Foundations of Physics (2024) 54:50
https://doi.org/10.1007/s10701-024-00783-5

Symplectic Quantization III: Non-relativistic Limit

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Received: 9 February 2024 / Accepted: 20 June 2024 / Published online: 9 July 2024

Abstract
First of all we shortly illustrate how the symplectic quantization scheme (Gradenigo and Livi, Found Phys 51(3):66, 2021) can be applied to a relativistic field theory with self-interaction. Taking inspiration from the stochastic quantization method by Parisi and Wu, this procedure is based on considering explicitly the role of an intrinsic time variable, associated with quantum fluctuations. The major part of this paper is devoted to showing how the symplectic quantization scheme can be extended to the non-relativistic limit for a Schrödinger-like field. Then we also discuss how one can obtain from this non-relativistic theory a linear Schrödinger equation for the single-particle wavefunction. This further passage is based on a suitable coarse-graining procedure, when self-interaction terms can be neglected, with respect to interactions with any external field. In the Appendix we complete our survey on symplectic quantization by discussing how this scheme applies to a non-relativistic particle under the action of a generic external potential.

Keywords Symplectic quantization · Non-relativistic limit · Non-relativistic quantum field theory · Quantum mechanics

1 Introduction

Symplectic quantization has been recently introduced [1, 2] as a quantization scheme suitable for quantum-relativistic field theories. It amounts to a deterministic version of the stochastic quantization method proposed several years ago by Parisi and Wu [4], and later adopted by several authors [5, 6, 8–10]. It is worth recalling that stochastic quantization aims at computing field correlators in the limit $\tau \to \infty$, where τ is interpreted as a *fictitious* time appearing in a Langevin-like equation. In this sense, there is no need to attribute to τ any physical interpretation, relying on the assumption that the stochastic evolution eventually samples a suitable measure in configuration space, epitomizing the statistical effect of quantum fluctuations. In fact, it was also argued that stochastic quantization amounts to a functional formulation of quantum field theory, that is formally equivalent to the canonical partition

Extended author information available on the last page of the article function of equilibrium statistical mechanics, provided a Wick rotation to an imaginary coordinate-time is adopted [6].

Conversely, the basic idea of symplectic quantization is that the statistical effect of quantum fluctuations can be sampled by a deterministic symplectic dynamics, driven by an *intrinsic* time τ, different from the coordinate time of quantum relativistic field theories. Accordingly, the main outcome of this approach is the possibility of looking at the dynamics of a quantum-relativistic field-theory as a relavant piece of physical information, while the system approaches asymptotically equilibrium. In this respect it is important to point out the basic hypothesis of symplectic quantization: the deterministic evolution equations of any quantum-relativistic field, ruled by the *intrinsic* time τ is *ergodic*, i.e. it samples a constant-action hyper-surface according to a uniform probability measure. Such an assumption stems from the consideration that quantum-relativistic fields are infinite dimensional functions and there is no a priori reason to be invoked contrary to ergodicity, once all symmetries of the model under scrutiny have been taken into account. The main ingredients of this approach are summarized in Sect. 2, in order to provide a preliminary framework for the main content of this paper. In fact, the main purpose of this paper is to show how the symplectic quantization procedure applies also to non-relativistic field theories, while recovering also the limit of standard quantum mechanics.

In Sect. 4 we illustrate how symplectic quantization can be extended to non-relativistic field theories by performing a suitable non-relativistic limit of the symplectic evolution equations, which transform into symplectic evolution equations for a Schrödinger field, which still keeps a dependence on the intrinsic time τ. We discuss also the connection of this non-relativistic scheme with the standard formulation of non-relativistic QFT: the Feynman path-integral turns out to be the Fourier-transform of the partition volume of the constant-action hypersurface $\Omega(\mathcal{A})$. This outcome elucidates also that symplectic quantization is the result of a deterministic dynamical evolution yielding a generalized microcanonical measure, whose entropy is given by

$$S_{\mathrm{sym}} = \log(\Omega(\mathcal{A})). \tag{1}$$

In Sect. 6 we go through the symplectic quantization scheme for a quantum field in interaction with a classical electromagnetic field and we show that, by a coarse-graining procedure over the fast dynamics ruled by τ (a sort of *adiabatic elimination* procedure) one can recover the standard quantum mechanical limit in terms of the Schrödinger equation, provided self-interaction terms can be neglected with respect to the interaction with external classical fields. In this perspective the symplectic quantization approach to non-relativistic field theories might be interpreted as a sort of formulation of quantum mechanics relying upon a hidden-variable theory. These are the momenta conjugated to fields yielding the dynamics engendered by the intrinsic time τ of quantum fluctuations. Note that this is not contradictory to the foundations of quantum mechanics and, in particular, with Bell's theorem. Actually, this does not rule out *non-local* theories with hidden variables (see, for instance [7]), like those characterized by global conservation constraints, as in the case of symplectic quantization. In Sect. 7 we report conclusions and perspectives of the matter contained in this paper. Finally, in the Appendix we provide further support to the generality of the symplectic approach, by reconsidering its interest also for "classical" quantum models.

2 Relativistic Formulation

Let us start from the Lagrangian density of a massive charged scalar field $\varphi(x)$, where $x = x^\mu = (x^0, \mathbf{x})$ with $x^0 = c\,t$, c the speed of light in vacuum, t the time, and $\mathbf{x} \in \mathbb{R}^3$ the vector of spatial coordinates. Including the quartic non-linear self-interaction the Lagrangian density reads

$$\begin{aligned}\mathcal{L}(\varphi, \partial_\mu \varphi) =& \partial^\mu \varphi^* \partial_\mu \varphi - \frac{m^2\,c^2}{\hbar^2}\,|\varphi|^2 - \frac{\lambda}{2}|\varphi|^4 \\ =& \frac{1}{c^2}|\partial_t \varphi|^2 - |\partial_{\mathbf{x}}\varphi|^2 - \frac{m^2\,c^2}{\hbar^2}\,|\varphi|^2 - \frac{\lambda}{2}|\varphi|^4\,.\end{aligned} \tag{2}$$

Since in what follows we want to consider the non-relativistic limit of the theory, we have to consider a complex, i.e. *charged*, field φ. The integral over the space-time continuum of the Lagrangian density in Eq. (2) is the relativistic-invariant action

$$S[\varphi(x)] = \int \mathcal{L}(\varphi, \partial_\mu \varphi)\, d^4x\,, \tag{3}$$

where $d^4x = dx^0\; d^3\mathbf{x} = c\; dt\; d^3\mathbf{x}$.

3 Symplectic Quantization

The fundamental idea of symplectic quantization is that $x^0 = c\,t$ is just a standard coordinate and that one has to consider an additional time variable, i.e. the *intrinsic time* τ parametrizing the sequence of quantum fluctuations [1, 2]. In practice, one assumes that the complex field depends also on this intrinsic time, i.e. $\varphi(x;\tau) = \varphi(x^0, \mathbf{x};\tau)$. Moreover, one postulates the existence of a generalized "Lagrangian" (which has however the units of an action) containing a sort of *kinetic energy term*:

$$\mathbb{L}[\varphi, \partial_\tau \varphi] = \int c_s^{-2}|\partial_\tau \varphi(x;\tau)|^2\, d^4x + S[\varphi(x;\tau)]\;, \tag{4}$$

where ∂_τ denotes the derivative with respect to τ and c_s is a suitable parameter with the dimensions of a velocity. Let us observe that the simple form of the generalized Lagrangian in Eq. (4) is closely inspired, almost dictated, by the analogous deterministic dynamics approach to Euclidean lattice field theory (see for instance [5]), where the presence of the quadratic term in $\partial_\tau \varphi$ is a necessary condition to guarantee the sampling of the equilibrium measure $\exp(-S_E/\hbar)$. The new physical constant c_s is associated with the intrinsic time-scale of the symplectic dynamics, which might be relevant for analyzing transient regimes, but which has no practical influence on the sampling of the equilibrium distribution of fields. Since there is in general no need for c_s to coincide with c, we will keep explicit the dependence on it. Moreover, in order to illustrate properly how to perform the non-relativistic limit in the symplectic quantization scheme, it is convenient to keep explicit the dependence also on the other physical constants $\hbar$ and c. Following a standard procedure, one can introduce the canonically conjugated momenta

$$\pi(x;\tau) = \frac{\delta \mathbb{L}}{\delta \partial_\tau \varphi^*(x;\tau)} = c_s^{-2} \partial_\tau \varphi(x;\tau) \tag{5}$$

$$\pi^*(x;\tau) = \frac{\delta \mathbb{L}}{\delta \partial_\tau \varphi(x;\tau)} = c_s^{-2} \partial_\tau \varphi^*(x;\tau) \tag{6}$$

and thus define the symplectic action, or generalized "Hamiltonian", as follows

$$\begin{aligned} &\mathbb{H}[\pi, \varphi] \\ &= \int \left[\pi^*(x;\tau)\partial_\tau \varphi(x;\tau) + \pi(x;\tau)\partial_\tau \varphi^*(x;\tau)\right] d^4x - \mathbb{L}[\varphi, \partial_\tau \varphi]. \end{aligned} \tag{7}$$

Making use of all of the previous equations one can finally write

$$\mathbb{H}[\pi, \varphi] = \int \frac{|\pi(x;\tau)|^2}{c_s^{-2}} \, d^4x - S[\varphi(x;\tau)] \ . \tag{8}$$

The symplectic dynamical equations then read

$$\partial_\tau \varphi(x;\tau) = \frac{\delta \mathbb{H}[\pi, \varphi]}{\delta \pi^*(x;\tau)} \ , \tag{9}$$

$$\partial_\tau \pi(x;\tau) = -\frac{\delta \mathbb{H}[\pi, \varphi]}{\delta \varphi^*(x;\tau)} \ . \tag{10}$$

For the field-theory (2) they have the explicit expressions

$$\partial_\tau \varphi(x;\tau) = \frac{\pi(x;\tau)}{c_s^{-2}} \ , \tag{11}$$

$$\partial_\tau \pi(x;\tau) = -\left(\Box + \frac{m^2 c^2}{\hbar^2} + \lambda |\varphi(x;\tau)|^2\right) \varphi(x;\tau), \tag{12}$$

where $\Box = c^{-2}\partial_t^2 - \partial_{\mathbf{x}}^2$. The purpose of the present discussion has been to introduce the formalism of symplectic quantization for relativistic fields in order to both connect with the previous papers on the subject [1, 2] and also to show how the approach to non-relativistic QFT can be directly deduced from the relativistic one, which needed therefore to be presented first.

4 Non-relativistic Limit

Here we discuss the non-relativistic limit in the framework of the symplectic quantization. The first step amounts to factorize the high-frequency component of the field $\varphi(x;\tau)$, thus epitomizing its dependence on the parameter c (speed of light):

$$\varphi(x;\tau) = \frac{\hbar}{\sqrt{2mc}}\, e^{-i\,\frac{mc^2}{\hbar}\,t}\, \Psi(\mathbf{x},t;\tau)\,, \tag{13}$$

where $\Psi(\mathbf{x},t;\tau)$ is the non-relativistic field, i.e. the Schrödinger field, which still exhibits a dependence on the intrinsic time τ. The adiabatic separation in slow and fast oscillating components of the field $\varphi(x;\tau)$ is a standard approach to perform $1/c$ expansions, see for instance [3]. Let us notice that within the symplectic quantization approach the distinction between *"classical"* and *"quantum"* fields is straightforward: anytime a field has quantum fluctuations it carries an explicit dependence on the intrinsic time τ, while a classical field does not exhibit any dependence on τ.

By substituting Eq. (13) into the relativistic action, Eq. (3), and performing the non-relativistic limit $c \to \infty$ one can conclude that the contribution $\frac{1}{2c^2}\Psi^* \frac{\partial^2\Psi}{\partial t^2} \to 0$ can be neglected, thus yielding the following expression for the non-relativistic action

$$\begin{aligned} S_{\mathrm{nr}}[\Psi(\mathbf{x},t;\tau)] &= \int dt\, d^3\mathbf{x}\, \mathcal{L}_{\mathrm{nr}}[\Psi(\mathbf{x},t;\tau)] \\ \mathcal{L}_{\mathrm{nr}}[\Psi] &= \frac{i\hbar}{2}\left(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*\right) - \frac{\hbar^2}{2m}|\partial_{\mathbf{x}}\Psi|^2 - \frac{g}{2}|\Psi|^4, \end{aligned} \tag{14}$$

where $g = \lambda\hbar^4/(4m^2c)$. Then, for consistency, we also define the *slow* non-relativistic component $\Pi(\mathbf{x},t;\tau)$ of the conjugated momentum field $\pi(x;\tau)$ as follows

$$\pi(x;\tau) = \frac{\hbar}{\sqrt{2mc}}\, e^{-i\,\frac{mc^2}{\hbar}\,t}\, \Pi(\mathbf{x},t;\tau). \tag{15}$$

In this way we obtain the expression of the generalized non-relativistic Hamiltonian

$$\begin{aligned} &\mathbb{H}_{\mathrm{nr}}[\Pi(\mathbf{x},t;\tau),\Psi(\mathbf{r},t;\tau)] \\ &\quad = \int \frac{|\Pi(\mathbf{x},t;\tau)|^2}{\mu} dt\, d^3\mathbf{x} - S_{\mathrm{nr}}[\Psi(\mathbf{x},t;\tau)], \end{aligned} \tag{16}$$

where $\mu = 2mc_s^{-2}/\hbar^2$. One can easily recognize in the previous formula a generalized *separable* Hamiltonian, containing a generalized "kinetic energy" term and a generalized "potential energy" term

$$\mathbb{H}_{\mathrm{nr}}[\Pi,\Psi] = \mathbb{K}[\Pi] + \mathbb{V}[\Psi], \tag{17}$$

with

$$\begin{aligned} \mathbb{K}[\Pi] &= \int \frac{|\Pi(\mathbf{x},t;\tau)|^2}{\mu}\, dt\, d^3\mathbf{x} \\ \mathbb{V}[\Psi] &= -S_{\rm nr}[\Psi(\mathbf{x},t;\tau)] \end{aligned} \tag{18}$$

The symplectic dynamical equations of the Schrödinger field $\Psi(\mathbf{x},t;\tau)$ read then

$$\partial_\tau \Psi(\mathbf{x},t;\tau) = \frac{\delta \mathbb{H}_{\rm nr}[\Pi,\Psi]}{\delta \Pi^*(\mathbf{x},t;\tau)} \ , \tag{19}$$

$$\partial_\tau \Pi(\mathbf{x},t;\tau) = -\frac{\delta \mathbb{H}_{\rm nr}[\Pi,\Psi]}{\delta \Psi^*(\mathbf{x},t;\tau)} \ . \tag{20}$$

For the non-relativistic limit of the field-theory, Eq. (2), they take the explicit expressions

$$\partial_\tau \Psi(\mathbf{x},t;\tau) = \frac{\Pi(\mathbf{x},t;\tau)}{\mu} \ , \tag{21}$$

$$\partial_\tau \Pi(\mathbf{x},t;\tau) = \left(i\hbar\partial_t + \frac{\hbar^2}{2m}\partial^2_{\mathbf{x}} - g|\Psi|^2 \right) \Psi(\mathbf{x},t;\tau) \ . \tag{22}$$

Before proceeding further in illustrating more details about the outcomes of the non-relativistic limit of the symplectic quantization scheme, we want to point out a crucial comment about its physical interpretation. The deterministic dynamics in the intrinsic time τ, engendered by Eqs. (21) and (22), makes the action $S_{\rm nr}[\Psi]$ play the role of an effective potential energy $-\mathbb{V}[\Psi]$, which is, accordingly, a fluctuating quantity. In the extended phase-space of the Schrödinger field $\Psi(\mathbf{x},t;\tau)$ and of its conjugated momentum $\Pi(\mathbf{x},t;\tau)$ these fluctuations are of quantum origin.

5 Connection with Standard Non-relativistic QFT

In this section we want to elaborate about the relation between the non-relativistic symplectic quantization scheme introduced in the previous section and the standard formulation of non-relativistic field theory (QFT), based on the path integral formalism. The key point is the assumption that the dynamics engendered by Eqs. (19) and (20) is ergodic in a generalized microcanonical ensemble. Note that despite one would like to have at disposal a proof rather than an assumption, ergodicity is quite difficult to be established on a rigorous ground. Anyway, there is a general consensus that such an assumption holds for models of infinite-dimensional variables, like field theories. Said differently, ergodicity amounts to assume the existence of a uniform probability measure $\mathcal{P}_{\mathcal{A}}(\Pi,\Psi)$ in the infinite-dimensional phase-space of fields and conjugate momenta reading as:

$$\mathcal{P}_{\mathcal{A}}[\Pi, \Psi] = \frac{\delta(\mathbb{H}_{\text{nr}}[\Pi, \Psi] - \mathcal{A})}{\Omega(\mathcal{A})}, \tag{23}$$

where the normalization factor is a generalized microcanonical partition function fixing to $\mathcal{A}$ the value of the generalized non-relativistic Hamiltonian $\mathbb{H}_{\text{nr}}$ (which has the actual physical dimensions of an action):

$$\Omega(\mathcal{A}) = \int \mathcal{D}[\Pi]\mathcal{D}[\Psi]\delta(\mathbb{H}_{\text{nr}}[\Pi, \Psi] - \mathcal{A}) \ , \tag{24}$$

The ergodic hypothesis amounts then to assume that the dynamical average along the trajectories of the symplectic quantization dynamics do correspond to ensemble averages with respect to the probability measure $\mathcal{P}_{\mathcal{A}}[\Pi, \Psi]$ in Eq. (23):

$$\begin{aligned} &\lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \mathcal{O}(\Psi_{\mathcal{A}}(\mathbf{x}, t; \tau'))\, d\tau' \\ &= \int \mathcal{D}[\Pi]\mathcal{D}[\Psi]\ \mathcal{O}(\Psi)\ \mathcal{P}_{\mathcal{A}}[\Pi, \Psi], \end{aligned} \tag{25}$$

where $\Psi_{\mathcal{A}}(\mathbf{x}, t; \tau')$ denotes a solution of symplectic dynamics equations relative to a generic initial condition, such that $\mathbb{H}_{\text{nr}}[\Pi(\mathbf{x}, t; 0), \Psi(\mathbf{x}, t; 0)] = \mathcal{A}$. In Eqs. (24) and (25) the symbols $\mathcal{D}[\Pi] = \mathcal{D}[\Pi(\mathbf{x}, t)]$ and $\mathcal{D}[\Psi] = \mathcal{D}[\Psi(\mathbf{x}, t)]$ denote functional integration over the fields $\Pi(\mathbf{x}, t)$ and $\Psi(\mathbf{x}, t)$, respectively. According to Eq. (25) the dynamical average can be replaced by the (microcanonical) ensemble average: for this reason the fields and their conjugated momenta in the integral on the r.h.s. do not keep any dependence on τ. This is in complete analogy to what is assumed in classical statistical mechanics, where the dynamics is replaced by (functional) integration over equilibrium configurations. In this case the partition function in Eq. (24) turns out to be just the sum over all classical configuration of the field and conjugated momenta, compatible with a given value of the generalized action.

It is worth stressing that the ergodic hypothesis is the core of the connection between symplectic quantization and ordinary quantum field theory also in the non-relativistic limit discussed here.

Then, since in the present framework the action $\mathcal{A}$ is the symplectic analog of the microcanonical internal energy, it is natural to introduce the dimensionless symplectic analog of the familiar microcanonical entropy as

$$S_{\text{sym}}(\mathcal{A}) = \ln \Omega(\mathcal{A}) \ . \tag{26}$$

In this way, the reduced Planck constant $\hbar$ fixes the constraint

$$\frac{1}{\hbar} = \frac{\partial S_{\text{sym}}(\mathcal{A})}{\partial \mathcal{A}} \ , \tag{27}$$

or, equivalently,

$$\frac{1}{\hbar} = \frac{1}{\Omega(\mathcal{A})} \frac{\partial \Omega(\mathcal{A})}{\partial \mathcal{A}} \,, \tag{28}$$

which relates the extensive quantity $\mathcal{A}$ to the intensive quantity $\hbar$. As previously discussed, in the symplectic quantization $\hbar$ plays the role of the intensive thermal energy $k_B T$.

At this point our purpose is to draw the precise connection between the microcanonical representation of symplectic quantization and the most familiar functional approach to quantum field theory, i.e. the path-integral representation. As we are going to show this bridge can be established by performing a suitable change of equilibrium statistical ensemble, which, in this case, does not amount to a standard Legendre transform. Actually, in the *microcanonical* representation of the symplectic quantization scheme the value of the action over the phase-space is fixed and one would like to transform this formalism to a sort of *canonical* representation fixing *on average* the action per unit volume. In equilibrium statistical mechanics the analogous procedure yields a canonical representation based on a Gibbs-Boltzmann weight. Since in symplectic quantization scheme (also in the present non-relativistic case) the functional with respect to which we would like to consider the integral transform is an action, i.e. a "non-positive" defined quantity, the only transformation which can be performed is the Fourier one. Accordingly, the Fourier transform of the partition volume in Eq. (24) reads

$$\begin{aligned} \mathcal{Z}(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\mathcal{A}\, e^{-iz\mathcal{A}}\, \Omega(\mathcal{A}) \\ &= \int \mathcal{D}[\Pi]\mathcal{D}[\Psi] e^{-iz\mathbb{H}_{\mathrm{nr}}[\Pi,\Psi]} \,, \end{aligned} \tag{29}$$

where z is the variable conjugated to $\mathcal{A}$ and has therefore physical dimensions of an inverse action, $[z] = [\mathcal{A}^{-1}]$. By then fixing $z = 1/\hbar$ and integrating over the quadratic dependence on conjugated momenta one obtains

$$\begin{aligned} \mathcal{Z}(\hbar) &= \int \mathcal{D}[\Pi]\mathcal{D}[\Psi]\, e^{-\frac{i}{\hbar}\mathbb{H}_{\mathrm{nr}}[\Pi,\Psi]} \\ &= \mathcal{N}(\hbar) \int \mathcal{D}[\Psi]\, e^{\frac{i}{\hbar} S_{\mathrm{nr}}[\Psi]}, \end{aligned} \tag{30}$$

which is the standard Feynman path integral representation of the non-relativistic QFT of the Schrödinger field $\Psi(\mathbf{x}, t)$, where $\mathcal{N}(\hbar)$ is a suitable normalization factor. We can conclude that within the symplectic quantization approach the Feynman path-integral is obtained, on the basis of the ergodic hypothesis mentioned in this paragraph, simply as the Fourier transform of a generalized microcanonical partition function based on the conservation of the symplectic action. Only on a *formal* ground, we can therefore think at Eq. (30) as *a sort of* partition function and its complex integrand as *a sort of* probability measure $\mathcal{P}_\hbar[\Pi, \Psi]$ which, provided the original ergodic hypothesis in Eq. (25) holds true, conserves *on average* the symplectic action, with $\mathcal{P}_\hbar[\Pi, \Psi]$ reading as

$$\mathcal{P}_\hbar[\Pi, \Psi] = \frac{1}{\mathcal{Z}(\hbar)} e^{-\frac{i}{\hbar}\mathbb{H}_{\rm nr}[\Pi,\Psi]}. \tag{31}$$

More precisely, we can write

$$\begin{aligned}
&\lim_{\tau\to+\infty} \frac{1}{\tau} \int_0^\tau \mathcal{O}(\Psi_\mathcal{A}(\mathbf{x}, t;\tau'))\, d\tau' \\
&= \int \mathcal{D}[\Pi]\mathcal{D}[\Psi]\ \mathcal{O}(\Psi)\ \mathcal{P}_\hbar[\Pi, \Psi] \\
&= \frac{1}{\mathcal{Z}(\hbar)} \int \mathcal{D}[\Pi]\mathcal{D}[\Psi]\ \mathcal{O}(\Psi)\ e^{-\frac{i}{\hbar}\mathbb{H}_{\rm nr}[\Pi,\Psi]} \\
&= \frac{\int \mathcal{D}[\Psi]\ \mathcal{O}(\Psi)\ e^{\frac{i}{\hbar}S_{\rm nr}[\Psi]}}{\int \mathcal{D}[\Psi]\ e^{\frac{i}{\hbar}S_{\rm nr}[\Psi]}}
\end{aligned} \tag{32}$$

which makes explicit the relation between symplectic quantization dynamical averages and the expectation values in ordinary quantum field theory.

Note that, as for Eqs. (8) and (16), the integration over the conjugated momenta in the last line of Eq. (32) is straightfoward only if the generalized Hamiltonian corresponding to the symplectic action is *separable*. Additional computational difficulties occur when one has to deal with models exhibiting symmetry properties whose symplectic quantization yields a non-separable generalized Hamiltonian of the form

$$A[\Pi, \Psi] = \mathbb{K}[\Pi, \Psi] + \mathbb{V}[\Phi]\,, \tag{33}$$

as is it happens, for instance, in non-abelian gauge theories [6] and, most notably, in gravity [2, 6].

We conclude this section by discussing the physical implications of the quantization constraint in Eq. (27), which draws a formal correspondence between $\hbar$, in the symplectic quantization approach, and the microcanonical definition of temperature T in ordinary statistical mechanics, $T^{-1} = \partial S(E)/\partial E$, where S is the microcanonical entropy and E is the total conserved energy. In order to do that we take inspiration from the Parisi-Wu stochastic quantization approach, where the quantum fluctuations of the fields are associated to a stochastic dynamics, which mimicks the contact with a fictitious thermal bath at temperature $\hbar$. Therefore, the main idea is that we need to set the conditions for the deterministic dynamics of Eq. (22) as if it would be performed in contact with a thermostat at temperature $\hbar$. This task can be achieved by assuming that a sort of equipartition theorem applies in the symplectic quantization scheme, as for an ergodic Hamiltonian dynamics of a classical system at temperature T. In fact, in a classical Hamiltonian system with total energy E, made on N degrees of freedom, whose kinetic energy is denoted by K, the equipartiton of the energy amounts to assume that

$$k_B T = \frac{E}{N} = \frac{2\langle K\rangle}{N}\,, \tag{34}$$

where

$$\langle K \rangle = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} ds\, K[\Pi(s)], \tag{35}$$

Accordingly, the angle brackets $\langle \bullet \rangle$ are a shorthand notation for a time-average, which, consistently with the ergodic hypothesis, is assumed to be independent of the initial conditions. We can export this procedure in the framework of the symplectic quantization scheme by introducing the following correlation function for generalized momenta

$$\langle \Pi^*(\mathbf{x},t)\Pi(\mathbf{y},t') \rangle = \lim_{\Delta\tau \to \infty} \frac{1}{\Delta\tau} \int_{\tau_0}^{\tau_0+\Delta\tau} ds\, \Pi^*(\mathbf{x},t;s)\Pi(\mathbf{y},t';s), \tag{36}$$

where $\Pi^*(\mathbf{x},t;s)$ and $\Pi(\mathbf{y},t';s)$ denote solutions of the Hamiltonian equations, Eq. (22), and τ_0 indicates a time scale over which the system has already reached stationarity. The analogous of Eq. (34) reads

$$\frac{1}{\mu}\langle \Pi^*(\mathbf{x},t)\Pi(\mathbf{y},t') \rangle = \frac{\hbar}{2}\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{y}), \tag{37}$$

where μ is a suitable dimensional constant. By Fourier-transforming this equation one obtains the *quantization* condition

$$\frac{1}{\mu}\langle \Pi^*(\mathbf{k},\omega)\Pi(\mathbf{k},\omega) \rangle = \frac{\hbar}{2}. \tag{38}$$

Note that for a lattice version of the theory, the condition in Eq. (37) should be replaced with

$$\frac{1}{\mu}\langle \Pi^*(\mathbf{x}_i,t_k)\Pi(\mathbf{x}_j,t'_l) \rangle = \frac{\hbar}{2}\,\frac{1}{ba^3}\,\delta_{kl}\,\delta_{ij}, \tag{39}$$

where a is the lattice spacing along spatial directions and b is the lattice spacing along time direction. Again, by Fourier-transforming this equation one obtains Eq. (38), where the continuum variables $\mathbf{k}$ and ω are replaced by their discretized version. These quantization conditions point out the special role played by $\hbar$ in the symplectic quantization scheme. In any practical computation by numerical methods of the deterministic dynamics in Eq. (22) one has to consider the class of initial conditions yielding a stationary evolution, compatible with such quantization constraint. More details about the possible recipes for identifying the right class of initial conditions for the numerical simulaton of the dynamics will be reported in a forthcoming paper by one of the authors [11].

6 Coarse-Graining: From Quantum Field Theory to Quantum Mechanics

In this section we illustrate how to obtain a non-relativistic theory for a quantum wave-function $\psi(\mathbf{x}, t)$, by applying a suitable coarse graining procedure to the Schrödinger field $\Psi(\mathbf{x}, t;\tau)$. We limit our considerations to local fields and single-particle wavefunctions, disregarding the case of many-body wavefunctions, since they are non-local objects unsuitable for describing in the same framework non-relativistic and relativistic systems, *a fortiori* in the presence of interactions.

We consider the paradigmatic case of a self-interacting quantum scalar field which is also in interaction with a *classical* electromagnetic field. Starting from the Lagrangian in Eq. (2), one can add the interaction with the electromagnetic classical field $A^\mu(\mathbf{x}, t) = A^\mu(x) = (\Phi(x), \mathbf{A}(x))$, with $\Phi(\mathbf{x}, t)$ the scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential, in a way which is both covariant and gauge-invariant, by introducing the covariant derivatives:

$$\partial_t \to \partial_t + i\frac{q}{\hbar}\Phi \tag{40}$$

$$\nabla \to \nabla - i\frac{q}{\hbar}\mathbf{A}. \tag{41}$$

The classical nature of the electromagnetic field excludes any dependence of $A^\mu(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, t)$ on the intrinsic time of quantum fluctuations τ. Since we assume that the elecromagnetic field is a static external field, we can also neglect "propagative" terms such as $F^{\mu\nu}F_{\mu\nu}$. Thereofere, the total Lagrangian, including the contributions of the external fields, reads

$$\begin{aligned}\mathcal{L}_{\text{tot}} = \mathcal{L}(\varphi, \partial_\tau\varphi) + \frac{q^2}{2mc^2}|\varphi|^2\Phi^2 - \frac{iq\hbar}{2mc^2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*)\Phi \\ - \frac{q^2}{2m}|\varphi|^2\mathbf{A}^2 + \frac{iq\hbar}{2m}(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*)\cdot\mathbf{A}\end{aligned} \tag{42}$$

In order to consider the non-relativistic limit, we rewrite $\varphi(\mathbf{x}, t;\tau)$ in terms of a high-frequency component $e^{-imc^2t/\hbar}$ and a *slow* component $\Psi(\mathbf{x}, t;\tau)$ according to the relation in Eq. (13). This replacement amounts to a sort of adiabatic elimination procedure, yielding the following Lagrangian of the field $\Psi(x, \tau)$

$$\begin{aligned}\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{nr}}[\Psi] + \frac{q^2}{2mc^2}|\Psi|^2\Phi^2 - q|\Psi|^2\Phi \\ - \frac{q^2}{2m}|\Psi|^2\mathbf{A}^2 + \frac{iq\hbar}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)\cdot\mathbf{A}\,,\end{aligned} \tag{43}$$

where the non-relativistic Lagrangian $\mathcal{L}_{\text{nr}}[\Psi]$ is the one define in Eq. (14). Without loss of generality we can assume a null vector potential, i.e. $\mathbf{A} = 0$. By replacing $1/c^2$ with $\epsilon_0\mu_0$ we can rewrite the complete Lagrangian in the form

$$
\begin{aligned}
&\mathcal{L}_{\mathrm{tot}}[\Psi(\mathbf{x},t;\tau)] \\
&\quad = \frac{i\hbar}{2}\big(\Psi^*(\mathbf{x},t;\tau)\partial_t\Psi(\mathbf{x},t;\tau) - \Psi(\mathbf{x},t;\tau)\partial_t\Psi^*(\mathbf{x},t;\tau)\big) \\
&\qquad - \frac{\hbar^2}{2m}|\partial_{\mathbf{x}}\Psi(\mathbf{x},t;\tau)|^2 - \frac{g}{2}|\Psi(\mathbf{x},t;\tau)|^4 \\
&\qquad + \left(\frac{\epsilon_0\mu_0 q^2}{2m}\Phi^2(\mathbf{x},t) - q\Phi(\mathbf{x},t)\right)|\Psi(\mathbf{x},t;\tau)|^2\,,
\end{aligned}
\tag{44}
$$

where we have made explicit the dependence of the quantum fields on τ, as opposed to the classical fields, which are assumed independent of it.

We can now introduce the coarse-graining argument which allows us to deduce quantum mechanics from quantum field theory. Let us recall that within the symplectic quantization approach we have to deal with a generalized Hamiltonian, that we have also called *symplectic action*, because its physical dimensions are those of an action. This generalized Hamiltonian is related to the Lagrangian in Eq. (44) and to the generalized conjugated momenta appearing in the equation of motion:

$$
\begin{aligned}
\partial_\tau\Pi(\mathbf{x},t;\tau) = &\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_{\mathbf{x}}^2 - g|\Psi|^2\right)\Psi(\mathbf{x},t;\tau) \\
&+ \left(\frac{\epsilon_0\mu_0 q^2}{2m}\Phi^2(\mathbf{x},t) - q\Phi(\mathbf{x},t)\right)\Psi(\mathbf{x},t;\tau).
\end{aligned}
\tag{45}
$$

Since for physical reasons we have to assume that the generalized Hamiltonian takes a finite value, also the "potential energy" contribution, i.e. the one depending on $\Psi(\mathbf{x},t;\tau)$, and the "kinetic energy" contribution, i.e. the one which depending on $\Pi(\mathbf{x},t;\tau)$, have to remain finite for any values of τ. Accordingly, the generalized momenta are bounded functions of τ. Therefore, averaging with respect to the intrinsic time τ both terms of Eq. (45) we obtain that the l.h.s. of this equation vanishes with $\tau \to +\infty$, namely

$$
\lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau \frac{\partial\Pi(\mathbf{x},t;\tau')}{\partial\tau'}\,d\tau' = \lim_{\tau\to+\infty}\frac{\Pi(\tau)-\Pi(0)}{\tau} \to 0
\tag{46}
$$

For what concerns the quantum field $\langle\Psi\rangle$, we assume that its intrinsic-time average

$$
\langle\Psi(\mathbf{x},t;\tau)\rangle_\tau = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau \Psi(\mathbf{x},t;\tau')\,d\tau'
\tag{47}
$$

defines the quantum mechanics wave-function

$$
\psi(\mathbf{x},t) = \langle\Psi(\mathbf{x},t;\tau)\rangle_\tau\,.
\tag{48}
$$

By assuming that the external field is such that $q^2\epsilon_0\mu_0\Phi^2/m \ll q\Phi$, Eq. (45) can be rewritten in terms of the wave-function as follows

$$
\begin{aligned}
& i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) \\
& \quad = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + q\psi(\mathbf{x}, t)\Phi(\mathbf{x}, t) + g\langle |\Psi|^2 \Psi \rangle .
\end{aligned}
\tag{49}
$$

The important information which must be retained from Eq. (49) is that *in general* the coarse-graining of the quantum fields over the fast dynamics over τ does not yield a simple quantum-mechanical theory. The average/coarse-graining of the non-linear interaction term with respect to the sequence of quantum fluctuations controlled by τ does not yield an expression which can be written in terms of the wave-function $\psi(\mathbf{x}, t)$ only, because

$$
\langle |\Psi|^2 \Psi \rangle \neq |\langle \Psi \rangle|^2 \langle \Psi \rangle . \tag{50}
$$

It is only in peculiar cases, related to quantum phase transitions, that one finds situations where the following replacement is correct

$$
\langle \hat{\Psi}^{+}(\mathbf{x}, t)\hat{\Psi}(\mathbf{x}, t)\hat{\Psi}(\mathbf{x}, t) \rangle = |\psi(\mathbf{x}, t)|^2 \psi(\mathbf{x}, t). \tag{51}
$$

where in Eq. (51) the second quantization formalism has been restored: the fluctuating field $\Psi(\mathbf{x}, t;\tau)$ has been replaced with the field operator $\hat{\Psi}(\mathbf{x}, t)$ and $\langle \bullet \rangle$ denotes the average over the Fock space rather than the intrinsic time average. A replacement as the one of Eq. (51) is possible for instance when the average is made with respect to a coherent state $|CS\rangle$, that is an eigenstate of the field operator $\hat{\Psi}(\mathbf{x}, t)$, such that $\hat{\Psi}(\mathbf{x}, t)|CS\rangle = \psi(\mathbf{x}, t)|CS\rangle$ (see, for instance [12]). However, we are not going to deal with these peculiar cases in this paper.

The main conclusion is that for local fields in the presence of self-interactions, in general the Schrödinger field $\Psi(\mathbf{x}, t;\tau)$ is intrinsically fluctuating and cannot be reduced to a wave-function. Only when the effect of the external classical field is dominant with respect to the self interaction, namely

$$
\| q\psi(\mathbf{x}, t)\Phi(\mathbf{x}, t) \| \gg \| g\langle |\Psi|^2 \Psi \rangle \|, \tag{52}
$$

where $\| \cdot \|$ denotes an L_2 norm in the appropriate Hilbert space, or when the self interaction is set to zero ($g = 0$), the coarse-graining with respect to the intrinsic time τ allows one to map the fluctuating quantum field to a wave-function, uniquely determined in each point of space-time as the solution of the Schrödinger equation:

$$
i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + q\Phi(\mathbf{x}, t) \right] \psi(\mathbf{x}, t) \tag{53}
$$

In summary, the above derivation is a well-defined protocol for obtaining a wave function $\psi(\mathbf{r}, t)$ (with $|\psi(\mathbf{x}, t)|^2$ representing a probability density in space-time) from a fluctuating quantum field $\Psi(\mathbf{r}, t;\tau)$ by a sort of adiabatic elimination procedure, combined with suitable assumptions about the role of self-interacting contributions in the original field theory.

7 Conclusions and Perspectives

In this paper we have explained how the symplectic quantization approach to quantum field theory introduced in [1, 2] can be extended to the non-relativistic field theory of a Schrödinger field and how, under a very reasonable *ergodic hypothesis* for the deterministic dynamics of this new approach, it is possible to readily obtain a connection with the standard functional approach to QFT based on the Feynman path-integral formalism. Moreover, we have illustrated how a *coarse-graining* procedure applied to the symplectic quantization scheme yields ordinary quantum mechanics from a theory of fluctuating quantum fields. This procedure is effective only for a free quantum field or in the presence of an external classical field much stronger than the self-interaction term of the quantum field. In Appendix A we discuss how the symplectic quantization formalism applies to a non-relativistic particle in a potential well. Last but not least we have pointed out how the symplectic quantization dynamics, which flows along the new intrinsic time τ of quantum fluctuations, can provide new numerical protocols for sampling quantum fluctuations of fields in real coordinate time t also in the non-relativistic case. Further investigations should certainly encompass numerical tests of the symplectic quantization scheme for non-relativistic quantum fields (a first numerical study of relativistic fields can be found in [11]) and the investigation of the interplay between this formalism and alternative functional approaches to study quantum fluctuations of non-relativistic fields (in particular bosonic ones) already present in the literature, e.g. the so-called Truncated Wigner Approximation [13] or the Stochastic Gross-Pitaevski equation [14, 15].

Appendix A: Non-relativistic Particle in a One-Dimensional External Potential

We complete our discussion on the non-relativistic limit of symplectic quantization by considering the one-dimensional non-relativistic problem of a particle in an external potential. The classical lagrangian of this non-relativistic system can be written as

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q) \ , \tag{A1}$$

where $q(t)$ denotes the coordinate of a particle of mass m as a function of coordinate time t and $\dot{q}$ is the derivative of the coordinate with respect to t, while $V(q)$ is the external potential acting on the particle. The action functional is given by

$$S[q(t)] = \int_{t_0}^{t_1} L(q, \dot{q})\, dt = \int_{t_0}^{t_1} \left(\frac{m}{2}\dot{q}^2 - V(q)\right) dt \ , \tag{A2}$$

where $t \in [t_I, t_F]$, i.e. with t_I and t_F the two limits of integration over the time t.

As previously discussed, the prescription of symplectic quantization is that any classical field, in this case the "position field" $q(t)$, is "quantized" by assuming for it a further dependence on the intrinsic time τ, so that for any given value of coordinate time t the field $q(t;\tau)$ is not fixed but fluctuates along the symplectic dynamics.

According to what already done for fields in the previous sections, we postulates the existence of a generalized "Lagrangian"

$$\begin{aligned}\mathbb{L}[q,\partial_\tau q] &= \int_{t_I}^{t_F} \frac{M}{2}(\partial_\tau q(t;\tau))^2\, dt + S[q(t;\tau)] \\ &= \int_{t_I}^{t_F} \left(\frac{M}{2}(\partial_\tau q)^2 + \frac{m}{2}\dot{q}^2 - V(q)\right) dt\ ,\end{aligned} \tag{A3}$$

where ∂_τ denotes again the derivative with respect to the intrinsic time τ and M is an appropriate dimensional constant. In this case

$$[M] = \text{mass}\ . \tag{A4}$$

The symplectic momentum reads

$$\pi(t;\tau) = \frac{\delta \mathbb{L}}{\delta \partial_\tau q(t;\tau)} = M\partial_\tau q(x;\tau) \tag{A5}$$

and the symplectic-action/generalized-Hamiltonian functional is

$$\begin{aligned}\mathbb{H}[\pi,q] &= \int_{t_I}^{t_F} \pi(t;\tau)\partial_\tau q(t;\tau)\, dt - \mathbb{L}[q,\partial_\tau q] \\ &= \int_{t_I}^{t_F} \frac{\pi(t;\tau)^2}{2M}\, dt - S[q,\partial_\tau q] \\ &= \int_{t_I}^{t_F} \left(\frac{\pi(t;\tau)^2}{2M} - \frac{m}{2}\dot{q}(t;\tau)^2 + V(q(t;\tau))\right) dt\ .\end{aligned} \tag{A6}$$

The symplectic dynamical equations are given by

$$\partial_\tau q(t;\tau) = \frac{\delta \mathbb{H}[\pi,q]}{\delta \pi(t;\tau)}\ , \tag{A7}$$

$$\partial_\tau \pi(t;\tau) = -\frac{\delta \mathbb{H}[\pi,q]}{\delta q(t;\tau)}\ . \tag{A8}$$

Explicitly, we have

$$\partial_\tau q(t;\tau) = \frac{\pi(t;\tau)}{M}\ , \tag{A9}$$

$$\partial_\tau \pi(t;\tau) = -m\ddot{q}(t;\tau) - \partial_q V(q(t;\tau))\ . \tag{A10}$$

Clearly, Eq. (A9) is equal to Eq. (A5).

As done in the discussion of quantum fields, we need to define the quantization constraints also for the symplectic quantization approach to ordinary quantum mechanics.

Following the discussion above in the main text, this is simply obtained by asking the following for the dynamical average of momenta

$$\frac{1}{2M}\langle \pi(t)\pi(t')\rangle = \frac{\hbar}{2}\,\delta(t-t'), \tag{A11}$$

which, in Fourier space, reads as

$$\frac{1}{2M}\langle \pi(\omega)\pi(-\omega)\rangle = \frac{\hbar}{2}, \tag{A12}$$

telling us that each frequency of the "momentum field" Fourier components, $\pi(\omega)$, carries a *half* quantum of action, the other half being carried by the "position field". Therefore, if one is about doing a numerical simulation on a discrete and finite time grid, where also frequencies are discretized, the quantization constraint of Eq. (A12) can be obtained by choosing as initial condition

$$\begin{aligned} &\frac{1}{2M}|\pi(\omega_i)|^2 = \hbar \quad \forall\, i, \\ &|q(\omega_i)|^2 = 0 \quad \forall\, i. \end{aligned} \tag{A13}$$

As previously discussed, given a generic observable $\mathcal{O}$ which depends on the coordinate q, i.e. $\mathcal{O}(q)$, its symplectic time average is given by

$$\langle \mathcal{O}(q)\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau}\mathcal{O}(q(t;\tau'))\,d\tau' \,. \tag{A14}$$

Again, under the assumption of symplectic ergodicity, this time average can be rewritten as the following micro-canonical statistical average

$$\langle \mathcal{O}(q)\rangle = \int \mathcal{D}[\pi]\mathcal{D}[q]\,\mathcal{P}[\pi,q]\,\mathcal{O}(q(t))\,, \tag{A15}$$

where

$$\mathcal{P}_{\mathcal{A}}[\pi,q] = \frac{\delta(\mathbb{H}[\pi,q]-\mathcal{A})}{\int \mathcal{D}[\pi]\mathcal{D}[q]\,\delta(\mathbb{H}[\pi,q]-\mathcal{A})} \tag{A16}$$

where $\mathcal{A}$ is a given fixed value of the action and $\mathbb{H}[\pi,q]$ is given by Eq. (A6) but clearly without the dependence with respect to the intrinsic time τ, i.e.

$$\mathbb{H}[\pi,q] = \int_{t_I}^{t_F}\left(\frac{\pi(t)^2}{2M} - \frac{m}{2}\dot{q}(t)^2 + V(q(t))\right)dt\,. \tag{A17}$$

As previously discussed, $\mathcal{A}$ is not arbitary. It is related to the reduced Planck constant $\hbar$ by the fundamental expression

$$\frac{1}{\hbar} = \frac{\partial}{\partial\mathcal{A}}\ln\left(\int \mathcal{D}[\pi]\mathcal{D}[q]\,\delta(\mathbb{H}[\pi,q]-\mathcal{A})\right), \tag{A18}$$

or, equivalently, one can use Eq. (28), where in this specific case

$$\Omega(\mathcal{A}) = \int \mathcal{D}[\pi]\mathcal{D}[q]\,\delta\big(\mathbb{H}[\pi,q] - \mathcal{A}\big)\,. \tag{A19}$$

1. Microcanonical Definition of $\hbar$

As a toy model, we consider a very simple one-dimensional system, where $q(t)$ and $\pi(t)$ are substantially time independent, i.e. the kinetic energy $(m/2)\dot{q}^2$ is negligible. Moreover, we assume that the potential energy is given by the quartic oscillator $V(q) = (\alpha/2)q^4$. It follows that

$$\begin{aligned}\Omega(\mathcal{A}) &= \int d\pi dq\,\delta\left((t_F - t_I)\left(\frac{\pi^2}{2M} + \frac{\alpha}{2}q^4\right) - \mathcal{A}\right)\\ &= \frac{1}{2}\sqrt{\frac{2M}{(t_F - t_I)}}\int dq \frac{1}{\sqrt{\mathcal{A} - \frac{\alpha}{2}q^4(t_F - t_I)}}\\ &= 1.748\sqrt{\frac{2M\mathcal{A}}{\alpha^{1/4}(t_F - t_I)^{3/4}}}\,.\end{aligned} \tag{A20}$$

Consequently, we obtain

$$\frac{1}{\Omega(\mathcal{A})}\frac{\partial\Omega(\mathcal{A})}{\partial\mathcal{A}} = \frac{1}{2\mathcal{A}} \tag{A21}$$

and from Eq. (28) we find

$$\mathcal{A} = \frac{\hbar}{2}\,. \tag{A22}$$

2. Ehrenfest Theorem

A fundamental relation which must be deduced from Eqs. (A9) and (A10) in order to asses the consistency of the above formalism with standard quantum mechanics is the Ehrenfest theorem. We start from the Ehrenfest relations, which can be obtained without effort from the Hamilton equations (A9) and (A10), simply by taking the average over intrinsinc time, see Eq. (A14). If the potential energy $V(q(t;\tau))$ is such that both $q(t;\tau)$ and $\pi(t;\tau)$ are bounded, we can write:

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau d\tau'\frac{\partial\pi(t;\tau')}{\partial\tau'} = \lim_{\tau\to\infty}\frac{\pi(t;\tau) - \pi(t;0)}{\tau} = 0, \tag{A23}$$

namely

$$\langle\partial_\tau\pi(t)\rangle = 0 = m\langle\ddot{q}(t)\rangle + \langle\partial_q V(q(t))\rangle, \tag{A24}$$

from which, by denotig as $m\dot{q}(t)$ the momentum conjugated to $q(t)$ with respect to ordinary time, we have

$$\frac{d}{dt}\langle p(t)\rangle = -\left\langle V'(q)\right\rangle, \tag{A25}$$

while we have that, in the present context, the equation

$$m\frac{d}{dt}\langle q(t)\rangle = \langle p(t)\rangle, \tag{A26}$$

comes simply as a definition.

Acknowledgements We warmly thank F. Bigazzi, P. Di Cintio, S. Franz, D. Seminara and S. Wimberger for useful discussions. G.G. is partially supported by the project MIUR-PRIN2022 *"Emergent Dynamical Patterns of Disordered Systems with Applications to Natural Communities"* code 2022WPHMXK and acknowledges the Physics Department of Sapienza, University of Rome, and the Physics and Astronomy Department "Galileo Galilei", University of Padova, for kind hospitality during some stages along the preparation of this manuscript. R.L. acknowledges partial support from project MIUR-PRIN2017 *Coarse-grained description for non-equilibrium systems and transport phenomena* (CO-NEST) n. 201798CZL. L.S. is partially supported by the European Union-NextGenerationEU within the National Center for HPC, Big Data and Quantum Computing [Project No. CN00000013, CN1 Spoke 10: Quantum Computing], by the BIRD Project *Ultracold atoms in curved geometries* of the University of Padova, by "Iniziativa Specifica Quantum" of Istituto Nazionale di Fisica Nucleare, by the European Quantum Flagship Project "PASQuanS 2", and by the PRIN 2022 Project "*Quantum Atomic Mixtures: Droplets, Topological Structures, and Vortices" of the Italian Ministry for University and Research*. L. S. also acknowledges the Project "Frontiere Quantistiche" (Dipartimenti di Eccellenza) of the Italian Ministry for Universities and Research.

Funding Open access funding provided by Gran Sasso Science Institute - GSSI within the CRUI-CARE Agreement.

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