-1} + 1)$, so by Young's inequality

$$|H(x, Du_k(x,t)) - H(x, Du(x,t))|$$
$$\leq |Dw(x,t)| \cdot \max\{|D_p H(x, Du_k(x,t))|, |D_p H(x, Du(x,t))|\}$$
$$\leq C_3(|Du_k(x,t)|^\gamma + |Du(x,t)|^\gamma + |Dw(x,t)|^\gamma + 1)$$
$$\leq C_4(|Du_k(x,t)|^\gamma + |Dw(x,t)|^\gamma + 1),$$

where C_3, C_4 depend on C_H only. Then, by maximal regularity applied to the linear equation (32),

$$\|w\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))}$$
$$\leq C_5\bigg(\|H(x, Du_k) - H(x, Du)\|_{L^q(Q_T)} + \|f - T_k(f)\|_{L^q(Q_T)}$$
$$+ \|u_0 - u_0\star\Gamma(1/k)\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}\bigg)$$
$$\leq C_6\|Dw\|_{L^{\gamma q}(Q_T)}^\gamma + C_6\bigg(\|Du_k\|_{L^{\gamma q}(Q_T)}^\gamma + \|f - T_k(f)\|_{L^q(Q_T)}$$
$$+ \|u_0 - u_0\star\Gamma(1/k)\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)} + 1\bigg),$$

where C_6 depends on C_H, d, q. Plugging now this inequality into (31) yields

$$\|Dw\|_{L^{\gamma q}(Q_T)}^\gamma \leq C_6 C_2^\gamma C^{\gamma-1}\big(\|f - T_k(f)\|_{L^q(Q_T)} + \|u_0 - u_0\star\Gamma(1/k)\|_{L^p(\mathbb{T}^d)}\big)^{\gamma-1}$$
$$\|Dw\|_{L^{\gamma q}(Q_T)}^\gamma + \cdots.$$

Hence, we choose $\bar{k}$ large enough so that

$$C_6 C_2^{\gamma} C^{\gamma-1} \big(\|f - T_{\bar{k}}(f)\|_{L^q(Q_T)} + \|u_0 - u_0 \star \Gamma(1/\bar{k})\|_{L^p(\mathbb{T}^d)} \big)^{\gamma-1} \leq \frac{1}{2}, \quad (33)$$

to get

$$\begin{aligned}
\|Dw\|_{L^{\gamma q}(Q_T)}^{\gamma} \\
&\leq \|Du_{\bar{k}}\|_{L^{\gamma q}(Q_T)}^{\gamma} + \|f - T_{\bar{k}}(f)\|_{L^q(Q_T)} + \|u_0 - u_0 \star \Gamma(1/\bar{k})\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)} + 1 \\
&\leq \|Du_{\bar{k}}\|_{L^{\gamma q}(Q_T)}^{\gamma} + 2\|f\|_{L^q(Q_T)} + 2\|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)} + 1.
\end{aligned} \quad (34)$$

Since $u_{\bar{k}}(0)$ is smooth and $T_{\bar{k}}(f) \in L^\infty(Q_T)$, we can apply Theorem 1.1 to $u_{\bar{k}}$ solving (17) to estimate $Du_{\bar{k}}$. Indeed, pick any $\bar{q} > q$. Then,

$$\|T_{\bar{k}}(f)\|_{L^{\bar{q}}(Q_T)} + \|u_0 \star \Gamma(1/\bar{k})\|_{W^{2-\frac{2}{\bar{q}},\bar{q}}(\mathbb{T}^d)} \leq \bar{k} + C_5 \bar{k}^{\left(\frac{d}{2}+1\right)\left(\frac{1}{q}-\frac{1}{\bar{q}}\right)} \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$$

in view of standard decay estimates for the heat equation (C_5 depends on d, q, $\bar{q}$ only, see, e.g., [68, Chapter 15]). Therefore, by Theorem 1.1,

$$\|Du_{\bar{k}}\|_{L^{\gamma\bar{q}}(Q_T)}^{\gamma} \leq C_{\bar{k}}, \quad (35)$$

where $C_{\bar{k}}$ depends on $\bar{k}$, $\|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$, q, d, C_H, T. Actually, Du can be proven to be bounded in $L^\infty(Q_T)$, see [20]. It is now straightforward to conclude. Indeed,

$$\|Du\|_{L^{\gamma q}(Q_T)} \leq \|Dw\|_{L^{\gamma q}(Q_T)} + \|Du_{\bar{k}}\|_{L^{\gamma q}(Q_T)},$$

and the assertion follows by (34) and (35). □

Remark 4.2 We claim that C appearing in the statement of Theorem 1.3 remains bounded when

- f varies in a bounded and *equi-integrable* set $\mathcal{F} \subset L^q(Q_T)$, and
- u_0 varies in a bounded set $\mathcal{U}_0 \subset W^{2-2/q,q}(\mathbb{T}^d)$.

Indeed, in addition to $\|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$, q, d, T, C_H, the constant C crucially depends on $\bar{k}$ appearing in (33). This is chosen in the proof large enough so that

$$\big(\|f - T_{\bar{k}}(f)\|_{L^q(Q_T)} + \|u_0 - u_0 \star \Gamma(1/\bar{k})\|_{L^p(\mathbb{T}^d)} \big) \leq c.$$

In turn, $c = (2C_6 C_2^{\gamma} C^{\gamma-1})^{-1}$ is independent of $f \in \mathcal{F}$ and $u_0 \in \mathcal{U}_0$, since they vary in bounded and equi-integrable sets in $L^q(Q_T)$ and $L^p(\mathbb{T}^d)$ respectively, cf. Remark 3.6. Note that by Sobolev embeddings, the closure of $\mathcal{U}_0$ in L^p (with p, q as above) is compact in $L^p(\mathbb{T}^d)$, and hence weakly compact and L^p-equi-integrable by the Dunford–Pettis theorem, see [10, Theorem 4.30].

Hence, we just need to verify that for $c > 0$, there exists k independent of $u_0 \in \mathcal{U}_0$ such that

$$\|u_0 - u_0 \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)} \leq c.$$

This follows again by the compactness of $\mathcal{U}_0$ in L^p, which can be covered by finitely many balls $B_{c/3}(u_j)$ in $L^p(\mathbb{T}^d)$. Choosing k large so that

$$\|u_j - u_j \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)} \leq c/3 \quad \text{for all } j,$$

we get, for $u_0 \in B_{c/3}(u_j)$,

$$\|u_0 - u_0 \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)}$$
$$\leq \|u_0 - u_j\|_{L^p(\mathbb{T}^d)} + \|u_j - u_j \star \Gamma(1/k)\|_{L^p(\mathbb{T}^d)} + \|u_j - u_0\|_{L^p(\mathbb{T}^d)} \|\Gamma(1/k)\|_{L^1(\mathbb{T}^d)} \leq c.$$

Remark 4.3 One can implement the same scheme to handle more general Hamilton–Jacobi equations of the form

$$\partial_t u - \sum_{i,j} a_{ij}(x,t) \partial_{ij} u(x,t) + H(x, Du) = f(x,t)$$

where $A \in C([0,T]; W^{2,\infty}(\mathbb{T}^d))$ and $\lambda I_d \leq A \leq \Lambda I_d$ for $0 < \lambda \leq \Lambda$. In particular, one has to appropriately adjust the proofs of the integral and Hölder estimates, following [20], and use the linear maximal regularity results in [61].

5 Applications to Mean Field Games

For a given couple $u_T, m_0 \in C^3(\mathbb{T}^d)$, consider the MFG system

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = g(m(x,t)) & \text{in } Q_T \\ \partial_t m - \Delta m - \operatorname{div}(D_p H(x, Du)m) = 0 & \text{in } Q_T \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{T}^d. \end{cases}$$

5.1 The Monotone (or Defocusing) Case

Proof of Theorem 1.4 We argue that under the restrictions on r it is possible to prove a priori bounds on second-order derivatives of solutions to (MFG) (and beyond, assuming additional regularity of the data). These are typically enough to prove existence theorems. One may indeed set up a fixed-point method, or a regularization procedure, which consists in replacing $g(m)$ by $g(m \star \chi_\varepsilon) \star \chi_\varepsilon$ (where χ_ε is a sequence of standard symmetric mollifiers). The existence of a solution $(m_\varepsilon, u_\varepsilon)$ is then standard (see e.g. [35]). Since the bounds on $(m_\varepsilon, u_\varepsilon)$ do not depend on $\varepsilon > 0$, it is therefore possible to pass to the limit and obtain a solution to (MFG).

The key a priori bound is stated in the next Lemma 5.1. Once bounds for u in $W_q^{2,1}(Q_T)$, $q > (d+2)/\gamma'$, are established, one can indeed improve the estimates via a rather standard bootstrap procedure involving parabolic regularity for linear equations. Indeed, $D_pH(x, Du)$ turns out to be bounded in L^p for some $p > d + 2$, which is the usual Aronson-Serrin condition yielding space-time Hölder continuity of m on the whole cylinder (see e.g. [41, Theorem III.10.1]). We can then use Theorem 1.1 to conclude that u in $W_q^{2,1}(Q_T)$ for any $q > d + 2$. This immediately implies by the embeddings of $W_q^{2,1}(Q_T)$ that u is bounded in $C^{1+\delta, \frac{1+\delta}{2}}(Q_T)$ for any $\delta \in (0, 1)$. Then, we can regard the Hamilton–Jacobi equation as a heat equation with a space-time Hölder continuous source, and by [41, Section IV.5.1] conclude that u_ε is bounded in $C^{2+\delta', \frac{1+\delta'}{2}}(Q_T)$ independently of ε. One then goes back to the Fokker–Planck equation to deduce that m also enjoys $C^{2+\delta', \frac{1+\delta'}{2}}(Q_T)$ bounds. □

Below we state and prove the crucial a priori estimate on solutions to (MFG).

Lemma 5.1 *Let (u, m) be a classical solution to (MFG). Under the assumptions of Theorem 1.4, there exists a constant $C > 0$ such that*

$$\|u\|_{W_q^{2,1}(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \le C, \quad q > \frac{d+2}{\gamma'}$$

for some positive constant C (depending only on the data).

***Proof* Step 1.** First order estimates. These are standard (see e.g. [35, Proposition 6.6]), and easily obtained by testing the Hamilton–Jacobi equation with $m - m_0$ and the Fokker–Planck equation with $u - u_T$; using the standing assumptions on H and f one obtains

$$\iint_{Q_T} |Du|^\gamma m \, dx dt + \iint_{Q_T} m^{r+1} \, dx dt \le C_1. \tag{36}$$

Step 2. Second order estimates. These are obtained by testing the Hamilton–Jacobi equation with Δm and the Fokker–Planck equation with Δu:

$$\iint_{Q_T} \mathrm{Tr}(D_{pp}^2 H (D^2 u)^2) m \, dx dt + \iint_{Q_T} g'(m)|Dm|^2 \, dx dt$$
$$= \int_{\mathbb{T}^d} \Delta u_T m(T) \, dx - \int_{\mathbb{T}^d} \Delta u(0) m_0 \, dx$$
$$- 2 \iint_{Q_T} \mathrm{Tr}(D_{px}^2 H D^2 u) m \, dx dt - \iint_{Q_T} \Delta_x H(x, Du) m \, dx dt. \tag{37}$$

On one hand, $\iint \mathrm{Tr}(D_{pp}^2 H(D^2 u)^2) m \ge \iint [C_H^{-1}(1 + |Du|^2)^{\frac{\gamma-2}{2}} |D^2 u|^2 - C_H] m$ by (H2), while on the other hand by Young and Cauchy–Schwarz inequality

$$-2 \iint_{Q_T} \text{Tr}(D^2_{px} H D^2 u) m \, dxdt - \iint_{Q_T} \Delta_x H(x, Du) m \, dxdt$$

$$\leq \frac{C_H^{-1}}{2} \iint_{Q_T} (1 + |Du|^2)^{\frac{\gamma-2}{2}} |D^2 u|^2 m$$

$$+ c \left(\iint_{Q_T} H(x, Du) m \, dxdt + \iint_{Q_T} m \, dxdt \right),$$

therefore, back to (37), integrating by parts we obtain

$$\frac{C_H^{-1}}{2} \iint_{Q_T} (1 + |Du|^2)^{\frac{\gamma-2}{2}} |D^2 u|^2 m \, dxdt + \iint_{Q_T} g'(m) |Dm|^2 \, dxdt$$

$$= \int_{\mathbb{T}^d} \Delta u_T m(T) \, dx - \int_{\mathbb{T}^d} u(0) \Delta m_0 \, dx + c \iint_{Q_T} |Du|^\gamma m \, dxdt + cT.$$

Plugging in (36), using lower bounds on g' and the fact that $u \geq \min u_T + \min H(\cdot, 0)$ by the comparison principle, we finally get

$$\iint_{Q_T} (1 + |Du|^2)^{\frac{\gamma-2}{2}} |D^2 u|^2 m \, dxdt + \iint_{Q_T} |D(m^{\frac{r+1}{2}})|^2 \, dxdt \leq C_2. \qquad (38)$$

Step 3. Setting $b(x, t) = -D_p H(x, Du(x, t))$, by the assumptions on H, (36) and (38) (recall also that $|D_p H(x, p)| \geq c^{-1}|p|^{\gamma-1} - c$), we have

$$\iint_{Q_T} |\text{div}(b)|^2 m \, dxdt \leq C \quad \text{if } \gamma \leq 2$$

$$\iint_{Q_T} |\text{div}(b)|^2 (1 + |b|)^{\frac{2-\gamma}{\gamma-1}} m \, dxdt \leq C \quad \text{if } \gamma > 2,$$

and since m solves a Fokker–Planck equation with drift b, applying Lemma 5.2 with $\mu = 2$ if $\gamma \leq 2$ and $\mu = \gamma$ if $\gamma > 2$ yields

$$\|m\|_{L^\infty(0,T;L^\eta(\mathbb{T}^d))} \leq C_\eta \qquad \text{for any } 1 \leq \eta \leq \begin{cases} \dfrac{d}{d-2} & \text{if } \gamma \leq 2 \\ \dfrac{d(\gamma-1)}{d(\gamma-1)-2} & \text{if } \gamma > 2. \end{cases}$$

Therefore, using again (38), we have bounds on $m^{\frac{r+1}{2}}$ in $L^\infty(0, T; L^{\eta\frac{2}{r+1}}(\mathbb{T}^d)) \cap L^2(0, T; W^{1,2}(\mathbb{T}^d))$, which, by parabolic interpolation [30, Proposition I.3.2] imply

$$\|m^r\|_{L^q(\mathbb{T}^d)} \leq C_2 \qquad \text{for any } q < \begin{cases} 1 + \dfrac{d}{r(d-2)} & \text{if } \gamma \leq 2 \\ 1 + \dfrac{(d+2)(\gamma-1)-2}{r[d(\gamma-1)-2]} & \text{if } \gamma > 2. \end{cases}$$

Under our assumptions on r, the exponent q can be chosen large enough to apply Theorem 1.1, that yields the assertion. □

The following lemma is needed to extract regularity information on m.

Lemma 5.2 *Let m be a classical solution to*

$$\partial_t m - \Delta m + \operatorname{div}(b(x, t)m) = 0 \qquad in \ Q_T$$

and assume that for $\mu \geq 2$

$$\left(\iint_{Q_T} |\operatorname{div}(b)|^2 (1 + |b|)^{\frac{2-\mu}{\mu-1}} m \, dxdt \right)^{\frac{1}{2}} \left(\iint_{Q_T} |b|^{\frac{\mu}{\mu-1}} m \, dxdt \right)^{\frac{\mu-2}{2\mu}} \leq K .$$

Then, there exists a constant C depending on K, μ and d, such that

$$\|m(t)\|_{L^p(\mathbb{T}^d)} \leq C + \|m(0)\|_{L^p(\mathbb{T}^d)} \qquad \forall t \in [0, T],$$

where $p = \frac{d(\mu-1)}{d(\mu-1)-2}$ if $d > 2$, while $p \in [1, \infty)$ if $d \leq 2$.

Proof The estimate can be obtained by testing the equation against m^{p-1} and using parabolic interpolation (see [34, Theorem 4.1] for further details). We briefly sketch it here for completeness. Testing the equation and integrating by parts yields

$$\int_{\mathbb{T}^d} m^p(t)dx + \frac{4(p-1)}{p} \int_0^t \int_{\mathbb{T}^d} |Dm^{\frac{p}{2}}|^2 dxds$$

$$= \int_{\mathbb{T}^d} m^p(0)dx - (p-1) \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(b)m^p \, dxdt.$$

We write

$$\int_0^t \int_{\mathbb{T}^d} \operatorname{div}(b)m^p \, dxdt = \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(b)m^{\frac{1}{2}} |b|^{\frac{2-\mu}{2(\mu-1)}} |b|^{\frac{\mu-2}{2(\mu-1)}} m^{\frac{\mu(p-1)+1}{\mu}} m^{\frac{\mu-2}{2\mu}} \, dxdt$$

Therefore, applying generalized Holder's inequality with exponents $(2, \mu, \frac{2\mu}{\mu-2})$ we deduce

$$\int_{\mathbb{T}^d} m^p(t)dx + \frac{4(p-1)}{p} \int_0^t \int_{\mathbb{T}^d} |Dm^{\frac{p}{2}}(s)|^2 \, dxds$$

$$\leq \int_{\mathbb{T}^d} m^p(0)dx + (p-1)K \left(\int_0^t \int_{\mathbb{T}^d} m^{\mu(p-1)+1}(s) \, dxds \right)^{\frac{1}{\mu}}. \qquad (39)$$

We then apply parabolic interpolation inequalities (see e.g. [30, Proposition I.3.1])

$$\int_0^t \int_{\mathbb{T}^d} |z|^\zeta dxds \leq C \left(\sup_{s \in [0,t]} \int_{\mathbb{T}^d} |z|^2(s)dx \right)^{\frac{2}{d}} \int_0^t \int_{\mathbb{T}^d} |Dz|^2 dxds \quad \text{where } \zeta = 2\frac{d+2}{d},$$

with $z = m^{\frac{p}{2}}$, $p = \frac{d(\mu-1)}{d(\mu-1)-2}$, so that $\zeta = 2\frac{\mu(p-1)+1}{p}$, to obtain

$$\int_0^t \int_{\mathbb{T}^d} m^{\mu(p-1)+1}(s)\,dxds = \int_0^t \int_{\mathbb{T}^d} m^{\frac{p}{2}\zeta}\,dxds$$

$$\leq C \left(\int_0^t \int_{\mathbb{T}^d} |Dm^{\frac{p}{2}}|^2 dxds \right) \left(\sup_{s\in[0,t]} \int_{\mathbb{T}^d} m^p(s)dx \right)^{\frac{2}{d}}.$$

Then, using also Young's inequality, the right-hand side of (39) can be controlled as follows:

$$\int_{\mathbb{T}^d} m^p(0)dx + C \left(\int_0^t \int_{\mathbb{T}^d} |Dm^{\frac{p}{2}}(s)|^2 \, dxds \right)^{\frac{1}{\mu}} \left(\sup_{s\in[0,t]} \int_{\mathbb{T}^d} m^p(s)dx \right)^{\frac{2}{d\mu}}$$

$$\leq \int_{\mathbb{T}^d} m^p(0)dx + \frac{4(p-1)}{p} \int_0^t \int_{\mathbb{T}^d} |Dm^{\frac{p}{2}}|^2 dxds + C_1 \left(\sup_{s\in[0,t]} \int_{\mathbb{T}^d} m^p(s)dx \right)^{\frac{2}{d(\mu-1)}}.$$

If $d > 2$, then $\frac{2}{d(\mu-1)} < 1$ and we are done. If $d \leq 2$, the proof is somehow simpler, by different nature of Sobolev embeddings of $W^{1,2}(\mathbb{T}^d)$; we refer to [36]. □

5.2 The Nonmonotone (or Even Focusing) Case

Proof of Theorem 1.5 We detail only the derivation of the a priori estimates for smooth solutions to (MFG), i.e. the existence of a constant $C > 0$ (depending only on the data) such that

$$\|u\|_{W_q^{2,1}(Q_T)} + \|Du\|_{L^{\gamma q}(Q_T)} \leq C, \qquad q > \frac{d+2}{\gamma'} \tag{40}$$

Since we fall in the maximal regularity regime for the Hamilton–Jacobi equation, the existence of a solution to the MFG system can then be derived as in Theorem 1.4.

We start from first order estimates as in Lemma 5.1, that now yield, using the assumptions on g,

$$\iint_{Q_T} L(x, D_p H(x, Du))m\,dxdt - \iint_{Q_T} m^{r+1}\,dxdt \leq C_1.$$

To bound the two terms in the left hand side separately, a further step is needed. Recall a crucial Gagliardo-Nirenberg type inequality proven in [23, Proposition 2.5], that reads

$$\left(\iint_{Q_T} m^{r+1}(x,t)\,dxdt \right)^{\delta} \leq C \left(\iint_{Q_T} |D_p H(x, Du(x,t))|^{\gamma'}\,dxdt + 1 \right). \tag{41}$$

for some $C > 0$ and $\delta > 1$, provided that $r < \gamma'/d$. Then, using this inequality, the assumptions on L and the fact that $\int_{\mathbb{T}^d} m(t) = 1$ for all t,

$$C_L^{-1} \iint_{Q_T} |D_p H(x, Du(x, t))|^{\gamma'} dx dt - C_L T$$

$$\leq \iint_{Q_T} L\big(x, D_p H(x, Du)\big) m \, dx dt \leq C \left(\iint_{Q_T} |D_p H(x, Du(x, t))|^{\gamma'} dx dt + 1 \right)^{1/\delta}$$
$$+ C_1.$$

Hence, back to (41), we get

$$\|m\|_{L^{r+1}(Q_T)} \leq C_2. \tag{42}$$

Therefore, $\|g(m)\|_{L^{\frac{r+1}{r}}(Q_T)} \leq C_2$. Note that by the assumptions on r, $q = \frac{r+1}{r}$ is large enough to apply Theorem 1.1, and (40) follows. $\qquad \square$

Appendix A: Some Embedding Theorems

Lemma A.1 *For $p > 1$, the space $\mathcal{H}_p^1(Q_T)$ is continuously embedded into $C([0, T]; W^{1-2/p, p}(\mathbb{T}^d))$, and $W_p^{2,1}(Q_T)$ is continuously embedded into $C([0, T]; W^{2-2/p, p}(\mathbb{T}^d))$.*

Proof We consider the embedding of $\mathcal{H}_p^1$ only (the other can be obtained similarly). Recall that the case $p = 2$ is classical (see e.g. [28, Theorem XVIII.2.1]). The general statement $p \neq 2$ can be proven via abstract methods for evolution problems, see [1,61]. We provide a short proof for reader's convenience. First, $u \in \mathcal{H}_p^1(Q_T)$ can be extended to a $v \in \mathcal{H}_p^1(\mathbb{T}^d \times (0, \infty))$ in the usual way: let $u(t) = u(T)$ for all $t \geq T$, and set $v(t) = \zeta(t) u(t)$, where ζ is a smooth function in $(0, \infty)$ which vanishes for $t \geq T + 1$ and is identically one for $t \in [0, T]$. Then, since $\mathcal{H}_p^1(\mathbb{T}^d \times (0, +\infty)) \simeq$

$W^{1,p}(0,+\infty;(W^{1,p'}(\mathbb{T}^d))') \cap L^p(0,+\infty;W^{1,p}(\mathbb{T}^d))$, apply [48, Corollary 1.14] to obtain that

$$\mathcal{H}_p^1(\mathbb{T}^d \times (0,+\infty)) \hookrightarrow C_b([0,+\infty);(W^{-1,p}(\mathbb{T}^d),W^{1,p}(\mathbb{T}^d))_{1-1/p,p}).$$

One then concludes by means of the Reiteration Theorem [48, Theorem 1.23] (see also [45]) that

$$(W^{-1,p}(\mathbb{T}^d),W^{1,p}(\mathbb{T}^d))_{1-1/p,p} \simeq W^{1-2/p,p}(\mathbb{T}^d),$$

which gives the statement. $\qquad\square$

Proposition A.2 *For $p > 1$, the parabolic space $\mathcal{H}_p^1(Q_T)$ continuously embedded into $L^\delta(0,T;W^{\alpha,\delta}(\mathbb{T}^d))$, where $\delta > p$ and*

$$\alpha = 1 + \frac{d+2}{\delta} - \frac{d+2}{p}.$$

Proof We adapt a strategy presented in [39,40] (see also [19] for the Bessel potential spaces setting). Let $\theta = p/\delta \in (0,1)$ and $\nu = (1-2/p)(1-\theta)+\theta$. We now use the (real) interpolation in the Sobolev–Slobodeckii scale to observe that $W^{\nu,p}(\mathbb{T}^d)$ can be obtained by interpolation between $W^{1,p}(\mathbb{T}^d)$ and $W^{1-2/p,p}(\mathbb{T}^d)$ (see [71, Theorem 2.4.2 p. 186 and eq. (16)]). Moreover, $W^{\nu,p}(\mathbb{T}^d)$ is continuously embedded into $W^{\nu+d/\delta-d/p,\delta}(\mathbb{T}^d)$, see [62]. Hence, for a.e. t,

$$c(d,p,\delta)\,\|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{\delta},\delta}(\mathbb{T}^d)} \le \|u(t)\|_{W^{\nu,p}(\mathbb{T}^d)} \le \|u(t)\|_{W^{1-2/p,p}(\mathbb{T}^d)}^{1-\theta}\,\|u(t)\|_{W^{1,p}(\mathbb{T}^d)}^{\theta}.$$

Then, $\alpha = \nu - \frac{d}{p} + \frac{d}{\delta} = 1 + \frac{d}{\delta} - \frac{d+2(1-\theta)}{p}$ and

$$\left(\int_0^T \|u(t)\|_{W^{\alpha,\delta}(\mathbb{T}^d)}^{\frac{p}{\theta}}\,dt\right)^{\theta} \le C_1 \left(\int_0^T \|u(t)\|_{W^{1-2/p,p}(\mathbb{T}^d)}^{(1-\theta)\frac{p}{\theta}}\,\|u(t)\|_{W^{1,p}(\mathbb{T}^d)}^{p}\,dt\right)^{\theta}$$

$$\le C_2 \sup_{t\in[0,T]} \|u(t)\|_{W^{1-2/p,p}(\mathbb{T}^d)}^{(1-\theta)p}\left(\int_0^T \|u(t)\|_{W^{1,p}(\mathbb{T}^d)}^{p}\,dt\right)^{\theta},$$

Recalling now that $\theta = p/\delta \in (0,1)$ we obtain

$$\left(\int_0^T \|u(t)\|_{W^{\alpha,\delta}(\mathbb{T}^d)}^{\delta}\,dt\right)^{\frac{1}{\delta}} \le C_3 \|u\|_{\mathcal{H}_p^1(Q_T)}.$$

$\qquad\square$

Lemma A.3 *For $0 < \alpha < 1$ and $1 \le p < \infty$, $W^{\alpha,p}(\mathbb{T}^d)$ is continuously embedded into $N^{\alpha,p}(\mathbb{T}^d)$.*

Proof We just need to localize analogous results on $\mathbb{R}^d$, which go back to [67, Lemma 9 p. 441], see also [65, Proposition 10-(b) Section 5.2] and [45, Theorem 17.38] for a proof via real interpolation methods. Let χ be a compactly supported cut-off function such that $\chi \equiv 1$ on the unit cube $[-2, 2]^d$. It is immediate to see that the extension operator is bounded

$$W^{k,p}(\mathbb{T}^d) \ni u \longmapsto \tilde{u} = \chi u \in W^{k,p}(\mathbb{R}^d)$$

for all nonnegative integers $k \geq 0$ and $p \geq 1$, so it is bounded from $W^{\alpha,p}(\mathbb{T}^d)$ to $W^{\alpha,p}(\mathbb{R}^d)$ by interpolation. Then,

$$\|u\|_{N^{\alpha,p}(\mathbb{T}^d)} \leq C_1 \|\tilde{u}\|_{N^{\alpha,p}(\mathbb{R}^d)} \leq C_2 \|\tilde{u}\|_{W^{\alpha,p}(\mathbb{R}^d)} \leq C_3 \|u\|_{W^{\alpha,p}(\mathbb{T}^d)} \,,$$

where the second inequality relies on the aforementioned embedding $W^{\alpha,p}(\mathbb{R}^d) \hookrightarrow N^{\alpha,p}(\mathbb{R}^d) \simeq B^\alpha_{p\infty}(\mathbb{R}^d)$. $\qquad\square$

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