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EXTENDING AND APPLYING A LOGIC FOR PRAGMATICS

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1. Introduction: The Philosophy of the Logic for Pragmatics

Consider the speech acts of asserting, denying, hypothesising, conjecturing, doubting that *p*. We talk of them as *pragmatic acts*: asserting, denying, hypothesising, conjecturing, doubting that *p* are acts that can (though need not) be performed by saying that one is doing so. Generally speaking, in a *logic* for these acts, the basic *logical structure* can be characterised in the following way:

$$Act(Content)$$

where 'Act' stands for a speech act intended at a certain level of idealisation, for instance: *to assert*, *to deny*, *to hypothesise*, *to conjecture*, *to doubt*, and so on; 'Content' is described by a formula of the chosen language with a certain degree of complexity.

Observe that a logic for a specific speech act can be done either by exploiting the expressive power of first order logic (FOL) or adopting either a fragment or an extension of FOL, *e.g.* propositional logic or propositional modal logic. In some cases, we assume that the *content* of a speech act is true or false according to classical *Tarskian semantics*; in other cases, we exploit the expressive resource of other semantic frameworks.

Intuitively, those acts are not always 'licit' or 'permissible'. Think, for instance, about an *assertion*. When do we have the right – so to speak – to assert something? It is rather natural to consider cases when we can do that; in other cases we cannot. So, since we are talking about acts we have to consider *when* they are justified (or unjustified) acts: when are we justified to assert, deny, hypothesise, conjecture, etc. that *p*? In pragmatic logical frameworks, propositions are classically viewed as the primary bearers of truth-values, whereas, strictly speaking, *acts* are not true or false, but justified (or unjustified) when they fulfill (or not) some felicity conditions.

One way to specify the adopted conditions is to establish a pragmatic justification function from a pragmatic formula to two primitive justification values: justified (J) and unjustified (U):

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$\pi(Act(Content)) = J \Leftrightarrow \phi$

$\pi(Act(Content)) = U \Leftrightarrow \psi$

The biconditional states the justification (or unjustification) conditions of a speech act: ϕ and ψ. We follow Dalla Pozza and Garola [12] in assuming that, broadly speaking, justification conditions define whether it is *rational* to *assert*, *deny*, *hypothesise*, *doubt*, etc. what is expressed by the content.

Dalla Pozza and Garola [12] formulate an example of pragmatic logic. Specifically, they introduce a logical framework in order to specify a *logic of assertion*. They state that the assertion of A is *justified* if and only if *there is a proof of A*, while the content of the assertion is true or false. The assertion is always justified or unjustified. So, putting their system into our general schema, we have:

Act: Assertion ($\vdash$);

Content: propositions (A, B, C, ...),

ϕ: there is a proof that the content asserted is true;

ψ: there is not a proof that the content asserted is true.

How are other kinds of speech acts included in the logical analysis of the pragmatics? There are fundamentally two ways to do it: either by *reduction* or by *extension*. According to the first strategy, one tries to define a new pragmatic operator by using pragmatic principles previously introduced. Reducing *denial* to a *negation of assertion* is – *prima facie* – a paradigmatic example of this strategy. In the second case, in contrast, one extends the basic framework to capture the features of the new operator. Indeed, the extension can be accomplished by (at least) two sub-strategies:

1. One can preserve the language used to describe the contents of acts and just modify the justification and unjustification conditions ($\phi*$ and $\psi*$).
2. One can maintain the justification and unjustification conditions, enriching the descriptions of the contents.

Conjecturing and *proving* are examples of (1) and (2). The aim of this paper is to provide an example for each extension strategy of the pragmatic logic. Firstly, we analytically introduce the logic for pragmatics LP, formulated in [12], as a logic for *assertions*. Then, in section 3, we embrace the first strategy in an attempt to show that the speech act of denial can be reduced to the act of assertion (we are assuming, indeed, the equivalence thesis). However, we will see that this attempt fails: the act of *denying A* cannot be reduced to the act of asserting $\neg A$. In section 4 and section 5, we propose two extensions of LP: the first one, which follows the first sub-strategy, previously cited, permits the treatment of the *hypotheses*, the second one, which is in harmony with the second sub-strategy, is such that the act of *proving* is formalisable.

2. Logic for pragmatics (LP) as a *logic for assertions*

In their logical system named *Logic for Pragmatics* (LP), [12] Dalla Pozza and Garola provided a formal treatment of assertion, by introducing some pragmatic connectives, which are required in order to formulate a pragmatic interpretation of intuitionistic propositional logic as a logic of assertions. LP propositions can be either true or false, while the judgements expressed as assertions can be justified (J) or unjustified (U).

Assertions are intended as "purely logical entities ... without making reference to the speaker's intention or beliefs". [12, 83]. LP is composed of two sets of formulas: *radical* and *sentential*. Every sentential formula contains at least a radical formula as a proper sub-formula.

Radical formulas are semantically interpreted by assigning them a (classical) truth-value. *Sentential formulas* (briefly, *assertions*), on the other hand, are pragmatically evaluated by assigning them a justification value (J, U), defined in terms of the intuitive notion of proof. The pragmatic language of LP is described below.

Alphabet.

The vocabulary of LP contains the following sets of signs.

Descriptive signs: the propositional letters p, q, r.

Logical signs for radical formulas: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$.

Logical signs for sentential formulas: the assertion sign $\vdash$ and the pragmatic connectives $\sim$ (negation), $\cap$ (conjunction), $\cup$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence).

Formation rules (FRs).

Radical formulas (*rfs*) are recursively defined by the following FRs.

FR1 (atomic formulas): every propositional letter is a *rf*.

FR2 (molecular formulas):

(i) let γ be a *rf*, then $\neg\gamma$ is a *rf*;

(ii) let γ_1 and γ_2 be *rfs*, then $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*.

Sentential formulas (*sfs*) are recursively defined by the following FRs.

FR3 (elementary formulas): Let γ be a *rf*, then $\vdash \gamma$ is a *sf*.

FR4 (complex formulas):

(i) let δ be a *sf*, then $\sim \delta$ is a *sf*;

(ii) let δ_1 and δ_2 be *sfs*, then $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, $\delta_1 \equiv \delta_2$ are *sfs*.

Every radical formula of LP has a truth-value. Every sentential formula has a justification value, which is defined in terms of the intuitive notion of proof and depends on the truth value of its radical sub-formulas. The semantics of LP is the same as for classical logic, and only provides an interpretation of the radical formulas, by assigning them a truth-value and interpreting propositional connectives as truth functions in a standard way.

To be precise, the semantic rules are the usual classical Tarskian ones and specify the truth-conditions (only for radical formulas) through an assignment function σ, thus regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and $1 =$ true and $0 =$ false; then:

1. $\sigma(\neg \gamma_1) = 1$ iff $\sigma(\gamma_1) = 0$
2. $\sigma(\gamma_1 \wedge \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ and $\sigma(\gamma_2) = 1$
3. $\sigma(\gamma_1 \vee \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ or $\sigma(\gamma_2) = 1$
4. $\sigma(\gamma_1 \rightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = 0$ or $\sigma(\gamma_2) = 1$

Whenever only classical metalinguistic procedures of proof are admitted in LP, the pragmatic connectives have a meaning that is explicated by the Brouwer, Heyting, Kolmogorov (*BHK*) intended interpretation of intuitionistic logical constants. The illocutionary force of an assertion plays an essential role in determining the *pragmatic* component of the meaning of an elementary formula, with the *semantic* component, namely, the meaning of p, interpreted as in a semantic theory.

Justification rules regulate the pragmatic evaluation π, specifying the justification-conditions for the sentential formulas in function of the σ-assignments of truth-values for their radical sub-formulas. A pragmatic interpretation of LP is an ordered pair $<\{J, U\}, \pi>$, where {J, U} is the set of justification values and π is a function of pragmatic evaluation in accordance with the following justification rules:

JR1 – Let γ be a radical formula. $\pi(\vdash \gamma) = J$ iff a proof exists that γ is true, i.e. that σ assigns the value 1 to γ. $\pi(\vdash \gamma) = U$ iff no proof exists that γ is true.

JR2 – Let δ be a sentential formula. Then, $\pi(\sim \delta) = J$ iff a proof exists that δ is unjustified, i.e., that $\pi(\delta) = U$.

JR3 – Let $\delta 1$ and $\delta 2$ be sentential formulas. Then:

1. $\pi(\delta_1 \cap \delta_2) = J$ iff $\pi(\delta_1) = J$ and $\pi(\delta_2) = J$;
2. $\pi(\delta_1 \cup \delta_2) = J$ iff $\pi(\delta_1) = J$ or $\pi(\delta_2) = J$;
3. $\pi(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi(\delta_2) = J$ whenever $\pi(\delta_1) = J$.

The soundness criterion (*SC*) is as follows:

(*SC*) Let γ be a radical formula, then $\pi(\vdash \gamma) = J$ implies that $\sigma(\gamma) = 1$.

SC states that if an assertion is justified, then the content of the assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behaviour and can be translated into the modal system S4, where '$\Box\gamma$' means that there is an (intuitive) proof (conclusive evidence) for γ.

The classical fragment of LP, CLP, is made up of all the *sfs* that do not contain pragmatic connectives. Axioms for CLP are the following:

A1 $\vdash (\gamma_1 \to (\gamma_2 \to \gamma_1))$

A2 $\vdash ((\gamma_1 \to (\gamma_2 \to \gamma_3)) \to ((\gamma_1 \to \gamma_2) \to (\gamma_1 \to \gamma_3)))$

A3 $\vdash ((\neg\gamma_2 \to \neg\gamma_1) \to ((\neg\gamma_2 \to \gamma_1) \to \gamma_2))$

The modus ponens rule in CLP is as follows:

$$\text{[MPP] if} \vdash \gamma_1, \vdash (\gamma_1 \to \gamma_2), \text{then} \vdash \gamma_2.$$

The intuitionistic fragment of LP, ILP, is made up of all the *sfs* containing only atomic radicals. The axioms of the intuitionistic fragment of ILP are as follows (where δ_1, δ_2, δ_3 contain atomic radicals):

A1. $\delta_1 \supset (\delta_2 \supset \delta_1)$

A2. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_3))$

A3. $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$

A4. $(\delta_1 \cap \delta_2) \supset \delta_1; (\delta_1 \cap \delta_2) \supset \delta_2$

A5. $\delta_1 \supset (\delta_1 \cup \delta_2); \delta_2 \supset (\delta_1 \cup \delta_2)$

A6. $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$

A7. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim \delta_2)) \supset (\sim \delta_1))$

A8. $\delta_1 \supset ((\sim \delta_1) \supset \delta_2)$

The modus ponens rule for ILP is as follows:

$$\text{[MPP'] If } \delta_1, \delta_1 \supset \delta_2, \text{ then } \delta_2$$

where, again, δ_1 and δ_2 contain atomic radicals. It is worth noting that the justification rules do not always allow for determining the justification value of a complex sentential formula when all the justification values of its components are known. For instance:

NR1 $\pi(\delta) = J$ implies $\pi(\sim \delta) = U$;

NR2 $\pi(\delta) = U$ does not necessarily imply $\pi(\sim \delta) = J$;

NR3 $\pi(\sim \delta) = J$ implies $\pi(\delta) = U$;

NR4 $\pi(\sim \delta) = U$ does not necessarily imply $\pi(\delta) = J$.

In addition, a formula δ is pragmatically valid or p-valid (invalid or p-invalid, respectively) if for every π and σ, the formula δ is justified (δ is unjustified, respectively).[1] In any case, no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, since pragmatic connectives are partial functions of justification. Moreover, a function ( )* mapping the set of *sfs* into an extension of the set of *rfs* obtained by means of the modal operator $\Box$ (*proved*), i.e. a modal translation of pragmatic assertive formulas, is (recursively) induced by the following correspondence:

$$(\vdash \gamma)^* = \Box\gamma$$

$$(\sim \delta)^* = \Box\neg(\delta)^*$$

$$(\delta_1 \cap \delta_2)^* = (\delta_1)^* \wedge (\delta_2)^*$$

$$(\delta_1 \cup \delta_2)^* = (\delta_1)^* \vee (\delta_2)^*$$

$$(\delta_1 \supset \delta_2)^* = \Box((\delta_1)^* \rightarrow (\delta_2)^*)$$

Radical and sentential formulas are related by means of the following 'bridge principles':[2]

(a) $(\vdash \neg \gamma) \supset (\sim \vdash \gamma)$

(b) $(\vdash (\gamma_1) \cap \vdash (\gamma_2)) \equiv \vdash (\gamma_1 \wedge \gamma_2)$

(c) $(\vdash (\gamma_1) \cup \vdash (\gamma_2)) \supset \vdash (\gamma_1 \vee \gamma_2)$

(d) $(\vdash (\gamma_1 \rightarrow \gamma_2)) \supset (\vdash \gamma_1 \supset \vdash \gamma_2)$

It is worth observing that **(a)** – **(d)** show the formal relationships between classical truth-functional connectives and pragmatic ones. Formula **(a)** states that from the assertion of not-γ the non-assertability of γ can be inferred. **(b)** states that the conjunction of two assertions is equivalent to the assertion of a conjunction; **(c)** states that from the disjunction of two assertions one can infer the assertion of a disjunction. And finally, **(d)** expresses the idea that from the assertion of a classical material implication follows the pragmatic implication between two assertions. Generally speaking, **(a)** – **(d)** express the relation between classical logic and pragmatic one.

How can we extend LP? A first option analysed is to extend this logic to the act of denying: if denying A is equivalent to asserting $\neg A$, once we have a tool for asserting A, it seems to be *prima facie* easy to extend the logic so to obtain a formal tool for the act of denying. In the next section we analyse this option.

[1] Other pragmatic criteria of validity are presented in [12].

[2] See [12].

3. Denying and LP

Suppose Bob asserts:

(1) Dover is north of London,

and Sarah disagrees.[3] Classically, Sarah may express disagreement by asserting the negation of what Bob said:

(2) Dover is *not* north of London.

In the classical theory of denial,[4] denying A is equivalent to asserting $\neg A$:

Classical denial. A is correctly denied iff $\neg A$ is correctly asserted.[5]

Roughly speaking, let us call Frege's thesis the idea that there is just one fundamental speech act, namley the assertion, and that *denial* could be reduced to it. Question:

(Q) Is it possible to extend LP so as to give rise to the speech act of denial?

In general terms, there are three ways to include *negation* in a general pragmatic logical structure, previously illustrated by *Act(Content)*, i.e.:

1. Negation of the content: from $Act(A)$ to $Act(\neg A)$
2. Negation of the act: from $Act(A)$ to $\sim Act(A)$
3. Negation of the justification function: from $\pi(Act(A)) = J$ to $\pi(Act(A)) = U$

We will proceed as follows: we will try to define a pragmatic denial ($\dashv$) by exploiting the three ways to negate an assertion, i.e., negating (i) the content; (ii) the act of assertion; (iii) and the justification value. We anticipate the result of our analysis: every attempt to expand LP with *denial* will meet some difficulties (on this see also [7]).

3.1. *Denying Content*

Our first option (i) consists of introducing the denial in LP by strictly following *Classical denial* and establishing that:

(1) $\pi(\dashv A) = J$ iff $\pi(\vdash \neg A) = J$

[3] We assume, here and throughout, that the disagreement in question isn't of the faultless kind: between Bob and Sarah only one can be correct as to whether Dover is north of London.

[4] For an introduction see [17].

[5] [18] calls this the *denial equivalence*.

That is, the denial of A is justified if and only if the assertion of $\neg A$ is justified. It is easy to realise that (1) does not work. Intuitively, the conditions by which content can be correctly asserted concern, as for LP, that there is a proof of what is asserted; this means that the epistemic standard is *high*, while – still intuitively – things are different for justifying denial. Particularly, (1) can be detached as:

(1') $\pi(\dashv A) = J \Rightarrow \pi(\vdash \neg A) = J$

and:

(1") $\pi(\vdash \neg A) = J \Rightarrow \pi(\dashv A) = J$

And while (1') is not plausible, (1") is plausible.

The reason by which (1') does not hold is that there are many situations where it is perfectly legitimate to deny certain content, say A, with no conclusive proof to assert its negation.

The most common cases can be subdivided into two families: cases in which it is extremely difficult to find a proof of $\neg A$ but it is reasonable to deny A on the basis of some indirect evidence and cases in which it is perhaps impossible to prove $\neg A$ but, nevertheless, A is deniable for contextual reasons. A couple of examples: let us consider the so-called "conspiracy theories" according to which many subjects are secretly in agreement to lead some, usually elaborate, plan. Even if a proof that the theory is false may not exist, it is perfectly reasonable to deny the thesis of a conspiracy theory for reasons connected to explanatory economy, abductive reasoning, and so on. Another example: let A be the sentence stating the existence of an entity, such as God of classical theism. Then, let us assume to be in a broadly naturalistic setting; now, since the existence (as the non-existence) of an alleged transcendent entity is, by definition, underdetermined from any possible physical experience, it follows that, as a matter of principle, there cannot be a proof of its non-existence (viz.,$\neg A$). However, given the assumed naturalistic stance, it is reasonable to deny A.

3.2. *Denying Act*

The second option exploits the expressive resources of pragmatic negation:

(2) $\pi(\dashv A) = J$ iff $\pi(\sim\vdash A) = J$

This means that the denial of A is justified if and only if it is justified that A is not justified. Even in this case we can detach the double conditional:

(2') $\pi(\dashv A) = J \Rightarrow \pi(\sim\vdash A) = J$

(2") $\pi(\sim\vdash A) = J \Rightarrow \pi(\dashv A) = J$

(2”) is not a logical and pragmatic principle: a proof of the fact that A cannot be proven is not a sufficient condition to deny A. From the impossibility to assert A, namely the impossibility of having conclusive evidence (or an intuitive proof) for A does not imply in any case the denial of A. When there is an obstacle in principle to assert A, an agent may coherently deny A or not, depending on the fact whether A is incompatible or not with an accepted framework. Moreover, also the idea that it can be proved that A is not proved is delicate: being proven seems to be a historical fact, which is not something susceptible of proof (at least, according to a standard meaning of proof). Within (2”) it is particularly important the intended interpretation of the act of proving and the act of denying; if we want to preserve the same intuitions followed in the previous analysis, we have to conclude that this is an unacceptable principle.

Observe, moreover, that (2’) is problematic. One can deny A without a proof of the fact that A is not proven. Let us think about situations similar to those discussed before: one can reject, for instance, a platonist ontology of mathematics even there is no proof of the fact that the existence of mathematical objects is unproven.

3.3. *Denying Justification*

Finally, the third option is as follows:

(3) $\pi(\dashv A) = J$ iff $\pi(\vdash A) = U$

That is, we are justified in denying A if and only if we do no have at disposal a proof of A. As before, (3) is better analysed once the right to left and the left to right versions are analysed.

(3’) $\pi(\dashv A) = J \Rightarrow \pi(\vdash A) = U$

and

(3”) $\pi(\vdash A) = U \Rightarrow \pi(\dashv A) = J$

(3’) seems to work: if it is rational to deny A then there should not be any proof of A. On the contrary, (3”) is controversial: the lack of a (conclusive) proof for A does not seem to be a sufficient condition to justify the denial of A. There are lots of highly speculative hypotheses in scientific practice without a *final proof* of their correctness. However, it does not follow that we are justified in denying them.

To conclude: it seems that neither (1) nor (2) and (3) can rightly specify denial from assertion. The reason lies in the same LP, i.e. that there should be a proof for the justification of the assertion; however, while the existence of a proof is *too strong* to deny A, its absence is definitely *too weak* as a requirement for denying A. To illustrate:

$$\begin{array}{lcl} & & \pi(\dashv A) = J \\ & \nearrow & \\ \pi(\vdash \neg A) = J & & \\ & \searrow & \\ & & \pi(\sim\vdash A) = J \end{array}$$

If so, the answer to our question (Q) (Is it possible to extend LP so as to give rise to the speech act of denial?) is negative. In LP we cannot follow *Frege's thesis* insight, reducing denial to assertion and negation. What about other acts? In the next section we consider the act of conjecturing or making an hypothesis.

4. Extending LP to a Logic of Hypotheses (HLP)

As for the act of denial, the act of conjecturing or making a hypothesis cannot be reduced to assertion. Consider, instead, the *hypothesis* as an act with a primitive illocutionary force, indicated by H, which is justified by means of a *scintilla of evidence*. What counts as evidence is contextually specified. The language of pragmatic logic for hypotheses (HLP) as follows:

Alphabet.

The vocabulary of HLP contains the following set of signs.
Descriptive signs: the propositional letters p, q, r.
Logical signs for radical formulas: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$.
Logical signs for sentential formulas: the sign for hypothesis H and connectives $\frown$ (negation), $\cap$ (conjunction), $\cup$ (disjunction), $\sqsupset$ (implication), $\equiv$ (equivalence).

Formation rules (FRs).

Radical formulas (*rfs*) are recursively defined by the following FRs:
FR5 (atomic formulas): every propositional letter is a *rf*
FR6 (molecular formulas):

(i) Let γ be a *rf*, then $\neg\gamma$ is a *rf*

(ii) Let γ_1 and γ_2 be *rfs*, then $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, and $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*.

Hypothetical formulas (*hpfs*; briefly *hypotheses*) are recursively defined by the following FRs:

FR7 (elementary formulas): Let γ be a *rf*, then Hγ is a *hpf*,
FR8 (complex formulas):

(i) let κ be a *hpf*, then $\frown \kappa$ is a *hpf*;

(ii) Let κ_1 and κ_2 be hpf, then $\kappa_1 \cap \kappa_2$, $\kappa_1 \cup \kappa_2$, $\kappa_1 \sqsupset \kappa_2$, and $\kappa_1 \equiv \kappa_2$ are *hpfs*.

Every radical formula of HLP has a truth value, which is assigned by classical semantic rules, as in section 2.

Hypothetical operators for hypothetical formulas formally behave in accordance with the justification rules expressed here below.[6] Observe that ε is a function of evidence from hypothetical formulas to justification values.

A pragmatic interpretation of HLP is an ordered pair $<\{J, U\}, \varepsilon>$, where $\{J, U\}$ is the set of justification values and ε is a function of pragmatic evaluation for hypothetical formulas such that the following *justification rules* are satisfied:

HJR1 Let γ be a radical formula. $\varepsilon(H\gamma) = J$ iff there is a scintilla of evidence that γ is true, while $\varepsilon(H\gamma) = U$ iff a scintilla of evidence does not exist that γ is true.

HJR2 Let κ be a hypothetical formula. Then, $\varepsilon(\frown \kappa) = J$ iff the scintilla of evidence that $\varepsilon(\kappa) = J$ is smaller than the scintilla of evidence that $\varepsilon(\frown \kappa) = U$ (i.e. briefly, iff we are more justified in doubting about κ than believing in it).[7]

HJR3 Let κ_1 and κ_2 be hypothetical formulas.

Then:

(i) $\varepsilon(\kappa_1 \cap \kappa_2) = J$ iff $\varepsilon(\kappa_1) = J$ and $\varepsilon(\kappa_2) = J$;

(ii) $\varepsilon(\kappa_1 \cup \kappa_2) = J$ iff $\varepsilon(\kappa_1) = J$ or $\varepsilon(\kappa_2) = J$;

(iii) $\varepsilon(\kappa_1 \sqsupset \kappa_2) = J$ iff there is a scintilla of evidence that $\varepsilon(\kappa_2) = J$ whenever $\varepsilon(\kappa_1) = J$[8]

HJR1 expresses, in particular, a soundness criterion for hypotheses:

let γ be a radical formula, then $\varepsilon(H\gamma) = J$ implies that there is a scintilla of evidence that γ is true.

Let us focus now on some notable principles concerning pragmatic hypothetical negation following from **HJR2**:

(HNR1) $\varepsilon(\kappa) = J$ does not imply that $\varepsilon(\frown \kappa) = U$.

(HNR2) $\varepsilon(\kappa) = U$ implies that $\varepsilon(\frown \kappa) = J$,

(HNR3) $\varepsilon(\frown \kappa) = J$ does not imply that $\varepsilon(\kappa) = U$,

(HNR4) $\varepsilon(\frown \kappa) = U$ implies that $\varepsilon(\kappa) = J$.

[6] Recent developments of LP are pointed out in [5] and [4].

[7] In other works on pragmatic logic, hypothetical negation has a slightly different meaning.

[8] We consider HJRs intuitive as criteria of justification.

Rules **HJR1-HJR3** can be supplemented with a fuzzy interpretation of the justification rules of hypotheses, which seem to be quite natural in order to easily handle our pre-theoretical insights of them. To be precise, let us introduce a new non-classical semantics on the set of radical formulas such that truth-values, indicated by | |, range in degree between 0 and 1 and satisfy the following rules.[9]

$|\neg\gamma| = 1 - |\gamma|$

$|\gamma_1 \vee \gamma_2| = Max(|\gamma_1|, |\gamma_2|)$

$|\gamma_1 \wedge \gamma_2| = Min(|\gamma_1|, |\gamma_2|)$

$|\gamma_1 \rightarrow \gamma_2| = 1 \quad \text{if } |\gamma_1| \leq |\gamma_2|$

$|\gamma_1 \rightarrow \gamma_2| = 1 - (|\gamma_1| - |\gamma_2|)$ otherwise.

A fuzzy approach to hypotheses might deal with those situations in which it is not easy to assign probability values to a specific hypothesis, because of some forms of fundamental uncertainty. For a discussion about justification values of hypothetical formulas and truth values of this fuzzy logic see [8].

We introduce now the definition of p-validity in HLP.

A hypothetical formula κ is *pragmatically valid* (or *p-valid*) iff, for every pragmatic evaluation ε, $\varepsilon(\kappa) = J$.

The following bridge principles can be proven to be p-valid formulas of HLP by using the fuzzy interpretation provided above.

(a°) $(\frown \mathrm{H}\gamma) \sqsupset (\mathrm{H}\neg\gamma)$

(b°) $\mathrm{H}(\gamma_1 \wedge \gamma_2) \sqsupset (\mathrm{H}(\gamma_1) \cap \mathrm{H}(\gamma_2))$

(c°) $\mathrm{H}(\gamma_1 \vee \gamma_2) \sqsupset (\mathrm{H}(\gamma_1) \cup \mathrm{H}(\gamma_2))$

(d°) $(\mathrm{H}\gamma_1 \sqsupset \mathrm{H}\gamma_2) \sqsupset \mathrm{H}(\gamma_1 \rightarrow \gamma_2)$

Principle **(a°)** shows the relationship between hypothetical and classical negation. **(b°)** indicates that the hypothesis of a conjunction entails the conjunction of hypotheses. **(c°)** states that a disjunctive hypothesis entails a disjunction of hypotheses. **(d°)** states that from an implication between hypotheses follows the hypothesis of the implication.

There are some other details for this logic but here we stop on the analysis of HLP.[10] For the rest of the section, we analyse a possible general principle connecting assertions and hypotheses.

[9] We follow [16].
[10] For a detailed analysis of HLP see [8].

Consider the following general principle:

(GP1) $\pi(\vdash \neg p) = J$ iff $\varepsilon(\mathrm{H}p) = U$

Let us consider separately the two implications of (GP1):

(GP1a) if $\pi(\vdash \neg p) = J$ then $\varepsilon(\mathrm{H}p) = U$

(GP1b) if $\varepsilon(\mathrm{H}p) = U$ then $\pi(\vdash \neg p) = J$

GP1a is intuitively plausible for any interpretation of the notion of a hypothesis, while **GP1b** is plausible if we assume that Hp may be justified (in a minimal sense) by the existence of a mere cognitive possibility of a situation (no matter how unlikely it might be) where p is true.[11] Following an epistemic interpretation for hypotheses, **GP1** holds and states what grounds for the justification of an assertion $\vdash p$ are also necessary and sufficient to consider the hypothesis H$(\neg p)$ unjustified. Once again, the fuzzy interpretation can make this principle clear.

Intuitively, consider a justified assertion $\vdash p$. The truth value of p is 1 because we have conclusive evidence (a proof) for p in this case. If, instead, $\vdash p$ is unjustified, then the truth value of p is either 0 (when we have a proof that p does not hold), or it is not determined (when there is contingently no proof of p). The interpretation of **GP1a** is trivial, since it simply states that from $(\vdash \neg p) = J$, hence $\sigma(\neg p) = 1$, it is possible to infer $\varepsilon(\mathrm{H}p) = U$, that is $|p| = 0$ in the fuzzy semantics for *rfs* of HLP. Let us consider **GP1b**. When $\varepsilon(\mathrm{H}p) = U$, $|p| = 0$ and therefore $|\neg p| = 1$. It follows that, in the classical semantics for *rfs*, $\sigma(\neg p) = 1$, hence $\pi(\vdash \neg p) = J$, which proves **GP1b**.

The second general principle connecting assertions and hypotheses is the following:

(GP2) The justification of $\vdash p$ implies the justification of Hp.

Namely, we need to justify an assertion that p is sufficient to justify the hypothesis that p.

5. Towards an extension of LP to a logic for the act of proving (PLP)

In this section, we briefly explore a possible extension of LP in order to explicate some logical features of the *act of proving*. As we said before, the elementary formulas of LP have the form $\vdash\gamma$ where Frege's symbol "$\vdash$" represents an *illocutionary force of assertion* and γ is a formula interpreted in classical truth-functional semantics. Thus, in accordance with Frege, propositions are classically true or false. However, unlike in Frege's proposal, in LP there are *pragmatic connectives* that build formulas from

[11] See on this [3].

elementary assertions. Such pragmatic expressions are interpreted in LP according to the BHK interpretation of intuitionistic connectives. Every *pragmatic expression* of LP has *justification conditions* in accordance with the BHK interpretation. An expression meeting such conditions is *justified*; otherwise, it is *unjustified*. As said before (see section 2) pragmatic expressions have a classical semantic value through Gödel, McKinsey and Tarski's translation of intuitionistic logic into the classical modal logic **S4**. In particular:

$$(\vdash \gamma)^* = \Box\gamma$$

$$(\sim \delta)^* = \Box\neg(\delta)^*$$

$$(\delta_1 \supset \delta_2)^* = \Box((\delta_1)^* \to (\delta_2)^*)$$

Notice that in LP there are no assertions whose contents are *modally characterised*. A way to overcome this limitation is to expand LP introducing a pragmatic logic for assertions having *modal* propositional contents (PLP). One could go in the direction of finding a way to mix two modal languages, $L_\Box$ and L_K, endowed with two independent boxes, $\Box$ and K, interpreted as "it is proved that" and "it is known that", respectively. Some *Bridge Principles* for PLP intended to logically connect the independent boxes, e.g. $\Box_1\alpha \to \Box_2\alpha$, should be added. For example, the following one seems to be an adequate element from the set of the *Bridge Principles* (BP):

$$\text{(BP)}\ \Box\alpha \to \neg K\neg\alpha$$

which can be intuitively read as "if it is the case that α is proved to be true, then it is not the case that α is known to be false". (BP) gives a logical connection between $\Box$ and K. The idea behind (BP) can be made clearer if we consider its equivalent formulation in terms of conjunction:

$$\text{(BP')}\ \neg(\Box\alpha \wedge K\neg\alpha)$$

(BP') identifies the relationship expressing a minimal condition held between proof and knowledge according to our pre-theoretical insights. That is, there must be a logical incompatibility between the proof that α is true and the knowledge that α is false.

Why could such an extension be useful? One example is to solve some paradoxes. Take a logical argument, known as the *Knowability Paradox* (the paradox is in [15]; for an introduction to the *Knowability Paradox* see [19]). It starts from the assumption that *every truth is knowable* and leads to the paradoxical conclusion that *every truth is actually known*. The idea that every truth is knowable is traditionally associated with a verificationist perspective, a perspective that assumes the intuitionistic logic as the "correct" one. The knowability paradox is usually formulated in a classical modal logic. Specifically, the *Knowability Paradox* (KPx) is based on two principles: the *Knowability Principle* (KP) and the *Principle of Non-Omniscience* (Non-Om). (KP) is usually expressed in the following way:

(KP) $\forall p(p \to \Diamond Kp)$

while (Non-Om) is formulated as:

(Non-Om) $\exists p(p \wedge \neg Kp)$

The expression "Kp" reads "p is, has been, or will be known by somebody". Specifically, the paradox arises because

(KP) $\to \neg$ (Non-Om)

is a theorem of classical modal logic, that is:

(KPx) $[\forall p(p \to \Diamond Kp)] \to [\forall p(p \to Kp)]$.

However, (KPx) is a *classical* (modal) theorem, and since (KP) has been traditionally associated with both contemporary verificationism and intuitionistic logic, it seems that (KP) is not the correct formalisation of knowability. In addition, the paradoxical reading of (KPx) could be avoided as soon as an attempt is made to express knowability into an adequate form which takes into account the verificationist and intuitionistic features of an anti-realist version of knowability. Further, in order to explain the verificationist and intuitionistic features in a setting *compatible* with classical systems, one could go in the direction of introducing a modal pragmatic language for assertions.

There are two reasons to go in this direction. The first one is general and technical, and it is related to the fact that any pragmatic language for assertions is an intuitionistic-like system, so it is specifically useful for the purpose. Indeed, since the pragmatic connectives are interpreted intuitionistic-like, any pragmatic language is essentially intuitionistic. One can give a verificationist interpretation of classical modal propositions in terms of assertions together with intuitionistic- like connections of them, defined via pragmatic connectives. The second reason is specifically related to the knowability paradox. Indeed, if alethic notions have an intuitionistic-like semantics, then "it is possible that", the dual notion, "it is necessary that", can be interpreted as "there is no proof that not". In such a way, the possibility of something being true is reduced to the (actual) absence of a proof of its falsity, and $\Diamond K\alpha$ becomes "(at this moment in time), there is no proof that $K\alpha$ is false".

A modal pragmatic language for assertions language could be obtained by an extension of the expressiveness of the pragmatic language for assertion LP, from propositional contents to modal propositional contents, and, in particular, to assertions on (classical) alethic and epistemic contents. Notice, *passim,* that there are a certain number of multi-modal approaches to the paradox in the literature, proposed from a variety of viewpoints. See, for example, [1], [2], [11], [13], [14], [20], and [21]. It should be a language preserving the main characteristic of LP; that is, the *integrated* perspective about truth and proof in accordance with the intuition that the notion of proof presupposes the classical notion of truth as a regulative concept, since a proof of a proposition amounts to a proof that its truth value is the value "true". Such a language should be based on the identification of the verificationist notion of truth with the notion of justified assertion – the proof of a classical (modal) truth. This verificationist interpretation of classical (modal) propositions in terms of assertions should be integrated with intuitionistic-like connections of them, defined via pragmatic connectives. In this way, a set of *Pragmatic Bridge Principles* (PBPx), explaining the relationships between the classical connectives, pragmatic ones and (classical) modal operators, could be obtained. Such a language could provide a fine-grained analysis of the notions of truth, proof, knowledge, and their relationships taking into account the fact that the *Knowability Principle* is mainly associated with a verificationist perspective in epistemology, and that an important connection between verificationism and intuitionistic logic is usually recognized. Indeed, as showed in [10] (see also [9] and [6]) it is possible to develop a pragmatic language with compatibilist perspective of classical and intuitionistic systems: a notable aspect concerning the communicability among different logics that becomes an advantage for the interpretation of the paradox from an anti-realistic perspective.

6. Conclusions

In this paper, we explored three possible developments for a system of logic for Pragmatics. The original framework was conceived to treat the illocutionary act of assertion. Since we are dealing with a *logic* of assertion we need to specify the conditions of justification (unjustification, respectively) of the act under scrutiny. Dalla Pozza and Garola advanced the idea according to which the justification is grounded in the available (maybe, in principle) evidence. Following this line of thought, we illustrated a pragmatic logic for the hypotheses: in our system the act of hypothesing that A is justified if there is at least a minimal evidence for A. Thus, it is quite natural to put the (pragmatic) logic of assertion and the (pragmatic) logic of hypothesis on the same line: the justification of these illocutionary acts depends on the evidence at disposal and we can consider the assertion as a limited case of the hypothesis, i.e., a hypothesis in which there is maximum evidence, namely, a proof.

This particular feature of the systems explains the asymmetry of these operators; for instance, the assertion of $\neg A$ is justified by the proof of $\neg A$. This means that the absence of proof of $\neg A$ does not convey any information about the justification of A. For this reason, one can see in these systems an intuitionistic flavour.

The second path we explored has to do with enlarging the language used to express the radicals. In the extension we explored a possible logic with propositional modal and epistemic logic. In this way, we get a relevant improvement of the expressive strength of our logic of assertion. As we previously stated, in pursuing these extensions, the analysis of the semantics and pragmatics of the act of proving in connection with the notion of knowability is particularly interesting.

Things are different for the third path of our research. The reflections about denial seem to suggest that, here, the justification (or, unjustification) of a denial does not depend on a certain amount of evidence. On the contrary, other features justify the denial of A. Our conjecture is that the logical ground of denial is a sort of *incompatibility* between the thesis at play (viz., A) and an accepted framework. Of course, it still remains to adequately characterise this kind of incompatibility. For the moment, we can say that this explains why the denial seems to be irreducible to the assertion.

These three possible extensions of the 'original' framework of pragmatic logic could be conducted separately, as different research programs. However, it is an open (and interesting) question to investigate whether there is a more comprehensive system of logic for pragmatic that include all the features displayed by our explorations. We leave this challenging task to be undertaken for future research.

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