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Well-posedness results for the Generalized Aw-Rascle-Zhang model

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Abstract

We establish existence, uniqueness, and stability results for the so-called Generalized Aw-Rascle-Zhang model, a second-order traffic model introduced by Fan, Herty, and Seibold. Our analysis is motivated by the companion paper (Marconi and Spinolo in Nonlocal Generalized Aw-Rascle-Zhang model: well-posedness and singular limit, 2025, arXiv:2505.10102).

Keywords: Generalized Aw-Rascle-Zhang model; Traffic models; Existence; Uniqueness; Well-posedness

Mathematics Subject Classification: 35L65

1 Introduction

This note deals with the Generalized Aw-Rascle-Zhang model introduced in [10], namely

$$\begin{cases} \partial_t \rho + \partial_x [V(\rho, u)\rho] = 0 \\ \partial_t u + V(\rho, u)\partial_x u = 0. \end{cases} \tag{1.1}$$

In the previous expression, the unknown $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ denotes the density of cars and V their velocity, whereas the unknown $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ represents the driving style of drivers (for instance, their empty road velocity, that is, the velocity they choose when the road is completely free). As such, it is a Lagrangian marker governed by the transport equation at the second line of (1.1), while the first line of (1.1) expresses the conservation of the total amount of cars. We refer to [10] for an in-depth discussion of the model, and to [11] for a comprehensive introduction to traffic flow models.

System (1.1) is formally equivalent to the system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x [V(\rho, u)\rho] = 0 \\ \partial_t [\rho u] + \partial_x [V(\rho, u)\rho u] = 0, \end{cases} \tag{1.2}$$

which, contrary to (1.1), admits a standard notion of solution in the sense of distributions. In the following, we will be only concerned with solutions (ρ, u) such that u is a Lipschitz continuous function, and hence we will use the formulation (1.1) rather than (1.2): the second equation of (1.1) will be satisfied as an identity between L^∞ functions. The use of (1.1) eases the analysis of the vanishing ρ case, which requires special care when one uses (1.2).

In this note, we deal with the Cauchy problem posed by coupling (1.1) with the initial data

$$\rho(0,\cdot)=\rho_0, \qquad u(0,\cdot)=u_0, \tag{1.3}$$

and, as in [10], we assume, in view of modeling considerations, that

$$0\le\rho_0\le 1, \quad u_0\in L^\infty(\mathbb{R}),\ u_0\ge 0 \tag{1.4}$$

and that[1]

$$V\in C^2(\mathbb{R}^2;\mathbb{R}), \quad V\ge 0, \quad \partial_1 V\le 0, \quad \partial_2 V\ge 0, \quad V(1,w)=0 \text{ for every } w, \tag{1.5}$$

where $\partial_1 V$ and $\partial_2 V$ denote the partial derivatives of V with respect to ρ and u, respectively. In the previous equations and in the following, we normalize to 1 the maximal possible car density, corresponding to bumper-to-bumper packing.

In the present note, we establish well-posedness results for (1.1), (1.3) in a suitable class of functions. Our analysis is motivated by the companion paper [19], where we consider a nonlocal version of (1.1), namely

$$\begin{cases}\partial_t\rho+\partial_x[V(\xi,u)\rho]=0\\ \partial_t u+V(\xi,u)\partial_x u=0,\end{cases} \qquad \text{where } \xi(t,x)=\int_x^{+\infty}\eta(x-y)\rho(t,y)dy \tag{1.6}$$

and $\eta:\mathbb{R}_+\to\mathbb{R}$ is a convolution kernel satisfying suitable assumptions. In [19] we establish the well-posedness of the Cauchy problem associated (1.3), (1.6). Note that the regularizing effect of the convolution makes the analysis of (1.6) very different from that one of (1.1). In particular, in [19] both the existence and uniqueness proofs rely on the classical method of characteristics, which cannot be applied in the case of (1.1) due to the low regularity of $V(\rho,u)$. In [19] we also discuss the nonlocal-to-local limit, which is obtained by replacing the convolution kernel η with a sequence η_ε converging to the Dirac Delta, formally reducing (1.6) to (1.1). In [19] we show that, under suitable conditions, this limit can be rigorously justified as we establish convergence of the solutions in the vanishing ε limit.

To introduce our well-posedness result concerning (1.1), we first define the notion of admissible solution. We require that ρ is an entropy admissible solution in the sense of Kružkov [17] of the conservation law at the first line of (1.1) and that the transport equation at the second line is satisfied as an identity between L^∞ functions (which makes sense in the regularity class we are considering).

Definition 1.1 The couple $(\rho,u)\in L^\infty_{\text{loc}}(\mathbb{R}_+\times\mathbb{R})\cap C^0(\mathbb{R}_+,L^1_{\text{loc}}(\mathbb{R}))\times W^{1\infty}(\mathbb{R}_+,\mathbb{R}))$ is an *entropy admissible solution* of (1.1) if the equation at the first line is satisfied in the sense of

[1] To simplify the exposition, we require that the conditions (1.5) are satisfied on the whole $\mathbb{R}^2$. In the following however we show that the entropy admissible solution $(\rho,u)\in[0,1]\times[0,\|u_0\|_{L^\infty}]$ and hence as a matter of fact it suffices to require that the conditions are satisfied in that range.

distributions, the second as an equality between L^∞ functions, and the following entropy condition holds:

$$\partial_t|\rho-k| + \partial_x\Big[\operatorname{sign}[\rho-k]\big[V(\rho,u)\rho - V(k,u)k\big]\Big] + \operatorname{sign}[\rho-k]k\partial_2 V(k,u)\partial_x u \le 0 \quad (1.7)$$

in the sense of distributions on $\mathbb{R}_+\times\mathbb{R}$, for every $k\in\mathbb{R}$.

Note that, since $\rho\in C^0(\mathbb{R}_+, L^1_{\text{loc}}(\mathbb{R}))$ and u is a Lipschitz continuous function, we can give a meaning to the values $\rho(t,\cdot)$ and $u(t,\cdot)$ at *every* t. An entropy admissible solution of the Cauchy problem (1.1), (1.3) is then an entropy admissible solution of (1.1) which attains the initial datum (1.3).

With the above definition in place, we can now state our well-posedness result.

Theorem 1.2 *Assume V satisfies (1.5) and that the initial data (ρ_0,u_0) satisfy (1.4), $\rho_0\in BV(\mathbb{R})$ and*

$$u_0\in W^{1\infty}(\mathbb{R}), \quad u_0' = z_0\rho_0, \quad z_0\in W^{1\infty}(\mathbb{R}) \quad (1.8)$$

and

$$z_0' = \rho_0\psi_0, \quad \textit{for some } \psi_0\in L^\infty(\mathbb{R}). \quad (1.9)$$

Then there is a unique entropy admissible solution of (1.1), (1.3) in the class of functions $(\rho,u)\in L^\infty_{\text{loc}}(\mathbb{R}_+;BV(\mathbb{R}))\times L^\infty(\mathbb{R}_+;W^{1\infty}(\mathbb{R}))$ such that $\partial_x u=\rho z$ with $z\in W^{1\infty}(\mathbb{R}_+\times\mathbb{R})$. Also, this unique solution satisfies

$$0\le\rho\le 1,\ \operatorname{TotVar}\rho(t,\cdot)\le D(t), \textit{ for every } t\ge 0, \quad (1.10)$$

$$0\le u_0\le\|u_0\|_{L^\infty}, \quad \|z\|_{L^\infty}\le\|z_0\|_{L^\infty}, \quad \partial_x z=\rho\psi, \quad \|\psi\|_{L^\infty}\le\|\psi_0\|_{L^\infty}, \quad (1.11)$$

where $D(t)$ is a suitable constant depending on $t, V, \|\rho_0\|_{L^1}, \|z_0\|_{L^\infty}, \|\psi_0\|_{L^\infty}$ and $\operatorname{TotVar}\rho_0$. *Given $T>0$, we also have the stability estimate*

$$\begin{aligned}&\|\partial_x u_1(t,\cdot)-\partial_x u_2(t,\cdot)\|_{L^1} + \|\rho_1(t,\cdot)-\rho_2(t,\cdot)\|_{L^1}\\ &\le K\Big[\|u_1(0,\cdot)-u_2(0,\cdot)\|_{C^0} + \|\rho_1(0,\cdot)-\rho_2(0,\cdot)\|_{L^1}\Big]\end{aligned} \quad (1.12)$$

for every $t\in[0,T]$. In the previous expression, (ρ_1,u_1) and (ρ_2,u_2) are two entropy admissible solutions and K is a suitable constant only depending on V and on the quantities $T, \|\rho_i(0,\cdot)\|_{L^1}, \|z_i(0,\cdot)\|_{L^\infty}, \|\psi_i(0,\cdot)\|_{L^\infty}$, $\operatorname{TotVar}\rho_i(0,\cdot)$ *for $i=1,2$.*

Some remarks are here in order. First, one can easily show[2] that, if (ρ_1,u_1) and (ρ_2,u_2) are as in the statement of the theorem, then

$$\|(u_1(s,\cdot)-u_2(s,\cdot))\|_{C^0}\le|u_{1\infty}-u_{2\infty}| + \|\partial_x u_1(s,\cdot)-\partial_x u_2(s,\cdot)\|_{L^1}, \quad (1.13)$$

[2] See (3.3).

where $u_{i\infty}$ denotes the asymptotic state of $u_i(0,\cdot)$ as $x \to -\infty$. These limits exist because $u_1(0,\cdot)$ and $u_2(0,\cdot)$ are both functions of bounded variation owing to (1.8). Combining (1.13) with (1.12) yields

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{C^0} + \|\rho_1(t,\cdot) - \rho_2(t,\cdot)\|_{L^1}$$
$$\le 2K\Big[\|u_1(0,\cdot) - u_2(0,\cdot)\|_{C^0} + \|\rho_1(0,\cdot) - \rho_2(0,\cdot)\|_{L^1}\Big].$$

Second, the regularity assumptions on u_0 imposed in (1.8) are obviously restrictive, but the identities $u_0' = \rho_0 z_0$ and $z_0' = \rho_0\psi_0$ can be partially justified as follows. Heuristically speaking, the identity $u_0' = \rho_0 z_0$ prescribes that the initial empty road velocity is constant when the initial density vanishes. This seems a quite an innocuous requirement from the modeling point of view, given that the empty road velocity of drivers can be arbitrarily defined on sets where the density is zero (i.e., there are no drivers). An analogous argument applies to the identity $z_0' = \rho_0\psi_0$.

Third, and explicit expressions for the constants $D(t)$ and K in (1.10) and (1.12), respectively, can be recovered if needed by following the proof. Fourth and last, in the proof of the uniqueness part, we adapt an argument due to Tveito and Winther [27], whereas the proof of the existence part is based on an iteration argument. We conclude this introduction with a comparison with some previous related works.

We first outline the most important features of the Generalized Aw-Rascle-Zhang model from the structural viewpoint by following, for instance, the analysis in [25, §3]. We regard (1.2) as a system of conservation laws in the variables $(\rho, m = \rho u)$. The eigenvalues of the Jacobian matrix of the flux are $\lambda_1 = \tilde V + \rho\partial_1\tilde V + m\partial_2\tilde V = V + \rho\partial_1 V$ and $\lambda_2 = \tilde V$, where we are using the notation $\tilde V(\rho, m) = V(\rho, m/\rho)$. The associated eigenvectors are

$$\mathbf{r}_1(\rho, m) = \begin{pmatrix} 1 \\ m/\rho \end{pmatrix} \qquad \mathbf{r}_2(\rho, m) = \begin{pmatrix} \partial_m\tilde V \\ -\partial_\rho\tilde V \end{pmatrix}. \tag{1.14}$$

Note that (1.2) is strictly hyperbolic if, for instance, ρ is positive and bounded away from and 0 and $\partial_1 V < 0$, a strictly stronger condition than the third one in (1.5). When (1.2) is strictly hyperbolic, the by-now classical Glimm-Bressan theory applies, see [7], and yields global-in-time existence and uniqueness for small total variation data. As a byproduct of the analysis in [6], the small total variation theory works for any strictly hyperbolic system, without the need to assume genuine nonlinearity or linear degeneracy of the characteristic fields.

Out of the small total variation framework, general global-in-time existence results have been obtained for strictly hyperbolic systems belonging to the so-called *Temple class* introduced in [26], see in particular [4, 5, 9, 14, 21, 22]. Note that the integral curves of the vector field $\mathbf{r}_1$ defined in (1.14) are straight lines and the second characteristic field $\mathbf{r}_2$ is linearly degenerate. Also, (1.2), being a 2×2 system, admits a couple of Riemann invariants providing a system of coordinates in the phase space. All in all, this means that (1.2) belongs to the Temple class, provided the strict hyperbolicity assumption is satisfied. Note, however, that, under (1.5), strict hyperbolicity fails when ρ vanishes and that system (1.2) does not possess any mechanism keeping ρ bounded away from 0.

Despite the absence of strict hyperbolicity, various authors have studied the system (1.2) employing ad-hoc methods that use the special structure of the system. In particular, in [16, 25] and [23] the authors have obtained global-in-time existence results under fairly general assumptions on the initial data by relying on either a random choice [16, 25] or on a wave-front tracking scheme [23]. See also [27] for an existence proof based on a finite difference scheme. Uniqueness is established in [27] by imposing on the initial data much more restrictive assumptions than the ones under which global in time existence can be proved. In this respect, note that a counter-example in [15] suggests that fairly restrictive assumptions on the data are necessary to ensure L^1 continuous dependence of the solution on the initial data.

The reason why we could not directly apply the results in [16, 23, 25, 27] in our case is because in those papers the authors viewed (1.2) as a model for a multiphase flow used in oil recovery and hence imposed conditions on the function V that are natural in view of these applications, but not so much in the vehicular traffic framework. For instance, they typically required that the function $\rho \mapsto \rho V(\rho, u)$ is s-shaped. However, note that the assumptions in [16, 23, 25, 27] imply that there is a curve inside the range of values that the solutions attain where the eigenvalues switch order, that is, $\lambda_1 < \lambda_2$ on one side of the curve, and $\lambda_1 > \lambda_2$ on the other side. This yields, in particular, a loss of strict hyperbolicity along the curve. Conversely, in the traffic flow framework, strict hyperbolicity is lost at the boundary of the phase space domain, which in principle should make the analysis easier. It seems, therefore, reasonable to expect that one could, for instance, adapt the techniques of [16, 23, 25], and obtain global-in-time existence results under assumptions on V that are natural in view of applications to traffic flows. In particular, in the case of the classical Aw-Rascle-Zhang model [3], a suitable invariant domain including vacuum regions has been detected and existence of solutions with bounded variation is obtained as a limit of Glimm approximations in [13]. In the same paper, the authors provide examples of the emergence of vacuum regions at positive time. Notice that the uniqueness of these solutions remains open. We refer to [12] for further references about the Aw-Rascle-Zhang model.

In this note, however, we follow a completely different path and establish existence by relying on an iteration scheme. Our argument requires fairly restrictive assumptions on the initial datum u_0, see (1.8) and (1.9). Note by the way that the nonlocal-to-local limit result in [19] provides yet another existence proof for (1.1), (1.3) under (1.8) and (1.9), but requires more restrictive conditions on V than (1.5). The reasons why we feel our approach is of some interest are threefold. First, our proof is much more elementary than the one in [16, 23, 25, 27] and requires neither the introduction of advanced techniques like the random choice or the wave front tracking algorithm, nor the quite laborious nonlocal-to-local limit argument in [19]. Second, the solutions obtained in our construction are more regular, and this allows us to match the uniqueness result we can obtain by adapting the technique in [27]. Third, as mentioned before, our interest in (1.1) is motivated by the nonlocal-to-local limit analysis in the companion work [19], where we also impose (1.8) and (1.9).

Wrapping up, the novelties of the present note compared to the existing literature are, to the best of our knowledge, the following: i) we provide a new and elementary global-in-time existence proof (under restrictive assumptions on the initial data) and ii) we adapt the uniqueness result in [27] to cover assumptions that are natural in the framework of traffic models. To conclude this overview, we refer to [24] and to the ongoing project [1] for other results on the Generalized Aw-Rascle-Zhang system.

Outline. The exposition is organized as follows. In §2 we establish the existence part of Theorem 1.2, in §3 the uniqueness and stability part.

Notation. We denote by $C(a_1,\dots,a_n)$ a constant that depends only on the quantities $a_1,\dots,a_n$. Its precise value can vary from one occurrence to another.

2 Existence

To establish the existence part of Theorem 1.2 we proceed as follows: in §2.1 we construct the approximating sequence, in §2.2 we pass to the limit, and in §2.3 we establish the proof of Lemma 2.1.

2.1 Construction of the approximation

First of all, we point out that without loss of generality, we can assume that the initial datum is compactly supported, that is

$$\rho_0(x) = 0 \text{ for a.e. } x \notin [-R,R] \text{ for some } R > 0. \tag{2.1}$$

Indeed, once we establish the existence part of Theorem 1.2 under the further assumption (2.1), we can remove it by relying on a fairly standard approximation argument. We then proceed according to the following steps.

Step 1: we set $\rho_1(t,x) := \rho_0(x)$, $u_1(t,x) = u_0(x)$. Next, we fix τ_0, to be determined in the following, and (ρ_n, u_n) satisfying

$$\begin{aligned} &0 \le \rho_n \le 1 \text{ on }]0,\tau_0[\times\mathbb{R}, \quad \|\rho(t,\cdot)\|_{L^1(\mathbb{R})} \le \|\rho_0\|_{L^1}, \\ &\mathrm{TotVar}\,\rho_n(t,\cdot) \le M_0 \quad \text{for every } t \in]0,\tau_0[, \end{aligned} \tag{2.2}$$

with M_0 to be determined in the following (see (2.14)), and

$$\|u_n\|_{L^\infty} \le \|u_0\|_{L^\infty}, \quad \partial_x u_n = \rho_n z_n, \quad \|z_n\|_{L^\infty} \le \|z_0\|_{L^\infty} \tag{2.3}$$

$$\partial_x z_n = \rho_n \psi_n, \quad \|\psi_n\|_{L^\infty} \le \|\psi_0\|_{L^\infty}. \tag{2.4}$$

On the time interval $t \in]0,\tau_0[$, we construct ρ_{n+1} as the entropy admissible solution[3] of the Cauchy problem

$$\begin{cases} \partial_t \rho_{n+1} + \partial_x[V(\rho_{n+1},u_n)\rho_{n+1}] = 0 \\ \rho_{n+1}(0,\cdot) = \rho_0. \end{cases} \tag{2.5}$$

Towards this end, we consider the vanishing viscosity approximation

$$\begin{cases} \partial_t \rho^\varepsilon_{n+1} + \partial_x[V(\rho^\varepsilon_{n+1},u_n)\rho^\varepsilon_{n+1}] = \varepsilon\partial_{xx}\rho^\varepsilon_{n+1} \\ \rho^\varepsilon_{n+1}(0,\cdot) = \rho_0 \end{cases} \tag{2.6}$$

[3] It has been pointed out to us by Graziano Guerra that, since u_n is only a Lipschitz continuous function, (2.5) does not exactly satisfy the assumptions imposed in the classical work by Kružkov [17] to establish existence and uniqueness of entropy admissible solutions. For this reason, we provide a sketch of the existence proof, which uses the special structure of u_n given by (2.3) and (2.4).

and we show that ρ^ε_{n+1} satisfies the estimates in (2.2) uniformly in ε as $\varepsilon \to 0^+$. The estimate

$$\|\rho^\varepsilon_{n+1}(t,\cdot)\|_{L^1} \leq \|\rho_0\|_{L^1} \tag{2.7}$$

is standard. Now, we show that

$$0 \leq \rho^\varepsilon_{n+1} \leq 1 \text{ on }]0,\tau_0[\times\mathbb{R}. \tag{2.8}$$

Given $\eta : \mathbb{R} \to \mathbb{R}$ smooth and convex and the corresponding flux $Q = Q(\rho, u)$ satisfying

$$Q(0,u) = 0, \quad \partial_1 Q(\rho,u) = \eta'(\rho)\left(\partial_1 V(\rho,u)\rho + V(\rho,u)\right),$$

multiplying (2.6) by $\eta'(\rho^\varepsilon_{n+1})$, we obtain

$$\begin{aligned}\partial_t\eta(\rho^\varepsilon_{n+1}) + \partial_x[Q(\rho^\varepsilon_{n+1},u_n)] - \big[\partial_2 Q(\rho^\varepsilon_{n+1},u_n) - \eta'(\rho^\varepsilon_{n+1})\rho^\varepsilon_{n+1}\partial_2 V(\rho^\varepsilon_{n+1},u_n)\big]\partial_x u_n \\ = \varepsilon\partial_{xx}\eta(\rho^\varepsilon_{n+1}) - \varepsilon\eta''(\rho^\varepsilon_{n+1})(\partial_x\rho^\varepsilon_{n+1})^2,\end{aligned}$$

and, by a simple approximation argument, we deduce that for every convex $\eta : \mathbb{R} \to \mathbb{R}$, possibly not smooth, it holds

$$\begin{aligned}&\partial_t\eta(\rho^\varepsilon_{n+1}) + \partial_x[Q(\rho^\varepsilon_{n+1},u_n)] - \big[\partial_2 Q(\rho^\varepsilon_{n+1},u_n) - \eta'(\rho^\varepsilon_{n+1})\rho^\varepsilon_{n+1}\partial_2 V(\rho^\varepsilon_{n+1},u_n)\big]\partial_x u_n \\ &\leq \varepsilon\partial_{xx}\eta(\rho^\varepsilon_{n+1})\end{aligned} \tag{2.9}$$

in the sense of distributions. Choosing $\eta(\rho) = [\rho]^- = \max\{-\rho, 0\}$, we have

$$\partial_2 Q(\rho^\varepsilon_{n+1},u_n) - \eta'(\rho^\varepsilon_{n+1})\rho^\varepsilon_{n+1}\partial_2 V(\rho^\varepsilon_{n+1},u_n) = 0,$$

therefore, integrating (2.9) with respect to x, we obtain $\partial_t\|\eta(\rho^\varepsilon_{n+1}(t))\|_{L^1} \leq 0$, and since $\eta(\rho^\varepsilon_{n+1}(0)) = 0$ it follows that $\rho^\varepsilon_{n+1} \geq 0$ on $]0,\tau_0[\times\mathbb{R}$.

Similarly, choosing $\eta(\rho) = [\rho - 1]^+ = \max\{\rho - 1, 0\}$ we have

$$\partial_2 Q(\rho^\varepsilon_{n+1},u_n) - \eta'(\rho^\varepsilon_{n+1})\rho^\varepsilon_{n+1}\partial_2 V(\rho^\varepsilon_{n+1},u_n) = 0$$

since $V(1,u) = 0$ for every u. Therefore, since $\eta(\rho^\varepsilon_{n+1}(0)) = 0$, it follows that

$$\rho^\varepsilon_{n+1} \leq 1 \quad \text{on} \quad]0,\tau_0[\times\mathbb{R}.$$

We have therefore proven (2.8) and it remains to show

$$\mathrm{TotVar}\rho^\varepsilon_{n+1} \leq M_0 \qquad \text{for every } t \in]0,\tau_0[. \tag{2.10}$$

We set $\partial_x\rho^\varepsilon_{n+1} := r^\varepsilon_{n+1}$, which yields

$$\begin{aligned}0 &= \partial_t r^\varepsilon_{n+1} + \partial_x[V(\rho^\varepsilon_{n+1},u_n)r^\varepsilon_{n+1}] + \partial_x[\partial_x V(\rho^\varepsilon_{n+1},u_n)\rho^\varepsilon_{n+1}] - \partial_{xx}r^\varepsilon_{n+1} \\ &= \partial_t r^\varepsilon_{n+1} + \partial_x\Big[[V(\rho^\varepsilon_{n+1},u_n) + \partial_1 V\rho^\varepsilon_{n+1}]r^\varepsilon_{n+1}\Big] + \partial_x[\partial_2 V\partial_x u_n\rho^\varepsilon_{n+1}] - \partial_{xx}r^\varepsilon_{n+1}.\end{aligned} \tag{2.11}$$

Recalling that $\partial_x u_n = \rho_n z_n$ we get

$$\begin{aligned}\partial_x[\partial_2 V \partial_x u_n \rho^\varepsilon_{n+1}] =& [\partial_{21} V r^\varepsilon_{n+1} + \partial_{22} V \partial_x u_n]\partial_x u_n \rho^\varepsilon_{n+1} \\ &+ \partial_2 V \rho^\varepsilon_{n+1}[\partial_x z_n \rho_n + z_n \partial_x \rho_n] + \partial_2 V \partial_x u_n r^\varepsilon_{n+1}\end{aligned}$$

and by plugging the above formula into (2.11), multiplying times $\text{sign}[r^\varepsilon_{n+1}]$ and integrating in space, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} |r^\varepsilon_{n+1}(t,x)|dx \\ &\le \|\partial_x u_n\|_{L^\infty}\|\rho^\varepsilon_{n+1}\|_{L^\infty}\left(\|\partial_{21} V\|_{C^0}\int_{\mathbb{R}} |r^\varepsilon_{n+1}(t,x)|dx + \|\partial_{22} V\|_{C^0}\int_{\mathbb{R}} |\partial_x u_n(t,x)|dx\right) \\ &+ \|\partial_2 V\|_{C^0}\|\rho^\varepsilon_{n+1}\|_{L^\infty}\left[\|\partial_x z_n\|_{L^\infty}\|\rho_n\|_{L^1} + \|z_n\|_{L^\infty}\underbrace{\int_{\mathbb{R}} |\partial_x \rho_n(t,x)|dx}_{\le M_0 \text{ by } (2.2)}\right] \\ &+ \|\partial_2 V\|_{C^0}\|\partial_x u_n\|_{L^\infty}\int_{\mathbb{R}} |r^\varepsilon_{n+1}(t,x)|dx \\ &\overset{(2.2),(2.3),(2.4)}{\le} C(V, \|z_0\|_{L^\infty})\int_{\mathbb{R}} |r^\varepsilon_{n+1}(t,x)|dx + C(V, \|z_0\|_{L^\infty}, \|\psi_0\|_{L^\infty}, \|\rho_0\|_{L^1}) \\ &+ C(V, \|z_0\|_{L^\infty})M_0.\end{aligned} \tag{2.12}$$

By applying the Comparison Theorem for ODEs, we conclude that

$$\sup_{t\in[0,\tau_0]}\int_{\mathbb{R}} |r^\varepsilon_{n+1}(t,x)|dx \le [\text{TotVar}\,\rho_0]\exp(\tilde{C}\tau_0) + [M_0+1][\exp(\tilde{C}\tau_0)-1], \tag{2.13}$$

for some $\tilde{C} = C(V, \|z_0\|_{L^\infty}, \|\psi_0\|_{L^\infty}, \|\rho_0\|_{L^1})$. We now set

$$M_0 := 4\text{TotVar}\rho_0 + 4 \tag{2.14}$$

and choose τ_0 in such a way that

$$[\exp(\tilde{C}\tau_0) - 1] \le 1/2 \tag{2.15}$$

and using (2.13), we arrive at

$$\text{TotVar}\,\rho^\varepsilon_{n+1}(t,\cdot) \le [1/2 + 1]\text{TotVar}\rho_0 + 1/2[M_0 + 1] \le M_0 \quad \text{for } t \in [0, \tau_0]. \tag{2.16}$$

Eventually, the uniform bound on the total variation of ρ^ε_{n+1} as $\varepsilon \to 0$ implies that the solutions ρ^ε_{n+1} are equicontinuous as elements of $C^0([0,\tau_0[;L^1(\mathbb{R}))$ (the proof is a straightforward adaptation of [8, Lemma 6.3.3]). Therefore, thanks to Ascoli-Arzelà theorem, we can extract sequence $\rho^{\varepsilon_k}_{n+1}$ that converges as $\varepsilon_k \to 0$ to an entropy admissible solution ρ_{n+1} of the problem (2.5) and this ends Step 1.

Step 2: we define u_{n+1}. Note that we cannot exactly rely on the classical method of characteristics owing to the low regularity of ρ_{n+1} and henceforth of the vector field $V(\rho_{n+1}, u_n)$. To circumvent this obstruction, we consider the Cauchy problem obtained by coupling the equation

$$\partial_t[\nu_{n+1}] + \partial_x[V(\rho_{n+1}, u_n)\nu_{n+1}] = 0 \tag{2.17}$$

with the initial datum $\nu_{n+1}(0,\cdot) = \psi_0\rho_0$. Given (2.5), existence and uniqueness results for (2.17) are available under very weak regularity assumptions on $V(\rho_{n+1}, u_n)$ and ρ_{n+1}, see [20, §4]. In the following for technical reasons we rely on [2, Theorem 2.5] applied with $p = \rho_{n+1}$, $b = V(\rho_{n+1}, u_n)$. We conclude that there is a unique solution of the Cauchy problem satisfying $|\nu_{n+1}| \le \|\psi_0\|_{L^\infty}\rho_{n+1}$. Note that setting $\psi_{n+1} := {}^{\nu_{n+1}}/_{\rho_{n+1}}$ if $\rho_{n+1} > 0$ and $\psi_{n+1} = 0$ otherwise, we obtain

$$\nu_{n+1} = \psi_{n+1}\rho_{n+1}, \qquad \|\psi_{n+1}\|_{L^\infty} \le \|\psi_0\|_{L^\infty} \tag{2.18}$$

In addition, note that $\|\nu_{n+1}(t,\cdot)\|_{L^1} \le \|\psi_{n+1}(t,\cdot)\|_{L^\infty}\|\rho_{n+1}(t,\cdot)\|_{L^1} \le \|\psi_0\|_{L^\infty}\|\rho_0\|_{L^1}$.

To define z_{n+1}, we point out that, owing to the identity $z_0' = \rho_0\psi_0$ and to the L^1 and L^∞ bounds on ρ_0 and ψ_0, respectively, the function z_0 has bounded total variation and hence the limit $z_\infty := \lim_{x\to-\infty} z_0(x)$ exists and is finite. We then set

$$z_{n+1}(t,x) := z_\infty + \int_{-\infty}^{x} \rho_{n+1}\psi_{n+1}(t,x)dx, \tag{2.19}$$

which is a $L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$ function owing to the L^1 and the L^∞ bounds on ρ_{n+1} and ψ_{n+1}, and furthermore satisfies the initial condition $z_{n+1}(0,\cdot) = z_0$ due to (1.9). Note that, integrating in the space variable (2.17), we have the identity

$$\partial_t z_{n+1} + V(\rho_{n+1}, u_n)\partial_x z_{n+1} = 0 \quad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.20}$$

We now want to establish the estimate

$$|z_{n+1}(t,x)| \le \|z_0\|_{L^\infty} \text{ for every } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.21}$$

If $V(\rho_{n+1}, u_n)$ were a regular vector field, we could apply the classical method of characteristic and (2.21) would straightforwardly follow from (2.20). Owing to the low regularity of $V(\rho_{n+1}, u_n)$, the proof of (2.21) is slightly more involved, and now we provide its details.

We fix a test function φ and use φz_{n+1} as a test function in the definition of distributional solution of (2.5). We conclude that $\nu_{n+1} = \rho_{n+1}z_{n+1}$ satisfies the very same continuity equation (2.17), now coupled with the initial datum $\rho_0 z_0$. We apply again [2, Theorem 2.5] and conclude that the Cauchy problem obtained by coupling (2.17) with the initial datum $\rho_0 z_0$ admits a unique solution satisfying $|\nu_{n+1}| \le M\rho_{n+1}$ for some $M > 0$, and that one can take $M = \|z_0\|_{L^\infty}$. By uniqueness, this solution must coincide with $\rho_{n+1}z_{n+1}$, and this implies that

$$|z_{n+1}(t,x)| \le \|z_0\|_{L^\infty} \text{ for } \rho_{n+1}\mathcal{L}^2 \text{ a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.22}$$

We now show that (2.22) implies (2.21). We fix $t > 0$, and point out that owing to (2.19) $z_{n+1}(t,\cdot)$ is a Lipschitz continuous function. Let $\Lambda_{n+1}(t)$ denote the open set of x-s such that $z_{n+1}(t,x) > \|z_0\|_{L^\infty}$. We decompose $\Lambda_{n+1}(t)$ as a countable union of disjoint intervals,

$$\Lambda_{n+1}(t) = \bigcup_{k=1}^{\infty} \mathcal{I}_k.$$

We now fix $k \in \mathbb{N}$ and assume by contradiction that $\mathcal{I}_k \neq \emptyset$. If $\mathcal{I}_k = \mathbb{R}$, then owing to (2.22) we have $\rho_{n+1}(t,\cdot) = 0$ a.e. on $\mathbb{R}$, and owing to (2.19) this yields $z_{n+1}(t,\cdot) = z_\infty$. Since z_∞ is the asymptotic state of z_0, then $|z_\infty| \le \|z_0\|_{L^\infty}$ and hence this contradicts the definition of $\Lambda_{n+1}(t)$. If $\mathcal{I}_k \neq \mathbb{R}$, then $\mathcal{I}_k =]a_k, b_k[$ and at least one between a_k and b_k is finite. Just to fix the ideas, let us assume that $a_k \in \mathbb{R}$, then we observe that owing to (2.22) we must have $\rho_{n+1} = 0$ a.e. on $\Lambda_{n+1}(t)$. We conclude that for every $x \in \mathcal{I}_k$ we have

$$\begin{aligned} |z_{n+1}(t,x)| &= \left| z_{n+1}(t,a_k) + \int_{a_k}^{x} v_{n+1}(t,x)dx \right| \le |z_{n+1}(t,a_k)| + \int_{a_k}^{x} |\psi_{n+1}| \underbrace{\rho_{n+1}(t,x)}_{=0} dx \\ &= |z_{n+1}(t,a_k)| \le \|z_0\|_{L^\infty}, \end{aligned}$$

where in the last equality we have used that $a_k \notin \Lambda_{n+1}(t)$. This contradicts the definition of $\Lambda_{n+1}(t)$ and therefore establishes (2.21).

We now set

$$u_{n+1}(t,x) := u_\infty + \int_{-\infty}^{x} z_{n+1}\rho_{n+1}(t,x)dx, \tag{2.23}$$

where u_∞ is the limit at $-\infty$ of the function u_0, which is of bounded total variation owing to the identity $u_0' = \rho_0 z_0$. By following the same argument as before, we get[4]

$$0 \le u_{n+1}(t,\cdot) \le \|u_0\|_{L^\infty} \tag{2.24}$$

and also

$$\partial_t u_{n+1} + V(\rho_{n+1}, u_{n+1})\partial_x u_{n+1} = 0 \quad \text{a.e } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.25}$$

Note furthermore that the previous analysis yields

$$\partial_x u_{n+1} = \rho_{n+1} z_{n+1} \tag{2.26}$$

and

$$\partial_x z_{n+1} = \rho_{n+1}\psi_{n+1}, \quad \|\psi_{n+1}\|_{L^\infty} \le \|\psi_0\|_{L^\infty}. \tag{2.27}$$

Step 3: by combining (2.7), (2.8) and (2.16) we get that ρ_{n+1} satisfies (2.2), by combining (2.26), (2.24) and (2.27) we get that u_{n+1} satisfies (2.3) and (2.4), which implies that we can iterate the construction for every $n \in \mathbb{N}$.

[4]Note that the bound from below in (2.24) is not mentioned in the statement of [2, Theorem 2.5], but it is a straightforward consequence of the proof, given that $u_0 \ge 0$.

2.2 Passage to the limit

We fix the same τ_0 as in (2.15). We recall the bounds (2.8) and (2.16) and control the time derivative by using the equation at the first line of (2.5) combined with the Volpert Chain Rule and the bound on $\partial_x u_n$ obtained by combining (2.21) and (2.26). We apply the Helly-Fréchet-Kolmogorov Compactness Theorem and conclude that there is $\{\rho_{n_k}\}$ such that

$$\rho_{n_k} \to \rho \quad \text{strongly in } L^1([0,\tau_0]\times\mathbb{R}), \tag{2.28}$$

for some limit function ρ such that $\text{TotVar}\,\rho(t,\cdot) \le M_0$ for a.e. $t \in [0,\tau_0]$. Note that the above convergence occurs in L^1 and not only in L^1_{loc} because owing to (2.1) for every $t \in [0,\tau_0]$, the function $\rho_{n+1}(t,\cdot)$ vanishes outside the interval $[-\tilde{R},\tilde{R}]$ for some $\tilde{R}$ depending on R and τ_0 but independent of n.

Next, we recall the bound (2.26), the identity $\partial_x z_{n+1} = \rho_{n+1}\psi_{n+1}$ (which follows from (2.19)) and the bound in (2.18). We also use the equation (2.20) to deduce a uniform bound on the time derivative. We apply the Arzelà Ascoli Theorem and conclude that there is a subsequence (which we do not relabel to simplify the notation) such that $\{z_{n_k}\}$ converges to some limit function z uniformly on the compact subsets of $[0,\tau_0]\times\mathbb{R}$. Note furthermore that $\partial_x z_{n_k}$ an $\partial_t z_{n_k}$ converge to $\partial_x z$ and $\partial_t z$ weakly* in $L^\infty([0,\tau_0]\times\mathbb{R})$ and that we have the identity $\partial_x z = \rho\psi$, where ψ is an accumulation point of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in the weak* topology and as such satisfies the bound in (1.11).

By combining (2.28) with the uniform convergence of $\{z_{n_k}\}$, we can pass to the limit in the identity (2.23) and conclude that the sequence $\{u_{n_k}\}$ converges uniformly on $[0,\tau_0]\times\mathbb{R}$ to the limit function

$$u(t,x) := u_\infty + \int_{-\infty}^{x} z\rho(t,x)dx. \tag{2.29}$$

By passing to the limit in (2.24) we obtain the first two inequalities in (1.11). Assume for a moment that we have shown that the sequence $\{u_{n_k-1}\}$ converges to the very same limit u given by (2.29)[5] Then we can pass to the limit in the distributional formulation (and in the entropy inequality) for the equation at the first line of (2.5) and obtain an entropy admissible solution of the equation at the first line of (1.1). By passing to the limit in the identity (2.25) we also obtain the equation at the second line of (1.1). In addition, note that the identity $\partial_x u = z\rho$ and the bound $\|z\|_{L^\infty} \le \|z_0\|_{L^\infty}$ follow directly from (2.29) and from (2.21) and the uniform convergence of $\{z_{n_k}\}$.

The above argument establishes the existence of an admissible solution of the Cauchy problem obtained by coupling (1.1) with (1.3) defined on the time interval $[0,\tau_0]$. To define a global in time solution, it suffices to point out that the value of the constant $\tilde{C}$ in (2.13) and henceforth the value of τ_0 in (2.15) *does not* depend on the total variation of the initial datum. By using the bounds (2.22) and $\|\rho(t,\cdot)\|_{L^1} \le \|\rho_0\|_{L^1}$ we conclude that the constants $\tilde{C}$ and τ_0 only depend on quantities that are preserved by the admissible solution, and hence we can iterate the construction and in this way establish global in time existence.

[5] By compactness, $\{u_{n_k-1}\}$ converges up to subsequences to some limit function v, but in principle u and v could be different.

To conclude the proof of the existence part of Theorem 1.2 we are thus left to show that the sequence $\{u_{n_k-1}\}$ converges to the same limit as $\{u_{n_k}\}$. Towards this end, we introduce the functional

$$\Phi_n(t) := \|[\rho_n - \rho_{n+1}](t,\cdot)\|_{L^1} + \|[\partial_x u_n - \partial_x u_{n-1}](t,\cdot)\|_{L^1}$$

and introduce the following result.

Lemma 2.1 *We have*

$$\lim_{n\to+\infty} \Phi_n(t) \to 0 \quad \textit{for every } t \geq 0. \tag{2.30}$$

We postpone the proof of Lemma 2.1 to the next paragraph and we now show that (2.30) implies that the sequence $\{u_{n_k-1}\}$ converges to the same limit as $\{u_{n_k}\}$ on $[0,\tau_0]\times\mathbb{R}$. We recall that, owing to (2.23), $\lim_{x\to-\infty} u_n(t,x) = u_\infty$ for every n and every $t \in [0,\tau_0]$, which in particular implies

$$\|u_n - u_{n-1}\|_{C^0} \leq \|\partial_x u_n - \partial_x u_{n-1}\|_{L^1}. \tag{2.31}$$

By combining (2.31) with (2.30), we then conclude that $\lim_{n\to+\infty} \|u_n(t,\cdot) - u_{n-1}(t,\cdot)\|_{C^0} = 0$ for every $t \geq 0$.

2.3 Proof of Lemma 2.1

We apply the stability result proven in [18, Theorem 2.6] (see also Remark 2.8 therein about entropy solutions) for conservation laws with space-time dependent fluxes $P = P(t,x,\rho)$, $Q = Q(t,x,\rho)$. In our setting, the fluxes take the form

$$(t,x,\rho) \mapsto P(t,x,\rho) \doteq \rho_{n+1} V(\rho_{n+1}, u_n(t,x)),$$
$$(t,x,\rho) \mapsto Q(t,x,\rho) \doteq \rho_{n+1} V(\rho_n, u_{n-1}(t,x)).$$

and the stability estimate in [18] implies that for every $t \geq 0$ we have

$$\begin{aligned}
\int_{\mathbb{R}} |\rho_{n+1}(t,x) - \rho_n(t,x)| dx \leq & \underbrace{\int_{\mathbb{R}} |\rho_{n+1}(0,x) - \rho_n(0,x)| dx}_{=0} \\
& + C(V) \int_0^t (\|z_n(s)\|_{L^\infty} + \|z_{n-1}(s)\|_{L^\infty}) \int_{\mathbb{R}} |\rho_{n+1}(s,x) - \rho_n(s,x)| dx ds \\
& + C(V) \underbrace{\int_0^t \int_{\mathbb{R}} (|\partial_x u_n||u_n - u_{n-1}| + |\partial_x u_n - \partial_x u_{n-1}|)\, dx ds}_{:=J} \\
& + C(V) M_0 \int_0^t \underbrace{\|(u_n(s,\cdot) - u_{n-1}(s,\cdot))\|_{L^\infty}}_{\leq \|[\partial_x u_n - \partial_x u_{n-1}](s,\cdot)\|_{L^1} \text{ by } (2.31)} ds
\end{aligned} \tag{2.32}$$

where M_0 is the same as in (2.2). We now control the term J as follows:

$$
\begin{aligned}
J &\leq \int_0^t \|[u_n - u_{n-1}](s,\cdot)\|_{L^\infty} \underbrace{\int_{\mathbb{R}} |\partial_x u_n|(s,x)dx}_{\leq \|z_0\|_{L^\infty}\|\rho_0\|_{L^1}} ds + \int_0^t \|[\partial_x u_n - \partial_x u_{n-1}](s,\cdot)\|_{L^1} ds \\
&\overset{(2.31)}{\leq} C(\|z_0\|_{L^\infty}, \|\rho_0\|_{L^1}) \int_0^t \|[\partial_x u_n - \partial_x u_{n-1}](s,\cdot)\|_{L^1} ds,
\end{aligned}
\tag{2.33}
$$

and conclude that

$$
\int_{\mathbb{R}} |\rho_{n+1}(t,x) - \rho_n(t,x)| dx \leq C(V, M_0, \|z_0\|_{L^\infty}, \|\rho_0\|_{L^1}) \int_0^t \Phi_n(s) ds \tag{2.34}
$$

We now point out that

$$
\begin{aligned}
&\|[\partial_x u_n - \partial_x u_{n-1}](t,\cdot)\|_{L^1} = \|[\rho_n z_n - \rho_{n-1} z_{n-1}](t,\cdot)\|_{L^1} \\
&\leq \|\rho_n[z_n - z_{n-1}](t,\cdot)\|_{L^1} + \|z_{n-1}\|_{L^\infty} \|[\rho_n - \rho_{n-1}](t,\cdot)\|_{L^1} \\
&\overset{(2.34)}{\leq} \|\rho_n[z_n - z_{n-1}]\|_{L^1} + C(V, M_0, \|z_0\|_{L^\infty}, \|\rho_0\|_{L^1}) \int_0^t \Phi_{n-1}(s) ds
\end{aligned}
\tag{2.35}
$$

and that

$$
\partial_t[\rho_n z_n] + \partial_x[V(\rho_n, u_{n-1})\rho_n z_n] = 0, \quad \partial_t[\rho_{n-1} z_{n-1}] + \partial_x[V(\rho_{n-1}, u_{n-2})\rho_{n-1} z_{n-1}] = 0,
$$

which implies (by using the equation for ρ_n and ρ_{n-1} and the Volpert Chain Rule)

$$
\begin{aligned}
0 &= \partial_t\big[\rho_n[z_n - z_{n-1}]\big] + \partial_t\big[z_{n-1}[\rho_n - \rho_{n-1}]\big] + \partial_x\big[V(\rho_n, u_{n-1})\rho_n[z_n - z_{n-1}]\big] \\
&\quad + \partial_x\big[z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}]\big] \\
&= \partial_t\big[\rho_n[z_n - z_{n-1}]\big] + \partial_t z_{n-1}[\rho_n - \rho_{n-1}] + z_{n-1}\partial_t[\rho_n - \rho_{n-1}] \\
&\qquad + \partial_x\big[V(\rho_n, u_{n-1})\rho_n[z_n - z_{n-1}]\big] \\
&\qquad + \partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}] \\
&\qquad + z_{n-1}\partial_x\big[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}\big] \\
&\overset{(2.5)}{=} \partial_t\big[\rho_n[z_n - z_{n-1}]\big] + \partial_t z_{n-1}[\rho_n - \rho_{n-1}] + \partial_x\big[V(\rho_n, u_{n-1})\rho_n[z_n - z_{n-1}]\big] \\
&\qquad + \partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}] \\
&\overset{(2.20)}{=} \partial_t\big[\rho_n[z_n - z_{n-1}]\big] - V(\rho_{n-1}, u_{n-2})\partial_x z_{n-1}[\rho_n - \rho_{n-1}] + \partial_x\big[V(\rho_n, u_{n-1})\rho_n[z_n - z_{n-1}]\big] \\
&\quad + \partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}].
\end{aligned}
$$

Recalling equation (2.5) for ρ_n and the Volpert Chain rule, we infer that the Lipschitz continuous function $z_n - z_{n-1}$ satisfies

$$
\begin{aligned}
0 &= \rho_n\big[\partial_t[z_n - z_{n-1}] + V(\rho_n, u_{n-1})\partial_x[z_n - z_{n-1}]\big] - V(\rho_{n-1}, u_{n-2})\partial_x z_{n-1}[\rho_n - \rho_{n-1}] \\
&\quad + \partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}],
\end{aligned}
$$

which by multiplying times $\mathrm{sign}(z_n - z_{n-1})$ yields

$$\begin{aligned}0 &= \rho_n\big[\partial_t|z_n - z_{n-1}| + V(\rho_n, u_{n-1})\partial_x|z_n - z_{n-1}|\big]\\ &- \mathrm{sign}(z_n - z_{n-1})V(\rho_{n-1}, u_{n-2})\partial_x z_{n-1}[\rho_n - \rho_{n-1}]\\ &+ \mathrm{sign}(z_n - z_{n-1})\partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}] = 0.\end{aligned}$$

Using again the equation for ρ_n, we then arrive at

$$\begin{aligned}0 &= \partial_t\big[\rho_n|z_n - z_{n-1}|\big] - \mathrm{sign}(z_n - z_{n-1})V(\rho_{n-1}, u_{n-2})\partial_x z_{n-1}[\rho_n - \rho_{n-1}]\\ &+ \partial_x\big[V(\rho_n, u_{n-1})\rho_n|z_n - z_{n-1}|\big]\\ &+ \mathrm{sign}(z_n - z_{n-1})\partial_x z_{n-1}[V(\rho_n, u_{n-1})\rho_n - V(\rho_{n-1}, u_{n-2})\rho_{n-1}]\end{aligned}$$

and by integrating the above inequality in space and time, we arrive at

$$\begin{aligned}&\int_{\mathbb{R}} \rho_n|z_n - z_{n-1}|(t,\cdot)dx\\ &\leq C(V)\|\partial_x z_{n-1}\|_{L^\infty}\left(\int_0^t\int_{\mathbb{R}}|\rho_n - \rho_{n-1}|dxds + \int_0^t\int_{\mathbb{R}}\rho_n|u_{n-1} - u_{n-2}|dxds\right)\\ &\overset{(2.8),(2.18)}{\leq} C(V, \|\psi_0\|_{L^\infty})\int_0^t\int_{\mathbb{R}}|\rho_n - \rho_{n-1}|dxds\\ &+ C(V, \|\psi_0\|_{L^\infty})\int_0^t \|[u_{n-1}(s,\cdot) - u_{n-2}(s,\cdot))\|_{L^\infty}\underbrace{\int_{\mathbb{R}}\rho_n(s,x)dx}_{\leq\|\rho_0\|_{L^1}}ds\\ &\overset{(2.31)}{\leq} C(V, \|\psi_0\|_{L^\infty}, \|\rho_0\|_{L^1})\int_0^t \Phi_{n-1}(s)ds\end{aligned} \tag{2.36}$$

By recalling (2.34) and (2.35), we eventually conclude that

$$\Phi_n(t) \leq C(V, M_0, \|z_0\|_{L^\infty}, \|\rho_0\|_{L^1})\int_0^t \Phi_n(s)ds + C(V, \|\psi_0\|_{L^\infty})\int_0^t \Phi_{n-1}(s)ds, \tag{2.37}$$

which, owing to the Grönwall Lemma, implies that

$$\Phi_n(t) \leq Ct\sup_{[0,t]}\Phi_{n-1}(s)\exp\{Ct\}$$

for some $C = C(V, M_0, \|z_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\psi_0\|_{L^\infty}) > 0$. Choosing $\delta > 0$ so that $C\delta\exp\{C\delta\} < \frac{1}{2}$, we obtain

$$\sup_{[0,\delta]}\Phi_n(t) \leq \frac{1}{2}\sup_{[0,\delta]}\Phi_{n-1}(t).$$

In particular, since $\Phi_1(t)$ is bounded, it follows that (2.30) holds for every $t \in [0,\delta]$. We prove that the same holds for $t \in [k\delta, (k+1)\delta]$ by induction on k: indeed, from (2.37), we deduce that

$$\Phi_n(t) \leq Ck\delta\sup_{t\in[0,k\delta]}(\Phi_n(t) + \Phi_{n-1}(t)) + C\int_{k\delta}^t(\Phi_n(s) + \Phi_{n-1}(s))ds.$$

By Grönwall Lemma it follows that

$$\Phi_n(t) \le \Big[Ck\delta \sup_{t\in[0,k\delta]} (\Phi_n(t)+\Phi_{n-1}(t)) + C\delta \sup_{[k\delta,t]} \Phi_{n-1}(s)\Big] \exp\{C\delta\}.$$

Then $Ck\delta \sup_{t\in[0,k\delta]}(\Phi_n(t)+\Phi_{n-1}(t)) \to 0$ by the inductive assumption and, since $C\delta\exp\{C\delta\} < \frac{1}{2}$, it follows that $\limsup_{n\to\infty}\Phi_n(t) = 0$.

3 Uniqueness and stability

We follow the same argument as in the proof of Lemma 2.1 and we provide the details for the sake of completeness. We fix $T > 0$ and $t \in [0,T]$ and we apply [18, Theorem 2.6] with

$$(t,x,\rho) \mapsto P(t,x,\rho) \doteq \rho V(\rho, u_1(t,x)), \qquad (t,x,\rho) \mapsto Q(t,x,\rho) \doteq \rho V(\rho, u_2(t,x)),$$

which yields

$$\begin{aligned}
\int_{\mathbb{R}} |\rho_1(t,x)-\rho_2(t,x)|dx \le & \int_{\mathbb{R}} |\rho_1(0,x)-\rho_2(0,x)|dx \\
& + C(V)\int_0^t (\|z_1(s)\|_{L^\infty} + \|z_2(s)\|_{L^\infty}) \int_{\mathbb{R}} |\rho_1(s,x)-\rho_2(s,x)|dxds \\
& + C(V,\|\rho_1\|_{L^\infty},\|\rho_2\|_{L^\infty}) \underbrace{\int_0^t\int_{\mathbb{R}} (|\partial_x u_1||u_1-u_2| + |\partial_x u_1 - \partial_x u_2|)\,dxds}_{:=L} \\
& + C(V,\|\rho_1\|_{L^\infty},\|\rho_2\|_{L^\infty}) M_0 \int_0^t \|(u_1(s,\cdot)-u_2(s,\cdot))\|_{L^\infty} ds
\end{aligned} \tag{3.1}$$

where M_0 is such that $\mathrm{TotVar}\,\rho_1(s,\cdot) \le M_0$ and $\mathrm{TotVar}\,\rho_2(s,\cdot) \le M_0$ for every $t\in[0,T]$. By using the identity $\partial_x u_1 = \rho_1 z_1$ and the estimate $\|\rho_1(t,\cdot)\|_{L^1} \le \|\rho_0\|_{L^1}$, we control the term L as follows:

$$L \le \|\rho_0\|_{L^1}\|z_1\|_{L^\infty}\int_0^t \|[u_1-u_2](s,\cdot)\|_{L^\infty} + \int_0^t \|[\partial_x u_1 - \partial_x u_2](s,\cdot)\|_{L^1} ds. \tag{3.2}$$

We now want to show that

$$\lim_{x\to-\infty} u_1(t,x) = \lim_{x\to-\infty} u_1(0,x), \qquad \lim_{x\to-\infty} u_2(t,x) = \lim_{x\to-\infty} u_2(0,x) \quad \text{for every } t>0 \tag{3.3}$$

Note that all the above limits exist and are finite because $u_1(t,\cdot)$ and $u_2(t,\cdot)$ are all functions of bounded variation owing to the identities $\partial_x u_i(t,\cdot) = \rho_i z_i(t,\cdot)$ and to the L^1 and L^∞ bounds on ρ_i and z_i, respectively. By contradiction, assume that there is $t>0$ such that one of the equalities in (3.3) fails, for instance $\lim_{x\to-\infty} u_1(t,x) \ne \lim_{x\to-\infty} u_1(0,x)(x)$. This yields the existence of $t>0$ and $d>0$ such that

$$\int_{-R}^{-R+1} |u_1(t,x)-u_1(0,x)|dx \ge d \quad \text{for every } R \text{ sufficiently large.} \tag{3.4}$$

On the other hand, by using the equation at the first line of (1.1), we have

$$
\begin{aligned}
\int_{-R}^{-R+1} |u_1(t,x) - u_1(0,x)| dx &= \int_{-R}^{-R+1} \left| \int_0^t \partial_\tau u_1(\tau,x) d\tau \right| dx \\
&\overset{(1.1)}{\leq} \int_0^t \int_{-R}^{-R+1} |V(\rho_1,u_1)\partial_x u_1(\tau,x)| d\tau dx \\
&\leq C(V) \int_0^t \int_{-R}^{-R+1} |\partial_x u_1(\tau,x)| d\tau dx \to 0 \text{ as } R \to +\infty.
\end{aligned}
$$

To establish the convergence at the last line of the above expression we have used the Lebesgue Dominated Convergence Theorem combined with the identity $\partial_x u_1(t,\cdot) = \rho_1 z_1(t,\cdot)$, which dictates (owing to the bounds on ρ_1 and z_1) that $\partial_x u_1$ is a bounded and summable function. The above convergence contradicts (3.4) and establishes (3.3). By using (3.3) we then get (1.13).

We now point out that

$$
\|\partial_x u_1 - \partial_x u_2\|_{L^1} = \|\rho_1 z_1 - \rho_2 z_2\|_{L^1} \leq \|\rho_1[z_1 - z_2]\|_{L^1} + \|z_2\|_{L^\infty} \|\rho_1 - \rho_2\|_{L^1} \tag{3.5}
$$

so we are actually left to control $\|\rho_1[z_1 - z_2]\|_{L^1}$. By combining the equations at the first and second lines of (1.1) with the identity $\partial_x u_i = \rho_i z_i$ we get

$$
\partial_t[\rho_1 z_1] + \partial_x[V(\rho_1,u_1)\rho_1 z_1] = 0, \quad \partial_t[\rho_2 z_2] + \partial_x[V(\rho_2,u_2)\rho_2 z_2] = 0. \tag{3.6}
$$

We also have

$$
\rho_1[\partial_t z_1 + V(\rho_1,u_1)\partial_x z_1] = 0, \quad \rho_2[\partial_t z_2 + V(\rho_2,u_2)\partial_x z_2] = 0, \tag{3.7}
$$

This yields

$$
\begin{aligned}
0 &\overset{(3.6)}{=} \partial_t\big[\rho_1[z_1 - z_2]\big] + \partial_t\big[z_2[\rho_1 - \rho_2]\big] + \partial_x\big[V(\rho_1,w_1)\rho_1[z_1 - z_2]\big] \\
&\quad + \partial_x\big[z_2[V(\rho_1,u_1)\rho_1 - V(\rho_2,u_2)\rho_2]\big] \\
&= \partial_t\big[\rho_1[z_1 - z_2]\big] + \partial_t z_2[\rho_1 - \rho_2] + z_2\partial_t[\rho_1 - \rho_2] + \partial_x\big[V(\rho_1,u_1)\rho_1[z_1 - z_2]\big] \\
&\quad + \partial_x z_2[V(\rho_1,u_1)\rho_1 - V(\rho_2,u_2)\rho_2] + z_2\partial_x\big[V(\rho_1,u_1)\rho_1 - V(\rho_2,u_2)\rho_2\big] \\
&\overset{(1.1)}{=} \partial_t\big[\rho_1[z_1 - z_2]\big] + \partial_t z_2[\rho_1 - \rho_2] + \partial_x\big[V(\rho_1,u_1)\rho_1[z_1 - z_2]\big] \\
&\quad + \partial_x z_2[V(\rho_1,u_1)\rho_1 - V(\rho_2,u_2)\rho_2].
\end{aligned} \tag{3.8}
$$

Note furthermore that

$$
\begin{aligned}
\partial_t z_2[\rho_1 - \rho_2] &= \rho_1\partial_t[z_2 - z_1] + \rho_1\partial_t z_1 - \partial_t z_2 \rho_2 \\
&\overset{(3.7)}{=} \rho_1\partial_t[z_2 - z_1] - \partial_x z_1 V(\rho_1,u_1)\rho_1 + \partial_x z_2 V(\rho_2,u_2)\rho_2 \\
&= \rho_1\partial_t[z_2 - z_1] + V(\rho_2,u_2)\rho_2\partial_x[z_2 - z_1] + \partial_x z_1[V(\rho_2,u_2)\rho_2 - V(\rho_1,u_1)\rho_1]
\end{aligned}
$$

and by plugging the above equation into (3.8), multiplying the result by $\text{sign}(z_1 - z_2)$ and arguing as in the proof of Lemma 2.1, we arrive at

$$
\begin{aligned}
0 = \partial_t\big[\rho_1|z_1 - z_2|\big] &+ \rho_1\partial_t|z_2 - z_1| + V(\rho_2, u_2)\rho_2\partial_x|z_2 - z_1| \\
&+ \text{sign}(z_1 - z_2)\partial_x z_1[V(\rho_2, u_2)\rho_2 - V(\rho_1, u_1)\rho_1] \\
&+ \partial_x\big[V(\rho_1, u_1)\rho_1|z_1 - z_2|\big] + \text{sign}(z_1 - z_2)\partial_x z_2[V(\rho_1, u_1)\rho_1 - V(\rho_2, u_2)\rho_2].
\end{aligned} \tag{3.9}
$$

Using the Integration by Parts Formula combined with the equation for ρ_1 (or more rigorously combining a suitable approximation argument with the definition of distributional solution for ρ_1), we get

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}} \rho_1\partial_t|z_2 - z_1|(s,x)dxds = &\int_{\mathbb{R}} \rho_1|z_2 - z_1|(t,x) - \int_{\mathbb{R}} \rho_1(0,x)|z_2 - z_1|(0,x) \\
&- \int_0^t \int_{\mathbb{R}} V(\rho_1, u_1)\rho_1\partial_x|z_2 - z_1|(s,x)dxds.
\end{aligned}
$$

By space and time integrating (3.9) and using the above identity we arrive at

$$
\begin{aligned}
&\int_{\mathbb{R}} \rho_1|z_1 - z_2|(t,x)dx \le \int_{\mathbb{R}} \rho_1|z_1 - z_2|(0,x)dx \\
&\quad + \int_0^t \int_{\mathbb{R}} |V(\rho_1, u_1)\rho_1 - V(\rho_2, u_2)\rho_2|[|\partial_x z_2| + |\partial_x z_1|](s,x)dxds \\
&\quad \le \int_{\mathbb{R}} \rho_1|z_1 - z_2|(0,x)dx \\
&\qquad + C(V, \|\partial_x z_1\|_{L^\infty}, \|\partial_x z_2\|_{L^\infty})\Bigg[\int_0^t \|[u_1 - u_2](s,\cdot)\|_{L^\infty} \underbrace{\int_{\mathbb{R}} \rho_1(s,x)dx}_{\le \|\rho_0\|_{L^1}} ds \\
&\qquad + \int_0^t \|[\rho_1 - \rho_2](s,\cdot)\|_{L^1} ds\Bigg] \\
&\quad \overset{(1.13)}{\le} \int_{\mathbb{R}} \rho_1|z_1 - z_2|(0,x)dx + C(V, \|\partial_x z_1\|_{L^\infty}, \|\partial_x z_2\|_{L^\infty}, \|\rho_1(0,\cdot)\|_{L^1})\Bigg[t|u_{1\infty} - u_{2\infty}| \\
&\qquad\quad + \int_0^t \|[\partial_x u_1 - \partial_x u_2](s,\cdot)\|_{L^1} ds + \int_0^t \|[\rho_1 - \rho_2](s,\cdot)\|_{L^1} ds.\Bigg]
\end{aligned} \tag{3.10}
$$

Next, we point out that

$$
\int_{\mathbb{R}} \rho_1|z_1 - z_2|(0,x)dx \overset{0\le\rho_1(0,\cdot)\le 1}{\le} \|\partial_x u_1(0,\cdot) - \partial_x u_2(0,\cdot)\|_{L^1} + \|z_2(0,\cdot)\|_{L^\infty}\|[\rho_1 - \rho_2](0,\cdot)\|_{L^1} \tag{3.11}
$$

and we set

$$
\Phi(t) = \|[\rho_1 - \rho_2](t,\cdot)\|_{L^1} + \|[\partial_x u_1 - \partial_x u_2](t,\cdot)\|_{L^1}.
$$

By combining (3.1), (3.2) (1.13), (3.5), (3.10) and (3.11), and using $\|\partial_x u_i\|_{L^\infty} \le \|\rho_i\|_{L^\infty} \|z_i\|_{L^\infty}$, we have

$$\Phi(t) \le \hat{C}\Big[\Phi(0) + |u_{1\infty} - u_{2\infty}|T + \int_0^t \Phi(s)ds\Big], \quad \text{for every } t \in [0, T],$$

where the constant $\hat{C}$ depends on V, the total variation of ρ_1, ρ_2 and $\|z_1\|_{L^\infty}$, $\|\rho_1(0,\cdot)\|_{L^1}$, $\|z_2\|_{L^\infty}$, $\|\partial_x z_2\|_{L^\infty}$, $\|\rho_1\|_{L^\infty}$ and $\|\rho_2\|_{L^\infty}$. Owing to the Grönwall Lemma this yields

$$\Phi(t) \le e^{\hat{C}t}[\Phi(0) + T|u_{1\infty} - u_{2\infty}|],$$

that is (1.12). The above estimate yields, in particular, the identities $u_1 = u_2$ and $\rho_1 = \rho_2$ if the solutions have the same initial data.

4 Conclusion and outlook

In this note, we have discussed the second-order traffic model (1.1) and established a new and elementary global existence proof. We have also demonstrated that the uniqueness argument in [27], originally introduced in the context of polymer flooding for oil recovery, can be extended to the traffic modeling framework.

An interesting development would be to extend the analysis, particularly the uniqueness result, to accommodate less regular initial data. As mentioned in the introduction, the wave-front tracking techniques developed in [23] in the oil recovery framework could likely be adapted to traffic modeling and used to establish global-in-time existence results. On the other hand, extending the currently available uniqueness result is probably a significantly harder problem and likely requires the introduction of new techniques.

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Data availability
No new data were generated or analyzed in this study.

Declarations

Competing interests
The authors declare that they have no competing interests.

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