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Research Article

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The first Grushin eigenvalue on cartesian product domains

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Abstract: In this paper, we consider the first eigenvalue $\lambda_1(\Omega)$ of the Grushin operator $\Delta_G := \Delta_{x_1} + |x_1|^{2s}\Delta_{x_2}$ with Dirichlet boundary conditions on a bounded domain Ω of $\mathbb{R}^d = \mathbb{R}^{d_1+d_2}$. We prove that $\lambda_1(\Omega)$ admits a unique minimizer in the class of domains with prescribed finite volume, which are the cartesian product of a set in $\mathbb{R}^{d_1}$ and a set in $\mathbb{R}^{d_2}$, and that the minimizer is the product of two balls $\Omega_1^* \subseteq \mathbb{R}^{d_1}$ and $\Omega_2^* \subseteq \mathbb{R}^{d_2}$. Moreover, we provide a lower bound for $|\Omega_1^*|$ and for $\lambda_1(\Omega_1^* \times \Omega_2^*)$. Finally, we consider the limiting problem as s tends to 0 and to $+\infty$.

Keywords: Grushin operator, Schrödinger operator, eigenvalue problem, minimization, cartesian product domain

MSC 2010: 35P15, 35P20, 47A75, 35J70, 34L15

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1 Introduction

We consider the Grushin operator Δ_G in $\mathbb{R}^d$ defined by

$$\Delta_G := \Delta_{x_1} + |x_1|^{2s}\Delta_{x_2}, \quad s > 0,$$

where $d \in \mathbb{N}$, $d \geq 2$, $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $x_1 \in \mathbb{R}^{d_1}$, and $x_2 \in \mathbb{R}^{d_2}$. Here x_1 and x_2 denote the first d_1 and last d_2 components of $x \in \mathbb{R}^d$, and Δ_{x_i} denotes the standard Laplacian with respect to x_i, $i = 1, 2$.

As one can immediately realize, Δ_G is not uniformly elliptic since it degenerates to Δ_{x_1} on the x_1-axis. In addition, if $s \in \mathbb{N}$, it can be written as

$$\Delta_G = \sum_{i=1}^{k} X_i^2,$$

where $k \in \mathbb{N}$ and $\{X_i\}_{i=1,\dots,k}$ is a family of smooth vector fields satisfying the Hörmander condition, i.e. $\{X_i\}_{i=1,\dots,k}$ generates a Lie algebra of maximum rank at any point (see [23]). However, in general (i.e. for $s \notin \mathbb{N}$) the Hörmander condition fails to hold since the generating vector fields are not smooth.

The operator Δ_G has been independently introduced by Baouendi [1] and Grushin [20, 21]. Later on, it has been generalized and further studied by several authors under different points of view. Here we mention, without the sake of completeness, Franchi and Lanconelli [15–17] for the Hölder regularity of weak solutions and for the embedding of the associated Sobolev spaces, Garofalo and Shen [19] for Carleman estimates and unique continuation results, D'Ambrosio [9] for Hardy inequalities, and Thuy and Tri [31] and Kogoj and Lanconelli [24] for semilinear problems. Finally, we mention Chen, Chen, Duan and Hu [4], Chen and Chen [5] Chen, Chen and Li [6], and Chen and Luo [7] for asymptotic bounds for eigenvalues.

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It is well known that the spectrum of the problem

$$\begin{cases} -\Delta_G u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded domain (i.e. a connected open set) Ω of $\mathbb{R}^d$ is made of eigenvalues of finite multiplicity that can be arranged in a divergent sequence:

$$0 < \lambda_1(\Omega) \leq \cdots \leq \lambda_j(\Omega) \leq \cdots \nearrow +\infty.$$

In the present paper, we are interested in moving some steps toward the understanding of the minimization problem of the first eigenvalue $\lambda_1(\Omega)$ among domains Ω with prescribed finite volume. Since the seminal works of Faber [13] and Krahn [25], it is known that the ball minimizes the first eigenvalue of the Dirichlet Laplacian among all domains with a fixed volume (see also [22] for a monograph on optimization problems for eigenvalues of elliptic operators). The same problem for degenerate operators is far from being understood and, to the best of our knowledge, no conclusive results for the optimization of Grushin eigenvalues are available in the literature, not even for the minimization of the first eigenvalue. In particular, an optimal shape for the first eigenvalue is not even conjectured, not even in the simplest case $d = 2$, $s = 1$, and in general it is not an euclidean ball (see Section 4)

It is worth mentioning that Lamberti, Luzzini and Musolino [27] showed that, when $s \in \mathbb{N}$, the symmetric functions of the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ depend real analytically upon suitable perturbations of the domain and proved an explicit Hadamard-type formula for their shape differential. This formula is then used to characterize critical domains under isovolumetric perturbations via an overdetermined problem, which for the first eigenvalue λ_1 with normalized eigenfunction u_1 consists of finding the domains such that the following problem is satisfied:

$$\begin{cases} -\Delta_G u_1 = \lambda_1 u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \\ \left(\dfrac{\partial u_1}{\partial \nu}\right)^2 |\nu_G|^2 = \text{const.} & \text{on } \partial\Omega. \end{cases}$$

Here, $\nu = (\nu_{x_1}, \nu_{x_2})$ denotes the outer unit normal field to $\partial\Omega$ and $\nu_G := (\nu_{x_1}, |x_1|^s \nu_{x_2})$. To the best of our knowledge, the understanding of this kind of overdetermined problems for degenerate operators is at the moment limited, and thus no information on critical domains can be extracted from them.

Our point of view in order to give a first and partial answer to the problem of minimizing the first Grushin eigenvalue is to consider the case in which Ω is the cartesian product of two bounded domains $\Omega_1 \subseteq \mathbb{R}^{d_1}$, $\Omega_2 \subseteq \mathbb{R}^{d_2}$. That is, for $V > 0$ fixed, we set

$$\mathcal{A}(V) := \{\Omega_1 \times \Omega_2 : \Omega_1 \subseteq \mathbb{R}^{d_1},\ \Omega_2 \subseteq \mathbb{R}^{d_2},\ \Omega_1, \Omega_2 \text{ are bounded domains},\ |\Omega_1||\Omega_2| = V\},$$

and we consider the minimization problem

$$\min_{\Omega\in\mathcal{A}(V)} \lambda_1(\Omega). \tag{1.2}$$

By separation of variables, problem (1.1) decouples into two problems. The first one is a problem for the standard Laplacian in $\mathbb{R}^{d_2}$ and the second one for the Schrödinger operator with potential $\mu|x_1|^{2s}$ in $\mathbb{R}^{d_1}$, where μ is the coupling constant. Our main result shows that problem (1.2) admits a unique minimizer which is the product of two balls Ω_1^* and Ω_2^* in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively (see Theorem 3.10). The main tool of the uniqueness proof relies on a differential inequality involving the second derivative of the first Schrödinger eigenvalue with respect to the coupling constant (see Proposition 3.8). As a further result, we provide some information on the localization of this unique minimum by proving a lower bound for $|\Omega_1^*|$, which in turn implies a lower bound for $\lambda_1(\Omega_1^* \times \Omega_2^*)$ (see Propositions 3.11 and 3.12). Then we study the asymptotic behavior of the problem when $s \to 0$ and $s \to +\infty$ and we deduce that our lower bounds are sharp in these limits. Finally, we provide some numerical computations in the planar case, that is, for $d_1 = d_2 = 1$. We first numerically solve the minimization problem for some value of $s > 0$ and then we also compute the first eigenvalue in the case of balls in $\mathbb{R}^2$ and we compare it with the first eigenvalue on rectangles.

The paper is organized as follows: Section 2 contains some preliminaries on the eigenvalue problem for the Grushin operator Δ_G. In Section 3, we prove our main results on the minimization problem on cartesian product domains. In particular, we prove that the minimization problem for the first Grushin eigenvalue admits a unique minimum, we provide some information on the localization of this minimum proving a lower bound, and we study the behavior of the problem when $s \to 0$ and $s \to +\infty$. Finally, in Section 4 we present the numerical computations.

2 Preliminaries on the eigenvalue problem

Let Ω be a bounded domain in $\mathbb{R}^d$. We retain the standard notation for the Lebesgue space $L^2(\Omega)$ of real-valued square integrable functions. We denote by $H^1_G(\Omega)$ the space of functions in $L^2(\Omega)$ such that $\nabla_{x_1} u \in (L^2(\Omega))^{d_1}$ and $|x_1|^s \nabla_{x_2} u \in (L^2(\Omega))^{d_2}$. The space $H^1_G(\Omega)$ is a Hilbert space with the following scalar product:

$$\langle u, v\rangle_{G,2} := \langle u, v\rangle_2 + \langle \nabla_{x_1} u, \nabla_{x_1} v\rangle_2 + \langle |x_1|^s \nabla_{x_2} u, |x_1|^s \nabla_{x_2} v\rangle_2 \quad \text{for all } u, v \in H^1_G(\Omega).$$

Here $\langle \cdot, \cdot \rangle_2$ denotes the standard scalar product in $L^2(\Omega)$. Moreover, if $u \in H^1_G(\Omega)$, we set

$$\nabla_G u := (\nabla_{x_1} u, |x_1|^s \nabla_{x_2} u)$$

and we refer to $\nabla_G u$ as the Grushin gradient of u. We denote by $H^1_{G,0}(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $H^1_G(\Omega)$. Analogs of the Rellich–Kondrachov embedding theorem and of the Poincaré inequality hold in $H^1_{G,0}(\Omega)$. That is, the following theorems hold (for proofs we refer to [18, Theorem 4.6] and [9, Theorem 3.7], respectively).

Theorem 2.1 (Rellich–Kondrachov). *Let Ω be a bounded domain in $\mathbb{R}^d$. Then the space $H^1_{G,0}(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Theorem 2.2 (Poincaré inequality). *Let Ω be a bounded domain in $\mathbb{R}^d$. Then there exists $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \le C \|\nabla_G u\|_{(L^2(\Omega))^d} \quad \text{for all } u \in H^1_{G,0}(\Omega).$$

We consider the eigenvalue problem for the Grushin operator with Dirichlet boundary conditions:

$$\begin{cases} -\Delta_G u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the unknowns λ (the eigenvalue) and u (the eigenfunction). Problem (2.1) is understood in the weak sense as follows:

$$\int_\Omega \nabla_G u \cdot \nabla_G v \, dx = \lambda \int_\Omega uv \, dx \quad \text{for all } v \in H^1_{G,0}(\Omega) \tag{2.2}$$

in the unknowns $\lambda \in \mathbb{R}$ and $u \in H^1_{G,0}(\Omega)$. By Theorem 2.1, Theorem 2.2 and by a standard procedure in spectral theory, problem (2.2) can be recast as an eigenvalue problem for a compact self-adjoint operator in $L^2(\Omega)$. In particular, the eigenvalues of equation (2.2) have finite multiplicity and can be represented by means of a divergent sequence:

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_j(\Omega) \le \cdots \nearrow +\infty.$$

Moreover, by the min-max principle (see [10, Section 4.5]), the following variational characterization holds:

$$\lambda_j(\Omega) = \min_{\substack{E \subseteq H^1_{G,0}(\Omega) \\ \dim E = j}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega |\nabla_G u|^2 \, dx}{\int_\Omega u^2 \, dx} \quad \text{for all } j \in \mathbb{N}.$$

We note that by Monticelli and Payne [29, Theorem 6.4] there exists a non-negative eigenfunction u_1 corresponding to the first eigenvalue $\lambda_1(\Omega)$. In addition, $\lambda_1(\Omega)$ is known to be simple if $s \in \mathbb{N}$ and Ω is connected and non-characteristic (see [5, Proposition A.2]), or if $\Omega \setminus \{x_1 = 0\}$ is connected (see [29, Theorem 6.4]).

3 The eigenvalue problem in cartesian product domains

Here we consider the eigenvalue problem for the Dirichlet Grushin operator (2.1) in cartesian product domains, so that it is possible to proceed by separation of variables. Let $\Omega_1 \subseteq \mathbb{R}^{d_1}$, $\Omega_2 \subseteq \mathbb{R}^{d_2}$ be two bounded domains and let $\Omega = \Omega_1 \times \Omega_2$. We claim that the solutions u of problem (2.1) can be written as

$$u(x_1, x_2) = f(x_1)g(x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

In this case, (2.1) becomes

$$-g(x_2)\Delta_{x_1} f(x_1) - |x_1|^{2s} f(x_1)\Delta_{x_2} g(x_2) = \lambda f(x_1)g(x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2,$$

which is equivalent to

$$g(x_2)(-\Delta_{x_1} f(x_1) - \lambda f(x_1)) = |x_1|^{2s} f(x_1)\Delta_{x_2} g(x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

Separating the equations and imposing the boundary conditions, for some $\mu > 0$ we have

$$\begin{cases} -\Delta_{x_2} g = \mu g & \text{in } \Omega_2, \\ g = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -\Delta_{x_1} f + \mu |x_1|^{2s} f = \lambda f & \text{in } \Omega_1, \\ f = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{3.2}$$

The eigenvalue problem is then split into two coupled eigenvalue problems, one for the Laplacian and the other for the Schrödinger operator with potential $\mu|x_1|^{2s}$. As is well known, problem (3.1) admits a sequence of eigenvalues

$$0 < \mu_1(\Omega_2) < \mu_2(\Omega_2) \le \cdots \le \mu_j(\Omega_2) \le \cdots \nearrow +\infty,$$

with corresponding eigenfunctions $\{g_j\}_{j\in\mathbb{N}}$ orthonormal in $L^2(\Omega_2)$, whereas problem (3.2), for each fixed $\mu > 0$, admits a sequence of eigenvalues

$$0 < E_1(\mu, \Omega_1) < E_2(\mu, \Omega_1) \le \cdots \le E_j(\mu, \Omega_1) \le \cdots \nearrow +\infty,$$

with eigenfunctions $\{f_j^\mu\}_{j\in\mathbb{N}}$ orthonormal in $L^2(\Omega_1)$. We note that, by the min-max principle, the first eigenvalue $E_1(\mu, \Omega_1)$ of problem (3.2) is given by

$$E_1(\mu, \Omega_1) = \min_{f \in H_0^1(\Omega_1)\setminus\{0\}} \frac{\int_{\Omega_1} |\nabla_{x_1} f|^2 + \mu |x_1|^{2s} f^2 \, dx_1}{\int_{\Omega_1} f^2 \, dx_1}.$$

Here and throughout this paper, by $H_0^1(\Omega_1)$ we denote the closure of $C_c^\infty(\Omega_1)$ with respect to the norm $(\|f\|_2^2 + \|\nabla_{x_1} f\|_2^2)^{1/2}$ of $H^1(\Omega_1)$. Therefore, a family of eigenvalues is given by $\{E_j(\mu_k(\Omega_2), \Omega_1)\}_{j,k\in\mathbb{N}}$ with associated eigenfunctions $\{g_k f_j^{\mu_k(\Omega_2)}\}_{j,k\in\mathbb{N}}$. The claim is proved since, recalling that $\{g_k\}_{k\in\mathbb{N}}$ and $\{f_j^{\mu_k(\Omega_2)}\}_{j\in\mathbb{N}}$ are complete systems, it is standard to show that

$$\{g_k f_j^{\mu_k(\Omega_2)}\}_{j,k\in\mathbb{N}}$$

is a complete system in $L^2(\Omega_1 \times \Omega_2)$. Then

$$\{\lambda_n(\Omega)\}_{n\in\mathbb{N}} = \{E_j(\mu_k(\Omega_2), \Omega_1)\}_{j,k\in\mathbb{N}}$$

and

$$\lambda_1(\Omega) = E_1(\mu_1(\Omega_2), \Omega_1) = \min_{f \in H_0^1(\Omega_1)\setminus\{0\}} \frac{\int_{\Omega_1} |\nabla_{x_1} f|^2 + \mu_1(\Omega_2)|x_1|^{2s} f^2 \, dx_1}{\int_{\Omega_1} f^2 \, dx_1}. \tag{3.3}$$

Remark 3.1. Since the first eigenvalue of the Dirichlet Laplacian $-\Delta_{x_2}$ on Ω_2 and the first eigenvalue of the Dirichlet Schrödinger operator $-\Delta_{x_1} + \mu|x_1|^{2s}$ on Ω_1 are well known to be simple, it is immediately seen that, in the case of a cartesian product domain $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 \subseteq \mathbb{R}^{d_1}$ and $\Omega_2 \subseteq \mathbb{R}^{d_2}$, the first eigenvalue $\lambda_1(\Omega)$ is simple without requiring any additional assumption. However, as already pointed out, the simplicity of the first Grushin eigenvalue is known to hold without requiring Ω to be a cartesian product domain under some additional assumptions (see [29, Theorem 6.4] and [5, Proposition A.2]).

3.1 Existence of a minimum

We now start to consider the minimization problem for the first eigenvalue in cartesian product domains with a prescribed volume. That is, we fix $V > 0$ and we consider the minimization problem

$$\min_{\Omega \in \mathcal{A}(V)} \lambda_1(\Omega), \tag{3.4}$$

where

$$\mathcal{A}(V) = \{\Omega_1 \times \Omega_2 : \Omega_1 \subseteq \mathbb{R}^{d_1},\ \Omega_2 \subseteq \mathbb{R}^{d_2},\ \Omega_1, \Omega_2 \text{ are bounded domains},\ |\Omega_1||\Omega_2| = V\}.$$

Remark 3.2. As is well known, if one removes a zero capacity set from either Ω_1 or Ω_2, the eigenvalues of problems (3.1) and (3.2) remain the same. Thus here in this paper, we do not allow this kind of irregularity in the domains and when speaking of uniqueness of minimizers we always mean uniqueness up to sets of zero capacity (see also [22, Section 3.2]).

Remark 3.3. Instead of considering the minimization problem (3.4), one can for instance consider the more simple problem of minimizing $\lambda_1(\Omega)$ in the class of cartesian products with each product domain having prescribed volume, that is, in

$$\mathcal{B}(V_1, V_2) = \{\Omega_1 \times \Omega_2 : \Omega_1 \subseteq \mathbb{R}^{d_1},\ \Omega_2 \subseteq \mathbb{R}^{d_2},\ \Omega_1, \Omega_2 \text{ are bounded domains},\ |\Omega_1| = V_1, |\Omega_2| = V_2\}$$

for some $V_1, V_2 > 0$. Since the ball in $\mathbb{R}^{d_2}$ with volume V_2, which we denote by $B_2(V_2)$, is the unique minimizer of the first eigenvalue of the Dirichlet Laplacian among all domains with volume V_2 (up to translation), it minimizes $\mu_1(\Omega_2)$. Accordingly, in order to minimize $\lambda_1(\Omega)$ for $\Omega = \Omega_1 \times \Omega_2 \in \mathcal{B}(V_1, V_2)$, we must have

$$\Omega_2 = B_2(V_2).$$

Also, by Benguria, Linde and Loewe [2, Theorems 3.7 and 4.2], the ball $B_1(V_1)$ in $\mathbb{R}^{d_1}$ centered at zero with volume V_1 is the unique minimizer of the first eigenvalue of the Schrödinger operator with potential $\mu_1(\Omega_2)|x|^{2s}$. Thus, the unique minimum of $\lambda_1(\Omega)$ in $\mathcal{B}(V_1, V_2)$ is attained by

$$\Omega = B_1(V_1) \times B_2(V_2),$$

up to a translation of $B_2(V_2)$. As one could expect, the minimization problem in $\mathcal{B}(V_1, V_2)$ is trivial and not general enough to capture the anisotropic nature of the problem.

We then return to the minimization problem (3.4). Let $\Omega = \Omega_1 \times \Omega_2 \in \mathcal{A}(V)$. We set

$$\tilde{\Omega}_j := |\Omega_j|^{-\frac{1}{d_j}} \Omega_j, \quad j = 1, 2,$$

so that

$$|\tilde{\Omega}_j| = 1, \quad j = 1, 2.$$

Let B_1 be the ball in $\mathbb{R}^{d_1}$ centered at zero with $|B_1| = 1$, and let B_2 be a ball in $\mathbb{R}^{d_2}$ with $|B_2| = 1$. Then

$$\mu_1(\Omega_2) = \mu_1(\tilde{\Omega}_2)|\Omega_2|^{-\frac{2}{d_2}} = \mu_1(\tilde{\Omega}_2)V^{-\frac{2}{d_2}}|\Omega_1|^{\frac{2}{d_2}} \geq \mu_1(B_2)V^{-\frac{2}{d_2}}|\Omega_1|^{\frac{2}{d_2}}.$$

By equation (3.3) we have

$$\begin{aligned}
\lambda_1(\Omega) &= \inf_{f \in H_0^1(\Omega_1)\setminus\{0\}} \frac{\int_{\Omega_1} |\nabla_{x_1} f|^2\, dx_1 + \mu_1(\Omega_2)\int_{\Omega_1} |x_1|^{2s} f^2\, dx_1}{\int_{\Omega_1} f^2\, dx_1} \\
&= \inf_{f \in H_0^1(\tilde{\Omega}_1)\setminus\{0\}} \left(|\Omega_1|^{-\frac{2}{d_1}} \frac{\int_{\tilde{\Omega}_1} |\nabla_{x_1} f|^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} + \mu_1(\tilde{\Omega}_2)V^{-\frac{2}{d_2}}|\Omega_1|^{\frac{2}{d_2}+\frac{2s}{d_1}} \frac{\int_{\tilde{\Omega}_1} |x_1|^{2s} f^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} \right) \\
&\geq \inf_{f \in H_0^1(\tilde{\Omega}_1)\setminus\{0\}} \left(|\Omega_1|^{-\frac{2}{d_1}} \frac{\int_{\tilde{\Omega}_1} |\nabla_{x_1} f|^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} + \mu_1(B_2)V^{-\frac{2}{d_2}}|\Omega_1|^{\frac{2}{d_2}+\frac{2s}{d_1}} \frac{\int_{\tilde{\Omega}_1} |x_1|^{2s} f^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} \right) \\
&= |\Omega_1|^{-\frac{2}{d_1}} \inf_{f \in H_0^1(\tilde{\Omega}_1)\setminus\{0\}} \left(\frac{\int_{\tilde{\Omega}_1} |\nabla_{x_1} f|^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} + \mu_1(B_2)V^{-\frac{2}{d_2}}|\Omega_1|^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}} \frac{\int_{\tilde{\Omega}_1} |x_1|^{2s} f^2\, dx_1}{\int_{\tilde{\Omega}_1} f^2\, dx_1} \right)
\end{aligned}$$

$$= |\Omega_1|^{-\frac{2}{d_1}} E_1\big(\mu_1(B_2) V^{-\frac{2}{d_2}} |\Omega_1|^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}}, \tilde{\Omega}_1\big)$$
$$\geq |\Omega_1|^{-\frac{2}{d_1}} E_1\big(\mu_1(B_2) V^{-\frac{2}{d_2}} |\Omega_1|^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}}, B_1\big).$$

Note that the last inequality in the above computations follows again by Benguria, Linde and Loewe [2, Theorems 3.7 and 4.2]. The previous inequality gives a lower bound for the first Grushin eigenvalue in the case of cartesian product domains. We note that the lower bound is attained if and only if Ω is the product of two balls, the first one being centered at zero. Thus, in order to minimize the first eigenvalue $\lambda_1(\Omega)$ for $\Omega \in \mathcal{A}(V)$ we need to find, if it exists, the volume $|\Omega_1|$ which minimizes the quantity

$$|\Omega_1|^{-\frac{2}{d_1}} E_1\big(\mu_1(B_2) V^{-\frac{2}{d_2}} |\Omega_1|^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}}, B_1\big).$$

In other words, we need to find the minimizing $t \in (0, +\infty)$ of the function

$$t^{-\frac{2}{d_1}} E_1\big(\mu_1(B_2) V^{-\frac{2}{d_2}} t^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}}, B_1\big). \tag{3.5}$$

For the sake of simplicity and clarity in the computations, by using the substitution

$$\sigma = \sigma(t) = \mu_1(B_2) V^{-\frac{2}{d_2}} t^{\frac{2}{d_1}+\frac{2}{d_2}+\frac{2s}{d_1}}, \quad t > 0, \tag{3.6}$$

we transform the minimization problem for $\lambda_1(\Omega)$ into studying the minimizers of

$$F(\sigma) := \sigma^{-\frac{d_2}{d_1+(1+s)d_2}} E_1(\sigma, B_1), \quad \sigma \in (0, +\infty). \tag{3.7}$$

We are able to prove the following proposition concerning the existence of a minimum for F.

Proposition 3.4. *The function F admits a minimum in* $(0, +\infty)$.

Proof. Since the first Schrödinger eigenvalue $E_1(\sigma, B_1)$ is simple for all $\sigma \in (0, +\infty)$, by classical analytic perturbation theory (see [11, 30]), it can be easily seen that F is an analytic function of $\sigma \in (0, +\infty)$, and thus in particular it is smooth. A more up-to-date formulation of abstract perturbation results can be found for example in [26, Theorem 2.27]. Moreover, we claim that the following assertions hold:
(i) $\lim_{\sigma \to 0^+} F(\sigma) = +\infty$.
(ii) $\lim_{\sigma \to +\infty} F(\sigma) = +\infty$.
Statements (i) and (ii) would immediately imply the validity of the lemma. Statement (i) holds because $E_1(\sigma, B_1)$ converges to the first eigenvalue of the Dirichlet Laplacian in B_1, which is strictly positive, when σ tends to zero.

Next, we consider statement (ii). Let f_σ be an eigenfunction corresponding to $E_1(\sigma, B_1)$. We still denote by f_σ its extension by zero in $\mathbb{R}^{d_1}$. We note that

$$\begin{aligned} E_1(\sigma, B_1) &= \int_{B_1} |\nabla_{x_1} f_\sigma|^2 \, dx_1 + \sigma \int_{B_1} |x_1|^{2s} f_\sigma^2 \, dx_1 \\ &= \int_{\mathbb{R}^{d_1}} |\nabla_{x_1} f_\sigma|^2 \, dx_1 + \sigma \int_{\mathbb{R}^{d_1}} |x_1|^{2s} f_\sigma^2 \, dx_1 \\ &\geq E_1(\sigma, \mathbb{R}^{d_1}) \\ &= \sigma^{\frac{1}{1+s}} E_1(1, \mathbb{R}^{d_1}). \end{aligned} \tag{3.8}$$

We have denoted by $E_1(\sigma, \mathbb{R}^{d_1})$ the first eigenvalue of the Schrödinger operator $-\Delta_{x_1} + \sigma |x_1|^{2s}$ in $\mathbb{R}^{d_1}$. The inequality in (3.8) holds because also $E_1(\sigma, \mathbb{R}^{d_1})$ can be variationally characterized as

$$E_1(\sigma, \mathbb{R}^{d_1}) = \min_{f \in H_0^1(\mathbb{R}^{d_1}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{d_1}} |\nabla_{x_1} f|^2 + \sigma |x_1|^{2s} f^2 \, dx_1}{\int_{\mathbb{R}^{d_1}} f^2 \, dx_1}, \tag{3.9}$$

and $H_0^1(\Omega_1) \subseteq H_0^1(\mathbb{R}^{d_1})$. Moreover, the last equality in (3.8) follows by a simple rescaling argument. Accordingly,

$$F(\sigma) = \sigma^{-\frac{d_2}{d_1+(1+s)d_2}} E_1(\sigma, B_1) \geq \sigma^{-\frac{d_2}{d_1+(1+s)d_2}+\frac{1}{1+s}} E_1(1, \mathbb{R}^{d_1}).$$

Since

$$-\frac{d_2}{d_1 + (1+s)d_2} + \frac{1}{1+s} = \frac{d_1}{(d_1 + (1+s)d_2)(1+s)} > 0,$$

statement (ii) holds. □

By the previous discussion and by Proposition 3.4, we deduce that the first Grushin eigenvalue admits at least a minimum in the class of cartesian product domains. Namely, we have the following proposition.

Proposition 3.5. *Let $V > 0$. There exists $\Omega^* = \Omega_1^* \times \Omega_2^* \in \mathcal{A}(V)$ such that*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega) \quad \textit{for all } \Omega \in \mathcal{A}(V).$$

Moreover, $\Omega_1^* \subseteq \mathbb{R}^{d_1}$, $\Omega_2^* \subseteq \mathbb{R}^{d_2}$ are two balls, the first one being centered in zero.

Our next step is to show that such a minimum is unique (up to translation of Ω_2). To this end, we need to develop some preliminary results.

3.2 The Schrödinger eigenvalue problem in the ball

In this section, we prove a differential inequality involving the second derivative of the first Schrödinger eigenvalue of (3.2) in a ball with respect to the coupling constant μ.

Let $R > 0$, $\mu > 0$. For the sake of brevity, we set

$$E_1(\mu, R) := E_1(\mu, B(0, R)),$$

where $E_1(\mu, B(0, R))$ is the first eigenvalue of problem (3.2) in $B(0, R) = \{x_1 \in \mathbb{R}^{d_1} : |x_1| < R\}$. By using spherical coordinates, since the first eigenfunction is radial, problem (3.2) for $E_1(\mu, R)$ can be written as

$$\begin{cases} -v'' - \dfrac{d_1 - 1}{r} v' + \mu r^{2s} v = E_1(\mu, R) v & \text{on } (0, R), \\ v(R) = 0. \end{cases} \tag{3.10}$$

Let v be the unique non-negative solution of (3.10) normalized in $L^2((0, R), r^{d_1 - 1} dr)$.

Remark 3.6. As already noted in the proof of Proposition 3.4, since $E_1(\mu, R)$ is simple, classical analytic perturbation theory implies that $E_1(\mu, R)$ and its corresponding eigenfunction v depend analytically upon $\mu > 0$. Accordingly, the computations performed in this section involving the derivatives of $E_1(\mu, R)$ and v with respect to μ are justified.

As a first step, we need some integral identities.

Lemma 3.7. *Let $R > 0$, $\mu > 0$. Let v be the unique non-negative solution of* (3.10) *normalized in $L^2((0, R), r^{d_1 - 1} dr)$. Then*

$$\int_0^R (v')^2 r^{d_1 - 1} \, dr + \mu \int_0^R r^{2s + d_1 - 1} v^2 \, dr = E_1(\mu, R), \tag{3.11}$$

$$\int_0^R (v')^2 r^{d_1 - 1} \, dr - \frac{R^{d_1}}{2} (v')^2(R) = s\mu \int_0^R r^{2s + d_1 - 1} v^2 \, dr, \tag{3.12}$$

$$\dot{E}_1(\mu, R) := \frac{\partial}{\partial \mu} E_1(\mu, R) = \int_0^R r^{2s + d_1 - 1} v^2 \, dr. \tag{3.13}$$

Proof. In order to prove (3.11), it suffices to multiply equation (3.10) by $r^{d_1 - 1} v$ and integrate by parts. Identity (3.12) follows by multiplying (3.10) by $r^{d_1} v'$, integrating by parts and using (3.11). Finally, equation (3.13) follows by classical abstract results in perturbation theory (see, e.g., [26, Theorem 2.30]) or, with the notation of quantum mechanics, by the Hellmann–Feynman theorem (see [14]). □

By using identities (3.11)–(3.13) of the previous proposition, we can recover the following differential identity for $E_1(\mu, R)$:

$$E_1(\mu, R) - \frac{R^{d_1}}{2}(v')^2(R) = \mu(1+s)\dot{E}_1(\mu, R). \tag{3.14}$$

We set

$$\dot{v} := \frac{\partial}{\partial \mu} v, \quad \ddot{E}_1(\mu, R) := \frac{\partial^2}{\partial \mu^2} E_1(\mu, R).$$

Then, by taking the derivative of (3.14) with respect to μ, we get

$$-R^{d_1} v'(R)\dot{v}'(R) = s\dot{E}_1(\mu, R) + \mu(1+s)\ddot{E}_1(\mu, R). \tag{3.15}$$

Our aim is to understand the sign of the right-hand side of the previous equation, and we have the following proposition.

Proposition 3.8. *Let $R > 0$, $\mu > 0$. Then*

$$s\dot{E}_1(\mu, R) + \mu(1+s)\ddot{E}_1(\mu, R) \geq 0.$$

Proof. Let v be the unique non-negative solution of problem (3.10) normalized in $L^2((0, R), r^{d_1-1}dr)$. We note that, since v is positive and $v(R) = 0$, we have

$$v'(R) \leq 0, \quad \dot{v}(R) = 0.$$

Moreover, by taking the μ-derivative of the normalization condition, we deduce that

$$\int_0^R r^{d_1-1} v\dot{v}\, dr = 0,$$

and accordingly $\dot{v}$ changes sign at least once. We will show that $\dot{v}$ changes sign only once in $(0, R)$, and that $\dot{v}'(R) \geq 0$. Differentiating (3.10) with respect to μ, we obtain

$$-\dot{v}'' - \frac{d_1 - 1}{r}\dot{v}' + \mu r^{2s}\dot{v} + r^{2s} v = \dot{E}_1(\mu, R)v + E_1(\mu, R)\dot{v} \quad \text{in } (0, R). \tag{3.16}$$

By equations (3.10) and (3.16), we get

$$(r^{d_1-1}(\dot{v}'v - \dot{v}v'))' = (r^{2s-d_1-1} - \dot{E}_1(\mu, R)r^{d_1-1})v^2 \quad \text{in } (0, R),$$

that is,

$$r^{d_1-1}(\dot{v}'v - \dot{v}v') = \int_0^r (t^{2s} - \dot{E}_1(\mu, R))t^{d_1-1}v^2(t)\, dt \quad \text{in } (0, R).$$

We set

$$\eta(r) := \int_0^r (t^{2s} - \dot{E}_1(\mu, R))t^{d_1-1}v^2(t)\, dt \quad \text{for all } r \in [0, R].$$

As one can immediately realize, $\eta(0) = \eta(R) = 0$. Moreover, since $t^{d_1-1}v^2(t)$ is non-negative for $t \in (0, R)$ and $(t^{2s} - \dot{E}_1(\mu, R))$ is strictly increasing for $t \in (0, R)$ and negative at $t = 0$, we obtain that η has exactly one critical point and

$$\eta(r) < 0 \quad \text{for all } r \in (0, R).$$

This implies that $\frac{\dot{v}}{v}$ is strictly decreasing in $(0, R)$ since

$$\left(\frac{\dot{v}}{v}\right)' = \frac{\dot{v}'v - \dot{v}v'}{v^2} < 0 \quad \text{in } (0, R),$$

and accordingly $\dot{v}$ can change sign only once and $\dot{v}'(R) \geq 0$. We have then proved that

$$-R^{d_1}v'(R)\dot{v}'(R) \geq 0,$$

and the statement follows by equation (3.15). □

3.3 Uniqueness of the minimum

We are now ready to prove the uniqueness of the minimum of problem (3.4) by means of the following proposition.

Proposition 3.9. *The function F defined in* (3.7) *has a unique minimum in* $(0, +\infty)$.

Proof. We take the derivative of F. Let $\sigma > 0$. Then

$$F'(\sigma) = \sigma^{-\frac{d_2}{d_1+(1+s)d_2}-1}\Big(-\frac{d_2}{d_1+(1+s)d_2}E_1(\sigma, B_1) + \sigma\dot{E}_1(\sigma, B_1)\Big).$$

By Proposition 3.4, F' has at least a zero, i.e. a critical point of F. Accordingly, let $\sigma^* > 0$ be a critical point of F. Computing the second derivative of F in σ^*, we get

$$\begin{aligned} F''(\sigma^*) &= (\sigma^*)^{-\frac{d_2}{d_1+(1+s)d_2}-1}\Big(-\frac{d_2}{d_1+(1+s)d_2}\dot{E}_1(\sigma^*, B_1) + \dot{E}_1(\sigma^*, B_1) + \sigma^*\ddot{E}_1(\sigma^*, B_1)\Big) \\ &= (\sigma^*)^{-\frac{d_2}{d_1+(1+s)d_2}-1}\Big(\frac{d_1}{(d_1+(1+s)d_2)(1+s)}\dot{E}_1(\sigma^*, B_1) + \frac{s}{1+s}\dot{E}_1(\sigma^*, B_1) + \sigma^*\ddot{E}_1(\sigma^*, B_1)\Big). \end{aligned}$$

By identity (3.13) on the μ-derivative of $E_1(\mu, R)$, we have that

$$\dot{E}_1(\sigma^*, B_1) > 0.$$

Moreover, by Proposition 3.8,

$$\frac{s}{1+s}\dot{E}_1(\sigma^*, B_1) + \sigma^*\ddot{E}_1(\sigma^*, B_1) \geq 0.$$

Thus,

$$F''(\sigma^*) > 0.$$

Since F is smooth in $(0, +\infty)$ and all its critical points have positive second derivative, F has only one critical point which is a minimum. □

By Proposition 3.5 and Proposition 3.9, we can immediately deduce that the first Grushin eigenvalue admits a unique minimum.

Theorem 3.10. *Let $V > 0$. There exists a unique set $\Omega^* = \Omega_1^* \times \Omega_2^*$ in $\mathcal{A}(V)$ (up to translations in $\mathbb{R}^{d_2}$) such that*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega) \quad \textit{for all } \Omega \in \mathcal{A}(V).$$

Moreover, $\Omega_1^* \subseteq \mathbb{R}^{d_1}$, $\Omega_2^* \subseteq \mathbb{R}^{d_2}$ are two balls, the first one being centered in zero.

3.4 Localizing the minimum

In this section, we obtain some information on the localization of the minimum we proved to be unique in the last section. Let B_1 be the ball in $\mathbb{R}^{d_1}$ centered in zero with $|B_1| = 1$, and let B_2 be a ball in $\mathbb{R}^{d_2}$ with $|B_2| = 1$. Moreover, we denote by τ_d the volume of the d-dimensional unit ball, that is,

$$\tau_d := \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})}.$$

Let $\sigma > 0$ and let f_σ be an eigenfunction corresponding to $E_1(\sigma, B_1)$ normalized in $L^2(B_1)$. We write $F'(\sigma)$ more explicitly. Since

$$\frac{d}{d\sigma}E_1(\sigma, B_1) = \int_{B_1} |x_1|^{2s} f_\sigma^2 \, dx_1,$$

we have

$$F'(\sigma) = -\Big(\frac{d_2}{d_1+(1+s)d_2}\sigma^{-\frac{d_2}{d_1+(1+s)d_2}-1}E_1(\sigma,B_1) - \sigma^{-\frac{d_2}{d_1+(1+s)d_2}}\int_{B_1}|x_1|^{2s}f_\sigma^2\,dx_1\Big)$$
$$= -\Big(\frac{d_2}{d_1+(1+s)d_2}\sigma^{-\frac{d_2}{d_1+(1+s)d_2}-1}\Big(\int_{B_1}|\nabla_{x_1}f_\sigma|^2\,dx_1 + \sigma\int_{B_1}|x_1|^{2s}f_\sigma^2\,dx_1\Big) - \sigma^{-\frac{d_2}{d_1+(1+s)d_2}}\int_{B_1}|x_1|^{2s}f_\sigma^2\,dx_1\Big)$$
$$= -\sigma^{-\frac{d_2}{d_1+(1+s)d_2}-1}\frac{d_2}{d_1+(1+s)d_2}\Big(\int_{B_1}|\nabla_{x_1}f_\sigma|^2\,dx_1 - \frac{d_1+sd_2}{d_2}\sigma\int_{B_1}|x_1|^{2s}f_\sigma^2\,dx_1\Big).$$

Let $\sigma^* > 0$ be the unique minimum point of F. Then $F'(\sigma^*) = 0$. That is,

$$\int_{B_1}|\nabla_{x_1}f_{\sigma^*}|^2\,dx_1 - \frac{d_1+sd_2}{d_2}\sigma^*\int_{B_1}|x_1|^{2s}f_{\sigma^*}^2\,dx_1 = 0.$$

We note that

$$\int_{B_1}|\nabla_{x_1}f_{\sigma*}|^2\,dx_1 \geq \mu_1(B_1)\|f_{\sigma*}\|^2_{L^2(B_1)} = \mu_1(B_1),$$

where $\mu_1(B_1)$ is the first Dirichlet Laplacian eigenvalue on the ball $B_1 \subseteq \mathbb{R}^{d_1}$, and

$$\int_{B_1}|x_1|^{2s}f_{\sigma^*}^2\,dx_1 \leq \tau_{d_1}^{-\frac{2s}{d_1}}\|f_{\sigma^*}\|^2_{L^2(B_1)} = \tau_{d_1}^{-\frac{2s}{d_1}},$$

where we have used the fact that $|x_1| \leq \tau_{d_1}^{-1/d_1}$ for all $x_1 \in B_1$. Thus,

$$\int_{B_1}|\nabla_{x_1}f_{\sigma*}|^2\,dx_1 - \frac{d_1+sd_2}{d_2}\sigma^*\int_{B_1}|x_1|^{2s}f_{\sigma^*}^2\,dx_1 \geq \mu_1(B_1) - \frac{d_1+sd_2}{d_2}\tau_{d_1}^{-\frac{2s}{d_1}}\sigma^*.$$

Since for all $\sigma \in (0,+\infty)$ such that

$$\sigma < \tau_{d_1}^{\frac{2s}{d_1}}\frac{d_2}{d_1+sd_2}\mu_1(B_1)$$

one has

$$\mu_1(B_1) - \frac{d_1+sd_2}{d_2}\tau_{d_1}^{-\frac{2s}{d_1}}\sigma > 0,$$

the unique minimum point of F must satisfy

$$\sigma^* \geq \tau_{d_1}^{\frac{2s}{d_1}}\frac{d_2}{d_1+sd_2}\mu_1(B_1).$$

In other words, recalling the substitution (3.6), we have proved the following lower bound.

Proposition 3.11. *Let $V > 0$. Let $\Omega_1^* \subseteq \mathbb{R}^{d_1}$, $\Omega_2^* \subseteq \mathbb{R}^{d_1}$ be the two balls given by Theorem 3.10. Then*

$$|\Omega_1^*| \geq \Big(\tau_{d_1}^{\frac{2s}{d_1}}\frac{d_2}{d_1+sd_2}\frac{\mu_1(B_1)}{\mu_1(B_2)}V^{\frac{2}{d_2}}\Big)^{\frac{d_1d_2}{2(d_1+(1+s)d_2)}}. \tag{3.17}$$

By the lower bound (3.17) of the previous proposition, it is also possible to provide a lower bound on

$$\min_{\Omega\in\mathcal{A}(V)}\lambda_1(\Omega).$$

Proposition 3.12. *Let $V > 0$. Let $\Omega_1^* \subseteq \mathbb{R}^{d_1}$, $\Omega_2^* \subseteq \mathbb{R}^{d_1}$ be the two balls given by Theorem 3.10. Then*

$$\lambda_1(\Omega_1^*\times\Omega_2^*) \geq \mu_1(B_2)^{\frac{1}{s+1}}V^{-\frac{2}{d_1+(1+s)d_2}}E_1(1,\mathbb{R}^{d_1})\Big(\tau_{d_1}^{\frac{2s}{d_1}}\frac{d_2}{d_1+sd_2}\frac{\mu_1(B_1)}{\mu_1(B_2)}\Big)^{\frac{d_1}{(s+1)(d_1+(1+s)d_2)}}. \tag{3.18}$$

Proof. Let σ^* be the unique minimizer of F. Consider the inequality (see (3.8))

$$E_1(\sigma^*,B_1) \geq (\sigma^*)^{\frac{1}{s+1}}E_1(1,\mathbb{R}^{d_1}) \quad \text{for all } \sigma > 0.$$

Then, recalling the substitutions made in (3.5)–(3.7) and using the lower bound (3.17), we have that the lower bound (3.18) holds and the statement is proved. □

3.5 Limits as $s \to 0^+$ and $s \to +\infty$

In this section, we study the behavior of the minimization problem when the parameter s tends either to 0 or to $+\infty$. We use the notation introduced in Section 3.1. In particular, B_1 denotes the ball in $\mathbb{R}^{d_1}$ centered in zero with $|B_1| = 1$, and B_2 denotes a ball in $\mathbb{R}^{d_2}$ with $|B_2| = 1$. Let $V > 0$. It will be convenient to introduce a new variable $\tilde{\sigma}$, defined by

$$\tilde{\sigma} = \tilde{\sigma}(t) = \mu_1(B_2) t^{\frac{2}{d_1} + \frac{2}{d_2}} V^{-\frac{2}{d_2}} \quad \text{for all } t > 0.$$

Then, if σ is the variable defined in (3.6), we have

$$\sigma(t) = \tilde{\sigma}(t) t^{\frac{2s}{d_1}} \quad \text{for all } t > 0.$$

The introduction of the variable $\tilde{\sigma}$ is motivated by the fact that, in this section, we will need to keep the dependence of the coupling constant σ explicit on s, since we are studying the behavior of the first eigenvalue when s tends to 0 or to $+\infty$. We then set

$$G_s(t) := t^{-\frac{2}{d_1}} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) \quad \text{for all } t > 0, \tag{3.19}$$

where we recall that $E_1(\mu, \Omega_1)$, for $\mu > 0$, denotes the first eigenvalue of problem (3.2) set in $\Omega_1 \subseteq \mathbb{R}^{d_1}$. We recall that, as noted in Section 3.1 (see in particular (3.5)), the unique minimal point of G_s represents the volume of the ball in $\mathbb{R}^{d_1}$ of the minimal set. We now start to consider here the limit as $s \to 0^+$.

Proposition 3.13. *Let G_s be the function defined in* (3.19). *Let*

$$G_0(t) := t^{-\frac{2}{d_1}} (\mu_1(B_1) + \tilde{\sigma}(t)) = t^{-\frac{2}{d_1}} \mu_1(B_1) + t^{\frac{2}{d_2}} V^{-\frac{2}{d_2}} \mu_1(B_2) \quad \textit{for all } t > 0.$$

Then

$$\lim_{s \to 0^+} G_s(t) = G_0(t) \quad \textit{for all } t > 0.$$

Proof. We start by noting that for all $f \in H_0^1(B_1)$, $\|f\|_{L^2(B_1)} = 1$, and $\sigma \in (0, +\infty)$, we have

$$\int_{B_1} |\nabla_{x_1} f|^2 \, dx_1 + \sigma \int_{B_1} |x_1|^{2s} f^2 \, dx_1 \leq \int_{B_1} |\nabla_{x_1} f|^2 \, dx_1 + \sigma \tau_{d_1}^{-\frac{2s}{d_1}}. \tag{3.20}$$

Then, taking the minimum over $f \in H_0^1(B_1)$ with $\|f\|_{L^2(B_1)} = 1$ to both sides of (3.20), we immediately obtain

$$E_1(\sigma, B_1) \leq \mu_1(B_1) + \sigma \tau_{d_1}^{-\frac{2s}{d_1}}. \tag{3.21}$$

On the other hand, if f_σ is the unique (up to sign) eigenfunction associated with $E_1(\sigma, B_1)$ satisfying $\|f_\sigma\|_{L^2(B_1)} = 1$, and if $B(0, \varepsilon)$ is the ball in $\mathbb{R}^{d_1}$ centered at zero and of radius $\varepsilon \in (0, \tau_{d_1}^{-1/d_1})$, then

$$\begin{aligned} E_1(\sigma, B_1) &= \int_{B_1} |\nabla_{x_1} f_\sigma|^2 \, dx_1 + \sigma \int_{B_1} |x_1|^{2s} f_\sigma^2 \, dx_1 \\ &\geq \int_{B_1 \setminus B(0,\varepsilon)} |\nabla_{x_1} f_\sigma|^2 \, dx_1 + \sigma \varepsilon^{2s} \int_{B_1 \setminus B(0,\varepsilon)} f_\sigma^2 \, dx_1 \\ &\geq \int_{B_1 \setminus B(0,\varepsilon)} f_\sigma^2 \, dx_1 \Bigg(\min_{\substack{f \in \tilde{H}_0^1(B_1 \setminus B(0,\varepsilon)) \\ f \neq 0}} \frac{\int_{B_1 \setminus B(0,\varepsilon)} |\nabla_{x_1} f|^2 \, dx_1}{\int_{B \setminus B(0,\varepsilon)} f^2 \, dx_1} + \sigma \varepsilon^{2s} \Bigg) \\ &= (\tilde{\mu}_1(\varepsilon) + \sigma \varepsilon^{2s}) \int_{B_1 \setminus B(0,\varepsilon)} f_\sigma^2 \, dx_1, \end{aligned} \tag{3.22}$$

where $\tilde{H}_0^1(B_1 \setminus B(0, \varepsilon))$ denotes the closure of $C_c^\infty(B_1)$ in $H^1(B_1 \setminus B(0, \varepsilon))$, and $\tilde{\mu}_1(\varepsilon)$ is the first eigenvalue of the Laplacian in $B_1 \setminus B(0, \varepsilon)$ with Dirichlet boundary conditions on ∂B_1 and Neumann boundary conditions on $\partial B(0, \varepsilon)$. It is well known that $\tilde{\mu}_1(\varepsilon) \to \mu_1(B_1)$ as $\varepsilon \to 0^+$ (see, e.g., [28] and the references therein). Moreover,

$$1 = \int_{B_1} f_\sigma^2 \, dx_1 = \int_{B_1 \setminus B(0,\varepsilon)} f_\sigma^2 \, dx_1 + \int_{B(0,\varepsilon)} f_\sigma^2 \, dx_1. \tag{3.23}$$

If $d_1 > 2$, we know from the Hölder inequality and the Sobolev inequality in the supercritical case (see, e.g., [12, Section 5.6.3, Theorem 6 (i)]) and from (3.21) that

$$\int_{B(0,\varepsilon)} f_\sigma^2\,dx_1 \leq C\varepsilon^2\|\nabla f_\sigma\|^2_{L^2(B_1)} \leq C\varepsilon^2 E_1(\sigma,B_1) \leq C\varepsilon^2\big(\mu_1(B_1)+\sigma\tau_{d_1}^{-\frac{2s}{d_1}}\big).$$

Moreover, if $d_1 = 1$, the subcritical Sobolev inequality (see, e.g., [12, Section 5.6.3, Theorem 6 (ii)]) and (3.21) imply that

$$\int_{B(0,\varepsilon)} f_\sigma^2\,dx_1 \leq C\varepsilon\|\nabla f_\sigma\|^2_{L^2(B_1)} \leq C\varepsilon E_1(\sigma,B_1) \leq C\varepsilon\big(\mu_1(B_1)+\sigma\tau_{d_1}^{-\frac{2s}{d_1}}\big).$$

Finally, if $d_1 = 2$, exploiting the critical Sobolev inequality (see [3, Section 4.7, Theorem 15]) together again with (3.21), we obtain

$$\begin{aligned}\int_{B(0,\varepsilon)} f_\sigma^2\,dx_1 &\leq C\varepsilon^2(1+|\log(\varepsilon)|)\|\nabla f_\sigma\|^2_{L^2(B_1)}\\ &\leq C\varepsilon^2(1+|\log(\varepsilon)|)E_1(\sigma,B_1)\\ &\leq C\varepsilon^2(1+|\log(\varepsilon)|)\big(\mu_1(B_1)+\sigma\tau_{d_1}^{-\frac{2s}{d_1}}\big);\end{aligned}$$

see also [8, Appendix B], where the above inequalities are derived with all details. In all cases, the constant C depends only on d_1 (in general it depends on the domain, which in this case is B_1). Thus, by the above inequalities, and by (3.22) and (3.23), for all $\sigma \in (0,+\infty)$, $s \in (0,+\infty)$ and $\varepsilon \in (0,\tau_{d_1}^{-1/d_1})$, we have

$$E_1(\sigma,B_1) \geq (\tilde{\mu}_1(\varepsilon)+\sigma\varepsilon^{2s})\big(1-C\omega(\varepsilon)\big(\mu_1(B_1)+\sigma\tau_{d_1}^{-\frac{2s}{d_1}}\big)\big)$$

for some continuous function $w : (0,\tau_{d_1}^{-1/d_1}) \to (0,+\infty)$ such that

$$\lim_{\varepsilon\to 0^+} w(\varepsilon) = 0.$$

Therefore, for all $t \in (0,+\infty)$, $s \in (0,+\infty)$ and $\varepsilon \in (0,\tau_{d_1}^{-1/d_1})$, we have

$$\begin{aligned}G_s(t) &\geq t^{-\frac{2}{d_1}}(\tilde{\mu}_1(\varepsilon)+\sigma(t)\varepsilon^{2s})\big(1-C\omega(\varepsilon)\big(\mu_1(B_1)+\sigma(t)\tau_{d_1}^{-\frac{2s}{d_1}}\big)\big),\\ G_s(t) &\leq t^{-\frac{2}{d_1}}\big(\mu_1(B_1)+\sigma(t)\tau_{d_1}^{-\frac{2s}{d_1}}\big).\end{aligned}$$

Since $\lim_{s\to 0^+}\sigma(t) = \tilde{\sigma}(t)$, we have

$$\limsup_{s\to 0^+} G_s(t) \leq G_0(t) \quad \text{for all } t > 0,$$

and, for all $\varepsilon \in (0,\tau_{d_1}^{-1/d_1})$,

$$\liminf_{s\to 0^+} G_s(t) \geq t^{-\frac{2}{d_1}}(\tilde{\mu}_1(\varepsilon)+\tilde{\sigma}(t))\big(1-C\omega(\varepsilon)(\mu_1(B_1)+\tilde{\sigma}(t))\big) \quad \text{for all } t > 0.$$

Thus,

$$\liminf_{s\to 0^+} G_s(t) \geq G_0(t) \quad \text{for all } t > 0.$$

We have then proved that

$$\lim_{s\to 0^+} G_s(t) = G_0(t) \quad \text{for all } t > 0.$$

and accordingly the statement follows. □

Remark 3.14. Clearly, $\mu_1(B_1)+\tilde{\sigma}$ is the first eigenvalue of problem (3.2) with $\mu = \tilde{\sigma}$ when we set $s = 0$. We easily see that $G_0(t)$ is optimized when

$$t^* = \Big(\frac{d_2\mu_1(B_1)}{d_1\mu_1(B_2)}\Big)^{\frac{d_1 d_2}{2d}} V^{\frac{d_1}{d}}, \tag{3.24}$$

and accordingly

$$G_0(t^*) = V^{-\frac{2}{d}} \frac{d}{d_1} \mu_1(B_1) \Big(\frac{d_1 \mu_1(B_2)}{d_2 \mu_1(B_1)} \Big)^{\frac{d_2}{d}}. \tag{3.25}$$

The optimum given by (3.24), (3.25) is the expected one, since, as $s \to 0^+$, the problem converges to the Dirichlet problem for the classical Laplacian in cartesian product domains. Note that the limit as $s \to 0^+$ of the lower bound (3.17) on the optimal t minimizing $G_s(t)$ for any s (i.e. the optimal $|\Omega_1^*|$) computed in Proposition 3.11 equals the t^* minimizing G_0. Therefore, the lower bound in Proposition 3.11 is sharp in the limit $s \to 0^+$.

Next, we pass to consider the limit as $s \to +\infty$. We first need a preliminary result on the asymptotic behavior of the first eigenvalue

$$E_1(s) := E_1(1, \mathbb{R}^{d_1})$$

of the Schrödinger operator $-\Delta_{x_1} + |x_1|^{2s}$ on $\mathbb{R}^{d_1}$ as $s \to +\infty$. The next Lemma 3.15 is probably known, but we include a detailed proof for the sake of completeness.

Lemma 3.15. *Let $B(0,1) \subseteq \mathbb{R}^{d_1}$ be the ball of radius one and centered at the origin. Let $\mu_1(B(0,1))$ be the first eigenvalue of the Dirichlet Laplacian on $B(0,1)$. Then*

$$\lim_{s\to+\infty} E_1(s) = \mu_1(B(0,1)) = \tau_{d_1}^{-\frac{2}{d_1}} \mu_1(B_1). \tag{3.26}$$

Proof. In order to prove (3.26), we provide sharp lower and upper bounds. We begin with the upper bound. Let $h \in (0,1)$. Let u_h be the first L^2-normalized eigenfunction of the Dirichlet Laplacian on $B(0,1-h)$. We shall still denote by u_h the extension by zero of u_h to $\mathbb{R}^{d_1}$. Clearly, the eigenvalue corresponding to u_h is

$$\frac{\mu_1(B(0,1))}{(1-h)^2}.$$

From the min-max principle (see (3.9)), we have that

$$E_1(s) \le \int_{B(0,1-h)} |\nabla_{x_1} u_h|^2 + |x_1|^{2s} u_h^2 \, dx_1 \le \frac{\mu_1(B(0,1))}{(1-h)^2} + (1-h)^{2s} \tag{3.27}$$

for all $s > 0$. Then

$$\limsup_{s\to+\infty} E_1(s) \le \frac{\mu_1(B(0,1))}{(1-h)^2}.$$

Since $h \in (0,1)$ is arbitrary, we also deduce that

$$\limsup_{s\to+\infty} E_1(s) \le \mu_1(B(0,1)).$$

Note that, by letting $h \to 0$ in (3.27), we also obtain that

$$E_1(s) \le \mu_1(B(0,1)) + 1$$

for all $s > 0$.

Now we pass to consider the lower bound. Let $s > 0$. Let f_s denote an L^2-normalized eigenfunction corresponding to $E_1(s)$. Then for all $h \in (0,1)$,

$$\begin{aligned}(1+h)^{2s} \int_{\mathbb{R}^{d_1} \setminus \overline{B(0,1+h)}} f_s^2 \, dx_1 &\le \int_{\mathbb{R}^{d_1} \setminus \overline{B(0,1+h)}} |x_1|^{2s} f_s^2 \, dx_1 \\ &\le E_1(s) \\ &\le \mu_1(B(0,1)) + 1,\end{aligned}$$

and thus

$$\int_{\mathbb{R}^{d_1} \setminus \overline{B(0,1+h)}} f_s^2 \, dx_1 \le \frac{\mu_1(B(0,1)) + 1}{(1+h)^{2s}},$$

which in turn implies that

$$f_s \to 0 \quad \text{in } L^2(\mathbb{R}^{d_1} \setminus \overline{B(0,1+h)}) \text{ for all } h \in (0,1).$$

Next we take $h, h' \in \mathbb{R}$ with $0 \le h' \le h \le 1$, and let $\rho \in C_c^\infty(\mathbb{R}^{d_1})$ be such that

$$0 \le \rho \le 1, \quad \operatorname{supp} \rho \subseteq B(0,1+h), \quad \rho_{|B(0,1+h')} = 1.$$

By the min-max principle for the first eigenvalue of the Dirichlet Laplacian in $B(0,1+h)$ with test function ρf_s and integrating by parts, we have

$$(1+h)^{-2}\mu_1(B(0,1)) \int_{\mathbb{R}^{d_1}} \rho^2 f_s^2 \, dx_1 \le \int_{\mathbb{R}^{d_1}} |\nabla_{x_1}\rho|^2 f_s^2 \, dx_1 - \int_{\mathbb{R}^{d_1}} \rho^2 f_s \Delta_{x_1} f_s \, dx_1.$$

By the eigenvalue equation $-\Delta_{x_1} f_s = E_1(s) f_s - |x_1|^{2s} f_s$, we can deduce that

$$\begin{aligned}(1+h)^{-2}\mu_1(B(0,1)) \int_{\mathbb{R}^{d_1}} \rho^2 f_s^2 \, dx_1 &\le \int_{B(0,1+h)\setminus\overline{B(0,1+h')}} |\nabla_{x_1}\rho|^2 f_s^2 \, dx_1 + E_1(s) \int_{\mathbb{R}^{d_1}} \rho^2 f_s^2 \, dx_1 - \int_{\mathbb{R}^{d_1}} |x_1|^{2s} \rho^2 f_s^2 \, dx_1 \\ &\le \int_{B(0,1+h)\setminus\overline{B(0,1+h')}} |\nabla_{x_1}\rho|^2 f_s^2 \, dx_1 + E_1(s) \int_{\mathbb{R}^{d_1}} \rho^2 f_s^2 \, dx_1.\end{aligned}$$

Since clearly

$$1 - \int_{\mathbb{R}^{d_1}\setminus\overline{B(0,1+h')}} f_s^2 \, dx_1 \le \int_{\mathbb{R}^{d_1}} \rho^2 f_s^2 \, dx_1 \le 1,$$

we have

$$(1+h)^{-2}\mu_1(B(0,1)) \le E_1(s) + \int_{B(0,1+h)\setminus\overline{B(0,1+h')}} |\nabla_{x_1}\rho|^2 f_s^2 \, dx_1 + (1+h)^{-2}\mu_1(B(0,1)) \int_{\mathbb{R}^{d_1}\setminus\overline{B(0,1+h')}} \rho^2 f_s^2 \, dx_1.$$

Since ρ^2 and $|\nabla_{x_1}\rho|^2$ are uniformly bounded, both

$$\int_{B(0,1+h)\setminus\overline{B(0,1+h')}} |\nabla_{x_1}\rho|^2 f_s^2 \, dx_1 \quad \text{and} \quad \int_{\mathbb{R}^{d_1}\setminus\overline{B(0,1+h')}} \rho^2 f_s^2 \, dx_1$$

converge to zero as $s \to +\infty$. That is, we proved the lower bound

$$\liminf_{s\to+\infty} E_1(s) \ge \frac{\mu_1(B(0,1))}{(1+h)^2},$$

which implies

$$\liminf_{s\to+\infty} E_1(s) \ge \mu_1(B(0,1)),$$

since $h \in (0,1)$ is arbitrary. The first equality of (3.26) is proved. The second equality follows simply by rescaling. □

Proposition 3.16. *Let G_s be the function defined in* (3.19). *Let*

$$G_\infty(t) := \begin{cases} \dfrac{\mu_1(B_1)}{t^{2/d_1}}, & t \in (0, \tau_{d_1}), \\ \dfrac{\mu_1(B_1)}{\tau_{d_1}^{2/d_1}}, & t \in [\tau_{d_1}, +\infty). \end{cases}$$

Then

$$\lim_{s\to+\infty} G_s(t) = G_\infty(t) \quad \textit{for all } t > 0.$$

Proof. We need to distinguish two cases: $t < \tau_{d_1}$ and $t \geq \tau_{d_1}$.

Assume first that $t < \tau_{d_1}$. We have immediately from the min-max principle that for all $\sigma > 0$,

$$E_1(\sigma, B_1) \geq \mu_1(B_1),$$

and in particular

$$\liminf_{s\to+\infty} E_1(\sigma, B_1) \geq \mu_1(B_1).$$

Moreover, for any $f \in H_0^1(B_1)$ with $\|f\|_{L^2(B_1)} = 1$,

$$\int_{B_1} |\nabla_{x_1} f|^2\, dx_1 + \tilde{\sigma}(t) \int_{B_1} (t^{\frac{1}{d_1}} |x_1|)^{2s} f^2\, dx_1 \leq \int_{B_1} |\nabla_{x_1} f|^2\, dx_1 + (t^{\frac{1}{d_1}} \tau_{d_1}^{-\frac{1}{d_1}})^{2s} \tilde{\sigma}(t).$$

Taking the minimum over all $f \in H_0^1(B_1)$ with $\|f\|_{L^2(B_1)} = 1$, we obtain

$$E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) \leq \mu_1(B_1) + (t^{\frac{1}{d_1}} \tau_{d_1}^{-\frac{1}{d_1}})^{2s} \tilde{\sigma}(t). \tag{3.28}$$

Since $t < \tau_{d_1}$, we have that

$$\limsup_{s\to+\infty} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) \leq \mu_1(B_1).$$

We have proved that

$$\lim_{s\to+\infty} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) = \mu_1(B_1),$$

and therefore, by recalling the definition (3.19), we have

$$\lim_{s\to+\infty} G_s(t) = \frac{\mu_1(B_1)}{t^{2/d_1}}.$$

Next, we pass to consider the case $t \geq \tau_{d_1}$. Let $L > t$ be fixed. We denote by $\mu_1(B(0, L^{-1/d_1}))$ the first eigenvalue of the Dirichlet Laplacian on the ball $B(0, L^{-1/d_1}) \subseteq \mathbb{R}^{d_1}$ centered in zero and of radius L^{-1/d_1}. Note that the volume of $B(0, L^{-1/d_1})$ is $\frac{\tau_{d_1}}{L} < 1$. In particular,

$$\mu_1(B(0, L^{-\frac{1}{d_1}})) = \Big(\frac{L}{\tau_{d_1}}\Big)^{\frac{2}{d_1}} \mu_1(B_1).$$

From the inclusion $H_0^1(B(0, L^{-1/d_1})) \subset H_0^1(B_1)$ (where we understand that any $u \in H_0^1(B(0, L^{-1/d_1}))$ is extended by zero to B_1), and from an analogous computation to the one in (3.28), we obtain

$$\begin{aligned} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) &\leq E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B(0, L^{-\frac{1}{d_1}})\big) \\ &\leq \mu_1\big(B(0, L^{-\frac{1}{d_1}})\big) + (t^{\frac{1}{d_1}} L^{-\frac{1}{d_1}})^{2s} \tilde{\sigma}(t) \\ &= \Big(\frac{L}{\tau_{d_1}}\Big)^{\frac{2}{d_1}} \mu_1(B_1) + (t^{\frac{1}{d_1}} L^{-\frac{1}{d_1}})^{2s} \tilde{\sigma}(t). \end{aligned}$$

Then, for all $L > t$,

$$\limsup_{s\to+\infty} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) \leq \Big(\frac{L}{\tau_{d_1}}\Big)^{\frac{2}{d_1}} \mu_1(B_1).$$

Therefore,

$$\limsup_{s\to+\infty} E_1\big(\tilde{\sigma}(t) t^{\frac{2s}{d_1}}, B_1\big) \leq \Big(\frac{t}{\tau_{d_1}}\Big)^{\frac{2}{d_1}} \mu_1(B_1),$$

which in turn implies

$$\limsup_{s\to+\infty} G_s(t) \leq \frac{\mu_1(B_1)}{\tau_{d_1}^{2/d_1}}.$$

In order to prove a lower bound, we can proceed as follows. We simply note that, by (3.8), for all $\sigma \in (0, +\infty)$,

$$E_1(\sigma, B_1) \geq E_1(\sigma, \mathbb{R}^{d_1}) = \sigma^{\frac{1}{s+1}} E_1(1, \mathbb{R}^{d_1}).$$

Now, since

$$(\sigma(t))^{\frac{1}{s+1}} = (\tilde{\sigma}(t))^{\frac{1}{s+1}} t^{\frac{2s}{d_1(s+1)}},$$

we have

$$\lim_{s\to+\infty} (\sigma(t))^{\frac{1}{s+1}} = t^{\frac{2}{d_1}}.$$

Moreover, by Lemma 3.15,

$$\lim_{s\to+\infty} E_1(1,\mathbb{R}^{d_1}) = \frac{\mu_1(B_1)}{\tau_{d_1}^{2/d_1}}.$$

Thus, we immediately deduce that

$$\liminf_{s\to+\infty} G_s(t) \geq \frac{\mu_1(B_1)}{\tau_{d_1}^{2/d_1}}.$$

This implies, along with the upper bound, that, for $t \geq \tau_{d_1}$,

$$\lim_{s\to+\infty} G_s(t) = \frac{\mu_1(B_1)}{\tau_{d_1}^{2/d_1}}.$$

Thus the statement is proved. □

Remark 3.17. Note that, in the limiting case $s \to +\infty$, we have a continuum of optimal t, namely all $t \geq \tau_{d_1}$ minimize $G_\infty(t)$. We also note that the lower bound on the optimal t provided for any s in Proposition 3.11 goes to τ_{d_1} as $s \to +\infty$. Therefore, also in this sense, that lower bound is sharp.

4 Some numerical computations

In this last section, we present some numerical computations in the planar case, that is, in the case $d_1 = d_2 = 1$. First, we consider the minimization problem of the first eigenvalue in the class of cartesian product domains (i.e. rectangles of $\mathbb{R}^2$). Then we also numerically compute the first eigenvalue in the case the domain is a ball in $\mathbb{R}^2$ and we make some comparisons with the case of rectangles. For simplicity we also set $V = 1$, but by a simple scaling argument one can also deduce similar results for the general case $V > 0$. Note that in this case

$$\tau_{d_1} = 2, \quad \mu_1(B_1)\tau^{-\frac{2}{d_1}} = \frac{\pi^2}{4} \approx 2.467.$$

The numerical scheme to solve the two decoupled one-dimensional problems has been implemented in Python with the help of Gabriele Santin (FBK-ICT).

Figure 1 shows the plot of $G_s(t)$ and the numerical computation of its minimum for some values of s. We recall that the function $G_s(t)$, defined in (3.19), equals $\lambda_1(\Omega_1 \times \Omega_2)$ when $\Omega_1 \subseteq \mathbb{R}^{d_1}$, $\Omega_2 \subseteq \mathbb{R}^{d_2}$ are two balls, the first one being centered in zero and with $|\Omega_1||\Omega_2| = 1$, $|\Omega_1| = t$. The unique minimum point t^* of $G_s(t)$ represents then the volume of Ω_1^*, where Ω_1^* and Ω_2^* are the balls which realize the minimum for the first eigenvalue (see Theorem 3.10).

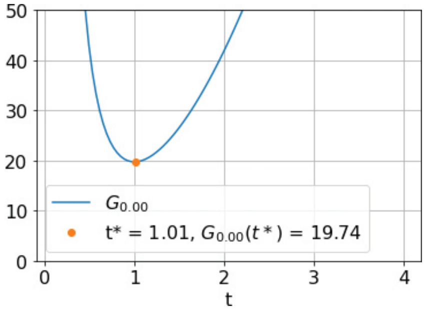
Figure 1a corresponds to the limiting case $s \to 0^+$. As expected, the minimum is attained at $t = 1$, which means, with the notation of Theorem 3.10, $|\Omega_1^*| = |\Omega_2^*| = 1$. Indeed, for the standard Laplacian in two dimensions, the minimum over the class of cartesian product domains is attained by a square. Figures 1b–1f show the numerical computation of the minimum for $s = 0.5$, $s = 1$, $s = 2$, $s = 3$, and $s = 150$.

We note that the numerical computations agree with the lower bounds of Propositions 3.11 and 3.12. Moreover, they also agree with the asymptotic behavior as $s \to +\infty$ computed in Proposition 3.16, since $G_s(t)$ tends to flatten to the value $\frac{\pi^2}{4}$ for $t > \tau_{d_1} = 2$ when s increases.

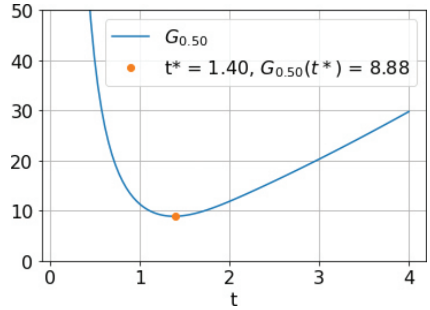
We conclude by comparing the eigenvalues on rectangles in $\mathbb{R}^2$ with those of the disk with the same area, centered at the origin (see Figure 2).

When $s = 0$, clearly the disk of unit area (i.e. radius $\pi^{-1/2}$) has lower first eigenvalue than any rectangle of the same area (Faber–Krahn inequality). In fact, for $s = 0$,

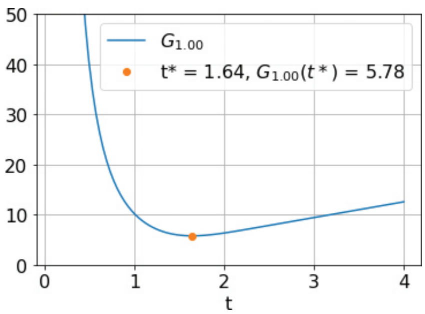
$$\lambda_1(B(0,\pi^{-\frac{1}{2}})) \approx 18.17 \quad \text{and} \quad \lambda_1(\Omega_1^* \times \Omega_2^*) \approx 19.74$$



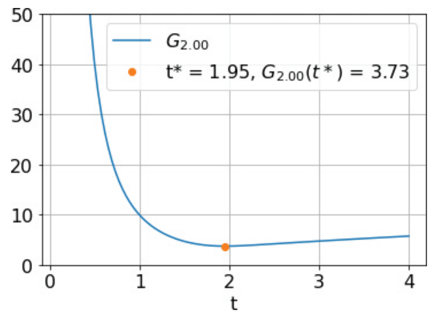
(a) $s = 0$



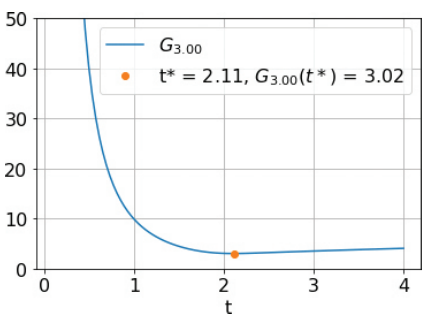
(b) $s = 0.5$



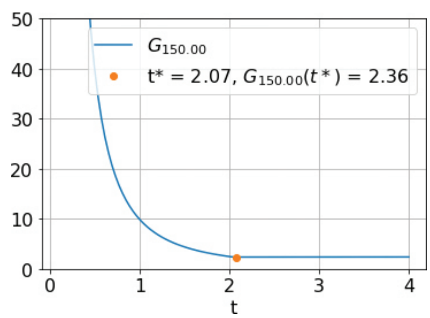
(c) $s = 1$



(d) $s = 2$



(e) $s = 3$



(f) $s = 150$

Figure 1: The unique minimizer t^* of G_s for $s = 0, 0.5, 1, 2, 3, 150$.

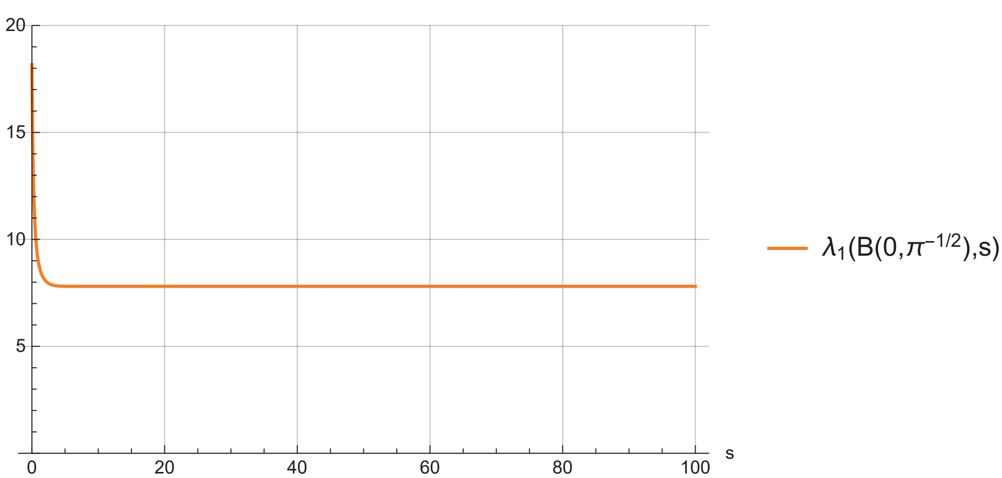


Figure 2: $\lambda_1(B(0, \pi^{-1/2}))$ as a function of s.

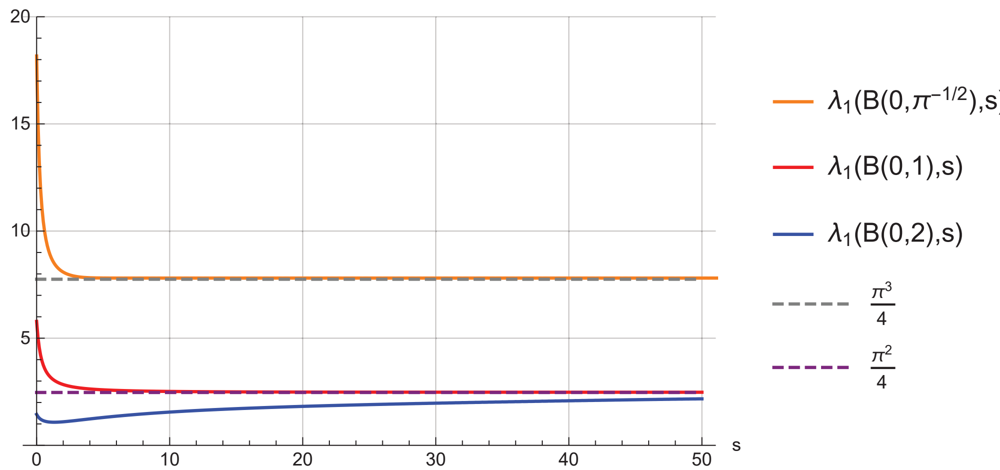


Figure 3: $\lambda_1(B(0,\pi^{-1/2}))$, $\lambda_1(B(0,1))$ and $\lambda_1(B(0,2))$ as a function of s.

Already when $s = 0.5$, we have

$$\lambda_1(B(0,\pi^{-\frac{1}{2}})) \approx 10.45 \quad \text{and} \quad \lambda_1(\Omega_1^* \times \Omega_2^*) \approx 8.88,$$

and, when $s = 1$, we have

$$\lambda_1(B(0,\pi^{-\frac{1}{2}})) \approx 8.90 \quad \text{and} \quad \lambda_1(\Omega_1^* \times \Omega_2^*) \approx 5.78.$$

We also note that, as $s \to +\infty$,

$$\lambda_1(B(0,\pi^{-1/2})) \approx 7.75 \approx \frac{\pi^3}{4},$$

while

$$\lambda_1(\Omega_1^* \times \Omega_2^*) \approx \frac{\pi^2}{4}.$$

The numerics suggest that there exists some $s_0 < \frac{1}{2}$ such that the disk of unit area is no more the minimizer among all domains of the same area, and we always find a rectangle doing better.

We have also computed the first eigenvalue on the disk of radius 1 and of radius 2 as functions of s (see Figure 3).

We note that, as $s \to +\infty$, the first eigenvalue of the disks of radius 1 and 2 seems to behave like $\frac{\pi^2}{4}$, exactly as $\lambda_1(\Omega_1 \times \Omega_2)$ with $|\Omega_1| \geq 2$ (which means, length of the side parallel to the x_1-axis greater than 2).

We note that $\frac{\pi^2}{4}$ is exactly the first Dirichlet eigenvalue of an interval of length 2. On the other hand, the value $\frac{\pi^3}{4}$, the expected limit of the first eigenvalue of the disk of area 1 as $s \to +\infty$, is the first Dirichlet eigenvalue of an interval of length $2\pi^{-1/2}$.

It seems that the behavior of the first eigenvalue of a domain, as $s \to +\infty$, is determined by the length of the longest segment parallel to the x_1-axis contained in $\Omega \cap \{|x_1| < 1\}$, which is 2 in the case of $B(0,1)$, $B(0,2)$ and of any rectangle $\Omega_1 \times \Omega_2$ with $|\Omega_1| \geq 2$, and is $2\pi^{-1/2}$ for $B(0,\pi^{-1/2})$. We will consider these issues from an analytical point of view in future works. At any rate, we are left with the following question.

Question: Does $\lambda_1(\Omega) \to \lambda_1((0,L))$ as $s \to +\infty$, where L is the length of the longest segment parallel to the x_1-axis contained in $\Omega \cap \{|x_1| < 1\}$, and $\lambda_1((0,L))$ is the first Dirichlet eigenvalue on $(0,L)$?

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