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Homomorphisms between problem spaces

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ABSTRACT

In procedural knowledge space theory (PKST), a "problem space" is a formal representation of the knowledge that is needed for solving all of the problems of a certain type. The competence state of a real problem solver is a subset of the problem space which satisfies a specific condition, named the "sub-path assumption". There could exist specific "symmetries" in a problem space that make certain parts of it "equivalent" up to those symmetries. Whenever an equivalence relation is introduced for elements in a problem space, the question almost naturally arises whether the collection of the induced equivalence classes forms, itself, a problem space. This is the main question addressed in the present article, which is restated as the problem of defining a homomorphism of one problem space into another problem space. Two types of homomorphisms are examined, which are named the "strong" and the "weak homomorphism". The former corresponds to the usual notion of "operation preserving mapping". The latter preserves operations in only one direction. Two algorithms are developed for testing the existence of homomorphisms between problem spaces. The notions and algorithms are illustrated in a series of three examples in which quite well-known neuro-psychological and cognitive tests are employed.

1. Introduction

Procedural knowledge space theory (Stefanutti, 2019; Stefanutti & Albert, 2003) is at the meeting point between the theory of problem spaces (Newell & Simon, 1972) and that of knowledge spaces (Doignon & Falmagne, 1985, 1999). In PKST, a "problem space" is a formal representation of the (procedural) knowledge that is needed for solving all of the problems of a certain type (e.g., all the different problems that can be conceived in the puzzle of the Tower of Hanoi, or all of the problems in the Tower of London test or, still, all of the problems of exiting a given labyrinth, etc.). In this respect, the problem space is regarded as the competence (state) of a perfect problem solver. The theory is fairly recent and its applications are not many, at the present time. Nonetheless, a few interesting ones are worth mentioning: Stefanutti (2019) derived the problem space and the corresponding knowledge space for the puzzle of the buckets of water; Stefanutti, de Chiusole, and Brancaccio (2021) applied PKST to the neuropsychological test of the Tower of London; an interesting application to the game of Go can be found in Sgaravatti (2022). Concerning problem-solving in education, applications can be found in Stefanutti (2014) and, albeit with a different perspective, also in Augustin, Hockemeyer, Kickmeier-Rust, and Albert (2010) and Kickmeier-Rust and Albert (2010), who developed computerized educational games.

The objective of PKST is twofold. On the one side, it aims at describing the "competence state" of a real problem solver, which could be different from that of a perfect problem solver. In PKST, the competence state of a real problem solver is a subset of the entire problem space which satisfies a specific condition, named the "sub-path assumption". Informally, this assumption states that a person who is capable of solving a given problem, by following a certain solution path, will also be able to solve all of the sub-problems that are encountered along that solution path. The collection of all the competence states of a given problem space is the "competence space".

The second objective of PKST is to describe and (possibly) predict the observable behavior of a problem solver. This not only includes the description and prediction of accuracy (i.e., correctness of the problem solution) but also the description and prediction of the single steps in the observable solution process carried out by an individual in attempting to solve a given problem.

Informally, a problem space consists of a set of problem states (or problem configurations) and a collection of rules that can be applied for transforming any given problem state into another one in the problem space. If the type of problem is "well-specified" (which is the case for all the examples provided in the present article), then both the problem states and the rules can be objectively stated, and the problem space can be constructed (e.g., by a computerized procedure) and displayed.

The concept of a "problem state" is a primitive in PKST, and therefore no assumptions are made concerning the details of its internal

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structure. Such assumptions should, eventually, be supplemented in the specific applications of the theory. Whatever internal structure a single problem state has, its representation could be more or less rich in details and more or less tight to the way the problem state appears to the eyes of the problem solver. However, in most cases, not all of the details need to be considered for solving the problem.

There could exist specific "symmetries" in a problem space that make certain problem states or certain rules "equivalent" up to those symmetries. In turn, equivalences among problem states and equivalences among transformation rules may give rise to equivalences among problems. Whenever an equivalence relation is introduced for the problem states or for the rules in a problem space, the question almost naturally arises whether the collection of the induced equivalence classes forms, itself, a problem space. This is the main question addressed in the present article. An answer to this question solves two distinct, though related open questions in PKST. The first one is methodological: In certain applications, problem spaces tend to be very large, giving rise to even larger competence spaces. Exploiting symmetries can dramatically reduce the size of the problem space. The second question is theoretical and it is about the "psychological equivalence" among problems. Two distinct problems are "psychologically equivalent" if an individual that is capable of solving the former, is also capable of solving the latter, and vice versa. The problem-solving behavior of an individual may provide evidence in favor or against the equivalence between certain problems, problem states, or operations. The problem space homomorphism provides a tool for obtaining a coarser and more abstract problem space from a concrete one. The abstract problem space on the one side and the concrete problem spaces on the other side are thus alternative models that can be tested against empirical evidence through approaches like that described in Stefanutti et al. (2021).

Two types of homomorphisms between problem spaces are examined. The strong homomorphism corresponds to the usual notion of "operation preserving mapping" that is found in abstract algebra. Most importantly, it always gives rise to a congruence partition. In certain practical applications, the strong homomorphism imposes unnecessary constraints. Due to such constraints, certain problem equivalences, that are required or expected to hold true in empirical observations, are forbidden by the strong homomorphism. This is a problem that calls for a weaker form of homomorphism. The weak homomorphism proposed in this article preserves operations in only one direction, giving rise to a weaker form of congruence partition.

This paper is organized as follows: Section 2 provides the necessary mathematical background on KST and PKST. In Section 3, the concept of problem space is characterized as a ternary relation satisfying a set of five axioms. Moreover, the concept of an incomplete problem space is introduced. Section 4 provides the definitions of two different types of homomorphisms between two problem spaces and presents some theoretical results about them. In Section 5, two algorithms are provided that exploit the obtained theoretical results for building and testing problem space homomorphisms. Section 6 shows, with three examples, the potential of the proposed approach for both research and applications. The examples refer to two planning and problem-solving ability tests, namely the Tower of London test and the tower of Hanoi task, and to a mental rotation task to study spatial abilities. A summary and some final remarks are given in Section 7.

2. Background

2.1. Homomorphism between algebraic structures

A homomorphism in algebra is a mapping between two algebraic structures of the same type, that preserves their operations and relations (see e.g., Rotman, 2012). If the algebraic structures are as simple as a single set of objects equipped with a single binary operation then, formally, given two algebraic structures $(A, \cdot)$ and $(B, \star)$, where A, B are non-empty sets, $\cdot$ is a binary operation for A, and $\star$ is a binary operation for B, the mapping $f : A \to B$ is a homomorphism if

$$f(a \cdot b) = f(a) \star f(b), \quad (1)$$

for every $a, b \in A$. The concept of homomorphism has been extended and used in several scientific fields, such as physics, computer science, biology, and psychology (e.g., Mohan, Devi, & Prakash, 2017; Schuurman & Yarkony, 2006; Yoeli & Ginzburg, 1964). In psychology, homomorphism is one of the key concepts for measurement. For instance, the ordinal scale (Krantz, Luce, Suppes, & Tversky, 1971) is the set of all the "order-preserving" homomorphisms from an empirical structure into a theoretical structure, preserving the relation "greater (or lesser) than".

A property of the homomorphism that will be useful in this work is that it induces a congruence on its domain. A congruence is an equivalence relation on an algebraic structure that is compatible with the operation of the structure. More precisely, a homomorphism $f : A \to B$ induces an equivalence relation $\sim_f$ on the set A, such that given any $a, b \in A$, $a \sim_f b$ if $f(a) = f(b)$. This equivalence relation is compatible with the operation $\cdot$ in the sense that $a \sim_f b$ and $c \sim_f d$ implies $a \cdot c \sim_f b \cdot d$. Because of this property, the binary operation $\star$ can be consistently defined at the level of the equivalence classes. Let $[a] = \{b \in A : b \sim_f a\}$ be the equivalence class of $a \in A$. Then, given any two $a, b \in A$, $[a \cdot b] = [a] \star [b]$.

2.2. Knowledge space theory

The theory of knowledge structures is a mathematical approach to the assessment of human knowledge (Doignon & Falmagne, 1999; Falmagne & Doignon, 2011). The main application of KST is to build adaptive assessment tools that can recover "what an individual knows" by asking a minimum number of questions. Let Q be a nonempty set of problems that can be created in a given field of knowledge (e.g., statistics, mathematics, physics, etc.). Then, individual knowledge is represented by a subset $K \subseteq Q$, called a *knowledge state*, and it represents the subset of problems that an individual masters. A *knowledge structure* $\mathcal{K}$ is the collection of all the existing knowledge states in the population. Formally, a knowledge structure $\mathcal{K}$ is any subset of 2^Q, containing at least Q and the empty set $\emptyset$. Among the most important types of knowledge structures there are the *knowledge spaces* which are closed under union, meaning that given any subfamily $\mathcal{F} \subseteq \mathcal{K}$, the union $\bigcup \mathcal{F}$ is in $\mathcal{K}$. A special case of a knowledge space is the learning space, also known as a well-graded knowledge space, that is characterized by the property such that for every non-empty state $K \in \mathcal{K}$ there is a problem $q \in K$ such that $K \setminus \{q\}$ is still in $\mathcal{K}$.

Several methods and procedures exist that can be applied to build the knowledge structure, given the set Q (see e.g., de Chiusole, Stefanutti, & Spoto, 2017; Düntsch & Gediga, 1996; Heller, Augustin, Hockemeyer, Stefanutti and Albert, 2013; Heller, Ünlü and Albert, 2013; Koppen, 1993; Spoto, Stefanutti, & Vidotto, 2016; Stefanutti, Albert, & Hockemeyer, 2005). The general idea behind all of them is to define relationships among the problems. Such relationships provide constraints that specify which subsets of Q are knowledge states. One approach to defining the relationships among problems is to take into account the skills implied in the solution of a problem itself. Many authors contributed to an extension of KST, named *competence-based knowledge space theory* (CbKST; Heller, Augustin et al., 2013; Heller, Ünlü et al., 2013; Stefanutti & Albert, 2003) that takes into account the skills. Given a set Π of skills, the *competence state* of an individual is a subset $C \subseteq \Pi$. The two sets Q and Π are related through a *skill map* (Doignon, 1994), that is a triple (Q, Π, τ) where $\tau : Q \to 2^\Pi$ is a mapping assigning a nonempty subset of skills in Π to each of the problems in Q. Doignon (1994) describes two alternative models for skill maps, that are named the *conjunctive model* and the *disjunctive model*. Given a problem q, under the conjunctive model, all the skills in $\tau(q)$ are necessary for solving problem q, whereas under the disjunctive model, any skill in $\tau(q)$ is sufficient.

2.3. Procedural KST

Procedural KST (Stefanutti, 2019) is a framework derived from KST for the formal modeling and assessment of human problem-solving. The fundamental notion in PKST is that of a "problem space" (Newell & Simon, 1972). The concepts in this section summarize the theoretical results obtained in Stefanutti (2019). They are explained with the help of a concrete example based on the Tower of Hanoi (ToH) tasks. The ToH tasks used in this example consist of three pegs and two disks of different diameters that can slide into any peg. Disks can be moved, only one at a time, from one peg to another, and the larger disk cannot be placed over the smaller one. The objective of the task is to match an "initial" and a "final" configuration of the problem in the minimum number of moves. The strategies involved in solving the ToH are extensively described in Section 6.1.

Let Ω be a set of moves (named "operations" henceforth) and $\omega_1\omega_2 \dots \omega_n$ be a sequence of operations in Ω. Given two sequences $\alpha = \omega_1\omega_2 \dots \omega_m$ and $\beta = \omega_{m+1}\omega_{m+2} \dots \omega_n$, their concatenation is the sequences $\alpha\beta = \omega_1\omega_2 \dots \omega_m\omega_{m+1}\omega_{m+2} \dots \omega_n$. The collection of all the sequences of operations of arbitrary finite length is

$$\Omega^* = \bigcup_{n\in\mathbb{Z}^+} \Omega^n,$$

where $\mathbb{Z}^+$ is the set of non-negative integer numbers. The collection Ω^* includes the "empty string" of operations ϵ.

A *problem space* is a triple $\mathbf{P} = (S, \Omega, \cdot)$, where S is a nonempty set of *problem states*, Ω is a nonempty set of *operations*, and $\cdot : S \times \Omega^* \to S$, is an operator satisfying, for all $s \in S$ and $\pi, \sigma \in \Omega^*$, the following two properties:

(P1) $s \cdot \epsilon = s$,
(P2) $(s \cdot \pi) \cdot \sigma = s \cdot \pi\sigma$.

The operator $\cdot$ is named *operation application*. A problem space can be represented using a directed graph (digraph) (S, E), whose vertices are the problem states in S, and the set E of edges is defined by

$$E = \{(s,t) \in S^2 : s \cdot \omega = t \text{ for some } \omega \in \Omega\}$$

Fig. 1 shows the directed graph for a portion of the problem space for the ToH with two disks. Each vertex in the directed graph corresponds to a problem state in the subset $S_{\text{ToH}} = \{a, b, c, d, e\}$ which contains five out of the nine possible problem states for the ToH with two disks. The collection Ω_{ToH} of operations consists of twelve operations, six for the larger disk and six for the smaller one. Naming "left", "center" and "right", the three pegs, the operations for each of the two disks are: (1) left to center; (2) center to right; (3) left to right; (4) center to left; (5) right to center; (6) right to left.

Let $\Omega_{\text{ToH}} = \{d_1, d_2, d_3, d_4, d_5, d_6, D_1, D_2, D_3, D_4, D_5, D_6\}$ be the set of operations, where, for $i \in \{1,2,3,4,5,6\}$ each d_i is a move related to the smaller disk, whereas each D_i is related to the larger disk. In Fig. 1 the directed edges are labeled by operations in Ω_{ToH}. Only the directed edges that transform a problem state s into another problem state t are represented in the figure, whereas self-connections are omitted for clarity.

A *problem* in the problem space $\mathbf{P}$ is a pair (s,t) of distinct states in S such that $t = s \cdot \pi$ for some sequence $\pi \in \Omega^*$. The collection of all the problems in $\mathbf{P}$ is thus

$$Q = \{(s,t) \in S^2 : s \neq t \text{ and } t = s \cdot \pi \text{ for some } \pi \in \Omega^*\}.$$

A path for the problem space $\mathbf{P}$ is any pair $s\pi \in \Pi = S \times (\Omega^* \setminus \{\epsilon\})$. A path $s\pi \in \Pi$ is said to *solve* problem $(s,t) \in Q$ if $s \cdot \pi = t$. The collection of all the paths solving a certain problem $(s,t) \in Q$ is denoted $\tau(s,t)$, and each of its members is named a *solution path*. Thus, (Q, Π, τ) with $\tau : Q \to 2^{\Pi} \setminus \{\emptyset\}$ is a mapping from the domain to the powerset of Π excluded the empty set. In Stefanutti (2019) solution paths are regarded as (procedural) "skills". This identity stems from

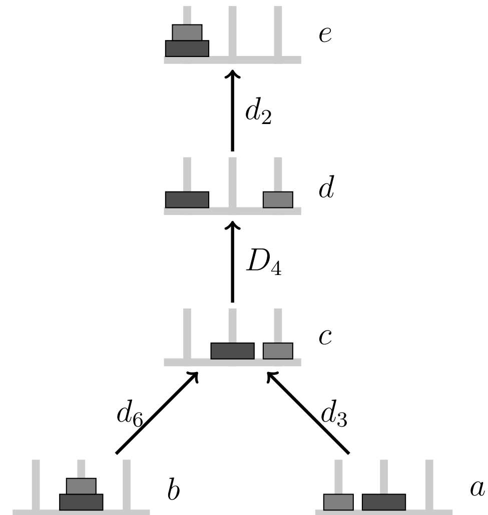

Fig. 1. Directed graph of a portion of the problem space for the two disks Tower of Hanoi.

the fact that the solution paths are part of the unobservable individual solution process. In many examples provided in this article, the solution paths are directly observable. The term "skill" is not appropriate for an observable solution path, which is part of the observable behavior. Nonetheless, borrowing the terminology of CbKST, the triple (Q, Π, τ) is named the *skill map* of the problem space $\mathbf{P}$.

A solution path $s\pi$ is a *sub-path* of another solution path $t\sigma$ (denoted by $s\pi \sqsubseteq t\sigma$) if there are $\alpha, \beta \in \Omega^*$ such that $\sigma = \alpha\pi\beta$ and $t \cdot \alpha = s$. The sub-path relation $\sqsubseteq$ is a partial order (i.e., reflexive, transitive, and antisymmetric) so that $(\Pi, \sqsubseteq)$ is a partially ordered set. A solution path $t\sigma \in \Pi$ is *cyclic* if it has a sub-path $s\pi \sqsubseteq t\sigma$ with $s \cdot \pi = s$. It is *acyclic* otherwise. The collection of all the acyclic solution paths in Π is denoted $\tilde{\Pi}$. If S and Ω are finite, then also $\tilde{\Pi}$ is finite. Moreover, like Π, also $\tilde{\Pi}$ is partially ordered by $\sqsubseteq$.

A subset $C \subseteq \Pi$ is said to *respect path inclusion* if the condition

$$s\pi \sqsubseteq t\sigma, t\sigma \in C \implies s\pi \in C$$

is satisfied for all $s\pi, t\sigma \in \Pi$. A subset of solution paths respecting path inclusion is named a *competence state* of the problem space $\mathbf{P}$. The collection $\mathcal{C}$ of all the competence states is the *competence space*, which turns out to be closed under both union and intersection (Stefanutti, 2019).

The collection of all the problems in Q that can be solved by an individual whose competence state is $C \in \mathcal{C}$, is

$$p(C) = \{(s,t) \in Q : \tau(s,t) \cap C \neq \emptyset\}.$$

It is worth noticing that $p : 2^{\Pi} \to 2^{Q}$ is a disjunctive problem function for the skill map τ. This collection is named the *knowledge state* delineated by competence state C. A problem (s,t) belongs to $p(C)$ if and only if C contains at least one of the solution paths solving (s,t). The collection $\mathcal{K}$ of all the knowledge states is the *knowledge space* derived from the problem space $\mathbf{P}$. The knowledge space turns out to be closed under union and an algorithm for its automatic derivation from a problem space was presented in Stefanutti (2019).

Consider for example the subset $Q_{\text{ToH}} = \{(a,e), (b,e), (c,e), (d,e)\}$ of all the problems with final state e obtainable from the problem space in Fig. 1. The collection of all the acyclic solution paths is $\Pi_{\text{ToH}} = \{ad_3D_4d_2,\ bd_6D_4d_2,\ cD_4d_2,\ dd_2\}$. For the collection of problems Q_{ToH}, the skill map $\tau_{\text{ToH}} : Q_{\text{ToH}} \to \Pi_{\text{ToH}}$ is then defined as follows:

$$\tau_{\text{ToH}}(a,e) = \{ad_3D_4d_2\},$$
$$\tau_{\text{ToH}}(b,e) = \{bd_6D_4d_2\},$$
$$\tau_{\text{ToH}}(c,e) = \{cD_4d_2\},$$

$\tau_{\mathrm{ToH}}(d,e) = \{dd_2\}$.

It can be seen that every problem has exactly one solution path (in general a problem may have more than one solution path).

At this point, the reader can easily check that the collection C_{ToH} of all the subsets of Π_{ToH} that respect path inclusion is:

$$C_{\mathrm{ToH}} = \{\emptyset, \{dd_2\}, \{cD_4d_2, dd_2\}, \{bd_6D_4d_2, cD_4d_2, dd_2\}, \times \{ad_3D_4d_2, cD_4d_2, dd_2\}, \Pi_{\mathrm{ToH}}\}.$$

Finally, by applying the problem function p to each competence state in C_{ToH} the resulting knowledge space is obtained:

$$\mathcal{K}_{\mathrm{ToH}} = \{\emptyset, \{(d,e)\}, \{(c,e),(d,e)\}, \{(b,e),(c,e),(d,e)\}, \{(a,e),(c,e),(d,e)\}, Q_{\mathrm{ToH}}\}.$$

Stefanutti et al. (2021) proposed a probabilistic model, called the Markov solution process model (MSPM), for the empirical validation of a knowledge space obtained from a problem space. Moreover, Brancaccio, de Chiusole, and Stefanutti (2023) extended to problem spaces the continuous Markov procedure for adaptive assessment in KST (Falmagne & Doignon, 1988).

3. Relational characterization of a problem space and incomplete problem spaces

This section is about the characterization of a problem space in terms of a ternary relation R satisfying five distinct axioms. Like in the previous sections, S denotes a nonempty and finite set of problem states, Ω denotes a nonempty and finite set of operations, and Ω^* is the set of all sequences of Ω. Let then $R \subseteq S \times \Omega^* \times S$ be a ternary relation. The five axioms for R are as follows:

(PS1) for all $a \in S$, $(a,\epsilon,a) \in R$ (*identity*);
(PS2) for all $a,b,c \in S$ and all $\pi \in \Omega^*$, if $(a,\pi,b),(a,\pi,c) \in R$ then $b = c$ (*uniqueness*);
(PS3) for all $a \in S$ and all $\pi \in \Omega^*$, there is $b \in S$ with $(a,\pi,b) \in R$ (*completeness*);
(PS4) for all $a,b,c \in S$, and all $\pi,\sigma \in \Omega^*$, if $(a,\pi,b),(b,\sigma,c) \in R$ then $(a,\pi\sigma,c) \in R$ (*transitivity*);
(PS5) for all $a,b \in S$, and all $\pi,\sigma \in \Omega^*$, if $(a,\pi\sigma,b) \in R$ then there exists $c \in S$ with $(a,\pi,c) \in R$ and $(c,\sigma,b) \in R$ (*decomposition*).

Theorem 1. *The triple (S,Ω,R) is a problem space if and only if all axioms from (PS1) to (PS5) are satisfied by R.*

Proof. Assume (S,Ω,R) satisfies (PS1) to (PS5). The proof is given by construction. Define the binary operator $\cdot : S \times \Omega^* \to S$ such that, for $a,b \in S$ and $\pi \in \Omega^*$, $a \cdot \pi = b$ iff $(a,\pi,b) \in R$. The binary operator $\cdot$ is well-defined. In fact, by (PS3), for every choice of $a \in S$ and $\pi \in \Omega^*$ there is $b \in S$ with (a,π,b), hence the operator maps the whole Cartesian product $S \times \Omega^*$ to S. Moreover, it follows from (PS2) that for every $a \in S$ and every $\pi \in \Omega^*$, there is a unique $b \in S$ with $(a,\pi,b) \in R$. Then, by (PS1), for every $a \in S$, $(a,\epsilon,a) \in R$, which implies $a \cdot \epsilon = a$. Finally, for $a,c \in S$, and $\pi,\sigma \in \Omega^*$, from (PS4) and (PS5) we have that $(a,\pi\sigma,c) \in R$ iff there is $b \in S$ such that $(a,\pi,b),(b,\sigma,c) \in R$, which is equivalent to state that $a \cdot \pi\sigma = c$ if and only if $a \cdot \pi = b$, $b \cdot \sigma = c$. By substitution we obtain that $a \cdot \pi\sigma = c$ iff $(a \cdot \pi) \cdot \sigma = c$, which holds iff $a \cdot \pi\sigma = (a \cdot \pi) \cdot \sigma$. The converse implication, from $(S,\Omega,\cdot)$ to the ternary relation R is then trivial. □

Theorem 1 is nothing else than a re-definition of a problem space, where the binary operator $\cdot$ is regarded as a ternary relation R that has properties of a functional relation (uniqueness and completeness). Thus, for $a,b \in S$ and $\pi \in \Omega^*$, we write $R(a,\pi) = b$ if and only if $(a,\pi,b) \in R$. The notation $(S,\Omega,\cdot)$ is named the *functional representation* of a problem space, whereas the notation (S,Ω,R) is referred to as the *relational representation*. One can easily convert the former into the latter by setting, for all $a,b \in S$ and all $\pi \in \Omega^*$, $(a,\pi,b) \in R$ iff $a \cdot \pi = b$.

In the definition of a problem space $(S,\Omega,\cdot)$ provided in Stefanutti (2019), the *dot* operator $\cdot$ maps the whole Cartesian product $S \times \Omega^*$ to the set S of problem states. In other words it is a *total function*. In practical applications, it might happen that not all operations $\omega \in \Omega$ are applicable to a given problem state $s \in S$. For instance, in the Tower of Hanoi problem, the operation that consists of moving a disk from the central peg to the right peg is not applicable if all disks are stacked on the left peg. This constitutes no problem, because such cases are represented through equalities of the form $s \cdot \omega = s$, and the operation ω is said to be *ineffective* in problem state s.

Nonetheless, in certain applications, or for purposes like those discussed in the next sections of this article, defining the dot operator as a partial function might prove to be more convenient. In this section, a weaker form of problem space is introduced, which is named an "incomplete problem space".

Definition 1. Given two nonempty and disjoint sets S and Ω, and a ternary relation $R \subseteq S \times \Omega^* \times S$, the triple (S,Ω,R) is an *incomplete problem space* if R satisfies (PS1) identity, (PS2) uniqueness, (PS4) transitivity, and (PS5) decomposition, and does not satisfy (PS3) completeness. For $a \in S$ and $\pi \in \Omega^*$, $R(a,\pi)$ is said to *exist* if and only if there exists $b \in S$ such that $(a,\pi,b) \in R$. Otherwise, we say that $R(a,\pi)$ *does not exist*.

Thus, an incomplete problem space is just a problem space in which completeness does not hold. Often, along the article, theoretical results are provided that hold for both incomplete problem spaces and problem spaces. In those cases, to lighten the text statement, it is simply stated that the result holds for a (incomplete) problem space.

In the relational representation of a problem space, the dot $\cdot$ operation is replaced by a ternary relation $R \subseteq S \times \Omega^* \times S$. In the following, some properties of binary relations are extended to the ternary relation R. Such properties are used in the subsequent sections.

Definition 2. Given two ternary relations $R_1, R_2 \subseteq S \times \Omega^* \times S$ their *composition* is the relation

$$R_1 \bullet R_2 = \{(a,\pi\sigma,c) \in S \times \Omega^* \times S : (a,\pi,b) \in R_1, (b,\sigma,c) \in R_2 \times \text{ for some } b \in S\}.$$

Definition 2 extends binary relation composition to the ternary relations that are of interest here in a rather obvious way. A straightforward property of the composition is associativity (which also holds with the composition of binary relations). Using the composition, the ith power of a ternary relation R is defined if $i > 0$, in which case it is

$$R^i = \begin{cases} R & \text{if } i = 1; \\ R \bullet R^{i-1} & \text{if } i > 1; \end{cases}$$

Definition 3. The *reduction* of the relation $R \subseteq S \times \Omega^* \times S$ is the subset

$$R^\circ = \{(a,\pi,b) \in R \ : \pi \in \Omega\},$$

whereas the *transitive closure* of R is the least transitive relation $U \subseteq S \times \Omega^* \times S$ including R.

The terms *reduction* and *transitive closure* are borrowed from the theory of graphs (see, e.g. Aho, Garey, & Ullman, 1972), where they have similar meanings with respect to directed graphs, rather than problem spaces.

It is worth noticing that the reduction may be, but need not be, transitive. Moreover, if the relation R is that of a problem space, then its reduction R° satisfies both (PS2) uniqueness and (PS3) completeness. Trivially, uniqueness holds in R° because it holds in R and $R^\circ \subseteq R$. Concerning completeness, given any $a \in S$ and $\pi \in \Omega$, by completeness of R there exists $b \in S$ such that $(a,\pi,b) \in R$. Hence, by definition, it also belongs to R°.

Theorem 2. *Given any two ternary relations $R, U \subseteq S \times \Omega^* \times S$, the relation U is the transitive closure of R if and only if $U = R^\bullet$, where*

$$R^\bullet = \bigcup_{i \in \mathbb{Z}^+} R^i,$$

and $\mathbb{Z}^+$ is the set of the positive integer numbers.

Proof. The proof will consists of showing that (i) $R^\bullet$ includes R, (ii) it is transitive and (iii) it is minimal. (i) The fact that $R \subseteq R^\bullet$ is trivial, since $R^\bullet$ is the union of all the R^i, including R itself. (ii) If $(a, \pi, b), (b, \sigma, c) \in R^\bullet$ then by definition of $R^\bullet$ there are some numbers $j, k > 0$ such that $(a, \pi, b) \in R^j$ and $(b, \sigma, c) \in R^k$. Since composition $\bullet$ is associative, $R^{k+j} = R^k \bullet R^j$, hence $(a, \pi\sigma, c) \in R^{k+j}$ and, $R^{k+j} \subseteq R^\bullet$ by definition. Therefore, $R^\bullet$ is transitive. Finally, (iii) $R^\bullet$ is minimal, if given any transitive relation X, $R \subseteq X$ implies $R^\bullet \subseteq X$. Given any of such X, induction on i can be used to show that $R^i \subseteq X$ for all the i. The case $R \subseteq X$ in which $i = 1$ holds by assumption. If $R^i \subseteq X$ holds, and $(a, \pi\sigma, c) \in R^{i+1}$, then $(a, \pi, b) \in R$ and $(b, \sigma, c) \in R^i$ for some $b \in S$, by definition of $\bullet$. Therefore, given the premises $(a, \pi, b), (b, \sigma, c) \in X$ and $(a, \pi\sigma, c) \in X$ by transitivity. It follows that $R^{i+1} \subseteq X$. Finally, since $R^i \subseteq X$ for all i implies $R^\bullet \subseteq X$. □

Theorem 3. *Let $\mathbf{P} = (S, \Omega, R)$ be a problem space and $I = \{(a, \epsilon, a) : a \in S\}$. Given any $U \subseteq S \times \Omega^* \times S$ the following two conditions are equivalent:*

(1) $R = U^\bullet \cup I$,
(2) $R^\circ \subseteq U \subseteq R$.

Proof. (2) $\implies$ (1). Let $(a, \pi, b) \in (U^\bullet \cup I) \vartriangle R$, with $\vartriangle$ the symmetric difference of two sets. Then, there are two cases: If $\pi \in \Omega$ then obviously $(a, \pi, b) \in R^\circ \subseteq R \cap (U^\bullet \cup I)$, otherwise, $\pi = \omega_1 \dots \omega_n$, with $\omega_1, \dots \omega_n \in \Omega$. Therefore there are $s_1, \dots s_{n-1} \in S$ such that $(a, \omega_1, s_1) \dots (s_{n-1}, \omega_n, b) \in R^\circ$. Thus, by transitivity $(a, \pi, b) \in (U^\bullet \cup I) \cap R$ and $(U^\bullet \cup I) \vartriangle R$ must be empty implying that $U^\bullet \cup I = R$. (1) $\implies$ (2). Given that $R = U^\bullet \cup I$ and $U \subseteq U^\bullet$, it immediately follows from $(a, \pi, b) \in U$ that $(a, \pi, b) \in R$. Otherwise, suppose that there exists a $(a, \pi, b) \in R^\circ \setminus U$. Since $\pi \in \Omega$, there cannot be (a, σ, c) and $(c, \alpha, b) \in U^\bullet \cup I$ such that $\pi = \sigma\alpha$. Thus, $(a, \pi, b) \notin U^\bullet \cup I$. □

It should be observed that Theorem 3 implies the uniqueness of the reduction of any ternary relation $R \subseteq S \times \Omega^* \times S$. In fact, suppose that there exist two subsets R' and R'' of R, that satisfy Definition 3. Then it follows from Condition (2) of Theorem 3, that $R' \subseteq R''$ and $R'' \subseteq R'$, which imply $R' = R''$, showing that the reduction is unique.

4. Problem space homomorphisms

Problem spaces are formal representations of the possible solution ways of a problem or of a family of problems. Especially in case the problem space is complex and huge, it is unlikely that human individuals solve problems by using a cognitive representation that exactly reproduces the "concrete problem space". It is more likely that such a representation undergoes various types of simplifications, missing all those details of the reference problem space that are not relevant to the problem solution.

Although complex, concrete problem spaces are often characterized by various types of "symmetries". To give an example, in the problem space of the Tower of London (ToL; Shallice, 1982) test every problem state could be described as a 6-tuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ where x_1, x_2 and x_3 represent the three available positions on the left peg, x_4 and x_5 represent the two available positions on the central peg, and x_6 represents the only available position on the right peg. Additionally, the set $\{0, 1, 2, 3\}$ is used to represent the three colored balls, with 'no ball' represented by 0. In the 6-tuple representing a ToL state, the specific assignment of colors – for instance, numbers 1 for black, 2 for red, and 3 for green – is completely inessential with respect to finding a solution. Therefore, the problem that consists of transforming the state $(2, 3, 0, 0, 0, 1)$ into the state $(1, 2, 3, 0, 0, 0)$ is "equivalent up to color permutation" to the problem of transforming the state $(3, 1, 0, 0, 0, 2)$ into the state $(2, 3, 1, 0, 0, 0)$.

Symmetries allow to construct of simpler and more abstract representations of the concrete problem space. The construction of an abstract representation cannot occur in total arbitrariness, however. There must be a guarantee that any solution of any problem in the abstract representation can be tracked down to a solution of a problem in the concrete problem space. This section is about the link between the concrete and the abstract problem space. It takes on the form of a special type of homomorphism between two problem spaces.

Definition 4. A *weak homomorphism* of a (incomplete) problem space (S_1, Ω_1, R_1) into another (incomplete) problem space (S_2, Ω_2, R_2) is a pair (ϕ, γ) of mappings $\phi : S_1 \to S_2$ and $\gamma : \Omega_1^* \to \Omega_2^*$ such that, for $a, b \in S_1$, $\pi, \sigma \in \Omega_1^*$,

(1) $(a, \pi, b) \in R_1 \implies (\phi(a), \gamma(\pi), \phi(b)) \in R_2$;
(2) $\gamma(\pi)\gamma(\sigma) = \gamma(\pi\sigma)$.

The pair of mappings (ϕ, γ) is a *strong homomorphism* if it is a weak homomorphism such that, for every $a, b \in S_1$ and every $\pi \in \Omega_1^*$,

(3) $(\phi(a), \gamma(\pi), \phi(b)) \in R_2 \implies \exists c \in S_1 : (a, \pi, c) \in R_1, \phi(b) = \phi(c)$.

Moreover, the homomorphism is an isomorphism if both ϕ and γ are bijections. Furthermore, the mapping γ is said to be a *monoid homomorphism* if $\gamma(\pi)\gamma(\sigma) = \gamma(\pi\sigma)$ for all $\pi, \sigma \in \Omega_1^*$.

Condition (3) of the definition of a strong homomorphism requires that the operation is preserved bidirectionally, as in a standard homomorphism. However, there are several empirical situations where certain problem equivalences are expected or required to hold, even though they violate Condition (3) of a strong homomorphism. The introduction of a weak homomorphism allows for preserving the operation only in one direction, from the larger problem space to the smaller one, while maintaining the required problem equivalences.

The following series of three examples further clarifies these scenarios and illustrates the impact on the problem space. Specifically, they illustrate simple cases where the presence of ineffective triples in a problem space makes it impossible to establish a homomorphism. However, by removing the ineffective triples, while maintaining the problem space essentially the same, a weak homomorphism can be established. Furthermore, the examples demonstrate a situation where the strong homomorphism does not hold due to the lack of completeness (PS3), and other situations where the strong homomorphism holds even between incomplete problem spaces.

Example 1. Let $\mathbf{P}_1 = (S, \Omega, R_1)$, $\mathbf{P}_2 = (S, \Omega, R_2)$, $\mathbf{P}_3 = (S', \Omega', R_3)$ be three problem spaces with $S = \{a, b, c, d\}$, $\Omega = \{\alpha, \beta\}$, $S' = \{e, f\}$, $\Omega' = \{\delta\}$ and

$$R_1^\circ = \{(a, \alpha, b), (a, \beta, a), (c, \alpha, c), (c, \beta, d), (b, \alpha, b), (b, \beta, b), (d, \alpha, d), (d, \beta, d)\},$$
$$R_2^\circ = \{(a, \alpha, b), (c, \beta, d)\},$$
$$R_3^\circ = \{(e, \delta, f)\}$$

are the reductions of R_1, R_2, and R_3, respectively. It should be observed that, up to deletion of ineffective moves (triples), problem space $\mathbf{P}_1$ and incomplete problem space $\mathbf{P}_2$ are essentially the same. Namely, $\mathbf{P}_2$ is obtained from $\mathbf{P}_1$ by deleting from R_1° all triples of the form (s, ω, s) with $s \in S$ and $\omega \in \Omega$.

Let $(\phi : S \to S', \gamma : \Omega \to \Omega')$ be a pair of mappings such that $\phi(a) = \phi(c) = e$, $\phi(b) = \phi(d) = f$, and $\gamma(\alpha) = \gamma(\beta) = \delta$. It is easily seen that (ϕ, γ) is a weak homomorphism between $\mathbf{P}_2$ and $\mathbf{P}_3$. In fact, one has $(\phi(a), \gamma(\alpha), \phi(b)) = (\phi(c), \gamma(\beta), \phi(d)) = (e, \delta, f) \in R_3$. It is not a strong homomorphism because $(\phi(a), \gamma(\beta), \phi(b)) = (e, \delta, f) \in R_3$, however there is no $z \in S$ with $(a, \beta, z) \in R_2$ and $\phi(z) = \phi(b)$.

On the other side, concerning $\mathbf{P}_1$ and $\mathbf{P}_3$, we observe that $(a, \beta, a) \in R_1$, however $(\phi(a), \gamma(\beta), \phi(a))$ is not in R_3, showing that (ϕ, γ) is not a homomorphism between $\mathbf{P}_1$ and $\mathbf{P}_3$. It can be shown, indeed, that no homomorphism at all can be established from $\mathbf{P}_1$ to $\mathbf{P}_3$. In fact, considering that (a, β, a) is in R_1, whatever value one assigns to $\phi(a)$, $(\phi(a), \gamma(\beta), \phi(a)) \in R_3$ would imply $e = f$, which is false.

Example 2. An applicative example of the concepts presented so far is the Tower of London test. This test involves three pegs of different heights and three balls of different colors that can be moved between the pegs according to certain rules. All possible configurations of the balls are displayed in Fig. 4, and a detailed explanation of the problem space of the test can be found in Section 6.2.

Regarding problem spaces $\mathbf{P}_1$ and $\mathbf{P}_2$ used in Example 1, we assume that problem states a and c are, respectively, the configuration shown in row 3 and column 1, and the configuration shown in row 3 and column 3 of Fig. 4. Similarly, b and d are, respectively, the configuration in row 4, column 1, and the configuration in row 4, column 3. The operation α refers to moving the red ball from the leftmost peg to the center peg, while β refers to moving the black ball from the leftmost peg to the center peg. Since, in configuration a, the black ball is already in the center peg, operation β is ineffective on it. Similarly, in c the red ball is below the black ones and cannot be moved, hence operation α is ineffective on c.

On the other hand regarding incomplete problem space $\mathbf{P}_3$, the problem state $e \in S'$ is the ToL configuration in which two balls are on the leftmost peg and one is on the center peg, regardless of the colors. Likewise, $f \in S'$ is the configuration in which two balls are on the center peg and one is on the leftmost peg, regardless of the colors. Lastly, the operation δ refers to moving a single ball from the leftmost peg to the center peg.

The homomorphism (ϕ, γ) establishes an equivalence up to color permutation between problem states and operations, respectively. This equivalence is likely to be psychologically relevant. However, as seen in the previous example, it is not possible to define it as long as ineffective moves are considered. Therefore, a incomplete problem space is needed, in which such moves do not exist.

The third example presents two situations in which a strong homomorphism can be defined between problem spaces without incurring into the issues presented previously.

Example 3. Let $\mathbf{P}_3 = (S', \Omega', R_3)$, $P_4 = (S, \Omega, R_4)$, $P_5 = (S, \Omega, R_5)$, be three incomplete problem spaces with $S = \{a, b, c, d\}$, $\Omega = \{\alpha, \beta\}$, $S' = \{e, f\}$, $\Omega' = \{\delta\}$ and

$$R_3^\circ = \{(e, \delta, f)\},$$
$$R_4^\circ = \{(a, \alpha, b), (a, \beta, d), (c, \alpha, b), (c, \beta, d)\},$$
$$R_5^\circ = \{(a, \alpha, b), (a, \beta, b), (c, \alpha, d), (c, \beta, d)\},$$

are the reductions of R_3, R_4, and R_5, respectively. Let $(\phi : S \to S', \gamma : \Omega \to \Omega')$ be such that $\phi(a) = \phi(c) = e$, $\phi(b) = \phi(d) = f$, and $\gamma(\alpha) = \gamma(\beta) = \delta$. Then (ϕ, γ) is a strong homomorphism from $\mathbf{P}_4$ to $\mathbf{P}_3$ and from $\mathbf{P}_5$ to $\mathbf{P}_3$. Concerning $\mathbf{P}_4$ and $\mathbf{P}_3$, one has $(\phi(a), \gamma(\alpha), \phi(b)) = (\phi(a), \gamma(\beta), \phi(d)) = (\phi(c), \gamma(\alpha), \phi(b)) = (\phi(c), \gamma(\beta), \phi(d)) = (e, \delta, f) \in R_4$. On the other side, from $(e, \delta, f) \in R_3$ we have that all the following triples are in R_3:

$$(\phi(a), \gamma(\alpha), \phi(b)), \qquad (\phi(a), \gamma(\alpha), \phi(d)),$$
$$(\phi(a), \gamma(\beta), \phi(b)), \qquad (\phi(a), \gamma(\beta), \phi(d)),$$
$$(\phi(c), \gamma(\alpha), \phi(b)), \qquad (\phi(c), \gamma(\alpha), \phi(d)),$$
$$(\phi(c), \gamma(\beta), \phi(b)), \qquad (\phi(c), \gamma(\beta), \phi(d)).$$

If $(\phi(x), \gamma(v), \phi(y))$ is any of such triples, then it has to be shown that there is $z \in S$ such that (x, v, z) is in R_4. For $(\phi(a), \gamma(\alpha), \phi(b))$ such triple is exactly (a, α, b); for $(\phi(a), \gamma(\alpha), \phi(d))$ it is again (a, α, b). Being now straightforward, the check for the rest of the triples is left to the reader. A similar check can be applied for verifying that (ϕ, γ) is a strong homomorphism from $\mathbf{P}_5$ to $\mathbf{P}_3$.

In the next Lemma it is shown that the neutral element ϵ of the monoid Ω^* is preserved under the weak homomorphism.

Lemma 1. *Given two (incomplete) problem spaces $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$, if (ϕ, γ) is a weak homomorphism from $\mathbf{P}_1$ to $\mathbf{P}_2$, then $\gamma(\epsilon) = \epsilon$.*

Proof. By contradiction, suppose that $\gamma(\epsilon) = \omega \neq \epsilon$. Since ϵ is the neutral element of the monoid Ω_1^*, it holds that $\epsilon\pi = \pi\epsilon = \pi$ for all $\pi \in \Omega_1^*$. Let $\sigma = \gamma(\pi)$. It follows from Definition 4 that $\gamma(\pi)\gamma(\epsilon) = \sigma\omega \neq \omega = \gamma(\pi\epsilon)$. Thus, $\pi\epsilon = \pi$, but $\gamma(\pi\epsilon) \neq \gamma(\pi)$ which contradicts the assumption that γ is a function. □

Obviously, Lemma 1 holds true also with a strong homomorphism. The next theorem provides a characterization of the strong homomorphism between problem spaces. This result reconciles our definition of a strong homomorphism with the standard definition of a homomorphism given in (1).

Theorem 4. *If $P_1 = (S_1, \Omega_1, \cdot)$ and $P_2 = (S_2, \Omega_2, \star)$ are problem spaces, then the pair of mappings $(\phi : S_1 \to S_2, \gamma : \Omega_1^* \to \Omega_2^*)$ is a strong homomorphism of P_1 into P_2 if and only if, for all $a, b \in S_1$ and $\pi, \sigma \in \Omega_1^*$,*

(i) $\phi(a \cdot \pi) = \phi(a) \star \gamma(\pi)$;
(ii) $\gamma(\pi\sigma) = \gamma(\pi)\gamma(\sigma)$.

Proof. Define the ternary relation R_1 by setting, for any $a, b \in S_1$, and $\pi \in \Omega_1^*$, $(a, \pi, b) \in R_1$ iff $a \cdot \pi = b$. Analogously, define R_2 by setting, for any $a', b' \in S_2$ and any $\pi' \in \Omega_2^*$, $(a', \pi', b') \in R_2$ iff $a' \star \pi' = b'$. Sufficiency: suppose that Conditions (i) and (ii) of Theorem 4 hold. We want to show that (ϕ, γ) is a strong homomorphism in the sense of Definition 4. Condition (2) of Definition 4 immediately follows from (ii). Given the premises $(a, \pi, b) \in R_1$ of Condition (1) of the same definition, it immediately follows from the definition of R_1 that $a \cdot \pi = b$ and hence $\phi(a \cdot \pi) = \phi(b)$. . Thus, for Condition (i), $\phi(a) \star \gamma(\pi) = \phi(b)$, and hence $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$. This shows that Condition (1) of Definition 4 holds. Given the premises $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$ of Condition (3) of Definition 4, it follows from the definition of R_2 that $\phi(a) \star \gamma(\pi) = \phi(b)$.

Thus, for Condition (i), $\phi(a \cdot \pi) = \phi(b)$. This last equality holds because $(S_1, \Omega_1, \cdot)$ is complete. Therefore there is $c \in S_1$ with $c = a \cdot \pi$ and $\phi(c) = \phi(b)$. Finally, from the definition of R_1 it follows that $(a, \pi, c) \in R_1$. Hence Condition (3) holds.

Necessity: suppose that Conditions (1), (2), and (3) of Definition 4 hold true, so that (ϕ, γ) is a strong homomorphism. Condition (ii) obviously follows from Condition (2). Concerning (i), assume that, for $a, b \in S_1$ and $\pi \in \Omega_1^*$ it holds that $\phi(a \cdot \pi) = \phi(b)$. Then there exists $c \in S_1$ such that $a \cdot \pi = c$, and $\phi(c) = \phi(b)$. Thus, it follows from Condition (1) of Definition 4 that $\phi(a) \star \gamma(\pi) = \phi(c)$, and since $\phi(c) = \phi(b)$, we obtain $\phi(a) \star \gamma(\pi) = \phi(b)$. Thus we have shown that, for any $a, b \in S_1$ and any $\pi \in \Omega_1^*$, $\phi(a \cdot \pi) = \phi(b)$ implies $\phi(a) \star \gamma(\pi) = \phi(b)$. Conversely, assume now that $\phi(a) \star \gamma(\pi) = \phi(b)$, that is $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$. Then, by Condition (3), there is $c \in S_1$ such that $(a, \pi, c) \in R_1$ and $\phi(c) = \phi(b)$. That is $a \cdot \pi = c$ and $\phi(c) = \phi(b)$, hence $\phi(a \cdot \pi) = \phi(c) = \phi(b)$. We have thus shown that, for any $a, b \in S_1$ and $\pi \in \Omega_1^*$, $\phi(a) \star \gamma(\pi) = \phi(b)$ implies $\phi(a \cdot \pi) = \phi(b)$. Overall, Conditions (1) and (3) imply that for all $a, b \in S_1$ and all $\pi \in \Omega_1^*$, the equivalence $\phi(a \cdot \pi) = \phi(b) \iff \phi(a) \star \gamma(\pi) = \phi(b)$ holds true. Lastly, it is immediate that $\phi(a) \star \gamma(\pi) = \phi(b)$ for some $b \in S_1$. In fact, by setting $\phi(a) \star \gamma(\pi) = c$ for some $c \in S_2$, we have $(\phi(a), \gamma(\pi), c) \in R_2$. Since $(S_1, \Omega_1, \cdot)$ is complete, there must be $b \in S_1$ with $(a, \pi, b) \in R_1$, that is $a \cdot \pi = b$. Hence $\phi(a \cdot \pi) = \phi(b)$. For Condition (1), it follows that $\phi(a) \star \gamma(\pi) = \phi(b)$. Hence $c = \phi(b)$. We thus conclude that $\phi(a \cdot \pi) = \phi(a) \star \gamma(\pi)$. □

Further, we characterize the (incomplete) problem space isomorphism.

Theorem 5. *Let (S_1, Ω_1, R_1) and (S_2, Ω_2, R_2) be two (incomplete) problem spaces, and let $\phi : S_1 \to S_2$ and $\gamma : \Omega_1^* \to \Omega_2^*$ be bijections. The pair (ϕ, γ) is an isomorphism between (S_1, Ω_1, R_1) and (S_2, Ω_2, R_2) if and only if, for all $a, b \in S_1$ and all $\pi \in \Omega_1^*$,*

$$(a, \pi, b) \in R_1 \iff (\phi(a), \gamma(\pi), \phi(b)) \in R_2,$$

and γ is a monoid homomorphism.

Proof. Assume that (ϕ, γ) is an isomorphism and that $(a, \pi, b) \in R_1$ for $a, b \in S_1$ and $\pi \in \Omega_1^*$. Then, obviously, $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$. On the contrary, assume $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$. Then there is $c \in S_1$ such that $(a, \pi, c) \in S_1$ and $\phi(c) = \phi(b)$. By the bijectivity of ϕ, this implies $c = b$. Hence $(a, \pi, b) \in R_1$. □

The strong homomorphism (ϕ, γ) induces equivalence relations $\sim_\phi$ and $\sim_\gamma$ on the two sets S_1, and Ω_1^* respectively. The obvious definitions are as follows: Given $a, b \in S_1$, $a \sim_\phi b$ iff $\phi(a) = \phi(b)$; given $\pi, \sigma \in \Omega^*$, $\pi \sim_\gamma \sigma$ iff $\gamma(\pi) = \gamma(\sigma)$. In the whole, these equivalence relations set up a *congruence*. Precisely, for any $a, b \in S_1$, and any $\pi, \sigma \in \Omega_1^*$, if $a \sim_\phi b$ and $\pi \sim_\gamma \sigma$ then $a \cdot \pi \sim_\phi b \cdot \sigma$. Similarly, if $\pi, \pi', \sigma, \sigma' \in \Omega_1^*$, $\pi \sim_\gamma \pi'$ and $\sigma \sim_\gamma \sigma'$ then $\pi\sigma \sim_\gamma \pi'\sigma'$. This allows to extend the dot operation $\cdot$ to equivalence classes. Namely, for $s \in S_1$ and $\pi \in \Omega_1^*$, equivalence classes are defined as $[s] = \{t \in S_1 : s \sim_\phi t\}$, and $[\pi] = \{\sigma \in \Omega_1^* : \sigma \sim_\gamma \pi\}$. Then $\cdot$ is extended as follows:

$$[a] \cdot [\pi] = [a \cdot \pi],$$

meaning that, for every state $a' \in [a]$, there exists a $\pi' \in [\pi]$ such that $a' \cdot \pi' \in [a \cdot \pi]$.

Things work rather differently with a weak homomorphism, for which a weaker form of "congruence" exists, as the following theorems state.

Theorem 6. *Let $P_1 = (S_1, \Omega_1, R_1)$ and $P_2 = (S_2, \Omega_2, R_2)$ be incomplete problem spaces, and (ϕ, γ) be a weak homomorphism of P_1 into P_2. Let moreover $a, a', b, b' \in S_1$, and $\pi, \pi' \in \Omega_1^*$. If $a \sim_\phi a'$, $\pi \sim_\gamma \pi'$, and both $R_1(a, \pi)$ and $R_1(a', \pi')$ exist, then $R_1(a, \pi) \sim_\phi R_1(a', \pi')$.*

Proof. If both $R_1(a, \pi)$ and $R_1(a', \pi')$ exist then

$$\phi(R_1(a, \pi)) = R_2(\phi(a), \gamma(\pi)) = R_2(\phi(a'), \gamma(\pi')) = \phi(R_1(a', \pi')).$$

Hence $R_1(a, \pi) \sim_\phi R_1(a', \pi')$. □

While with (strong) homomorphisms, it always holds true that, for any $a \in S$ and any $\pi \in \Omega_1^*$, $[a]\cdot[\pi] = [a\cdot\pi]$, this does not hold in general with weak homomorphisms.

Define the ternary relation $\mathcal{R}_1 \subseteq S_1/\phi \times \Omega_1^*/\gamma \times S_1/\phi$, by setting, for any $a, b \in S_1$ and any $\pi \in \Omega_1^*$,

$$([a], [\pi], [b]) \in \mathcal{R}_1 \iff ([a] \times [\pi] \times [b]) \cap R_1 \neq \emptyset.$$

We say that $\mathcal{R}_1([a], [\pi])$ *exists* iff there is $[b] \in S_1/\phi$ with $([a], [\pi], [b]) \in \mathcal{R}_1$. It should be noticed that $([a], [\pi], [b]) \in \mathcal{R}_1$ if and only if $(a', \pi', b') \in R_1$ for some $a' \in [a]$, $\pi' \in [\pi]$, and $b' \in [b]$.

Theorem 7. *For $a \in S_1$ and $\pi \in \Omega_1^*$, $\mathcal{R}_1([a], [\pi])$ exists iff $R_1(a', \pi')$ exists for some $a' \in [a]$ and some $\pi' \in [\pi]$. In that case,*

$$\mathcal{R}_1([a], [\pi]) = [R_1(a', \pi')].$$

Proof. Assume $\mathcal{R}_1([a], [\pi])$ exists. Then there is $[b] \in S_1/\phi$ such that $([a], [\pi], [b]) \in \mathcal{R}_1$. This holds true iff there are $a' \in [a]$, $b' \in [b]$, and $\pi' \in [\pi]$ such that $(a', \pi', b') \in R_1$. Therefore $R_1(a', \pi')$ must exist. Assume now that there exist $a' \in [a]$ and $\pi' \in [\pi]$ such that $R_1(a', \pi')$ exists. Then there is $b \in S_1$ such that $(a', \pi', b) \in R_1$. Hence $([a'], [\pi'], [b]) \cap R_1 \neq \emptyset$. Thus $([a'], [\pi'], [b]) \in \mathcal{R}_1$ and, hence $([a], [\pi], [b]) \in \mathcal{R}_1$. We thus conclude that $\mathcal{R}_1([a], [\pi])$ exists. Moreover, suppose by contradiction there exists $[b'] \in S_1/\phi$ with $([a], [\pi], [b']) \in \mathcal{R}_1$. There are thus $(a, \pi, b) \in R_1 \cap ([a] \times [\pi] \times [b])$, and $(a, \pi, b') \in R_1 \cap ([a] \times [\pi] \times [b'])$. By the uniqueness property of R_1, we must have $b = b'$. Hence $[b] = [b']$. Thus we obtain $\mathcal{R}_1([a], [\pi]) = [b] = [R_1(a, \pi)]$. □

5. Algorithms for testing problem space homomorphisms

In this section, two algorithms are presented. Given two problem spaces $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$, the former algorithm tests whether a given pair $(\phi : S_1 \to S_2, \gamma : \Omega_1^* \to \Omega_2^*)$ of mappings is a (either weak or strong) homomorphism of $\mathbf{P}_1$ into $\mathbf{P}_2$. This is specifically accomplished by examining the reductions of R_1 and R_2. The advantage of working with reductions is that they are smaller and thus easier to handle. If (ϕ, γ) is a weak homomorphism, then the latter algorithm transforms it into a strong homomorphism by adding the smallest possible number of triples to the reduction of R_1.

5.1. Testing the problem space homomorphism

In this section an algorithm is proposed which tests if a pair (ϕ, γ), where $\phi : S_1 \to S_2$, and $\gamma : \Omega_1^* \to \Omega_2^*$, is a homomorphism between two (incomplete) problem spaces $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$. Such algorithm works by testing the homomorphism on the two reductions $R_1^\circ \subseteq R_1$ and $R_2^\circ \subseteq R_2$. The following two collections are defined, which will be used in the sequel. The pre-image of ϕ is the mapping ϕ^{-1} with S_2 as domain and 2^{S_1} as codomain such that, for any $a \in S_2$:

$$\phi^{-1}(a) = \{a' \in S_1 : \phi(a') = a\},$$

and the pre-image of γ is the mapping γ^{-1} with Ω_2^* as domain and $2^{\Omega_1^*}$ as codomain such that for any $\pi \in \Omega_2^*$

$$\gamma^{-1}(\pi) = \{\pi' \in \Omega_1^* : \gamma(\pi') = \pi\}.$$

Theorem 8 establishes that the existence of a weak homomorphism between the two reductions is necessary and, if γ is a monoid homomorphism, also sufficient for having a weak homomorphism between the two problem spaces.

Theorem 8. *Let $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$, $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ be two incomplete problem spaces. Consider the pair (ϕ, γ) of mappings $\phi : S_1 \to S_2$, and $\gamma : \Omega_1^* \to \Omega_2^*$ where γ is a monoid homomorphism. The implication*

$$(a, \pi, b) \in R_1^\circ \implies (\phi(a), \gamma(\pi), \phi(b)) \in R_2^\circ \tag{2}$$

holds true for all $(a, \pi, c) \in S_1 \times \Omega_1^* \times S_1$, if and only if (ϕ, γ) is a weak homomorphism from $\mathbf{P}_1$ to $\mathbf{P}_2$.

Proof. Assume that Condition (2) holds true. To show that (ϕ, γ) is a weak homomorphism, it suffices to show that

$$(a, \pi, c) \in R_1 \implies (\phi(a), \gamma(\pi), \phi(c)) \in R_2 \tag{3}$$

holds for all $(a, \pi, c) \in S_1 \times \Omega^* \times S_1$. Suppose at first that $\pi \in \Omega_1$. In this case since $R_1^\circ \subset R_1$, from Condition (2) it follows that $(\phi(a), \gamma(\omega), \phi(b)) \in R_2^\circ$. Hence, $(\phi(a), \gamma(\omega), \phi(b)) \in R_2$ immediately follows from $R_2^\circ \subseteq R_2$.

Suppose now that $\pi \in \Omega_1^* \setminus (\Omega_1 \cup \{\epsilon\})$. In that case, there exist $\sigma \in \Omega_1^*$, and $\omega \in \Omega_1$ such that $\pi = \sigma\omega$. By (PS5) exists a $b \in S_1$ with $(a, \sigma, b), (b, \omega.c) \in R_1$. Assume that $(\phi(a), \gamma(\sigma), \phi(b)) \in R_2$, from Condition (2) and the fact that $R_2^\circ \subseteq R_2$ it follows that $(\phi(b), \gamma(\omega), \phi(c)) \in R_2$. By the transitivity property (PS4), this implies $(\phi(a), \gamma(\sigma\omega), \phi(c)) \in R_2$.

The converse implication that Condition (2) holds true if (ϕ, γ) is a weak homomorphism immediately follows from the fact that $R_1^\circ \subset R_1$ and $R_2^\circ \subset R_2$ and from the definition of a weak homomorphism itself. □

Theorem 9, on the other hand, establishes that a pair of functions is a strong homomorphism between two problem spaces if and only if it is a strong homomorphism between their reductions. The theorem immediately follows from the definitions of strong and weak homomorphisms, and from that of a reduction.

Theorem 9. *Let $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$, $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ be two incomplete problem spaces. The pair (ϕ, γ) of mappings $\phi : S_1 \to S_2$, and $\gamma : \Omega_1^* \to \Omega_2^*$, is a strong homomorphism from $\mathbf{P}_1$ into $\mathbf{P}_2$ if and only if it is a weak homomorphism, such that the implication*

$$(\phi(a), \gamma(\pi), \phi(b)) \in R_2^\circ \implies \exists c \in S_1 : (a, \pi, c) \in R_1^\circ, \phi(b) = \phi(c) \tag{4}$$

holds true for all $a, b \in S_1$ and all $\pi \in \Omega_1^*$.

The algorithm which tests the homomorphism between two incomplete problem spaces $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ consists of three steps:

1. The two reductions R_1° and R_2° are obtained from R_1 and R_2 respectively.
2. The algorithm iterates through R_1°, and for each triple $(a_1, \omega_1, b_1) \in R_1^\circ$ the following condition is tested:
$$|\{(a_2, \omega_2, b_2) \in R_2^\circ \ : \ (\phi(a_1), \gamma(\omega_1), \phi(b_1)) = (a_2, \omega_2, b_2)\}| = 1. \tag{5}$$
If Condition (5) is found to be false for anyone of the elements in R_1° then, the algorithm terminates and it is concluded that (ϕ, γ) is neither a weak nor a strong homomorphism. Otherwise, step 3 is entered.
3. The algorithm iterates through R_2°, and for each triple $(a_2, \omega_2, b_2) \in R_2^\circ$ the following condition is tested for each $(a', \omega') \in \phi^{-1}(a_2) \times \gamma^{-1}(\omega_2)$:
$$\{b \in \phi^{-1}(b_2) : (a', \omega', b) \in R_1^\circ\} \neq \emptyset \tag{6}$$
If Condition (6) is found to be false for any triple in R_2° then, the algorithm terminates and it is concluded that (ϕ, γ) is a weak homomorphism. Otherwise, if Condition (6) is found to be true for all triples in R_2°, then it is concluded that (ϕ, γ) is a strong homomorphism.

Step 1 consist in the construction of the reductions R_1° and R_2° of the two relations R_1 and R_2 respectively. Step 2 tests if each element in the reduction R_1° is mapped to a single element in R_2°. By Theorem 8 this is a sufficient condition for (ϕ, γ) being a weak homomorphism. Finally, Step 3 tests that given any (s_2, ω_2, t_2), for each pair $(s_1, \omega_1) \in \phi^{-1}(s_2) \times \gamma^{-1}(\omega_2)$ there is a problem state t_1, with $\phi(t_1) = t_2$ such that (s_1, ω_1, t_1). If this condition is false, the condition for the strong homomorphism does not hold true.

5.2. Local completion of a problem space

Sometimes, only a weak homomorphism from an incomplete problem space to another one can be found. The minimum number of changes that must be applied to the original problem space to obtain a strong homomorphism is studied in this section. The problem space resulting from these changes is called a "local completion" of the original problem space. The term "local" is used to express that these changes apply to certain localized areas of the problem space. In addition, an algorithm that constructs the local completion of a problem space is described.

Definition 5. Let $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ and $\mathbf{P}_3 = (S_1, \Omega_1, R_3)$ be three (incomplete) problem spaces and (ϕ, γ) be a weak homomorphism from $\mathbf{P}_1$ to $\mathbf{P}_2$. Then, $\mathbf{P}_3 = (S_1, \Omega_1, R_3)$ is a *local completion* of $\mathbf{P}_1$ if the following conditions hold:

(LC1) (ϕ, γ) is a strong homomorphism from $\mathbf{P}_3$ to $\mathbf{P}_2$,
(LC2) $R_1 \subseteq R_3$.

Moreover, the local completion $\mathbf{P}_3$ is said to be *minimal* if there is no other local completion strictly included in it. Furthermore, $\mathbf{P}_3$ is said to be a *least local completion* if the distance $|R_3^\circ \Delta R_1^\circ|$, where R_3° and R_1° are the reductions of R_3 and R_1, is minimum in the set of all the local completions of R_1.

It should be noticed that the local completion of a problem space need not be unique. As a counterexample, in Examples 1 and 3, the pair (ϕ, γ) is a weak homomorphism from $\mathbf{P}_2$ to $\mathbf{P}_3$, whereas it is a strong homomorphism from $\mathbf{P}_4$ to $\mathbf{P}_3$ and also from $\mathbf{P}_5$ to $\mathbf{P}_3$. It can be easily seen that both $\mathbf{P}_4$ and $\mathbf{P}_5$ are local completions of $\mathbf{P}_2$, since $R_2 = R_4 \cap R_5$.

Theorem 10. *If (ϕ, γ) is a weak homomorphism from $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ to $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$, and $\mathbf{P}_1$ is a problem space, then (ϕ, γ) is also a strong homomorphism.*

Proof. It is sufficient to show that for all $a, b \in S_1$ and all $\pi \in \Omega_1$ if $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$, then there exists $c \in S_1$ such that $(a, \pi, c) \in R_1$ and $\phi(c) = \phi(b)$. The existence of $(a, \pi, c) \in R_1$ for some $c \in S$ follows immediately from completeness (PS3). In addition, $(\phi(a), \gamma(\pi), \phi(c)) \in R_2$ since (ϕ, γ) is a weak homomorphism by premises. Finally, since $(\phi(a), \gamma(\pi), \phi(c)) \in R_2$ and, $(\phi(a), \gamma(\pi), \phi(b)) \in R_2$, $\phi(c) = \phi(b)$ by uniqueness (PS2). Therefore, (ϕ, γ) is a strong homomorphism. □

A trivial consequence of Theorem 10 is that a problem space that satisfies the completeness property also satisfies the local completion conditions with respect to any homomorphism (ϕ, γ). Although, the inverse implication does not necessarily hold.

Example 4. Let $\mathbf{P}_1 = (S, \Omega, R)$, $\mathbf{P}_2 = (S', \Omega', R')$, be two incomplete problem spaces with $S = \{a, b, c, d\}$, $\Omega = \{\alpha, \beta, \delta\}$, $S' = \{e, f, g\}$, $\Omega' = \{\pi, \sigma\}$ and

$$R = \{(a, \alpha, b), (a, \beta, d), (c, \delta, d)\},$$
$$R' = \{(e, \pi, f), (g, \sigma, f)\}.$$

Let $(\phi : S \to S', \gamma : \Omega \to \Omega')$ be such that $\phi(a) = e$, $\phi(b) = \phi(d) = f$, $\phi(c) = g$, $\gamma(\alpha) = \gamma(\beta) = \pi$, and $\gamma(\delta) = \sigma$. The reader can easily see that (ϕ, γ) is a strong homomorphism from $\mathbf{P}_1$ to $\mathbf{P}_2$. Thus, $\mathbf{P}_1$ is a local completion (of itself) with respect to (ϕ, γ). Nevertheless, it is not a problem space since $R(a, \delta)$, $R(b, \alpha)$, and $R(b, \beta)$ do not exist.

The rest of the section concerns the development of an algorithm that performs local completions, and it is named *local completion algorithm* (LC algorithm). Let $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ and $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ be two incomplete problem spaces and (ϕ, γ) a weak homomorphism from $\mathbf{P}_1$ to $\mathbf{P}_2$.

The algorithm consists of two nested loops. At the outset, it sets $U_{0,0} = R_1^\circ$. The outer loop iterates through the triples in R_2°. In each iteration $i > 0$ of the outer loop, a new triple $(a, \omega, b) \in R_2^\circ$ is considered. The inner loop iterates through the pairs in $\phi^{-1}(a) \times \gamma^{-1}(\omega)$. In each iteration j, a new pair $(a', \omega') \in \phi^{-1}(a) \times \gamma^{-1}(\omega)$ is considered, and the following condition is tested:

$$\{b' \in \phi^{-1}(b) : (a', \omega', b') \in U_{i,j}\} = \emptyset. \tag{7}$$

If condition (7) is found to be true, then an element $b' \in \phi^{-1}(b)$ is arbitrarily chosen, and the collection $U_{i,j+1} = U_{i,j} \cup \{(a', \omega', b')\}$ is constructed. Otherwise, $U_{i,j+1} = U_{i,j}$. The inner loop terminates when Condition (7) has been tested for all the pairs in the Cartesian product $\phi^{-1}(a) \times \gamma^{-1}(\omega)$. The outer loop of the algorithm terminates when all the triples in R_2° have been considered. The set $U_{n,m}$ obtained out of the last iteration $n = |R_2^\circ|$ of the outer loop and the last iteration $m = |\phi^{-1}(a) \times \gamma^{-1}(\omega)|$ of the inner loop is a ternary relation such that $R_1^\circ \subseteq U_{n,m} \subseteq S_1 \times \Omega_1 \times S_1$.

Theorem 11. *The transitive closure of the collection $U_{n,m}$ obtained by an application of the LC algorithm to the reductions R_1° and R_2°, is a least local completion of R_1.*

Proof. In the first place we show that the condition of strong homomorphism given in Definition 4 is equivalent to the following one:

$$\forall (a, \pi, b) \in R_2, \forall (a', \pi') \in \phi^{-1}(a) \times \gamma^{-1}(\pi), \exists b' \in \phi^{-1}(b) : (a', \pi', b') \in R_1 \tag{8}$$

Consider any $a', b' \in S_1$ and any $\pi' \in \Omega_1$ such that $(\phi(a'), \gamma(\pi'), \phi(b')) \in R_2$. The condition of strong homomorphism requires the existence of an element $c \in S_1$ such that $(a', \pi', c) \in R_1$ and $\phi(c) = \phi(b')$. By setting $\phi(a') = a$, $\phi(b') = b$, and $\gamma(\pi') = \pi$, we have that $(a, \pi, b) \in R_2$ and for every $(a', \pi') \in \phi^{-1}(a) \times \gamma^{-1}(\pi)$ there is a $c \in S_1$ such that, $(a', \pi', c) \in R_1$ and $\phi(c) = \phi(b') = b$. This immediately implies Condition (8). The converse follows similarly. Given any $(a, \pi, b) \in R_2$, by setting $a = \phi(a')$, $b = \phi(b')$, and $\pi = \gamma(\pi')$, we have that $(\phi(a'), \gamma(\pi'), \phi(b')) \in R_2$. Since, by Condition (8), for every $(a', \pi') \in \phi^{-1}(a) \times \gamma^{-1}(\pi)$ there exists a $c \in \phi(b')$ such that $(a', \pi', c) \in R_1$ and $\phi(c) = \phi(b')$, it follows that (ϕ, γ) is a strong homomorphism.

Considering the LC algorithm, we now observe that given any triple $(a, \omega, b) \in R_2^{\circ}$ and any pair $(a', \pi') \in \phi^{-1}(a) \times \gamma^{-1}(\omega)$, Condition (7) holds true if and only if

$$\exists b' \in \phi^{-1}(b) : (a', \omega', b') \in U_{i,j} \tag{9}$$

is false. Whenever this happens the LC algorithm adds exactly one triple (a', ω', b') to $U_{i,j}$, such that $U_{i,j} \cup \{(a', \omega', b')\}$ satisfies Condition (7). Therefore, when the algorithm terminates, Condition (9) will be satisfied for all triples $(a, \omega, b) \in R_2^{\circ}$ and all pairs $(a', \pi') \in \phi^{-1}(a) \times \gamma^{-1}(\omega)$. Thus, Condition (8) is satisfied for R_1° and R_2°, and by Theorem 9 it must also be satisfied for R_1 and R_2.

Finally, since at every single step $j > 0$ of the inner loop, at most one triple is added to $U_{i,j}$, the cardinality of $U_{n,m}$ must be the smallest possible, therefore the solution must be a least local completion. □

6. Some application examples

In this section, three examples of different applications are provided. The first application is based on the classical test of the Tower of Hanoi. It aims to show the impact that an abstract representation has on the complexity of the problem space (i.e., number of problem states and operations) and, as a consequence, on the complexity of the knowledge space (i.e., number of knowledge states). The second one is based on the Tower of London test and shows the use of the problem space homomorphism to formally represent a psychological hypothesis. Finally, the third example is based on the mental rotation task, in which a discrete representation of a continuous task is presented and described.

6.1. Tower of Hanoi task

The *Tower of Hanoi* (ToH) is a task with a long tradition in studies on problem-solving (see e.g., Ewert & Lambert, 1932; Gagne & Smith, 1962; Simon, 1975; Stefanutti & Albert, 2003). Its problem space is well known, and many substructures and symmetries are evident in it. The ToH consists of three pegs and n disks of different diameters that can slide onto any peg. The problem solver may move only one disk at a time, and no larger disk may be placed over a smaller one. The ToH task aims at matching the initial and the final configurations of the problem in the minimum number of moves. Fig. 2 shows the problem space representation of the ToH with 4 disks. The basic strategy to solve a problem with a number n of disks is a simple recursive strategy (called *goal recursion strategy*), and its solution path can be decomposed into nested subproblems. Given a tower of n disks T_n, the stack of $(n-1)$ disks is the $(n-1)$th sub-tower T_{n-1}. The $(n-1)$th sub-tower is simply obtained by removing the larger disk from T_n. Then, the goal recursion strategy can be stated as follows: To move a tower T_n from peg i to peg j having a temporary peg k,

1. if T_{n-1} is not empty, then move T_{n-1} from i to k;
2. move the larger disk from i to j;
3. if T_{n-1} is not empty, then move T_{n-1} from k to j;

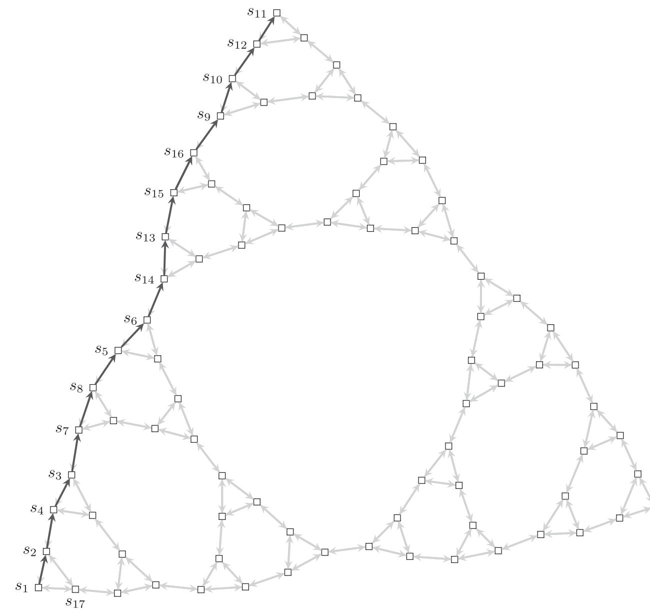

Fig. 2. Directed graph of the Tower of Hanoi problem space with four disks. The solution path discussed in example (10) is highlighted in black in the figure.

Such strategy is recursive, as it calls itself for moving the tower T_{n-1} in both steps (1) and (3). As pointed out by Stefanutti and Albert (2003) this strategy consists of two kinds of operations: *moving a sub-tower* with $n-1$ disks (denoted t_{n-1}) and *moving a disk* (denoted d). Consider, for instance, the incomplete problem space $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ of the 4-disks ToH represented in Fig. 2. It should be observed that the condition $(a, \omega, a) \notin R_1$ holds true for all $a \in S_1$ and all $\omega \in \Omega_1$.

Problem state s_1 in Fig. 2 represents the configuration in which the whole 4-disks tower is stacked on the leftmost peg, whereas problem state s_{11} represents the configuration in which the 4-disk tower is stacked on the rightmost peg. The shortest solution path that connects s_1 to s_{11} (black edges in Fig. 2) is represented by the sequence:

$$s_1\, t_1\, s_2\, d_{2,4}\, s_4\, t_1\, s_3\, d_{3,7}\, s_7\, t_1\, s_8\, d_{8,5}\, s_5\, t_1\, s_6\, d_{6,14}\, s_{14}\, t_1\, s_{13}\, \ldots$$
$$d_{15,16}\, s_{15}\, t_1\, s_{16}\, d_{16,9}\, s_9\, t_1\, s_{10}\, d_{10,12}\, s_{12}\, t_1\, s_{11}, \tag{10}$$

where $t_1 \in \Omega_1$ is the operation that moves the sub-tower with one disk, and $d_{ij} \in \Omega_1$ is the operation that consists of moving a disk from one peg to another, thus transforming problem state s_i to problem state s_j.

On the other hand, consider problem space $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ depicted in Fig. 3. The solution path that solves (s'_1, s'_{11}) (black edges in Fig. 3) is represented by the sequence:

$$s'_1\, t_2\, s'_4\, d'_{47}\, s'_7\, t_2\, s'_5\, d'_{514}\, s'_{14}\, t_2\, s'_{15}\, d'_{159}\, s'_9\, t_2\, s'_{11}, \tag{11}$$

where $t_2 \in \Omega_2$ is the operation moving the sub-tower with two disks and $d'_{ij} \in \Omega_2$ is the operation that consists of moving a disk from one peg to another, thus transforming problem state s'_i to problem state s'_j.

A weak homomorphism (ϕ_1, γ_1) is defined between $\mathbf{P}_1$, and $\mathbf{P}_2$, where: (i) the problem states in $\mathbf{P}_1$ that are linked to one another through operation t_1 (e.g., s_1, s_2 and s_{17}) are mapped to the same problem state in $\mathbf{P}_2$ (e.g., $s'_1 = \phi_1(s_1) = \phi_1(s_2) = \phi_1(s_{17})$); (ii) $\gamma_1(t_1) = \epsilon$; (iii) each of the three operations $d_{14,13}, d_{53,57}, d_{54,52}$ is mapped to itself by γ_1; and (iv) the remaining operations, namely those in $\Omega_1 \backslash \{d_{14,13}, d_{53,57}, d_{54,52}, t_1\}$ are mapped to operation $t_2 \in \Omega_2$, which is interpreted as "moving a two-disks sub-tower".

A similar procedure can be applied for constructing a homomorphism between the incomplete problem space $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ and the incomplete problem space $\mathbf{P}_3 = (S_3, \Omega_3, R_3)$, where t_3 is the operation of moving a three-disks sub-tower. In general, by applying the goal

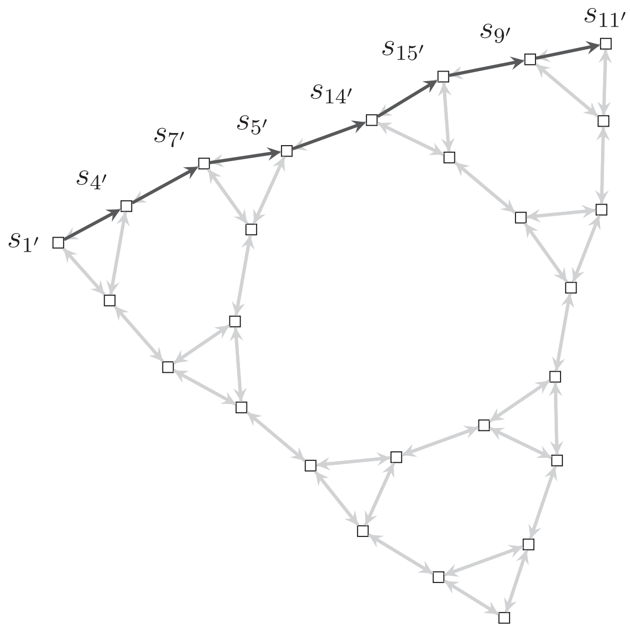

Fig. 3. Directed graph of the abstract representation of Tower of Hanoi problem space with four disks. The solution path discussed in example (11) is highlighted in black in the figure.

Table 1
Cardinalities of the goal and knowledge spaces of the ToH using different n in the goal recursion strategy. Column one displayed the number of disks n involved in the operation t_n, "move the tower with n disks." Columns two and three displayed the cardinality of the problem spaces and knowledge spaces, respectively.

t_n	$\|S_n\|$	$\|\mathcal{K}_n\|$
1	81	21,343,456,836
2	27	6,084
3	9	36

recursion strategy to the ToH it is possible to define a homomorphism $(\phi_{n-1}, \gamma_{n-1})$ between the ToH problem space with n disks and the problem space with $(n-1)$ disks. Recalling Example (10), the sequence $s_1\, t_1\, s_2\, d_{2,4}\, s_4\, t_1\, s_3$ performs the following task: It moves the sub-tower with one disk from the leftmost peg to the central peg (t_1), then it moves the top disk from the leftmost peg to rightmost one (d_{24}), and finally it moves the sub-tower with one disk from the central to rightmost peg (t_1) creating a two-disk tower. The same sequence of moves described above is mapped by the homomorphism to the sequence $s'_1\, t_2\, s'_4$ in Example (11). This sequence performs the task of moving the two-disks tower from the leftmost peg to the rightmost one (t_2). In this representation, the three operations that in $\mathbf{P}_1$ move, one at a time, two disks to form a two-disk tower are joined into a single operation. In this sense, $\mathbf{P}_2$ can be seen as a problem space for the three disks ToH.

A knowledge space $\mathcal{K}_1$ for the collection of problems $Q_1 = \{(s_i, s_1) : s_i \in S_1 \setminus \{s_1\}\}$ was derived from the problem space $\mathbf{P}_1$. In the collection Q_1 the problem state $s_1 \in S_1$ is the goal for all the problems. Similarly, knowledge spaces $\mathcal{K}_2$ and $\mathcal{K}_3$ were derived from the problem spaces $\mathbf{P}_2$ and $\mathbf{P}_3$ on the collections $Q_2 = \{(s_i, s'_1) : s'_i \in S_2 \setminus \{s'_1\}\}$ and $Q_3 = \{(s_i, s''_1) : s_i \in S_3 \setminus \{s''_1\}\}$, respectively. It is worth noticing that $\phi_1(s_1) = s'_1$ and $\phi_2(s'_1) = s''_1$ thus, the goal is preserved across the different representations.

Table 1 shows the characteristics of the problem and knowledge spaces obtained from the application of the goal recursion strategy. In particular, it displays the number n of disks involved in a single operation t_n (first Column), the number of problem states $|S_n|$ and knowledge states $|\mathcal{K}_n|$ (second and third Columns).

Table 1 shows that the number of knowledge states drastically reduces by moving from the representation which is closer to the physical one ($t_n = 1$) to the most abstract representation of the concrete problem space ($t_n = 3$). In particular, the cardinality of the knowledge space derived from the concrete problem space exceeds twenty billions. In contrast, the knowledge space derived from the two-disk tower homomorphism has only 6084 knowledge states. The more abstract representation gives only 36 knowledge states.

Several research questions can arise in comparing those problem spaces. For example, is the detailed representation of the task given by $\mathbf{P}_1$ necessary? Or would a simpler model better explain the solution behavior of an individual? Is problem space $\mathbf{P}_3$ sufficient to discriminate between individuals with different problem-solving abilities? The problem space homomorphism ensures that any observed solution in the concrete problem space can be linked to a solution in the abstract one. Thus, it allows testing these kinds of research questions by empirically validating the knowledge spaces and comparing them by means of statistical model selection criteria.

6.2. Tower of London test

The tower of London test is a neuropsychological test proposed by Shallice (1982) as a variant of the ToH to study patients with frontal lobe lesions. The ToL consists of three pegs with different heights and three balls of different colors that can slide onto any peg. Fig. 4 depicts all the possible ToL configurations.

Like the ToH, the ToL test aims at matching the initial and final configuration in the least number of moves. The concrete problem space $\mathbf{P}_{\text{ToL}} = (S_{\text{ToL}}, \Omega_{\text{ToL}}, R_{\text{ToL}})$ of the ToL consists of $6\times6 = 36$ different problem states obtained as the Cartesian product of the six different permutations of the three colors times the six spatial arrangements of the balls in the pegs. Each problem state can be uniquely referred to as a pair ab of numbers, where a stands for one of the six spatial arrangements whereas b stands for one of the six color permutations. The reader is referred to Stefanutti et al. (2021) for the complete list of problem states codings.

Which characteristics of the ToL problem determine its difficulty is a long-debated topic. The original hypothesis by Shallice was that two problems with the same initial configuration have the same difficulty if they can be solved using the same minimum number of moves. Within the PKST approach, it is possible express this hypothesis in a formal way. With this aim a few definitions are needed. Let $\mathbf{P} = (S, \Omega, R)$ be a problem space.

Definition 6. A solution paths $(s, \pi, t) \in R$ is a *shortest path at* $s \in S$ if and only if $l(\pi) \le l(\sigma)$ for all $\sigma \in \Omega^*$ such that $(s, \sigma, t) \in R$, where $l(\pi)$ and $l(\sigma)$ denote the length of π and σ respectively. The *distance* $\delta(s, t)$ from a problem state $s \in S$ to another problem state $t \in S$ is the length of the string π in a shortest path (s, π, t). By convention $\delta(s, t) = \infty$ if $(s, \pi, t) \notin R$ for any $\pi \in \Omega^*$.

Let Q_{ToL} be the collection of all the problems that can be formulated in the Tower of London problem space $\mathbf{P}_{\text{ToL}}$. Shallice's hypothesis can be formalized as follows:

Definition 7. Two problems $(s, t), (s, g) \in Q_{\text{ToL}}$ are equivalent if given the shortest paths $(s, \pi, t) \in R_{\text{ToL}}$ and $(s, \sigma, g) \in R_{\text{ToL}}$ then $l(\pi) = l(\sigma)$.

In order to explore the implications of Definition 7 on the problem space, consider the following example:

Example 5. Considering the problem sub-space $\mathbf{P}_1 = (S_1, \Omega_1, R_1)$ of the ToL show in Fig. 5.

Setting state 31 as initial configuration, the collection of all the problems in $\mathbf{P}_1$ is

$$Q_1 = \{(31, 21), (31, 51), (31, 22), (31, 52), (31, 12), (31, 42), (31, 32)\}.$$

Under Definition 7, three equivalence classes are obtained for the problems in Q_1. First, problems $(31, 21)$ and $(31, 51)$ that are solved by

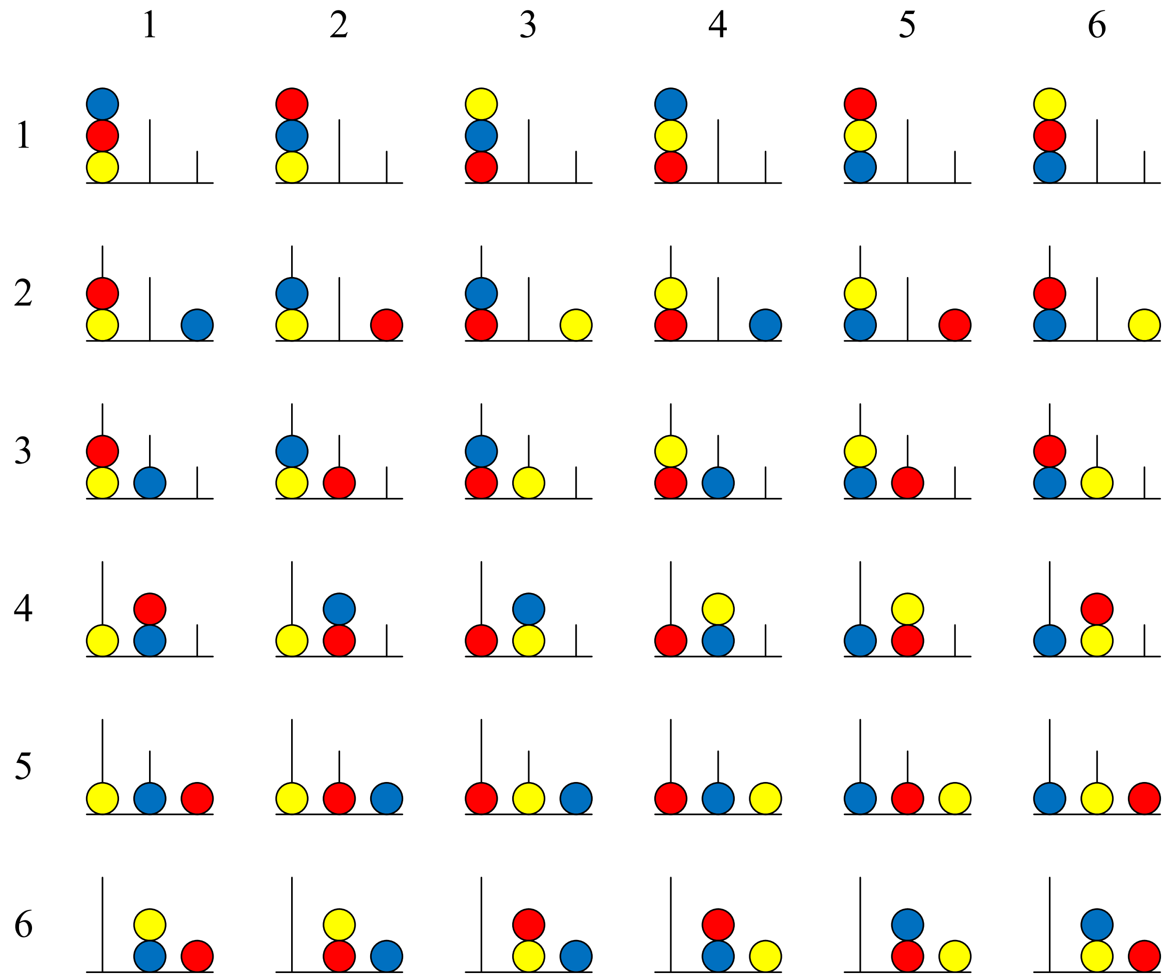

Fig. 4. The $6 \times 6 = 36$ different problem states of the Tower of London test. The six different spatial arrangements of the balls vary across rows and the six different permutations of three colors vary across columns.

single operations. Second, problems $(31, 22)$ and $(31, 52)$ are solved by strings of two operations. And finally problems $(31, 12)$, $(31, 42)$, and $(31, 32)$ are solved by the shortest paths of length three.

Consider now a problem space $\mathbf{P}_2 = (S_2, \Omega_2, R_2)$ such that $S_2 = \{s_0, s_1, s_2, s_3\}$, $\Omega_2 = \{\omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}\}$ and $R_2 = \{(s_0, \omega_{11}, s_1), (s_1, \omega_{12}, s_2), (s_2, \omega_{13}, s_3), (s_3, \omega_{14}, s_3)\}^{\bullet}$. We recall that $\bullet$ denotes the transitive closure of the relation. Consider moreover the pair of function (ϕ, γ) with $\phi : S_1 \to S_2$ and $\gamma : \Omega_1^* \to \Omega_2^*$ defined in the following way:

$$\begin{aligned} &\phi(21) = \phi(51) = s_1, && \gamma(\omega_1) = \gamma(\omega_2) = \omega_{11},\\ &\phi(22) = \phi(52) = s_2, && \gamma(\omega_3) = \gamma(\omega_4) = \omega_{12},\\ &\phi(12) = \phi(32) = \phi(42) = s_3, && \gamma(\omega_5) = \gamma(\omega_6) = \gamma(\omega_7) = \gamma(\omega_9) = \omega_{13},\\ &\phi(31) = s_0 && \gamma(\omega_{10}) = \gamma(\omega_8) = \omega_{14}. \end{aligned}$$

The provided definition of γ is restricted to the set Ω_1, however it can be easily extended to the whole Ω_1^* by applying the fundamental rule of the homomorphism. Thus, we also require that, given any $\pi, \sigma \in \Omega_1^*$, $\gamma(\pi\sigma) = \gamma(\pi)\gamma(\sigma)$.

We show that: (1) (ϕ, γ) is a strong homomorphism and (2) this last induces on the problems of problem space $\mathbf{P}_1$ the equivalence classes required by Definition 7, thus showing that Shallice hypothesis is respected.

Concerning point (1), $\gamma(\pi\sigma) = \gamma(\pi)\gamma(\sigma)$ is true by definition. Moreover, we have

$$(\phi(31), \gamma(\omega_1), \phi(21)) = (\phi(31), \gamma(\omega_2), \phi(51)) = (s_0, \omega_{11}, s_1),$$
$$(\phi(21), \gamma(\omega_3), \phi(52)) = (\phi(51), \gamma(\omega_4), \phi(22)) = (s_1, \omega_{12}, s_2),$$
$$(\phi(22), \gamma(\omega_9), \phi(32)) = (\phi(22), \gamma(\omega_6), \phi(13)) = (\phi(52), \gamma(\omega_7), \phi(32)) = \dots$$
$$(\phi(52), \gamma(\omega_5), \phi(42)) = (s_2, \omega_{13}, s_3),$$
$$(\phi(13), \gamma(\omega_{10}), \phi(32)) = (\phi(42), \gamma(\omega_8), \phi(32)) = (s_3, \omega_{14}, s_3).$$

This shows that the weak homomorphism conditions are satisfied. Additionally, since $\mathbf{P}_1$ is a problem space, by Theorem 10, (ϕ, γ) is a strong homomorphism.

As for point (2), it is now easy to see that the following equivalence classes on the set of problem Q_1 are induced by the homomorphism (ϕ, γ), $\{(31, 21), (31, 51)\}$, $\{(31, 22), (31, 52)\}$, and $\{(31, 32), (31, 13), (31, 42)\}$. These are exactly the classes required by Definition 7. To conclude in this example two problems are equivalent if they have the same length and this assumption of problem equivalence is properly represented by problem space $\mathbf{P}_2$ through homomorphism (ϕ, γ).

The general rules that have been used for constructing both homomorphisms and the resulting problem space in Example 5 can now be stated as follows:

> Given a fixed initial problem state $s_1 \in S_{\text{ToL}}$, define (ϕ, γ) such that, for any $s_2, s_3, \in S_{\text{ToL}}$ and any $\pi, \sigma \in \Omega_{\text{ToL}}$ such that $R_{\text{ToL}}(s_2, \pi) = t_2$ and $R_{\text{ToL}}(s_3, \sigma) = t_3$:
>
> **(A1)** $\phi(s_2) = \phi(s_3)$ if and only if $\delta(s_1, s_2) = \delta(s_1, s_3)$;
>
> **(A2)** $\gamma(\pi) = \gamma(\sigma)$ if and only if $\delta(s_1, s_2) = \delta(s_1, s_3)$ and $\delta(s_1, t_2) = \delta(s_1, t_3)$.

A homomorphism that respects both conditions (A1) and (A2) induces in $\mathbf{P}_{\text{ToL}}$ an equivalence relation on the set of problems that is in agreement with Shallice's hypothesis. Moreover, Condition (A1) requires that all problem states at the same distance from the initial state belong to the same equivalence class. Condition (A2) requires that any two operations are equivalent if and only if they are applicable to equivalent problem states and produce equivalent problem states.

6.3. Mental rotation tasks

The mental rotation tasks usually consist of "mentally transforming" a visual stimulus to match a given configuration. A common task for assessing mental rotation ability for a three-dimensional (3D) object is the Shepard–Metzler tasks (S–M tasks; Shepard & Metzler, 1971). In the S–M task, participants are required to establish whether two images represent the same object rotated at different angles, or not. An example could be to match a 3D object with its rotation by 180 degrees around the x axis. Shepard and Metzler (1971) assumed that individuals perform mental rotation in solving this task. Obviously, mental rotation is not accessible to direct inspection, and it is only deduced from the observation of indirect measures. It is highly plausible that, if mental rotation really occurs, then the path inclusion assumption should stay true. In fact, a 180 degrees "mental rotation" should be the composition of two separate 90 degrees "mental rotations". Thus, each of the two 90 degrees "mental rotations" should be a sub-path of the 180 degrees one.

Of course, the sub-path assumption could be false. For example, a 180 degrees rotation may be easier to identify compared to the 90 degrees one. Therefore, in that case, the validity of "mental rotation" as stated above would be threatened. Nonetheless, PKST could still be useful for empirically testing some of the necessary conditions of the mental rotation hypothesis.

Several computational models of mental rotation can be found in the literature (Peebles, 2019; Yurt & Sunbul, 2012). However, there is no formal model of this cognitive task to the best of our knowledge. Other than what discussed above the problem space could be used as a discrete model of mental rotation at a certain level of granularity, irrespective of the real (continuous or discrete) nature of the mental rotation itself.

The problem space presented in this section was constructed using the 3D object proposed in the S-M task. This problem space is a discrete representation of the continuous one under the assumption that, given an initial configuration, the object can be rotated on only one axis at a time by an angle α. The granularity of the model depends on the size of α.

Consider a simple example with the following characteristics: $\alpha = 45°$; the object can rotate on the x and y axes; and the maximum rotation angle is 90° in each axis.

Given the initial configuration of the object in Fig. 6, the incomplete problem space $\mathbf{P} = (S, \Omega, R)$ is obtained. The set Ω contains operations of two types. Operations of the first type are rotations about the x axis denoted as $x_s(\alpha)$, where s is the problem state to which the rotation is applied and α is the angle of the rotation. The second type of operations are rotations about the y axis are denoted $y_s(\alpha)$. Since $\alpha = 45°$ is fixed in this example, it will be omitted. Thus the operation will be denoted as x_s and y_s, respectively.

Fig. 6 depicts problem space $\mathbf{P}$, in which a clear symmetry appears between the left and right branches. From this observation the hypothesis arises that mental rotation skills could be independent of the rotation axis. More precisely, if an individual can solve a problem (s,t), and another problem (s,t') can be solved by a sequence of operations which is "symmetric" with respect to the one that solves (s,t), then that individual can also solve (s,t'). A sequence π of operations is "symmetric" with respect to another sequence σ if (i) the two have the same length and (ii) every single rotation about a given axis in σ corresponds to a rotation about the other axis in π (e.g. if $\sigma = xyx$ then $\pi = yxy$).

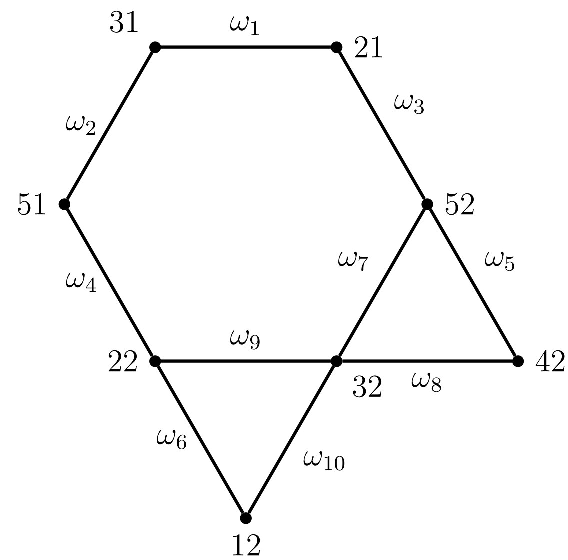

Fig. 5. Directed graph of the problem sub-space presented in Example 5.

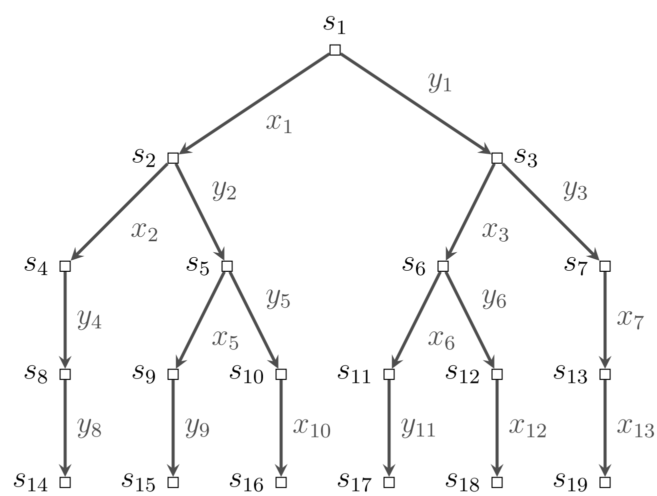

Fig. 6. Directed graph of the problem space for the mental rotation task with angle of rotation $\alpha = 45°$.

This hypothesis gives rise to an abstract problem space $\mathbf{P}' = (S', \Omega', R')$. A problem space homomorphism (ϕ, γ) linked $\mathbf{P}'$ and $\mathbf{P}$. Function ϕ maps the problem states of the left branch and their symmetrical counterparts in the right branch to the same problem state in $\mathbf{P}'$ as follows:

$$\phi(s_2) = \phi(s_3), \quad \phi(s_9) = \phi(s_{12}), \quad \phi(s_4) = \phi(s_7),$$
$$\phi(s_{10}) = \phi(s_{11}), \quad \phi(s_5) = \phi(s_6), \quad \phi(s_{14}) = \phi(s_{19}),$$
$$\phi(s_8) = \phi(s_{13}) \quad \phi(s_{15}) = \phi(s_{18}), \quad \phi(s_{16}) = \phi(s_{17}).$$

Given two sequences $\pi, \sigma \in \Omega^*$, $\gamma(\pi) = \gamma(\sigma)$ if and only if σ and π are symmetric to one another.

Considering sequence $x_1y_2y_5 \in \Omega^*$ which solves problem (s_1, s_{10}) and sequence $y_1x_3x_6 \in \Omega^*$ which solves problem (s_1, s_{11}), it follows from the two equalities $\phi(s_{10}) = \phi(s_{11})$ and $\gamma(x_1y_2y_5) = \gamma(y_1x_3x_6)$ that $(s_1, x_1y_2y_5, s_{10}), (s_1, y_1x_3x_6, s_{11}) \in R$, and $(\phi(s_1), \gamma(x_1y_2y_5), \phi(s_{10}))$, $(\phi(s_1), \gamma(y_1x_3x_6), \phi(s_{11})) \in R'$. A similar check is straightforwardly applied to all the problems in order to verify that (ϕ, γ) is a weak homomorphism.

The example presented in this section is just one of the many research questions that could be addressed using this approach. In fact, the problem space and the knowledge space derived from it can be the foundation for a computerized assessment tool, in which the participants can rotate an object to its goal configuration in a number of finite steps. Every single step would consist of a rotation by a constant angle around a given axis. Moreover, the impact of different angle sizes or different rotation axes can be manipulated and studied. The usefulness of a notion of homomorphism between problem spaces can be particularly appreciated in the specific context of this example, where problems are of a continuous nature. These last can be modeled by an entire hierarchy of discrete problem spaces at different levels of granularity, and related to one another by homomorphic functions.

7. Final remarks

The concept of problem space refers to an abstract representation that the problem solver might have of a given task (Newell & Simon, 1972) or as a concrete representation of the physical structure of the task that can be objectively constructed and displayed (see, e.g., Langley, Magnani, Schunn, & Thagard, 2005; Stefanutti, 2019; Zhang & Norman, 1994). In this article, these two alternative notions of problem space were linked by means of a homomorphism between the two problem spaces. The concept of a homomorphism between different problem space representations is not new (see e.g., Banerji & Ernst, 1972; Chandrasekaran, 2011; Luger, 1976; Luger & Bauer, 1978). Nonetheless, to our knowledge, there is no previous work in which a problem space homomorphism is developed within a formal theoretical framework.

A homomorphism between two problem spaces is a pair of mappings that preserve the relation among problem states and the sequence of operations. The homomorphism allows tracing each abstract solution path back to a concrete one. Additionally, it links observed responses in the concrete space to responses in the abstract space, which enables the empirical validation of the abstract problem space through the probabilistic model.

When some operations cannot be applied to all the problem states and, therefore, the problem space is incomplete, the homomorphism conditions are too strong. Therefore, a weaker kind of homomorphism was introduced which preserves the relation R when the problem spaces are incomplete.

Two algorithms were developed in this article. The former one tests whether a given pair of mappings (ϕ, γ) is a homomorphism between two problem spaces. The latter algorithm considers a weak homomorphism (ϕ, γ) and provides its local completion.

This work addresses two open questions in PKST. The first question arises when problem spaces have a large number of states, such as in mental rotation tasks, chess, or Go. In all these cases the problem space is too large to be constructed. To overcome this limitation, one approach is to focus on a theoretically interesting portion of the problem space, as shown by Stefanutti et al. (2021) for the Tower of London, by Sgaravatti (2022) for the game of Go, and by Kickmeier-Rust and Albert (2010) and Stefanutti (2014) in educational applications of problem spaces. The alternative approach presented in this article consists of building an abstract representation of the problem space, which is smaller than the original one but homomorphic to it. Through this alternative approach, larger portions of the problem space can be considered. Section 6.1 provides a detailed example of this approach in the context of the Tower of Hanoi.

The second open question concerns the psychological representation of problem spaces. It is reasonable to assume that some characteristics of the problems are not relevant to the human problem-solving process. Once all irrelevant aspects have been removed, and the result is homomorphic to the original problem space, what remains is a more abstract representation. For example, if specific symmetries in problem spaces (e.g., the colors of the balls in the ToL) are hypothesized to be not relevant to the human problem-solving process, they could be removed in an abstract representation of the problem space. Such a more abstract model could then be tested against empirical evidence, for instance, through the approach described in Stefanutti et al. (2021).

The definition of a problem space homomorphism between different problem space representations could provide a rigorous formal representation of specific hypotheses concerning the solution process. An example of this type is offered in Section 6.2, with the Tower of London test and the "Shallice hypothesis". Moreover, studying the relationship between different problem spaces can give rise to new interesting research questions. In the examples provided in the previous section, it was the case of the mental rotation task where, different types of symmetries can be highlighted, by inspecting the problem space. Here the research question was whether such symmetries give rise to rotation problems that are equivalent (in difficulty) with respect to the spatial skills of a human individual.

Finally, the formal definition of problem space homomorphism, together with the algorithms to test it, could be the foundation for developing statistical procedures that start from a potentially large concrete problem space and tries to simplify it by extracting symmetries and equivalences from the response behavior of a sample of human individuals.

CRediT authorship contribution statement

Andrea Brancaccio: Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis, Conceptualization. **Luca Stefanutti:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization.

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