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A shape preserving quasi-interpolation operator based on a new transcendental RBF

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Abstract

It is well-known that the univariate Multiquadric quasi-interpolation operator is constructed based on the piecewise linear interpolation by $|x|$. In this paper, we first introduce a new transcendental RBF based on the hyperbolic tangent function as an smooth approximant to $\phi(r) = r$ with higher accuracy and better convergence properties than the MQ RBF $\sqrt{r^2 + c^2}$. Then the Wu-Schaback's quasi-interpolation formula is rewritten using the proposed RBF. It preserves convexity and monotonicity. We prove that the proposed scheme converges with a rate of $O(h^2)$. So it has a higher degree of smoothness. Some numerical experiments are given in order to demonstrate the efficiency and accuracy of the method.

Keywords: Radial basis functions (RBFs), quasi-interpolation, hyperbolic tangent function.

AMS subject classification: 65D05, 65D12, 65D20.

1 Introduction

Given a set of n distinct (scattered) points $\{x_j\}_{j=0}^n \in \Omega \subseteq \mathbb{R}^d$ and corresponding data values $\{f_j\}_{j=0}^n \in \mathbb{R}$, a standard way to interpolate a function $f \in C^1 : \Omega \to \mathbb{R}$ is by using

$$\mathcal{L}f(x) = \sum_{j=0}^{n} \lambda_j \mathcal{X}(x - x_j), \tag{1}$$

with the coefficients λ_j determined by the interpolation conditions $\mathcal{L}f(x_j) = f_j,\ j = 0, \ldots, n$, where $\mathcal{X}(\cdot)$ is an interpolation kernel. Many authors use Radial Basis Functions (RBFs) to solve the interpolating problem (1), that is $\mathcal{X}(x - x_j) = \phi(\|x - x_j\|)$, ($\|\cdot\|$ is the Euclidean norm) with $\phi : [0, \infty) \to \mathbb{R}$, is some radial function [46]. Then, the coefficients λ_j are determined solving a symmetric linear system $A\lambda = f$, where $A = \left[\phi(\|x_i - x_j\|)\right]_{0 \le i,j \le n}$. RBF method provides excellent interpolants for high dimensional scattered data sets. The corresponding theory had been extensively studied by many researchers (see e.g [2, 30, 31, 32, 36, 37, 44, 46, 50, 49]). That is why in the last few decades, RBFs have been widely applied in a number of fields such as multivariate function approximation, neural networks and solution of differential and integral equations (see e.g [6, 7, 10, 16, 19, 24, 25, 33, 39, 45, 52]). The Multiquadric (MQ) RBF

$$\phi_j(x) = \sqrt{\|x - x_j\|^2 + c^2}, \tag{2}$$

proposed by Hardy [17], is undoubtedly the most popular RBF that is used in many applications and is representative of the class of global infinitely differentiable RBFs. Hardy [18] summarized the achievement of study of MQ from 1968 to 1988 and showed that it can be applied in hydrology, geodesy, photogrammetry, surveying and mapping, geophysics, crustal movement, geology, mining and so on. In the survey paper [11], Franke pointed out that MQ interpolation was the best among 29 scattered data interpolation methods in terms of timing, storage, accuracy, visual pleasantness of surface reconstruction and ease to implement. The existence of the solution of the associated interpolation problem was shown later on by Micchelli [32]. Although the MQ interpolation is always solvable, the resulting matrix quickly becomes ill-conditioned as the number of points increases. Researchers concentrated on a weaker form of (1), known as *quasi-interpolation*, that holds only for polynomials of some low degree m, i.e.,

$$\mathcal{L}p_m(x_j) = p_m(x_j), \quad \forall p_m \in \Pi_m^d,$$

for all $0 \le j \le n$, where Π_m^d denotes the space of polynomials of degree less and equal to m in $\mathbb{R}^d$. Beatson and Powell [1, 35] first proposed a univariate quasi-interpolation formula where $\mathcal{X}$ in (1), is a linear combination of MQ RBF and low degree

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polynomials. Their idea is based on the fact that the MQ degenerates to $|x-x_j|$, for $c=0$ and $d=1$, hence quasi-interpolation (1) is the usual piecewise linear interpolation which reproduces linear polynomials as c tends to zero. However, their operator requires the approximation of the derivatives of the function at endpoints, which is not convenient for practical purposes. Thus, Wu and Schaback [51] constructed another univariate MQ quasi-interpolation operator with without the use of derivatives at the endpoints. Given data

$$a=x_0<x_1<\cdots<x_n=b \qquad h:=\max_{2\le j\le n}(x_j-x_{j-1}),$$

Wu-Schaback's MQ quasi-interpolation formula is

$$(\mathcal{L}_{MQ}f)(x)=f_0\alpha_0(x)+f_1\alpha_1(x)+\sum_{j=2}^{n-2}f_j\psi_j(x)+f_{n-1}\alpha_{n-1}(x)+f_n\alpha_n(x) \tag{3}$$

where

$$\begin{aligned}
\alpha_0(x)&=\frac{1}{2}+\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},\\
\alpha_1(x)&=\frac{\phi_2(x)-\phi_1(x)}{2(x_2-x_1)}-\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},\\
\alpha_{n-1}(x)&=\frac{(x_n-x)-\phi_{n-1}(x)}{2(x_n-x_{n-1)}}-\frac{\phi_{n-1}(x)-\phi_{n-2}(x)}{2(x_{n-1}-x_{n-2})},\\
\alpha_n(x)&=\frac{1}{2}+\frac{\phi_{n-1}(x)-(x_n-x)}{2(x_n-x_{n-1})},\\
\psi_j(x)&=\frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)}-\frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1)}}, \qquad 2\le j\le n-2.
\end{aligned}$$

The main advantage of this formula is that it does not require the solution of any linear system. Instead, the formula uses the function values f_j at x_j as its coefficients. The drawback is that instead of $c=O(h)$, one needs to use a smaller shape parameter $c^2|\log c|=O(h^2)$ in order to achieve quadratic convergence, resulting in a lower smoothness. Note that for $c=0$, the basis functions given in quasi-interpolant $\mathcal{L}_{MQ}f$ are cardinal with respect to $\{x_j\}_{j=0}^n$. For a general quasi-interpolation operator $\mathcal{L}$ we can state the following definitions.

Definition 1.1. The quasi-interpolation operator $\mathcal{L}$ constructed at the data points $\{(x_j,f_j)\}$, is called to be monotonicity preserving, if the first order divided difference $f[x_j,x_{j+1}]$ is nonnegative (non-positive) implies that $(\mathcal{L}f)'$ is also nonnegative (non-positive).

Definition 1.2. The quasi-interpolation operator $\mathcal{L}$ constructed at the data points (x_j,f_j), is called to be convexity preserving if the second order divided difference $f[x_{j-1},x_j,x_{j+1}]$ is nonnegative (non-positive, zero) implies that $(\mathcal{L}f)''$ is also nonnegative (non-positive, zero).

Since $\sqrt{x^2+c^2}$ tends to $|x|$ as c tends to zero, and radial basis interpolation (as well as the quasi-interpolation) based on $|x|$ is piecewise linear, Wu and Schaback claimed that the shape preserving properties of piecewise linear interpolation can be expected to hold for quasi-interpolation with multiquadrics, too. Actually, they first showed that the quasi-interpolation operator of Beatson and Powell is indeed convexity preserving. Then they proved that the quasi-interpolation operator (3) is monotonicity and convexity preserving. In 2004, Ling [26] proposed a multilevel quasi-interpolation operator and proved that it converges with a rate of $O(h^{2.5})\log h$ as $c=O(h)$. In 2009, Feng and Li [9] constructed a shape preserving quasi-interpolation operator by shifts of cubic MQ functions proving that it can produce an error of $O(h^2)$ as $c=O(h)$. Wang et al. [42] proposed an improved univariate MQ quasi-interpolation operator, by using Hermite interpolating polynomials, with convergence rate heavily depending on the shape parameter c. Jiang et al. [22] proposed two new multilevel univariate MQ quasi-interpolation operators with higher approximation order. Ling proposed a multidimensional quasi-interpolation operator using the dimension-splitting multiquadric basis function approach [27], and Wu et al. modified their idea by using multivariate divided difference and the idea of the superposition [48]. Gao and Wu [12] studied the quasi-interpolation for the linear functional data rather than the discrete function values. Moreover, MQ quasi-interpolation has been successfully applied in a wide range of fields. For example, in 2007, Wang and Wu [43] applied the operator (3) to tackle approximate implicitization of parametric curves. In 2011, Wu [47] presented a new approach to construct the so-called shape preserving interpolation curves based on MQ quasi-interpolation (3). A vast discussion on approximating k-th derivatives ($k\ge 0$) by using MQ quasi-interpolation can be found in [29] where the authors improved the approximation behaviors near the boundary by introducing a polynomial, which is shown to be an effective technique for MQ quasi-interpolation schemes. Combining the techniques of trigonometric spline quasi-interpolation [28] with MQ quasi-interpolation, a quasi-interpolation called MQ trigonometric spline quasi-interpolation scheme for periodic data was constructed in [13]. It is based on a periodic kernel and applying trigonometric divided differences to the periodic kernel. The MQ trigonometric spline quasi-interpolation not only possesses many fair properties of MQ quasi-interpolant such as smoothness, simplicity, efficiency, and capabilities of approximating high-order derivatives but it turns out to be also quasi-interpolant as well as its derivatives are periodic. Gao and Zhang [14] extended MQ trigonometric spline quasi-interpolation for numerical differentiation of noisy data. Moreover, motivated from the need of dealing with integral functionals in data science applications and other applications of MQ quasi-interpolation, in [15] there have been constructed three new MQ quasi-interpolation schemes for integral functionals. Finally, Hon and Wu [20], Chen and Wu [4, 5], Jiang and Wang [21], and other researches provided some successful examples using MQ quasi-interpolation operators to solve different types of partial differential equations.

The outline of the paper is as follows. In the next section, we introduce a new quasi-interpolation operator based on the hyperbolic tangent function, that is the function

$$g(x) = x\tanh\left(\frac{x}{c}\right), \quad c > 0 \tag{4}$$

which leads to a smooth and shape preserving interpolation operator with $O(h^2)$ rate of convergence. In section 3, we discuss its accuracy providing an error estimate. Numerical experiments are presented in section 4 with the aim of comparing the accuracy of our quasi-interpolation scheme with that of Wu and Schaback's, and also verifying the convergence rate of new quasi-interpolation operator by examples. The last section summarizes the conclusion and some further works.

2 Quasi-interpolation operator based on a new transcendental RBF

In this section, we first analyse a new approximation of $|x|$ based on the hyperbolic tangent, with better accuracy than the MQ RBF $\sqrt{x^2+c^2}$. The general question is, are there any good approximations of the absolute value function which are smooth? One simple approximation is MQ RBF $\sqrt{x^2+c^2}$. Carlos Ramirez et al. [38] proved that $\sqrt{x^2+c^2}$ is the most computationally efficient and smooth approximation of $|x|$, while S. Voronin et al. [40] proved the following Lemma.

Lemma 2.1. *The approximation of $|x|$ by the multiquadrics $g(x) = \sqrt{x^2+c^2}, c \in \mathbb{R}_+$ satisfies*

$$\left||x| - \sqrt{x^2+c^2}\right| \le c \quad ,$$
$$|x| \le \sqrt{x^2+c^2} \quad .$$

As noticed by Gauss in [41], the hyperbolic tangent can be written using the continued fraction

$$\tanh(x) = \frac{x}{1 + \frac{x^2}{3+\frac{x^2}{5+\cdots}}}.$$

This fact shows immediately that the function $g(x) = x\tanh\left(\frac{x}{c}\right)$ is a nonnegative function that indeed can be used to approximate $|x|$.

Since for the hyperbolic tangent

$$\lim_{c\to 0^+}\tanh\left(\frac{x}{c}\right) = \begin{cases} 1, & x>0,\\ 0, & x=0\\ -1, & x<0.\end{cases} .$$

we then have the approximation

$$x\tanh\left(\frac{x}{c}\right) \approx |x|.$$

Now, we show that the approximation of $|x|$ by $x\tanh\left(\frac{x}{c}\right)$ is more accurate than that given by the multiquadric.

Lemma 2.2. *The approximation of $|x|$ by $g(x) = x\tanh\left(\frac{x}{c}\right), c \in \mathbb{R}_+$ satisfies*

$$\left||x| - x\tanh\left(\frac{x}{c}\right)\right| \le 0.28c < c, \tag{5}$$
$$x\tanh\left(\frac{x}{c}\right) \le |x|. \tag{6}$$

Proof. The proof of (5) is trivial for $x = 0$. Letting $h(x) = |x| - x\tanh\left(\frac{x}{c}\right)$ that, for $x > 0$, becomes $h(x) = x - x\tanh\left(\frac{x}{c}\right)$. The maxima and minima of h are those that annihilate

$$h'(x) = \left(\frac{x}{c}\right)\left(\tanh\left(\frac{x}{c}\right)\right)^2 - \tanh\left(\frac{x}{c}\right) + \left(1 - \frac{x}{c}\right).$$

Setting $\frac{x}{c} = t$, we have

$$t\tanh^2(t) - \tanh(t) + (1-t) = 0$$

which reduces to solve $s(t) = t(\tanh(t)+1) - 1 = 0$. The function s on $t \ge 0$ is strictly increasing, with $s(0) = -1$. Hence there exits only one zero in (0,1). By numerically solving it, with the Matlab function `fzero` using `options = optimset('Display','iter');` the method converges in 7 iterations to the only solution in $(0,1)$ to the value $t^* = 0.639232271380537$ (in particular $s(t^*) = -1.110223024625157e-16$) so that $x^* = 0.639232271380537c$.

Notice that when $x < 0$, that is $t < 0$ and $s(t) < -1$, showing that the value t^* is the only extremal value of h. Hence, the claimed value, approximated to two digits, follows

$$h(x^*) = 0.278464542761074c \simeq 0.28c.$$

To prove (6), we have

$$x\tanh\left(\frac{x}{c}\right) \le |x| \iff x^2\tanh^2\left(\frac{x}{c}\right) \le x^2,$$
$$\iff \tanh^2\left(\frac{x}{c}\right) \le 1.$$

□

Theorem 2.3. *The approximation of* $|x|$ *by* $x\tanh(\frac{x}{c})$ *is more accurate than that with* $\sqrt{x^2+c^2}$.

Proof. It is clear that
$$\cosh\left(\frac{x}{c}\right) > \frac{x}{c}.$$
Since $\cosh(x)$ is an even function we have
$$\cosh^2\left(\frac{x}{c}\right) > \frac{x^2}{c^2},$$
then
$$x^2\mathrm{sech}^2\left(\frac{x}{c}\right) < c^2,$$
which in turn gives
$$x^2 - x^2\tanh^2\left(\frac{x}{c}\right) < c^2.$$
Then
$$x^2 - x^2\tanh^2\left(\frac{x}{c}\right) < c^2 = \left(x^2+c^2\right) - x^2.$$
□

Moreover, the function $x\tanh\left(\frac{x}{c}\right)$ converges to $|x|$ faster than $\sqrt{x^2+c^2}$ to $|x|$ by decreasing c, as stated in the next Theorem.

Theorem 2.4. *If* $c \longrightarrow 0^+$ *then* $x\tanh\left(\frac{x}{c}\right) - |x| = o\left(\sqrt{x^2+c^2} - |x|\right)$.

Proof. We have
$$\begin{aligned}\lim_{c\longrightarrow 0^+}\frac{x\tanh\left(\frac{x}{c}\right)-|x|}{\sqrt{x^2+c^2}-|x|} &= \lim_{c\longrightarrow 0^+}\frac{x^2\tanh^2(\frac{x}{c})-x^2}{c^2} \times \lim_{c\longrightarrow 0^+}\frac{\sqrt{x^2+c^2}+|x|}{x\tanh\left(\frac{x}{c}\right)+|x|}\\ &= -x^2\lim_{c\longrightarrow 0^+}\frac{\mathrm{sech}^2\left(\frac{x}{c}\right)}{c^2}\times\frac{2|x|}{2|x|} = -x^2\lim_{t\longrightarrow+\infty}\frac{t^2}{\cosh^2(tx)} = 0\end{aligned}$$
where $t = \frac{1}{c}$.

□

In order to illustrate the superiority of the new hyperbolic approximation to $|x|$, L_∞ error norm
$$\max_{1\le i\le n}|g(x_i) - |x_i||,$$
and the rate of convergence
$$r_c = \frac{\log\left(\frac{E_{c_i}}{E_{c_{i-1}}}\right)}{\log\left(\frac{c_i}{c_{i-1}}\right)},$$
for both approximants $x\tanh\left(\frac{x}{c}\right)$ and $\sqrt{x^2+c^2}$ are reported in Table 1, for $n = 100, 200, 400$ equally spaced points in $[-10, 10]$. Table 1 shows that $x\tanh\left(\frac{x}{c}\right)$ approximates $|x|$ much better than $\sqrt{x^2+c^2}$ while Table 1 and the logarithmic scale plots 1 show that the approximant $x\tanh\left(\frac{x}{c}\right)$ has exponential rate of convergence to $|x|$ as $c \to 0$ instead of $O(c^2)$ provided by $\sqrt{x^2+c^2}$.

2.1 New transcendental RBF

Let us introduce the following globally supported and infinitely differentiable transcendental RBF
$$\phi(r) = r\tanh\left(\frac{r}{c}\right),$$
abbreviated by RTH, where $r = \|x - x_j\|$ and $\|\cdot\|$ is the Euclidean norm $\mathbb{R}^d$.

The parameter $c > 0$ is called *shape parameter*.

Compared with interpolation, the quasi-interpolation method does not require the solution of any linear system, avoiding the ill-conditioning which arises when solving a linear system. The smaller the shape parameter c the faster ϕ approaches to r. So the shape parameter c can be chosen sufficiently small for getting accurate numerical solutions.

Theorem 2.5. *The RTH RBF is conditionally negative definite of order* 1 *on every* $\mathbb{R}^d$.

Proof. We show that $\psi(r) = -\phi(r)$ is conditionally positive definite of order 1. We have $\psi(r) = f(s) = -\sqrt{s}\tanh\left(\frac{\sqrt{s}}{c}\right)$, where $s = r^2$. Now for
$$g(s) = -f'(s) = \frac{1}{2}s^{-\frac{1}{2}}\tanh\left(\frac{\sqrt{s}}{c}\right) + \frac{1}{2c}\left(1-\tanh^2\left(\frac{\sqrt{s}}{c}\right)\right),$$
we have
$$(-1)^l g^{(l)}(s) \ge 0, \quad \text{for all } l \in \mathbb{N}_0 \text{ and all } s > 0.$$
So $-f'(s)$ is completely monotone on $(0, \infty)$. Now, since $f \notin \Pi_m^d$, the claim is proved according to Micchelli's theorem [32]. □

Table 1: L_∞ errors and convergence rates for both approximants of $|x|$ for different values of c.

		$\left\lvert \lvert x\rvert - \sqrt{x^2+c^2}\right\rvert$		$\left\lvert \lvert x\rvert - x\tanh\left(\frac{x}{c}\right)\right\rvert$	
n	c	L_∞ error	r_c	L_∞ error	r_c
100	0.1	4.1127e-02	—	2.3656e-02	—
	0.05	1.1698e-02	1.813823944	3.4922e-03	2.759998057
	0.025	3.0478e-03	1.940421754	6.2490e-05	5.804367034
	0.0125	7.7050e-04	1.983901373	1.9342e-08	11.65767264
	0.00625	1.9317e-04	1.995923901	1.8457e-15	23.32106557
200	0.1	6.1665e-02	—	2.6930e-02	—
	0.05	2.0637e-02	1.579218611	1.1875e-02	1.181286716
	0.025	5.8753e-03	1.812498837	1.7723e-03	2.744232777
	0.0125	1.5314e-03	1.939811357	3.2376e-05	5.774554268
	0.00625	3.8718e-04	1.983774825	1.0436e-08	11.59914019
400	0.1	7.8030e-02	—	2.7348e-02	—
	0.05	3.0867e-02	1.337963627	1.3456e-02	1.023185719
	0.025	1.0337e-02	1.578247724	5.9488e-03	1.177579030
	0.0125	2.9442e-03	1.811869965	8.9273e-04	2.736302862
	0.00625	7.6754e-04	1.939561834	1.6476e-05	5.759785971

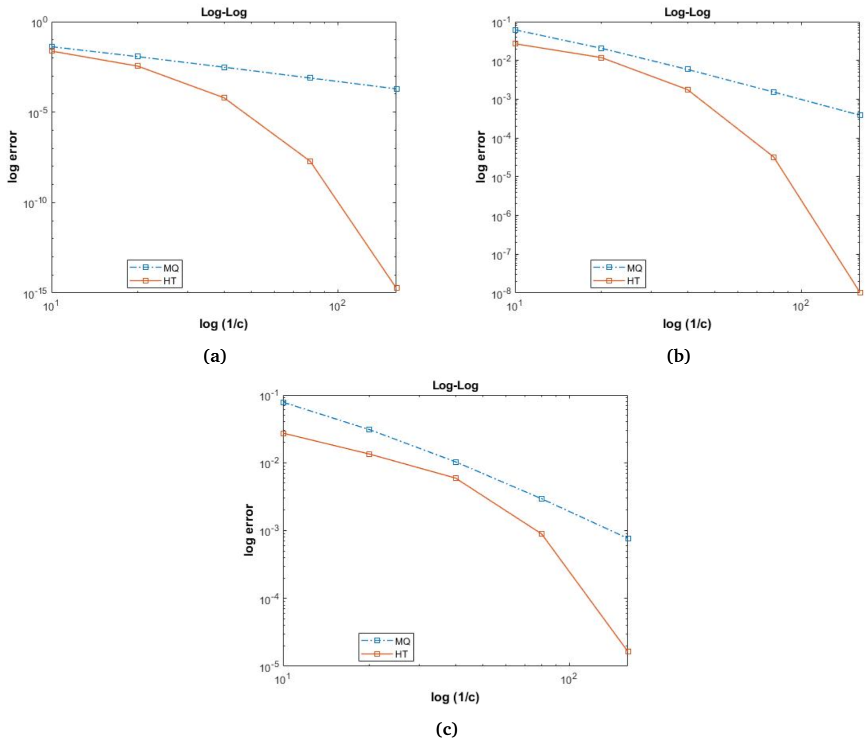


Figure 1: $\log|\text{error}|$ versus $\log(1/c)$ for $n = 100$ (a), $n = 200$ (b), and $n = 400$ (c).

Remark 1. Since ϕ is conditionally negative definite of order 1 and $\phi(0) = 0$, then the matrix $A = \left[\phi(\|x_i - x_j\|)\right]_{1\le i,j\le n}$ has one positive and $n-1$ negative eigenvalues and in particular it is invertible.

In the sequel, we consider $d = 1$, since our work is confined to the univariate case. We have seen before that the RTH RBF is an smooth approximant to $\tau(r) = r$ with higher accuracy and better convergence properties than the MQ RBF $\sqrt{r^2+c^2}$, by decreasing shape parameter c. In Figure 2, we have plotted both RTH basis

$$\phi_j(x) = (x - x_j)\tanh\left(\frac{x - x_j}{c}\right), \tag{7}$$

and MQ basis (2) centered at $x_j = 0$. It can be noted from Figure 2 that the RTH RBF approaches to $|x|$ faster than the MQ RBF, even with larger shape parameters. Moreover, in RTH RBF $\phi_j(x_j) = 0$ independent of the value of c, but MQ requires that $c = 0$. This property of the RTH RBF leads to getting more accurate results in corresponding quasi-interpolants. The first and second derivatives of the RTH RBF (7) are of the form

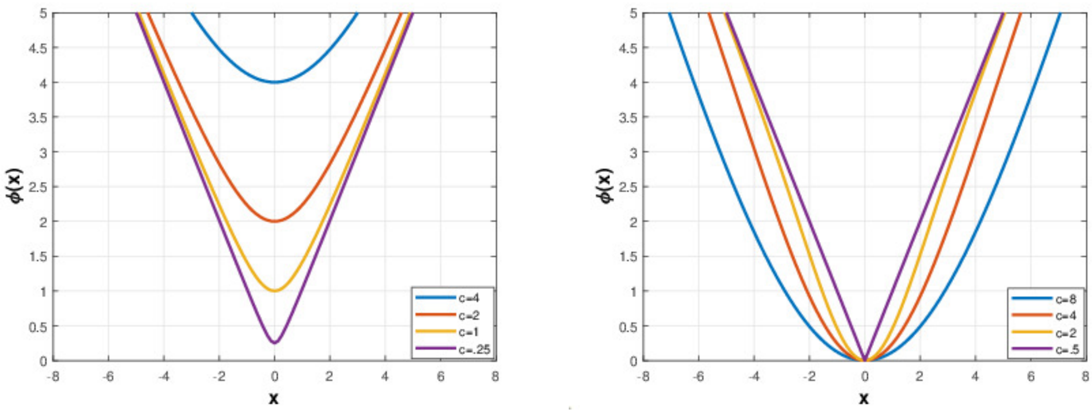
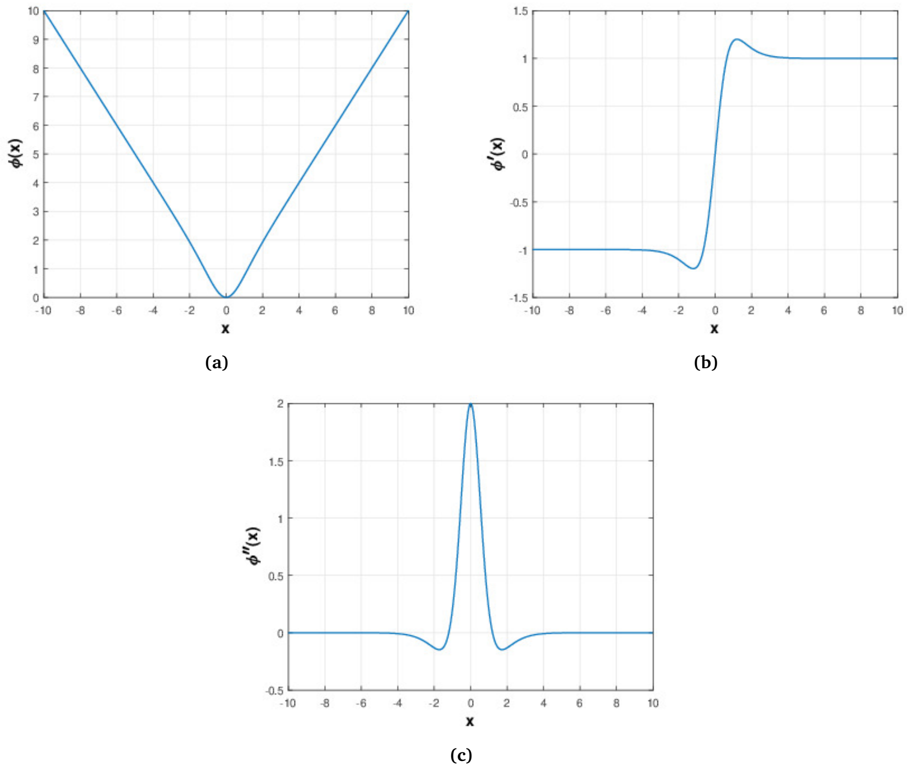


Figure 2: Plots of Multiquadric RBF (left), and RTH RBF (right) for different values of shape parameter c.

$$\begin{aligned}
\phi_j'(x) &= \tanh\left(\frac{x-x_j}{c}\right) + \frac{(x-x_j)}{c}\operatorname{sech}^2\left(\frac{x-x_j}{c}\right),\\
\phi_j''(x) &= \left(\frac{2}{c} - 2\left(\frac{x-x_j}{c^2}\right)\tanh\left(\frac{x-x_j}{c}\right)\right)\operatorname{sech}^2\left(\frac{x-x_j}{c}\right),
\end{aligned}$$

and are plotted in Figure 3 for $c=1$ and centerd at $x_j=0$. In Tables 2 and 3, we summarized the properties of both MQ and



(a) **(b)** **(c)**

Figure 3: (a) RTH RBF, (b) first derivative, (c) second derivative. The shape parameter is $c=1$

RTH RBFs, where $\xi = 1.199678640$, is obtained numerically by calculating the roots of the second derivative.

2.2 Quasi-interpolation operator

The quasi-interpolation operator of a function $f:[a,b]\to\mathbb{R}$ with RTH RBF on the scattered points

$$a = x_0 < x_1 < \cdots < x_n = b \qquad h := \max_{2\le j\le n}(x_j - x_{j-1}), \tag{8}$$

Table 2: Comparing both RBFs.

Name	$\phi_j(x)$	$\lim_{x\to x_j}\phi_j(x)$	$\lim_{c\to 0}\phi_j(x)$	$\lim_{x\to\pm\infty}\phi_j^{'}(x)$	condition
MQ RBF	$\sqrt{c^2+(x-x_j)^2}$	c	$\lvert x-x_j\rvert$	± 1	$x\in(-\infty,\infty)$
RTH RBF	$\big(x-x_j\big)\tanh\Big(\frac{x-x_j}{c}\Big)$	0	$\lvert x-x_j\rvert$	± 1	$x\in(-\infty,\infty)$

Table 3: Comparing both RBFs.

Name	$\phi_j(x)$	$\phi_j^{'}(x)$	$\phi_j^{''}(x)$	condition
MQ RBF	$\sqrt{c^2+(x-x_j)^2}$	Strictly increasing	≥ 0	$x\in(-\infty,\infty)$
RTH RBF	$\big(x-x_j\big)\tanh\Big(\frac{x-x_j}{c}\Big)$	Strictly increasing	≥ 0	$x\in[-c\xi,c\xi]$

has the form

$$(\mathcal{L}_{RTH}f)(x)=f_0\alpha_0(x)+f_1\alpha_1(x)+\sum_{j=2}^{n-2}f_j\psi_j(x)+f_{n-1}\alpha_{n-1}(x)+f_n\alpha_n(x) \tag{9}$$

where

$$\begin{aligned}
\alpha_0(x)&=\frac{1}{2}+\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},\\
\alpha_1(x)&=\frac{\phi_2(x)-\phi_1(x)}{2(x_2-x_1)}-\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},\\
\alpha_{n-1}(x)&=\frac{(x_n-x)-\phi_{n-1}(x)}{2(x_n-x_{n-1})}-\frac{\phi_{n-1}(x)-\phi_{n-2}(x)}{2(x_{n-1}-x_{n-2})},\\
\alpha_n(x)&=\frac{1}{2}+\frac{\phi_{n-1}(x)-(x_n-x)}{2(x_n-x_{n-1})},\\
\phi_j(x)&=\big(x-x_j\big)\tanh\Big(\frac{x-x_j}{c}\Big),\quad j=1,\dots,n-1,\ c\in\mathbb{R}_+,\\
\psi_j(x)&=\frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)}-\frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1})},\qquad 2\leq j\leq n-2.
\end{aligned}$$

The formula (9) can be rewritten as

$$\begin{aligned}
(\mathcal{L}_{RTH}f)(x)\quad=\quad&\frac{1}{2}\sum_{j=1}^{n-1}f[x_{j-1},x_j,x_{j+1}](x_{j+1}-x_{j-1})\phi_j(x)+\\
&\frac{f_0+f_n}{2}+\frac{1}{2}f[x_0,x_1](x-x_0)-\frac{1}{2}f[x_{n-1},x_n](x_n-x).
\end{aligned} \tag{10}$$

Let $\phi_{-1}(x)=|x-x_{-1}|$, $\phi_0(x)=|x-x_0|$, $\phi_n(x)=|x-x_n|$ and $\phi_{n+1}(x)=|x-x_{n+1}|$, then for $x\in[x_0,x_n]$, the operator $\mathcal{L}_{RTH}$ can be rearranged as

$$(\mathcal{L}_{RTH}f)(x)=\sum_{j=0}^{n}f_j\psi_j(x), \tag{11}$$

where

$$\psi_j(x)=\frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)}-\frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1})},\quad j=0,\dots,n,$$

and $x_{-1}<x_0,\ x_{n+1}>x_n$.

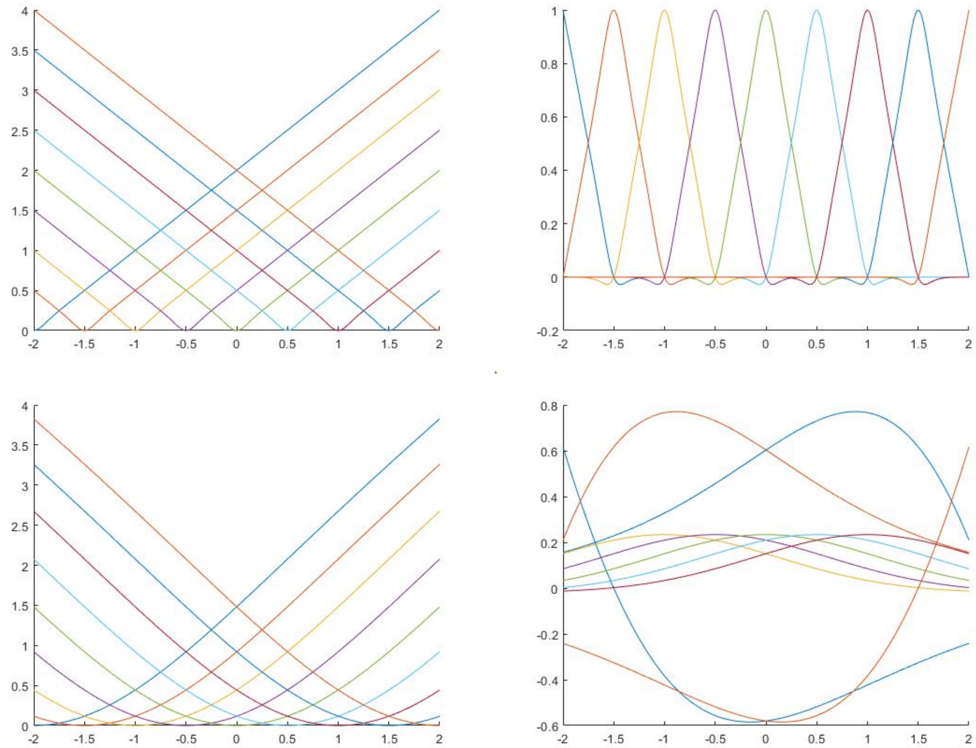


Figure 4: First row: the basis functions ϕ_j for $c = 0.1$ and the corresponding functions ψ_j on $[-2, 2]$. Second row: the same functions for $c = 2.1$

Remark 2. From relation (10), it is clear that the quasi-interpolation operator $\mathcal{L}_{RTH}$ reproduces the linear polynomials on $[x_0, x_n]$, that is

$$\sum_{j=0}^{n}(ax_j + b)\psi_j(x) = ax + b, \quad a, b \in \mathbb{R}, \tag{12}$$

from which we also get $\sum_{j=0}^{n} \psi_j(x) = 1$ at any point $x \in [x_0, x_n]$ (see also Figure 4).

The first and second derivatives of $(\mathcal{L}_{RTH} f)(x)$ in (10) can be calculated as follows

$$(\mathcal{L}_{RTH} f)'(x) = \frac{1}{2}\sum_{j=1}^{n-1} f[x_{j-1}, x_j, x_{j+1}](x_{j+1} - x_{j-1})\phi_j'(x) + \frac{1}{2}\left(f[x_0, x_1] + f[x_{n-1}, x_n]\right). \tag{13}$$

$$(\mathcal{L}_{RTH} f)''(x) = \frac{1}{2}\sum_{j=1}^{n-1}\left[\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right]\phi_j''(x). \tag{14}$$

In order to prove the shape preserving property of the quasi-interpolation operator (9), we give some important definitions and theorems from differential geometry (cf. e.g. [34]).

Definition 2.1. A differentiable plane curve $\alpha : (a, b) \to \mathbb{R}^2$ is said to be regular if its derivative never vanishes. That is

$$\forall t \in (a, b), \qquad \alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}\right) \neq (0, 0).$$

Theorem 2.6. *Let C be a regular plane curve given by $\alpha(t)$. Then the curvature κ of C at t is given by*

$$\kappa[\alpha](t) = \left\|\alpha'(t) \times \alpha''(t)\right\| / \left\|\alpha'(t)\right\|^3.$$

Definition 2.2. Let $f \in C^2[a, b]$. The curvature of the plane curve $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{\frac{3}{2}}}.$$

Definition 2.3. An isometry of $\mathbb{R}^2$ is a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$d(F(p), F(q)) = d(p, q),$$

for all points p, q in $\mathbb{R}^2$.

Theorem 2.7. *Every isometry of $\mathbb{R}^2$ is the composition of translations, reflections and rotations.*

Definition 2.4. Two curves $\alpha, \beta : I \longrightarrow \mathbb{R}^2$ are congruent provided there exists an isometry F of $\mathbb{R}^2$ such that $\beta = F(\alpha)$; that is, $\beta(t) = F(\alpha(t))$ for all t in I.

Intuitively speaking, congruent curves are the same except for position in space.

Theorem 2.8. *If $\alpha, \beta : I \longrightarrow \mathbb{R}^2$ are plane curves such that $\kappa_\alpha = \kappa_\beta$, then α and β are congruent.*

Theorem 2.9. *The quasi-interpolation operator $\mathcal{L}_{RTH}$ constructed by data points $\{(x_j, f_j)\}$, is monotonicity and convexity preserving for c small enough.*

Proof. We show that the MQ and RTH quasi-interpolants are congruent for c small enough. According to the Theorem 2.8, we prove that

$$\lim_{c\to 0} |\kappa_{\mathcal{L}_{MQ}}(x) - \kappa_{\mathcal{L}_{RTH}}(x)| = 0.$$

Let $x \neq x_j$, otherwise both quasi-interpolants (3) and (9) do not have first and second derivatives as c approaches 0. Now, according to definition 2.2, we have

$$\kappa_{\mathcal{L}_{MQ}}(x) = \frac{|(\mathcal{L}_{MQ}f)''(x)|}{\left(1 + \left((\mathcal{L}_{MQ}f)'(x)\right)^2\right)^{\frac{3}{2}}}.$$

Since for MQ RBF,

$$\phi_j''(x) = \frac{c^2}{\left(c^2 + (x - x_j)^2\right)^{3/2}},$$

then

$$\lim_{c\longrightarrow 0} \phi_j''(x) = 0.$$

Moreover

$$(\mathcal{L}_{MQ}f)''(x) = \frac{1}{2}\sum_{j=1}^{n-1}\left[\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right]\phi_j''(x),$$

then

$$\lim_{c\longrightarrow 0} \kappa_{\mathcal{L}_{MQ}}(x) = 0,$$

which leads to

$$\forall \epsilon > 0 \quad \exists \delta_1 > 0; \quad |c| < \delta_1 \Rightarrow |\kappa_{\mathcal{L}_{MQ}}(x)| < \epsilon.$$

Similarly, for RTH RBF, we have

$$\lim_{c\longrightarrow 0} \phi_j''(x) = 0,$$

then

$$\lim_{c\longrightarrow 0} \kappa_{\mathcal{L}_{RTH}}(x) = 0,$$

which leads to

$$\forall \epsilon > 0 \quad \exists \delta_2 > 0; \quad |c| < \delta_2 \Rightarrow |\kappa_{\mathcal{L}_{RTH}}(x)| < \epsilon.$$

The proof completes by considering $\delta = \min\{\delta_1, \delta_2\}$. □

3 Accuracy of the quasi-interpolation operator $\mathcal{L}_{RTH}$

In this section, we give an approximation order for the quasi-interpolation operator $\mathcal{L}_{RTH}$.

Theorem 3.1. *Assume f'' is Lipschitz continuous. The quasi-interpolation operator $\mathcal{L}_{RTH}f$, at the point set (8) as $h \to 0$, converges as follows*

$$\|f - \mathcal{L}_{RTH}f\|_\infty \leq kh^2, \tag{15}$$

where k is independent of h and c.

Proof. Let $t(y)$ be the local Taylor approximation of f at y, that is

$$t(y) = f(x) + f'(x)(y-x), \quad x \in [a,b]$$

According to Remark 2, we get

$$\sum_{j=0}^{n}(x-x_j)\psi_j(x) = 0, \quad \sum_{j=0}^{n}\psi_j(x) = 1.$$

Then we get

$$\begin{aligned}\sum_{j=0}^{n} t(x_j)\psi_j(x) &= \sum_{j=0}^{n}\big[f(x) + f'(x)(x_j - x)\big]\psi_j(x)\\ &= f(x)\sum_{j=0}^{n}\psi_j(x) + f'(x)\sum_{j=0}^{n}(x-x_j)\psi_j(x)\\ &= f(x).\end{aligned}$$

Since $f''(x)$ is Lipschitz continuous, then for every $x_1,\ x_2 \in [a,b]$, $|f''(x_1) - f''(x_2)| \leq c_0|x_1 - x_2|$, where $0 < c_0 = \text{ess}\sup_{a\leq x\leq b}|f'''(x)|$.

Now according to (10) and the reproduction of linear functions, it is enough to bound the difference $\mathcal{L}_{RTH}f - \mathcal{L}f$, with $\mathcal{L}f$ the linear approximation of f.

$$\begin{aligned}|\mathcal{L}_{RTH}f(x) - \mathcal{L}f(x)| &= \left|\sum_{j=0}^{n}\big(f(x_j) - t(x_j)\big)\psi_j(x)\right|\\ &\leq \frac{1}{2}\left|\sum_{j=1}^{n-1}\big(f[x_{j-1},x_j,x_{j+1}] - t[x_{j-1},x_j,x_{j+1}]\big)(x_{j+1}-x_{j-1})\phi_j(x)\right|\\ &\leq \frac{1}{4}\sum_{j=1}^{n-1}|f''(\xi) - f''(\eta)||\phi_j(x)(x_{j+1}-x_{j-1})|, \quad (\xi,\eta \in (x_{j-1},x_{j+1}))\\ &\leq \frac{1}{2}c_0 h\sum_{j=1}^{n-1}|x-x_j|(x_{j+1}-x_{j-1})\\ &\leq \frac{1}{2}c_0 h\sum_{\substack{j=1\\ |x-x_j|\leq h}}^{n-1}|x-x_j|(x_{j+1}-x_{j-1}) + \frac{1}{2}c_0 h\sum_{\substack{j=1\\ |x-x_j|>h}}^{n-1}|x-x_j|(x_{j+1}-x_{j-1})\\ &\leq 4c_0h^3 + c_0h\left(\int_{|x-t|>h}|x-t|\,dt + O(h)\right)\\ &\leq k_1h^3 + k_2h^2\\ &\leq kh^2\end{aligned}$$

□

4 Numerical results

In this section, we compare the accuracy of the quasi-interpolation operator $\mathcal{L}_{RTH}$ with that of Wu and Schaback, $\mathcal{L}_{MQ}$ (defined in (3)) for the approximation of five functions. We take equidistant center points and choose different shape parameters c and also different step sizes h. The maximum absolute error norm is then computed for comparing approximation accuracy. The rate of convergence is also computed by

$$r_h = \frac{\ln\left(\frac{E_{h_i}}{E_{h_{i-1}}}\right)}{\ln\left(\frac{h_i}{h_{i-1}}\right)},$$

where E_{h_i} indicates the error of the quasi-interpolant $\mathcal{L}_{RTH}f$ corresponding to the parameter h_i. In all tests, we choose $m = 220$ equidistant evaluation points. All experiments have been done using an Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz, 2.11 GHz.

4.1 Test problem 1

In the first test problem, we apply the RTH quasi-interpolation to approximate the function (cf. [8])

$$f_1(x) = \frac{\sinh(x)}{1+\cosh(x)}, \quad x \in [-3,3].$$

The results are shown in Tables 4-6. In Tables 4, 5, and 6, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|\mathcal{L}_{RTH}f - f\|_\infty$ and $\|\mathcal{L}_{MQ}f - f\|_\infty$. In Table 7, we set $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate r_h of $\mathcal{L}_{RTH}f$ with the variation of h.

Table 4: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 1.

c	0.2	0.1	0.05	0.02	0.01
h	0.1	0.1	0.1	0.1	0.1
$\|\mathcal{L}_{MQ}f - f\|_\infty$	9.3×10^{-3}	3.1×10^{-3}	1.1×10^{-3}	3.9×10^{-4}	2.7×10^{-4}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	2.9×10^{-3}	6.2×10^{-4}	7.1×10^{-5}	2.2×10^{-4}	1.4×10^{-4}

Table 5: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 1.

c	0.02	0.01	0.005	0.002	0.001
h	0.01	0.01	0.01	0.01	0.01
$\|\mathcal{L}_{MQ}f - f\|_\infty$	1.8×10^{-4}	5.3×10^{-5}	1.6×10^{-5}	4.7×10^{-6}	2.4×10^{-6}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	3.0×10^{-5}	6.3×10^{-6}	7.1×10^{-7}	2.2×10^{-6}	1.4×10^{-6}

Table 6: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 1.

c	0.002	0.001	0.0005	0.0002	0.0001
h	0.001	0.001	0.001	0.001	0.001
$\|\mathcal{L}_{MQ}f - f\|_\infty$	2.7×10^{-6}	7.5×10^{-7}	2.2×10^{-7}	5.6×10^{-8}	3.2×10^{-8}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	3.0×10^{-7}	6.3×10^{-8}	7.1×10^{-9}	2.2×10^{-8}	2.4×10^{-8}

Table 7: Convergence rates of $\mathcal{L}_{RTH}f$ by using $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$; Test problem 1.

c	0.01	0.01	0.01	0.01	0.01
h	0.2	0.1	0.05	0.025	0.0125
$\|\mathcal{L}_{RTH}f - f\|_\infty$	9.5×10^{-4}	2.4×10^{-4}	5.6×10^{-5}	5.6×10^{-6}	2.5×10^{-6}
r_h	-	1.9848	2.0995	3.3219	1.1634

4.2 Test problem 2

In this experiment we apply the RTH quasi-interpolation to approximate the function (again considered in [8])

$$f_2(x) = \sin\left(\frac{x}{2}\right) - 2\cos(x) + 4\sin(\pi x), \quad x \in [-4,4]. \tag{16}$$

The comparison results are shown in Tables 8-10. In Tables 8, 9, and 10, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|\mathcal{L}_{RTH}f - f\|_\infty$ and $\|\mathcal{L}_{MQ}f - f\|_\infty$. In Table 11, we set $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate r_h of $\mathcal{L}_{RTH}f$ with the variation of h.

Table 8: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 2.

c	0.2	0.1	0.05	0.02	0.01
h	0.1	0.1	0.1	0.1	0.1
$\|\mathcal{L}_{MQ}f - f\|_\infty$	1.2×10^{0}	4.5×10^{-1}	1.7×10^{-1}	7.2×10^{-2}	5.6×10^{-2}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	4.5×10^{-1}	1.2×10^{-1}	1.4×10^{-2}	4.8×10^{-2}	5.1×10^{-2}

Table 9: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 2.

c	0.02	0.01	0.005	0.002	0.001
h	0.01	0.01	0.01	0.01	0.01
$\|\mathcal{L}_{MQ}f - f\|_\infty$	3.0×10^{-1}	9.2×10^{-3}	2.9×10^{-3}	9.1×10^{-4}	6.1×10^{-4}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	6.4×10^{-3}	1.3×10^{-3}	1.5×10^{-4}	4.8×10^{-4}	5.1×10^{-4}

Table 10: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 2.

c	0.002	0.001	0.0005	0.0002	0.0001
h	0.001	0.001	0.001	0.001	0.001
$\|\mathcal{L}_{MQ}f - f\|_\infty$	4.9×10^{-4}	1.4×10^{-4}	4.1×10^{-5}	1.1×10^{-5}	6.6×10^{-6}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	6.5×10^{-5}	1.4×10^{-5}	1.5×10^{-6}	4.8×10^{-6}	5.0×10^{-6}

Table 11: Convergence rates of $\mathcal{L}_{RTH}f$ by using $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$; Test problem 2.

c	0.01	0.01	0.01	0.01	0.01
h	0.2	0.1	0.05	0.025	0.0125
$\|\mathcal{L}_{RTH}f - f\|_\infty$	2.3×10^{-1}	5.1×10^{-2}	1.2×10^{-2}	1.1×10^{-3}	5.0×10^{-4}
r_h	-	2.1730	2.0874	3.4474	1.1375

4.3 Test problem 3

Consider the function (see again [8])

$$f_3(x) = 10e^{-x^2} + x^2, \quad x \in [-3, 3], \tag{17}$$

for approximating by the RTH quasi-interpolation operator. The comparison results are shown in Tables 12-14. In Tables 12, 13, and 14, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|\mathcal{L}_{RTH}f - f\|_\infty$ and $\|\mathcal{L}_{MQ}f - f\|_\infty$. In Table 15, we set $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate r_h of $\mathcal{L}_{RTH}f$ on varying h.

Table 12: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 3.

c	0.2	0.1	0.05	0.02	0.01
h	0.1	0.1	0.1	0.1	0.1
$\|\mathcal{L}_{MQ}f - f\|_\infty$	4.9×10^{-1}	1.9×10^{-1}	7.4×10^{-2}	3.1×10^{-2}	2.4×10^{-2}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	2.2×10^{-1}	5.4×10^{-2}	6.4×10^{-3}	1.1×10^{-2}	1.2×10^{-2}

Table 13: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 3.

c	0.02	0.01	0.005	0.002	0.001
h	0.01	0.01	0.01	0.01	0.01
$\|\mathcal{L}_{MQ}f - f\|_\infty$	9.7×10^{-2}	2.9×10^{-2}	9.09×10^{-3}	2.5×10^{-3}	1.5×10^{-3}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	2.8×10^{-3}	5.9×10^{-4}	6.7×10^{-5}	2.1×10^{-4}	2.2×10^{-4}

Table 14: Comparison of approximation accuracy of RTH and MQ quasi-interpolation; Test problem 3.

c	0.002	0.001	0.0005	0.0002	0.0001
h	0.001	0.001	0.001	0.001	0.001
$\|\mathcal{L}_{MQ}f - f\|_\infty$	2.1×10^{-4}	6.0×10^{-5}	1.8×10^{-5}	4.8×10^{-6}	2.9×10^{-6}
$\|\mathcal{L}_{RTH}f - f\|_\infty$	2.8×10^{-5}	5.9×10^{-6}	6.7×10^{-7}	2.1×10^{-6}	1.2×10^{-6}

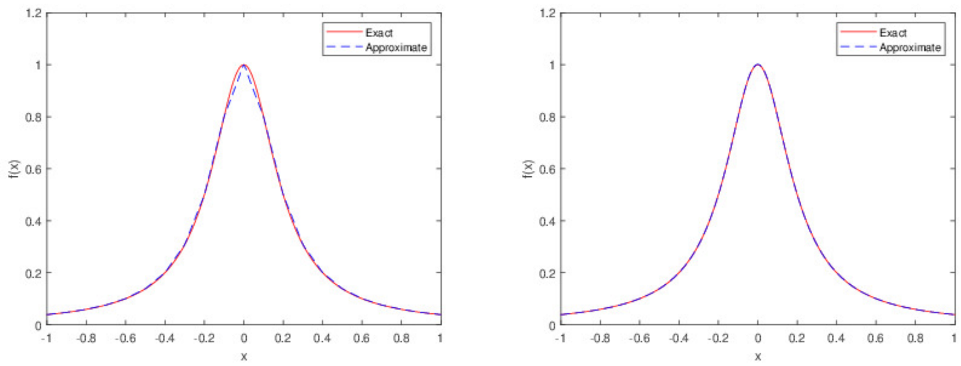
Remark 3. By analyzing the results in Tables 4-6, 8-10, and 12-14, we see that the accuracy of the RTH quasi-interpolation scheme is dependent on the shape parameter c and on step size h. Furthermore, the accuracy of the RTH quasi-interpolation operator is better than that of MQ for the same values of c and h. From Tables 7, 11, 15, we see that the convergence rate of $\mathcal{L}_{RTH}$ reaches up to 2 which justifies our theoretical findings of Section 3. By these numerical experiments, we can say that the quasi-interpolation $\mathcal{L}_{RTH}$ is a very attractive alternative, in terms of accuracy and convergence, to $\mathcal{L}_{MQ}$.

Table 15: Convergence rates of $\mathcal{L}_{RTH}f$ by using $c = 0.01$, $h = 0.2, 0.1, 0.05, 0.025, 0.0125$; Test Problem 3.

c	0.01	0.01	0.01	0.01	0.01
h	0.2	0.1	0.05	0.025	0.0125
$\|\mathcal{L}_{RTH}f - f\|_\infty$	8.6×10^{-2}	2.2×10^{-2}	5.2×10^{-3}	4.9×10^{-4}	2.2×10^{-4}
r_h	-	1.9668	2.0809	3.4076	1.1552

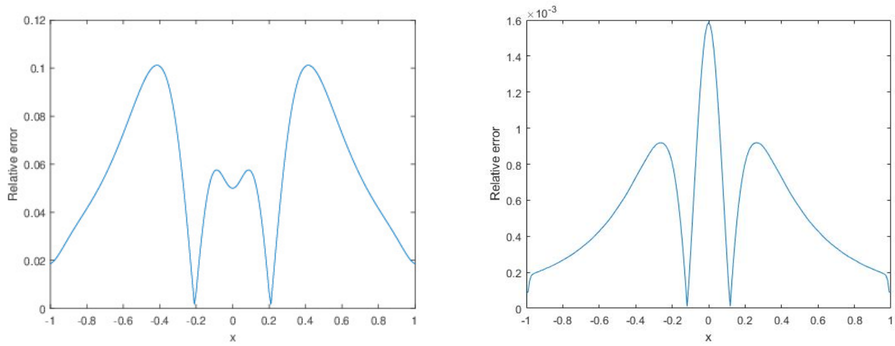
4.4 Test problem 4 (Runge function)

Let us consider the Runge function on $[-1,1]$, that is $f_4(x) = \dfrac{1}{1+25x^2}$. Figure 5 shows the exact and approximate values of f_4 for $c = 0.01,\ h = 0.1,\ 0.02$. In Figure 5, we see that the Runge phenomenon has disappeared by decreasing h. Relative errors are shown in Figure 6 using the RTH quasi-interpolation operator.

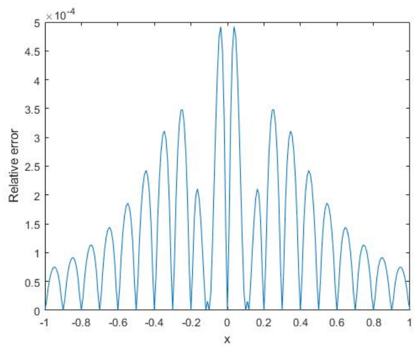


(a) (b)

Figure 5: RTH quasi-interpolation of $f_4(x) = \frac{1}{1+25x^2}$; $h = 0.1$ (a), $h = 0.02$ (b), and $c = 0.01$.



(a) (b)



(c)

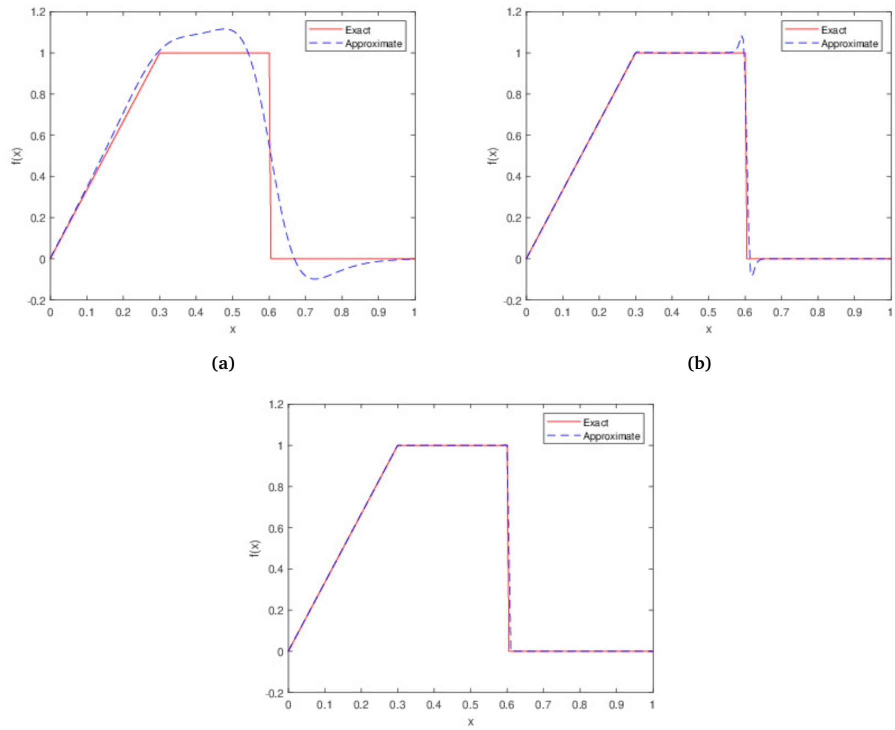
Figure 6: Relative errors: $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.01$; Test problem 4.

4.5 Test problem 5 (Gibbs Phenomenon)

It is well-known that any global or high order approximation method suffers from the Gibbs phenomenon if the function has a jump discontinuity in the given domain. In this test problem, we show that the RTH quasi-interpolation operator substantially mitigates the Gibbs phenomenon (cf. [3]).

$$f_5(x) = \begin{cases} \frac{10}{3}x, & 0 \le x \le 0.3, \\ 1, & 0.3 \le x \le 0.6, \\ 0, & 0.6 < x \le 1. \end{cases}$$

Figure 7 shows the exact and approximate values of f_5. In Figure 7, we see that the Gibbs oscillations are considerably attenuated by decreasing c. Relative errors are reported in Figure 8.



(a) **(b)** **(c)**

Figure 7: Approximations of f_5 with RTH quasi-interpolation; $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.01$.

4.6 Test problem 6 (A piecewise analytic function)

As a final example, we consider the piecewise analytic function (cf. [23])

$$f_6(x) = \begin{cases} \sin(x), & x < 0, \\ \cos(x), & x > 0, \end{cases}$$

with $x \in [-1, 1]$. Figure 9 shows the exact and approximate values of f_6, where Gibbs oscillations are considerably attenuated by decreasing c. Relative errors are shown in Figure 10.

4.7 Test problem 7 (Derivatives approximation)

In this test problem, we approximate the first and second derivatives of the following function [29]

$$f_7(x) = \cos(x)e^{2x^2}, \quad x \in [0, 1], \tag{18}$$

based on the RTH quasi-interpolation schemes (13)-(14). The relative error functions for the first and second derivatives are plotted in Figures 11-12, respectively. It can be noted from Figures 11-12 that the the accuracy is poor near the boundary. We should handle this problem by introducing a polynomial like the one discussed in [29]. We leave this to our further works.

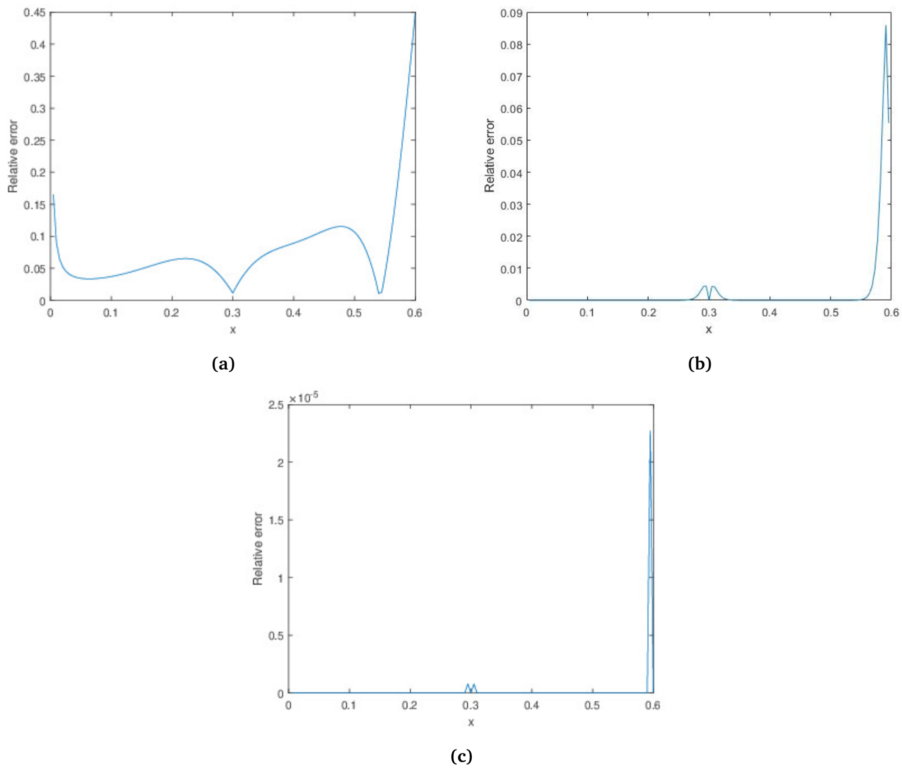


Figure 8: Relative errors: $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.01$; Test problem 5.

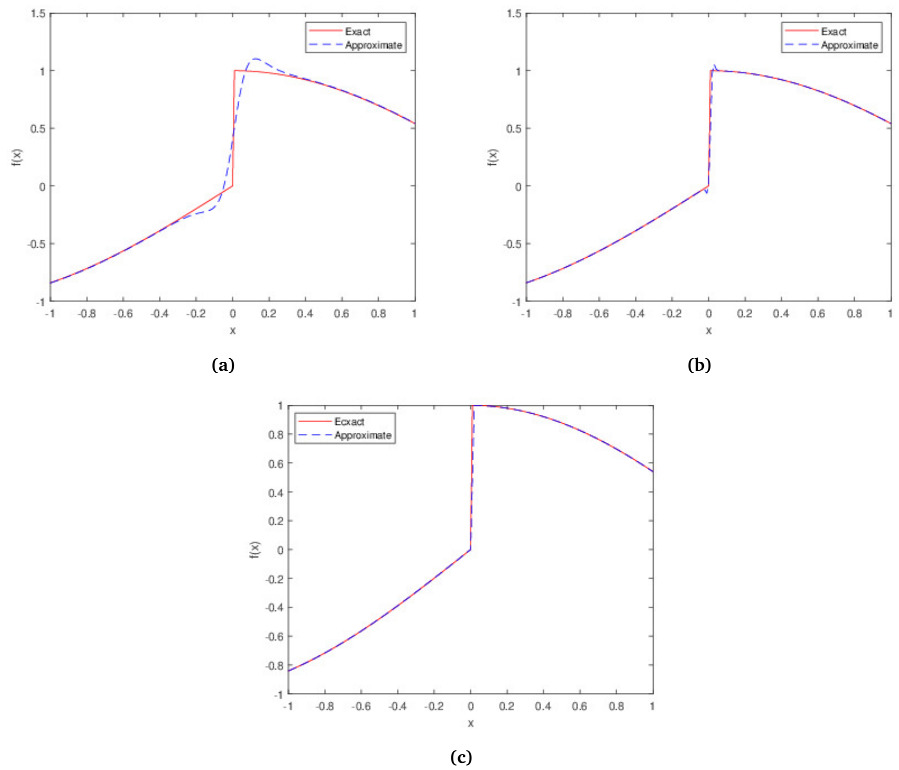
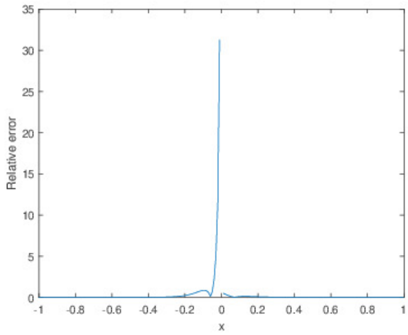
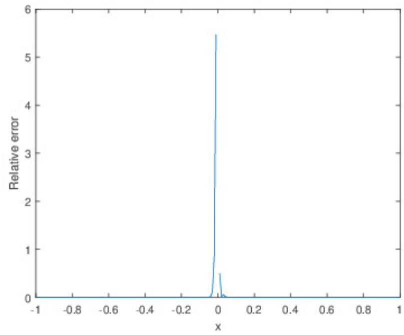


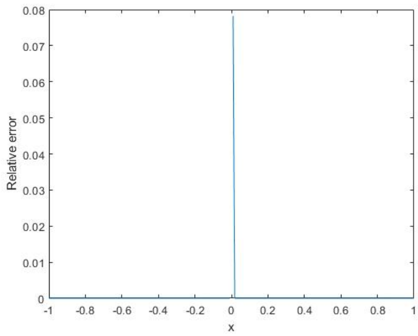
Figure 9: RTH quasi-interpolation of the piecewise analytic function f_6; $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.02$.



(a)

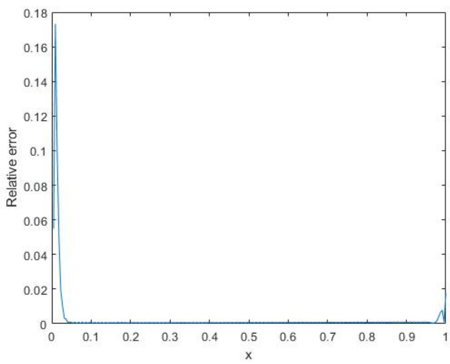


(b)

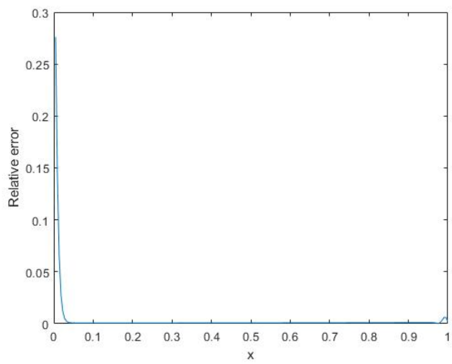


(c)

Figure 10: Relative errors: $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.01$; Test problem 6.



(a)



(b)

Figure 11: Relative errors of the first derivative approximation by RTH quasi-interpolation for $h = 0.01$ (a), $h = 0.001$ (b), and $c = 0.01$; Test problem 7.

5 Conclusion

In this paper, an efficient shape preserving quasi-interpolation operator with high degree of smoothness and very accurate results is proposed. It is based on the reformulation of Wu-Schaback quasi-interpolation operator by a new transcendental RBF of the form $\phi(r) = r \tanh\left(\frac{r}{c}\right)$. The quasi-interpolation operator, called $\mathcal{L}_{RTH}$ has nice convergence properties, being $\|\mathcal{L}_{RTH} - f\|_\infty \leq k h^2$, with h being the step size and k a positive constant independent on the shape parameter c and the step size h (cf. Theorem 3.1). Numerical experiments reveal that the proposed quasi-interpolation operator not only gives very accurate results but also it does not suffer of the Runge and Gibbs phenomena (see Test problems 4-6). As a future work we go further into the approximation properties to the high order derivatives by using RTH quasi-interpolation. We also work in the application of the operator to real worlds problems, in particular to irregular surfaces approximation and image segmentation.

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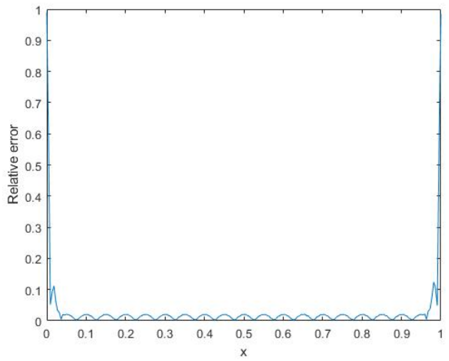
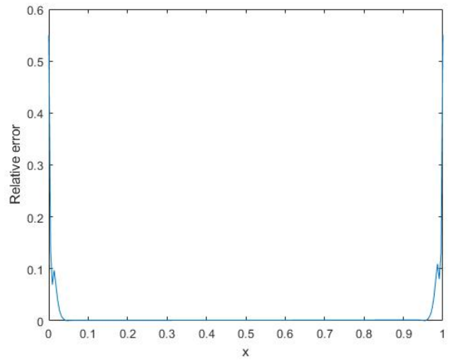


(a)



(b)

Figure 12: Relative errors of the second derivative approximation by RTH quasi-interpolation for $h = 0.01$ (a), $h = 0.001$ (b), and $c = 0.01$; Test problem 7.

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