Leveraging Optimal Transport to Design Optimal Mechanisms for the Facility Location Problem[*]

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In this paper, we investigate the k-Facility Location Problem on the line within the Bayesian Mechanism Design framework and analyze the percentile mechanisms, a class of truthful mechanisms that locates the facilities based on the order of the agents' reports. We first connect the k-FLP to the Wasserstein projection problems and use this connection to retrieve the limit of the ratio between the expected cost of a percentile mechanism and the expected optimal cost. Moreover, we characterize its limit and convergence speed. We infer an upper bound on the Bayesian approximation ratio when $n > k$, contrasting the classic worst-case analysis where percentile mechanisms have an unbounded approximation ratio whenever $k > 2$. This allows us to introduce criteria to determine which percentile mechanism is better suited to address a given agent distribution. We then establish the existence of an optimal percentile mechanism and characterize it via a system of k equations. Finally, we estimate the optimality loss that occurs if we retrieve the optimal percentile mechanism using an approximation of the agents' distribution. All results hold for the Social, Maximum, and l_p costs.

CCS Concepts: • **Theory of computation → Facility location and clustering**; **Algorithmic mechanism design**; **Algorithmic game theory**.

Additional Key Words and Phrases: Automated Mechanism Design, Bayesian Mechanism Design, Optimal Transport, Facility Location Problem

1 Introduction

The scope of Mechanism Design is defining procedures that aggregate a group of agents' private information for optimizing a social objective. Nevertheless, merely optimizing the social objective based on the agents' reported preferences often leads to undesired manipulation due to the agents' self-interested behaviour. For this reason, one of the most important properties a mechanism should possess is *truthfulness*, which guarantees that no agent benefits from misreporting its private information. This stringent property is often incompatible with the optimization of the social objective, so we have to compromise on a sub-optimal solution. To quantify the efficiency loss, Nisan and Ronen introduced the notion of *approximation ratio*, which is the highest ratio between the social objective achieved by a truthful mechanism and the optimal social objective achievable over all the possible agents' reports [48].

One of the most famous examples of these problems is the k-Facility Location Problem (k-FLP). In its most basic form, the k-FLP consists in locating k *homogeneous* facilities among n self-interested agents. Every agent

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[*] A previous version of this paper appeared in the Proceedings of the 17th Symposium of Algorithmic Game Theory (SAGT 2024) [9]. In this improved version of the paper, we complement our studies by considering percentile vectors that have 0-1 entries and by studying the problem under different metrics, such as the Maximum Cost and the l_p costs.

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needs to access one of the facilities, since all the facilities provide the same service, they would prefer to have at least one placed as close as possible to their position. Despite its simplicity, this problem and its variants have found a wide range of applications in fields such as disaster relief [12], supply chain management [46], healthcare [1], clustering [38], and public facilities accessibility [14]. The study of the k-FLP from an algorithmic mechanism design viewpoint was initiated by Procaccia and Tennenholtz. In their seminal work [50], they considered the problem of locating one facility amongst a group of agents located on a line. They were the first to design an allocation process that places the facility while keeping in mind that every agent is self-interested, i.e. that the agents would manipulate the process in its favour if able. Ever since the study of the social aspects of the k-FLP has received extensive attention. Subsequently, a variety of methods with fixed approximation ratios for positioning one or two facilities on different types of structures such as trees [29, 30], circles [42, 43], general graphs [2, 25], and metric spaces [9, 24, 58] were introduced. Despite the overarching generality of the underlying space, these positive outcomes pertain to scenarios with a limited number of agents or when the facilities to place are at most 2.

The approximation ratio results are, indeed, much more negative when we move to three or more facilities. Fotakis and Tzamos [32] showed that for every $k \geq 3$, there does not exist any deterministic, anonymous, and truthful mechanisms with a bounded approximation ratio for k-FLP on the line, even for simple instances with $k+2$ agents. The same result has been later shown to be true for higher dimensions problems [60]. Nonetheless, it is still possible to define truthful mechanisms with a bounded approximation ratio when the number of agents is equal to the number of facilities plus one, as shown in [26], by considering randomized mechanisms [31], or when the facilities have a capacity limit [8].

Our study concerns a class of truthful mechanisms for the generic k-FLP, the *percentile mechanisms*, first introduced in [57]. Although every percentile mechanism has an unbounded approximation ratio, we show that this is not the case if the agents' type is sampled from a probability distribution. This framework is also known as Bayesian Mechanism Design [20, 35]. Under these premises, we show that, depending on the distribution that describes the agents' position, it is possible to select a percentile mechanism that asymptotically behaves optimally, i.e. it minimizes the Social Cost. Furthermore, given the agents' distribution, we provide a characterization of the optimal percentile mechanism via a system of equations. Interestingly, these results hold for more generic settings; in particular, we show that all our results hold when we consider different costs, such as the Maximum Cost or the l_p costs. Our results mark an important connection between Optimal Transportation and Mechanism Design, which allows us to (i) define a general way to study mechanisms from a Bayesian perspective, and (ii) provide an alternative framework to retrieve optimal truthful mechanisms, which is the defining problem of Automated Mechanism Design [53].

1.1 Our Contribution

In this paper, we conduct a comprehensive investigation of the k-Facility Location Problem (k-FLP) from a Bayesian Mechanism Design perspective, where we assume that agents' positions on the line follow a distribution μ [35, 36]. We focus specifically on the class of percentile mechanisms [57] and explore the conditions under which the Bayesian approximation ratio of these mechanisms – defined as the ratio between the expected cost induced by a mechanism and the expected optimal cost – is bounded. We establish that each percentile mechanism exhibits distinct performance depending on the measure μ and we identify the optimal percentile mechanism tailored to a distribution μ. In particular, we show that, for any given probability measure μ, there exists a percentile mechanism whose Bayesian approximation ratio goes to 1, proving that considering just percentile mechanisms is not restrictive.

Our study begins by establishing a connection between the k-FLP and a projection problem in the Wasserstein space. Through this connection, we import tools and techniques from Optimal Transport theory to approach the k-FLP. In particular, we demonstrate that when the number of agents on the line tends to infinity, the ratio between the expected cost induced by the mechanism and the expected optimal cost converges to a bounded value. Moreover, we characterize both the limit value of the ratio and the speed of convergence. Finally, leveraging the characterization of the limit and its convergence rate, we derive a bound on the performances of percentile mechanisms for any finite number of agents.

We then tackle the problem of retrieving the best percentile mechanism tailored to a distribution μ and the number of facilities k. We show that there always exists a percentile vector, namely $\vec{v}_\mu \in (0,1)^k$, that induces the optimal percentile mechanism, *i.e.* a mechanism whose expected social cost is asymptotic to the optimal expected cost as the number of agents increases. We characterize this vector as the solution to a system of k equations and employ it to compute the optimal percentile vector associated with common probability measures, such as the Uniform and Gaussian distributions. Lastly, we show that the optimal percentile vector is invariant under positive affine transformations of the probability measures describing agents. In particular, $\vec{v}_\mu$ does not depend on the specific mean and variance of the distribution μ.

To conclude the paper, we present a study on the stability of the optimal percentile vector. Specifically, let $\tilde{\mu}$ be an approximation of the true agents' distribution μ. Additionally, let $\vec{v}_{\tilde{\mu}}$ and $\vec{v}_\mu$ represent the optimal percentile vectors associated with $\tilde{\mu}$ and μ, respectively. We demonstrate that when the agents are distributed according to μ, the Bayesian approximation ratio limit of the percentile mechanism induced by $\vec{v}_{\tilde{\mu}}$ deviates from 1 by an amount proportional to the infinity Wasserstein distance between μ and $\tilde{\mu}$. Notably, the more precise the approximation of μ, the better the asymptotic performance of the optimal percentile mechanism induced by $\vec{v}_\mu$ when the agents are distributed according to μ. All the results discussed hold for the most common social objectives, including the Social Cost, the Maximum Cost, and the l_p costs. To maintain focus in the main discussion, we first introduce all the main results for the Social Cost and then extend the results to the l_p and Maximum Costs in a dedicated Section.

A preliminary version of this paper appeared in the Proceedings of the 17th Symposium of Algorithmic Game Theory 2024 (SAGT 24) [9]. In this revised and enhanced version, we have complemented the study by considering the case in which the percentile vector inducing the percentile mechanism has one or more 0-1 entries and extended our study to include the Maximum Cost and the l_p cost. Owing to this extension, we have been able to retrieve the optimal percentile mechanism for both the l_p and Maximum Costs, in particular we retrieve a closed formula that computes the optimal percentile mechanism with respect to the Maximum Cost regardless of the number of facilities k.

1.2 Related Work

The study of k-FLP research from an algorithmic mechanism design viewpoint was initiated by Procaccia and Tennenholtz in [50]. When $k = 1, 2$ there are several truthful mechanisms, such as the median mechanism [15] and its generalizations [13, 47], that achieve small constant worst-case approximation ratios. When $k > 2$, however, these efficiency guarantees are much more negative. Indeed, due to an impossibility result, it is well-known that there does not exist a truthful mechanism that has a bounded approximation ratio while being deterministic and anonymous [32, 60]. It is worthy of notice however, that this impossibility result does not apply to randomized mechanisms [31], to instances where the number of agents is precisely equal to the number of facilities plus one, as shown in [26], and to instances where the facilities have a uniform capacity limit as shown in [8].

The *Percentile Mechanisms* are a class of mechanisms for the k-FLP that, similarly to the median mechanism, places the facilities at the place of k agents depending on their order on the line, [57]. Due to the dictatorial-like nature of these mechanisms, it is easy to build an *ad hoc* instance of the k-FLP on which the optimal social cost is as small as we like, but the cost attained by the mechanisms greater than a positive constant. These instances, however, do depend on the percentile mechanism and, in most of the applied contexts, carry little practical sense.

Bayesian Mechanism Design is an alternative paradigm to evaluate the performances of a mechanism. While in classic Mechanism Design, the designer does not have any information about the agents' type, in Bayesian Mechanism Design, every agent's type is drawn from a known probability distribution, [20, 35]. Consequentially, this defines a distribution over the set of inputs over which the mechanism is defined, allowing us to introduce the notion of expected cost of a mechanism. The Bayesian Mechanism Design framework has been used to study routing games [34], facility location problems [62], combinatorial mechanisms based on ϵ-greedy mechanisms [44], and, more extensively, auction mechanism design problems [19, 37, 61]. To the best of our knowledge, the only other two papers studying the k-FLP in a Bayesian Mechanism Design framework, are [10], where the k-Capacitated Facility Location Problem is considered, and [62], in which the authors study how to use the Lugosi-Mendelson median [45] to define approximately truthful mechanisms for the 1-FLP.

Over the past few decades, Optimal Transport (OT) methods have gradually found their application within the broad landscape of Theoretical Computer Science. Notable examples include Computer Vision [7, 49, 51, 52], Computational Statistics [41], Clustering [6], and Machine Learning in general [21, 33, 55, 56]. However, there has been limited advancement in applying OT theory to the field of mechanism design. To the best of our knowledge, aside the k-Capacitated FLP [10], the only field related to mechanism design that has been explored using OT theory is auction design [22]. In their work, the authors demonstrated that the optimal auction mechanism for independently distributed items can be characterized by the Dual Formulation of an OT problem. Moreover, they utilized this relationship to derive the optimal mechanism for various item classes, thereby establishing a fruitful application of OT theory in the context of mechanism design.

2 Preliminaries

In this section, we fix the necessary notations on the k-Facility Location Problem (k-FLP), Bayesian Mechanism Design, and Optimal Transport (OT). Furthermore, we recall the definition of the percentile mechanisms.

2.1 The k-Facility Location Problem

Given a set of self-interested agents $\mathcal{N} = [n] := \{1, 2, \dots, n\}$, we denote with $\{x_i\}_{i\in[n]}$ the positions of the agents over $\mathbb{R}$. Without loss of generality, assume that the agents are indexed such that the positions x_i's are in non-decreasing order. We denote with $\vec{x} := (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ the vector containing the elements of $\mathcal{X}$. In this setting, if the k facilities are located at the entries of the vector $\vec{y} := (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$, an agent positioned in x_i incurs a cost of $c_i(x_i, \vec{y}) = \min_{j\in[k]} |x_i - y_j|$ to access a facility.[1] Finally, a cost function is a map $C : \mathbb{R}^n \times \mathbb{R}^k \to [0, +\infty)$ that associates to every couple $(\vec{x}, \vec{y})$ the overall cost of placing the facilities at $\vec{y}$ when the agents are located at the entries of $\vec{x}$. Given a vector $\vec{x} \in \mathbb{R}^n$ containing the agents' positions, the *k-Facility Location Problem* with respect to the cost C, consists in finding the locations for k facilities that minimize the function $\vec{y} \to C(\vec{x}, \vec{y})$. There are two typical cost functions. One is the *Social Cost* (*SC*), also known as the *Utilitarian Cost*, which is defined as the sum of all the agents' costs, i.e.,

$$SC(\vec{x}, \vec{y}) = \sum_{i\in[n]} c_i(x_i, \vec{y}).$$

The other is the *Maximum Cost* (*MC*), also known as the *Egalitarian Cost*, which is defined as the maximum cost amongst all agents' costs, i.e.,

$$MC(\vec{x}, \vec{y}) := \max_{i\in[n]} c_i(x_i, \vec{y}).$$

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[1] In what follows, we will use $\vec{y} = (y_1, \dots, y_k)$ and the set of points $\{y_j\}_{j\in[k]}$, intercheangeably. In particular, we say that the facilities are located at $\vec{y}$ instead of saying that the facilities are located at the entries of $\vec{y}$.

These two costs are special cases of the l_p *costs* (C_{l_p}), which are defined as the l_p norm of the agents' costs, i.e.,

$$C_{l_p}(\vec{x},\vec{y}) := \Big(\sum_{i=1}^{n} c_i(x_i,\vec{y})^p\Big)^{\frac{1}{p}},$$

see [28]. Indeed, the Social Cost and the Maximum Cost are the l_1 Cost and l_∞ Cost, respectively. Given the agent positions $\vec{x}$ and the number of facilities k, we denote $SC_{opt}(\vec{x})$, $MC_{opt}(\vec{x})$, and $(C_{l_p})_{opt}(\vec{x})$ the minimal cost for the Social, the Maximum, and the l_p costs, respectively. Given that multiplying the cost function by a constant does not alter the approximation ratio of the mechanisms, we rescale the Social Cost and the l_p costs by the number of agents. With a slight abuse of notation, from now on, we set

$$SC(\vec{x},\vec{y}) = \frac{1}{n}\sum_{i\in[n]} c_i(x_i,\vec{y}) \qquad \text{and} \qquad C_{l_p}(\vec{x},\vec{y}) := \Big(\frac{1}{n}\sum_{i=1}^{n} c_i(x_i,\vec{y})^p\Big)^{\frac{1}{p}},$$

where n represents the total number of agents. As we will see, this amendment simplifies the exposition of our results.

2.2 Mechanism Design and the Worst-Case analysis

In the context of Mechanism Design, a *k-facility location mechanism* is a function $f:\mathbb{R}^n \to \mathbb{R}^k$ that takes the agents' reports $\vec{x}$ in input and returns a set of k locations $\vec{y}$ for the facilities. In general, an agent may misreport its position if it results in a set of facility locations such that the agent's incurred cost is smaller than the cost they would incur by reporting truthfully. A mechanism f is said to be *truthful* (or *strategyproof*) if, for every agent, its cost is minimized when it reports its true position. That is,

$$c_i(x_i, f(\vec{x})) \le c_i(x_i, f(\vec{x}_{-i}, x_i'))$$

for any misreport $x_i' \in \mathbb{R}$, where $\vec{x}_{-i}$ is the vector $\vec{x}$ without its i-th component. Albeit deploying a truthful mechanism instead of computing the optimal location prevents agents from misreporting their positions, it comes with a loss in terms of efficiency. To evaluate this efficiency loss, Nisan and Ronen introduced the notion of approximation ratio [48]. Given a truthful mechanism f, its approximation ratio with respect to the Social Cost is defined as

$$ar(f) := \sup_{\vec{x}\in\mathbb{R}^n} \frac{SC_f(\vec{x})}{SC_{opt}(\vec{x})}, \tag{1}$$

where $SC_f(\vec{x})$ is the Social Cost of placing the facilities at $f(\vec{x})$ and $SC_{opt}(\vec{x})$ is the optimal Social Cost achievable when the agents' report is $\vec{x}$. In what follows, we will refer to the worst-case approximation ratio defined in (1) as the approximation ratio. Similarly, we define the approximation ratio of f with respect to the Maximum Cost and the l_p costs. Depending on the cost we consider, the same mechanism achieves different approximation ratio. In what follows, it will be clear from the context which cost function we use to evaluate the approximation ratio. Evaluating a mechanism f from its approximation ratio is also known as the worst-case analysis of f.

2.3 Bayesian Analysis

In Bayesian Mechanism Design, we assume that the agents' types follow a probability distribution and study the performance of mechanisms from a probabilistic viewpoint. Every agent's type is then described by a random variable X_i. In what follows, we assume that every X_i is identically distributed according to a law (*i.e.* a probability distribution) μ and independent from the other random variables. A mechanism is said to be truthful if, for every agent i, it holds

$$\mathbb{E}_{\vec{X}_{-i}}[c_i(x_i, f(x_i, \vec{X}_{-i}))] \le \mathbb{E}_{\vec{X}_{-i}}[c_i(x_i, f(x_i', \vec{X}_{-i}))] \qquad \forall x_i \in \mathbb{R}, \tag{2}$$

where x_i is the agent i's true type, $\vec{X}_{-i}$ is the $(n-1)$-dimensional random vector that describes the other agents' type, and $\mathbb{E}_{\vec{X}_{-i}}$ is the expectation with respect to the joint distribution of $\vec{X}_{-i}$. Given $\beta \in \mathbb{R}$, let $\vec{X}_n$ be the random vector describing the agents' type; a mechanism f is a β-approximation if $\mathbb{E}[SC_f(\vec{X}_n)] \leq \beta\,\mathbb{E}[SC_{opt}(\vec{X}_n)]$ holds, so that the lower β is, the better the mechanism is. To unify the notation, we define the Bayesian approximation ratio for the Social Cost as the ratio between the expected Social Cost of a mechanism and the expected Social Cost of the optimal solution. More formally, given a mechanism f, its Bayesian approximation ratio with respect to the Social Cost is defined as follows

$$B^{(n)}_{ar,1}(f) := \frac{\mathbb{E}[SC_f(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]}, \tag{3}$$

where the expected value is taken over the joint distribution of the vector $\vec{X}_n := (X_1, \dots, X_n)$. Notice that, if $B^{(n)}_{ar,1}(f) < +\infty$, then f is a $B^{(n)}_{ar,1}(f)$-approximation. Similarly, we define the Bayesian approximation ratio of f with respect to the Maximum Cost and the l_p costs. For the sake of consistency, we denote the Bayesian approximation ratio of f with respect to the l_p cost with $B^{(n)}_{ar,p}(f)$ and the Bayesian approximation ratio with respect to the Maximum Cost with $B^{(n)}_{ar,\infty}(f)$. Since we consider only truthful mechanisms, in what follows we use $\vec{x}$ to denote the vector containing the agents' reports and the agents' real position interchangeably. Moreover, we use the capital letter $\vec{X}_n$ to denote the random vector describing the agents' types.

2.4 The Percentile Mechanisms

The class of *percentile mechanisms* has been introduced in [57]. Given a vector $\vec{v} = (v_1, v_2, \dots, v_k)$, such that $0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq 1$, the percentile mechanism induced by $\vec{v}$, namely $\mathcal{PM}_{\vec{v}}$, proceeds as follows: (i) The mechanism designer collects all the reports of the agents, namely $\{x_1, x_2, \dots, x_n\}$. Without loss of generality, let us assume that the reports are already ordered in non-decreasing order, i.e. $x_i \leq x_{i+1}$. (ii) The designer places the k facilities at the positions $y_j = x_{i_j}$, where $i_j = \lfloor (n-1)v_j \rfloor + 1$. Notice that, if the values x_i are sampled from a distribution, the output of any percentile mechanism is composed by the $(\lfloor (n-1)v_j \rfloor + 1)$-th order statistics of the sample. As established in [57], percentile mechanisms are truthful if the cost of an agent placed at x_i is $c_i = \min_{j\in[k]} |x_i - y_j|$, where y_j are the position of the facilities. Thus, when $k > 2$, the approximation ratio of any percentile mechanism becomes unbounded regardless of the cost considered since the percentile mechanisms are also anonymous and deterministic [32]. In other words, for every percentile vector $\vec{v}$ and regardless of the objective function considered, $ar(\mathcal{PM}_{\vec{v}}) = +\infty$. Moreover, it is worth noting that since percentile mechanisms are truthful in the classic setting, they also retain their truthfulness within the Bayesian framework [35]. Therefore, throughout this paper, there is no need to distinguish between the agents' types and their reported information.

2.5 Basic Notions on Optimal Transport

In the following, we denote with $\mathcal{P}(\mathbb{R})$ the set of probability measures over $\mathbb{R}$. Given a measure $\gamma \in \mathcal{P}(\mathbb{R})$, we denote with $spt(\gamma) \subset \mathbb{R}$ the support of γ, that is, the smallest closed set $C \subset \mathbb{R}$ such that $\gamma(C) = 1$. Furthermore, we denote with $\mathcal{P}_k(\mathbb{R})$ the set of probability measures over $\mathbb{R}$ whose support consists of k points. That is, $\nu \in \mathcal{P}_k(\mathbb{R})$ if and only if $\nu = \sum_{j=1}^k \nu_j \delta_{x_j}$, where $x_j \in \mathbb{R}$ for every $j \in [k]$, $\nu_j \geq 0$ are real values such that $\sum_{j=1}^k \nu_j = 1$, and δ_{x_j} is the Dirac's delta centered in x_j. Given two measures $\alpha, \beta \in \mathcal{P}(\mathbb{R})$ and $p \in [1, +\infty]$, the p-th order Wasserstein distance between α and β is defined as

$$W_p(\alpha, \beta) = \left(\min_{\pi \in \Pi(\alpha,\beta)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p d\pi \right)^{\frac{1}{p}}, \tag{4}$$

where $\Pi(\alpha,\beta)$ is the set of probability measures over $\mathbb{R}\times\mathbb{R}$ whose first marginal is equal to α and the second marginal is equal to β [39]. When $p=+\infty$, we set $W_\infty(\alpha,\beta)=\min_{\pi\in\Pi(\alpha,\beta)}\max_{(x,y)\in spt(\pi)}|x-y|$. It is well-known that, for every $p\in[1,+\infty]$, W_p is a metric over $\mathcal{P}(\mathbb{R})$. For a complete introduction to the Optimal Transport theory, we refer the reader to [59] and [54].

Basic Assumptions. In the remainder of the paper, we tacitly assume that the underlying distribution μ satisfies the following properties: (i) The measure μ is absolutely continuous. We denote with ρ_μ its density. (ii) The support of μ is an interval, which can be bounded or not, and that ρ_μ is strictly positive on the interior of the support. (iii) The density function ρ_μ is differentiable on the support of μ. As an important consequence of this set of assumptions, we have that the cumulative distribution function (c.d.f.) of μ, namely F_μ, is locally bijective. In particular, we have that its pseudo-inverse function, namely $F_\mu^{[-1]}$, is well-defined on $(0,1)$.

3 The Bayesian Analysis of the Percentile Mechanism

In this section, we study the percentile mechanisms in the Bayesian Mechanism Design framework. Specifically, we consider a scenario where the agents' reports are drawn from a shared distribution μ, which satisfies the basic assumptions outlined in Section 2.

3.1 The k-Facility Location Problem as a Wasserstein Projection problem

Given a vector $\vec{x}:=(x_1,x_2,\dots,x_n)$ containing the reports of n agents, we define the measure $\mu_{\vec{x}}:=\frac{1}{n}\sum_{i=1}^n\delta_{x_i}$. Using the map $\vec{x}\to\mu_{\vec{x}}$, we are able to associate any agents' profile to a probability measure in $\mathcal{P}_n(\mathbb{R})\subset\mathcal{P}(\mathbb{R})$. Let us now consider the following minimization problem

$$\min_{\lambda\in\mathcal{P}_k(\mathbb{R})} W_p(\mu_{\vec{x}},\lambda), \tag{5}$$

where $p\in[1,+\infty]$. Due to the metric properties of W_p, problem (5) is also known as the Wasserstein projection problem on $\mathcal{P}_k(\mathbb{R})$. Since $\mathcal{P}_k(\mathbb{R})$ is closed with respect to any W_p metric (see Chapter 7 in [3]), any Wasserstein projection problem admits at least a solution. When $\mu_{\vec{x}}$ is clear from the context, we denote with $\nu^{(k,n)}$ the solution to problem (5). In general, given a measure ζ, we will say that ν is the projection of ζ over $\mathcal{S}\subset\mathcal{P}(\mathbb{R})$ with respect to W_p if $\nu\in\mathcal{S}$ and $W_p(\zeta,\nu)\le W_p(\zeta,\rho)$ for every $\rho\in\mathcal{S}$. According to this notation, we have that $\nu^{(k,n)}$ is the projection of $\mu_{\vec{x}}$ over $\mathcal{P}_k(\mathbb{R})$ with respect to W_p.

The starting point of our Bayesian analysis of the percentile mechanisms consists in connecting the k-FLP to a suitable Wasserstein projection problem. In particular, we show that the objective value of problem (5) is the same as the objective value of the k-FLP.

Theorem 3.1. *Let $\vec{x}=(x_1,\dots,x_n)$ be the reports of n agents. Let $\vec{y}=(y_1,\dots,y_k)$ be the solution to the k-FLP with respect to the Social Cost. Then the set $\{y_j\}_{j\in[k]}$ is the support of a measure $\nu^{(k,n)}$ that solves problem* (5) *with $p=1$. Moreover, we have that*

$$SC_{opt}(\vec{x})=W_1(\mu_{\vec{x}},\nu^{(k,n)})=\min_{\lambda\in\mathcal{P}_k(\mathbb{R})}W_1(\mu_{\vec{x}},\lambda).$$

Vice-versa, if $\nu\in\mathcal{P}_k(\mathbb{R})$ is a solution to problem (5)*, then its support $\{y_j\}_{j\in[k]}$ is a solution to the k-FLP with respect to the Social Cost. Similarly, if $p=\infty$, the support of the solution to problem* (5) *is the optimal location with respect to the Maximum Cost. Vice-versa, the entries of every optimal location vector $\vec{y}$ for the k-FLP problem with respect to the Maximum Cost is the support of a minimizer of problem* (5) *for $p=\infty$. Finally, the same implications hold for every $p\in(1,+\infty)$. In this case, the solution is taken with respect to the l_p Cost.*

Proof. Let $\vec{x}$ be the vector containing the reports of n agents, and let $\vec{y}$ be the vector containing the optimal location for k facilities with respect to the Social Cost when the agents are located according to $\vec{x}$. Without loss of generality, we assume that the closest facility to each agent x_i is unique so that the sets A_j, defined as

$$A_j := \Big\{ x_i \;:\; \min_{l\in[k]} |x_i - y_l| = |x_i - y_j| \Big\},$$

are well-defined and disjoint.

First, we show that, given an optimal facility location $\vec{y}$, it is possible to retrieve a measure $\nu \in \mathcal{P}_k(\mathbb{R})$ that solves the projection problem (5) for $p = 1$ and whose support is $\{y_j\}_{j\in[k]}$.

For every y_j, let us set $\nu_j = \frac{\ell_j}{n}$, where $\ell_j := |A_j|$ is the number of agents whose closest facility is located at y_j. We then set $\nu = \sum_{j\in[k]} \nu_j \delta_{y_j}$. Since A_j are disjoint sets, we have $\nu \in \mathcal{P}_k(\mathbb{R})$. Let us now consider the transportation plan, namely π, between $\mu_{\vec{x}}$ and ν defined as

$$\pi_{i,j} := \pi_{x_i,y_j} = \begin{cases} \frac{1}{n} & \text{if} \quad x_i \in A_j \\ 0 & \text{otherwise.} \end{cases}$$

Since according to π every agent goes to its closest facility, π is optimal, thus we have $W_1(\mu_{\vec{x}}, \nu) = \sum_{i\in[n],j\in[k]} |x_i - y_j|\pi_{i,j} = \frac{1}{n}\sum_{j\in[k]}\sum_{x_i\in A_j} |x_i - y_j|$. We now show that ν solves problem (5) for $p = 1$. Toward a contradiction, let us assume that $\tilde{\nu} = \sum_{j=1}^{k} \tilde{\nu}_j \delta_{\tilde{y}_j} \in \mathcal{P}_k(\mathbb{R})$ is such that $W_1(\mu_{\vec{x}}, \tilde{\nu}) < W_1(\mu_{\vec{x}}, \nu)$. Let us define the partition of agents $\tilde{A}_j$ related to the set of points $\{\tilde{y}_j\}_{j\in[k]}$.[2] Therefore, we have

$$\frac{1}{n}\sum_{j\in[k]}\sum_{x_i\in\tilde{A}_j} |x_i - \tilde{y}_j| = W_1(\mu_{\vec{x}}, \tilde{\nu}_j) < W_1(\mu_{\vec{x}}, \nu) = \frac{1}{n}\sum_{j\in[k]}\sum_{x_i\in A_j} |x_i - y_j|, \tag{6}$$

which contradicts the optimality of $\vec{y}$. Hence the first implication of the theorem is proven.

For the inverse implication, it suffices to repeat the same argument backwards. Indeed, let ν be a solution to the W_1 Projection problem. Toward a contradiction, let us assume that the support of ν is not a solution to the k-FLP. Then, given a solution to the k-FLP problem, we can use the argument used in the first part of the proof to build a new measure that has a lower cost than ν, which would contradict the optimality of the initial solution.

Through a similar argument, we get the same conclusion for every $p \in (1, +\infty]$. □

By restricting the set on which the projection problem is defined, we retrieve a similar characterization for the cost of any k-facility location mechanism.

Theorem 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a k-facility location mechanism. Then, the Social Cost of f is the minimal distance between $\mu_{\vec{x}}$ and a closed subset of $\mathcal{P}_k(\mathbb{R})$ with respect to W_1, that is*

$$SC_f(\vec{x}) = \min_{\{\lambda_j\}_{j\in[k]}} W_1(\mu_{\vec{x}}, \lambda), \tag{7}$$

where $\lambda = \sum_{j\in[k]} \lambda_j \delta_{y_j}$ and $\vec{y} = (y_1, y_2, \ldots, y_k) = f(\vec{x})$. Similarly, the l_p Cost or the Maximum Cost of the mechanism f can be expressed as a projection problem described in (7) by replacing W_1 with W_p or W_∞, respectively.

Proof. Let f be a mechanism, $\vec{x}$ the vector containing the reports of n agents, and let us denote with $\vec{y}$ the vector containing the positions returned by the mechanism f, so that $\vec{y} = f(\vec{x})$. For every $j \in [k]$, let us denote with A_j the set of agents that are closer to the facility placed at y_j. Without loss of generality, we assume that every A_j is disjoint from the other so that $A_j \cap A_r = \emptyset$ for every $j \neq r$. Let us now define $\nu^{(n)}$ as

$$\nu^{(n)} = \sum_{j\in[k]} \nu_j^{(n)} \delta_{y_j}$$

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[2] Again, without loss of generality, we can assume that the facility that is closest to a given agent is unique.

where $\nu_j^{(n)} = \frac{\ell_j}{n}$ and $\ell_j = |A_j|$. We now show that $\nu^{(n)}$ is a solution to problem (7). Indeed, the discrete probability measure π is defined as

$$\pi_{i,j} := \pi_{x_i,y_j} = \begin{cases} \frac{1}{n} & \text{if} \quad x_i \in A_j \\ 0 & \text{otherwise,} \end{cases}$$

is a transportation plan between $\mu_{\vec{x}}$ and $\nu^{(n)}$. Furthermore, since according to π every agent goes to its closest facility, we have

$$\sum_{i\in[n]} \sum_{j\in[k]} |x_i - y_j|\pi_{i,j} = W_1(\mu_{\vec{x}}, \nu^{(n)}).$$

Finally, if $\tilde{\nu}$ is such that $spt(\tilde{\nu}) = spt(\nu) = \{y_j\}_{j\in[k]}$ and $W_1(\mu_{\vec{x}}, \tilde{\nu}) < W_1(\mu_{\vec{x}}, \nu^{(n)})$, we infer that there exists at least one agent that can be reallocated to a closer facility, which would contradict the definition of A_j. This concludes the proof for the Social Cost case.

Through a similar argument, we get to the same conclusion for both the l_p costs and the Maximum Cost. □

Notice that the projection problem (7) is a further restricted version of the projection problem (5). Indeed, in (5), the support of the solution can be any subset of $\mathbb{R}$ containing k elements, while in (7), the support of the solution is fixed by the mechanism f.

3.2 The Bayesian Analysis of Percentile Mechanisms for Social Cost

In this section, we use the results presented in Theorem 3.1 and Theorem 3.2 to study the limiting behaviour of the Bayesian approximation ratio of any percentile mechanism with respect to the Social Cost. First, we consider the case in which the percentile vector $\vec{v}$ does not have any binary entry, i.e. $\vec{v} \in (0,1)^k$. Secondly, we extend our obtained results to encompass the general case where $\vec{v} \in [0,1]^k$. Indeed, these two cases yield different results and need to be approached differently.

The techniques we develop to study the Social Cost can be used to handle the Maximum Cost and the l_p Cost. However, depending on the cost function under discussion, the set of assumptions needed to make the Bayesian ratio converge changes. For this reason, we restrict our analysis only to the Social Cost and defer the study of the l_p and Maximum Costs to 3.4.

From Theorem 3.1, the k-FLP with respect to the Social Cost can be cast as a projection problem in the space of probability distributions with respect to W_1. It is well-known that, in order to ensure that the W_1 distance between two measures is finite, both the measures must have a finite first moment [59]. We recall that a measure μ has a finite first moment if

$$\int_{\mathbb{R}} |x|d\mu < +\infty. \tag{8}$$

Lemma 3.1. *Let $\vec{X}_n := (X_1, X_2, \dots, X_n)$ be the random vector describing the reports of n i.i.d. agents distributed as μ. If μ satisfies* (8)*, then, for every $k \in \mathbb{N}$, we have that $\mathbb{E}[SC_{opt}(\vec{X}_n)]$ converges to $W_1(\mu, \nu^{(k)})$ as $n \to \infty$, where $\nu^{(k)}$ is the solution to the following projection problem*

$$\min_{\lambda\in\mathcal{P}_k(\mathbb{R})} W_1(\mu, \lambda). \tag{9}$$

In particular, we have that $\mathbb{E}[SC_{opt}(\vec{X}_n)]$ is strictly positive for n large enough.

Proof. Let $\nu^{(k,n)}$ be the solution to problem (5) and let $\nu^{(k)}$ be the solution to problem (9). Owing to the triangular inequality and to the properties of the projection problem, we have

$$W_1(\mu_{\vec{x}}, \nu^{(k,n)}) \leq W_1(\mu_{\vec{x}}, \nu^{(k)}) \leq W_1(\mu_{\vec{x}}, \mu) + W_1(\mu, \nu^{(k)})$$

and, similarly

$$W_1(\mu, \nu^{(k)}) \leq W_1(\mu, \nu^{(k,n)}) \leq W_1(\mu, \mu_{\vec{x}}) + W_1(\mu_{\vec{x}}, \nu^{(k,n)}),$$

which implies $|W_1(\mu, \nu^{(k)}) - W_1(\mu_{\vec{x}}, \nu^{(k,n)})| \leq W_1(\mu, \mu_{\vec{x}})$ and thus

$$\mathbb{E}[|W_1(\mu, \nu^{(k)}) - W_1(\mu_{\vec{x}}, \nu^{(k,n)})|] \leq \mathbb{E}[W_1(\mu, \mu_{\vec{x}})].$$

Since $\lim_{n\to\infty} \mathbb{E}[W_1(\mu, \mu_{\vec{x}})] = 0$ (see Theorem 5.3 in [16]), we infer that $\mathbb{E}[W_1(\mu_{\vec{x}}, \nu^{(k,n)})]$ converges to $W_1(\mu, \nu^{(k)})$ as $n \longrightarrow \infty$. Finally, since $W_1(\mu, \nu^{(k)})$ is strictly positive, for n large enough, we have $\mathbb{E}[W_1(\mu_{\vec{x}}, \nu^{(k,n)})]$ is strictly positive as well. □

Let us now consider the limit expected cost of the mechanism, we observe that the set characterizing the projection problem (7) is dependent on the output of the percentile mechanism. Hence, the argument used to prove Lemma 3.1 cannot be directly applied in this case. However, by employing a more sophisticated construction and leveraging the convergence properties of the k-th order statistics, it is possible to identify the limit of $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]$ and ensure convergence by imposing mild assumptions on the percentile vector $\vec{v}$.

Lemma 3.2. *Let μ be a measure that satisfies* (8). *Given $k \in \mathbb{N}$, let $\vec{v} \in (0,1)^k$ be a percentile vector. Then, $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]$ converges to $W_1(\mu, \nu_{Q_{\vec{v}}})$, where $\nu_{Q_{\vec{v}}}$ is defined as*

$$\nu_{Q_{\vec{v}}} := \sum_{i=1}^{k} (F_\mu(z_i) - F_\mu(z_{i-1}))\delta_{F_\mu^{[-1]}(v_i)}, \tag{10}$$

where $z_i = \frac{(F_\mu^{[-1]}(v_i)+F_\mu^{[-1]}(v_{i+1}))}{2}$ for $i = 1, \dots, k-1$, $z_0 = \inf_{x\in spt(\mu)} x$, and $z_k = \sup_{x\in spt(\mu)} x$, F_μ is the cumulative distribution function of μ, and $F_\mu^{[-1]}$ is the pseudo-inverse function related to μ.

Proof. First, we notice that the measure (10) is well-defined since there exists a j such that $v_j \neq 0, 1$. Let $\vec{v} = (v_1, \dots, v_k)$ be a percentile vector, $\vec{x}$ be the vector containing the reports of the agents, $\nu^{(k,n)}$ be the solution to problem (7), and let $\vec{y}$ be the vector containing the facility positions returned by the percentile mechanisms. Without loss of generality, we assume that $y_\ell \leq y_{\ell+1}$ for every $\ell \in [k-1]$, so that $\nu^{(k,n)} = \sum_{j\in[k]}(\nu^{(k,n)})_j\delta_{y_j}$. To lighten-up the notation, we set $\nu_{Q_{\vec{v}}} := \nu_Q$, so that $\nu_Q := \sum_{j\in[k]}(\nu_Q)_j\delta_{F_\mu^{[-1]}(v_j)}$, where $(\nu_Q)_j := (F_\mu(z_j) - F_\mu(z_{j-1}))$, where $z_0 = -\infty$, $z_k = +\infty$, and $z_i = \frac{y_i+y_{i+1}}{2}$ for every $i = 2, \dots, k-1$. We now show that ν_Q is the solution to the following minimization problem

$$\min_{\{\lambda_j\}_{j\in[k]}} W_1(\lambda, \mu), \tag{11}$$

where $\lambda = \sum_{j=1}^{k} \lambda_j \delta_{F_\mu^{[-1]}(v_j)}$. We can rewrite the W_1 distance between μ and ν_Q as follows

$$W_1(\mu, \nu_Q) = \sum_{j=1}^{k} \int_{F_\mu^{[-1]}(\sum_{i=1}^{j}(\nu_Q)_i)}^{F_\mu^{[-1]}(\sum_{i=1}^{j+1}(\nu_Q)_i)} |x - F_\mu^{[-1]}(v_j)| d\mu.$$

By definition of ν_Q we have that $\sum_{i=1}^{j}(\nu_{Q_{\vec{v}}})_i = F_\mu(z_j)$, thus

$$W_1(\mu, \nu_Q) = \sum_{j=0}^{k} \int_{z_j}^{z_{j+1}} |x - F_\mu^{[-1]}(v_j)| d\mu = \int_{-\infty}^{+\infty} \min_{j\in[k]} |x - F_\mu^{[-1]}(v_j)| d\mu, \tag{12}$$

where we used the fact that $z_0 = -\infty$, $z_k = +\infty$, and $z_i = \frac{F_\mu^{[-1]}(v_i)+F_\mu^{[-1]}(v_{i+1})}{2}$ for every $i = 2, \dots, k-1$. Thus every point in the support of μ is assigned to its closest facility, thus ν_Q is a solution to (11).

We are now ready to study the convergence of $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]$. For every $n \in \mathbb{N}$, let us define $\gamma_n = \sum_{j\in[k]}(\nu_Q)_j\delta_{y_j}$[3], then, by a similar argument to the one used to prove Lemma 3.1, we have

$$W_1(\mu_{\vec{x}}, \nu^{(k,n)}) \le W_1(\mu_{\vec{x}}, \gamma_n) \le W_1(\mu_{\vec{x}}, \mu) + W_1(\mu, \nu_Q) + W_1(\nu_Q, \gamma_n).$$

For every $n \in \mathbb{N}$, let us now define $\eta_n := \sum_{j\in[k]}(\nu^{(k,n)})_j\delta_{F_\mu^{[-1]}(v_j)}$. We then have

$$W_1(\mu, \nu_Q) \le W_1(\mu, \eta_n) \le W_1(\mu, \mu_{\vec{x}}) + W_1(\mu_{\vec{x}}, \nu^{(k,n)}) + W_1(\nu^{(k,n)}, \eta_n).$$

Since $W_1(\nu_Q, \gamma_n), W_1(\nu^{(k,n)}, \eta_n) \ge 0$, we infer that

$$\mathbb{E}[|W_1(\mu, \nu_Q) - W_1(\mu_{\vec{x}}, \nu^{(k,n)})|] \le \mathbb{E}[W_1(\mu, \mu_{\vec{x}})] + \mathbb{E}[W_1(\nu_Q, \gamma_n)] + \mathbb{E}[W_1(\nu^{(k,n)}, \eta_n)]. \tag{13}$$

From [16], we have that $\lim_{n\to\infty}\mathbb{E}[W_1(\mu, \mu_{\vec{x}})] = 0$. Thus, if $n \to \infty$, if we prove that both $\mathbb{E}[W_1(\nu_Q, \gamma_n)]$ and $\mathbb{E}[W_1(\nu^{(k,n)}, \eta_n)]$ go to zero as $n \to \infty$ we conclude the proof, since it would imply that $\mathbb{E}[W_1(\mu_{\vec{x}}, \nu^{(k,n)})]$ converges to $W_1(\mu, \nu_Q)$ as $n \to \infty$.

First, we show that $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \to 0$ as $n \to \infty$. Given a set of reports $\vec{x} = (x_1, \dots, x_n)$, let $\vec{y} = (y_1, \dots, y_k)$ be the facility locations returned by the percentile mechanism $\mathcal{PM}_{\vec{v}}$. Since every y_j is the $(\lfloor (n-1)v_j \rfloor + 1)$-th statistic of the sample vector $\vec{x}$, we have that

$$\mathbb{E}[W_1(\nu_Q, \gamma_n)] \le \sum_{j\in[k]}(\nu_Q)_j\mathbb{E}[|X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j)|] \le \sum_{j\in[k]}\mathbb{E}[|X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j)|],$$

where $X_{\lfloor (n-1)v_j\rfloor+1}$ is the $(\lfloor (n-1)v_j \rfloor + 1)$-th order statistic of the sample vector $\vec{x}$. Owing to Bahadur's representation formula (see [11] and [40]), we can express the random variable associated with the empiric v_j-th quantile of a distribution as follows

$$X_{\lfloor (n-1)v_j\rfloor+1} = F_\mu^{[-1]}(v_j) + \frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))} + R_n, \tag{14}$$

where R_n is a rest function, for which holds $R_n \le O(n^{-\frac{3}{4}})$ with probability 1, ρ_μ is the density of μ, and $S_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(X_i)_{\{X_i\le t\}}$ where

$$\mathbb{I}(X_i)_{\{X_i\le t\}} = \begin{cases} 1 & \text{if} \quad X_i \le t, \\ 0 & \text{otherwise.} \end{cases}$$

By rearranging the terms in (14), we get

$$X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j) = \frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))} + R_n, \tag{15}$$

hence, by taking the absolute value and the expected value of both sides of (15), we have

$$\mathbb{E}[|X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j)|] \le \mathbb{E}\left[\left|\frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}\right|\right] + \mathbb{E}[|R_n|]. \tag{16}$$

To conclude, we need to prove that the right-hand side of equation (16) goes to zero. First, we observe that $\frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}$ converges to 0 almost surely. Indeed, by definition we have that $S_n(F_\mu^{[-1]}(v_j)) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(X_i)_{\{X_i\le F_\mu^{[-1]}(v_j)\}}$. Denoted with $Y_i = \mathbb{I}(X_i)_{\{X_i\le F_\mu^{[-1]}(v_j)\}}$, we have that every Y_i is a Bernoulli variable that takes value 1 with probability $F_\mu(F_\mu^{[-1]}(v_j)) = v_j$ and takes value 0 with probability $1 - v_j$. In particular, the

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[3] We recall that y_j is the j-th point in the support of $\nu^{(k,n)}$

expected value of every Y_i is v_j. Moreover, since every X_i is independent from the other X_ls, we have that every Y_i is independent from the other Y_ls. By the Law of Large Numbers [27], we then have that the expected value of $|S_n(F_\mu^{[-1]}(v_j)) - v_j|$ converges to 0 almost surely as n goes to infinity. Lastly, notice that, from our assumptions on μ, we have that $\rho_\mu(F_\mu^{[-1]}(v_j)) > 0$, hence we conclude that $\frac{S_n(F_\mu^{[-1]}(v_j))-v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}$ converges to 0 almost surely.

Owing to the fact that $\frac{S_n(F_\mu^{[-1]}(v_j))-v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}$ converges to 0 almost surely, to the i.i.d. assumption on the X_is, and to condition (8), we have that $\frac{S_n(F_\mu^{[-1]}(v_j))-v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}$ converges to 0 with respect to the L^1 norm, since

$$\mathbb{E}\left[\left|\frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}\right|\right] \leq K\mathbb{E}\Big[\Big|\sum_{i=1}^{n}\frac{(Z_i - p)}{n}\Big|\Big], \tag{17}$$

where $\{Z_i\}_{i\in[n]}$ is a group of i.i.d. Bernoulli random variables that are equal to 1 with probability p and equal to 0 with probability $1-p$. By Jensen's inequality, we get

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n}\frac{(Z_i - p)}{n}\Big|\Big] \leq \frac{1}{n}\Big(\mathbb{E}[\sum_{i=1}^{n}|Z_i - p|^2]\Big)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}Var(Z_1)$$

and thus

$$\mathbb{E}\left[\left|\frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}\right|\right] \leq O(n^{-\frac{1}{2}}).$$

If we prove that R_n converges to 0 with respect to the L^1 norm, we infer that $\mathbb{E}[W_1(\nu^{(k)},\gamma_n)] \to 0$. We already know that R_n converges to 0 almost surely.

We now show that R_n is uniformly integrable, *i.e.*

$$\lim_{a\to+\infty}\sup_n \int_a^{+\infty}|R_n|dx = 0 \tag{18}$$

and that $\sup_n \mathbb{E}[|R_n|] < +\infty$, which allows us to conclude that R_n converges with respect to the L^1 norm using the Vitali convergence theorem (see Theorem 4.5.4 in [17]). From Bahadur's formula (15), we have that

$$R_n = X_{\lfloor(n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j) - \frac{S_n(F_\mu^{[-1]}(v_j)) - v_j)}{\rho_\mu(F_\mu^{[-1]}(v_j))},$$

which proves that $\sup_n \mathbb{E}[|R_n|] < +\infty$, since $\sup_n \mathbb{E}[|X_{\lfloor(n-1)v_j\rfloor+1}|] < +\infty$[4] and the other quantities are bounded random variables. We now prove (18). Since $\frac{S_n(F_\mu^{[-1]}(v_j))-v_j)}{\rho_\mu(F_\mu^{[-1]}(v_j))}$ and $F_\mu^{[-1]}(v_j)$ are both bounded, it suffices to show that the sequence $X_{\lfloor(n-1)v_j\rfloor+1}$ is uniformly integrable. It is well-known that the density of the $(\lfloor(n-1)v_j\rfloor+1)$-th order statistic of an absolutely continuous random variable is given by the formula

$$\frac{n!}{(n-\lfloor(n-1)v_j\rfloor)!(\lfloor(n-1)v_j\rfloor)!}\rho_\mu(x)[F_\mu(x)]^{\lfloor(n-1)v_j\rfloor}[1-F_\mu(x)]^{n-\lfloor(n-1)v_j\rfloor} \tag{19}$$

(see Theorem 5.4.4 in [18]). To prove that the sequence $X_{\lfloor(n-1)v_j\rfloor+1}$ is uniformly integrable we then need to prove that

$$\lim_{a\to+\infty}\sup_n \int_a^{+\infty}\frac{n!\ x\ \rho_\mu(x)[F_\mu(x)]^{\lfloor(n-1)v_j\rfloor}[1-F_\mu(x)]^{n-\lfloor(n-1)v_j\rfloor}}{(n-\lfloor(n-1)v_j\rfloor)!(\lfloor(n-1)v_j\rfloor)!}dx = 0.$$

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[4] This follows from the fact that sample quantiles are asymptotically unbiased.

Since X has finite expected value, namely m, and every c.d.f. is increasing, we get

$$\begin{aligned}\int_a^{+\infty} \frac{n!\; x\; \rho_\mu(x)[F_\mu(x)]^{\lfloor (n-1)v_j \rfloor}[1-F_\mu(x)]^{n-\lfloor (n-1)v_j \rfloor}}{(n-\lfloor (n-1)v_j \rfloor-1)!(\lfloor (n-1)v_j \rfloor)!}dx \\ \leq \frac{n!\; m}{(n-\lfloor (n-1)v_j \rfloor-1)!(\lfloor (n-1)v_j \rfloor)!}[1-F_\mu(a)]^{n-\lfloor (n-1)v_j \rfloor}. \end{aligned} \tag{20}$$

From the properties of the binomial coefficient, we have that

$$\frac{n!}{(n-\lfloor (n-1)v_j \rfloor-1)!(\lfloor (n-1)v_j \rfloor)!}[1-F_\mu(a)]^{n-\lfloor (n-1)v_j \rfloor} \leq \alpha_n[1-F_\mu(a)]^{(1-q)n},$$

where $q \in (0,1)$ is such that $\max_{j\in[k]} v_j < q$ and

$$\alpha_n = \begin{cases} \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} & \text{if} \quad n \text{ is even,} \\ \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!} & \text{otherwise.} \end{cases} \tag{21}$$

To conclude the proof, we show that there exists a value a' such that for every $a > a'$ the sum of $\alpha_n(1-F_\mu(a))^{(1-q)n}$ converges; hence $\alpha_n(1-F_\mu(a))^{(1-q)n}$ is infinitesimal for every $a > a'$. Indeed, using the ratio test criteria, we get that, for odd n

$$\lim_{n\to\infty} \frac{(n+1)!}{(\frac{n+1}{2})!(\frac{n+1}{2})!} \frac{(\frac{(n+1)}{2})!(\frac{(n-1)}{2})!}{n!} \frac{[1-F_\mu(a)]^{(1-q)(n+1)}}{[1-F_\mu(a)]^{(1-q)n}} = 2[1-F_\mu(a)]^{(1-q)},$$

thus, there exists a large enough value a for which

$$2[1-F_\mu(a)]^{(1-q)} < 1.$$

Similarly, we show that also for even n we infer the same conclusion, proving the uniform integrability of R_n. In particular, R_n converges to zero with respect to the L^1 norm. Moreover, notice that $\mathbb{E}[R_n] \leq O(n^{-\frac{3}{4}})$, which proves that $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \leq O(n^{-\frac{1}{2}})$, thus $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \to 0$. In particular, R_n converges to zero with respect to the L^1 norm and $\mathbb{E}[R_n] \leq O(n^{-\frac{3}{4}})$, which proves that $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \leq O(n^{-\frac{1}{2}})$, thus $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \to 0$.

We now prove that $\mathbb{E}[W_1(\nu^{(k,n)}, \eta_n)] \to 0$ as $n \to \infty$. From a similar argument, we have that

$$\mathbb{E}[W_1(\nu^{(k,n)}, \eta_n)] \leq \sum_{j\in[k]} \nu_j^{(k,n)} \mathbb{E}[|X_{\lfloor (n-1)v_j \rfloor+1} - F_\mu^{[-1]}(v_j)|] \leq \sum_{j\in[k]} \mathbb{E}[|X_{\lfloor (n-1)v_j \rfloor+1} - F_\mu^{[-1]}(v_j)|],$$

which, by the same argument we used to show $\mathbb{E}[W_1(\nu_Q, \gamma_n)] \leq O(n^{-\frac{1}{2}})$ as $n \to \infty$, allows us to conclude that

$$\lim_{n\to\infty} \mathbb{E}[W_1(\nu^{(k,n)}, \eta_n)] \to 0.$$

To conclude, we notice that $\mathbb{E}[|SC_{\vec{v}}(\vec{X}_n) - W_1(\mu, \nu_Q)|] \leq \mathbb{E}[W_1(\mu_{\vec{x}}, \mu)] + O(n^{-\frac{1}{2}})$. □

By combining the convergence results shown in Lemma 3.1 and Lemma 3.2, we infer that the Bayesian approximation ratio of any $\mathcal{PM}_{\vec{v}}$ converges to a bounded quantity.

Theorem 3.3. *Let $\vec{X}_n$ be a random vector of n i.i.d. variables distributed according to μ and let $\vec{v} \in (0,1)^k$ be a percentile vector. Then, if μ satisfies* (8)*, we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]} = \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}.$$

Proof. From Lemma 3.1, we have that $\mathbb{E}[SC_{opt}(\vec{X}_n)]$ converges to $W_1(\mu,\nu^{(k)})$, where $\nu^{(k)}$ is the solution to problem (9). Since μ is absolutely continuous and $\nu^{(k)}$ is a discrete measure, we have that $W_1(\mu,\nu^{(k)})>0$. From Lemma 3.2, have that $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]$ converges to $W_1(\mu,\nu_{Q_{\vec{v}}})$, where $\nu_{Q_{\vec{v}}}$ is defined as in (10). Finally, since both quantities are well-defined and finite, we have

$$\lim_{n\to\infty}\frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]}=\frac{W_1(\mu,\nu_{Q_{\vec{v}}})}{W_1(\mu,\nu^{(k)})}<+\infty,$$

which concludes the proof. □

Theorem 3.3 ensures that the limit of the Bayesian approximation ratio of any percentile mechanisms is upper-bounded by a quantity that depends only on μ, k, and $\vec{v}$. For an illustration, we compute this quantity for a generic k and $\vec{v}$, and μ is the uniform distribution over $[0,1]$.

Example 3.1. *Let $\vec{v}=(v_1,\dots,v_k)$ be a percentile vector and let the underlying distribution μ be the uniform distribution over $[0,1]$. The measure $\nu_{Q_{\vec{v}}}$ is then defined as $\nu_{Q_{\vec{v}}}:=\sum_{i=1}^k\frac{v_{i+1}-v_{i-1}}{2}\delta_{v_i}$, where $v_0=0$ and $v_{k+1}=1$. It is easy to see that the projection of μ over $\mathcal{P}_k(\mathbb{R})$ is $\nu^{(k)}:=\frac{1}{k}\sum_{j=1}^k\delta_{\frac{2j-1}{2k}}$. From a simple computation, we infer that*

$$W_1(\mu,\nu_{Q_{\vec{v}}})=\sum_{i=1}^k\Big[\frac{(v_{i+1}-v_i)^2+(v_i-v_{i-1})^2}{2}\Big]$$

and $W_1(\mu,\nu^{(k)})=\frac{1}{4k}$. Moreover, since $v_i\le v_{i+1}$ and $v_j\in[0,1]$, we have that $(v_{i+1}-v_i)^2\le v_{i+1}-v_i$, and obtain

$$\lim_{n\to\infty}B_{ar,1}^{(n)}(\mathcal{PM}_{\vec{v}})\le 4k\sum_{i=1}^k\Big[\frac{(v_{i+1}-v_{i-1})}{2}\Big]\le 4k.$$

That is, when the agents are distributed according to an uniform distribution, the Bayesian approximation ratio of any percentile mechanism for the k-FLP is upper bounded by $4k$.

We now characterize the convergence rate of the Bayesian approximation ratio. To do so, we need to require that μ has a compact support or that there exists a $\delta>0$ such that

$$\int_{\mathbb{R}}|x|^{2+\delta}d\mu<+\infty. \tag{22}$$

In both frameworks, we have that the convergence rate is at most of the order $n^{-\frac{1}{2}}$.

Theorem 3.4. *Let $\vec{X}_n$ be a random vector of n i.i.d. variables distributed according to μ and let $\vec{v}\in(0,1)^k$ be a percentile vector. If either μ is supported on a compact set or μ satisfies (8), we have that*

$$\left|\frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]}-\frac{W_1(\mu,\nu_{Q_{\vec{v}}})}{W_1(\mu,\nu^{(k)})}\right|\le O(n^{-\frac{1}{2}}). \tag{23}$$

Thus the convergence rate of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ is $O(n^{-\frac{1}{2}})$. Moreover, for every $\vec{v}\in(0,1)^k$, there exists a constant value $C>0$ such that, for every $n>k$, we have

$$B_{ar,1}^{(n)}(\mathcal{PM}_{\vec{v}})\le\frac{W_1(\mu,\nu_{Q_{\vec{v}}})}{W_1(\mu,\nu^{(k)})}+\frac{C}{\sqrt{n}}, \tag{24}$$

where $\nu_{Q_{\vec{v}}}$ is defined in (10) and $\nu^{(k)}$ is a minimizer of (9).

Proof. It suffices to show that both $\left|\mathbb{E}[SC_{opt}(\vec{X}_n)] - W_1(\mu, \nu^{(k)})\right| \leq O(n^{-\frac{1}{2}})$ and $\left|\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] - W_1(\mu, \nu_{Q_{\vec{v}}})\right| \leq O(n^{-\frac{1}{2}})$ hold. Indeed, in this case, we have that

$$\begin{aligned}
\left|\frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]} - \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}\right| &= \left|\frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]W_1(\mu, \nu^{(k)}) - W_1(\mu, \nu_{Q_{\vec{v}}})\mathbb{E}[SC_{opt}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]W_1(\mu, \nu^{(k)})}\right| \\
&\leq \left|\frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]W_1(\mu, \nu^{(k)}) - W_1(\mu, \nu_{Q_{\vec{v}}})W_1(\mu, \nu^{(k)})]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]W_1(\mu, \nu^{(k)})}\right| \\
&\quad + \left|\frac{W_1(\mu, \nu_{Q_{\vec{v}}})W_1(\mu, \nu^{(k)}) - W_1(\mu, \nu_{Q_{\vec{v}}})\mathbb{E}[SC_{opt}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]W_1(\mu, \nu^{(k)})}\right| \\
&\leq \frac{\left|\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] - W_1(\mu, \nu_{Q_{\vec{v}}})\right|}{\mathbb{E}[SC_{opt}(\vec{X}_n)]} + \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{\mathbb{E}[SC_{opt}(\vec{X}_n)]}\frac{\left|W_1(\mu, \nu^{(k)}) - \mathbb{E}[SC_{opt}(\vec{X}_n)\right|}{W_1(\mu, \nu^{(k)})} \\
&\leq O(n^{-\frac{1}{2}}),
\end{aligned}$$

which concludes the proof. The inequality $\left|W_1(\mu, \nu^{(k)}) - \mathbb{E}[SC_{opt}(\vec{X}_n)\right| \leq O(n^{-\frac{1}{2}})$ follows from the result in [16], since

$$\left|W_1(\mu, \nu^{(k)}) - \mathbb{E}[SC_{opt}(\vec{X}_n)\right| \leq \mathbb{E}[|W_1(\mu, \nu^{(k)}) - SC_{opt}(\vec{X}_n)|] \leq \mathbb{E}[W_1(\mu_{\vec{x}}, \mu)] \leq O(n^{-\frac{1}{2}}).$$

The identity $\mathbb{E}\left[\left|SC_{\vec{v}}(\vec{X}_n) - W_1(\mu, \nu_{Q_{\vec{v}}})\right|\right] \leq O(n^{-\frac{1}{2}})$ has been partially shown in the proof of Lemma 3.2. Indeed, we have shown that

$$\left|\mathbb{E}[SC_{\vec{v}}(\vec{x})] - W_1(\mu, \nu_{Q_{\vec{v}}})\right| \leq \mathbb{E}\left[\left|SC_{\vec{v}}(\vec{X}_n) - W_1(\mu, \nu_{Q_{\vec{v}}})\right|\right] \leq \mathbb{E}[W_1(\mu_{\vec{x}}, \mu)] + O(n^{-\frac{1}{2}}),$$

which, in conjunction with the estimate $\mathbb{E}[W_1(\mu_{\vec{x}}, \mu)] \leq O(n^{-\frac{1}{2}})$, concludes the first half of the proof.

Lastly, we notice that, when $n > k$, we have that $\mathbb{E}[SC_{opt}(\vec{X}_n)] > 0$, hence $B_{ar,1}^{(n)}(\mathcal{PM}_{\vec{v}})$ is well-defined. By definition of $O(n^{-\frac{1}{2}})$, there exists a constant $C > 0$ such that

$$\left|B_{ar,1}^{(n)}(\mathcal{PM}_{\vec{v}}) - \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}\right| \leq \frac{C}{\sqrt{n}},$$

thus

$$B_{ar,1}^{(n)}(\mathcal{PM}_{\vec{v}}) \leq \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})} + \frac{C}{\sqrt{n}},$$

which concludes the proof. □

It is noteworthy that the value $\frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}$ in the bound (24), is a constant that does not depend on n, but depends only on the specifics of the problem, that is μ, k, and $\vec{v}$.

3.3 The case $v_j \in \{0, 1\}$ for at least one $j \in [k]$

The condition $\vec{v} \in (0, 1)^k$ plays a central role in the proof of Lemma 3.2 since it allows us to study the convergence of the $(\lfloor (n-1)v_j \rfloor + 1)$-th order statistic through Bahadur's formula. In this section, we show that, if we drop $\vec{v} \in (0, 1)^k$, we are still able to show that the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ converges, but we lose the convergence speed guarantee in most of the cases.

For the sake of clarity, we divide the discussion into two sub-cases: first, we consider the case in which μ has a compact (or equivalently, bounded) support, and then we move to the case in which the support of μ is unbounded. For both sub-cases, we restrict our study to the Social Cost since the same argument can be applied to the other costs.

3.3.1 *The case $spt(\mu)$ is compact.*

In this setting, the measure (10) is well-defined, since $F_\mu^{[-1]}(0) = a$ and $F_\mu^{[-1]}(1) = b$, where $a < b$ are the extreme points of the support of μ, i.e. $spt(\mu) = [a, b]$. However, unlike the case when $\vec{v} \in (0, 1)^k$, we need to consider the possibility that ρ_μ might vanish at the boundaries of the support, i.e., $\rho_\mu(a) = 0$ and/or $\rho_\mu(b) = 0$. If this happens, we can no longer use Bahadur's formula. Although without using Bahadur's formula we are unable to estimate the convergence speed of $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]$, we characterize the limit of the Bayesian approximation ratio of any $\mathcal{PM}_{\vec{v}}$ when n goes to ∞.

Theorem 3.5. *Let μ be a measure such that $spt(\mu)$ is a compact interval. Then, given a percentile vector $\vec{v} \in [0, 1]^k$, the limit of the expected Social Cost of $\mathcal{PM}_{\vec{v}}$ is equal to the objective value of problem* (11)*, that is $\lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{v}}})$, where $\nu_{Q_{\vec{v}}}$ is the measure defined in* (10)*. In particular, we have that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]} = \frac{W_1(\mu, \nu_{Q_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}, \tag{25}$$

where $\nu^{(k)}$ is the solution to problem $\min_{\rho\in\mathcal{P}_k(\mathbb{R})} W_1(\mu, \rho)$.

Proof. We only need to establish that $\lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{v}}})$, since identity (25) is a straightforward consequence. We notice that, if $v_j = 0$, the output of mechanism $\mathcal{PM}_{\vec{v}}$ always contains the first order statistic of the sample, which we denote with $X_{1:n}$. Similarly, if $v_j = 1$ for some $j \in [k]$, the output of $\mathcal{PM}_{\vec{v}}$ always contains the last-order statistic of the sample, which we denote with $X_{n:n}$.

Following the same arguments used in the proof of Lemma 3.2, we have that equation (13) holds. Thus, to conclude the proof, it suffices to show that

$$\lim_{n\to\infty} \sum_{j\in[k]} \mathbb{E}[|X_{\lfloor (n-1)v_j \rfloor+1} - F_\mu^{[-1]}(v_j)|] = 0.$$

During the proof of Lemma 3.2, we have already shown that

$$\lim_{n\to\infty} \mathbb{E}[|X_{\lfloor (n-1)v_j \rfloor+1} - F_\mu^{[-1]}(v_j)|] = 0$$

for every $v_j \neq 0, 1$. We then just need to focus on the cases $\lim_{n\to\infty} \mathbb{E}[|X_{1:n} - F_\mu^{[-1]}(0)|] = \lim_{n\to\infty} \mathbb{E}[|X_{1:n} - a|] = 0$ and $\lim_{n\to\infty} \mathbb{E}[|X_{n:n} - F_\mu^{[-1]}(1)|] = \lim_{n\to\infty} \mathbb{E}[|X_{n:n} - b|] = 0$. Since these two cases are similar, we only focus on the first one, i.e. $\lim_{n\to\infty} \mathbb{E}[|X_{1:n} - a|] = 0$. Since the density of $X_{1:n}$ is well known to be

$$n\ \rho_\mu(x)(1 - F(x))^{n-1},$$

we rewrite the expected value $\mathbb{E}[|X_{1:n} - a|]$ as follows

$$\begin{aligned}
\mathbb{E}[|X_{1:n} - a|] &= \int_a^b |x - a|\ n\ \rho_\mu(x)(1 - F(x))^{n-1} dx \\
&= \int_a^{a+\varepsilon} |x - a|\ n\ \rho_\mu(x)(1 - F(x))^{n-1} dx + \int_{a+\epsilon}^b |x - a|\ n\ \rho_\mu(x)(1 - F(x))^{n-1} dx \\
&\leq \int_a^{a+\varepsilon} \epsilon\ n\ \rho_\mu(x)(1 - F(x))^{n-1} dx + n\,(1 - F(a + \epsilon))^{n-1} \int_{a+\epsilon}^b |x - a| \rho_\mu(x) dx \\
&\leq \epsilon + Cn(1 - F(a + \epsilon))^{n-1},
\end{aligned}$$

where $C = \int_{a+\epsilon}^{b} |x-a|\rho_\mu(x)dx$ is a finite constant. Since $F(a+\epsilon) > 0$ for every $\epsilon > 0$, we infer

$$\lim_{n\to\infty} \mathbb{E}[|X_{1:n} - a|] \leq \epsilon + \lim_{n\to\infty} Cn(1 - F(a+\epsilon))^{n-1} = \epsilon.$$

Since $\mathbb{E}[|X_{1:n} - a|] \geq 0$ for every n and $\lim_{n\to\infty} \mathbb{E}[|X_{1:n} - a|] \leq \epsilon$ for every $\epsilon > 0$, we infer $\lim_{n\to\infty} \mathbb{E}[|X_{1:n} - a|] = 0$, which concludes the proof. □

Lastly, if the density $\rho_\mu(x)$ is strictly greater than a positive constant, we use Bahadur's formula once again and retrieve a convergence speed guarantee for the Bayesian approximation ratio.

Theorem 3.6. *Let μ be a measure whose support is compact, let $\vec{v} \in [0,1]^k$ a percentile vector, and let $\delta > 0$ be such that $\rho_\mu(x) > \delta$ for every $x \in spt(\mu)$. Then, we have $|\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] - W_1(\mu, \nu_{Q_{\vec{v}}})| \leq O(n^{-\frac{1}{2}})$. In particular, the convergence rate of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ is $O(n^{-\frac{1}{2}})$.*

Proof. The proof of this theorem is the same as the one of Lemma 3.2. Indeed, since $\rho_\mu(x) > \delta > 0$ on $spt(\mu)$, we are able to use Bahadur's formula and thus estimate the convergence rate of the numerator of the Bayesian approximation ratio. By the same arguments used in the proof of Theorem 3.4, we have that the convergence speed of the Bayesian approximation ratio is $O(n^{-\frac{1}{2}})$. □

3.3.2 The case $spt(\mu)$ is unbounded. When the support of the measure is unbounded, we have that either the first or the last order statistic of the random variable $X \sim \mu$ do diverge. As a consequence, the measure $\nu_{Q_{\vec{v}}}$ (see (10)) might not be well-defined. To avoid such scenarios, we assume that the percentile mechanism does have at least one entry that is neither 0 nor 1.

First, we consider the case in which the support of μ is $\mathbb{R}$. Notice that the density of μ, namely ρ_μ, is infinitesimal when x goes to infinity, i.e.

$$\lim_{|x|\to\infty} \rho_\mu(x) = 0, \tag{26}$$

thus, we cannot use Bahadur's formula to study the convergence of either the first or the last-order statistic. Notice that (26) holds for any absolutely continuous measure supported over $\mathbb{R}$. So, in this setting, there is no way to use Bahadur's formula and retrieve any convergence speed guarantee for the Bayesian approximation ratio of any $\mathcal{PM}_{\vec{v}}$ if $\vec{v}$ has at least a 0 or 1 entry. Even though we cannot estimate its convergence speed, the Bayesian approximation ratio converges to a finite constant when n goes to ∞.

Theorem 3.7. *Let $\vec{v} \in [0,1]^k$ be a percentile vector such that $v_j \notin \{0,1\}$ for at least one index $j \in [k]$ and let μ be a probability measure whose support is $\mathbb{R}$. Then, we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]} = \frac{W_1(\mu, \nu_{\tilde{Q}_{\vec{v}}})}{W_1(\mu, \nu^{(k)})}, \tag{27}$$

where $\nu_{\tilde{Q}_{\vec{v}}}$ is defined as

$$\nu_{\tilde{Q}_{\vec{v}}} = \sum_{i\in N_{\vec{v}}} (F_\mu(z_i) - F_\mu(z_{i-1}))\delta_{F_\mu^{[-1]}(v_i)}, \tag{28}$$

where $N_{\vec{v}} := \{j \in [k], \quad s.t. \quad v_j \notin \{0,1\}\}$ and $z_i = \frac{(F_\mu^{[-1]}(v_i) + F_\mu^{[-1]}(v_{i+1}))}{2}$ for $i = 1, \dots, k'-1$, $z_0 = -\infty$, and $z_{k'} = +\infty$, where k' is the cardinality of $N_{\vec{v}}$, i.e. $k' = |N_{\vec{v}}|$.

Proof. Let us denote with $\vec{w} \in \mathbb{R}^{k'}$ the vector obtained removing all the 0 and 1 entries from $\vec{v}$. Then, for every given instance $\vec{x}$ of the k-FLP, we have that $\mathcal{PM}_{\vec{w}}(\vec{x}) \subset \mathcal{PM}_{\vec{v}}(\vec{x})$, thus we have $SC_{\vec{v}}(\vec{x}) \leq SC_{\vec{w}}(\vec{x})$ on every instance $\vec{x}$, hence

$$\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] \leq \mathbb{E}[SC_{\vec{w}}(\vec{X}_n)].$$

By taking the lim sup on both sides, we infer

$$\limsup_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] \leq \limsup_{n\to\infty} \mathbb{E}[SC_{\vec{w}}(\vec{X}_n)] = \lim_{n\to\infty} \mathbb{E}[SC_{\vec{w}}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{w}}}), \tag{29}$$

where the last equality follows from Lemma 3.2, since $\vec{w} \in (0,1)^k$. To conclude, we show that $\liminf_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{x})] \geq W_1(\mu, \nu_{Q_{\vec{w}}})$, which, combined with (29), concludes the proof. For the sake of clarity, we focus on the case in which there is only one entry of v that is in $\{0,1\}$, since the case in which there are multiple 0-1 entries is analogue. Without loss of generalization, we assume $v_1 = 0$, since the case $v_k = 1$ is symmetric.

For every $T > 0$, let us denote with E_T the event $\{X_{1:n} \leq -T\}$. Then, we have

$$\lim_{n\to\infty} P(E_T) = 1. \tag{30}$$

Indeed, by definition, we have $P(X_{1:n} \leq -T) = 1 - (1 - F_\mu(-T))^n$. Since $F_\mu(-T) > 0$ for every T, we infer (30).

Given an instance $\vec{x}$, let $\nu^{(k,n)}$ be the solution to $\min_{\lambda \in S_{f(\vec{x})} \subset \mathcal{P}_k(\mathbb{R})} W_1(\mu_{\vec{x}}, \lambda)$. We define ν'_n as follows

$$\nu'_n = (\nu^{(k,n)})_1 \delta_{x_1} + \sum_{j=2}^{k} (\nu^{(k,n)})_j \delta_{F_\mu^{[-1]}(w_j)},$$

so that ν'_n is the probability measure whose support is $\{x_1\} \cup \{F_\mu^{[-1]}(w_j)\}_{j=2,\ldots,k}$ and associates to x_1 probability $(\nu^{(k,n)})_1$ and probability $(\nu^{(k,n)})_j$ to the point $F_\mu(w_j)$ for $j = 2, \ldots, k$.

Since W_1 is metric, we can use the triangular inequality to infer

$$W_1(\mu, \nu'_n) \leq W_1(\mu, \mu_{\vec{x}}) + W_1(\mu_{\vec{x}}, \nu^{(k,n)}) + W_1(\nu^{(k,n)}, \nu'_n),$$

or, equivalently

$$W_1(\mu, \nu'_n) - W_1(\mu, \mu_{\vec{x}}) - W_1(\nu^{(k,n)}, \nu'_n) \leq W_1(\mu_{\vec{x}}, \nu^{(k,n)}). \tag{31}$$

If we take the expected value on both sides of (31) and take the liminf, we get

$$\liminf_{n\to\infty} \mathbb{E}[W_1(\mu, \nu'_n)] - \liminf_{n\to\infty} \mathbb{E}[W_1(\nu^{(k,n)}, \nu'_n)] \leq \liminf_{n\to\infty} \mathbb{E}[W_1(\mu_{\vec{x}}, \nu^{(k,n)})],$$

since, from [16], we have $\liminf_{n\to\infty} \mathbb{E}[W_1(\mu, \mu_{\vec{x}})] = \lim_{n\to\infty} \mathbb{E}[W_1(\mu, \mu_{\vec{x}})] = 0$.

First, we show that $\liminf_{n\to\infty} \mathbb{E}[W_1(\nu^{(k,n)}, \nu'_n)] = 0$. Since, for every instance $\vec{x}$, both $\nu^{(k,n)}$ and ν'_n have a mass equal to $(\nu^{(k,n)})_1$ at x_1 and since the optimal transportation plan between two measures on the line is monotone (see [59]), we have the following bound

$$\mathbb{E}[W_1(\nu^{(k,n)}, \nu'_n)] \leq \sum_{j=2}^{k} \mathbb{E}[|X_{\lfloor (n-1)v_j \rfloor + 1} - F_\mu^{[-1]}(w_j)|].$$

Then, by the same argument used during the proof of Lemma 3.2, the right side of the latter inequality converges to 0 when $n \to \infty$.

Secondly, we study the term $\liminf_{n\to\infty} \mathbb{E}[W_1(\mu, \nu'_n)]$. Given T, we rewrite the expectation as follows

$$\mathbb{E}[W_1(\mu, \nu'_n)] = \mathbb{E}[W_1(\mu, \nu'_n) | E_T^c] P(E_T^c) + \mathbb{E}[W_1(\mu, \nu'_n) | E_T] P(E_T).$$

We notice that $\mathbb{E}[W_1(\mu, \nu'_n) | E_T^c]$ is bounded, since $W_1(\mu, \nu'_n) \leq W_1(\mu, \delta_{F_\mu^{[-1]}(v_2)})$, which is finite, since μ has finite first moment and $v_2 \notin \{0,1\}$. Since for every T we have that $\lim_{n\to\infty} P(E_T) = 1$, we infer that $\lim_{n\to\infty} P(E_T^c) = 0$, thus

$$\lim_{n\to\infty} \mathbb{E}[W_1(\mu, \nu'_n) | E_T^c] P(E_T^c) = 0.$$

We then focus our attention on the term $\lim_{n\to\infty}\mathbb{E}[W_1(\mu,\nu'_n)|E_T]P(E_T)$. By definition, we have

$$\begin{aligned}\mathbb{E}[W_1(\mu,\nu'_n)|E_T] \geq{}& \mathbb{E}\left[\int_{-\infty}^{\frac{X_{1:n}+F_\mu^{[-1]}(v_2)}{2}} |x-X_{1:n}|d\mu \quad \middle| E_T\right] + \mathbb{E}\left[\int_{\frac{X_{1:n}+F_\mu^{[-1]}(v_2)}{2}}^{\frac{F_\mu^{[-1]}(v_2)+F_\mu^{[-1]}(v_3)}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right]\\ &+\sum_{j=3}^{k-1}\mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{j-1})+F_\mu^{[-1]}(v_j)}{2}}^{\frac{F_\mu^{[-1]}(v_j)+F_\mu^{[-1]}(v_{j+1})}{2}} |x-F_\mu^{[-1]}(v_j)|d\mu \quad \middle| E_T\right] + \mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{k-1})+F_\mu^{[-1]}(v_k)}{2}}^{+\infty} |x-F_\mu^{[-1]}(v_k)|d\mu \quad \middle| E_T\right]\end{aligned}$$

Since $\mathbb{E}\left[\int_{-\infty}^{\frac{X_{1:n}+F_\mu^{[-1]}(v_2)}{2}} |x-X_{1:n}|d\mu \quad \middle| E_T\right]$ is a positive term, we have that

$$\begin{aligned}&\mathbb{E}[W_1(\mu,\nu'_n)|E_T]\\ &\quad\geq \mathbb{E}\left[\int_{\frac{X_{1:n}+F_\mu^{[-1]}(v_2)}{2}}^{\frac{F_\mu^{[-1]}(v_2)+F_\mu^{[-1]}(v_3)}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right] + \sum_{j=3}^{k-1}\mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{j-1})+F_\mu^{[-1]}(v_j)}{2}}^{\frac{F_\mu^{[-1]}(v_j)+F_\mu^{[-1]}(v_{j+1})}{2}} |x-F_\mu^{[-1]}(v_j)|d\mu \quad \middle| E_T\right]\\ &\quad\quad+\mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{k-1})+F_\mu^{[-1]}(v_k)}{2}}^{+\infty} |x-F_\mu^{[-1]}(v_k)|d\mu \quad \middle| E_T\right]\\ &\quad\geq \mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_2)-T}{2}}^{\frac{F_\mu^{[-1]}(v_2)+F_\mu^{[-1]}(v_3)}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right] + \sum_{j=3}^{k-1}\mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{j-1})+F_\mu^{[-1]}(v_j)}{2}}^{\frac{F_\mu^{[-1]}(v_j)+F_\mu^{[-1]}(v_{j+1})}{2}} |x-F_\mu^{[-1]}(v_j)|d\mu \quad \middle| E_T\right]\\ &\quad\quad+\mathbb{E}\left[\int_{\frac{F_\mu^{[-1]}(v_{k-1})+F_\mu^{[-1]}(v_k)}{2}}^{+\infty} |x-F_\mu^{[-1]}(v_k)|d\mu \quad \middle| E_T\right]\\ &\quad= \mathbb{E}[W_1(\mu,\nu_{\tilde{Q}_{\tilde{v}}})|E_T] - \mathbb{E}\left[\int_{-\infty}^{\frac{F_\mu^{[-1]}(v_2)-T}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right]\\ &\quad= W_1(\mu,\nu_{\tilde{Q}_{\tilde{v}}}) - \mathbb{E}\left[\int_{-\infty}^{\frac{F_\mu^{[-1]}(v_2)-T}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right],\end{aligned}$$

so that

$$\liminf_{n\to\infty}\ \mathbb{E}[W_1(\mu,\nu'_n)] \geq \liminf_{n\to\infty}\left(W_1(\mu,\nu_{\tilde{Q}_{\tilde{v}}}) - \mathbb{E}\left[\int_{-\infty}^{\frac{F_\mu^{[-1]}(v_2)-T}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \quad \middle| E_T\right]\right)P(E_T). \tag{32}$$

Finally, since μ has finite first moment, we have that

$$\lim_{T\to\infty}\int_{-\infty}^{\frac{F_\mu^{[-1]}(v_2)-T}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu = 0,$$

hence, for any given $\epsilon>0$, there exists a T_ϵ for which

$$\int_{-\infty}^{\frac{F_\mu^{[-1]}(v_2)-T_\epsilon}{2}} |x-F_\mu^{[-1]}(v_2)|d\mu \leq \epsilon \tag{33}$$

holds. Therefore, combining equation (33) with (32), we find

$$\liminf_{n\to\infty}\mathbb{E}[W_1(\mu,\nu^{(k,n)})] \geq \liminf_{n\to\infty}(W_1(\mu,\nu_{\tilde{Q}_{\tilde{v}}})-\epsilon)P(E_{T_\epsilon}) = W_1(\mu,\nu_{\tilde{Q}_{\tilde{v}}})-\epsilon,$$

which, in conjunction with (29), concludes the proof. □

Lastly, we address the scenarios in which the support of the probability measure μ is unbounded on one side, i.e. when $spt(\mu) = (-\infty, A]$ or $spt(\mu) = [A, +\infty)$ for a constant $A \in \mathbb{R}$. To handle these cases, we combine the arguments used for the case $spt(\mu) = \mathbb{R}$ with the arguments deployed for the case where $spt(\mu) = [a, b]$. Indeed, following the arguments used during the proof of Theorem 3.7, if $spt(\mu) = (-\infty, A]$, we have that the first order statistic of X does not converge if the number of samples n goes to infinity. Similarly, following the arguments used during the proof of Theorem 3.6, if $v_j = 1$ for some $j \in [k]$, the n-th order statistic of X does converge to A if the number of samples n goes to infinity. Thus, if $\vec{v}$ is such that $v_j = 0$ for some $j \in [k]$, we have that $\mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{w}}})$, where $\vec{w}$ is the vector containing all the non null entries of $\vec{v}$. Moreover, if $v_j = 0$ for some $j \in [k]$ or if $v_j = 1$ and $\rho_\mu(A) = 0$, we do not have any guarantee on the convergence speed of $SC_{\vec{v}}(\vec{X}_n)$. A similar argument allows us to deal with the case $spt(\mu) = [A, +\infty)$.

To close this section, we show that given a percentile vector $\vec{v} \in [0, 1]^k$, either $\vec{v} \in (0, 1)^k$ or there exists a percentile vector $\vec{w} \in (0, 1)^k$ such that $w_j = v_j$ if $v_j \neq 0, 1$ and

$$\lim_{n\to\infty} \mathbb{E}[SC_{\vec{w}}(\vec{X}_n)] < \lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)]. \tag{34}$$

Therefore, in the Bayesian framework, we can restrict our attention to percentile vectors in $(0, 1)^k$ without incurring any efficiency loss.

Theorem 3.8. *Let $\vec{v} \in [0, 1]^k \backslash (0, 1)^k$ be a percentile vector. Then, there exists a percentile vector $\vec{w} \in (0, 1)^k$ such that $w_j = v_j$ if $v_j \neq 0, 1$ and for which* (34) *holds.*

Proof. Without loss of generality, let us assume that the entries of the percentile vector $\vec{v}$ are all different, i.e. $v_j \neq v_i$ if $i \neq j$. Let us define $N_0 = \{j \in [k] \;\text{ s.t. }\; v_j = 0\}$ and $N_1 = \{j \in [k] \;\text{ s.t. }\; v_j = 1\}$. Since $\vec{v} \in [0, 1]^k \backslash (0, 1)^k$ either N_0 or N_1 is not empty. We denote with k_0 and k_1 the cardinality of N_0 and N_1, respectively. For every $\epsilon \in (0, 1)$, we defined $\vec{w}_\epsilon \in (0, 1)^k$ as follows

$$(\vec{w}_\epsilon)_j = \begin{cases} \epsilon \;\text{ if }\; j \in N_0, \\ 1 - \epsilon \;\text{ if }\; j \in N_1, \\ v_j \;\text{ otherwise.} \end{cases} \tag{35}$$

To prove this theorem, we show that there exists a value of ϵ for which (34) holds when we set $\vec{w} = \vec{w}_\epsilon$.

First, let us consider the case in which $spt(\mu) = \mathbb{R}$. By Theorem 3.7, we infer

$$\lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] = W_1(\mu, \nu_{\tilde{Q}_{\vec{v}}}),$$

where $\nu_{\tilde{Q}_{\vec{v}}}$ is defined as in (28). Let us now consider ϵ such that $\epsilon \neq v_j$ and $1 - \epsilon \neq v_j$ for every $j \in [k]$. Then, according to Lemma 3.2, we have

$$\lim_{n\to\infty} \mathbb{E}[SC_{\vec{w}_\epsilon}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{w}_\epsilon}}),$$

where $\nu_{Q_{\vec{w}_\epsilon}}$ is defined as in (10). Since $spt(\nu_{\tilde{Q}_{\vec{v}}}) \subset spt(\nu_{Q_{\vec{w}_\epsilon}})$ and (12) hold, we infer

$$W_1(\mu, \nu_{Q_{\vec{w}_\epsilon}}) = \int_{\mathbb{R}} \min_{y_j \in spt(\nu_{Q_{\vec{w}_\epsilon}})} |x - y_j| d\mu < \int_{\mathbb{R}} \min_{y_j \in spt(\nu_{\tilde{Q}_{\vec{v}}})} |x - y_j| d\mu = W_1(\mu, \nu_{\tilde{Q}_{\vec{v}}}).$$

Thus, $W_1(\mu, \nu_{Q_{\vec{w}_\epsilon}}) < W_1(\mu, \nu_{\tilde{Q}_{\vec{v}}})$, which concludes the proof for the case $spt(\mu) = \mathbb{R}$.

Let us now consider the case in which the support of μ is a compact interval, namely $[a, b]$. From Theorem 3.5, we have that

$$\lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(\vec{X}_n)] = W_1(\mu, \nu_{Q_{\vec{v}}}),$$

where $\nu_{Q_{\vec{v}}}$ is defined as in (10). Without loss of generality, let us assume that $v_1 = 0$ and $v_j \in (0,1)$ for $j = 2, \dots, k$. By definition of $\nu_{Q_{\vec{v}}}$, we have that

$$W_1(\mu, \nu_{Q_{\vec{v}}}) = \int_a^{z_1} |x - y_1| d\mu + \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} |x - y_j| d\mu,$$

where $y_1 \leq y_2 \leq \dots \leq y_k$ are points in the support of $\nu_{Q_{\vec{v}}}$, $z_0 = a$, $z_k = b$, and $z_j = \frac{y_{j-1}+y_j}{2}$ for $j = 2, \dots, k-1$. Let us consider the following integral

$$\int_a^{z_1} |x - y_1| d\mu.$$

Since y_1 is the leftmost point in the support of $\nu_{Q_{\vec{v}}}$ and there exists an index j such that $v_j = 0$, we have $y_1 = a$. Let y_1' be the median of μ restricted to $[a, z_1]$, then we have

$$\int_a^{z_1} |x - y_1| d\mu = \int_a^{z_1} |x - a| d\mu > \int_a^{z_1} |x - y_1'| d\mu. \tag{36}$$

Moreover, we have

$$\int_a^{z_1} |x - y_1| d\mu + \int_{z_1}^{z_2} |x - y_2| d\mu > \int_a^{z_1'} |x - y_1'| d\mu + \int_{z_1'}^{z_2} |x - y_2| d\mu,$$

where $z_1' = \frac{y_2 + y_1'}{2}$, hence we infer

$$\begin{aligned} W_1(\mu,\nu_{Q_{\vec{v}}}) &= \int_a^{z_1} |x - y_1| d\mu + \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} |x - y_j| d\mu \\ &> \int_a^{z_1'} |x - y_1'| d\mu + \int_{z_1'}^{z_2} |x - y_2| d\mu + \sum_{j=3}^{k} \int_{z_{j-1}}^{z_j} |x - y_j| d\mu = W_1(\mu, \nu'), \end{aligned}$$

where

$$\nu' = (F_\mu(z_1'))\delta_{y_1'} + (F_\mu(z_2) - F_\mu(z_1'))\delta_{y_2} + \sum_{j=3} (\nu_{Q_{\vec{v}}})_j \delta_{y_j}.$$

We notice that $\nu' = \nu_{Q_{\vec{w}_\epsilon}}$ if we set $\epsilon = F_\mu(y_1')$. Indeed, we have $(w_\epsilon)_1 = F_\mu(y_1')$ and $(w_\epsilon)_j = v_j$ for every $j = 2, \dots, k$. Using again Lemma 3.2, we have

$$\lim_{n\to\infty} \mathbb{E}[SC_{\vec{w}_\epsilon}(X_n)] = W_1(\mu, \nu_{Q_{\vec{w}_\epsilon}}) < W_1(\mu, \nu_{Q_{\vec{v}}}) = \lim_{n\to\infty} \mathbb{E}[SC_{\vec{v}}(X_n)].$$

Through a similar argument, we deal with the case $v_j = 1$ for an index $j \in [k]$ and the case in which $\vec{v}$ has both a 0 and 1 entry.

Lastly, we consider the case in which $spt(\mu)$ is unbounded from the left (i.e. $spt(\mu) = (-\infty, R]$ for $R \in \mathbb{R}$) or from the right (i.e. $spt(\mu) = [R, +\infty)$ for $R \in \mathbb{R}$). In this case, however, it suffices to combine the argument used for the cases $spt(\mu) = [a, b]$ and $spt(\mu) = \mathbb{R}$. ☐

3.4 The Bayesian Analysis of the Percentile Mechanism for the l_p and Maximum Cost

We now extend the results we presented for the Social Cost to encompass the l_p and the Maximum Cost.

3.4.1 The l_p-Cost case. First, we consider the l_p Cost. As for the Social Cost, to ensure the convergence of the Bayesian approximation ratio, the underlying distribution μ has to be regular enough. Given $p \in (1, +\infty)$, μ has finite p-th moment if

$$\int_{\mathbb{R}} |x|^p d\mu < +\infty. \tag{37}$$

Condition (37) ensures us that the problem $\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_p(\mu, \lambda)$ is well-defined and that its objective value is finite. Indeed, if μ satisfies condition (37), both Lemma 3.1 and Lemma 3.2 can be extended to the l_p Cost case. As a main consequence, for any percentile mechanisms, the limit of the Bayesian approximation ratio with respect to the l_p Cost exists and is finite.

Theorem 3.9. *If μ satisfies the hypothesis of Theorem 3.3 and* (37)*, the limit of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ with respect to the l_p Cost converges to $\frac{W_p(\mu, \nu_{Q_{\vec{v}}})}{W_p(\mu, \nu^{(k)})}$ when $n \to \infty$, where $\nu_{Q_{\vec{v}}}$ is defined in* (10) *and $\nu^{(k)}$ is a solution to the problem* $\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_p(\mu, \lambda)$.

Proof. The proofs of both Lemma 3.1 and Lemma 3.2 are based on the triangular inequality of W_1, then, since W_p is a distance as well, we can extend both the results to the l_p Cost. Moreover, to have a well-defined projection problem with respect to the W_p norm, μ must satisfy (37). Since the minimal W_p distance between an absolutely continuous measure and $\mathcal{P}_k(\mathbb{R})$ is positive, we conclude the proof by the same argument used in the proof of Theorem 3.3. □

Similarly to the Social Cost case, to control the convergence speed of the Bayesian approximation ratio, the underlying distribution μ must satisfy a stricter version of condition (37).

Theorem 3.10. *Let $\vec{v} \in (0,1)^k$ be a percentile vector. If μ has finite $(2p+\delta)$-th moment for $\delta > 0$, that is, $\int_{\mathbb{R}} |x|^{2p+\delta} d\mu < +\infty$, and satisfies the hypothesis of Theorem 3.3, we have that the convergence rate of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ is $O(n^{-\frac{1}{2p}})$. In particular, we have that*

$$B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}}) \leq \frac{W_p(\mu, \nu_{Q_{\vec{v}}})}{W_p(\mu, \nu^{(k)})} + \frac{C}{n^{\frac{1}{2p}}}, \tag{38}$$

where $\nu_{Q_{\vec{v}}}$ is defined in (10) *and $\nu^{(k)}$ is a solution to the problem*

$$\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_p(\mu, \lambda). \tag{39}$$

Proof. It follows from the same argument used in Proof of Theorem 3.4. Indeed, from [16], we have that $\leq O(n^{-\frac{1}{2p}})$ under these assumptions. To prove that also the denominator convergences with speed $O(n^{-\frac{1}{2p}})$, we use the same argument we used in Proof of Lemma 3.2. In this case, using Bahadur's formula again [23], we just need to estimate

$$\mathbb{E}\left[\left|\frac{S_n(F_\mu^{[-1]}(v_j)) - v_j}{\rho_\mu(F_\mu^{[-1]}(v_j))}\right|^p\right] \leq C\mathbb{E}\left[\left|S_n(F_\mu^{[-1]}(v_j)) - v_j\right|^p\right],$$

where C is a positive constant that depends on the derivative of the probability distribution. Indeed, the rest of Bahadur's formula is uniformly integrable, and its convergence speed with respect to the L^1 norm is still $O(n^{-\frac{1}{2}})$. By expressing explicitly S_n, we get

$$\mathbb{E}\left[\left|S_n(F_\mu^{[-1]}(v_j)) - v_j\right|^p\right] = \mathbb{E}\left[\left|\frac{1}{n}\sum_{k=1}^n (Y_k - v_j)\right|^p\right] \leq \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^n (Y_k - v_j)^p\right],$$

where the last inequality comes from Jensen's inequality and where Y_k are i.i.d. Bernoulli random variables that are equal to 1 with probability v_j and 0 with probability $1 - v_j$. Using Holder's inequality, we retrieve

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n}(Y_k - v_j)^p\right] \leq \frac{1}{n}\left(\mathbb{E}\left[\left(\sum_{k=1}^{n}(Y_k - v_j)^p\right)^2\right]\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}Var((Y_k - v_j)^p).$$

We then conclude the proof by taking the p-th root estimation we got. The upper bound on $B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}})$ follows by the same argument used in the proof of Theorem 3.4. □

Notice that all the contents of Section 3.3 can be extended to the l_p cost functions proven that the measure μ is regular enough, that is it satisfies $\int_{\mathbb{R}}|x|^{2p+\delta}d\mu < +\infty$. We summarize the main results in the following theorem.

THEOREM 3.11. *Let μ be a measure such that $spt(\mu)$ is a compact interval. Then, given a percentile vector $\vec{v} \in [0,1]^k$, we have that*

$$\lim_{n\to\infty}\frac{\mathbb{E}[(C_{l_p})_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[(C_{l_p})_{opt}(\vec{X}_n)]} = \frac{W_p(\mu,\nu_{Q_{\vec{v}}})}{W_p(\mu,\nu^{(k)})}, \tag{40}$$

where $\nu^{(k)}$ is the solution to problem $\min_{\rho\in\mathcal{P}_k(\mathbb{R})} W_p(\mu,\rho)$. Moreover, if $\delta > 0$ is such that $\rho_\mu(x) > \delta$ for every $x \in spt(\mu)$, we have $|\mathbb{E}[(C_{l_p})_{\vec{v}}(\vec{X}_n)] - W_p(\mu,\nu_{Q_{\vec{v}}})| \leq O(n^{-\frac{1}{2}})$. In particular, the convergence rate of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ is $O(n^{-\frac{1}{2}})$.

PROOF. Notice that if μ has support compact, than the results from Section 5 of [16] allow us to infer that also the denominator of the Bayesian approximation ratio converges as $O(n^{-\frac{1}{2}})$. The remainder of the proof follows by the argument used to prove Theorem 3.5 and 3.6. □

Finally, Theorem 3.8 holds also for this framework, in particular, if $\vec{v} \in [0,1]^k \setminus (0,1)^k$ is a percentile vector, there exists a percentile vector $\vec{w} \in (0,1)^k$ such that $w_j = v_j$ if $v_j \neq 0, 1$ whose associated percentile mechanism has a lower Bayesian approximation ratio with respect to any l_p costs.

3.4.2 The Maximum Cost case. Finally, we consider the Maximum Cost. In this case, we have to assume that μ is compactly supported since otherwise $W_\infty(\mu,\nu) = +\infty$ for every $\nu \in \mathcal{P}_k(\mathbb{R})$ and, therefore, the limit of the Bayesian approximation ratio of any percentile mechanism is not well-defined. When μ has compact support, it is possible to extend Lemma 3.1 and 3.2, and thus also Theorem 3.3, to the Maximum Cost case. Under these stronger assumptions, also the convergence speed guarantee presented in Theorem 3.4 extends to the Maximum Cost case.

THEOREM 3.12. *Let $\vec{X}_n$ be a random vector of n i.i.d. variables distributed according to a compactly supported measure μ and let $\vec{v} \in (0,1)^k$ be a percentile vector. Then, the limit of the Bayesian approximation ratio of every $\mathcal{PM}_{\vec{v}}$ is finite and it converges to $\frac{W_\infty(\mu,\nu_{Q_{\vec{v}}})}{W_\infty(\mu,\nu^{(k)})}$ when $n \to \infty$, where $\nu_{Q_{\vec{v}}}$ is defined in (10) and $\nu^{(k)}$ is a solution to the problem $\min_{\lambda\in\mathcal{P}_k(\mathbb{R})} W_\infty(\mu,\lambda)$. Furthermore, the convergence rate of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ is $O(n^{-\frac{1}{2}})$. In particular, we have that*

$$B^{(n)}_{ar,\infty}(\mathcal{PM}_{\vec{v}}) \leq \frac{W_\infty(\mu,\nu_{Q_{\vec{v}}})}{W_\infty(\mu,\nu^{(k)})} + \frac{C}{\sqrt{n}}. \tag{41}$$

PROOF. First, we study the denominator. From [16], we have that $\mathbb{E}[W_\infty(\mu_{\vec{x}},\mu)] \leq O(n^{-\frac{1}{2}})$, then, by the same argument used in Lemma 3.1, we conclude that

$$\mathbb{E}[MC(\vec{X}_n) - W_\infty(\mu,\nu^{(k)})] \leq O(n^{-\frac{1}{2}}).$$

In particular, $\mathbb{E}[MC(\vec{X}_n)]$ converges to $W_\infty(\mu, \nu^{(k)})$. Since $W_\infty(\mu, \nu^{(k)})$ is strictly positive, we infer that also $\mathbb{E}[MC(\vec{X}_n)]$ is strictly positive for n large enough.

To conclude, we consider the numerator. Since we have that

$$\mathbb{E}\Big[\max_{j\in[k]} |X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j)|\Big] \leq \sum_{i=1}^{k} \mathbb{E}[|X_{\lfloor (n-1)v_j\rfloor+1} - F_\mu^{[-1]}(v_j)|],$$

we can use the same argument used in Lemma 3.2 (notice that the uniform integrability follows from the fact that also $X_{\lfloor (n-1)v_j\rfloor+1}$ is bounded) and conclude that

$$\mathbb{E}\Big[MC_{\vec{v}}(\vec{X}_n) - W_\infty(\mu, \nu_{Q_{\vec{v}}})\Big] \leq O(n^{-\frac{1}{2}}),$$

hence $\mathbb{E}[MC_{\vec{v}}(\vec{X}_n)]$ converges to $W_\infty(\mu, \nu_{Q_{\vec{v}}})$. Finally, by the same argument used in the proof of Theorem 3.3 and Theorem 3.4, we are able to retrieve the limit of the Bayesian approximation ratio and conclude the proof. The bound on $B_{ar,\infty}^{(n)}(\mathcal{PM}_{\vec{v}})$ follows by the same argument used in the proof of Theorem 3.4. □

Notice that all the contents of Section 3.3 can be extended to the Maximum Cost functions proven that the measure μ has compact support. Notice, however, that the results of Section 3.3.2 cannot be extended as we need μ to have bounded support. We summarize the main results in the following theorem.

Theorem 3.13. *Let μ be a measure such that $spt(\mu)$ is a compact interval. Then, given a percentile vector $\vec{v} \in [0,1]^k$, we have that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[MC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[MC_{opt}(\vec{X}_n)]} = \frac{W_\infty(\mu, \nu_{Q_{\vec{v}}})}{W_\infty(\mu, \nu^{(k)})}, \tag{42}$$

where $\nu^{(k)}$ is the solution to problem $\min_{\rho\in\mathcal{P}_k(\mathbb{R})} W_\infty(\mu,\rho)$. Moreover, if $\delta > 0$ is such that $\rho_\mu(x) > \delta$ for every $x \in spt(\mu)$. Then, we have $|\mathbb{E}[MC_{\vec{v}}(\vec{X}_n)] - W_\infty(\mu, \nu_{Q_{\vec{v}}})| \leq O(n^{-\frac{1}{2}})$.

Proof. The proof follows by the argument used to prove Theorem 3.5 and 3.6. □

Before analysing how to retrieve the optimal percentile mechanism tailored to a distribution μ, we note that the tools used for studying the percentile mechanism also allow us to examine the Bayesian approximation ratio of other classic mechanisms for the k-FLP. To keep the discussion on track, we defer to Appendix B our computations to explore the Bayesian approximation ratio of the Median, Leftmost, and Left-Rightmost mechanisms in detail. A summary of our findings for these mechanisms is provided in Table 1.

4 The Optimal Percentile Mechanism

In this section, we demonstrate that for any given distribution μ and number of facilities k, it is always possible to define a percentile mechanism whose asymptotic expected cost converges to the expected optimal cost. As a preliminary result, we show that the limit of the Bayesian approximation ratio of the percentile mechanisms and the percentile vector $\vec{v}_\mu$ defined in (44) are immune to scale changes, regardless of which cost we are considering. This is particularly useful when the mechanism designer only knows the class of distribution to which the agents' distribution belongs. For example, the designer might know that the agents' type follows a Gaussian distribution but is unaware of its mean and/or its standard deviation. In the following, we show that the optimal percentile and the limit of the Bayesian approximation ratio of the percentile mechanisms are the same regardless of the mean or standard deviation of the distribution.

Theorem 4.1. *Let X be the random variable that describes the agents' type distribution. If $\vec{v}_\mu$ is the optimal percentile vector associated with X, then $\vec{v}_\mu$ is also the optimal percentile vector for any random variable of the form*

		SC				MC				C_{l_2}			
		$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar	$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar	$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar
$k = 1$	med	1	1	1	1	n/a	n/a	1	2	1	1.05	1	$\sqrt{2}$
	lt	n/a	1.4	2	$n-1$	n/a	n/a	2	2	n/a	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{n}\ ^{*}$
	rt	n/a	n/a	2	$n-1$	n/a	n/a	2	2	n/a	n/a	$\sqrt{2}$	$\sqrt{n}\ ^{*}$
$k = 2$	lrt	n/a	2.4	2	$n-2$	n/a	n/a	2	2	n/a	1.87	$\sqrt{2}$	$\sqrt{n-1}\ ^{*}$

Table 1. The limit of the Bayesian approximation ratio for three distributions, vs. the worst-case approximation ratio in the classic mechanism design setting (the *ar* columns). We report our results for the Normal Distribution $\mathcal{N}$, the Exponential Distribution $\mathcal{E}$, and the Uniform Distribution $\mathcal{U}$. The label n/a stands for *not applicable* since the distribution does not satisfy the minimal assumptions to be evaluated. The bounds in the *ar* columns are either known tight bounds (without a star) or lower bounds proven in B (with a star). All the computations are deferred to Appendix B.

$X' := \sigma X + m$, where $m \in \mathbb{R}$ and $\sigma > 0$. In particular, the Bayesian approximation ratio of any percentile mechanism $\mathcal{PM}_{\vec{v}}$ does not depend on the mean nor the standard deviation of the distribution μ.

Proof. Let $L(x) = \sigma x + m$ be a linear function such that $\sigma > 0$. Since $\sigma > 0$, L is a bijective and monotone-increasing function. We denote the inverse function of L with H. It is well-known that if X is a random variable whose law is μ, then the law of $L(X)$ is $L_\# \mu$. [5] We now show that if $\nu^{(k)}$ is the projection of μ over $\mathcal{P}_k(\mathbb{R})$, then $L_\# \nu^{(k)}$ is a projection of $L_\# \mu$ over $\mathcal{P}_k(\mathbb{R})$. First, notice that, since $\nu^{(k)} \in \mathcal{P}_k(\mathbb{R})$, then $L_\# \nu^{(k)} \in \mathcal{P}_k(\mathbb{R})$. Toward a contradiction, let γ be the projection of $L_\# \mu$ over $\mathcal{P}_k(\mathbb{R})$. By definition of γ, we have

$$W_1(\gamma, L_\# \mu) < W_1(L_\# \nu^{(k)}, L_\# \mu).$$

Let us now define $\eta = H_\# \gamma \in \mathcal{P}_k(\mathbb{R})$, by definition we have $\gamma = L_\# \eta$. Furthermore, by the properties of the Wasserstein distance, we have

$$W_1(L_\# \eta, L_\# \mu) = \sigma W_1(\eta, \mu) \tag{43}$$

and

$$W_1(L_\# \nu^{(k)}, L_\# \mu) = \sigma W_1(\nu^{(k)}, \mu).$$

In particular, we get

$$W_1(\eta, \mu) < W_1(\nu^{(k)}, \mu),$$

which contradicts the optimality of $\nu^{(k)}$. We therefore conclude that $L_\# \nu^{(k)}$ is a projection of $L_\# \mu$ over $\mathcal{P}_k(\mathbb{R})$.

Notice that $y \in spt(\nu^{(k)})$ if and only if $\sigma y + m \in spt(L_\# \nu^{(k)})$. From Corollary 4.2, we have that the j-th entry of the optimal vector related to μ is $F_\mu(y_j)$. Similarly, the j-th entry of the optimal vector related to $L_\# \mu$ is $F_{L_\# \mu}(\sigma y_j + m) = F_\mu(y_j)$, thus, by Theorem 4.2, the optimal percentile vector with respect to $\sigma X + m$ is also optimal with respect to X and vice-versa. Finally, we notice that the Bayesian approximation ratio of every $\mathcal{PM}_{\vec{v}}$ is the same for both μ and $L_\# \mu$. It follows from (43) and $W_1(L_\# \mu, L_\# \nu_{Q_{\vec{v}}}) = \sigma W_1(\mu, \nu_{Q_{\vec{v}}})$.

Through the same argument, we show that the same result holds for the l_p and Maximum Costs. □

Theorem 4.1 formalizes the following observation: the optimal facility locations and the output of any percentile mechanism do not depend on the scale. Indeed, given a percentile vector $\vec{v}$ and the number of agents n, if the agents' positions are sampled from a random variable X, the output of $\mathcal{PM}_{\vec{v}}$ is the vector containing the $(\lfloor (n-1)v_j \rfloor + 1)$-th order statistics of the sample. Since the ordering of the values is unaffected by positive affine

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[5] We recall that $L_\# \mu$ is the pushforward of the measure μ, defined as $L_\# \mu(A) = \mu(L^{-1}(A))$, see [59].

transformations, scaling any sample just magnifies (or shrinks) the cost of the output according to σ. Similarly, if we scale the agents' positions, the optimal facility locations will scale accordingly. Hence the ratio of the two costs is immune to scale changes. Finally, we notice that if we allow σ to be negative, Theorem 4.1 is no longer valid. Indeed, multiplying a generic random variable by a negative quantity alters its quantiles and thus the optimal percentile vector.

We now study the optimal percentile mechanism tailored to a probability distribution and a specific objective cost. Since the process of determining the optimal percentile mechanism for both the Social Cost and the l_p Cost follows a similar approach, we address these cases together, deferring the analysis of the Maximum Cost to a dedicated section.

4.1 The Social Cost and the l_p Cost

From Theorem 3.3, we observe that if $W_1(\mu, \nu_{Q_{\vec{v}}}) = W_1(\mu, \nu^{(k)})$, the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}}$ converges to 1 when $n \longrightarrow \infty$. We now show that, for any $k \in \mathbb{N}$ and any underlying distribution μ, there always exists a percentile vector whose associated mechanism asymptotically behaves optimally. In other words, the limit of the Bayesian approximation ratio of the induced mechanism is equal to 1. Given an underlying distribution μ, we denote with $\vec{v}_\mu$ its related optimal percentile vector.

Theorem 4.2. *Let μ be the underlying distribution and $\{y_j\}_{j\in[k]}$ be the support of the solution of problem* (9)*. Then, the vector $\vec{v}_\mu$ defined as*

$$(v_\mu)_j = F_\mu(y_j), \tag{44}$$

is an optimal percentile vector with respect to the Social Cost. Similarly, if $\{y_j\}_{j\in[k]}$ is the support of the solution to the projection problem with respect to W_p, formula (44) *defines the optimal percentile vector with respect to the l_p Cost.*

Proof. We prove this statement only for the Social Cost case, the other cases follow by the same argument. By Lemma 3.2, we have that

$$\mathbb{E}[SC_{\vec{v}_\mu}(\vec{X}_n)] \longrightarrow W_1(\mu, \nu_{Q_{\vec{v}_\mu}})$$

when $n \longrightarrow \infty$. We observe that $spt(\nu_{Q_{\vec{v}_\mu}}) = spt(\nu^{(k)})$. By definition of $\nu^{(k)}$, we have that

$$W_1(\mu, \nu^{(k)}) \leq W_1(\mu, \nu_{Q_{\vec{v}_\mu}}).$$

Furthermore, due to equation (12), we have

$$W_1(\mu, \nu_{Q_{\vec{v}_\mu}}) = \int_{-\infty}^{+\infty} \min_{j\in[k]} |x - y_j| d\mu = \min_{\{\lambda_j\}_{j\in[k]}} W_1(\lambda, \mu) \leq W_1(\nu^{(k)}, \mu),$$

where $\lambda = \sum_{j=1}^k \lambda_j \delta_{y_j}$ and y_j are the points in the support of both $\nu^{(k)}$ and $\nu_{Q_{\vec{v}_\mu}}$. We then infer that

$$W_1(\mu, \nu^{(k)}) = W_1(\mu, \nu_{Q_{\vec{v}_\mu}}), \tag{45}$$

which concludes the proof. □

Although showing that $W_1(\mu, \nu_{Q_{\vec{v}_\mu}}) = W_1(\mu, \nu^{(k)})$ is enough to prove Theorem 4.2, it is also worth of notice that $\nu_{Q_{\vec{v}_\mu}} = \nu^{(k)}$ holds. Indeed, toward a contradiction, let us assume that $\nu_{Q_{\vec{v}_\mu}} \neq \nu^{(k)}$. Then there exists a $\bar{j} \in [k]$ such that $(\nu_{Q_{\vec{v}_\mu}})_i = \nu_i^{(k)}$ for every $i = 1, \dots, \bar{j} - 1$ and $(\nu_{Q_{\vec{v}_\mu}})_{\bar{j}} \neq \nu_{\bar{j}}^{(k)}$. Since the optimal transportation plan between two measures supported over a line is monotone, we have that

$$W_1(\mu, \nu^{(k)}) = \sum_{j=0}^{k} \int_{l_j}^{l_{j+1}} |x - y_j| d\mu,$$

where $l_0 = -\infty$ and $l_r = F_\mu^{[-1]}(\sum_{i=1}^r \nu_i^{(k)})$ for every $r \in [k]$. Since $(\nu_{Q_{\vec{v}_\mu}})_{\bar{j}} \neq \nu_{\bar{j}}^{(k)}$, we have that $l_{\bar{j}} \neq \frac{y_{\bar{j}}+y_{\bar{j}+1}}{2}$. Thus we have

$$W_1(\mu, \nu^{(k)}) \neq \int_{-\infty}^{+\infty} \min_{j \in [k]} |x - y_j| d\mu,$$

which contradicts the definition of $\nu^{(k)}$ and (12), we then conclude that $\nu_{Q_{\vec{v}_\mu}} = \nu^{(k)}$.

Given $k \in \mathbb{N}$ and a probability measure μ, it is possible to leverage the properties of the optimal solution to retrieve a system of k equations that characterizes the optimal percentile mechanism. Indeed, let us denote with $y_1, \ldots, y_k$ the support of the solution to $\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_1(\mu, \lambda)$ and let $z_i = \frac{y_i+y_{i+1}}{2}$ for $i = 1, \ldots, k-1$, $z_0 = -\infty$, and $z_k = +\infty$. Let us now focus on y_1. Since every agent's cost is defined by its distance to the closest facility, we know that every agent in (z_0, z_1) will access the facility located in y_1. Due to the optimality of the solution, we infer that y_1 is locally optimal over the set (z_0, z_1). Otherwise, we could reduce the cost of the solution by replacing y_1 with the optimal facility location for the problem restricted to (z_0, z_1). Since we are considering the Social Cost, the local optimality of y_1 is expressed by the identity $2(F_\mu(y_1) - F_\mu(z_0)) = F(z_1) - F_\mu(z_0)$, since y_1 has to be the median of μ when the measure is restricted to (z_0, z_1).

When we consider the Social Cost, we can express the local optimality condition for every facility location and replacing every z_i with its definition and retrieve the following theorem.

Theorem 4.3. *Given $k \in \mathbb{N}$ and $\mu \in \mathcal{P}(\mathbb{R})$, let ν be a solution to Problem* (9)*. Then the optimal percentile vector $\vec{v}_\mu \in (0,1)^k$ satisfies the following system of k equations*

$$\begin{cases} 2F_\mu(y_1) = F_\mu\Big(\frac{y_1+y_2}{2}\Big) \\ 2\bigg(F_\mu(y_2) - F_\mu\Big(\frac{y_2+y_1}{2}\Big)\bigg) = F_\mu\Big(\frac{y_2+y_3}{2}\Big) - F_\mu\Big(\frac{y_1+y_2}{2}\Big) \\ \qquad \cdots \\ 2\bigg(F_\mu(y_{k-1}) - F_\mu\Big(\frac{y_{k-1}+y_{k-2}}{2}\Big)\bigg) = F_\mu\Big(\frac{y_k+y_{k-1}}{2}\Big) - F_\mu\Big(\frac{y_{k-1}+y_{k-2}}{2}\Big) \\ 2\bigg(F_\mu(y_k) - F_\mu\Big(\frac{y_k+y_{k-1}}{2}\Big)\bigg) = 1 - F_\mu\Big(\frac{y_k+y_{k-1}}{2}\Big) \end{cases}, \tag{46}$$

where $y_1 \leq y_2 \leq \cdots \leq y_k$ are the points in the support of ν.

Proof. Given $k \in \mathbb{N}$ and a probability measure μ, let us denote with $y_1, \ldots, y_k$ the support of the solution to $\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_1(\mu, \lambda)$ and let $z_i = \frac{y_i+y_{i+1}}{2}$ for $i = 1, \ldots, k-1$, $z_0 = -\infty$, and $z_k = +\infty$. Let us now focus on y_1. Since every agent's cost is defined by its distance to the closest facility, we know that every agent in (z_0, z_1) will access the facility located in y_1. Otherwise, we could reduce the cost of the solution by assigning an agent to a closer facility.

Moreover, due to the optimality of the solution, we infer that y_1 is locally optimal over the set (z_0, z_1), thus y_1 is the median of μ restricted to (z_0, z_1). Otherwise, we could reduce the cost of the solution by replacing y_1 with the optimal facility location for the problem restricted to (z_0, z_1). Therefore, we have that $2(F_\mu(y_1) - F_\mu(z_0)) = F(z_1) - F_\mu(z_0)$, since y_1 has to be the median of μ when the measure is restricted to (z_0, z_1). Let us now consider y_i. By a similar argument, we have that every agent located in (z_{i-1}, z_i) will access the facility located at y_i, thus y_i must be located at the median of the measure obtained by restricting μ to (z_{i-1}, z_i). In particular, infer that $F_\mu(y_i) - F_\mu(z_{i-1}) = F_\mu(z_i) - F_\mu(y_i)$ or, equivalently

$$2\bigg(F_\mu(y_i) - F_\mu\Big(\frac{y_i + y_{i-1}}{2}\Big)\bigg) = F_\mu\Big(\frac{y_{i+1} + y_i}{2}\Big) - F_\mu\Big(\frac{y_i + y_{i-1}}{2}\Big),$$

which concludes the proof. □

	SC			C_{l_2}		
	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
$\mathcal{N}$	(0.5)	$(0.25, 0.75)$	$(0.15, 0.5, 0.85)$	(0.5)	$(0.16, 0.84)$	$(0.27, 0.5, 0.73)$
$\mathcal{E}$	(0.5)	$(0.33, 0.83)$	$(0.25, 0.67, 0.92)$	$(1-e^{-1})$	$(0.45, 0.92)$	$(0.35, 0.8, 0.97)$
$\mathcal{U}$	(0.5)	$(0.25, 0.75)$	$(0.16, 0.5, 0.83)$	(0.5)	$(0.25, 0.75)$	$(0.16, 0.5, 0.83)$

Table 2. The asymptotically optimal percentile vectors for the Normal ($\mathcal{N}$), Exponential ($\mathcal{E}$), and Uniform distribution ($\mathcal{U}$). Every row contains the optimal percentile vectors of a distribution for 1, 2, and 3 facilities and with respect to either the Social or the l_2 cost.

Since the projection problem (9) admits a solution, the system (46) always admits at least a solution. Through a similar argument, it is possible to characterize the optimal percentile vector with respect to the l_2 Cost. In this case, the local optimality of the solution to problem (9) ensures us that every y_i is located at the mean of the probability distribution μ restricted on the intervals (z_{i-1}, z_i).

Owing to the hypothesis on μ and to Theorem 3.8, we infer that the optimal percentile mechanism does not have any 0-1 entry, and thus infer the following theorem.

Theorem 4.4. *Given a probability distribution μ such that condition (8) is satisfied, let $\vec{v}_\mu$ be the optimal percentile vector with respect to the Social Cost associated with μ. Then, for the Social Cost, there exists a constant C such that, for every $n > k$, we have*

$$B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) \leq 1 + \frac{C}{\sqrt{n}}.$$

Moreover, for every $1 < p < \infty$, there exists a constant $C_p > 0$ such that, for every $n > k$, we have

$$B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}_\mu}) \leq 1 + \frac{C_p}{n^{\frac{1}{2p}}}.$$

If μ has a compact support, then we have $B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}_\mu}) \leq 1 + \frac{C}{\sqrt{n}}$.

Proof. From Theorem 3.8, we have that the optimal percentile vector $\vec{v}_\mu$ belongs to $(0,1)^k$. In particular, we have that $\vec{v}_\mu$ meets the conditions of Theorem 3.3 and 3.4, hence there exists a constant $C > 0$ such that

$$B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) \leq \lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) + \frac{C}{\sqrt{n}}.$$

To conclude the proof, it suffices to notice that, by definition of optimal percentile vector, we have that

$$\lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) = 1.$$

□

In Appendix A, we show how to use Theorem 4.3 to compute the optimal percentile mechanism for the uniform, normal, and exponential distributions. In Table 2, we summarize our finding.

4.2 Computing the Optimal Percentile Vectors for the Maximum Cost

Given a measure μ whose support is compact, let us consider the following projection problem

$$\min_{\lambda\in\mathcal{P}_k(\mathbb{R})} W_\infty(\mu,\lambda). \tag{47}$$

Due to the definition of W_∞, the solution to problem (47) has an explicit formula. In particular, the objective value of (47) only depends on the support of the measure μ and on k, the number of facilities we need to locate.

Lemma 4.1. *Let $[a,b] \subset \mathbb{R}$ be a bounded interval. For every measure μ supported over $[a,b]$, we have that: (i) the projection of μ over $\mathcal{P}_k(\mathbb{R})$ is supported over $y_j = a + \frac{b-a}{2k}(2j-1)$ and (ii) the objective value achieved by any solution to problem* (47) *is equal to $\frac{b-a}{2k}$. In particular, if two measures have the same support, their projections also have the same support. Moreover, the asymptotic optimal Maximum Cost is the same for both probability measures.*

Proof. Let $\nu^{(k)}$ be the projection of μ over the space $\mathcal{P}_k(\mathbb{R})$ with respect to W_∞ and let $\vec{y} = (y_1, \dots, y_k)$ be the vector whose entries contains the support of the measure $\nu^{(k)}$. Without loss of generality, we assume $y_\ell \le y_{\ell+1}$. Let A_j be the set of points that are closer to y_j than any other facility. Notice that each A_j is an interval of $\mathbb{R}$. It is easy to see that $W_\infty(\mu, \nu^{(k)}) = \max_{j\in[k]} \frac{|A_j|}{2}$, where $|A_j|$ is the length of the interval A_j. Hence we have that the minimum W_∞ distance is achieved when all the A_j's are intervals of length $\frac{b-a}{k}$ and the y_j is placed in the middle of A_j, so that $W_\infty(\mu, \nu^{(k)}) = \frac{b-a}{2k}$ and

$$y_j = \frac{b-a}{2k}(2j-1),$$

which concludes the proof. □

Lemma 4.1, combined with formula (44), allows us to characterize the optimal percentile vector, with respect to the Maximum Cost, for every compactly supported measure μ.

Theorem 4.5. *Let μ be a distribution such that $spt(\mu) = [a,b]$. Then, the optimal percentile vector for k facilities for the Maximum Cost when the agents are distributed according to μ is $v_j = F_\mu(a + \frac{b-a}{2k}(2j-1))$.*

Proof. It follows by combining the results of Lemma 4.1 with equation (44). □

To conclude, we show that the limit of the Bayesian approximation ratio with respect to the Maximum Cost of every percentile mechanism is bounded by a constant that depends only on k.

Theorem 4.6. *Let μ be a measure supported over a compact interval, that is $spt(\mu) = [a,b]$. Then, if μ satisfies the hypothesis of Theorem 3.3, we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}[MC_{\vec{v}}(\vec{X}_n)]}{\mathbb{E}[MC_{opt}(\vec{X}_n)]} = \frac{2kW_\infty(\mu, \nu_{Q_{\vec{v}}})}{b-a} = \frac{2k \max_{j\in[k]}\big\{\max\{|z_{j-1}-y_j|, |z_j - y_j|\}\big\}}{b-a}, \tag{48}$$

where $y_j = F_\mu^{[-1]}(v_j)$ and $z_j = \frac{y_j + y_{j+1}}{2}$ for $j = 1, \dots, k-1$, $z_0 = a$, and $z_k = b$. In particular, for any $\vec{v} \in [0,1]^k$, we have $\lim_{n\to\infty} B^{(n)}_{ar,\infty}(\mathcal{PM}_{\vec{v}}) \le 2k$ with respect to the Maximum Cost. In particular, there exists a constant $C > 0$ such that

$$B^{(n)}_{ar,\infty}(\mathcal{PM}_{\vec{v}}) \le 2k + \frac{C}{\sqrt{n}},$$

whenever $n > k$.

Proof. The first identity in (48) follows from Theorem 3.3 and Lemma 4.1. The second identity in (48) follows from the definition of W_∞. Moreover, since

$$\max\left\{|z_{j-1} - F_\mu^{[-1]}(v_j)|, |z_j - F_\mu^{[-1]}(v_j)|\right\} \le b - a,$$

we infer $\lim_{n\to\infty} B^{(n)}_{ar,\infty}(\mathcal{PM}_{\vec{v}}) \le 2k$. □

4.3 Computing the Optimal Percentile Mechanism from an approximation of μ

To conclude, we study the stability of the optimal percentile mechanism, which ensures that similar distributions lead to similar optimal percentile vectors. This property is extremely important since, in many cases, the mechanism designer has only access to an approximation or a prediction of agents' distribution, which we denote with $\tilde{\mu}$. Thus, the designer is able to compute the optimal percentile with respect to the approximation measure, that is $\vec{v}_{\tilde{\mu}}$, rather than the real optimal percentile vector $\vec{v}_{\mu}$. We now show that it is possible to estimate the difference between the limit of the Bayesian approximation ratio of $\mathcal{PM}_{\vec{v}_{\tilde{\mu}}}$ and 1, i.e. the limit of the Bayesian approximation ratio of the optimal percentile mechanism. In particular, we show that the closer μ and $\tilde{\mu}$ are with respect to the W_∞ distance, the closer the asymptotic cost of $\mathcal{PM}_{\vec{v}_{\tilde{\mu}}}$ gets to the optimal cost.

Theorem 4.7. *Let $\tilde{\mu}$ and μ be two probability measures supported over a compact interval I. Let $\vec{v}_{\tilde{\mu}}$ be the percentile vector obtained by solving the system* (44) *by using $\tilde{\mu}$ instead of μ. Then, we have*

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}[SC_{\vec{v}_{\tilde{\mu}}}(\vec{X}_n)]}{\mathbb{E}[SC_{opt}(\vec{X}_n)]}-1\right| \le \frac{W_\infty(\mu,\tilde{\mu})+2W_1(\mu,\tilde{\mu})}{W_1(\mu,\nu^{(k)})} \le 3\frac{W_\infty(\mu,\tilde{\mu})}{W_1(\mu,\nu^{(k)})}, \tag{49}$$

where $\mathbb{E}$ is the expected value with respect to the real agents' distribution μ. Similarly, we have that

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}[(C_{l_p})_{\vec{v}_{\tilde{\mu}}}(\vec{x})]}{\mathbb{E}[(C_{l_p})_{opt}(\vec{x})]}-1\right| \le 3\frac{W_\infty(\mu,\tilde{\mu})}{W_p(\mu,\nu^{(k)})} \qquad \text{and} \qquad \lim_{n\to\infty}\left|\frac{\mathbb{E}[MC_{\vec{v}_{\tilde{\mu}}}(\vec{X}_n)]}{\mathbb{E}[MC_{opt}(\vec{X}_n)]}-1\right| \le 3\frac{W_\infty(\mu,\tilde{\mu})}{W_\infty(\mu,\nu^{(k)})}, \tag{50}$$

for any $p \in (1,+\infty)$. Lastly, for every $n > k$ and $p \in [1,+\infty]$, there exists a constant $C > 0$ such that

$$|B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}_{\mu}}) - B^{(n)}_{ar,p}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| \le 3\frac{W_\infty(\mu,\tilde{\mu})}{W_p(\mu,\nu^{(k)})} + \frac{C}{\sqrt{n}},$$

where $\vec{v}_{\mu}$ is the optimal percentile vector associated with μ with respect to the Social, the Maximum, or the l_p costs.

Proof. Let us denote with $\tilde{\nu}^{(k)}$ the projection of $\tilde{\mu}$ over $\mathcal{P}_k(\mathbb{R})$ and with $\nu^{(k)}$ the projection of μ over $\mathcal{P}_k(\mathbb{R})$. We denote with $\{y_j\}_{j\in[k]}$ the support of $\nu^{(k)}$ and with $\{\tilde{y}_j\}_{j\in[k]}$ the support of $\tilde{\nu}^{(k)}$. Accordingly, we denote with $\vec{v}_{\tilde{\mu}}$ and $\vec{v}_{\mu}$ the optimal percentile vectors associated with $\tilde{\mu}$ and μ, respectively. By Lemma 3.2, we have that the numerator of the Bayesian approximation ratio converges to $W_1(\mu,\nu_{Q_{\vec{v}_{\tilde{\mu}}}})$, where $\nu_{Q_{\vec{v}_{\tilde{\mu}}}}$ is defined as in (10). Let us now consider, $\beta_{\vec{v}_{\tilde{\mu}}}$ defined as

$$\beta_{\vec{v}_{\tilde{\mu}}} := \sum_{j\in[k]}(\tilde{\nu}^{(k)})_j\delta_{z_j},$$

where $\vec{z} = (z_1,\dots,z_k)$ is the support of $\nu_{Q_{\vec{v}_{\tilde{\mu}}}}$. Then we have

$$\begin{aligned} W_1(\mu,\nu_{Q_{\vec{v}_{\tilde{\mu}}}}) &\le W_1(\mu,\beta_{\vec{v}_{\tilde{\mu}}}) \le W_1(\mu,\tilde{\mu}) + W_1(\tilde{\mu},\beta_{\vec{v}_{\tilde{\mu}}}) \le W_1(\mu,\tilde{\mu}) + W_1(\tilde{\mu},\tilde{\nu}^{(k)}) + W_1(\tilde{\nu}^{(k)},\beta_{\vec{v}_{\tilde{\mu}}}) \\ &\le W_1(\mu,\tilde{\mu}) + W_1(\tilde{\mu},\nu^{(k)}) + W_1(\tilde{\nu}^{(k)},\beta_{\vec{v}_{\tilde{\mu}}}) \le 2W_1(\mu,\tilde{\mu}) + W_1(\mu,\nu^{(k)}) + W_1(\tilde{\nu}^{(k)},\beta_{\vec{v}_{\tilde{\mu}}}). \end{aligned} \tag{51}$$

By definition of $\beta_{\vec{v}_{\tilde{\mu}}}$ and $\tilde{\nu}^{(k)}$, we have that

$$W_1(\tilde{\nu}^{(k)},\beta_{\vec{v}_{\tilde{\mu}}}) \le \sum_{j\in[k]}(\tilde{\nu}^{(k)})_j|F^{[-1]}_{\tilde{\mu}}(\tilde{p}_j) - F^{[-1]}_{\mu}(\tilde{p}_j)|.$$

Since $W_1(\mu,\tilde{\mu}) \le W_\infty(\mu,\tilde{\mu})$ and $W_\infty(\mu,\tilde{\mu}) = \max_{\ell\in[0,1]}|F^{[-1]}_{\mu}(\ell) - F^{[-1]}_{\tilde{\mu}}(\ell)|$, we infer

$$W_1(\tilde{\nu}^{(k)},\beta_{\vec{v}_{\tilde{\mu}}}) \le W_\infty(\mu,\tilde{\mu}),$$

which concludes the proof for the Social Cost. Through a similar argument, we deal with the Maximum and the l_p costs.

To conclude, we notice that, from Theorem 3.8 we infer that the optimal percentile vectors $\vec{v}_\mu$ and $\vec{v}_{\tilde{\mu}}$ belong to the set $(0,1)^k$. Then we have

$$\begin{aligned}
&|B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) - B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| \leq \\
&\qquad |1 - \lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| + |B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) - 1| + |\lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}}) - B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| \\
&\qquad \leq 3\frac{W_\infty(\mu,\tilde{\mu})}{W_1(\mu,\nu^{(k)})} + |B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) - 1| + |\lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}}) - B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})|.
\end{aligned}$$

Since μ has compact support and $\vec{v}_\mu, \vec{v}_{\tilde{\mu}} \in (0,1)^k$, we infer that $|B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) - 1| \leq O(n^{-\frac{1}{2}})$ and $|\lim_{n\to\infty} B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}}) - B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| \leq O(n^{-\frac{1}{2}})$. Thus there exists a constant $C > 0$ such that

$$|B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_\mu}) - B^{(n)}_{ar,1}(\mathcal{PM}_{\vec{v}_{\tilde{\mu}}})| \leq 3\frac{W_\infty(\mu,\tilde{\mu})}{W_1(\mu,\nu^{(k)})} + \frac{C}{\sqrt{n}}$$

which concludes the proof for the Social Cost. The proof for the Maximum and l_p costs follow by a similar argument. ☐

5 Conclusion and Future Works

In this paper, we studied the percentile mechanisms in the Bayesian Mechanism Design framework. We have shown that, under mild assumptions, the ratio between the expected cost of the mechanisms and the expected optimal cost converges to a constant as the number of agents goes to infinity. We have characterized both the limit value and the convergence speed. We then showed that for every underlying distribution μ, there exists an optimal percentile vector $\vec{v}_\mu$ that does not depend on the mean or the variance of the distribution. The scale invariance property allows us to compute the optimal percentile vector even when the designer only knows the class to which the probability measure belongs. Finally, we have shown that determining the optimal percentile mechanism from an approximation of the underlying distribution leads to a mechanism whose performance is quasi-optimal as long as the approximation is close to the real distribution with respect to W_∞.

A natural open question is whether our formalism could be adopted to extend our results to higher dimensional cases. In [57], the percentile mechanisms are generalized to higher dimensions by dealing with each dimension separately. This suggests that our approach can be extended to also handle higher-dimensional problems thanks to the fact that also the Wasserstein Distance can be separate along each cardinal direction [4, 5]. Moreover, our framework can be extended beyond the classic k-FLP. In particular, it is foreseeable to use our results to tackle the case in which the facilities have a capacity constraint. Another interesting direction is to adapt our reformulation of the problem through OT theory to design and study randomized mechanisms for the k-FLP.

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A Optimal Percentile Vectors of classic probability distributions.

In this appendix, we study the optimal percentile vector with respect to the Social and l_2 Costs. In particular, we retrieve the formula of the optimal percentile mechanisms when the underlying distribution μ is Uniform, Exponential, or Gaussian, and we need to locate 1, 2, or 3 facilities. First, we focus on the cases $k = 1$ and $k = 2$. We then deal with $k = 3$ and show how to approach the general case $k > 3$. In particular, we show that for the Social Cost and the l_2 Cost, it is possible to retrieve the optimal percentile vector by solving a system of k equations.

A.1 One Facility

When $k = 1$, let X be the random variable describing the agents' type, and y be the optimal location of the facility. Then y is the real value that minimizes $\mathbb{E}[|X - y|]$ or $\mathbb{E}[|X - y|^2]$, depending on whether we want to minimize the Social or the l_2 Cost. It is well-known that the median mechanism is optimal with respect to the Social Cost function when we have a discrete input. Similarly, the value y that minimizes $\mathbb{E}[|X - y|]$ is the median of X. Thus, as we are about to see, the median mechanism retains its optimality also in the Bayesian framework. Unfortunately, a similar result does not hold for the l_2 Cost. Indeed, in this case, $\mathbb{E}[|X - y|^2]$ is minimized when y is equal to the mean of X. Hence, the optimal percentile vector depends on the structure of the measure μ.

Theorem A.1. *If the underlying distribution μ is symmetric, the optimal percentile mechanism for both the Social and l_2 Costs is induced by $\vec{v} = (0.5)$, that is, the median mechanism. If μ is not symmetric, the optimal percentile mechanism with respect to the Social Cost is still the median mechanism. For the l_2 cost, however, the optimal percentile mechanism is induced by $\vec{v} = (F_\mu(m))$, where m is the mean of μ. In particular, the median mechanism (i.e $\vec{v} = (0.5)$) is optimal for both the Uniform and the Gaussian distributions, while the optimal percentile for any exponential distribution is $\vec{v}_{\mathcal{E}} = (1 - e^{-1})$.*

Proof. Notice that if μ is symmetric, the mean and the median of μ are the same. The median is the value that minimizes the function $c \longrightarrow \mathbb{E}[|X - c|]$. Similarly, the mean is the value that minimizes the function $c \longrightarrow \mathbb{E}[|X - c|^2]$. Thus, by Theorem 4.2, we have that the percentile mechanism induced by $\vec{p} = (0.5)$ is asymptotically optimal with respect to both the Social and l_2 Costs.

If μ is not symmetric, we still have that the percentile mechanism induced by $\vec{p} = (0.5)$ is optimal with respect to the Social Cost. However, since the median and the mean are now different values, the optimal percentile vector with respect to the l_2 norm is $\vec{p} = (F_\mu(m))$, where m is the mean of μ. By recalling that the mean of the exponential distribution $\mathcal{E}(1)$ is 1, we infer the last statement of the Theorem. □

A.2 Two Facilities

When $k = 2$, the symmetry of the distribution is sufficient to retrieve the optimal percentile mechanism with respect to the Social Cost. Moreover, as for the one facility case, the optimal percentile vector does not depend on the measure μ.

Theorem A.2. *If μ is symmetric, the optimal percentile mechanism with respect to the Social Cost is induced by the vector $\vec{v} = (0.25, 0.75)$.*

Proof. Since μ is symmetric, we have that the optimal locations for two facilities are symmetric with respect to the median (or, equivalently, the mean) of μ. In particular, the median of μ is equidistant from both facilities. It then suffices to retrieve the location of the facility on the right to the median to retrieve the other one. Since the facility locations are globally optimal, every facility location has to be locally optimal, thus the facility on the left of the median minimizes the functional $\mathbb{E}[|X - c|]$, where the expected value is taken with respect to μ restricted on the set $\{X \leq med(\mu)\}$. By symmetry, the other facility will be located at the median of μ restricted over $\{X \geq med(\mu)\}$, from which we conclude that the optimal percentile vector is $\vec{p} = (0.25, 0.75)$. ◻

Similarly, we characterize the optimal percentile mechanisms with respect to the l_2 Cost.

Theorem A.3. *Let μ be a symmetric probability distribution and let m be its mean. Then, the optimal percentile mechanism with respect to the l_2 Cost is $\vec{v} = (F_\mu(m_-), F_\mu(m_+))$, where m_- is the mean of μ restricted to the set $x \leq m$ and m_+ is the mean of μ restricted to the set $x > m$. In particular, the optimal percentile vector for the Gaussian distribution is $\vec{v}_{\mathcal{N}} = (\Phi(-1), \Phi(+1)) \sim (0.16, 0.84)$ while the optimal vector for the uniform distribution is $\vec{v}_{\mathcal{U}} = (0.25, 0.75)$.*

Proof. Due to Theorem 4.1, without loss of generality, we assume that the underlying distribution has a null mean, i.e., $\int_{\mathbb{R}} x d\mu = 0$. By the same argument used in the proof of Theorem A.2, we infer that the optimal percentile vector has the form $\vec{p} = (F_\mu(m_-), F_\mu(m_+))$, where m_- is the mean of μ restricted to the set $x \leq m$ and m_+ is the mean of μ restricted to the set $x > m$. The last part of the statement follows from the fact that, for the standard Gaussian distribution $\mathcal{N}(0,1)$, we have that $m_+ = 1$ and $m_- = -1$. ◻

For asymmetric distributions, the computation of the optimal percentile mechanism is less straightforward and requires solving a system of equations. In the following, we compute the optimal vector for the exponential distribution.

Theorem A.4. *Let μ be an exponential probability distribution. Then, the optimal percentile vector with respect to the Social Cost for two facilities is $\vec{v}_{\mathcal{E}} = (0.33, 0.83)$. The optimal percentile vector with respect to the l_2 Costs is $\vec{v}_{\mathcal{E}} \sim (0.45, 0.92)$.*

Proof. First, we consider the Social Cost case. Let y_1 and y_2 be the points in the support of the projection of μ over $\mathcal{P}_2(\mathbb{R})$. Let us denote with z the middle point of y_1 and y_2, that is $z = \frac{1}{2}(y_1 + y_2)$. We then have that agents on the left of z will be using the facility at y_1, while the ones on the right, will use the one in y_2. In particular, the expected Social Cost is

$$\int_0^z |x - y_1| e^{-x} dx + \int_z^\infty |x - y_2| e^{-x} dx.$$

We now show that y_1 is the median of μ restricted to $\{x \leq z\}$. Assume, toward a contradiction, that y_1 is not the median of μ restricted over $\{x \leq z\}$. There exists a value y_1' such that

$$\int_0^z |x - y_1| e^{-x} dx > \int_0^z |x - y_1'| e^{-x} dx.$$

Hence we get

$$\begin{aligned}\int_0^z |x - y_1| e^{-x} dx + \int_z^\infty |x - y_2| e^{-x} dx &> \int_0^z |x - y_1'| e^{-x} dx + \int_z^\infty |x - y_2| e^{-x} dx \\ &\geq \min_{\lambda_1, \lambda_2} W_1(\mu, \lambda),\end{aligned}$$

where $\lambda = \lambda_1 \delta_{y_1'} + \lambda_2 \delta_{y_2}$, which contradicts the optimality of the facility locations. We therefore conclude that y_1 is the median of μ restricted over $(-\infty, z)$, i.e.

$$2F_\mu(y_1) = F_\mu(z).$$

Similarly, we get that

$$2(1 - F_\mu(y_2)) = (1 - F_\mu(z)).$$

		SC				MC				C_{l_2}			
		$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar	$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar	$\mathcal{N}$	$\mathcal{E}$	$\mathcal{U}$	ar
$k=1$	med	1	1	1	1	n/a	n/a	1	2	1	1.05	1	$\sqrt{2}$
	lt	n/a	1.4	2	$n-1$	n/a	n/a	2	2	n/a	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{n}$ *
	rt	n/a	n/a	2	$n-1$	n/a	n/a	2	2	n/a	n/a	$\sqrt{2}$	$\sqrt{n}$ *
$k=2$	lrt	n/a	2.4	2	$n-2$	n/a	n/a	2	2	n/a	1.87	$\sqrt{2}$	$\sqrt{n-1}$ *

Table 3. The limit of the Bayesian approximation ratio for three distributions, vs. the worst-case approximation ratio in the classic mechanism design setting (the *ar* columns). We report our results for the Normal Distribution $\mathcal{N}$, the Exponential Distribution $\mathcal{E}$, and the Uniform Distribution $\mathcal{U}$. The label n/a stands for *not applicable* since the distribution does not satisfy the minimal assumptions to be evaluated.

Since $F_\mu(t) = 1 - e^{-t}$, we get the following system of equations

$$\begin{cases} 2(1 - e^{-y_1}) = (1 - e^{-z}) \\ 2e^{-y_2} = e^{-z} \\ z = \frac{1}{2}(y_1 + y_2) \end{cases} . \tag{52}$$

From the second and third equations, we retrieve $y_2 = y_1 + 2\log(2)$, which, plugged in the first equation, allows us to conclude that $y_1 = log(3) - \log(2)$ and $y_2 = \log(6)$. From equation (44), we conclude the proof for the Social Cost.

Let us now consider the l_2 Cost case and let us denote again with y_1 and y_2 the support of the projection of μ with respect to W_2. Again, let us denote with z the middle point between y_1 and y_2. By an argument similar to the one used in the Social Cost case, we have that y_1 is the mean value of μ restricted on $\{x \leq z\}$, while y_2 is the mean value of μ restricted to $\{x \geq z\}$. This leads us to define the following system of equations

$$\begin{cases} y_1 = \displaystyle\int_0^z x \frac{e^{-x}}{1 - e^{-z}} dx \\ y_2 = \displaystyle\int_z^\infty x \frac{e^{-x}}{e^{-z}} dx \\ z = \dfrac{1}{2}(y_1 + y_2) \end{cases} . \tag{53}$$

By solving the system, we retrieve $y_1 = 0.59$ and $y_2 = 2.59$, which, combined with equation (44), allows us to conclude the proof. □

For the sake of completeness, in Table 3, we report the limit of the Bayesian approximation ratio of the classic truthful mechanisms to locate 1 or 2 facilities, i.e., the median (*med*), the left-most (*lt*), the right-most (*rt*), and the left-right-most (*lrt*) mechanisms when the underlying distribution is a Gaussian, an Exponential, and a Uniform distribution. For comparison, we include the worst-case approximation ratio for these cost functions as the last column for each cost function. For a definition of *med*, *lt*, *rt*, and *lrt* we refer the reader to Appendix B.

A.3 Three and More Facilities

Lastly, we consider the case $k \geq 3$. For both the Social and l_2 Costs, we show that the optimal percentile vector can be retrieved by solving a system of equations obtained by expressing the local optimality properties of the solution to problem (9). For the sake of simplicity, we detail the procedure for the Social Cost and describe how the same technique applies to the l_2 Cost.

Let us now consider a probability distribution μ and its c.d.f., namely F_μ. Given k, let us denote with $y_1, \dots, y_k$ the support of the solution to $\min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_1(\mu, \lambda)$. Let us now define $z_i = \frac{y_i + y_{i+1}}{2}$ for $i = 1, \dots, k-1$, $z_0 = -\infty$, and $z_k = +\infty$. Let us now focus on y_1. Since every agent's cost is defined by its distance to the closest facility, we know that every agent in (z_0, z_1) will access the facility located in y_1. Due to the optimality of the solution, we infer that y_1 is locally optimal over the set (z_0, z_1). Otherwise, we could reduce the cost of the solution by replacing y_1 with the optimal facility location for the problem restricted to (z_0, z_1). Since we are considering the Social Cost, the local optimality of y_1 is expressed by the identity $2(F_\mu(y_1) - F_\mu(z_0)) = F(z_1) - F_\mu(z_0)$, since y_1 has to be the median of μ when the measure is restricted to (z_0, z_1). By expressing the local optimality condition for every facility location and replacing every z_i with its definition, we get the following system of k equations

$$\begin{cases} 2F_\mu(y_1) = F_\mu\left(\frac{y_1+y_2}{2}\right) \\ 2\left(F_\mu(y_2) - F_\mu\left(\frac{y_1+y_2}{2}\right)\right) = F_\mu\left(\frac{y_2+y_3}{2}\right) - F_\mu\left(\frac{y_1+y_2}{2}\right) \\ \qquad \dots \\ 2\left(F_\mu(y_{k-1}) - F_\mu\left(\frac{y_{k-2}+y_{k-1}}{2}\right)\right) = F_\mu\left(\frac{y_{k-1}+y_{k-2}}{2}\right) - F_\mu\left(\frac{y_{k-1}+y_{k-2}}{2}\right) \\ 2\left(F_\mu(y_k) - F_\mu\left(\frac{y_k+y_{k-1}}{2}\right)\right) = 1 - F_\mu\left(\frac{y_k+y_{k-1}}{2}\right) \end{cases}. \tag{54}$$

Notice that, since the projection problem (9) admits a solution, the system (54) always admits at least a solution.

Through a similar argument, it is possible to characterize the optimal percentile vector with respect to the l_2 Cost. In this case, the local optimality of the solution to problem (9) ensures us that every y_i is located at the mean of the probability distribution μ restricted on the intervals (z_{i-1}, z_i).

To conclude this section, we leverage these characterizations to explicitly compute the optimal percentile vector for the 3-FLP with respect to some classic probability distributions. As for the previous cases, we distinguish between symmetric and asymmetric distributions, since by exploiting the symmetry of the measure it is possible to simplify the problem.

Theorem A.5. *Let μ be a symmetric distribution whose mean is equal to 0. Then, the second entry of the optimal percentile mechanism with respect to both the Social and the l_p Costs is 0.5. The other two entries are $F_\mu(\pm 2z)$, where z minimizes the following function $z \longrightarrow \int_0^z x^p d\mu + \int_z^{+\infty} |x - 2z|^p d\mu$, where $p = 1$ if we consider the Social Cost and $p = 2$ for the l_2 Cost. In particular, for the Social Cost, the optimal percentile vectors for the Uniform and Gaussian distribution are $\vec{v}_{\mathcal{U}} \sim (0.16, 0.5, 0.83)$ and $\vec{v}_{\mathcal{N}} \sim (0.15, 0.5, 0.85)$, respectively. For the l_2 Cost, we have that the optimal percentile vector for the Uniform and Gaussian distribution are $\vec{v}_{\mathcal{N}}^{(2)} \sim (0.17, 0.5, 0.83)$, and $\vec{v}_{\mathcal{U}}^{(2)} \sim (0.16, 0.5, 0.83)$, respectively.*

Proof. By a symmetry argument, we have that one of the three facilities has to be located at the mean of the measure. Similarly, the other two facilities must be symmetrically located with respect to the mean. Let us now denote with $2y$ the position of the facility on the right of 0. Therefore y is the minimum of the function

$$z \longrightarrow \int_0^z x^p d\mu + \int_z^\infty |x - 2z|^p d\mu,$$

which concludes the first half of the proof.

Let us now consider the Social Cost, hence $p = 1$, and the Gaussian distribution, since the other cases are similar. In this case, up to a constant, we need to minimize the following functional

$$
\begin{aligned}
z \longrightarrow & \int_0^z x e^{-\frac{x^2}{2}} dx + \int_z^\infty |x - 2z| e^{-\frac{x^2}{2}} dx \\
= & \int_0^z x e^{-\frac{x^2}{2}} dx + \int_z^{2z} (2z - x) e^{-\frac{x^2}{2}} dx + \int_{2z}^\infty (x - 2z) e^{-\frac{x^2}{2}} dx
\end{aligned}
$$

with respect to z. By computing the integrals in the latter equation, the function to optimize boils down to

$$z \longrightarrow 1 - 2e^{-\frac{z^2}{2}} + 2e^{-2z^2} + 2z(2\Phi(2z) - \Phi(z)) - 2z,$$

where $\Phi(z)$ is the cumulative distribution function of the Gaussian distribution. By differentiating the last equation and setting the derivative equal to zero, we retrieve the following equation

$$(2\Phi(2z) - \Phi(z)) - 1 = 0.$$

Solving the equation leads us to $z = 0.62$. Owing to the symmetry of the probability distribution, we are able to retrieve the optimal percentile vector with respect to the Social Cost.

Through similar computations, it is possible to retrieve the optimal percentile vector for the Gaussian distribution with respect to the l_2 Cost and for the uniform distribution for both the Social and the l_2 Costs. □

Finally, when the underlying distribution is not symmetric, the optimal percentile vector can still be found by solving the system described in (54). For the sake of completeness, we retrieve the optimal percentile vector for the Exponential distribution.

Theorem A.6. *When $k = 3$, the optimal percentile vector for the Exponential distribution with respect to the Social Cost is $\vec{v}_{\mathcal{E}} = (0.25, 0.67, 0.92)$. The optimal percentile vector with respect to the l_2 cost is $(0.35, 0.8, 0.97)$.*

Proof. First, we consider the optimal percentile vector with respect to the Social Cost. Let y_1, y_2, and y_3 denote the support of the projection of μ over $\mathcal{P}_3(\mathbb{R})$ with respect to the Social Cost. Let us denote with z_1 and z_2 the middle point between y_1 and y_2, and the middle point of y_2 and y_3, respectively. Following the same argument used in the proof of Theorem A.4, we have that y_1 is the median of μ restricted to $[0, z_1]$. Similarly, we have that y_2 is the median of μ restricted to $[z_1, z_2]$ and y_3 is the median of μ restricted on $[z_3, \infty)$. These three conditions allows us to write the following system

$$
\begin{cases}
2(1 - e^{-y_1}) = 1 - e^{-z_1} \\
2((1 - e^{-y_1}) - (1 - e^{-z_1})) = (1 - e^{-z_2}) - (1 - e^{-z_2}) \\
e^{-z_2} = 2e^{-y_2} \\
z_1 = \frac{1}{2}(y_1 + y_2) \\
z_2 = \frac{1}{2}(y_2 + y_3)
\end{cases}
$$

whose solutions are $y_1 = \log(\frac{4}{3})$, $y_2 = \log(3)$, and $y_3 = \log(12)$. Using equation (4.5) again, we conclude the proof for the Social Cost case.

We now move to the l_2 cost. In this case, the optimal percentile vector is the unique solution to the following system of equations

$$\begin{cases} y_1(1-e^{-z_1}) = 1-(z_1+1)e^{-z_1} \\ y_2(e^{-z_1}-e^{-z_2}) = (z_1+1)e^{-z_1}-(z_2+1)e^{-z_2} \\ y_3e^{-z_2} = (z_2+1)e^{-z_2} \\ z_1 = \frac{1}{2}(y_1+y_2) \\ z_2 = \frac{1}{2}(y_2+y_3) \end{cases}. \tag{55}$$

By solving the system, we infer that $y_1 = 0.43$, $y_2 = 1.61$, and $y_3 = 3.61$, hence the optimal percentile vector is $(1-e^{-0.43}, 1-e^{-1.61}, 1-e^{-3.61}) \sim (0.35, 0.8, 0.97)$. ◻

B Bayesian Analysis of the Classic Truthful Mechanisms

In this Appendix, we study the convergence of the Bayesian approximation ratio of the classic truthful mechanisms. In particular, we consider

- the leftmost mechanism (lt), which returns the leftmost position amongst the ones given as input by the agents, i.e. $lt(\vec{x}) = \min_{i\in[n]} x_i$;
- the rightmost mechanism (rt), which returns the rightmost position amongst the ones given as input by the agents, i.e. $rt(\vec{x}) = \max_{i\in[n]} x_i$;
- the median mechanism, that returns the median of the set of agents positions, i.e. $med(\vec{x}) = x'_{\lfloor \frac{n}{2} \rfloor}$, where $\vec{x}'$ is the vector obtained by rearranging the entries of x increasingly;
- the left-rightmost mechanism (lrt), which places two locations, one at the leftmost and one at the rightmost position amongst the ones given as input by the agents.

Notice that those mechanisms are special cases of the percentile mechanisms, so all the previous results apply to them. First, we consider the mechanisms that locate just one facility; i.e. lt, rt, and med. The only mechanism whose asymptotic Bayesian approximation ratio converges when the measure μ has an unbounded support is the med mechanism.

Theorem B.1. *Let $\vec{X}_n$ be a vector of n i.i.d. random variables that are distributed as μ. Let us assume that μ satisfies the condition of Lemma 3.2. Then, we have that the median mechanism is asymptotically optimal with respect to the Social Cost. Moreover, for the l_2 Cost, we have that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[(C_{l_2})_{med}(X_n)]}{\mathbb{E}[(C_{l_2})_{opt}(X_n)]} = \left(\frac{\int_{\mathbb{R}} |x-med(\mu)|^2 d\mu}{\int_{\mathbb{R}} |x-m(\mu)|^2 d\mu} \right)^{\frac{1}{2}}$$

where $med(\mu)$ and $m(\mu)$ are the median and the mean of μ, respectively. In particular, we infer $\lim_{n\to\infty} B^{(n)}_{ar,2}(med) \leq 1 + \frac{|med(\mu)-m(\mu)|}{Var(\mu)}$, where $Var(\mu)$ is the variance of μ.

Proof. By definition, $\delta_{med(\mu)}$ is the projection of μ with respect to the Social Cost over $\mathcal{P}_1(\mathbb{R})$. Hence, from Theorem 3.3, we conclude that the median mechanism is asymptotically optimal. Similarly, if we consider the l_2 Cost, we have that $\delta_{m(\mu)}$ is the projection of μ over $\mathcal{P}_1(\mathbb{R})$. Thus,

$$\lim_{n\to\infty} \frac{\mathbb{E}[(C_{l_2})_{med}(X_n)]}{\mathbb{E}[(C_{l_2})_{opt}(X_n)]} = \left(\frac{\int_{\mathbb{R}} |x-med(\mu)|^2 d\mu}{\int_{\mathbb{R}} |x-m(\mu)|^2 d\mu} \right)^{\frac{1}{2}}.$$

The last part of the Theorem follows by applying the Minkowski inequality to the numerator. ◻

We now turn our attention to the leftmost and rightmost mechanisms. It is worth noting that the leftmost and rightmost mechanisms share the same properties by their symmetric definitions. To analyze these mechanisms, we assume that the support of μ is bounded on one side. Specifically, we consider the left side for the leftmost mechanism (lt) and the right side for the rightmost mechanism (rt).

Corollary B.1. *The leftmost mechanism lt has a finite limit of the Bayesian approximation ratio if and only if there exists $a \in \mathbb{R}$ such that $\mu((-\infty, a]) = 0$ and $\mu((-\infty, a+\delta]) > 0$ for every $\delta > 0$. In that case, we have that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[SC_{lt}(X_n)]}{\mathbb{E}[SC_{opt}(X_n)]} = \frac{m(\mu) - a}{\int_a^\infty |x - med(\mu)| d\mu},$$

where $med(\mu)$ is the median of μ. For the l_2 Cost, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}[(C_{l_2})_{lt}(X_n)]}{\mathbb{E}[(C_{l_2})_{opt}(X_n)]} = \left(\frac{Var(\mu)^2 + m(\mu)^2 - am(\mu) + a^2}{Var(\mu)^2}\right)^{\frac{1}{2}}.$$

In particular, if $a = 0$, we have $\lim_{n\to\infty} B^{(n)}_{ar,2}(lt) = \frac{(Var(\mu)^2 + m(\mu)^2)^{\frac{1}{2}}}{Var(\mu)} \leq 1 + \frac{m(\mu)}{Var(\mu)}$. A similar result holds for the rightmost mechanism.

Proof. We prove the Corollary only for the leftmost mechanism, the result for the rightmost mechanism follows by the same argument. Since the leftmost mechanism returns the smallest value of the vector $\vec{X}_n$, we have that $lt(\vec{X}_n)$ converges to a. We therefore have that $\lim_{n\to\infty} \mathbb{E}[SC_{lt}(\vec{X}_n)] = \int_a^\infty |x-a| d\mu = \int_a^\infty (x-a) d\mu = m(\mu) - a$, which concludes the proof in the Social Cost case. Let us now consider the l_2 Cost. In this case, we have that

$$\lim_{n\to\infty} \mathbb{E}[(C_{l_2})_{lt}(\vec{X}_n)] = \Big(\int_a^\infty (x-a)^2 d\mu\Big)^{\frac{1}{2}} = \Big(\int_a^\infty x^2 d\mu - 2am(\mu) + a^2\Big)^{\frac{1}{2}}$$
$$= \Big(Var^2(\mu) + m(\mu)^2 - 2am(\mu) + a^2\Big)^{\frac{1}{2}},$$

which concludes the thesis. □

Finally, we consider the mechanisms for locating two facilities: the left-rightmost mechanism (lrt). For lrt, we need to assume that the support of μ is bounded at least from one side.

Corollary B.2. *Given a measure $\mu \in \mathcal{P}(\mathbb{R})$ whose support is bounded at least from one side. Then, the limit of the Bayesian approximation ratio of the left-rightmost mechanism with respect to the Social Cost is equal to*

$$\frac{\int_a^{\frac{a+b}{2}} (x-a) d\mu + \int_{\frac{a+b}{2}}^b (b-x) d\mu}{\min_{\lambda \in \mathcal{P}_2(\mathbb{R})} W_1(\mu, \lambda)}. \tag{56}$$

Similarly, for the l_2 Cost we have

$$\frac{\int_a^{\frac{a+b}{2}} (x-a)^2 d\mu + \int_{\frac{a+b}{2}}^b (b-x)^2 d\mu}{\min_{\lambda \in \mathcal{P}_2(\mathbb{R})} W_2(\mu, \lambda)}. \tag{57}$$

In both formulae, we used the convention $\frac{a+\infty}{2} = +\infty$ and $\int_{+\infty}^{+\infty} f(x) dx = 0$.

Proof. The Social Case result follows by noticing that

$$\lim_{n\to\infty} \mathbb{E}[SC_{lrt}(\vec{X}_n)] = \int_a^{\frac{a+b}{2}} (x-a) d\mu + \int_{\frac{a+b}{2}}^b (b-x) d\mu,$$

and by applying Theorem 3.3. The result with respect to the l_2 Cost case follows a similar observation. □

Finally, we consider the Bayesian approximation ratio of the classic mechanisms with respect to the Maximum Cost. In particular, we have the following.

Corollary B.3. *Let μ be a measure supported over a compact interval $[a, b]$. The mechanisms med, rt, and lt have a limit of the Bayesian approximation ratio with respect to the Maximum Cost equal at most to* 2 *regardless of the measure μ. Similarly, the mechanism lrt has an approximation ratio with respect to the Maximum Cost equal at most to* 2.

Proof. Without loss of generality, let us assume that μ is supported over $[0, 1]$ (see Theorem 4.1). When we have one facility to locate, the optimal Maximum Cost is equal to $\frac{1}{2}$, while it is easy to check that the limiting cost of the lt and rt cost is 1, which concludes the proof for these two mechanisms. The cost of the median mechanism is $\max\{med(\mu), 1 - med(\mu)\}$. Since there always exists a measure whose median is close as we like to 1, we conclude that $\lim_{n\to\infty} B^{(n)}_{ar,\infty}(med) \leq 2$. Finally, the lrt mechanism has a cost of $\frac{1}{2}$, while the optimal Maximum Cost for two facilities is $\frac{1}{4}$, hence we have $\lim_{n\to\infty} B^{(n)}_{ar,\infty}(lrt) = 2$. □

Using the Theorems presented so far, we are able to retrieve the limit of the Bayesian approximation ratio of the classic mechanisms with respect to the Social, the Maximum, and the l_2 Costs for some classic probability distributions. For the sake of completeness, we compute the cost of the left-rightmost mechanism for the exponential distribution and the Social Cost, since it is the less straightforward computation to do. All the results of these computations are left in Table 3.

Let us now consider the formula (56) where μ is exponential. Since μ is bounded only from the left, we have that the numerator of equation (56) boils down to

$$\int_0^{+\infty} xe^{-x}dx = 1,$$

since, in this case $b = +\infty$ and thus $\frac{a+b}{2} = +\infty$ as well. To compute the denominator of (56), recall that the optimal solution of the Wasserstein projection problem with respect to W_1 is a measure supported over $\ln(\frac{2}{3})$ and $\ln(6)$ (see the proof of Theorem A.4). Thus, the denominator of (56) boils down to

$$\int_0^{\ln(2)} \left|x - \ln\left(\frac{2}{3}\right)\right| e^{-x}dx + \int_{\ln(2)}^{+\infty} |x - \ln(6)|e^{-x}dx \sim 0.35 + 0.07 = 0.42.$$

Thus, we conclude that the limit of the Bayesian approximation ratio of the left-rightmost mechanism with respect to the exponential distribution for the Social Cost is $\frac{1}{0.42} \sim 2.4$.

The Lower Bounds for the Leftmost and the Left-rightmost Mechanisms

To conclude, we prove the lower bounds on the worst-case approximation ratio of the leftmost (lt) and left-rightmost (lrt) mechanisms with respect to the l_2 cost (see the bounds marked by a star in Table 3). We show these lower bounds for both mechanisms by considering a specific instance. We assume all the agents' reports are in the interval $[0, 1]$. That is, $x_i \in [0, 1]$ for every $i \in [n]$.

Let us begin with the leftmost (lt) mechanism. Let us consider $x_1 = 0$ and $x_i = 1$ for every $i = 2, \ldots, n$. The output of the leftmost mechanism is $x_0 = 0$, so that the l_2 cost of the mechanism, in this instance is

$$(C_{l_2})_{lt}(\vec{x}_n) = \sqrt{\sum_{i=2}^{n} |1 - 0|^2 \frac{1}{n}} = \sqrt{\frac{n-1}{n}}.$$

By definition, the best possible location for the facility, which we denote with y, is in the mean of the reports, thus $y = \frac{n-1}{n}$. Therefore, the optimal l_2 cost is equal to

$$(C_{l_2})_{opt}(\vec{x}_n) = \sqrt{\left(\frac{n-1}{n}\right)^2 \frac{1}{n} + \sum_{i=2}^{n} \frac{1}{n^2}\frac{1}{n}}$$

$$= \sqrt{\frac{(n-1)^2 + (n-1)}{n^3}} = \sqrt{\frac{n(n-1)}{n^3}} = \frac{\sqrt{n-1}}{n}.$$

Therefore, we have that

$$ar(lt) \geq \frac{\frac{\sqrt{n-1}}{\sqrt{n}}}{\frac{\sqrt{n-1}}{n}} = \sqrt{n}.$$

We now show the lower bound for the left-rightmost mechanism. Let us consider the following report: $x_1 = 0$, $x_n = 1$, and $x_i = 0.5$ for $i = 2, \dots, n-1$. The output of the left-rightmost mechanism is $(0, 1)$, thus the l_2 cost of the mechanism is

$$(C_{l_2})_{rlt}(\vec{x}_n) = \sqrt{\sum_{i=2}^{n-1} \left(\frac{1}{2}\right)^2 \frac{1}{n}} = \frac{\sqrt{n-2}}{2\sqrt{n}.}$$

Let us now compute the optimal l_2 cost of $\vec{x}$. Since the agents' reports are supported over three points, one of the two facilities must supply both the agents in 0.5 and either the agent in 0 or the agent in 1. Due to the symmetry of the instance, we consider the optimal solution that isolates the agent in 0, so that the optimal location of the two facilities is $y_1 = 0$ and $y_2 = 0.5 + \frac{1}{2(n-1)}$ (notice that the location shared between the agents in 0.5 and the agent in 1 must be placed at the mean of $(\vec{x}_n)_{-1}$. Therefore, we get that the optimal cost with respect to the l_2 cost of $\vec{x}_n$ is

$$(C_{l_2})_{opt}(\vec{x}_n) = \sqrt{\sum_{i=2}^{n-1} \left(0.5 + \frac{1}{2(n-1)} - 0.5\right)^2 \frac{1}{n} + \left(\frac{n-2}{2(n-1)}\right)^2 \frac{1}{n}}$$

$$= \frac{1}{2\sqrt{n}} \sqrt{\frac{n-2}{(n-1)^2} + \frac{(n-2)^2}{(n-1)^2}}$$

$$= \frac{\sqrt{n-2}}{2\sqrt{n(n-1)}},$$

which, combined with $(C_{l_2})_{rlt}(\vec{x}_n) = \frac{\sqrt{n-2}}{2\sqrt{n}}$, allows us to conclude that $ar(lrt) \geq \sqrt{n-1}$.

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