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Some Remarks on the Variational Iteration Method

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Abstract

In this paper, we reconstruct the He's iterative scheme in order to show that the He's variational iteration method can be handled without using the correction functional and restricted variations. We apply the Laplace transform to determine the general Lagrange multiplier without invoking variational theory. We conclude with an interesting comparison between the two methods of successive approximations and the He's variational iteration.

Keywords: Laplace transform, variational iteration method, successive approximations

1 Introduction

In the last few decades, considerable work has been devoted to develop new methods to analytically/numerically solve linear and/or nonlinear differential and/or integral equations. There has been a great amount of research done to address the issues of nonlinearity and singularity phenomena that arise in many scientific and engineering problems.

The variational iteration method proposed by Ji-Huan He [1] is one of the methods which have received mach attention. It is based on modification of the general Lagrange multiplier [13], restricted variations and correction functional which has found a wide range application for the solution of nonlinear ordinary and partial differential equations. In fact, this method can be considered as an interesting application of Banach's Fixed-point Theorem in Banach spaces or Contraction Mapping Theorem in metric spaces which can give rise to solution which converges rapidly in a large class of nonlinear problems.

The variational iteration method was successfully applied to autonomous ordinary differential equations [3], to nonlinear polycrystalline solids [6], to the construction of solitary solutions and compaction-like solutions for nonlinear dispersive equations [4], to Schrödinger-KdV, generalized KdV and shallow water equations [8], to Burgers and coupled Burgers equations [9], to the linear Helmholtz partial differential equation [10], to the nonlinear fractional differential equations with Caputo differential derivative [11], to the nonlinear differential-difference equations [12] among other places. Reader is referred to [14] for further applications of the method.

Recently, Ramos showed that the He's variational iteration method can be derived by means of adjoint operators, Green's function, integration by parts and the method of weighted residuals without making any recourse whatsoever to Lagrange multipliers, correction functionals and restricted variations [7]. Similarly, in this paper a new approach is proposed to construct the He's iterative scheme by the use of Laplace transform without making any recourse whatsoever to Lagrange multipliers, correction functionals and restricted variations. Using this technique, the construction of Lagrange multiplier reduces to a simple inverse Laplace transform instead of solving the Euler-Lagrange equation. We also show that the claim "the *variational iteration method is nothing else by the Picard-Lindelöf theory for initial-value problems in ordinary differential equations*" [7],

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is incorrect except in special cases. Indeed, we show that such a claim is correct just for first-order ordinary differential equations and a few higher order ordinary differential equations.

The paper is organized as follows. In Section 2, we elaborate our idea in construction of Lagrange multiplier by the use of Laplace transform. We then illustrate the new approach by a general example in Section 3. Section 4 is devoted to comparison of the method of successive approximations and the He's variational iteration method through an exhaustive argument which investigate their coincidence in a variety of ordinary differential equations. In Section 5 some examples are presented in order to compare the elapsed time of both methods when they give rise to the same results. The paper is concluded with a summary in the last section.

2 Another approach to the He's variational iteration method

Consider the following nonlinear ordinary differential equation:

$$L[u(t)]+N[u(t)]=h(t), \quad t\geq 0 \tag{1}$$

where $u(t)$ is an unknown function, L is a linear differential operator, N is a nonlinear operator and $h(t)$ is a given function. Following [5], the basic character of the He's variational iteration method is to construct a correction functional for (1), which reads

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\tau)(L[u_n(\tau)]+ N[\tilde{u}_n(\tau)]-h(\tau))d\tau.$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_n is the n th approximate solution and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n=0$. Therefore, for determining the unknown function λ , an Euler-Lagrange's differential equation with natural boundary conditions must be solved.

In the following we intend to establish the He's iterative scheme in a different manner and to determine the unknown function λ without requiring any correction functionals, Lagrange multipliers or restricted variations.

Assume that $\lambda(t)$ is an unknown function which will be determined later and multiply (1) by $\lambda(0)$, integrate it and then add $u(t)$ to both sides to obtain

$$u(t)=u(t)+ \int_0^t \lambda(t-\tau)(L[u(\tau)]+N[u(\tau)]-h(\tau))d\tau, \tag{2}$$

this can be solved iteratively as

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(t-\tau)(L[u_n(\tau)]+ N[u_n(\tau)]-h(\tau))d\tau. \tag{3}$$

The iterative scheme (3) is exactly the same as that of the He's variational iteration method [1-12] For determining the unknown function $\lambda(t)$, we take the Laplace transform of (3), thereby obtaining

$$\ell[u_{n+1}]=\ell[u_n]+ \ell[\lambda](\ell[L[u_n]]+\ell[N[u_n]-h]). \tag{4}$$

Theorem 1 Let

$$L[u(t)]=c_m(t)u^{(m)}(t)+\cdots+c_1(t)u'(t)+ c_0(t)u(t),$$

where for $i=0,\ldots,m$ the function $c_i(t)$ is analytic at $t=0$. Then we have

$$\ell[L[u]]=F(s)\ell[u]+G(s)+H(s),$$

where $F(s)$ and $H(s)$ are polynomials of s of degree m and $m-1$, respectively.

Proof. Since for $i=0,\ldots,m$ the function $c_i(t)$ is analytic at $t=0$ we can write

$$c_i(t)=\sum_{j=0}^{\infty}a_{ij}t^j,$$

and therefore

$$\ell[L[u]]=\sum_{i=0}^{m}\sum_{j=0}^{\infty}a_{ij}\ell[t^j u^{(i)}]$$

$$=\sum_{i=0}^{m}\sum_{j=0}^{\infty}(-1)^j a_{ij}\frac{d^j\ell[u^{(i)}]}{ds^j}$$

this can be rewritten as

$\ell[L[u]]$

$$=\sum_{i=0}^{m}\sum_{j=0}^{\infty}(-1)^j a_{ij}\frac{d^j}{ds^j}\left(s^i\ell[u]-\sum_{k=1}^{i}s^{i-k}u^{(k-1)}(0)\right).$$

But this is equivalent to

$$\ell[L[u]]=\sum_{i=0}^{m}\sum_{j=0}^{\infty}(-1)^j a_{ij}\frac{d^j(s^i\ell[u])}{ds^j}-\sum_{i=0}^{m}\sum_{j=0}^{\infty}(-1)^j a_{ij}\sum_{k=1}^{i}u^{(k-1)}(0)\frac{d^j s^{i-k}}{ds^j}.$$

Therefore we obtain

$$\ell[L[u]]=F(s)\ell[u]+G(s)+H(s),$$

where

$$\begin{cases} F(s)=\sum_{i=0}^{m}\left(\sum_{k=i}^{m}(-1)^{i+k}\frac{k!}{i!}a_{k,k-i}\right)s^i, \\ G(s)=\sum_{i=0}^{m}\sum_{j=0}^{\infty}(-1)^j a_{ij}\left(\sum_{k=0}^{j-1}\binom{j}{k}\frac{d^k s^i}{ds^k}\frac{d^{j-k}(\ell[u])}{ds^{j-k}}\right), \\ H(s)=\sum_{i=1}^{m}\left(\sum_{k=0}^{i-1}(-)^{k+1}k!a_{ik}u^{(i-1+k)}(0)\right)s^{i-1}. \end{cases}$$

□

Remark 1. Similar result is established when for any $i=0,\ldots,m$ the function $c_i(t)$ is analytic at $t=t_0$.

Therefore, if $\ell[u(t)]$ satisfies the hypotheses of theorem 1, we have

$$\ell[L[u_n]]=F(s)\ell[u_n]+G_n(s)+H_n(s).$$

Hence (4) can be rewritten as

$$\ell[u_{n+1}]=(1+F(s)\ell[\lambda])\ell[u_n]+\ell[\lambda](G_n(s)+H_n(s)+\ell[N[u_n]-h]). \quad (5)$$

For simplicity, we find $\lambda(t)$ such that

$$1+F(s)\ell[\lambda]=0, \quad (6)$$

this yields

$$\lambda(t)=\ell^{-1}\left[\frac{-1}{F(s)}\right]. \quad (7)$$

According to the proof of theorem 1, $F(s)$ is a polynomial of s of degree m with real coefficients and therefore it can be decomposed into the prime factors of degree either one or two. Consequently, by using a partial fraction expansion on $1/F(s)$, we can simply implement the inverse Laplace transform of (5), obtaining

$$u_{n+1}(t)=K[u_n(t)]+\int_0^t\lambda(t-\tau)(N[u_n(\tau)]-h(\tau))d\tau, \quad (8)$$

where K is a linear operator defined as

$$K[u_n(t)]=\ell^{-1}\left[-\frac{G_n(s)+H_n(s)}{F(s)}\right].$$

Remark 2. In practice we have a variety of choices for determining the unknown function λ. In other words, we are able to choose just some parts of $L[u(t)]$ which contain the differential term with the highest order.

Remark 3. If (1) is of the form

$$u^{(m)}(t)=f(t,u(t),u'(t),\ldots,u^{(m-1)}(t)), \quad t\geq 0,$$

then (8) can be written as

$$u_{n+1}(t)=\sum_{i=0}^{m-1}\frac{t^i}{i!}u_n^{(i)}(0)-$$
$$\frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1}\times$$
$$f(\tau,u_n(\tau),u_n'(\tau),\ldots,u_n^{(m-1)}(\tau))d\tau. \qquad (9)$$

3 A general example

To better illustrate our new approach, consider the following nonlinear differential equation which characterizes several physical phenomena,

$$c_m u^{(m)}(t)+\cdots+c_1u'(t)+c_0u(t)+N[u(t)]=$$
$$h(t), \qquad t\geq 0,$$

where $c_0,\ldots,c_m \in R$. At first, we have

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(t-\tau)(c_m u_n^{(m)}(\tau)+$$
$$\cdots+c_1u_n'(\tau)+c_0u_n(\tau)+N[u_n(\tau)]-h(\tau))d\tau,$$

and then by using the Laplace transform we obtain

$$\ell[u_{n+1}]=(1+F(s)\ell[\lambda])\ell[u_n]+$$
$$\ell[\lambda](H_n(s)+\ell[N[u_n]-h]),$$

where

$$F(s)=\sum_{i=0}^{m}c_i s^i,$$

$$G_n(s)=0,$$

$$H_n(s)=-\sum_{i=1}^{m}c_i\sum_{k=1}^{i}u_n^{(k-1)}(0)s^{i-k}.$$

Therefore $\lambda(t)$ determined as follows

$$\lambda=\ell^{-1}\left[\frac{-1}{\sum_{i=0}^{m}c_i s^i}\right].$$

For the simple case when $m=2$ we have

$$F(s)=c_2s^2+c_1s+c_0,$$

$$H_n(s)=-(c_2s+c_1)u_n(0)-c_2u_n'(0)$$

and hence

$$\lambda=\ell^{-1}\left[\frac{-1}{c_2s^2+c_1s+c_0}\right]. \qquad (10)$$

Table 1 $\lambda(t)$ for different values of c_0, c_1, c_2

c_2	c_1	c_0	λ (t)
0	1	0	-1
1	0	0	$-t$
0	1	1	$-e^{-t}$
1	0	ω^2	$-\sin t/\omega$

Disregarding sign of $c_1^2-4c_2c_0$, (10) has always a unique solution. Some special cases are reported in Table 1 which are the same as those have obtained by the He's variational iteration method [1-12]. Therefore, we could provide the He's variational iteration method without invoking the Lagrange multiplier, correction functionals or restricted variations.

4 A comparison of the method with the successive approximations

In [7] it is claimed that " *shown that the variational iteration method is nothing else by the Picard-Lindelöf theory for initial-value problems in ordinary differential equations and Banach's fixed-point theory for initial-value problems in partial differential equations, and the convergence of these iterative procedures is ensured provided that the resulting mapping is Lipschitz continuous and contractive.* ". The purpose of this section is to show that such a claim is correct just for the first-order ordinary differential equations and a few higher order ordinary differential equations.

Obviously, for the following initial-value problem

$$\begin{cases} u'(t) = f(t,u(t)), & t \ge 0, \\ u(0) = \alpha, \end{cases}$$

the He's iterative scheme and the iterative scheme of successive approximations coincide as

$$v_{n+1}(t) = \alpha + \int_0^t f(\tau, v_n(\tau))d\tau, \qquad n \ge 0,$$

provided that $v_0(t)$ satisfies the initial conditions. Now let us consider the following initial-value problem

$$\begin{cases} u''(t) = f(t,u(t),u'(t)), & t \ge 0, \\ u(0) = \alpha, \quad u'(0) = \beta \end{cases}$$

and then convert it to a system of first-order differential equations as

$$\begin{cases} U' = F(t,U(t)), & t \ge 0, \\ U(0) = U_0, \end{cases}$$

where

$$U_0 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad U(t) = \begin{bmatrix} u_1(t) := u(t) \\ u_2(t) := u'(t) \end{bmatrix},$$

$$F(t,U(t)) = \begin{bmatrix} u_2(t) \\ f(t,u_1(t),u_2(t)) \end{bmatrix}$$

and then establish the successive approximations for $n \ge 0$ as

$$\begin{cases} u_{1,n+1}(t) = \alpha + \int_0^t u_{2,n}(\tau)d\tau, \\ u_{2,n+1}(t) = \beta + \int_0^t f(\tau, u_{1,n}(\tau), u_{2,n}(\tau))d\tau. \end{cases} \tag{11}$$

On the other hand, with regard to (9), the corresponding He's iterative scheme can be constructed as follows

$$v_{n+1}(t) = \alpha + \beta t + \int_0^t (t-\tau) f(\tau, v_n(\tau), v_n'(\tau))d\tau, \quad n \ge 0, \tag{12}$$

provided that $v_0(t)$ satisfies the initial conditions.

Theorem 2. The iterative scheme (12) coincides with the iterative scheme (11) in the sense of

$$\forall n \ge 1, \qquad \begin{cases} v_n(t) = u_{1,2n}(t) = u_{1,2n+1}(t), \\ v_n'(t) = u_{2,2n}(t) = u_{2,2n-1}(t), \end{cases}$$

when $u_{1,0}(t) = v_0(t)$, $u_{2,0}(t) = v_0'(t)$ and f is only a function of t and u.

Proof. The proof follows by induction. For the case when $n = 1$, at first we can write

$$u_{1,1}(t) = \alpha + \int_0^t u_{2,0}(\tau)d\tau = \alpha + \int_0^t v_0'(\tau)d\tau = v_0(t).$$

Therefore, we have

$$u_{2,2}(t) = u_{2,1}(t) = \beta + \int_0^t f(\tau, v_0(\tau))d\tau = v_1'(t)$$

and deduce that

$$\begin{cases} u_{1,2}(t) = \alpha + \int_0^t u_{2,1}(\tau)d\tau = \alpha + \int_0^t v_1'(\tau)d\tau = v_1(t), \\ u_{1,3}(t) = \alpha + \int_0^t u_{2,2}(\tau)d\tau = \alpha + \int_0^t v_1'(\tau)d\tau = v_1(t). \end{cases}$$

Now, suppose that the assertion holds for an order n. By the induction hypothesis, we have

$$u_{2,2n+2}(t) = u_{2,2n+1}(t) = \beta + \int_0^t f(\tau, v_n(\tau))d\tau = v_{n+1}'(t),$$

from which we immediately obtain

$$\begin{cases} u_{1,2n+2}(t) = \alpha + \int_0^t u_{2,2n+1}(\tau)d\tau = \alpha + \int_0^t v_{n+1}'(\tau)d\tau = v_{n+1}(t), \\ u_{1,2n+3}(t) = \alpha + \int_0^t u_{2,2n+2}(\tau)d\tau = \alpha + \int_0^t v_{n+1}'(\tau)d\tau = v_{n+1}(t). \end{cases}$$ □

Theorem 3. The iterative scheme (12) coincides with the iterative scheme (11) in the sense of

$$\forall n \geq 1, \quad \begin{cases} v_n(t) = u_{1,n+1}(t), \\ v_n'(t) = u_{2,n}(t), \end{cases}$$

when $u_{1,0}(t) = v_0(t)$, $u_{2,0}(t) = v_0'(t)$, and f is only a function of t and u'.

Proof. The proof follows by induction and is similar to the proof of Theorem 2. □

Corollary 1. Theorems 2 and 3 are not fulfilled if f is a function of t, u and u'.

Proof. We can write

$$u_{1,1}(t) = \alpha + \int_0^t u_{2,0}(\tau)d\tau$$
$$= \alpha + \int_0^t v_0'(\tau)d\tau = v_0(t).$$

Therefore, we have

$$u_{2,1}(t) = \beta + \int_0^t f(\tau, v_0(\tau), v_0'(\tau))d\tau$$
$$= v_1'(t),$$

which yields $v_1(t) = u_{1,2}(t)$, but

$$u_{2,2}(t) = \beta + \int_0^t f(\tau, v_0(\tau), v_1'(\tau))d\tau$$

implies that $v_1'(t) \neq u_{2,2}(t) \neq v_2'(t)$. Furthermore, from

$$u_{1,3}(t) = \alpha + \beta t +$$
$$\int_0^t (t-\tau) f(\tau, v_0(\tau), v_1'(\tau))d\tau,$$

we have $v_1(t) \neq u_{1,3}(t) \neq v_2(t)$. Moreover,

$$u_{2,3}(t) = \beta + \int_0^t f(\tau, v_1(\tau), \beta +$$
$$\int_0^\tau f(s, v_0(s), v_1'(s))ds)d\tau,$$

implies that $v_2'(t) \neq u_{2,3}(t) \neq v_3'(t)$ and so on. □

Now we consider the following initial-value problem

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)), & t \geq 0, \\ u(0) = \alpha, \quad u'(0) = \beta, \quad u''(0) = \gamma \end{cases}$$

and convert it to a system of first-order differential equations as

$$\begin{cases} U' = F(t, U(t)), & t \geq 0, \\ U(0) = U_0, \end{cases}$$

where

$$U_0 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \quad U(t) = \begin{bmatrix} u_1(t) := u(t) \\ u_2(t) := u'(t) \\ u_3(t) := u''(t) \end{bmatrix},$$

$$F(t, U(t)) = \begin{bmatrix} u_2(t) \\ u_3(t) \\ f(t, u_1(t), u_2(t), u_3(t)) \end{bmatrix}$$

and then establish the successive approximations for $n \geq 0$ as

$$\begin{cases} u_{1,n+1}(t) = \alpha + \int_0^t u_{2,n}(\tau)d\tau, \\ u_{2,n+1}(t) = \beta + \int_0^t u_{3,n}(\tau)d\tau, \\ u_{3,n+1}(t) = \gamma + \\ \int_0^t f(\tau, u_{1,n}(\tau), u_{2,n}(\tau), u_{3,n}(\tau))d\tau. \end{cases} \tag{13}$$

On the other hand, with regard to (9), the corresponding He's iterative scheme can be constructed as follows

$$v_{n+1}(t) = \alpha + \beta t + \gamma\frac{t^2}{2} + \frac{1}{2}\int_0^t (t-\tau)^2 \times$$
$$f(\tau, v_n(\tau), v_n'(\tau), v_n''(\tau))d\tau, \quad n \geq 0 \tag{14}$$

provided that $v_0(t)$ satisfies the initial conditions.

Theorem 4. The iterative scheme (14) coincides with the iterative scheme (13) in the sense of

$$\begin{cases} v_n(t) = u_{1,3n}(t) = u_{1,3n+1}(t) = u_{1,3n+2}(t), \\ v_n'(t) = u_{2,3n-1}(t) = u_{2,3n}(t) = u_{2,3n+1}(t), \\ v_n''(t) = u_{3,3n-2}(t) = u_{3,3n-1}(t) = u_{3,3n}(t), \end{cases}$$

for $n \geq 0$, when

$$u_{1,0}(t) = v_0(t), \; u_{2,0}(t) = v_0'(t), \; u_{3,0}(t) = v_0''(t)$$

and f is only a function of t and u.

Proof. The proof follows by induction and is similar to the proof of Theorem 2. □

Remark 4. Similar theorems can be established when f is only a function of t and u' or f is only a function of t and u''.

Corollary 2. Theorems 4 is not fulfilled if f is a function of t, u and u'.

Proof. Similar to the proof of Corollary 1 we have

$$\begin{cases} v_0(t) = u_{1,1}(t) = u_{1,2}(t) \\ v_0'(t) = u_{2,1}(t) \end{cases},$$

$$\begin{cases} v_1(t) = u_{1,3}(t) = u_{1,4}(t) \\ v_1'(t) = u_{2,2}(t) = u_{2,3}(t) \\ v''(t) = u_{3,1}(t) = u_{3,2}(t)_1 \end{cases}, \quad \begin{cases} v_2(t) = u_{1,6}(t) \\ v_2'(t) = u_{2,5}(t) \\ v_2''(t) = u_{3,4}(t) \end{cases}$$

and

$$\begin{cases} v_1(t) \neq u_{1,5}(t) \\ v_1'(t) \neq u_{2,4}(t), \\ v_1''(t) \neq u_{3,3}(t) \end{cases} \quad \begin{cases} u_{1,7}(t) \neq v_2(t) \neq u_{1,8}(t) \\ u_{2,6}(t) \neq v_2'(t) \neq u_{2,7}(t) \\ u_{3,5}(t) \neq v_2''(t) \neq u_{3,6}(t) \end{cases}$$

and so on.

Remark 5 Similar corollaries can be established when f is a function of t, u and u'' or f is a function of t, u' and u'' or f is a function of t, u, u' and u''.

Remark 6 Generalization these results, it can be claimed that the He's iterative scheme coincide with the iterative scheme of successive approximations in some senses just for some types of the m-th order ordinary differential equations but not for all of them. Hence, we can deal with the convergence of the He's iterative scheme with the aid of the Generalized Picard-Lindelöf Theorem in some (but not all) cases.

5. Symbolic computations

In this section we intend to compare the He's iterative scheme and the iterative scheme of successive approximations for the case when both of them in some senses give rise to the same results.

The results of some tests are summarized in Table 2 in which μ returns the number of seconds of elapsed time of second iteration of the He's iterative scheme while v is the number of seconds of elapsed time of fourth iteration of the iterative scheme of successive approximations. For all of the model problems, initial-condition is considered as $u(0) = \alpha$ and $u'(0) = 0$. Furthermore, we assume $v_0(t) = u_{1,0}(t) = \alpha \cos \omega t$ and $u_{2,0}(t) = -\alpha\, \omega \sin \omega t$. We must point out that both of the iterative schemes are tested under the same environments and conditions.

Table 2. Elapsed time of two iterative schemes for several model problems

model problem	elapsed time	model problem	elapsed time
$u'' + ku + \varepsilon u^2 = 0$	$\mu \cong 13, v \cong 17$	$u'' + ku + \varepsilon u^2 = \cos(t)$	$\mu \cong 37, v \cong 75$
$u'' + ku + \varepsilon u^3 = 0$	$\mu \cong 29, v \cong 39$	$u'' + ku + \varepsilon u^3 = \cos(t)$	$\mu \cong 94, v \cong 194$
$u'' + ku + \varepsilon u^2 + \delta u^3 = 0$	$\mu \cong 45, v \cong 107$	$u'' + ku + \varepsilon u^2 + \delta u^3 = \cos(t)$	$\mu \cong 362, v \cong 849$

Remark 7. Results obtained in this section imply that in cases which the He's iterative scheme coincides with the iterative scheme of successive approximations (in the senses posed in the previous section), the He's iterative scheme can often be faster than the other one.

6 Summary

The most important contribution of the paper is based on two facts. The first one is that the general Lagrange multiplier can be determined by the use of Laplace transform and the second is that the method of successive approximations and the variational iteration method are two completely different methods which can produce the same iterations in some senses just for some types of ordinary differential equations. We point out that the method of successive approximations can be used to solve the explicit ordinary differential equations of the form $u^{(m)}(t)=f(t,u(t),\dots,u^{(m-1)}(t))$ while the He's variational iteration method can be applied to solve not only the explicit but also the implicit ordinary differential equations.

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