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The Minimalist Foundation and Bishop's Constructive Mathematics

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20.1 Introduction

A central aspect of Bishop's constructive mathematics in [10, 12] emphasized in [15] is that of being a generalization of classical mathematics. Indeed, contrary to other constructive approaches, such as Brouwer's intuitionistic mathematics or Markov's recursive mathematics, in his mathematical development Bishop did not use any principle incompatible with classical mathematics as that formalizable in Zermelo–Fraenkel set theory. In this way Bishop produced an analysis of mathematical concepts that is finer than in other approaches.

Bishop himself in [10, 11] and in unpublished notes sketched a foundation for his mathematics. Many proposals of a formal system apt to founding his constructive mathematics followed afterwards in the style of axiomatic set theory in [1, 2, 3, 23, 56] and in that of type theory by Martin-Löf in [52, 58].

Most notably, the so-called notion of 'setoid' over Martin-Löf's type theory appears to be close to the idea of 'set' sketched in [10] as well as the notion of 'type-theoretic function', which appears to be an adequate representation of Bishop's notion of 'operation' because it explicitly shows its computational contents or 'numerical meaning'.

Then the model of setoids formalized over Martin-Löf's type theory appears to be a suitable framework to formalize Bishop's constructive mathematics. A whole study of its categorical structure as a quotient completion had been started and it is still ongoing (see, for example, [45, 46, 49, 58, 61]), and many different kinds of setoid models have been considered (see, for example, [6, 27]).

The main drawback of the formalization of mathematics in the setoid models is that it is very far from the language used in the informal mathematical practice of constructive proofs, including that in Bishop's literature and even more so that of classical mathematics. This is because the formalization in this model, and more generally in Martin-Löf's type theory, requires us to handle lots of computational details useful for the extraction of programs from proofs but apparently useless to develop the constructive proofs themselves.

To overcome this problem, in [73] it was proposed that Martin-Löf's type theory should be extended with some abstract concepts, like that of 'proof-irrelevant proposition' and that of 'subset', as soon as they satisfy the *forget–restore principle* introduced by the second author of the present chapter. This principle states that one can abstract away from irrelevant computational information when this information can be restored in the process of extracting a program from a constructive proof.

Pushing forward the idea of the forget–restore principle, in [47] we introduced the notion of *two-level foundation* for constructive mathematics. Such a foundation should consist of:

- one theory acting as the *extensional level* written in a language close to the usual mathematical practice of proofs;
- another theory acting as the *intensional level* written in a type-theoretic language suitable for extraction of programs from proofs;
- an interpretation of the extensional level in (a model of) the intensional level showing that the extensional level has been obtained from the intensional one following the forget–restore principle.

The introduction of a two-level foundation was also motivated by the need to build a new foundation for constructive mathematics. Indeed, since 2005 with [47], we embarked on the project of building a *Minimalist Foundation* where the mathematics developed in it turns out to be compatible with the different approaches to constructivism, and also with classical mathematics. To formalize Bishop's mathematics we intended to build an intuitionistic and predicative foundation finer than the formal systems available in the literature, and characterized by the lack of whatsoever choice principle, including the so-called *axiom of unique choice*.

In [39] a full formal system, called the Minimalist Foundation, here named **MF** for short, was proposed.

In parallel, a new approach to constructivism, called 'dynamic', was also put forward in [65, 67, 68, 69, 70, 71]. This was inspired by the constructive approach originating explicitly with Brouwer at the beginning of the twentieth century and revived in the 1960s and 1970s (see, for example, [10, 51], among others). The first chapter of [72] will contain a detailed introduction to dynamic constructivism.

In the following we are going to describe what aspects of our minimalist approach and **MF** have in common with Bishop's one, called `BISH`, and what differ.

The main common aspects include the following:

- the compatibility with classical mathematics via a language close to that of usual mathematical practice;

- the need to compile this language in a strictly algorithmic language to extract the computational contents of constructive proofs.

Both aspects are fulfilled in **MF** by crucially employing its two-level structure.

Indeed, compatibility with the standard Zermelo–Fraenkel foundation for classical mathematics is fulfilled at the extensional level of **MF**, while the extraction of programs from proofs is at its intensional level. In particular, the intensional level can be interpreted in realizability semantics extending the *Kleene realizability of intuitionistic arithmetic* as shown by Ishihara, Maietti, Maschio, and Streicher in [31]. This fact has two main consequences which emphasize the constructivity of the whole **MF**.

The first is that the intensional level of **MF** is consistent with *full axiom of choice* and *formal Church thesis* as advocated in [47]. This characteristic is generally not satisfied by the other constructive intensional foundations in the literature such as the extension of Martin-Löf type theory called Homotopy Type Theory in [76] (because it satisfies the function extensionality principle).

The second consequence is that the extensional level of **MF** turns out to be consistent with the formal Church thesis (see [44]) via its interpretation at the intensional level.

Furthermore, the intensional level of **MF** could serve as a base for a minimalist proof-assistant whose formalized proofs can, a priori, be reused in proof-assistants based on the many extensions. This would be a practical application of the fact that **MF** can well serve as a basic theory to compare the different approaches to mathematics and their proofs.

We can underline some major peculiarities of **MF** not present in Bishop's conception of mathematics `BISH`.

One main difference is about the concept of function. As in `BISH`, in **MF** we have both the notion of *operation* with the meaning of representing a computable function, and that of functional relation. However, contrary to `BISH` and other type-theoretic foundations for `BISH`, in **MF** these two notions are kept well distinct. In fact, while operations between two sets do form a set, functions do not generally do. This distinction is guaranteed by the lack of the general validity of choice principles in both levels of **MF** (see [41]). Indeed it is enough to add a rule of unique choice to both levels of **MF** to guarantee the validity of the axiom of unique choice which makes the two notions coincide.

There is a major consequence of the absence of choice principles from **MF** combined with its predicative nature (even à la Feferman see [31, 43, 44]) when adopting **MF** to develop topology. It is that the constructive pointfree approach to topology introduced by Martin-Löf and the second author in the 1980s in [64] under the name of *formal topology* constitutes not only a valid alternative to pointwise approaches for constructive analysis by Brouwer (see [75]) and Bishop (see [10, 12]), but it appears to be compulsory. The main reason, as sketched in [40, 42], is that in **MF** real numbers, either as Dedekind cuts or as Cauchy sequences, cannot be proven to be sets. Also, choice sequences of Baire and Cantor spaces do not form a set. All this is a consequence of the fact that in **MF** functional relations between two sets do not generally form a set. Instead, a priori, using Martin-Löf's type theory in [58] as a foundation, both pointwise approaches and pointfree ones could seem legitimate. In fact, one can define a pointwise topology on Dedekind real numbers, because these are in bijective correspondence with Cauchy sequences, and the latter can be represented in Martin-Löf's type theory as a setoid. Also, in the predicative foundation of Aczel's constructive set theory **CZF** in [1, 2, 3] both Dedekind reals and Cauchy reals form a set and a pointwise approach is possible.

In this chapter we recall the basic definitions of formal topology necessary to introduce the previously mentioned example of real numbers and Baire and Cantor spaces by underlying how they are formalized in **MF**.

The way constructive topology is formalized in **MF** agrees well with our minimalist attitude, especially if we want to work in a constructive foundation compatible with classical predicativity where we can distinguish the real (effective) structure of a topology from a corresponding ideal (infinitary) structure of formal points.

A major benefit from developing pointfree topology in **MF** in the form of formal topology is that we gain in clarity and in an analysis of topological concepts finer than in other foundations.

Finally, we conclude by describing an extension of **MF**, actually of its extensional level, which appears closer to `BISH`.

This extension of **MF** is characterized by the validity of choice principles including the axiom of unique choice and the axiom of countable choices. It should also be interpretable in Martin-Löf's type theory to form a two-level foundation by extending the interpretation in [39]. But a proof of this is left to future work.

20.2 Why Adopt a Minimalist Foundation?

A plurality of philosophical reasons for a constructive approach to mathematics has been proposed, both before and after Brouwer and Bishop.

Presently, various logical systems to formalize constructive mathematics are available in the literature. They range from axiomatic set theories, as Aczel's **CZF** in [1, 2, 3] or Friedman's **IZF** in [8], to the internal theory of categorical universes as topoi or pretopoi in [33, 35, 37], to type theories such as Martin-Löf's type theory in [58] or Coquand's Calculus of Inductive Constructions in [19, 21]. No existing constructive foundation has yet superseded the others as the standard one, as Zermelo–Fraenkel axiomatic set theory did for classical mathematics.

Various machine-aided proof development systems are also available to implement mathematics (see, for example, [78]). Many of those for constructive mathematics are based on type systems which are also paradigms of (functional) programming languages with the possibility of extracting the computational contents of constructive mathematical proofs. Some of these, for example Coq in [18] or Matita in [5], are based on impredicative typed systems, while some others, for example Agda in [13] and Nuprl in [4], are based on predicative ones.

Beginning with [47], we embarked on the project of developing a foundation with minimal assumptions. The main reason for this choice is to support our general attitude to preserve all effective notions and conceptual distinctions as much as possible, with no a priori exception. The result is a foundation which is minimalist also in the sense that it becomes a common core among the most relevant constructive foundations. Thus we expect that such a minimalist foundation should be useful not only to constructive mathematicians but also to logicians, for example as a base system to do constructive reverse mathematics, and also to computer scientists, as a base for a *minimalist* proof-assistant suitable for formalizing reusable proofs and for program extraction from proofs.

20.2.1 Founding Constructive Mathematics on a Two-Level Theory

In our opinion, a constructive foundation should make evident those key aspects which differentiate constructive mathematics from classical mathematics. For example, a typical characteristic of constructive proofs, contrary to classical ones, is the possibility of extracting programs computing witnesses of true existential statements occurring in them.

Even better, any proof in a constructive system should be seen as a program. Hence, a foundation for constructive mathematics should be at the same time a theory of sets, in which to formalize mathematical theorems, and a programming language, in which to extract the computational contents of mathematical proofs.

In [47] we argued that such a constructive foundation (validating Heyting arithmetics at least) should be a two-level theory consisting of the following.

- A level, called *extensional*, which should be an extensional set theory (with undecidable equality of sets and elements) formulated in a language close to that used in the common practice of developing mathematics.
- Another level, called *intensional*, which should be an intensional theory (with decidable equality of sets and elements) enjoying extraction of programs from proofs; according to [47] this level should possibly be a proofs-as-programs theory, that is, a theory consistent with the axiom of choice

$$\text{(AC)} \qquad \forall x \in A\, \exists y \in B\, R(x, y) \;\longrightarrow\; \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

for A, B sets and $R(x, y)$ a logical relation, and the formal Church thesis for functions between natural numbers denoted with the symbol $Fun(Nat, Nat)$

$$\text{(CT)} \qquad \forall f \in Fun(Nat, Nat) \qquad \exists e \in Nat$$
$$(\,\forall x \in Nat\; \exists y \in Nat\; T(e, x, y)\; \&\; U(y) = f(x)),$$

where Nat is the set of natural numbers and $T(e, x, y)$ is the Kleene predicate expressing that y is the computation executed by the program numbered e on the input x and $U(y)$ is the output of the computation y.

- Then, in order to guarantee the extraction of programs even from proofs written at the extensional level, we required that the extensional level should be obtained as an abstraction of the intensional level according to the forget–restore principle proposed by the second author of the present chapter in [73].

The link between the two levels was then made more technical in [39], by requiring that the extensional level should be interpreted in the intensional one by means of a quotient completion of the latter, that is, the extensional level should be seen as (a fragment of) the internal language of a quotient completion built on the intensional one.

This kind of link captures what happens in the practice of computer-aided formalization of mathematics in an intensional type theory, which makes use of the so-called model of 'setoids' built on it (see [6, 27]). Actually another motivation behind the notion of two-level foundation in [39, 47] is the desire to make explicit the extensional theory validated in the quotient model chosen to formalize mathematical proofs in intensional-type theory.

Our two-level structure where the intensional level is consistent with axiom of choice and formal Church thesis fully agrees with Bishop's need to exhibit the computational contents of constructive proofs, in particular of existential statements whose witness can be chosen computationally (see [10, Chapter 1] and [11]).

20.3 The Minimalist Foundation

In [39] we presented a two-level formal system which satisfies the requirements in [47] of a two-level foundation for constructive mathematics. We call this system *the two-level minimalist foundation*, or **MF** for short. We are aware, however, that a specific formal system, which is static by definition, cannot fully capture the dynamics of the minimalist approach to constructivism, started in [47, 67, 68, 69].

The two levels of **MF** are both given by a type theory à la Martin-Löf: the intensional level, called **mTT**, is an intensional type theory including aspects of Martin-Löf theory in [58] (and extending the set-theoretic version in [47] with collections), and its extensional level, called **emTT**, is an extensional type theory including aspects of extensional Martin-Löf's theory in [53]. Then a quotient model of setoids à la Bishop in [6, 10, 27, 59] is used in [39] to interpret the extensional level in the intensional one. A categorical study of this quotient model has been carried out in [45, 46, 49] and is related to the construction of Hyland's effective topos in [28, 29].

In the following, we explain the main characteristics of the extensional level **emTT** and of **mTT** viewed more as a many-sorted logic than as a type theory. This is because both levels of **MF** are given by a type theory that includes a primitive notion of proposition, which allows us to control the validity of choice principles.

Need for Two Types of Entities: Sets and Collections A minimalist foundation for constructive mathematics should certainly be based on intuitionistic predicate logic and include at least the axioms of Heyting arithmetic. Hence we could expect to build it starting from a many-sorted logic, such as Heyting arithmetic of finite types in [75], where sorts, which we call *types*, include the basic sets we need to represent our mathematical entities.

However, in order to develop topology in an intuitionistic and predicative way, we need a foundation with two kinds of types: sets and collections. The main reason is that the *power of a non-empty set*, namely the discrete topology over a non-empty set, fails to be a set in a predicative foundation, and it is only a *collection*.

Need for Two Types of Propositions In parallel with the presence of sets and collections, to keep the system predicative we also need to distinguish two types of propositions: those closed under quantifications on sets, called here *small propositions* as in [39] (and *proper propositions* in [72]), from those closed under any kind of quantification, called here simply *propositions* as in [39] (and *improper propositions* in [72]). Both kinds of propositions include propositional equalities which are small propositions only if they refer to elements of a set.

Need for Two Types of Functions It is well known that by *adding the principle of excluded middle* to some constructive foundations, such fas Aczel's **CZF** or Martin-Löf's type theory, one can derive that *power-collections become sets* and thus get an impredicative theory. In both such theories this is due to the fact that the collection of functions from a set A to the boolean set $\{0, 1\}$, called *exponentiation of the boolean set over A*, forms a set, too. Therefore, if we wish to have compatibility with classical theories where the power of a non-empty set is not a set as in Feferman's predicative theories in [23], we need to avoid exponentiation of functions.

A drastic solution is to drop all axioms yielding any form of exponentiation. What we propose is to allow exponentiation only of a certain kind, as happens in

[23]. To this purpose, we introduce a primitive notion of *operation*, represented by certain functional terms

$$f(x) \in B \; [x \in A]$$

in a set B with a free variable in the set A. These operations can be defined as *type-theoretic functions* of a type theory, like in Martin-Löf's type theories in [53, 58]. Clearly any operation $f(x) \in B \; [x \in A]$ must give rise to a functional relation $f(x) =_B y \; [x \in A, y \in B]$, namely what is usually called function. What we do not wish to guarantee is the converse. Our idea is then that *only exponentiation of operations from a set A to a set B forms a set*.

20.3.1 The Main Types of the Extensional Level of the Minimalist Foundation

The formal system **emTT** of the extensional level of the Minimalist Foundation in [39] is written in the style of Martin-Löf's type theory in [58] by means of the following four kinds of judgements:

$$A \; type \; [\Gamma] \qquad A = B \; type \; [\Gamma] \qquad a \in A \; [\Gamma] \qquad a = b \in A \; [\Gamma];$$

that is, the type judgement (expressing that something is a specific type), the type equality judgement (expressing that two types are equal), the term judgement (expressing that something is a term of a certain type), and the term equality judgement (expressing the *definitional equality* between terms of the same type), respectively, all under a context Γ.

The word *type* is used as a meta-variable to indicate four kinds of entities: collections, sets, propositions, and small propositions, namely

$$type \in \{coll, set, prop, prop_s\}.$$

Therefore, in **emTT** types are actually formed by using the following judgements:

$$A \; set \; [\Gamma] \qquad B \; coll \; [\Gamma] \qquad \phi \; prop \; [\Gamma] \qquad \psi \; prop_s \; [\Gamma]$$

saying that A is a set, that B is a collection, that ϕ is a proposition, and that ψ is a small proposition.

Here, contrary to [39] where we use only capital latin letters as meta-variables for all types, we use greek letters ψ, ϕ as meta-variables for propositions and capital italic latin letters A, B as meta-variables for sets or collections, and small italic latin letters a, b, c as meta-variables for terms, that is, elements of the various types.

Observe that for a set A, when we say that

$$a \in A \; [\Gamma]$$

is derivable in **emTT**, we actually mean that the term a *is an element of the set* A *under the context* Γ and hence the symbol $\in$ stands for a *set membership*. As usual in type theory, *equality of sets* is given *primitively* and is not defined by equating sets with the same elements. This is indeed a main difference between a set theory defined as a typed system in the style of Martin-Löf's type theory in [58] and an axiomatic set theory à la Zermelo–Fraenkel.

We now proceed by briefly describing the various kinds of types in **emTT**, starting from small propositions and propositions, then sets, and finally collections.

Small propositions in **emTT** include all the logical constructors of intuitionistic predicate logic with equality and quantifications restricted to sets:

$$\phi\ prop_s \ \equiv\ \ \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \\ \forall x \in A\ \phi(x) \mid \exists x \in A\ \phi(x) \mid x =_A y$$

provided that A *is a set*. Here we use the more familiar $x =_A y$ for the *extensional* equality type $\mathsf{Eq}(A, a, b)$ of Martin-Löf type theory in [53].

Then, *propositions* of **emTT** include all the logical constructors of intuitionistic predicate logic with equality and quantifications on all kinds of types, namely sets and collections. Of course, small propositions are also propositions:

$$\phi\ prop \ \equiv\ \ \phi\ prop_s \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \\ \forall x \in B\ \phi(x) \mid \exists x \in B\ \phi(x) \mid x =_B y.$$

In order to close sets under comprehension, for example to include the set of positive natural numbers $\{x \in \mathrm{N} \mid x \geq 1\}$, and to define operations on such sets, we need to think of propositions as types of their proofs: small propositions are seen as sets of their proofs while generic propositions are seen as collections of their proofs. That is, we add to **emTT** the following rules:

$$(\textbf{prop}_s\textbf{-into-set})\ \frac{\phi\ prop_s}{\phi\ set} \qquad\qquad (\textbf{prop-into-coll})\ \frac{\phi\ prop}{\phi\ coll}.$$

The difference between the notion of set and collection will be explained later in this section.

A key feature of the extensional typed system **emTT** is proof-irrelevance of propositions. This means that in **emTT** a proof of a proposition, if it exists, is *unique* and equal to a canonical proof term called true thanks to the following rules:

$$(\textbf{prop-mono})\ \frac{\phi\ prop\ [\Gamma] \qquad p \in \phi\ [\Gamma] \quad q \in \phi\ [\Gamma]}{p = q \in \phi\ \ [\Gamma]}$$

$$(\textbf{prop-true})\ \frac{\phi\ prop \qquad p \in \phi}{\mathsf{true} \in \phi}.$$

Proof-irrelevance of propositions justifies the introduction of a judgement asserting that *a proposition* ϕ *is true under a context* Γ *assuming propositions* $\psi_1, \ldots, \psi_m$ *are true* as in [53, 54]. This judgement can be directly interpreted in **emTT** as follows:

$$\phi\ true\ [\Gamma; \psi_1\ true, \ldots, \psi_m\ true] \ \equiv\ \mathsf{true} \in \phi\ [\Gamma,\ y_1 \in \psi_1, \ldots, y_m \in \psi_m].$$

In **emTT** *sets* are characterized as inductively generated types and they include the following:

$$A\ set \ \equiv\ \ \phi\ prop_s \ |\ N_0 \ |\ N_1 \ |\ List(A) \ | \\ \Sigma_{x\in A} B(x) \ |\ A + B \ |\ \Pi_{x\in A}\, B(x) \ |\ A/\rho,$$

where the notation N_0 stands for the empty set, N_1 stands for the singleton set, $List(A)$ stands for the set of lists on the set A, $\Sigma_{x\in A}B(x)$ stands for the indexed sum of the family of sets $B(x)\ set\ [x \in A]$ indexed on the set A, $A + B$ stands for the disjoint sum of the set A with the set B, $\Pi_{x\in A}B(x)$ for the product type of the family of sets $B(x)\ set\ [x \in A]$ indexed on the set A, and A/ρ stands for the quotient set provided that ρ is a small equivalence relation $\rho\ prop_s\ [x \in A, y \in A]$. Moreover, we call N the set of natural numbers represented by $List(N_1)$.

The notion of set in **emTT** agrees with that in [10] and in [51]. According to them, sets must have an effective nature which is mostly forgotten in any axiomatic approach where a universe of sets closed under certain properties is implicitely assumed as the underlying range of the set variables. In fact, each set A must be specified by providing a finite number of rules to construct all its elements (see the rules of **emTT** forming elements of sets in [39]). It is understood that the rules defining a set are inductive, that is, their application can be iterated any finite number of times. The infinite is only potential, and in a certain sense it is always reduced to a finite description, at a higher order: not a finite number of elements, but a finite number of rules to generate (the infinite number of) them. In particular, the elements of the product type $\Pi_{x\in A}B(x)$ are only terms

$$b(x) \in B(x)\ [x \in A].$$

In the case the family $B(x)\ set\ [x \in A]$ is just a constant set B indexed on the set A, we indicate the product type simply as

$$A \to B \ \equiv\ \Pi_{x\in A}B$$

and its elements are just *operations*

$$b(x) \in B\ [x \in A].$$

Hence, in **emTT** *operations between two sets form a set*, but generic functions between them do not.

Finally, *collections* in **emTT** include the following types:

$$B\,coll \;\equiv\; \quad A\,set \;\mid\; \phi\,prop \;\mid\; \mathcal{P}(1) \;\mid\; A \to \mathcal{P}(1) \;\mid\; \Sigma_{x\in B}\,C(x),$$

where $\mathcal{P}(1)$ and $A \to \mathcal{P}(1)$ stand for the power-collections of the singleton and of a set A respectively, and $\Sigma_{x\in B}\,C(x)$ stands for the indexed sum of the family of collections $C(x)\;col\;[x \in B]$ indexed on the collection B. Actually, for a set A, we will use the common abbreviation of power-collection

$$\mathcal{P}(A) \;\equiv\; A \to \mathcal{P}(1).$$ [1]

Elements of the power-collections rely on the notion of subset, which in **emTT** is inspired by that in [73] put on top of Martin-Löf's type theory. A *subset of a set* A is defined as the equivalence class of a *small* predicates $\phi(x)$ depending on one argument in A with respect to the equivalence relation of equiprovability. This is the minimum we must require in order to close subsets under comprehension. Indeed, for any small predicate $\phi(x)\;prop_s\;[x \in A]$ on a set A we can define its subset comprehension as

$$\{\, x \in A \;\mid\; \phi(x)\,\} \in \mathcal{P}(A).$$

Moreover, two equiprovable small predicates give rise to the same subset, that is, in **emTT** we can derive

$$\frac{\phi_1(x) \;\leftrightarrow\; \phi_2(x)\;true\;[x \in A]}{\{\, x \in A \;\mid\; \phi_1(x)\,\} =_{\mathcal{P}(A)} \{\, x \in A \;\mid\; \phi_2(x)\,\}\;\; true}.$$

In the following we indicate subsets of a set A with capital letters $U, V, W \ldots$.

Associated with the notion of subset we have also a *subset membership* indicated with the symbol ϵ, which we distinguish from the primitive set membership $\in$ used to say that an element belongs to a certain set. Given a subset $U \subseteq A$ of a set A, that is, $U \in \mathcal{P}(A)$, for any $a \in A$ we define a new small proposition

$$a \,\epsilon\, U\;prop_s.$$

We can prove in **emTT** that

$$U = \{\, x \in A \;\mid\; x \,\epsilon\, U\,\} \in \mathcal{P}(A)$$

and also that, for any small predicate $\phi(x) \in \mathsf{prop_s}\;[x \in A]$ on the set A and for any element $a \in A$,

$$a \,\epsilon\, \{\, x \in A \;\mid\; \phi(x)\,\} \;\leftrightarrow\; \phi(a)\;\; true.$$

[1] The notation $A \to \mathcal{P}(1)$ for the power-collection $\mathcal{P}(A)$ is used to remember that its elements are operations from a set A to the power-collection on the singleton.

The subset equality is equivalent to usual extensional equality with respect to membership ϵ, namely we can derive in **emTT** that

$$\forall x \in A\, (x \,\epsilon\, U \leftrightarrow x \,\epsilon\, W) \;\leftrightarrow\; U =_{\mathcal{P}(A)} W \;\; true$$

and, of course, that

$$\{\, x \in A \mid \phi(x) \,\} =_{\mathcal{P}(A)} \{\, x \in A \mid \psi(x) \,\} \quad \leftrightarrow \quad \forall_{x \in A}\, (\phi(x) \leftrightarrow \psi(x)) \;\; true.$$

In particular, $\mathcal{P}(1)$ denotes the power-collection of the singleton N_1 and its elements are equivalence classes of small propositions closed under the equivalence relation of equiprovability.

The fact that subset equality corresponds to usual extensional equality of sets suggests that we can view the subset theory in **emTT** as a *local set theory* where subsets of a set A can be considered *local sets* in [9] in the style of Zermelo–Fraenkel set theory. Then, membership and extensional equality via elements becomes a local property restricted to a given set A. To this purpose, observe that among subsets of A, there is A itself thought of as the subset

$$\{\, x \in A \mid \mathtt{tt} \,\},$$

where $\mathtt{tt}$ is any tautology. Moreover, we can define *quantifiers relativized to a subset*: this means that, if $U \subseteq A$ and φ is a small predicate (or propositional operation) with an argument in A, we write $\exists x \,\epsilon\, U\; \varphi$ as an abbreviation for the formula $\exists x \in A\; (x \,\epsilon\, U \;\&\; \varphi)$, and $\forall x \,\epsilon\, U\; \varphi$ as an abbreviation for the formula $\forall x \in A\; (x \,\epsilon\, U \rightarrow \varphi)$. A consequence of these definitions is that all laws of many-sorted intuitionistic logic regarding quantifiers extend to quantifiers relativized to a subset.

Note that the membership relation ϵ between terms and subsets is crucial in **emTT** to obtain an embedding of subsets into sets, which associates the set

$$\Sigma_{x \in A}\; x \,\epsilon\, U \;\; set$$

to a subset $U \subseteq A$. In this way an operation from $U \subseteq A$ to a set B can be represented as an operation in $\Sigma_{x \in A}\; x \,\epsilon\, U \rightarrow B$.

The **emTT** distinction between set and collection is analogous to the distinction between set and class in axiomatic set theory. But while in axiomatic set theory the distinction is mainly due to problems with consistency (or size), here it is motivated by quality of information and preservation of predicativity. Indeed, sets are kept distinct from collections to be able to maintain a distinction between computable, effective domains (represented by sets) and non-computable ones (represented by collections). This distinction is also extended to propositions in **emTT** by selecting small propositions as those propositions closed only under quantifications over sets and only under propositional equality only on sets. Then, to avoid an impredicative power-collection of a set, a subset must be defined as an equivalence class of small predicates and not of generic ones.

An important conceptual reason why even the power-collection $\mathcal{P}(1)$ of the singleton is only a collection and not a set is that in **emTT** we intend the notion of small proposition to be *open*. The same is done for that of proposition, of set, and of collection. Indeed, although we have fixed the system **emTT**, new sets or collections can be introduced at any time. This implies in particular that the collection of small propositions (quotiented under equiprovability) is not a set. Indeed, each time we fix our propositions or sets by fixing a formal system, both notions become inductively generated. However, we cannot support an induction principle inside the formal system, given that the number of inductive hypotheses should change any time we introduce a new set or proposition. This is different from the induction principle on the set of natural numbers, which has only two hypothesis: what we do on the number zero, and with any successor number.

20.3.2 The Main Types of the Intensional Level of the Minimalist Foundation

Here we briefly describe the main types of the formal system **mTT** of the intensional level of the Minimalist Foundation in [39] by simply pointing out the differences with those of **emTT**.

In essence **mTT** is a dependent type theory which provides a *predicative version* of Coquand's Calculus of Constructions in [19]. It is written in the style of *intensional* Martin-Löf's type theory in [58] by means of the following four kinds of judgements:

$$A\ type\ [\Gamma] \qquad A = B\ type\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad a = b \in A\ [\Gamma].$$

Like **emTT**, **mTT** includes *small propositions* and *propositions* which are closed under the same type constructors as those in **emTT** except that the propositional equality type is written $\mathsf{Id}(A, a, b)$ and has proper rules specifying its elements. There are also the rules stating that small propositions are propositions, that small propositions are sets and that propositions are collections. A main difference with respect to **emTT** is that in **mTT** the rules (**prop-mono**) and (**prop-true**) are omitted. As a consequence, all propositions in **mTT** are seen as types of their proofs which are *not in general unique* as usual in intensional type theory. Moreover, as in the intensional version of Martin-Löf's type theory, in **mTT** the definitional equality of terms of the same type given by the judgement

$$a = b \in A\ [\Gamma],$$

which should be computable, is *no longer equivalent* to the propositional equality type

$$\mathsf{Id}(A, a, b)\ prop\ [\Gamma],$$

which is not necessarily computable and not necessarily equipped with only one proof.

Sets in **mTT** are closed under the same constructors as those in **emTT** with the exception of the quotient set constructor A/ρ. As in **emTT**, in **mTT** there is also the rule stating that sets are collections.

Finally, *collections* in **mTT** include the same constructors as those of **emTT** except that the power-collection of the singleton $\mathcal{P}(1)$ is replaced by the universe of small propositions $\mathsf{prop_s}$ and the power-collection constructor $A \to \mathcal{P}(1)$ on a set A is replaced by the collection $A \to \mathsf{prop_s}$ of predicates or propositional operations depending on the set A.

The dependent type theory **mTT** was designed in order to serve as a base for a proof-assistant.

20.3.3 On the Extraction of Programs from Proofs in MF

Here we describe how Bishop's desire to compile a foundation for constructive mathematics in a programming language is fulfilled for **MF**.

First of all, **MF** was structured as a two-level theory to interpret constructive proofs done at its extensional level **emTT** to proofs done at its intensional level **mTT** from which to extract the computational contents in the form of programs. However, the extraction of the computational contents of proofs in **mTT** cannot be performed in **mTT** itself as shown in [41] but in a stronger theory or in the realizability semantics in [31]. One could then think of enlarging the intensional level to become the stronger theory needed, but this would not satisfy the forget–restore principle according to which the entities at the extensional level should be obtained by abstraction from the intensional ones, or more concretely as quotients of intensional entities.

A priori, the intensional level **mTT** itself could serve as a programming language to compile proofs done at the extensional level. Indeed, **mTT** is a dependent type theory where we can construct a correct and terminating program as a typed term meeting a certain specification defined as its type. But to extract programs from constructive proofs it is desirable that from a proof of an existential statement under hypothesis

$$p(x) \in \exists y \in B\ R(x, y)\ [x \in A]$$

for generic types A and B, one may extract a functional program $f \in A \to B$ whose graph is contained in the graph of $R(x, y)$, namely for which we can prove that there exists a proof-term $q(x)$ such that we can derive

$$q(x) \in R(x, f(x)) \; [x \in A]$$

This property is called *choice rule*.

In all the versions of Martin-Löf dependent type theory in [53, 58] the choice rule is valid thanks to the identification of the **MLTT**-existential quantifier with the *the strong indexed sum of a set family*, which characterizes the so-called *propositions-as-sets isomorphism*. Then the *axiom of choice*

$$\text{(AC)} \qquad \forall x \in A \, \exists y \in B \; R(x, y) \quad \longrightarrow \quad \exists f \in A \to B \, \forall x \in A \; R(x, f(x))$$

is valid for generic types A and B.

However, in **mTT** the existential quantifier is not identified with the strong indexed sum type whilst it is still a type of its proofs. The result is that the choice rule in Definition 20.1 is not valid.

Definition 20.1 The dependent type theory **mTT** satisfies the *choice rule* if for every small proposition $R(x, y) \; prop_s \; [x \in A, y \in B]$ derivable in **mTT**, for any derivable judgement in **mTT** of the form

$$p(x) \in \exists_{y \in B} \; R(x, y) \; [x \in A],$$

there exists in **mTT** a typed term

$$f(x) \in B[x \in A]$$

for which we can find a proof-term $q(x)$ and derive in **mTT**

$$q(x) \in R(x, f(x)) \; [x \in A].$$

Proposition 20.2 *In* **mTT** *the* choice rule *is not valid.*

Proof See [41]. □

Hence, when proving a statement of the form

$$\forall_{x \in A} \exists y \in B \; R(x, y)$$

in the dependent typed theory **mTT**, we cannot always extract a functional term $f \in A \to B$ computing the witness of the existential quantification depending on a $x \in A$ within the theory itself but we need to find it in a more expressive proofs-as-programs theory.

For **mTT** we can use Martin-Löf's type theory, for short **MLTT**, in [58] as the more expressive theory where to perform the mentioned witness extraction. Indeed, we can interpret **mTT** inside **MLTT** as shown in [39] by preserving the meaning of its entities.

This extraction is done by first embedding the proof-term

$$p \in \forall_{x \in A}\ \exists y \in B\ R(x, y)$$

derived in **mTT** and then using **MLTT**-projections to extract f.

The other possibility is to perform this witness extraction in the realizability model of **mTT** in [31]. This realizability model guarantees that the intensional level **mTT** of **MF** is a 'proofs-as-programs theory' in the sense of [47], namely that **mTT** is consistent with the axiom of choice (AC) and the formal Church thesis (CT) by identifying $Fun(Nat, Nat)$ in **mTT** with the type of *functional relations between natural numbers*.

$$\Sigma_{R \in \mathcal{P}(\mathrm{N},\mathrm{N})}\ \forall_{x \in A}\ \exists! y \in B\ \langle x, y \rangle \varepsilon R,$$

where $\exists! y \in B\ \langle x, y \rangle \varepsilon R\ \equiv\ \forall_{y_1 \in B}\ \forall_{y_2 \in B}\ R(x, y_1)\ \&\ R(x, y_1)\ \rightarrow\ \mathsf{Id}(B, y_1, y_2)$

Actually in [31] **mTT** is shown to be consistent with (AC) and the *formal Church thesis for operations between natural numbers*

$$(\mathrm{CT_{tt}})^2 \qquad \forall f \in \mathrm{N} \rightarrow \mathrm{N}$$
$$\exists e \in \mathrm{N} \qquad \forall x \in \mathrm{N}\ \exists y \in \mathrm{N}\ \ (T(e, x, y)\ \wedge\ U(y) =_N f(x))$$

stating that all operations between natural numbers are recursive.

The consistency of **mTT** with (AC) and ($\mathrm{CT_{tt}}$) implies the consistency of **mTT** with (AC) and (CT) since we can easily show the following.

Lemma 20.3 *In* **mTT** *extended with (AC) the formal Church thesis for functional relations CT is equivalent to the formal Church thesis for operations* CT_{tt}.

Therefore we conclude as follows.

Proposition 20.4 *The intensional level* **mTT** *of* **MF** *in [39] is consistent with (CT) and the axiom of choice in the form*

$$\text{(AC)} \qquad \forall x \in A\ \exists y \in B\ R(x, y)\ \ \longrightarrow\ \ \exists f \in A \rightarrow B\ \forall x \in A\ R(x, f(x))$$

for A, B *collections and* $R(x, y)$ *any proposition in* **mTT**.

Proof It follows from Lemma 20.3 and [31]. □

As a consequence of the interpretation of the extensional level of **MF** into its intensional level in [39], we can also deduce that the extensional level of **MF** is consistent with the formal Church thesis.

Proposition 20.5 *The extensional level* **emTT** *of* **MF** *in [39] is consistent with* (CT).

Proof A proof of this proposition can be obtained in various ways. For example, it also follows from the realizability interpretations of **mTT** in [43, 44], or from the fact that **emTT** can be interpreted by preserving the meaning of its entities both in the internal theory of a topos and then in Hyland's Effective Topos in [28] or in Aczel's set theory in [1]. It would be also possibile to interpret **emTT** in the predicative version of Hyland's Effective Topos in [48]. □

Observe that the consistency requirement of a proofs-as-programs theory with (AC) and (CT) just guarantees that from proofs of existential statements on natural numbers under hypothesis

$$\exists y \in \mathrm{N}\ R(x,y)\ true\ [x \in Nat]$$

we can extract of a computable choice function

$$f \in \mathrm{N} \to \mathrm{N}$$

producing a witness under hypothesis such that we can find a proof of

$$R(x, f(x))\ true\ [x \in \mathrm{N}].$$

It is then clear that our proofs-as-programs requirement does not fully capture the idea of a foundational theory that is at the same time a programming language satisfying the choice rule, such as **MLTT** in [58].

It is still an open problem whether **MLTT** enjoys our proofs-as-programs requirement or, equivalently, whether **MLTT** is consistent with the formal Church thesis (CT).

Our purpose with the proofs-as-programs requirement in [47] was to single out a property characterizing theories which are interpretable in extensions of Kleene realizability semantics for Heyting Arithmetics with finite types (see [75]).

20.3.4 Benefits of Distinguishing Operations from Functions

Inspired by Brouwer's difference between lawlike and choice sequences in [75], in **MF** contrary to `BISH` we can define *choice sequences* from the set of natural numbers N to a set B as *functions* (in the sense of functional relations, that is, total and single-valued relations), and *lawlike sequences* as *operations*.

Definition 20.6 (Choice and lawlike sequences) Given a set A, a *choice sequence* from the set N of natural numbers to A is a function defined by a small functional relation $\alpha(x,y)\ prop_s\ [x \in \mathrm{N}, y \in A]$ in **emTT**.

A *lawlike sequence* from the set N of natural numbers to A is an operation

$$f \in \mathrm{N} \to A$$

in **emTT**, or equivalently, thanks to the rules in [39] defining elements in $\mathrm{N} \to A$, an **emTT**-term $f(x) \in A\ [x \in \mathrm{N}]$.

It is possible to keep a distinction between choice sequences and lawlike sequences because in **emTT** the *axiom of unique choice*

$$(\mathrm{AC!}_{N,N}) \qquad \forall x \in \mathrm{N}\, \exists! y \in \mathrm{N}\, R(x,y) \;\longrightarrow\; \exists f \in \mathrm{N} \to \mathrm{N}\, \forall x \in \mathrm{N}\, R(x, f(x)),$$

which turns a function between natural numbers into an operation, is not valid, as shown in [40, 41]. Our distinction allows us to clarify and compare results about choice sequences in the literature, since choice sequences are sometimes identified with our functions, for example in [63], and sometimes with our operations, for example in [75].

Another consequence of the distinction between operations and functions is that we can refine the notion of decidable subset of the set of natural numbers N. In constructive mathematics it is common to say that a subset $U \subseteq \mathrm{N}$ is decidable if $\forall x\ (\, x \,\epsilon\, U \vee x \,\not\epsilon\, U\,)$ holds. In our theory we can distinguish three notions.

Definition 20.7 A subset U of the set N is said to be:

- *complemented*, if $\forall x\ (\, x \,\epsilon\, U \vee (x \,\not\epsilon\, U)\,)$ holds, and in this case U is classified by a function from N to the boolean set Bool

$$\chi_U(x,y) \;\equiv\; (x \,\epsilon\, U \;\&\; y =_{\mathsf{Bool}} 1) \;\vee\; (x \,\not\epsilon\, U \;\&\; y =_{\mathsf{Bool}} 0);$$

- *detachable*, if the subset U is classified by an operation, namely we can derive

$$\exists_{f \in \mathrm{N} \to \mathsf{Bool}}\ \forall x \in \mathrm{N}\, ((x \,\epsilon\, U \;\&\; f(x) =_{\mathsf{Bool}} 1) \;\vee\; (x \,\not\epsilon\, U \;\&\; f(x) =_{\mathsf{Bool}} 0));$$

- *decidable*, if U is classified by a computable operation, namely we can derive

$$\begin{aligned} \exists_{f \in \mathrm{N} \to \mathsf{Bool}}\ & (\forall x \in \mathrm{N}\, ((x \,\epsilon\, U \;\&\; f(x) =_{\mathsf{Bool}} 1) \;\vee\; (x \,\not\epsilon\, U \;\&\; f(x) =_{\mathsf{Bool}} 0)) \\ & \&\quad \exists e \in \mathrm{N} \quad \forall x \in \mathrm{N}\, \exists y \in \mathrm{N}\, (T(e,x,y) \wedge U(y) =_{\mathrm{N}} f(x))), \end{aligned}$$

  where $T(e,x,y)$ is the Kleene predicate expressing that y is the computation executed by the program numbered e on the input x and $U(y)$ is output of the computation y.

Observe that, classically, all subsets are complemented. Of course, in the presence of the axiom of unique choice, functions and operations coincide and hence complemented and detachable subsets coincide, too, as for example in Martin-Löf's type theory.

All the three kinds of subsets coincide in the Kleene realizability interpretation of Heyting arithmetic. This interpretation is in some sense the intended interpretation of the arithmetic fragment of a constructive foundation. Hence, the identification of the name 'decidable' with our notion of complemented subsets (that we do not follow here, however), has its own (plausible) justification.

In [40, 42], we observed that if we extend **emTT** with the principle of excluded middle then we can prove the existence of a power-set of detachable subsets, which do not necessarily coincide with all subsets, that is, with complemented ones. This option of restricting exponentiation as a set to lawlike sequences opens the way to build a theory compatible with classical predicativity as those in [23].

20.4 Why Adopt the Pointfree Approach to Develop Topology in MF?

Bishop, like Brouwer, developed constructive analysis by adopting a pointwise approach which presented some difficulties solved by them in different ways (see [60, 75]). When developing topology in **MF** we need to adopt the pointfree approach. The most important reason is that, when working in **MF**, the pointwise approach is not suitable because relevant examples of classical topologies (real numbers both as Dedekind cuts or Cauchy sequences, Baire space, Cantor space, etc.) do not give rise to a pointwise topology since their points do not form a set.

A solution is to work with the pointfree topology associated to each of these spaces. The constructive approach to pointfree topology given by formal topology has provided evidence that most important results of constructive analysis (see, for example, [51, 60]) can be reached in a compatible way with classical mathematics as in Bishop's constructive approach, but without assuming further principles, such as the Fan Theorem adopted by Brouwer in his pointwise approach and in [14, 15, 30, 75].

Before entering into details, we briefly review a constructive notion of topological space and then the main concepts of formal topology.

20.4.1 A Predicative Constructive Notion of Topological Space

Considering that in a predicative foundation the discrete topology on a given non-empty set is not a set but a collection, we need to review the concept of topological space by distinguishing what belongs to the realm of sets from what belongs to the realm of collections.

At first, one could think of simply keeping the traditional definition of topological space $(X, \mathcal{O}X)$ by just declaring the topology $\mathcal{O}X$ to be only a subcollection of the power of X which is a suplattice, that is, a complete join-semilattice, with finite distributive meets. This approach is compulsory in order to include the discrete topology among topologies. Even more, as shown in [22], there is no non-trivial suplattice, and hence no non-trivial topology, which is a set.

One should then define suplattices as collections closed under sups of *set-indexed* families. However, as in [7] and [64], suplattices are easier to handle by restricting to the notion of *set-based* suplattice, namely a semilattice that is generated by taking sups from a set(-indexed family) of elements, called *generators*.

Topologically this means that we need to assume that the collection of opens of a space has a base that is a set. To make this assumption rigorous, we require that for a given set of points X we have a set S together with a family of subsets $\mathsf{ext}\,(a) \subseteq X\ [a \in S]$ acting as a base for the topology on X. Elements a of S act as names of basic opens of X; they are called *formal basic neighbourhoods* or simply *observables*.

Then, following [66], we define a subset of X to be open if it is equal to $\mathsf{ext}\, U \equiv \cup_{a\epsilon U}\, \mathsf{ext}\, a$ for some subset $U \subseteq S$. It is immediate to see that open subsets are closed under unions of set-indexed families.

Then we need to require closure of open subsets under intersection. To this purpose, it is convenient to start from *basic neighbourhoods*, that is, subsets of X of the form $\mathsf{ext}\, a$ for some $a \in S$. For all $a, b \in S$, the intersection $\mathsf{ext}\, a \cap \mathsf{ext}\, b$ is open, that is, it is equal to $\mathsf{ext}\, W$ for some $W \subseteq S$, if and only if

B0 $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}\,(a \downarrow b)$ for all $a, b \in S$,

where

$$a \downarrow b \equiv \{c \in S :\ \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}.$$

In fact, $\mathsf{ext}\,(a \downarrow b)$ is by its definition the greatest open subset contained in $\mathsf{ext}\, a \cap \mathsf{ext}\, b$. Then, from B0, by two applications of distributivity in $\mathcal{P}X$, we can easily obtain

B1 $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\,(U \downarrow V)$ for all $U, V \subseteq S$,

where

$$U \downarrow V \equiv \cup_{a\epsilon U} \cup_{b\epsilon V}\ a \downarrow b.$$

Finally, to obtain that the whole space is open we need to add the requirement

B2 $X = \mathsf{ext}\, S$.

It is clear that for any family of subsets of X indexed by the set S, that is, for any $\mathsf{ext}\, a \subseteq X$ for $a \in S$, satisfying B1 and B2, the collection of subsets $\mathsf{ext}\, U \subseteq X$ for $U \in \mathcal{P}S$ is closed under set-indexed unions and finite intersections.

Therefore we can give the following constructive version of topological spaces (see [66]).

Definition 20.8 A *concrete space* is a structure $\mathcal{X} = (X, \mathsf{ext}, S)$ where X, S are sets and $\mathsf{ext}(a) \subseteq X\ [a \in S]$ is a set-indexed family of subsets satisfying:

B1 $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\,(U \downarrow V)$ for all $U, V \subseteq S$,

B2 $X = \mathsf{ext}\, S$.

In an impredicative foundation with power-sets, this is just a reformulation of the common notion of topological space.

The notion of concrete space is present in [10], under the name of neighbourhood space. The discrete topology on a set X is obviously an example of concrete space with X itself as base and $\mathsf{ext}\,(x) \equiv \{x\}$ for $x \in X$.

A useful example of concrete space is given by the set $\mathbb{Q}$ of rational numbers with the topology produced by the base of open intervals.

In more detail, the base is the set $\mathbb{Q} \times \mathbb{Q}$ of pairs $\langle p, q\rangle$ of rational numbers, and the basic neighbourhood with index $\langle p, q\rangle$ is the subset

$$\mathsf{ext}\,(\,\langle p, q\rangle\,) \equiv \{r \in \mathbb{Q} \mid p < r < q\,\}$$

for all $p, q \in \mathbb{Q}$.

In other constructive and predicative foundations, such as Aczel's **CZF** and Martin-Löf's type theory in [58], another example of concrete space is that of real numbers. It is not so in our **MF**, as we shall see later. Even when the topology of real numbers provides an example of concrete space, it is well known from Brouwer that a constructive pointwise development of analysis fails to get important properties (see [75]), such as compactness of the closed interval $[0, 1]$, unless further principles, for example the Fan Theorem, are assumed or some basic topological notions are changed as in Bishop's approach (see [10, 12, 15]). An alternative approach to constructive topology, and analysis, is offered by formal topology.

20.4.2 The Predicative Constructive Pointfree Approach of Formal Topology

The approach of formal topology to pointfree topology was introduced by Per Martin-Löf and the second author in the 1980s; the first published account is [64]. The intended foundation was then Martin-Löf's type theory MLTT in [58]. However, as underlined in the introduction of [64], to the second author it was already clear that it was necessary to work with an explicit notion of subset, and with a primitive notion of proposition using the judgement that a proposition is true without any reference to proof-terms in [54]. Such a conception of subsets and propositions was later specified in [73] as a tool to be added on top of type theory. As noticed in [36, 37], working with existential quantifiers with no proof-terms means that the axiom of choice no longer holds. This is different from **MLTT** where existential quantifications are identified with indexed sums, according to the proposition-as-set isomorphism, thus making the axiom of choice (AC) derivable. Moreover, this explains why in formal topology, as developed by the second author, every use of the axiom of choice was explicit. Given that the notion of subset in [73] and a primitive notion of proof-irrelevant propositions have been incorporated in our Minimalist Foundation, all the main definitions and results on formal topology (by the second author) can be carried in it. Actually the combination of the tool of *extensional* subsets with the *intensional* **MLTT** partly anticipated the notion of two-level theory in [47], because subsets are not formally included in **MLTT**.

The main idea of formal topology is to replace the notion of concrete space with an abstract axiomatization of the structure of open subsets, and then to recover its points in a *formal way* as suitable subsets of opens. The precise definition is reached by describing the structure of the set S of basic neighbourhoods in a concrete space (X, ext, S) with no mention of the set X of points.

While in the concept of concrete space (X, ext, S) points in X are given at the same time as the formal basic neighbourhoods in S and both form a set, in formal topology only the structure of opens is described, starting from the set S of formal basic neighbourhoods and from a new primitive relation $a \lhd U$, called *formal cover*, between formal basic neighbourhoods $a \in S$ and subsets $U \subseteq S$. A formal cover relation is the abstract counterpart of

$$\mathsf{ext}\, a \subseteq \mathsf{ext}\, U,$$

which expresses in a concrete space that the open $\mathsf{ext}\, U$ is a covering of the basic neighbourhood $\mathsf{ext}\, a$. Then, the notion of *formal topology* extends that of formal cover with the addition of a primitive predicate $Pos(a)$ for $a \in S$, which is the abstract counterpart of the assertion that the basic neighbourhood $\mathsf{ext}\, a$ is inhabited. Details of the definitions are now presented.

Definition 20.9 (Formal cover) A *formal cover* $\mathcal{A} = (S, \lhd)$ is given by a set S and a relation $\lhd \subseteq S \times \mathcal{P}(S)$ between elements and subsets of S that satisfies the following rules for every $a \in S$ and $U, V \subseteq S$:

$$\frac{a \,\epsilon\, U}{a \lhd U}\ reflexivity \qquad\qquad \frac{a \lhd U \quad U \lhd V}{a \lhd V}\ transitivity$$

$$\frac{a \lhd U \quad a \lhd V}{a \lhd U {\downarrow_{\mathcal{A}}} V}\ convergence,$$

where $U \lhd V \overset{def}{\Longleftrightarrow} (\forall b \,\epsilon\, U)\ (b \lhd V)$ and $U \downarrow_{\mathcal{A}} V = \{a \in S : (\exists u \,\epsilon\, U)(a \lhd u)\ \&\ (\exists v \,\epsilon\, V)(a \lhd v)\}$.

This definition provides a predicative counterpart of the impredicate notion of pointfree topology called *locale* in [32, 35]. In fact, to any formal cover $\mathcal{A} = (S, \lhd)$

we can associate an operator $\mathcal{A}$ on $\mathcal{P}(S)$, that is, an operation $\mathcal{A} : \mathcal{P}(S) \to \mathcal{P}(S)$ (that by abuse of notation we call the formal cover itself!), by putting

$$\mathcal{A}U \stackrel{def}{=} \{a \in S \mid a \lhd U\} \tag{20.1}$$

for any $U \subseteq S$. Then, reflexivity and transitivity of the cover means that the operator $\mathcal{A}$ is a *saturation* (or closure operator) and convergence means that $\mathcal{A}$ satisfies

$$\mathcal{A}(U \downarrow_{\mathcal{A}} V) \;=\; \mathcal{A}U \cap \mathcal{A}V.$$

The collection $Sat(\mathcal{A})$ of all fixed points of the operator $\mathcal{A}$ (i.e., all subsets U of S satisfying $\mathcal{A}(U) = U$) with the order given by inclusion form a locale. See [17] for an account and discussion on the several variants of the definitions of formal cover.

Then a formal topology is defined as follows.

Definition 20.10 A *formal topology* $\mathcal{S} = (S, \lhd, \mathsf{Pos})$ is a formal cover $(S, \lhd)$ equipped with a *positivity* predicate, that is a predicate $\mathsf{Pos}(a)$ for $a \in S$ which satisfies the conditions

$$\text{(monotonicity)}\ \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \,\epsilon\, U)\ \mathsf{Pos}(u)} \qquad \text{(positivity)}\ \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}.$$

Formal topologies provide a predicative counterpart of the impredicative notion of *open locale* in [34].

Formal covers, as well as formal topologies, can be inductively generated from a set-indexed family of axioms of the form $a \lhd U$.

Definition 20.11 Given a set S, an *axiom-set* is a pair I, C, given by a family of sets $I(a)$ for each $a \in S$ and a family of subsets $C(a, i) \subseteq S$ for $a \in S$ and $i \in I(a)$ (with the intended meaning that $a \lhd C(a, i)$ holds).

The definition of inductively generated formal cover was introduced in [20] and for our purposes we just recall the following.

Definition 20.12 Given a preordered set $(S, \leq)$ and an axiom-set I, C, the *inductively generated formal cover (formal topology)* $(S, \lhd_{I,C})$ is a formal cover (formal topology) satisfying:

(i) $a \lhd_{I,C} C(a, i)$ for every $a \in S$ and $i \in I(a)$;
(ii) if $\lhd'$ is another formal cover (formal topology) such that $a \lhd' C(a, i)$ for all $a \in S$ and $i \in I(A)$, then $a \lhd_{I,C} U \;\to\; a \lhd' U$ holds for all $a \in S$ and $U \subseteq S$.

Observe that in generating the formal topology the preorder on the set of basic neighbourhoods $(S, \leq)$ is essential to produce a distributive lattice of open subsets (see [17] for a detailed explanation).

In the Minimalist Foundation we can define some inductively generated formal topologies but not all, as shown in [50]. For example, in **MF** we can represent the pointfree topology of real numbers or that of the Cantor space by reproducing the argument used in [77] to define inductively generated formal covers, and formal topologies, in an extension of Martin-Löf's type theory with ordinals.

Therefore we assume the existence of an inductively generated cover when needed both at the extensional and at the intensional levels of **MF**. A proper two-level extension of the Minimalist Foundation with generic inductively generated formal topologies satisfying the requirements in [48] was built in [50] and called $\mathbf{MF}_{ind}$. In particular, in [50] the intensional level is shown to be consistent with the formal Church thesis and the axiom of choice by extending the Kleene realizability interpretation of intuitionistic arithmetic as was done in [31] but in a constructive metatheory.

We now recall the notion of a formal point. Given any formal topology $\mathcal{S}$, a formal point over $\mathcal{S}$ is a subset α of the set S such that it makes sense to think of $a \epsilon \alpha$ as meaning that the observable a is an approximation of α. To obtain a precise definition, one considers the case in which $\mathcal{S}$ is the topology of a concrete space X and takes some pointfree properties of the subset $\{ a \in A \mid x \epsilon \mathsf{ext}(a) \}$, which is the trace on S of a concrete point $x \in X$, as the conditions to define a subset $\alpha \subseteq S$ to be a formal point.

Definition 20.13 (Formal point) Let $\mathcal{A} \equiv (A, \lhd)$ be a formal cover. An inhabited subset α of A is a *formal point* if, for any $a, b \in A$ and any $U \subseteq A$, it satisfies the following conditions:

$$(\alpha \textit{ is filtering})\ \frac{a \epsilon \alpha \quad b \epsilon \alpha}{(\exists c \epsilon \{a\} \downarrow_{\mathcal{A}} \{b\})\ c \epsilon \alpha} \qquad (\alpha \textit{ splits the cover})\ \frac{a \epsilon \alpha \quad a \lhd U}{(\exists u \epsilon U)\ u \epsilon \alpha}.$$

Then, one can take the collection of $Pt(\mathcal{A}) \equiv \{\alpha \in \mathcal{P}(S) \mid \alpha \text{ formal point}\}$ and make it a formal space as follows.

Definition 20.14 (Formal topology and formal space) For any formal cover $\mathcal{A} \equiv (S, \lhd)$, the collection $Pt(\mathcal{A})$ of formal points of $\mathcal{A}$ with the topology generated by the basic neighbourhoods of the form $\mathsf{Ext}(a) \equiv \{ \alpha \in Pt(\mathcal{A}) \mid a \epsilon \alpha \}$ for $a \in S$ defines the *formal space* of points of $\mathcal{A}$ (that by abuse of notation we still call $Pt(\mathcal{A})$).

In an impredicative foundation, where power-collections are sets, it is clear that $Pt(\mathcal{A})$ defines a concrete space for any formal cover $\mathcal{A}$. Hence, as is well known, impredicatively one can prove the existence of an adjunction between formal covers and concrete spaces (see [32, 35]). This impredicative adjunction associates to a formal cover its formal space, and conversely to a concrete space (X, ext, S) the formal cover $(S, \lhd_X)$ defined by

$$a \lhd_X U \;\equiv\; \mathsf{ext}\, a \subseteq \mathsf{ext}\, U.$$

But, not all formal covers arise from concrete spaces in this way. Moreover, the formal cover induced by a formal space is not necessarily equal to the starting formal cover $\mathcal{A}$, that is, not all formal covers are *spatial*. And even more, the formal space of a formal cover arising from a concrete space is not necessarily equivalent to the starting concrete space, that is, not all concrete spaces are *sober*.

In a constructive and predicative foundation like our minimalist one, such an adjunction is no longer available, because the collection $Pt(\mathcal{A})$ is *not necessarily* a set.

Here we will see at least three relevant examples of proper formal spaces, that is, formal spaces whose formal points cannot form a set in our **MF**: the formal space of real numbers, Cantor, and Baire spaces. In all these examples, we will see how our foundation allows us to distinguish points which are given effectively, namely concrete points identified with lawlike sequences, from points which are only ideally so, that is, formal points, which are identified with choice sequences. It is a predicative foundation which allows one, and, in the same time compels one, to take care of this distinction between an effective or real structure, such as that of open basic neghbourhoods, from an ideal or non-effective structure such as that of formal points.

So in a constructive approach to topology such as our minimalist one, formal topologies and formal points are not just an option to describe something which is there in any case. They are introduced also as the only way to treat those spaces which otherwise would be constructively unreachable.

20.4.3 Examples of Pointfree Topologies whose Formal Points do not Form a Set

The first example of topology whose formal points do not form a set in the Minimalist Foundation is the formal topology of real numbers, such as Dedekind cuts.

Definition 20.15 (Formal topology of real numbers) The *formal topology of real numbers* $\mathcal{R} \equiv (\mathbb{Q} \times \mathbb{Q}, \lhd_{\mathcal{R}}, \mathsf{Pos}_{\mathcal{R}})$ is an inductively generated formal topology defined as follows. The base is $\mathbb{Q} \times \mathbb{Q}$ and the basic neighbourhoods are pairs of rational numbers, $\langle p, q\rangle$ with $p, q \in \mathbb{Q}$. A preorder on $\mathbb{Q} \times \mathbb{Q}$ is defined as follows

$$\langle p, q\rangle \leq \langle p', q'\rangle \;\equiv\; p' \leq p \leq q \leq q'$$

for p, q, p', q' in $\mathbb{Q}$. The cover is defined inductively by the following rules (which are a formulation in our context of Joyal axioms, cf. [32], pp. 123–124):

$$\frac{q \le p}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{\langle p, q\rangle \in U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{p' \le p < q \le q' \qquad \langle p', q'\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

$$\frac{p \le r < s \le q \qquad \langle p, s\rangle \lhd_{\mathcal{R}} U \quad \langle r, q\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \mathsf{wc}\ \frac{wc(\langle p, q\rangle) \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

where in the last axiom we have used the abbreviation

$$wc(\langle p, q\rangle) \equiv \{\, \langle p', q'\rangle \in \mathbb{Q} \times \mathbb{Q} \mid p < p' < q' < q\}$$

(wc stands for ‘well-covered’).

The positivity predicate is $\mathsf{Pos}_{\mathcal{R}}(\langle p, q\rangle) \equiv p < q$, expressing that the pair of rationals represents a non-empty interval.

As shown in [57], formal points of the formal topology $\mathcal{R}$ are in bijection with the collection of Dedekind cuts on the rationals. The proof carries over to our foundation after observing that $\mathcal{R}$ can be defined in **emTT** as described in [50].

Definition 20.16 A *Dedekind cut* on the rationals is a pair (L, U) with inhabited $L, U \subseteq \mathbb{Q}$ satisfying the following properties:

(disjointness)	$\forall q \in \mathbb{Q}\ \neg(q \,\epsilon\, U \,\&\, q \,\epsilon\, L)$
(L-openess)	$\forall p \,\epsilon\, L\ \exists q \,\epsilon\, L\ p < q$
(U-openess)	$\forall q \,\epsilon\, U\ \exists p \,\epsilon\, U\ p < q$
(L-monotonicity)	$\forall q \,\epsilon\, L\ \forall p \in \mathbb{Q}\ (p < q \to p \,\epsilon\, L)$
(U-monotonicity)	$\forall p \,\epsilon\, U\ \forall q \in \mathbb{Q}\ (p < q \to q \,\epsilon\, U)$
(locatedness)	$\forall q \in \mathbb{Q}\, \forall p \in \mathbb{Q}\ (p < q \to p \,\epsilon\, L \vee q \,\epsilon\, U)$.

Proposition 20.17 *In* **emTT** *the formal points of the inductively generated formal topology $\mathcal{R}$ are in bijection with the collection of Dedekind cuts on the rationals.*

Proof Given a formal point $\alpha \in Pt(\mathcal{R})$ we can build the following Dedekind cut:

$$L_\alpha \equiv \{\, p \in \mathbb{Q} \mid \exists_{q \in \mathbb{Q}} \langle p, q\rangle \,\epsilon\, \alpha \,\} \qquad U_\alpha \equiv \{\, q \in \mathbb{Q} \mid \exists_{p \in \mathbb{Q}} \langle p, q\rangle \,\epsilon\, \alpha \,\}.$$ [3]

Conversely, given a Dedekind cut (L, U) we can define the following formal point

$$\alpha_{(L,U)} \equiv \{\langle p, q\rangle \in \mathbb{Q} \times \mathbb{Q} \mid p \,\epsilon\, L \,\&\, q \,\epsilon\, U\}.$$

In [57] it is proved that formal points of $\mathcal{R}$, or Dedekind cuts, are also in bijective correspondence with Cauchy sequences à la Bishop in [10]. This correspondence does not work in **emTT**: only Cauchy sequences à la Bishop can be shown to be formal points of $\mathcal{R}$. To make this point clear, we recall the notion of Cauchy sequence à la Bishop. In the following with N^{+} we mean the set of positive natural numbers. □

[3] Note that the base of our topology $\mathcal{R}$ does not contain $+\infty, -\infty$ like that in [57].

Definition 20.18 (Cauchy sequence à la Bishop) A function $R(n,x)$ $prop_s$ $[n \in \mathrm{N}^+, x \in \mathbb{Q}]$, indicated with the usual notation $(x_n)_{n \in N^+}$, is a *Cauchy sequence* in **emTT** if we can prove for any $n, m \in \mathrm{N}^+$

$$\mid x_n - x_m \mid \leq \; 1/n + 1/m.^4$$

As in [57], we can also prove in **emTT** that any Cauchy sequence $(x_n)_{x \in \mathrm{N}}$ determines a formal point α of the formal topology $\mathcal{R}$ if we define it by:

$$\alpha \; \equiv \; \{\langle p, q\rangle \in \mathbb{Q} \times \mathbb{Q} \mid \; \exists \, n \in \mathrm{N}^+ \;\; p < x_n - 2/n < x_n + 2/n < q\}.$$

Conversely, given a formal point α, we can prove in **emTT** that within α there exists a countable number of strictly decreasing intervals as follows:

$$\begin{aligned} \forall n \in \mathrm{N}^+ \;\; \exists \, \langle x_n, y_n\rangle \in \mathbb{Q} \times \mathbb{Q} \;\; & (((\langle x_n, y_n\rangle \; \epsilon \; \alpha \;\; \& \;\; \mid x_n - y_n \mid < (2/3)^n) \\ & \& \; \exists \, \langle x_{n+1}, y_{n+1}\rangle \in \mathbb{Q} \times \mathbb{Q} \;\; (x_n \leq x_{n+1} < y_{n+1} \leq y_n \;\; \& \\ & \qquad (\langle x_{n+1}, y_{n+1}\rangle \; \epsilon \; \alpha \;\; \& \;\; \mid x_{n+1} - y_{n+1} \mid < (2/3)^{n+1}))). \end{aligned}$$

This is proved by induction; in fact, for $n \in \mathrm{N}$ and $\langle x_n, y_n\rangle \; \epsilon \; \alpha$ we can find a covering

$$\langle x_n, y_n\rangle \lhd_{\mathcal{R}} \{ \, l_{n_1}, l_{n_2} \, \}$$

such that $l_{n_1} \; \equiv \; \langle x_n, z_{n_1}\rangle$ and $l_{n_2} \; \equiv \; \langle z_{n_2}, y_n\rangle$ with $z_{n_1} \; \equiv \; x_n + (2 \cdot (y_n - x_n))/3$ and $z_{n_2} \; \equiv \; x_n + (y_n - x_n)/3$. Since the formal point α splits the cover, we can prove

$$\forall n \in N \;\; \exists \, i \; (i \varepsilon \{1, 2\} \; \& \; l_{ni} \; \epsilon \; \alpha).$$

However, such l_{ni} is *not necessarily unique*, because the formal point can be contained in both!

Classically, one can define a function $\mathcal{L}(n)$ for $n \in N$ by cases by putting

$$\mathcal{L}(n) \; \equiv \; \begin{cases} l_{n1} & \text{if } \; l_{n_1} \; \epsilon \; \alpha \\ l_{n2} & \text{if } \; l_{n_2} \; \epsilon \; \alpha \; \& \; \neg l_{n_1} \; \epsilon \; \alpha. \end{cases}$$

Constructively this does not work because α is not complemented. But, if we work in a foundation such as Martin-Löf's type theory MLTT, actually in the setoid model over it, by using the axiom of dependent choice on $\mathbb{Q} \times \mathbb{Q}$, we can even extract an operation $l(n) \in \mathbb{Q} \times \mathbb{Q} \; [n \in \mathrm{N}^+]$ such that for any $n \in \mathrm{N}^+$

$$l(n) \; \epsilon \; \alpha$$

and, after naming $l(n) =_{Q \times Q} \langle x_n, y_n\rangle$ the values of the operation on each natural

[4] This is formally written as $\forall \, p \in \mathbb{Q} \, \forall \, q \in \mathbb{Q} \;\; (R(n,p) \; \& \; R(m,q) \;\; \rightarrow \mid q - p \mid \leq \; 1/n + 1/m)$, where the definition of module is the usual one.

number, the conditions $x_n - y_n \leq (2/3)^n$ and $x_n \leq x_{n+1} < y_{n+1} \leq y_n$ hold for each natural number n. Then, a Cauchy sequence can be defined by taking the first components $(x_n)_{n \in N}$ or the second components $(y_n)_{n \in N}$. Hence, any Dedekind cut or formal point of $\mathcal{R}$ corresponds to a lawlike Cauchy sequence à la Bishop in **MLTT**.

Since in our foundation no axiom of choice is available, this proof cannot be carried out. At a closer look, it does not appear constructively justified to be able to extract a choice of the interval where the formal point is, with no extra information.

What actually happens in **MLTT** is that the splitting of points is *already given* with an operation choosing an interval where the point is, and hence from this choice a definition by cases can be given similarly to that done classically. This example explains why in Martin-Löf's type theory real numbers as formal points are only the lawlike ones, namely those for which we can extract a lawlike Cauchy sequence.

This is not true in our foundation. In fact, the property that a formal point splits the cover is expressed through an existential quantifier $\exists_{x \in A}\ \phi(x)\ [w \in \Gamma]$ under a context Γ, which does not necessarily provide an operation $\mathsf{wit}(d) \in A\ [w \in \Gamma]$ (depending on the context Γ) returning a witness for the existential statement when this holds. Such an operation is available only in a Kleene realizability interpretation of our foundation. As expected, in **emTT** real numbers as formal points of the formal topology $\mathcal{R}$ *cannot coincide* with lawlike Cauchy sequences (see [40, 42]). Even more, *real numbers such as formal points of $\mathcal{R}$, and hence such as Dedekind cuts, do not form a set*. Analogously, *real numbers as Cauchy sequences à la Bishop do not form a set*. These results are obtained through a realizability interpretation of **emTT** which interprets **emTT**-sets as countable subsets of natural numbers and **emTT**-collections as entities which are not necessarily countable.

Now we describe two other examples of formal topologies whose formal points do not form a set in **emTT**. These are Cantor and Baire formal topologies, which are defined as instances of the more general notion of formal topology on the tree over a set. In order to define such formal topologies, we need to represent the tree over a set A, which we identify with the nodes labelled by lists of elements in a set A, using the abbreviation $A^* \equiv List(A)$. We write $[l, x]$ for the list obtained by appending $x \in A$ to the list $l \in A^*$ and $[l, t]$ for the list obtained by appending the list $t \in A^*$ to the list $l \in A^*$.

Definition 20.19 The *tree formal topology over a set A* is the formal topology $A^N \equiv (A^*, \lhd_{A^N}, \mathsf{Pos}_{A^N})$ where $\lhd_{A^N}$ is inductively generated by the following rules

$$\text{rfl}\ \frac{l \,\epsilon\, V}{l \lhd_{A^N} V} \qquad \leq \frac{s \sqsubseteq l \qquad l \lhd_{A^N} V}{s \lhd_{A^N} V} \qquad \text{tr}\ \frac{\forall x \in A\ \ [l, x] \lhd_{A^N} V}{l \lhd_{A^N} V},$$

where $s \sqsubseteq l \equiv \exists_{t \in A^*}\, s =_{A^*} [l, t]$, i.e. l is an initial segment of s.

The positivity predicate is true on any element, that is, $\mathsf{Pos}_{A^N}(l) \equiv \mathtt{tt}$ for any $l \in A^*$.

Among tree formal topologies, we distinguish Cantor and Baire formal topologies as follows

Definition 20.20 (Cantor and Baire formal topologies) The tree formal topology when $A \equiv \{0, 1\}$, namely

$$\{0,1\}^N \equiv (\{0,1\}^*, \lhd_{\{0,1\}^N}, \mathsf{Pos}_{\{0,1\}^N})$$

is called *Cantor formal topology*.

The tree formal topology when $A \equiv \mathrm{N}$, namely

$$\mathrm{N}^N \equiv (\mathrm{N}^*, \lhd_{N^N}, \mathsf{Pos}_{N^N})$$

is called *Baire formal topology*.

Note that the Cantor formal topology is definable in **MF** but the Baire formal topology is not as specified in [50].

Formal points of such topologies coincide with choice sequences of Definition 20.6.

Proposition 20.21 *Formal points $Pt(A^N)$ of the tree formal topology over a set A are in bijective correspondence with choice sequences on the tree A^*.*

Proof Given a formal point α, we define a function $R_\alpha(n, x)\; prop_s\; [n \in \mathrm{N}, x \in A]$ as follows:

$$R_\alpha(n, a) \equiv \exists\, l \,\epsilon\, \alpha \;\; l_{n+1} =_A a,$$

where l_n is the nth component of l. □

Conversely, given a function $R(n, x)\; prop_s\; [n \in \mathrm{N}, x \in A]$ the subset

$$\alpha_R \equiv \{\, l \in A^* \mid \forall n \in \mathrm{N}\, (n < \mathsf{lh}(l) \rightarrow R(n, l_{n+1}))\},$$

where $\mathsf{lh}(l)$ is the length of l, turns out to be a formal point.

An alternative proof follows after noting, as observed in [74], that the tree formal topology over a set A is isomorphic to the exponential formal topology of the discrete formal topology of N over the discrete formal topology on the set A (see [38] for a predicative treatment of exponentiation). Therefore its formal points are in bijection with functions, because every function between discrete topologies is continuous. This explains why we denote the tree formal topology with the symbol A^N.

The realizability interpretation in [40, 42], showing that real numbers (both as

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Dedekind cuts or Cauchy sequences) do not form a set, also shows that *choices sequences as formal points of Cantor or Baire formal topology do not form a set either* and hence they only form a proper collection. Therefore predicatively we can only work with the pointfree topologies of usual Cantor and Baire spaces.

The equivalence of each of our pointfree topologies with the corresponding pointwise topology of their formal spaces as in Definition 20.14, namely spatiality, is not generally valid in **emTT**. Indeed, spatiality of our tree formal topologies amounts to the well-known principle of Bar Induction, as first observed in [24].

Definition 20.22 (Bar Induction in topological form) In **emTT** extended with the inductive definitions necessary to define the formal topologies $\lhd_{A^N}$ for any set A, the principle of Bar Induction is the following statement: for any given set A

(BI(A))

$$\forall l \in A^* \, \forall V \in \mathcal{P}(A^*) \;\; (\forall \alpha \, \epsilon \, Pt(\lhd_{A^N}) \;\; (\, l \, \epsilon \, \alpha \;\rightarrow\; \alpha \between V \;) \rightarrow l \lhd_{A^N} V)$$

where

$$V \between W \;\equiv\; \exists a \in A \, (a \, \epsilon \, V \;\wedge\; a \, \epsilon \, W)$$

expresses that two subsets V, W of a set A overlap (see [66]).

The above formulation of BI(A) means that the topology put on the formal points of the tree A^* that are its choice sequences, coincides with the pointfree one. Hence, Bar Induction implies that we can reason topologically on choice sequences by induction on finite sequences, given that the pointfree topologies are inductively generated (see [67, 72]).

The usual Fan Theorem in [75] is then an instance of Bar Induction (see [24, 25]).

Definition 20.23 (Fan Theorem) We call *Fan Theorem* the formulation BI($\{\,0, 1\,\}$) of BI(A) on Cantor formal topology, namely when $A \;\equiv\; \{\,0, 1\,\}$.

Spatiality of Cantor formal topology allows us to derive compactness of Cantor space in [24].

In [40, 42] it is shown that **emTT** is compatible with the described principle of Bar Induction BI(A) for any set A, and the identification of lawlike sequences with recursive ones. Indeed, there exists a realizability interpretation showing that real numbers and choice sequences do not form a set validating *the formal Church thesis for operations between natural numbers* ($\mathrm{CT}_{\mathrm{tt}}$).

Hence, the realizability interpretation in [40, 42] shows that **emTT** is compatible with constructive foundations where Bar Induction or the Fan Theorem are used as in Brouwer's constructive pointwise approach of topology, by keeping a computable interpretation of operations between natural numbers with the validity of $\mathrm{CT}_{\mathrm{tt}}$.

Actually a motivation to develop our **MF** was exactly to study a development of topology in the presence of these extra axioms.

Observe that compatibility with Bar Induction and $\mathrm{CT}_{\mathrm{tt}}$ is not possible for Martin-Löf's type theory, because the axiom of choice, and hence also the axiom of unique choice, is valid in there. To see this, first observe that in our **MF** we can prove the well-known result by Kleene [75] that formal Church thesis for choice sequences is contradictory with the Fan Theorem, and hence also with Bar Induction. Observe then that this result can be formulated by saying that *there is no model of* **emTT** $+ \lhd_{A^N}$ + FT + $\mathrm{CT}_{\mathrm{tt}}$ + $\mathrm{AC}_{\mathrm{N,N}}$. Therefore a theory validating the axiom of unique choice cannot keep together a computational interpretation of operations and Bar Induction. Hence, the consistency of **emTT** with Bar Induction and $\mathrm{CT}_{\mathrm{tt}}$ explains why the axiom of unique choice, and a fortiori the axiom of choice, is not valid in **emTT**.

20.5 Extending MF with choice principles

The aim of this section is to show an extension of **MF** with choice principles in the spirit of `BISH`.

Ideally one would choose – as the most adequate basic foundation for Bishop's mathematics – a theory which fully axiomatizes the setoid model on Martin-Löf's type theory in [58]. But Martin-Löf's type theory is intended as a full-scale theory for formalizing constructive mathematics only if it is left opened to further extensions with all the needed inductive definitions. Currently only the first-order fragment of the setoid model has been axiomatized categorically in [62]. It would be desirable to have a type-theoretic presentation of the internal theory of the setoid model over **MLTT** as an extension of **emTT**, or better as an extension of the internal type theory of an arithmetic locally cartesian closed pretopos in [37].

Here we just consider an extension of **emTT** with the forms of axiom of choice which are acceptable constructively. Indeed, it is well known [26] that the full axiom of choice is not constructively acceptable in foundations with extensional principles.

The two-level structure of **MF** makes very evident the connection established in [55] between Zermelo's choice axiom and the type-theoretical axiom of choice in showing that the formula of the full axiom of choice

$$\text{(AC)} \qquad \forall x \in A\ \exists y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

for A, B sets and $R(x, y)$ a small proposition, written at the extensional level of **MF** gets interpreted at the intensional level of **MF** as *Martin-Löf's extensional axiom of choice* in [55] represented by the formula

(AC_{ext})

$$\forall x \in A\ \exists y \in B\ R(x,y) \quad \longrightarrow$$
$$\exists f \in A \to B \quad (\mathbf{Ext(f)}^{\cong_\mathbf{B}}_{\simeq_\mathbf{A}} \ \ \& \ \ \forall x \in A \ \ R(\, x\, ,\, f(x))),$$

where A , B are sets and

$$\mathbf{Ext}(f)^{\cong_B}_{\simeq_A} \ \equiv \ \forall x_1 \in A\, \forall x_2 \in A\, (x \simeq_A y \ \to \ f(x_1) \simeq_B f(x_2)),$$

provided that $R(x,y)$ is a small proposition preserving given equivalence relations $\simeq_A$ and $\simeq_B$ in the sense that we can find in **mTT** a proof of

$$\forall x_1 \in A\, \forall x_2 \in A\, \forall y \in B\, (x_1 \simeq_A x_2 \ \to \ (R(x_1,y) \leftrightarrow R(x_2,y)))$$

and of

$$\forall x \in A\, \forall y_1 \in B\, \forall y_2 \in B\, (y_1 \simeq_B y_2 \ \to \ (R(x,y_1) \leftrightarrow R(x,y_2))).$$

As expected, the extensional axiom of choice (AC_{ext}) is not constructively acceptable since it implies the law of the excluded middle (see [16, 39, 55]).

Therefore, to develop constructive mathematics we cannot use the axiom of choice on all sets if we work in an extensional foundation such as **emTT**. On the contrary, as shown in Section 20.3.3, the intensional level **mTT** is consistent with the full axiom of choice

$$\text{(AC)} \qquad \forall x \in A\, \exists y \in B\, R(x,y) \ \longrightarrow \ \exists f \in A \to B\, \forall x \in A\, R(x, f(x)),$$

which is called *intensional* by Martin-Löf in [55].

The different status of (AC) in the extensional and intensional levels of **MF** reflects the different status of the axiom of choice in axiomatic set theory and in type theory. This fact was another motivation for building a two-level theory where to distinguish the various kinds of choice principles. Indeed, as described previously, the two forms of (AC) can only be visible in an intensional theory, like **mTT** or **MLTT**, but the fact that they express the same formula in different contexts is only visible in the two-level structure of **MF**.

20.5.1 The Extension emtt_{ac} *of* MF

We extend the extensional level of **MF** to form a theory that we call emtt_{ac} for short.

This theory emtt_{ac} is simply obtained from **emTT** by adding the axiom of unique choice for all types and the axiom of choice for what we call *intensional sets*.

Intensional sets are sets of **emTT** whose interpretations in **mTT** given in [39] turn out to be 'faithful copies' of them in **mTT** (with their terms and their propositional equality), and they include the following:

$$A\ set_i \ \equiv \quad N_0 \ \mid \ N_1 \ \mid \ List(A) \ \mid \Sigma_{x\in A} B(x) \ \mid \ A + B,$$

provided that A and B are intensional sets, as well as $B(x)$ $[x \in A]$ is a family of intensional sets on an intensional set A.

Then, formally the theory emtt$_{ac}$ is obtained by extending **emTT** with:

- the *axiom of unique choice* written as follows

  (AC!$_R$)

$$\frac{R(a,b)\ prop\ \ [a \in A, b \in B]}{\forall x \in A\ \exists! y \in B\ R(x,y)\ \longrightarrow\ \exists f \in A \to B\ \forall x \in A\ R(x, f(x))\ true}$$

  for all A and B types in **emTT** where as usual

$$\exists! y \in A\ R(x,y)\ \equiv\ \exists y \in B\ R(x,y)\ \wedge\ \forall y_1, y_2 \in B\ (R(x,y_1)\ \wedge\ R(x,y_1)\ \to\ y_1 =_B y_2);$$

- the axiom of choice on intensional sets

  (iAC$_R$)

$$\frac{A\ set_i\quad B\ set\qquad\qquad R(a,b)\ prop_s\ \ [a \in A, b \in B]}{\forall x \in A\ \exists y \in B\ R(x,y)\ \longrightarrow\ \exists f \in A \to B\ \forall x \in A\ R(x, f(x))\ true}.$$

By adding the axiom of unique choice we guarantee that the graph of each functional relation is also a graph of an operation and hence has a computational meaning in accordance with `BISH`.

Following the interpretation in [39] emtt$_{ac}$ can be interpreted in **mTT** extended with a proof term $\mathtt{p}_{ac}$

(AC$_R$)

$$\frac{A\ set\quad B\ set\qquad R(a,b)\ prop_s\ \ [a \in A, b \in B]}{\mathtt{p_{ac}} \in \forall x \in A\ \exists y \in B\ R(x,y)\ \longrightarrow\ \exists f \in A \to B\ \forall x \in A\ R(x, f(x))}.$$

In turn this extension of **mTT** naturally interprets in Martin-Löf's type theory **MLTT** where we can also extract programs from its proofs. Hence, **MLTT** could serve for the intensional level of a two-level foundation with emtt$_{ac}$ for the extensional level. But a direct extension of the interpretation in [39] for emtt$_{ac}$ into **MLTT** is left to future work.

20.6 Concluding Remarks

In our opinion the existence in **MF** of proper formal spaces, such as the space of real numbers both as Dedekind cuts or Cauchy sequences, constitutes an advantage which could be useful for performing reverse Bishop constructive mathematics. In fact, a positive and practical motivation of the minimalist approach is to provide a finer grid to look at reality, in particular topology, and thus preserve pieces of information, structures, conceptual distinctions (for example, all that is necessary to be able to instruct a computer) which would be lost, and actually are not even considered, in a classical or impredicative foundation. For instance, our Minimalist Foundation allows us to distinguish infinitary or ideal topological concepts not enjoying induction principles, like Brouwer's choice sequences, from inductive or real ones, for example, lawlike sequences. On the other hand, the minimalist attitude means that all results in the Minimalist Foundation about topology are also valid for the most relevant constructive and classical foundations.

Another important motivation supporting the pointfree approach to topology is given by a recent result showing that pointfree topology can be seen as a generalization of topology with points. To obtain this, one first has to introduce the notion of *positive topology*, that is an enrichment of formal topologies by the addition of a suitable primitive notion of closed subset. Then one can show that the category of concrete spaces can be embedded in the category of positive topologies. A full book [72] on positive topologies and their developments will soon be published.

For the future, we plan to investigate topologies of real numbers and of choice sequences using the more powerful tool provided by positive topologies.

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References

[1] Aczel, P. 1978. The type theoretic interpretation of constructive set theory. In: *Logic Colloquium '77 (Proceedings of the Conference, Wrocław, 1977)*. Studies in Logic and the Foundations of Mathematics, vol. 96. Amsterdam, New York: North-Holland.

[2] Aczel, P. 1982. The type-theoretic interpretation of constructive set theory: choice principles. In: Troelstra, A., and van Dalen, D (eds.), *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Studies in Logic and the Foundations of Mathematics, vol. 110. Amsterdam, New York: North-Holland.

[3] Aczel, P. 1986. The type-theoretic interpretation of constructive set theory: inductive definitions. In: *Logic, Methodology and Philosophy of Science, VII (Salzburg, 1983)*. Studies in Logic and the Foundations of Mathematics, vol. 114. Amsterdam, New York: North-Holland.

[4] Allen, S. F., Bickford, M., Constable, R. L., et al. 2006. Innovations in computational type theory using Nuprl. *J. Appl. Logic*, **4**(4), 428–469.

[5] Asperti, A., Ricciotti, W., Coen, C. Sacerdoti, and Tassi, E. 2011. The Matita interactive theorem prover. In: Bjørner, N., and Sofronie-Stokkermans, V. (eds.), *Proceedings of the 23rd International Conference on Automated Deduction (CADE-2011), Wroclaw, Poland*. Lecture Notes in Computer Science, vol. 6803. Cham: Springer.

[6] Barthes, G., Capretta, V., and Pons, O. 2003. Setoids in type theory. *J. Funct. Programming*, **13**(2), 261–293. Special issue on Logical Frameworks and Metalanguages.

[7] Battilotti, G., and Sambin, G. 2006. Pretopologies and a uniform presentation of sup-lattices, quantales and frames. Pages 30–61 of: Banaschewski, B., Coquand, T., and Sambin, G. (eds), *Special Issue: Papers presented at the 2nd Workshop on Formal Topology (2WFTop 2002)*. Annals of Pure and Applied Logic, vol. **137**. Amsterdam: Elsevier.

[8] Beeson, M. 1985. *Foundations of Constructive Mathematics.* Berlin: Springer-Verlag.

[9] Bell, J. L. 1988. *Toposes and Local Set Theories: An Introduction.* Oxford: Clarendon Press.

[10] Bishop, E. 1967. *Foundations of Constructive Analysis*. New York: McGraw-Hill.

[11] Bishop, E. 1970. Mathematics as a numerical language. Pages 53–71 of: Kino, A., Myhill, J., and Vesley, R. E. (eds.), *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968*. Studies in Logic and the Foundations of Mathematics, vol. 60. Amsterdam: Elsevier.

[12] Bishop, E., and Bridges, D. S. 1985. *Constructive Analysis*. Berlin: Springer.

[13] Bove, A., Dybjer, P., and Norell, U. 2009. A brief overview of Agda – a functional language with dependent types. Pages 73–78 of: Berghofer, S., Nipkow, T., Urban, C., and Wenzel, M. (eds.), *Theorem Proving in Higher Order Logics, 22nd International Conference, TPHOLs 2009*. LNCS, vol. 5674. Berlin: Springer.

[14] Bridges, D. 2008. A reverse look at Brouwer's fan theorem. Pages 316–325 of: *One Hundred Years of Intuitionism (1907-2007)*. Basel: Birkäuser.

[15] Bridges, D., and Richman, F. 1987. *Varieties of Constructive Mathematics.* London Mathematical Society Lecture Note Series, vol. 97. Cambridge: Cambridge University Press.

[16] Carlström, J. 2004. $\mathrm{EM} + \mathrm{Ext}_{-} + \mathrm{AC}_{\mathrm{int}}$ is equivalent to $\mathrm{AC}_{\mathrm{ext}}$. *Math. Logic Q.*, **50**(3), 236–240.

[17] Ciraulo, F., Maietti, M. E., and Sambin, G. 2013. Convergence in formal topology: a unifying presentation. *Logic Anal.*, **5**(2), 1–45.

[18] Coq Development Team. 2010. *The Coq Proof Assistant Reference Manual: release 8.3*. Orsay: INRIA.

[19] Coquand, T. 1990. Metamathematical investigation of a calculus of constructions. Pages 91–122 of: Odifreddi, P. (ed.), *Logic in Computer Science*. New York: Academic Press.

[20] Coquand, T., Sambin, G., Smith, J., and Valentini, S. 2003. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, **124**(1–3), 71–106.

[21] Coquand, T., and Paulin-Mohring, C. 1990. Inductively defined types. Pages 50–66 of: Martin-Löf, P., and Mints, G. (eds.), *Proceedings of the International Conference on Computer Logic (Colog '88)*. LNCS, vol. 417. Berlin: Springer.

[22] Curi, G. 2010. On some peculiar aspects of the constructive theory of point-free spaces. *Math. Logic. Q.*, **56**(4), 375–387.

[23] Feferman, S. 1979. Constructive theories of functions and classes. Pages 159–224 of: *Logic Colloquium '78 (Mons, 1978)*. Studies in Logic and the Foundations of Mathematics Amsterdam, New York: North-Holland.

[24] Fourman, M., and Grayson, R. J. 1982. Formal spaces. Pages 107–122 of: *The L.E.J. Brouwer Centenary Symposium* (Noordwijkerhout, 1981). Studies in Logic and the Foundations of Mathematics, vol. 110. Amsterdam, New York: North-Holland.

[25] Gambino, N., and Schuster, P. 2007. Spatiality for formal topologies. *Math. Struct. Comput. Sci.*, **17**(1), 65–80.

[26] Goodman, N., and Myhill, J. 1978. Choice implies excluded middle. *Z. Math. Logik Grundlag. Math.*, **24**, 461.

[27] Hofmann, M. 1997. *Extensional Constructs in Intensional Type Theory.* Distinguished Dissertations. Berlin: Springer.

[28] Hyland, J. M. E. 1982. The effective topos. Pages 165–216 of: *The L.E.J. Brouwer Centenary Symposium* (Noordwijkerhout, 1981). Studies in Logic and the Foundations of Mathematics, vol. 110. Amsterdam, New York: North-Holland.

[29] Hyland, J. M. E., Johnstone, P. T., and Pitts, A. M. 1980. Tripos theory. *Bull. Austral. Math. Soc.*, **88**, 205–232.

[30] Ishihara, H. 2005. Constructive reverse mathematics: compactness properties. Pages 245–267 of: Crosilla, L., and Schuster, P. (eds.), *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*. Oxford Logic Guides, no. 48. Oxford University Press.

[31] Ishihara, H., Maietti, M. E., Maschio, S., and Streicher, Th. 2018. Consistency of the intensional level of the Minimalist Foundation with Church's thesis and axiom of choice. *Arch. Math. Log.*, **57**(7–8), 595–668.

[32] Johnstone, P. T. 1982. *Stone Spaces*. Cambridge: Cambridge University Press.

[33] Joyal, A., and Moerdijk, I. 1995. *Algebraic Set Theory*. London Mathematical Society Lecture Note Series, vol. 220. Cambridge: Cambridge University Press.

[34] Joyal, A., and Tierney, M. (1984). An extension of the Galois theory of Grothendieck. *Memoirs of the American Mathematical Society*, **309**. Providence, RI: American Mathematical Society.

[35] Mac Lane, S., and Moerdijk, I. 1992. *Sheaves in Geometry and Logic. A First Introduction to Topos Theory.* Berlin: Springer Verlag.

[36] Maietti, M. E. 1998 (February). The Type Theory of Categorical Universes. Ph.D. thesis, University of Padova.

[37] Maietti, M. E. 2005a. Modular correspondence between dependent type theories and categories including pretopoi and topoi. *Math. Struct. Comp. Sci.*, **15**(6), 1089–1149.

[38] Maietti, M. E. 2005b. Predicative exponentiation of locally compact formal topologies over inductively generated ones. Pages 202–222 of: *From Sets and Types to Topology and Analysis: Practicable Foundation for Constructive Mathematics*. Oxford Logic Guides, vol. 48. Oxford: Oxford University Press.

[39] Maietti, M. E. 2009. A minimalist two-level foundation for constructive mathematics. *Ann. Pure Appl. Logic*, **160**(3), 319–354.

[40] Maietti, M. E. 2012. *Consistency of the Minimalist Foundation with Church Thesis and Bar Induction*. Preprint (available via `www.math.unipd.it/~maietti/pubb.html`).

[41] Maietti, M. E. 2017. On choice rules in dependent type theory. Pages 12–23 of: *Proc. Theory and Applications of Models of Computation – Proceedings 14th Annual Conference, TAMC 2017, Bern, Switzerland, April 20–22, 2017*.

[42] Maietti, M. E. 2018. The continuum from the predicative point of view of the Minimalist Foundation. Invited talk at "Das Kontinuum – 100 years later", University of Leeds, 11–14/9.

[43] Maietti, M. E., and Maschio, S. 2014. An extensional Kleene realizability semantics for the Minimalist Foundation. Pages 162–186 of: *Proc. 20th International Conference on Types for Proofs and Programs, TYPES 2014, May 12–15, 2014, Paris, France*.

[44] Maietti, M. E., and Maschio, S. 2016. A predicative variant of a realizability tripos for the Minimalist Foundation. *IfCoLog J. Logics Appl.*, **3**(4), 873–888.

[45] Maietti, M. E., and Rosolini, G. 2013a. Elementary quotient completion. *Theory Appl. Categ.*, **27**, 445–463.

[46] Maietti, M. E., and Rosolini, G. 2013b. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, **7**(3), 371–402.

[47] Maietti, M. E., and Rosolini, G. 2015. Unifying exact completions. *Appl. Categ. Structures*, **23**, 43–52.

[48] Maietti, M. E., and Sambin, G. 2005. Toward a minimalist foundation for constructive mathematics. Pages 91–114 of: Crosilla, L., and Schuster, P. (eds.), *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*. Oxford Logic Guides, no. 48. Oxford: Oxford University Press.

[49] Maietti, M. E., and Maschio, S. 2021. A predicative variant of Hyland's Effective Topos. *J. Symbol. Logic*, **86**(2), 433–447.

[50] Maietti, M. E., Maschio, S., and Rathjen, M. 2021. A realizability semantics for inductive formal topologies, Church's Thesis and Axiom of Choice. *Log. Meth. Comp. Sci.*, **17**(2), 21-1–21-21.

[51] Martin-Löf, P. 1970. *Notes on Constructive Mathematics*. Stockholm: Almqvist & Wiksell.

[52] Martin Löf, P. 1975. An intuitionistic theory of types: predicative part. Pages 73–118 of: *Logic Colloquium '73 (Bristol)*. Studies in Logic and the Foundations of Mathematics, vol. 80. Amsterdam: North-Holland.

[53] Martin-Löf, P. 1984. *Intuitionistic Type Theory*. Notes by G. Sambin of a Series of Lectures given in Padua, June 1980. Naples: Bibliopolis.

[54] Martin-Löf, P. 1985. On the meanings of the logical constants and the justifications of the logical laws. Pages 203–281 of: *Proceedings of the Conference on Mathematical Mogic (Siena, 1983/1984)*, vol. 2. Reprinted in: *Nordic J. Philosophical Logic*, **1**(1), no. 1, 11–60.

[55] Martin-Löf, P. 2006. 100 years of Zermelo's axiom of choice: What was the problem with it? *Comp. J.*, **49**(3), 10–37.

[56] Myhill, J. 1975. Constructive set theory. *J. Symbol. Logic*, **40**(3), 347–382.

[57] Negri, S., and Soravia, D. 1999. The continuum as a formal space. *Arch. Math. Logic*, **38**(7), 423–447.

[58] Nordström, B., Petersson, K., and Smith, J. 1990. *Programming in Martin Löf's Type Theory.* Oxford: Clarendon Press.

[59] Palmgren, E. 2005a. *Bishop's set theory*. Slides for lecture at the TYPES summer school.

[60] Palmgren, E. 2005b. Continuity on the real line and in formal spaces. Pages 165–175 of: Crosilla, L., and Schuster, P. (eds.), *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*. Oxford Logic Guides, no. 48. Oxford: Oxford University Press.

[61] Palmgren, E., and Wilander, O. 2014. Constructive categories and setoids of setoids in type theory. *Logical Meth. Comp. Sci.*, **10**(3), 1–14.

[62] Palmgren, Erik. 2012. Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets. *Ann. Pure Appl. Logic*, **163**(10), 1384–1399.

[63] Rathjen, M. 2005. Constructive set theory and Brouwerian principles. *J. UCS*, **11**(12), 2008–2033.

[64] Sambin, G. 1987. Intuitionistic formal spaces – a first communication. Pages 187–204 of: Skordev, D. (ed.), *Mathematical Logic and its Applications*. New York: Plenum.

[65] Sambin, G. 2002. Steps towards a dynamic constructivism. Pages 261–284 of: Gärdenfors, P., Wolenski, J., and Kijania-Placek, K. (eds.), *In the Scope of Logic, Methodology and Philosophy of Science. Volume One of the XI International Congress of Logic, Methodology and Philosophy of Science, Cracow, August 1999*. Alphen aan den Rijn: Kluwer.

[66] Sambin, G. 2003. Some points in formal topology. *Theoretical Computer Science*, **305**, 347–408.

[67] Sambin, G. 2008. Two applications of dynamic constructivism: Brouwer's continuity principle and choice sequences in formal topology. Pages 301–315 of: van Atten, M., Boldini, P., Bourdeau, M., and Heinzmann, G. (eds.), *One*

Hundred Years of Intuitionism (1907–2007): The Cerisy Conference. Basel: Birkhäuser.

[68] Sambin, G. 2011. A minimalist foundation at work. Pages 69–96 of: DeVidi, D., Hallett, M., and Clark, P. (eds.), *Logic, Mathematics, Philosophy, Vintage Enthusiasms. Essays in Honour of John L. Bell*. The Western Ontario Series in Philosophy of Science, vol. 75. Berlin: Springer.

[69] Sambin, G. 2012. Real and ideal in constructive mathematics. Pages 69–85 of: Dybjer, P., Lindström, S., Palmgren, E., and Sundholm, G. (eds.), *Epistemology versus Ontology, Essays on the Philosophy and Foundations of Mathematics in honour of Per Martin-Löf*. Logic, Epistemology and the Unity of Science, vol. 27. NewYork/Dordrecht: Springer.

[70] Sambin, G. 2017. C for Constructivism. Beyond clichés. *Lett. Mat. Int.*, **5**, 87–91. https://doi.org/10.1007/s40329-017-0169-1.

[71] Sambin, G. 2019. Dynamics in foundations: what does it mean in the practice of mathematics? Pages 455–494 of: Centrone, S., Kant, D., and Sarikaya, D. (eds.), *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*. Synthese Library, vol. 407. Cham: Springer. `https://doi.org/10.1007/978-3-030-15655-8_21`.

[72] Sambin, G. 2022. *Positive Topology and the Basic Picture. New Mathematics Emerging from Dynamic Constructivism*. Oxford: Oxford University Press. (In the press.)

[73] Sambin, G., and Valentini, S. 1998. Building up a toolbox for Martin-Löf's type theory: subset theory. Pages 221–244 of: Sambin, G., and Smith, J. (eds.), *Twenty-Five Years of Constructive Type Theory*. Proceedings of a Congress held in Venice, October 1995. Oxford: Oxford University Press.

[74] Sigstam, I. 1995. Formal spaces and their effective presentations. *Arch. Math. Logic*, **34**, 211–246.

[75] Troelstra, A. S., and van Dalen, D. 1988. *Constructivism in Mathematics, an Introduction*, vol. I and II. *Studies in Logic and the Foundations of Mathematics*. Amsterdam: North-Holland.

[76] Univalent Foundations Program, 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `https://homotopytypetheory.org/book`.

[77] Valentini, S. 2007. Constructive characterizations of bar subsets. *Ann. Pure Appl. Logic*, **145**(3), 368–378.

[78] Wiedijk, F. 2006. *The Seventeen Provers of the World*. LNCS, vol. 3600. Berlin: Springer.