Topology and its Applications 307 (2022) 107768

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# A Conley-type Lyapunov function for the strong chain recurrent set

Olga Bernardi [a], Anna Florio [b], Jim Wiseman [c,*]

[a] *Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Via Trieste 63, 35121 Padova, Italy*
[b] *Sorbonne Université, Université de Paris, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75005 Paris, France*
[c] *Department of Mathematics, Agnes Scott College, Decatur, GA, USA*

A R T I C L E I N F O

*Article history:*
Received 3 December 2020
Received in revised form 1 July 2021
Accepted 1 July 2021
Available online 7 July 2021

*MSC:*
37B20
37B35
37B65
37C10

*Keywords:*
Strong chain recurrence
Lyapunov functions
Strongly stable sets

A B S T R A C T

Let $\phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space $(X, d)$. In this article we constructively prove the existence of a continuous Lyapunov function for $\phi$ which is strictly decreasing outside $\mathcal{SCR}_d(\phi)$. Such a result generalizes Conley's Fundamental Theorem of Dynamical Systems for the strong chain recurrent set.



## 1. Introduction

This article continues a project of the authors, started in [3] and proceed in [4] and [5], concerning the study of the links between recurrent sets and Lyapunov functions.

Let $\phi$ be a continuous flow on a compact metric space $(X, d)$. The aim of the present paper is to give a constructive proof of the existence of a continuous Lyapunov function for $\phi$ which is strictly decreasing outside the strong chain recurrent set $\mathcal{SCR}_d(\phi)$.

Such a result generalizes Conley's Fundamental Theorem of Dynamical Systems—see the seminal book [6][Section 6.4, Page 39]—since we look at $\mathcal{SCR}_d(\phi)$, instead of the chain recurrent set $\mathcal{CR}(\phi)$.

* Corresponding author.
*E-mail addresses:* obern@math.unipd.it (O. Bernardi), anna.florio@imj-prg.fr (A. Florio), jwiseman@agnesscott.edu (J. Wiseman).

https://doi.org/10.1016/j.topol.2021.107768
0166-8641/

For dynamics given by the iteration of a homeomorphism, the problem has already been solved by Fathi and Pageault in [8][Theorem 2.6], by using Fathi's formalism in weak KAM theory. In particular, they proved the following result.

**Theorem.** *Let $f : X \to X$ be a homeomorphism on a compact metric space $(X, d)$. Then there exists a Lipschitz continuous Lyapunov function for $f$ which is strictly decreasing outside $\mathcal{SCR}_d(f)$.*

Yokoi independently proved the existence of a continuous Lyapunov function (a priori non-Lipschitz) for a homeomorphism on a compact metric space, that strictly decreases outside the $\mathcal{SCR}_d(f)$ (see [9][Theorem 5.2]).

Combining the variational approach established by Fathi and Pageault in [8] and some of Conley's techniques presented in [6], the authors Bernardi and Florio have attacked the same problem in the framework of continuous flows, uniformly Lipschitz continuous on compact subsets of $[0, +\infty)$. We recall that a continuous flow $\phi : X \times \mathbb{R} \to X$ on a compact metric space $(X, d)$ is uniformly Lipschitz continuous on compact subsets of $[0, +\infty)$ if for any $T > 0$ there exists $M_T > 0$ such that for every $x, y \in X$

$$d(\phi_t(x), \phi_t(y)) \leq M_T\, d(x, y) \qquad \forall t \in [0, T]\,.$$

In [4][Theorem 4.1], the following result is proved.

**Theorem.** *Let $\phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space $(X, d)$, uniformly Lipschitz continuous on the compact subsets of $[0, +\infty)$. Then there exists a Lipschitz continuous Lyapunov function for $\phi$ which is strictly decreasing outside $\mathcal{SCR}_d(\phi)$.*

The proof of the above result is constructive. Nevertheless, the authors did not manage to generalize the result for only continuous flows, getting rid of the further hypothesis about uniformly Lipschitz regularity of the flow with respect to time. This is due to the fact that, in building the Lyapunov function for the flow, some "regularizing" process of an initial function (coming from the variational approach) is needed. In particular, some Lipschitz-like control over time is required. Even if any flow of a Lipschitz continuous vector field satisfies the regularity hypothesis of the above theorem, there are examples of (dynamically interesting) flows that fail to fulfill such a hypothesis, see e.g. Example 3.4.

In this article—for a continuous flow—we prove the existence of a Lyapunov function, which is strictly decreasing outside the strong chain recurrent set, in the most general framework. In order to obtain it, we exploit the Conley-type decomposition of the strong chain recurrent set, previously shown in [3][Theorem 2]. The constructive method to prove the existence of the required Lyapunov function is then inspired by Conley's original ideas presented in the proof of his celebrated theorem in the chain recurrent case.

Thus, our main result reads as follows.

**Theorem 1.1.** *If $\phi : X \times \mathbb{R} \to X$ is a continuous flow on a compact metric space $(X, d)$, then there exists a continuous Lyapunov function for $\phi$ which is strictly decreasing outside $\mathcal{SCR}_d(\phi)$.*

**Acknowledgments.** Olga Bernardi thanks prof. Alberto Abbondandolo who introduced her to Conley's theory of Lyapunov functions and posed her the problem solved in this article. O. Bernardi and J. Wiseman have been supported by the PRIN project 2017S35EHN 003 2019-2021 "Regular and stochastic behavior in dynamical systems".

## 2. Preliminaries

Let $\phi : X \times \mathbb{R} \to X$, $(x, t) \mapsto \phi_t(x)$ be a continuous flow on a compact metric space $(X, d)$. In this section we recall the notions of Lyapunov function, strong chain recurrent point, stable set and strongly stable set.

**Definition 2.1.** A continuous function $h : X \to \mathbb{R}$ is a Lyapunov function for $\phi$ if $h(\phi_t(x)) \leq h(x)$ for every $t \geq 0$ and $x \in X$.

**Definition 2.2.** Given $x, y \in X$, $\varepsilon > 0$ and $T > 0$, a strong $(\varepsilon, T)$-chain from $x$ to $y$ is a finite sequence $(x_i, t_i)_{i=1,\dots,n} \subset X \times \mathbb{R}$ such that $t_i \geq T$ for all $i$, $x_1 = x$ and, setting $x_{n+1} = y$, we have

$$\sum_{i=1}^{n} d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon.$$

A point $x \in X$ is said to be strong chain recurrent if for any $\varepsilon > 0$ and $T > 0$ there exists a strong $(\varepsilon, T)$-chain from $x$ to $x$.
The set of strong chain recurrent points is denoted by $\mathcal{SCR}_d(\phi)$.

**Definition 2.3.** A closed set $B \subset X$ is stable if it has a neighborhood base of forward invariant sets.

We refer to [1][Page 1732] and [2][Paragraph 1.1]. If $B$ is a stable set, then for every $x \in X$ either $\omega(x) \cap B = \emptyset$ or $\omega(x) \subseteq B$, see [3][Lemma 4.1]. Moreover, the complementary of a stable set $B$ is defined as

$$B^\bullet := \{x \in X : \ \omega(x) \cap B = \emptyset\}.$$

The set $B^\bullet \subset X$ is invariant and disjoint from $B$ but it is not necessarily closed even if $B$ is closed (see also Paragraph 1.5 in [7]).

**Definition 2.4.** A closed set $B \subset X$ is strongly stable if there exist a family $(U_\eta)_{\eta \in (0,1)}$ of closed nested neighborhoods of $B$ and a function

$$(0, 1) \ni \eta \mapsto T(\eta) \in (0, +\infty)$$

bounded on compact subsets of $(0, 1)$, such that:

$(i)$ for any $0 < \eta < \lambda < 1$, $\{x \in X : \ d(x, U_\eta) < \lambda - \eta\} \subseteq U_\lambda$;
$(ii)$ $B = \bigcap_{\eta \in (0,1)} \omega(U_\eta)$;
$(iii)$ for any $0 < \eta < 1$, $\mathrm{cl}\{\phi_{[T(\eta),+\infty)}(U_\eta)\} \subseteq U_\eta$.

We refer to [3][Definition 4.2]. Every strongly stable set $B$ is closed, forward invariant and stable, see [3][Remark 4.1].
In [3][Theorem 4.2], the subtle relation between strongly stable sets and $\mathcal{SCR}_d(\phi)$ has been explained:

**Theorem 2.1.** *If $\phi : X \times \mathbb{R} \to X$ is a continuous flow on a compact metric space, then*

$$\mathcal{SCR}_d(\phi) = \bigcap \{B \cup B^\bullet : \ B \textit{ is strongly stable}\}. \tag{1}$$

The proof of the main result of this paper is based on the above theorem.

## 3. Proof of Theorem 1.1

### 3.1. A Lyapunov function for $(B, B^\bullet)$

Let $\phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space $(X, d)$.

In this section, for every pair $(B, B^\bullet)$ with $B$ strongly stable, we construct a Lyapunov function for $\phi$ which is strictly decreasing on $X \setminus (B \cup B^\bullet)$.

**Lemma 3.1.** *Let $B \subset X$ be a strongly stable set and $(U_\eta)_{\eta \in (0,1)}$ be a family of closed nested neighborhoods of $B$ as in Definition 2.4.*

*If $B^\bullet \neq \emptyset$ then there exists an $\eta_0 \in (0, 1)$ such that*

$$B^*_{\eta_0} := \{x \in X : \ \forall t \geq 0, \ \phi_t(x) \notin U_{\eta_0}\} \tag{2}$$

*is nonempty. The set $B^*_{\eta_0}$ is forward invariant, $B^*_{\eta_0} \subseteq B^\bullet$ and $B \cap cl(B^*_{\eta_0}) = \emptyset$.*

**Proof.** Let $x$ be an element of $B^\bullet$, so $\omega(x) \cap B = \emptyset$. Since $B = \bigcap_{\eta \in (0,1)} \omega(U_\eta)$, there exists $\eta_0$ such that $\omega(x) \cap \omega(U_{\eta_0}) = \emptyset$. Since $U_{\eta_0}$ is eventually forward invariant, this means that $\phi_t(x) \notin U_{\eta_0}$ for all $t \geq 0$. □

We observe that, since the $U_\eta$'s are nested,

$$B^*_{\eta_0} = \{x \in X : \ \forall t \geq 0, \ \phi_t(x) \notin \bigcup_{\eta \in (0,\eta_0]} U_\eta\}.$$

**Theorem 3.1.** *Let $B^\bullet \neq \emptyset$ and $B^*_{\eta_0}$ be as in formula (2) of Lemma 3.1.*

*Then there exists a continuous Lyapunov function $h : X \to \mathbb{R}$ for $\phi$ such that*

*(i)* $h^{-1}(0) = B$.
*(ii)* $h^{-1}(1) = cl(B^*_{\eta_0})$.
*(iii)* *$h$ is strictly decreasing on the set $X \setminus (B \cup B^\bullet)$.*

**Proof.** We first define the function $l : X \to \mathbb{R}$ as follows:

$$l(x) := \begin{cases} \dfrac{\eta}{\eta_0} & \text{if } \eta = \inf_{\lambda \in (0,\eta_0)} \lambda \text{ such that } x \in U_\lambda, \\ \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

The function $l$ is continuous: this follows from the fact that each $U_\eta$ is closed and by property $(iii)$ of Definition 2.4. Moreover, $B = l^{-1}(0)$, $\mathrm{cl}(B^*_{\eta_0}) \subseteq l^{-1}(1)$ and $l(X) \subseteq [0, 1]$. Define now the function $k : X \to \mathbb{R}$ by

$$k(x) := \sup_{t \geq 0}\{l(\phi_t(x))\}. \tag{4}$$

Since both $B$ and $\mathrm{cl}(B^*_{\eta_0})$ are forward invariant, it follows that $B = k^{-1}(0)$ and $\mathrm{cl}(B^*_{\eta_0}) \subseteq k^{-1}(1)$. Moreover, $k(X) \subseteq [0, 1]$. We show now that the function $k$ is continuous.

$(a)$ $k$ is continuous on $\mathrm{cl}(B^*_{\eta_0})$. Since $l$ is continuous and since $\mathrm{cl}(B^*_{\eta_0}) \subseteq l^{-1}(1)$, for every $\varepsilon > 0$ there exists a neighborhood $V$ of $\mathrm{cl}(B^*_{\eta_0})$ such that

$$|l(y) - l(x)| < \varepsilon$$

$\forall y \in V$ and $\forall x \in \mathrm{cl}(B^*_{\eta_0})$. In particular, for every $y \in V$ we have $1 - l(y) < \varepsilon$. Observe that $l \le k \le 1$. Thus we deduce that

$$|k(y) - k(x)| = 1 - k(y) \le 1 - l(y) < \varepsilon$$

$\forall y \in V$ and $\forall x \in \mathrm{cl}(B^*_{\eta_0})$. This concludes the proof of the continuity of $k$ on cl $(B^*_{\eta_0})$.

$(b)$ $k$ is continuous on $B$. Since $l$ is continuous and since $B = l^{-1}(0)$, for every $\varepsilon > 0$ there exists a neighborhood $V$ of $B$ such that $l_{|V} < \varepsilon$. Corresponding to $V$ and up to restricting $V$, there exists $\eta \in (0, \eta_0)$ such that $U_\eta \subseteq V$. From property $(iii)$ of Definition 2.4, there is $T(\eta) > 0$ so that

$$\phi_{[T(\eta),+\infty)}(U_\eta) \subseteq U_\eta \subseteq V. \tag{5}$$

Observe that $\phi_{[T(\eta),+\infty)}(U_\eta)$ is a neighborhood of $B$. From (5), the function $k$ is $\varepsilon$-bounded on $\phi_{[T(\eta),+\infty)}(U_\eta)$. This proves the continuity of $k$ on $B$.

$(c)$ $k$ is continuous on $X \setminus (B \cup \mathrm{cl}\ (B^*_{\eta_0}))$.
Fix $x \in X \setminus (B \cup \mathrm{cl}\ (B^*_{\eta_0}))$. Thus, there exists $\tau(x) \ge 0$ such that $\phi_{\tau(x)}(x) \in \bigcup_{\eta \in (0,\eta_0)} U_\eta$. Let $\bar\eta \in (0, \eta_0)$ be such that $\bar\eta = \inf_{\lambda \in (0,\eta_0)} \lambda$ so that $\phi_{\tau(x)}(x) \in U_{\bar\eta}$. Then, by property $(i)$ of Definition 2.4 and the continuity of the flow, fixed $\bar\eta < \tilde\eta < \eta_0$, there exists a neighborhood $V$ of $x$ such that $\phi_{\tau(x)}(V) \subseteq U_{\tilde\eta}$. Therefore, by property $(iii)$ of Definition 2.4, we have

$$\phi_{[\tau(x)+T(\tilde\eta),+\infty)}(V) \subseteq U_{\tilde\eta}$$

and consequently

$$l|_{\phi_{[\tau(x)+T(\tilde\eta),+\infty)}(V)} \le \frac{\tilde\eta}{\eta_0}.$$

Hence, for every $y \in V$ we have

$$k(y) = \sup_{t \ge 0}\{l(\phi_t(y))\} = \max_{t \in [0,\tau(x)+T(\tilde\eta)]}\{l(\phi_t(y))\}.$$

By the continuity of $y \mapsto \max_{t \in [0,\tau(x)+T(\tilde\eta)]}\{l(\phi_t(y))\}$, we conclude that $k$ is continuous at $x \in X \setminus (B \cup \mathrm{cl}\ (B^*_{\eta_0}))$.

By its definition in (4), it immediately follows that $k$ is a Lyapunov function for $\phi$.
Define now the function $h : X \to \mathbb{R}$ as

$$h(x) := \int_0^{+\infty} e^{-s} k(\phi_s(x))\, ds. \tag{6}$$

Since $k$ is continuous, the function $h$ is continuous too; moreover, $h$ is decreasing along trajectories. Moreover, on one hand $h(x) = 0$ if and only if $k(\phi_s(x)) = 0$ for every $s \ge 0$, i.e. if and only if $x \in B$. On the other hand, $h(x) = 1$ if and only if $k(\phi_s(x)) = 1$ for every $s \ge 0$. That is if and only if $\phi_s(x) \notin U_{\eta_0}$ for every $s \ge 0$, which is the definition of $B^*_{\eta_0}$.
We finally prove that the function $h$ is strictly decreasing on

$$x \in X \setminus (B \cup B^\bullet) = \{x \in X \setminus B :\ \omega(x) \subseteq B\}.$$

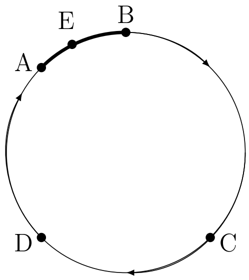


**Fig. 1.** The dynamics of Example 3.1.

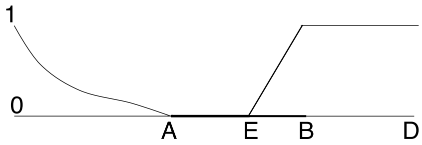


**Fig. 2.** The Lyapunov function for the dynamics of Example 3.1.

Indeed, for any $x \in X \setminus (B \cup B^{\bullet})$ and $t > 0$, we have

$$h(\phi_t(x)) - h(x) = \int_0^{+\infty} e^{-s} \left(k(\phi_{s+t}(x)) - k(\phi_s(x))\right) ds < 0$$

since the integrand is not identically zero. Indeed, if this is not the case, then, since $k$ is a Lyapunov function, $k(\phi_T(x)) = k(x)$ for every $T \geq 0$. But, since $x \in X \setminus (B \cup B^{\bullet})$ and arguing as at point $(c)$ of the continuity argument for $k$, we can find $\tau > 0$ such that $k(\phi_\tau(x)) < k(x)$, obtaining the required contradiction. This concludes the proof. □

We recall that the corresponding result for an attractor-repeller pair is contained in [6][Section B, page 33]. Our proof follows the main lines of Conley's original idea. However we notice that Conley's Lyapunov function for an attractor-repeller pair is identically zero on the attractor and identically 1 on the repeller. The same does not hold for $(B, B^{\bullet})$, with $B$ strongly stable. In such a case—see also Example 3.1 below—the Lyapunov function of Theorem 3.1 assumes all the values between 0 and 1 in $B^{\bullet}$.

**Example 3.1.** (Example 4.4 in [3])
On the circle $\mathbb{R}/\mathbb{Z}$ endowed with the standard quotient metric, consider the dynamical system of Fig. 1. In the sequel, we denote by $\widehat{XY}$ (resp. $cl(\widehat{XY})$) the clockwise-oriented open (resp. closed) arc from $X$ to $Y$. The arc $cl(\widehat{AB})$ and the points $C$ and $D$ are fixed; on the other points we have a clockwise flow. Then $cl(\widehat{AE})$ is strongly stable with $B^{\bullet} = \widehat{ED} \cup \{D\}$. In such a case, a set $B^*_{\eta_0}$ as in formula (2) of Lemma 3.1 is—for example—$B^*_{\eta_0} = cl(\widehat{BD})$ and the corresponding Lyapunov function for $(B, B^{\bullet})$ constructed in Theorem 3.1 equals 0 if $x \in cl(\widehat{AE})$ and $\frac{d(x,E)}{d(x,E)+d(x,B)}$ if $x \in \widehat{EB} \subseteq B^{\bullet}$. Then, such a function assumes all the values between 0 and 1 in $B^{\bullet}$ (see Fig. 2).

### *3.2. Reduction to a countable set of* $(B, B^{\bullet})$

Conley's proof of Fundamental Theorem of Dynamical Systems uses the facts that attractor-repeller pairs are closed and that there are countably many such pairs. We notice that the pairs $(B, B^{\bullet})$, involved in the decomposition of $\mathcal{SCR}(\phi)$, aren't as well behaved. Firstly, different $B$'s can give the same $B \cup B^{\bullet}$: the obvious example is the identity flow on a connected compact metric space. Moreover, $B \cup B^{\bullet}$ isn't

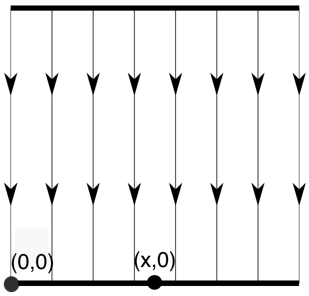


**Fig. 3.** Dynamics of Example 3.2.

necessarily closed, there can be uncountably many different $B \cup B^\bullet$'s and, finally, the structure $B \cup B^\bullet$ is not conserved by finite intersection.

The first part of this section is devoted to showing—by simple examples—these facts.

**Example 3.2.** On the compact space $[0,1]^2 \subset \mathbb{R}^2$ endowed with the standard metric, consider the dynamical system of Fig. 3. Segments $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are fixed; moreover, for every $x \in [0,1]$, on each open vertical interval $\{x\} \times (0,1)$ we have a north-south flow. In such a case, $B = \{(0,0)\}$ is strongly stable with complementary $B^\bullet = ((0,1] \times [0,1]) \cup \{(0,1)\}$, and the union $B \cup B^\bullet$ is not closed. Moreover, every closed segment $B_x$ connecting $(0,0)$ to $(x,0)$, $x \in [0,1]$, is strongly stable, and the collection $(B_x)_{x\in[0,1]}$ gives uncountably many different $B_x \cup B_x^\bullet$'s.

**Example 3.3.** Let us consider again Example 3.1. It is straightforward to see that there exist $B_0$ and $B_1$ strongly stable sets such that

$$B_0 \cup B_0^\bullet = cl(\widehat{AD}) \qquad \text{and} \qquad B_1 \cup B_1^\bullet = cl(\widehat{DC}).$$

However $cl(\widehat{AD}) \cap cl(\widehat{DC}) = \{D\} \cup cl(\widehat{AC})$ is not the union of a strongly stable set and its complementary.

The aim of the second part of the section is proving the next theorem.

**Theorem 3.2.** *Let $\phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space $(X,d)$. Then there exists an—at most countable—collection of strongly stable sets $\{B_n\}$ such that*

$$\mathcal{SCR}_d(\phi) = \bigcap_n B_n \cup B_n^\bullet. \tag{7}$$

We will use the following lemmas in the proof of the theorem.

**Lemma 3.2.** *Let $\{U_\alpha\}$ be a—possibly uncountable—collection of open sets of a separable space. Then there exists an—at most countable—subcollection $\{U_n\}$ such that*

$$\bigcup_n U_n = \bigcup_\alpha U_\alpha.$$

**Proof.** Take a countable basis for the topology and let $\{V_n\}$ be the set of basis elements that are contained in $\bigcup_\alpha U_\alpha$. For each $V_n$, set $U_n = U_\alpha$ for some $U_\alpha$ containing $V_n$, or $U_n = \emptyset$ if no such $U_\alpha$ exists. We claim that $\bigcup_n U_n = \bigcup_\alpha U_\alpha$.

The inclusion $\bigcup_n U_n \subseteq \bigcup_\alpha U_\alpha$ is immediate. To see that $\bigcup_n U_n \supseteq \bigcup_\alpha U_\alpha$, let $x$ be any element of $\bigcup_\alpha U_\alpha$. Then $x$ is in $U_\alpha$ for some $\alpha$, and so there exists a basis element $V_n$ such that $x \in V_n \subseteq U_\alpha$. Thus $U_n$ is not empty and contains $x$. $\square$

**Lemma 3.3.** *Let $\phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space $(X, d)$. Then*

$$\bigcap\{cl(B \cup B^\bullet) : B \text{ is strongly stable}\} = \bigcap\{B \cup B^\bullet : B \text{ is strongly stable}\}.$$

**Proof.** The inclusion

$$\bigcap\{cl(B \cup B^\bullet) : B \text{ is strongly stable}\} \supseteq \bigcap\{B \cup B^\bullet : B \text{ is strongly stable}\}$$

is immediate.

In order to show the other inclusion, we need to recall some results from [3][Section 3].

For fixed $\varepsilon > 0$, $T > 0$ and $Y \subset X$, let

$$\Omega(Y, \varepsilon, T) := \{x \in X : \text{there is a strong } (\varepsilon, T)\text{-chain from a point of } Y \text{ to } x\}$$

and

$$\bar{\Omega}(Y, \varepsilon, T) := \bigcap_{\eta > 0} \Omega(Y, \varepsilon + \eta, T).$$

Moreover, let

$$\bar{\Omega}(Y) := \bigcap_{\varepsilon>0,\ T>0} \Omega(Y, \varepsilon, T) = \bigcap_{\varepsilon>0,\ T>0} \bar{\Omega}(Y, \varepsilon, T).$$

Let now $x$ be a point in $cl(B \cup B^\bullet) \setminus (B \cup B^\bullet)$ for some strongly stable set $B$. We will show that there exists a strongly stable set $\tilde{B}$ (clearly depending on the point $x$) such that $x \notin cl(\tilde{B} \cup \tilde{B}^\bullet)$.

Since $x \notin B \cup B^\bullet$, by Theorem 2.1, $x \notin \mathcal{SCR}_d(\phi)$. Equivalently,

$$x \notin \bar{\Omega}(x) = \bigcap_{\varepsilon>0,\ T>0} \bar{\Omega}(x, \varepsilon, T).$$

In particular, there exist $\varepsilon > 0$ and $T > 0$ such that $x \notin \bar{\Omega}(x, \varepsilon, T)$.

We proceed by proving that—corresponding to these $\varepsilon > 0$ and $T > 0$—there exists a closed ball $\tilde{C}$ centered in $x$ such that

$$\tilde{C} \cap \bar{\Omega}(\tilde{C}, \varepsilon, T) = \emptyset. \tag{8}$$

Suppose, for the sake of contradiction, that $C \cap \bar{\Omega}(C, \varepsilon, T) \neq \emptyset$ for every closed ball $C$ centered at $x$, in particular, for a sequence $(C_n)_{n\in\mathbb{N}}$ of closed balls centered at $x$ of radius $1/n \to 0$. This means that for every $\eta > 0$ and $n \in \mathbb{N}$ there are points $x_n^\eta$ and $y_n^\eta \in C_n$ such that there exists a $(\varepsilon + \eta, T)$-chain from $y_n^\eta$ to $x_n^\eta$. Since, for every $\eta > 0$,

$$\lim_{n\to+\infty} x_n^\eta = x = \lim_{n\to+\infty} y_n^\eta,$$

we conclude that $x \in \bar{\Omega}(x, \varepsilon, T)$. This fact contradicts the hypothesis that $x \notin \bar{\Omega}(x, \varepsilon, T)$ and therefore there necessarily exists a closed ball $\tilde{C}$ centered in $x$ satisfying formula (8).

In order to conclude, define

$$\tilde{B} := \omega(\bar{\Omega}(\tilde{C}, \varepsilon, T)),$$

which is—see Example 4.3 in [3]—a strongly stable set. Moreover (see Corollary 3.1 in [3]),

$$\tilde{B} \subseteq \bar{\Omega}(\tilde{C}, \varepsilon, T).$$

As a consequence, since (by formula (8)) $x \notin \bar{\Omega}(\tilde{C}, \varepsilon, T)$, then $x \notin \tilde{B}$. We finally recall that—by Lemma 3.6 in [3]—$\omega(\tilde{C}) \subseteq \omega(\bar{\Omega}(\tilde{C}, \varepsilon, T))$, so that for every point $y \in \tilde{C}$, we have

$$\omega(y) \subseteq \omega(\tilde{C}) \subseteq \omega(\bar{\Omega}(\tilde{C}, \varepsilon, T)) = \tilde{B}.$$

This means that $y \notin \tilde{B}^\bullet$ for all $y \in \tilde{C}$.

Finally, since the point $x \notin \tilde{B}$ and $x \in int(\tilde{C})$, we conclude that $x \notin cl(\tilde{B} \cup \tilde{B}^\bullet)$ and the desired inclusion is proved. □

As a direct consequence of the previous lemmas, we can give the proof of Theorem 3.2.

**Proof of Theorem 3.2.** Consider the collection of open sets

$$\{U_\alpha\} := \{X \backslash cl(B_\alpha \cup B_\alpha^\bullet) : B_\alpha \text{ is strongly stable}\}.$$

Then, by Lemma 3.2, there exists an—at most countable—subcollection $\{U_n\}$ such that

$$\bigcup_\alpha U_\alpha = \bigcup_n U_n.$$

Consequently, applying also Theorem 2.1 and Lemma 3.3, we obtain

$$\begin{aligned} \mathcal{SCR}_d(\phi) &= \bigcap_\alpha B_\alpha \cup B_\alpha^\bullet = \bigcap_\alpha cl(B_\alpha \cup B_\alpha^\bullet) \\ &= X \backslash \bigcup_\alpha U_\alpha = X \backslash \bigcup_n U_n = \bigcap_n cl(B_n \cup B_n^\bullet) \\ &\supseteq \bigcap_n B_n \cup B_n^\bullet \supseteq \bigcap_\alpha B_\alpha \cup B_\alpha^\bullet = \mathcal{SCR}_d(\phi), \end{aligned}$$

which gives exactly formula (7). □

### *3.3. End of proof of Theorem 1.1*

**Proof of Theorem 1.1.** With reference to Theorems 3.1 and 3.2, let denote by $h_n$ the Lyapunov function for the $n$th-pair $(B_n, B_n^\bullet)$. As in [6][Page 39], define the function

$$h(x) = \sum_{n=0}^{+\infty} \frac{h_n(x)}{3^n}. \tag{9}$$

The function $h$ is continuous. Moreover—as a consequence of Theorems 2.1 and 3.1—$h$ is a Lyapunov function for $\phi$ which is strictly decreasing outside $\mathcal{SCR}_d(\phi)$. □

**Remark 3.1.** We observe that the proof in [3] of the decomposition of the strong chain recurrent set (here Theorem 2.1) actually uses only the property of $\phi$ being a semiflow. Moreover, the construction of the function in Theorem 3.1 and the countability result in Theorem 3.2 also work under this hypothesis. Consequently, Theorem 1.1 can be rephrased, more generally, for a continuous semiflow on a compact metric space.

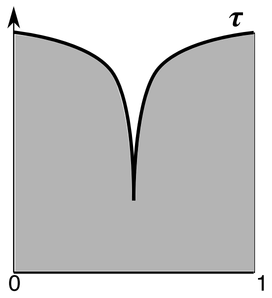


**Fig. 4.** The metric space of Example 3.4.

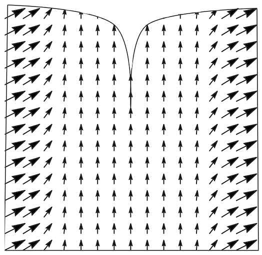

**Fig. 5.** The flow lines of Example 3.4.

As recalled in the Introduction, if the flow is uniformly Lipschitz continuous on compact subsets of $[0, +\infty)$, then the result of Theorem 1.1 is proved in [4][Theorem 4.1]. The proof we present here does not need any additional regularity assumption on the continuous flow.

We finally present an example of a continuous flow which is not uniformly Lipschitz continuous on compact time subsets. That is, an example of a flow to which the result in [4] cannot be applied, but for which Theorem 1.1 holds.

**Example 3.4.** Let us think of $\mathbb{T}$ as $[0, 1]$ with 0 identified to 1. Define $\tau : [0, 1] \to \mathbb{R}$ as

$$\tau(x) := \begin{cases} \sqrt{x - \frac{1}{2}} + \frac{\sqrt{2}-1}{\sqrt{2}} & x \in \left[\frac{1}{2}, 1\right] \\ \sqrt{\frac{1}{2} - x} + \frac{\sqrt{2}-1}{\sqrt{2}} & x \in \left[0, \frac{1}{2}\right] \end{cases}$$

Consider $X = \{(x, y) \in \mathbb{R}^2 : \ x \in [0, 1], \ 0 \le y \le \tau(x)\}$ endowed with the standard metric from $\mathbb{R}^2$, see Fig. 4. Identify the graph of $\tau$ with $[0, 1]$, i.e. such that $(x, \tau(x)) \sim (x, 0)$. Let us consider the flow $\phi$ on $X$ whose flow lines are described in Fig. 5.

We first observe that $\phi$ is a continuous flow, not uniformly Lipschitz continuous on compact subsets of $[0, +\infty)$. Indeed, since the function $\tau$ is not Lipschitz continuous at $x = 1/2$, for any time $T$ greater than the maximum of $\tau$, we cannot find a uniform constant $M_T$ such that, for every $x \in [0, 1]$, we have

$$d(\phi_t(1/2, 0), \phi_t(x, 0)) \le M_T d((1/2, 0), (x, 0)) \qquad \forall t \in [0, T] .$$

So, the condition of uniform Lipschitz continuity is not satisfied.

The set of periodic points corresponds to the central vertical strip (of positive measure), so that $Per(\phi) \neq X$. We also observe that every point in $X \setminus Per(\phi)$ cannot be strong chain recurrent: its dynamics goes "from left to right" and, since the vertical strip of periodic points has positive measure, the strip cannot be traversed by a finite number of jumps whose sum is arbitrarily small. Therefore,

$$\mathcal{SCR}_d(\phi) = Per(\phi) \neq X.$$

## References

[1] E. Akin, Topological dynamics, in: Mathematics of Complexity and Dynamical Systems, vol. 1–3, Springer, New York, 2012, pp. 1726–1747.
[2] E. Akin, M. Hurley, J. Kennedy, Generic homeomorphisms of compact manifolds, in: Proceedings of the 1998 Topology and Dynamics Conference (Fairfax, VA), in: Topology Proc., vol. 23, 1998, pp. 317–337.
[3] O. Bernardi, A. Florio, A Conley-type decomposition of the strong chain recurrent set, Ergodic Theory Dynam. Systems 39 (5) (2019) 1261–1274.
[4] O. Bernardi, A. Florio, Existence of Lipschitz continuous Lyapunov function strict outside the strong chain recurrent set, Dyn. Syst. 34 (1) (2019) 71–92.
[5] O. Bernardi, A. Florio, J. Wiseman, The generalized recurrent set, explosions and Lyapunov functions, J. Dyn. Differential Equations 32 (4) (2020) 1797–1817.
[6] C. Conley, Isolated invariant sets and the Morse index, CBMS Regional Conference Series in Mathematics 38 (1978), iii+89.
[7] C. Conley, The gradient structure of a flow: I, Ergodic Theory and Dynamical Systems 8 (8*) (1988) 11–26.
[8] A. Fathi, P. Pageault, Aubry-Mather theory for homeomorphisms, Ergodic Theory Dynam. Systems 35 (4) (2015) 1187–1207.
[9] K. Yokoi, On strong chain recurrence for maps, Ann. Polon. Math. 114 (2) (2015) 165–177.