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Large time well-posedness for a Dirac–Klein–Gordon system

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ABSTRACT

In this paper, we prove well-posedness for a Dirac equation perturbed with a moving potential W that satisfies a Klein–Gordon equation. This represents a “toy model” for atoms with relativistic corrections, as the wave function of the electrons interacts with an electric field generated by a nucleus with a given charge density. One of the main ingredients we need is a new family of Strichartz estimates for time-dependent perturbations of the Dirac equation: these represent a result of independent interest.

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1. Introduction

The aim of this paper is to study the well-posedness of a system representing a toy model for the Helium atom with relativistic corrections that the first and second authors started in [6]. The Helium atom is represented by a nucleus in position $q(t) \in \mathbb{R}^3$ and by two electrons. The wave function u satisfies the following Cauchy problem :

$$\begin{cases} i\partial_t u = \mathcal{D}_m u + W[q]u + |\langle u, \beta u\rangle|^{\frac{p-1}{2}}\beta u, & u(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4 \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

Here, $\mathcal{D}_m$ with $m > 0$, denotes the massive Dirac operator: this is classically represented as

$$\mathcal{D}_m = -i\sum_{k=1}^{3} \alpha_k \partial_k + \beta m = -i(\alpha \cdot \nabla) + \beta m$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and the 4×4 Dirac matrices are given by

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \tag{2}$$

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where I_2 is the 2×2 identity matrix and σ_k for $k = 1, 2, 3$ are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

In what follows, we will take for simplicity $m = 1$ and we will denote $\mathcal{D} = \mathcal{D}_1$.

The potential $W[q]$ is the electrodynamic field generated by the nucleus, assuming the nucleus is not punctual but is spread according to a function $\chi : \mathbb{R}^3 \to \mathbb{R}$ representing the charge distribution, centered in $q(t)$, where $q : \mathbb{R} \to \mathbb{R}^3$ is the position of the nucleus in time. In other words, $W[q]$ is the solution W to the equation

$$\begin{cases} \partial_{tt} W + W - \Delta W = \chi(x - q(t)), & W(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R} \\ \qquad W(0,x) = w_0, \quad \partial_t W(0,x) = w_1. \end{cases} \tag{4}$$

The function χ is assumed to be real and satisfying suitable conditions that we will state later on.

In the first time, we assume that q is given; in this setting, the Dirac equation and the Klein–Gordon equation are decoupled.

The nonlinear term we are considering in (1), that is sometimes referred to as *Soler nonlinearity*, is classical in this setting (see e.g. [10]), as it is the main example of covariant nonlinearities for the Dirac equation, that is such that the equation is left invariant under Lorentz transforms.

Before stating our main results, let us comment on our system. As we mentioned above, the map u represents the wave function of the electrons, the map W represents the electric field generated by a nucleus centered in the position $q(t)$ at time t and with a charge density distribution given by $\chi(x - q(t))$. In fact, in relativistic electronic structure theory, the nuclei, composed of small constituents (i.e. nucleons or quarks) and bound together by the strong force, should be represented by an extended distribution function χ instead of a Dirac delta distribution (see e.g. [1]). Here, we propose to study a non-linearity that we can deal with by making use of standard Strichartz estimates, that is $|\langle u, \beta u\rangle|^{(p-1)/2}\beta u$. Another possible choice would be to consider the Hartree type nonlinear term $(|x|^{-\gamma} * |u|^2)u$, $\gamma \geqslant 1$, but this would force us either to provide a more refined version of Strichartz estimates in Besov spaces (see e.g. [15]) or to require more regularity on the initial data (see e.g. [7]). Note that to be perfectly consistent with the physics literature, the Klein–Gordon equation should be replaced by the wave equation; however, the wave equation admits less dispersion than the Klein–Gordon one, and this fact represents a substantial obstacle to the strategy we pursue here. To the best of our knowledge, there is no result about the Dirac–wave equation system in dimension 3 that suits our problem.

Our first main result is the following (we postpone the overview of the notation to the end of the introduction):

Theorem 1.1 (*Global Well-Posedness For* (1))**.** *Let p and $s \leqslant 2$ be such that:*

$$\begin{cases} s \geqslant \frac{3}{2} - \frac{1}{p-1}, & \text{if } p > 3 \text{ is an odd integer,} \\ \frac{p-1}{2} > s \geqslant \frac{3}{2} - \frac{1}{p-1}, & \text{if } p > 3 \text{ is not an odd integer.} \end{cases}$$

Let $q(t)$ satisfy the following:

$$\|\ddot{q}\|_{L^1(\mathbb{R})} \leqslant \frac{1}{2}, \qquad q \in L^\infty(\mathbb{R}). \tag{5}$$

Then, if $\|w_0\|_{W^{s+3,1}}$, $\|w_1\|_{W^{s+2,1}}$, $\|\chi\|_{W^{2+s,1}}$, $\left\|\langle x\rangle^{3+}\chi\right\|_{L^\infty}$ and $\|u_0\|_{H^s}$ are small enough, the unique solution $W[q]$ to (4) belongs to the space

$$C^0(\mathbb{R}, L^\infty(\mathbb{R}^3)) + L^1(\mathbb{R}, H^{s,+\infty}(\mathbb{R}^3))$$

and system (1) admits a unique global solution u in the space

$$C^0(\mathbb{R}, H^s(\mathbb{R}^3)) \cap L^{p-1}(\mathbb{R}, L^\infty(\mathbb{R}^3)). \tag{6}$$

Remark 1.1. It is easy to see that the most frequently used charge density distribution such as the Gauss-type, and the Fermi-type satisfy our assumptions on χ (for detail, see [1, Section 4.4 and Section 4.5]).

Remark 1.2. For the proof of Theorem 1.1, we follow the argument developed in [10, Theorem II] where the case $p = 3$ is excluded for the global well-posedness. The reason is technical, and it is ultimately due to the failure of a Gagliardo–Nirenberg inequality at the critical level (see [10, Remark 7]). We do not know whether it will be possible to be able to cover the case $p = 3$ as well by making use of some more efficient nonlinear argument; anyway, as this is not really the focus of our paper, we do not mean to strive on the optimality of p.

The proof of Theorem 1.1 is quite standard *provided* one has suitable Strichartz estimates at disposal; to the best of our knowledge, they are not available in the form we need, and we thus need to prove them. To begin with, let us give the following

Definition 1.2 (*Dirac Admissible Triple*)**.** The triple (p, r, s) is *Dirac admissible* if and only if there exists a constant C such that for any $u_0 \in H^s$

$$\|e^{it\mathcal{D}} u_0\|_{L^p_t L^r_x} \leqslant C\|u_0\|_{H^s}.$$

Remark 1.3. The standard choice of Dirac admissible triple (p, r, s) is the non-endpoint Schrödinger admissible one [9]:

$$\frac{2}{p} + \frac{3}{r} = \frac{3}{2}, \quad 2 < p \leqslant +\infty, \quad 2 \leqslant r \leqslant 6, \quad s = \frac{1}{2} + \frac{1}{p} - \frac{1}{r}.$$

Actually, to deal with the nonlinear term in system (1), it is helpful to work with a different triple, that is the one given by

$$\left(p - 1, +\infty, \frac{3}{2} - \frac{1}{p-1}\right), \qquad p > 3;$$

in fact, the estimates, in this case, can be retrieved by the classical ones and the application of a Gagliardo–Nirenberg inequality (see [10], Theorem 1.5).

We thus prove the following

Theorem 1.3 (*Strichartz Estimates*)**.** *Let $T \in (0, +\infty]$. Let $u = S_V(t)u_0$ be a solution to*

$$\begin{cases} i\partial_t u = \mathcal{D}u + V(t, x)u, & u(t, x) : (0, T) \times \mathbb{R}^3_x \to \mathbb{C}^4 \\ u(0, x) = u_0(x) \end{cases} \tag{7}$$

where $V(t, x)$ is an operator. Let $N > \frac{3}{2}$ and $s \geqslant 0$. Assume that

- *system (7) is well-posed on H^s,*
- *there is a constant $\varepsilon > 0$ small enough such that*

$$\|V\|_{T,s,N} := \left\| \langle x \rangle^N (1 - \Delta)^{s/2} V (1 - \Delta)^{-s/2} \langle x \rangle^N \right\|_{L^\infty((0,T), L^2 \to L^2)} \leqslant \varepsilon. \tag{8}$$

Then the following estimate holds:

$$\|S_V(t)u_0\|_{L^\infty((0,T);H^s)} \lesssim \|u_0\|_{H^s}. \tag{9}$$

Furthermore, if (p, r, s) is any Dirac admissible triple then the following Strichartz estimates hold:

$$\|S_V(t)u_0\|_{L^p((0,T);L^r)} \lesssim \|u_0\|_{H^s}. \tag{10}$$

Remark 1.4. Strichartz estimates for potential perturbations of the Dirac equation have been widely investigated (see e.g. [3–5,9]). Theorem 1.3 improves on existing results, as here V is a time-dependent *operator*, not necessarily a multiplication one; this result is thus of independent interest.

As a second step, we couple system (1) with a nuclear dynamics of Hellmann–Feynman type, that is we now consider the following more involved system:

$$\begin{cases} i\partial_t u = \mathcal{D}u + W[q]u + |\langle u,\beta u\rangle|^{\frac{p-1}{2}}\beta u; \qquad u(t,x):\mathbb{R}_t\times\mathbb{R}^3_x \to \mathbb{C}^4 \\ M\ddot{q} = \left\langle u \left| \dfrac{x-q}{|x-q|^3} \right| u \right\rangle := \displaystyle\int_{\mathbb{R}^3} \langle u(x),u(x)\rangle_{\mathbb{C}^4} \frac{x-q}{|x-q|^3}dx; \\ \qquad u(0,x) = u_0(x); \\ \qquad q(0,x) = 0, \quad \dot{q}(0,x) = v_0. \end{cases} \tag{11}$$

for some $M \gg 1$ and with the same notations as for system (1).

This coupling comes from the fact that the electrons act on the nucleus via a potential

$$\left\langle u \left| \frac{1}{|x-q|} \right| u \right\rangle.$$

Note that now the Dirac equation and the Klein–Gordon equation are *coupled* through the dynamics of q. We keep the electrostatic approximation here because the nucleus is far heavier ($M \gg 1$) than the electrons and thus carries some inertia. Hence we assume that its dynamics are driven by the classical dynamics of a charged particle in a given field. Note that this type of system has been studied in [7] in the nonrelativistic case with electrostatic approximations for the nucleus and the electrons: the authors proved global well-posedness for the system. We stress the fact that for a nonrelativistic system, the Coulomb potential is not scaling-critical, which makes all the difference with the problem at stake.

For system (11), we prove the following:

Theorem 1.4 (*Large-Time Well-Posedness For* (11))**.** *Let p and $s \leqslant 2$ be such that:*

$$\begin{cases} s \geqslant \frac{3}{2} - \frac{1}{p-1}, & \text{if } p>3 \text{ is an odd integer,} \\ \frac{p-1}{2} > s \geqslant \frac{3}{2} - \frac{1}{p-1}, & \text{if } p>3 \text{ is not an odd integer.} \end{cases}$$

Let χ, w_0, w_1, q_1, q_2 be as in the assumptions of Theorem 1.1 with the additional assumption that $\| \langle x\rangle^{3+}\nabla\chi\|_{L^\infty}$ be sufficiently small. For all $R>0$, such that

$$\|u_0\|_{H^s} + \|w_0\|_{W^{s+3,1}} + \|w_1\|_{W^{s+2,1}} + \|\chi\|_{W^{s+1,1}} \leqslant R,$$

there exists a constant $C_2 = C_2(R)$ such that the unique solution $W[q]$ to (4) belongs to the space

$$C^0([0,T], L^\infty(\mathbb{R}^3)) + L^1([0,T], H^{s,+\infty}(\mathbb{R}^3))$$

and system (11) admits a unique solution (u,q) in the space

$$C^0([0,T], H^s(\mathbb{R}^3)) \times C^2([0,T],\mathbb{R}^3)$$

for any $T \leqslant C_2 \min(\sqrt{M}, |v_0|^{-1})$.

Remark 1.5. The regularity assumption $s > 3/2$ on the initial condition u_0 is needed in order to prove well-posedness for the dynamics of the nuclei or, more precisely, to prove that the map $F(q) = \left\langle u|\frac{x-q}{|x-q|^3}|v\right\rangle$ is Lipschitz continuous and thus to be able to apply Picard fixed point Theorem; therefore, it represents an unavoidable threshold. This fact has been already noticed and discussed in [6] (see Remark 1.5 there). On the other hand, the (additional) upper bound $s \leqslant 2$ turns out to be necessary in view of providing suitable estimates on the function W (see e.g. Proposition 3.5). This upper bound is thus due to technical reasons; again, this condition could be lifted at the price of losing derivatives on χ. We omit the details.

We remark that we call the Theorem “large-time well-posedness” because the time T of well-posedness goes to ∞ as M and $|v_0|^{-1}$ go to ∞, which corresponds to taking a nucleus that is infinitely heavier than the electrons.

Overview of the paper and sketch of the proof. As the paper is quite articulated, let us give a short overview of the main ideas of our argument. The main difficulty with Eq. (1) and system (11) is driven by the Klein–Gordon equation on W: in fact, this Klein–Gordon equation cannot be solved directly by making use of standard Strichartz estimates, as indeed in our assumptions the function $\chi(x-q(t)) \notin L^p_t(\mathbb{R}, L^r(\mathbb{R}^3))$ for any $1 < p < +\infty$ and $1 < r \leqslant +\infty$. To be more precise, we cannot find a functional space $L^p_t(\mathbb{R}, L^r(\mathbb{R}^3))$ for some $1 < p < +\infty$ and $r > 1$ such that $W \in L^p_t(\mathbb{R}, L^r(\mathbb{R}^3))$. To overcome this problem, we will not solve the (full) Klein–Gordon equation for W by standard Strichartz estimates; instead, we shall decompose the potential W into a sum of a “dispersive part” (that means that it enjoys some “nice” dispersive estimates) and a “non-dispersive” part. The dispersive part will be studied by means of standard Strichartz estimates for the Klein–Gordon equation, while the non-dispersive one will be treated as a perturbation of the free Dirac equation (we postpone to the beginning of Section 3 a more detailed overview of this decomposition). Therefore, we will need some Strichartz estimates for the Dirac equation perturbed with a non-stationary potential which, to the best of our knowledge, are not known: this will be the first step of our argument. Once Strichartz estimates are available, the proof of Theorems 1.1 and 1.4 becomes fairly straightforward. In particular, the proof of 1.4 requires some additional effort in order to handle the classical dynamics on q: to show that it is well posed, we need to assume sufficient regularity on the initial condition u_0.

The plan of the paper is thus the following:

- In Section 2 we shall prove Strichartz estimates for the Dirac equation with a moving potential, the proof relying on the well-established path

$$\text{virial identity} \Rightarrow \text{weak dispersive estimates} \Rightarrow \text{Strichartz estimates.}$$

- In Section 3 we shall deal with the Klein–Gordon equation: we provide the aforementioned decomposition of the solution, prove some useful estimates on the single terms and prove global well-posedness.
- Section 4 will be devoted to the proofs of Theorems 1.1–1.4, that is the well-posedness for systems (1) and (11).

Notation. We use the standard notation L^p for Lebesgue spaces, often distinguishing with a subscript x (resp. t) the norm in space on $\mathbb{R}^3_x$ (resp. in time on $\mathbb{R}_t$); with the subscript X_T we will denote norms on a time interval $(0,T)$ with $T \in (0,+\infty]$, that is e.g. $L^p_T = L^p_t((0,T))$. We will denote with $W^{s,p}$ the Sobolev spaces:

$$\|f\|_{W^{s,p}} := \left(\sum_{|\alpha| \leqslant s} \|D^\alpha f\|^p_{L^p} \right)^{1/p},$$

for $s \in \mathbb{N}$ and $p \geq 1$, and for $s \in (0,+\infty) \setminus \mathbb{N}$, let $s = m + r$ with $m \in \mathbb{N}$ and $r \in (0,1)$, then

$$\|f\|_{W^{s,p}} := \left(\|f\|^p_{W^{m,p}} + \sum_{|\alpha|=m} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{4rp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

We will denote with $H^{s,p}$ the spaces equipped with the norms

$$\|f\|_{H^{s,p}} := \|H^s f\|_{L^p}$$

where $H = \sqrt{1-\Delta}$, for $s \geqslant 0$ and $p \geqslant 1$, with the usual convention for the case $p = 2$ that is $H^s = H^{s,2}$. According to interpolation theory,

$$\|f\|_{H^{s,p}} \lesssim \|f\|_{W^{s,p}}, \qquad 1 \leqslant p \leqslant +\infty, \tag{12}$$

and according to Calderón–Zygmund inequality,

$$\|f\|_{W^{s,p}} \lesssim \|f\|_{H^{s,p}}, \qquad 1 < p < +\infty. \tag{13}$$

The Strichartz norms will be denoted as

$$\|f\|_{XY} = \|f\|_{X_t Y_x} = \|f\|_{X(\mathbb{R}_t; Y(\mathbb{C}^4_x))}$$

where X and Y might be Lebesgue, Sobolev or weighted Sobolev spaces; the local-in-time versions will be written as $X_T Y_x = X((0,T); Y(\mathbb{R}^3_x))$ for some $T \in (0, +\infty]$. As declared, we will often omit the subscripts t and x when the context will make it unambiguous.

Let $\langle x \rangle = \sqrt{1+|x|^2}$. We will make use of the following weighted norms: by $L^2(\langle x \rangle^N)$ and $H^1(\langle x \rangle^N)$ we denote respectively the spaces induced by the norms

$$\|u\|_{L^2(\langle x \rangle^N)} := \left\| \langle x \rangle^N u \right\|_{L^2}, \qquad \|u\|_{H^1(\langle x \rangle^N)} := \|u\|_{L^2(\langle x \rangle^N)} + \|\nabla u\|_{L^2(\langle x \rangle^N)} \tag{14}$$

where N is a real number (that may be negative). Notice that the $H^1(\langle x \rangle^N)$ norm of u is equivalent to the H^1 norm of $\langle x \rangle^N u$, which in turns makes it equivalent to the L^2 norm of $\mathcal{D} \langle x \rangle^N u$.

We recall that the norm that will play the starting role, as defined in (8), is given by

$$\|V\|_{T,s,N} := \left\| \langle x \rangle^N (1-\Delta)^{s/2} V (1-\Delta)^{-s/2} \langle x \rangle^N \right\|_{L^\infty((0,T), L^2 \to L^2)}$$

for $s, N \in \mathbb{R}$. When $T = \infty$, we denote it as $\|V\|_{s,N}$.

Finally, we recall that the functional space $X + Y$ is defined through the norm

$$\|u\|_{X+Y} := \inf_{u = u_1 + u_2} (\|u_1\|_X + \|u_2\|_Y).$$

2. Linear estimates for the Dirac equation: proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, that is of Strichartz estimates for solutions to Eq. (7) under suitable assumptions on the potential V. The strategy is classical in this framework, and it is based on virial identity. Therefore we will proceed as follows: after introducing our functional setting and some inequalities in Section 2.1, in Section 2.2 we shall build the virial identity and prove a weak dispersive estimate, while in 2.3 we shall derive the Strichartz estimates via the standard argument.

2.1. Preliminaries

We begin with some norm inequalities that we shall need in the sequel.

Proposition 2.1. *For all $N \in \mathbb{R}$, the norm*

$$\|u\|_{\tilde{H}^1(\langle x \rangle^N)} := \|\mathcal{D}u\|_{L^2(\langle x \rangle^N)} + C \|u\|_{L^2(\langle x \rangle^N)}$$

is equivalent to the $H^1(\langle x \rangle^N)$ one defined in (14) for C large enough.

Proof. By definition of the Dirac operator, we have that

$$\|\mathcal{D}u\|^2_{L^2(\langle x\rangle^N)} = \|u\|^2_{L^2(\langle x\rangle^N)} + \left\|\sum_{j=1}^{3} \langle x\rangle^N \alpha_j \partial_j u\right\|^2_{L^2} + 2\mathrm{Re}\left\langle -\sum_{j=1}^{3} i\langle x\rangle^N \alpha_j\partial_j u, \langle x\rangle^N \beta u\right\rangle_{L^2}.$$

On the one hand, it is easy to see that

$$\left\|\sum_{j=1}^{3} \langle x\rangle^N \alpha_j \partial_j u\right\|_{L^2} = \left\|\langle x\rangle^N \nabla u\right\|_{L^2}.$$

On the other hand, by Young's inequality, for any $a > 0$

$$\left|\mathrm{Re}\left\langle -\sum_{j=1}^{3} i\langle x\rangle^N \alpha_j\partial_j u, \langle x\rangle^N \beta u\right\rangle_{L^2}\right| \leqslant \frac{1}{a}\left\|\langle x\rangle^N \nabla u\right\|^2_{L^2} + a\|u\|^2_{L^2(\langle x\rangle^N)}.$$

As a consequence, we get

$$\|\mathcal{D}u\|^2_{L^2(\langle x\rangle^N)} \geqslant \left(1-\frac{2}{a}\right)\|u\|^2_{H^1(\langle x\rangle^N)} + (1-2a)\|u\|^2_{L^2(\langle x\rangle^N)}$$

and

$$\|\mathcal{D}u\|^2_{L^2(\langle x\rangle^N)} \leqslant \left(1+\frac{2}{a}\right)\|u\|^2_{H^1(\langle x\rangle^N)} + (1+2a)\|u\|^2_{L^2(\langle x\rangle^N)}.$$

from which we deduce the result taking some $a > \frac{1}{2}$ and C large enough. □

Proposition 2.2. *For all N in $\mathbb{R}$, $s > 0$, $\alpha \in \mathbb{N}^3$ and for all $u \in L^2(\langle x\rangle^N)$, if $|\alpha| < s$ the following inequality holds*

$$\left\|D^\alpha H^{-s} u\right\|_{L^2(\langle x\rangle^N)} \lesssim \|u\|_{L^2(\langle x\rangle^N)}.$$

Proof. We prove the statement for $N \in \mathbb{N}$, the rest of the cases will be covered by standard interpolation. By Plancherel theorem, we know that

$$\left\|\langle x\rangle^N D^\alpha H^{-s} u\right\|_{L^2} = \left\|H^N \xi^\alpha \langle\xi\rangle^{-s}\widehat{u}\right\|_{L^2} \lesssim \sum_{|\gamma|\leqslant N}\left\|D^\gamma_\xi \xi^\alpha\langle\xi\rangle^{-s}\widehat{u}\right\|_{L^2}$$
$$\lesssim \sum_{k=0}^{N}\left\|\langle\xi\rangle^{|\alpha|-k-s} D_\xi^{N-k}\widehat{u}\right\|_{L^2} \leqslant \sum_{k=0}^{N}\left\|\langle x\rangle^{N-k} u\right\|_{L^2} \lesssim \left\|\langle x\rangle^N u\right\|_{L^2}$$

and this concludes the proof. □

2.2. Weak dispersive estimates

The aim of this subsection is to prove a weak dispersive estimate for solutions to (7), that is to say, that we prove the following proposition.

Proposition 2.3. *Let $T \in (0, +\infty]$, $N > \frac{3}{2}$ and $s \geqslant 0$. Assume that $V \in C((0,T), H^s \to H^s)$ is such that*

$$\|V\|_{T,s,N} \leqslant \varepsilon, \tag{15}$$

for $\varepsilon > 0$ small enough. Then the following estimate holds

$$\|u\|_{L^2_T H^s(\langle x\rangle^{-N})} \leqslant C(\varepsilon)\|u_0\|_{H^s} \tag{16}$$

for some constant $C(\varepsilon)$ depending on ε.

Remark 2.1. Notice that this result in particular implies

$$\|S_0(t)u_0\|_{L^2_T L^2(\langle x\rangle^{-N})} \lesssim \|u_0\|_{L^2} \tag{17}$$

for any $N > \frac{3}{2}$, as indeed condition (15) is obviously satisfied when $V = 0$

The remaining of this subsection is dedicated to the proof of Proposition 2.3, which is divided into various steps. The first one consists in reducing the problem to the case of the regularity $s = 1$; the second step consists in establishing a virial identity, namely an identity of the form

$$\int_0^t \Theta_1(\psi, v)(\tau)d\tau = \Theta(\psi, v)(t) - \Theta(\psi, v)(0)$$

where the quantities involved depend on v the solution to the Dirac linear equation with time-dependant potential, and a function ψ called a multiplier.

The rest of the proof consists in proving that Θ_1 controls an adequate norm on v to the square given an appropriate (family of) multiplier ψ and that $\Theta(\psi, v)(t)$ is controlled by the norm of the initial datum (regardless of the time t) to the square thanks to the symmetries of the equation.

In the third step, we control the terms appearing in Θ_1 that depend on the potential, and that we will consider as perturbative. The estimate do not depend on the choice of the multiplier ψ but on its norm.

In the fourth step, we control Θ thanks to the initial datum.

In the fifth step, we give out a one-parameter family of multiplier $(\psi_R)_{R>0}$ such that

$$\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}$$

is controlled by the supremum on R of the non perturbative term in Θ_1.

In the sixth and final step, we combine all the previous estimates to prove the proposition.

Proof of Proposition 2.3. Step 1 : Reducing the regularity to $s = 1$. We introduce the function $v = H^{s-1}u$ that satisfies the equation

$$i\partial_t v = \mathcal{D}v + \widetilde{V}v \tag{18}$$

with $\widetilde{V} = H^{s-1}VH^{1-s}$ and $v_0 = H^{s-1}u_0$. Then from $\|V\|_{T,s,N} \leqslant \epsilon$, we infer that

$$\|\widetilde{V}\|_{T,1,N} \leqslant \epsilon.$$

The advantage of using the function v is in that we now aim to prove an estimate at the H^1 level on it (which in fact is the "natural setting" for the weak dispersive estimates with our strategy), and the H^1 norm of v is equivalent to the H^s norm of u.

Step 2 : Virial identity. As it is often the case when dealing with the Dirac equation, in order to build a useful virial identity we consider the squared system, that is

$$-\partial_t^2 v = \mathcal{D}^2 v + \mathcal{D}\widetilde{V}v + i\partial_t(\widetilde{V}v). \tag{19}$$

Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be some real, regular function independent of time to be chosen later; we then set

$$\Theta := 2\mathrm{Re}\,\langle[-\Delta, \psi]v, \partial_t v\rangle + 2\mathrm{Re}\left\langle[-\Delta, \psi]v, i\widetilde{V}v\right\rangle, \tag{20}$$

so that

$$\partial_t \Theta = 2\mathrm{Re}\left\langle [-\Delta,\psi]v, \partial_t^2 v + i\partial_t(\widetilde{V}v)\right\rangle + 2\mathrm{Re}\left\langle [-\Delta,\psi]\partial_t v, i\widetilde{V}v\right\rangle$$

Plugging (19) in the above yields

$$\partial_t \Theta = -2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}^2 v\right\rangle + A + B \tag{21}$$

with

$$A = 2\mathrm{Re}\left\langle [-\Delta,\psi]\partial_t v, i\widetilde{V}v\right\rangle$$

and

$$B = -2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}\widetilde{V}v\right\rangle$$

(identity (21) or its time derivative is often referred to as *virial identity*). Moreover, the following identity holds

$$-2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}^2 v\right\rangle = \left\langle \Delta^2\psi v, v\right\rangle - 4\left\langle \partial_k v, \partial_k\partial_j\psi\partial_j v\right\rangle. \tag{22}$$

Indeed, the proof of (22) is classical, but we include it anyway for the sake of completeness. As $\mathcal{D}^2 = 1-\Delta$, we have

$$2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}^2 v\right\rangle = -2\mathrm{Re}\left\langle [-\Delta,\psi]v, \Delta v\right\rangle.$$

We recall that $[-\Delta,\psi]$ is skew-symmetric, and Δ is self-adjoint. Then we get

$$-2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}^2 v\right\rangle = -\left\langle [-\Delta,\psi]v, \Delta v\right\rangle - \left\langle \Delta v, [-\Delta,\psi]v\right\rangle = \left\langle [\Delta,[\Delta,\psi]]v, v\right\rangle.$$

We have $[\Delta,\psi] = \Delta\psi + 2\nabla\psi\nabla$, which gives

$$[\Delta,[\Delta,\psi]] = \Delta^2\psi + 4\nabla\Delta\psi\cdot\nabla + 4\nabla\otimes\nabla\psi\cdot\nabla\otimes\nabla.$$

We compute

$$a := \left\langle \nabla\otimes\nabla\psi\cdot\nabla\otimes\nabla v, v\right\rangle = \left\langle \partial_j\partial_k\psi\partial_j\partial_k v, v\right\rangle.$$

We use that ψ is real to get

$$a = \left\langle \partial_j\partial_k v, (\partial_j\partial_k\psi)v\right\rangle.$$

We use that ∂_j is skew-symmetric and the Leibniz rule to get

$$a = -\left\langle \partial_k v, \partial_j^2\partial_k\psi v\right\rangle - \left\langle \partial_k v, \partial_j\partial_k\psi\partial_j v\right\rangle.$$

In other words

$$a = -\left\langle \nabla\Delta\psi\cdot\nabla v, v\right\rangle - \left\langle \partial_k v, \partial_j\partial_k\psi\partial_j v\right\rangle.$$

Summing up, we get

$$-2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}^2 v\right\rangle = \left\langle \Delta^2\psi v, v\right\rangle - 4\left\langle \partial_k v, \partial_j\partial_k\psi\partial_j v\right\rangle.$$

Step 3 : Estimating the perturbative terms in the left-hand side regardless of the choice of the multiplier. The following estimates hold

$$\begin{aligned} \|A\|_{L^1_T} &\lesssim \|\psi\|_2\|\widetilde{V}\|_{T,1,N}\,\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})} \\ \|B\|_{L^1_T} &\lesssim \|\psi\|_2\|\widetilde{V}\|_{T,1,N}\,\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}. \end{aligned} \tag{23}$$

where $\|\psi\|_2 = \|\nabla\psi\|_{L^\infty} + \|\Delta\psi\|_{L^\infty}$.

We start with the estimate for the term A. We have

$$A = 2\mathrm{Re}\left\langle [-\Delta,\psi]\partial_t v, i\widetilde{V}v\right\rangle.$$

Recalling that $[-\Delta,\psi] = -\Delta\psi - 2\nabla\psi\cdot\nabla$ is skew-symmetric, we get

$$|A| \leqslant \|\partial_t v\|_{L^2(\langle x\rangle^{-N})}\left\|[-\Delta,\psi]\widetilde{V}v\right\|_{L^2(\langle x\rangle^{N})} \lesssim \|\psi\|_2\|\partial_t v\|_{L^2(\langle x\rangle^{-N})}\left\|\widetilde{V}v\right\|_{H^1(\langle x\rangle^{N})}.$$

Obviously,

$$\left\|\widetilde{V}v\right\|_{H^1(\langle x\rangle^N)} \lesssim \left\|\langle x\rangle^N H\widetilde{V}H^{-1}\langle x\rangle^N\right\|_{L^2\to L^2}\|v\|_{H^1(\langle x\rangle^{-N})}.$$

We now control $\|\partial_t v\|_{L^2(\langle x\rangle^{-N})}$. From the equation on v, we get

$$\|\partial_t v\|_{L^2(\langle x\rangle^{-N})} = \left\|\mathcal{D}v + \widetilde{V}v\right\|_{L^2(\langle x\rangle^{-N})} \leqslant \|\mathcal{D}v\|_{L^2(\langle x\rangle^{-N})} + \left\|\widetilde{V}v\right\|_{L^2(\langle x\rangle^{-N})}$$

from which, by Proposition 2.1, we get

$$\|\partial_t v\|_{L^2(\langle x\rangle^{-N})} \lesssim \|v\|_{H^1(\langle x\rangle^{-N})} + \left\|\langle x\rangle^N H\widetilde{V}H^{-1}\langle x\rangle^N\right\|_{L^2\to L^2}\|v\|_{H^1(\langle x\rangle^{-N})}.$$

Using the fact that $\widetilde{V}$ should be small, we get

$$|A| \lesssim \|\psi\|_2\left\|\langle x\rangle^N H\widetilde{V}H^{-1}\langle x\rangle^N\right\|_{L^2\to L^2}\|v\|^2_{H^1(\langle x\rangle^{-N})}.$$

The Cauchy–Schwarz inequality on the integral on time gives the result for A.

We now estimate the term B. We have

$$B = -2\mathrm{Re}\left\langle [-\Delta,\psi]v, \mathcal{D}\widetilde{V}v\right\rangle.$$

This gives by Cauchy–Schwarz inequality,

$$|B| \lesssim \|\psi\|_2\|v\|_{H^1(\langle x\rangle^{-N})}\left\|\mathcal{D}\widetilde{V}v\right\|_{L^2(\langle x\rangle^N)}.$$

As we have seen previously,

$$\left\|\mathcal{D}\widetilde{V}v\right\|_{L^2(\langle x\rangle^N)} \lesssim \left\|\langle x\rangle^N H\widetilde{V}H^{-1}\langle x\rangle^N\right\|_{L^2\to L^2}\|v\|_{H^1(\langle x\rangle^{-N})}. \tag{24}$$

Using Cauchy–Schwarz inequality on the integral on time, we get the result.

Step 4 : Estimating the right-hand side thanks to the initial datum

The following estimate holds

$$\|\Theta\|_{L^\infty_T} \lesssim \|\psi\|_2\left(\|v_0\|^2_{H^1} + \|\widetilde{V}\|_{T,1,N}\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}\right). \tag{25}$$

First, let us prove that

$$\|\Theta\|_{L^\infty_T} \lesssim \|\psi\|_2\|v\|^2_{L^\infty_T H^1}. \tag{26}$$

Starting from identity (20), by Hölder inequality we get

$$\|\Theta\|_{L^\infty_T} \lesssim \|[-\Delta,\psi]v\|_{L^\infty_T L^2}\left(\|\partial_t v\|_{L^\infty_T L^2} + \left\|\widetilde{V}v\right\|_{L^\infty_T L^2}\right).$$

We have on the one hand

$$\|[\Delta,\psi]v\|_{L^\infty_T L^2} \lesssim \|\psi\|_2\|v\|_{L^\infty_T H^1},$$

and on the other hand, by $\|\widetilde{V}\|_{T,1,N} \leqslant \epsilon$, we have

$$\left\|\widetilde{V}v\right\|_{L^\infty_T L^2} \lesssim \left\|\langle x\rangle^N H\widetilde{V}v\right\|_{L^\infty_T L^2} \leqslant \left\|\widetilde{V}\right\|_{T,1,N}\|\langle x\rangle^{-N}Hv\|_{L^\infty_T L^2} \lesssim \|v\|_{L^\infty_T H^1}.$$

Finally, since $i\partial_t v = \mathcal{D}v + \widetilde{V}v$, we obtain

$$\|\partial_t v\|_{L^\infty_T L^2} \lesssim \|v\|_{L^\infty_T H^1}$$

and thus (26) follows. To conclude, we now prove that

$$\|v\|^2_{L^\infty_T H^1} \lesssim \|v_0\|^2_{H^1} + \|\widetilde{V}\|_{T,1,N}\, \|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}\,.$$

We proceed as usual. Recall that the equation is well-posed in any H^s with propagation of regularity hence the computations below make sense. We differentiate

$$\|v(t)\|^2_{H^1} = \langle Hv, Hv\rangle$$

to get

$$\partial_t \|v(t)\|^2_{H^1} = 2\mathrm{Re}\,\langle Hv, H\partial_t v\rangle = 2\mathrm{Im}\,\langle Hv, Hi\partial_t v\rangle\,.$$

Note that as the Dirac operator $\mathcal{D}$ is self-adjoint on L^2, we get that $\langle Hv, \mathcal{D}Hv\rangle = \langle Hv, \mathcal{D}Hv\rangle = \overline{\langle \mathcal{D}Hv, Hv\rangle}$ is real, so $\mathrm{Im}\,\langle Hv, H\mathcal{D}v\rangle = 0$. Then using the equation on v,

$$\partial_t \|v(t)\|^2_{H^1} = 2\mathrm{Im}\left\langle Hv, H\widetilde{V}v\right\rangle,$$

from which we get

$$\|v(t)\|^2_{L^\infty_T H^1} \leqslant \|v_0\|^2_{H^1} + 2\left\|\left\langle Hv, H\widetilde{V}v\right\rangle\right\|_{L^1_T}.$$

Finally, by the inequality (24) we get the result.

Step 5 : Estimating the norm of the solution thanks to a one parameter family of multipliers Let us now introduce the family of multipliers $(\psi_R)_{R>0}$, which is completely standard in this contest (see e.g. [3]).

For all $R > 0$ we define the radial function ψ_R such that $\psi_R(0) = 0$ and

$$\psi_R'(r) = \begin{cases} \frac{r}{\langle R\rangle} & \text{if } r \leqslant R \\ \frac{R}{\langle R\rangle}\left(\frac{3}{2} - \frac{1}{2}\frac{R^2}{r^2}\right) & \text{if } r > R \end{cases}$$

with $r = |x|$.

The choice of the multipliers and straightforward computations yields the following properties.

We have

$$\Delta\psi_R = \frac{3}{\langle R\rangle}\mathbb{1}_{r\leqslant R} + \frac{R}{\langle R\rangle}\frac{3}{r}\mathbb{1}_{r>R}, \tag{27}$$

$$\Delta^2\psi_R = -\frac{3}{R\langle R\rangle}\delta(r-R), \tag{28}$$

$$\partial_k\partial_j\psi_R = \delta_j^k\frac{\psi_R'}{r} + \mathbb{1}_{r>R}\frac{3R}{2\langle R\rangle}\frac{x_j x_k}{r^3}\left(\frac{R^2}{r^2} - 1\right), \tag{29}$$

$$\|\psi_R\|_2 \leqslant \frac{9}{2}. \tag{30}$$

We have for all $R \geqslant 0$,

$$-2\mathrm{Re}\left\langle[-\Delta, \psi_R]v, \mathcal{D}^2 v\right\rangle \geqslant \frac{3}{R\langle R\rangle}\int_{S_R}|v|^2 + \frac{4}{\langle R\rangle}\int_{B_R}|\nabla v|^2 \tag{31}$$

where S_R is the sphere of radius R and B_R is the ball of radius R.

We have

$$\left\langle -\Delta^2\psi_R v, v\right\rangle = \int \frac{3}{R\langle R\rangle}\delta(r-R)|v|^2 = \frac{3}{R\langle R\rangle}\int_{S_R}|v|^2,$$

and

$$\langle \partial_k v, \partial_k \partial_j \psi_R \partial_j v\rangle = \int_{B_R} \frac{|\nabla v|^2}{\langle R\rangle} + \frac{R}{\langle R\rangle}\int_{B_R^c}\Big[\frac{1}{r}\Big(\frac{3}{2}-\frac{1}{2}\frac{R^2}{r^2}\Big)|\nabla v|^2 + \frac{x_k x_j}{r^3}\frac{3}{2}\Big(\frac{R^2}{r^2}-1\Big)\overline{\partial_k v}\partial_j v\Big].$$

Let

$$a := \int_{B_R^c} \frac{x_k x_j}{r^3}\frac{3}{2}\Big(\frac{R^2}{r^2}-1\Big)\overline{\partial_k v}\partial_j v.$$

We have

$$a = \int_{B_R^c} \frac{\overline{x_k \partial_k v} x_j \partial_j v}{r^3}\frac{3}{2}\Big(\frac{R^2}{r^2}-1\Big).$$

Because $\frac{R^2}{r^2}-1$ is negative, we get

$$a \geqslant \int_{B_R^c} \frac{|\nabla v|^2}{r}\frac{3}{2}\Big(\frac{R^2}{r^2}-1\Big).$$

We now sum up and get

$$\langle \partial_k v, \partial_k \partial_j \psi_R \partial_j v\rangle \geqslant \int_{B_R} \frac{|\nabla v|^2}{\langle R\rangle} + \frac{R}{\langle R\rangle}\int_{B_R^c}\frac{1}{r}\Big(\frac{3}{2}\frac{R^2}{r^2}-\frac{1}{2}\frac{R^2}{r^2}\Big)|\nabla v|^2.$$

From the positivity of $\int_{B_R^c}\frac{1}{r}\Big(\frac{3}{2}\frac{R^2}{r^2}-\frac{1}{2}\frac{R^2}{r^2}\Big)|\nabla v|^2$, we get

$$\langle \partial_k v, \partial_k \partial_j \psi_R \partial_j v\rangle \geqslant \int_{B_R} \frac{|\nabla v|^2}{\langle R\rangle}.$$

We have for any $\alpha > \frac{3}{2}$ and $\beta > \frac{1}{2}$,

$$\|v\|^2_{L^2_T L^2_x(\langle x\rangle^{-\alpha})} \lesssim \sup_R \int_{-T}^{T} \frac{1}{R\langle R\rangle}\int_{S_R}|v|^2, \tag{32}$$
$$\|\nabla v\|^2_{L^2_T L^2_x(\langle x\rangle^{-\beta})} \lesssim \sup_R \int_{-T}^{T} \frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2.$$

Let $w(x) = \int_{-T}^{T}|v|^2(x)$. We have

$$\|v\|^2_{L^2_T L^2_x(\langle x\rangle^{-\alpha})} = \int \frac{w}{\langle |x|\rangle^{2\alpha}}\,\mathrm{d}x = \int \mathrm{d}r\,\langle r\rangle^{-2\alpha}\int_{S_r} w$$

from which we get

$$\|v\|^2_{L^2(\langle x\rangle^{-\alpha})} \leqslant \int dr \frac{r\langle r\rangle}{\langle r\rangle^{2\alpha}}\sup_R \frac{1}{R\langle R\rangle}\int_{S_R} w.$$

Since $\alpha > \frac{3}{2}$, we have $2\alpha - 2 > 1$ and thus $\frac{r\langle r\rangle}{\langle r\rangle^{2\alpha}}$ is integrable which gives the first result.

Let $z = \int_{-T}^{T}|\nabla v|^2$, we have

$$\|\nabla v\|^2_{L^2(\langle x\rangle^{-\beta})} = \int \frac{z}{\langle r\rangle^{2\beta}} = \int dr \frac{1}{\langle r\rangle^{2\beta}}\int_{S_r} z.$$

We write $\frac{1}{\langle r\rangle^{2\beta}} = \int_r^{+\infty} 2\beta \frac{\tau}{\langle \tau\rangle^{2\beta+2}}d\tau$. We get

$$\|\nabla v\|^2_{L^2(\langle x\rangle^{-\beta})} = 2\int d\tau \frac{\tau}{\langle \tau\rangle^{2\beta+2}}\int_{B_\tau} z,$$

and thus

$$\|\nabla v\|^2_{L^2(\langle x\rangle^{-\beta})} = 2\int d\tau \frac{\tau}{\langle\tau\rangle^{2\beta+1}}\frac{1}{\langle\tau\rangle}\int_{B_\tau} z,$$

from which we deduce

$$\|\nabla v\|^2_{L^2(\langle x\rangle^{-\beta})} \leqslant 2\int d\tau \frac{\tau}{\langle\tau\rangle^{2\beta+1}}\sup_R \frac{1}{\langle R\rangle}\int_{B_R} z.$$

Since $2\beta > 1$, we get that $\frac{\tau}{\langle\tau\rangle^{2\beta+1}}$ is integrable from which we can conclude.

What we deduce from all the different estimates involved in this step is that with our choice of family of multipliers, we have

$$\|v\|_{L^2_T H^1(\langle x\rangle^{-N})} \lesssim \sup_R \Big(-2\mathrm{Re}\,\big\langle[-\Delta,\psi_R]v,\mathcal{D}^2 v\big\rangle\Big). \tag{33}$$

Step 6 : Combining all previous steps. Let $N > \frac{3}{2}$, and assume that

$$\|\widetilde{V}\|_{T,1,N} \leqslant \varepsilon$$

for some constant ε small enough. Then the following estimate holds

$$\|v\|_{L^2_T H^1(\langle x\rangle^{-N})} \leqslant C(\varepsilon)\|v_0\|_{H^1}.$$

Recall that Θ is defined by (20). Here we choose $\psi = \psi_R$, then we use $\Theta_R = \Theta$ for this case.

We specialize Θ_R with our choice of $\psi = \psi_R$. We have

$$\int_{-T}^{T} \partial_t \Theta_R = \Theta_R(T) - \Theta_R(-T). \tag{34}$$

On the one hand we have

$$\int_{-T}^{T} \partial_t \Theta_R \geqslant -\int_{-T}^{T} 2\mathrm{Re}\,\big\langle[-\Delta,\psi_R]v,\mathcal{D}^2 v\big\rangle - \|A\|_{L^1_T} - \|B\|_{L^1_T}\,. \tag{35}$$

and on the other hand

$$\Theta_R(T) - \Theta_R(-T) \leqslant 2\,\|\Theta_R\|_{L^\infty_T}\,. \tag{36}$$

By Eq. (31) we have

$$-\int_{-T}^{T} 2\mathrm{Re}\,\big\langle[-\Delta,\psi_R]v,\mathcal{D}^2 v\big\rangle \geqslant \int_{-T}^{T}\Big(3\frac{1}{R\,\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big). \tag{37}$$

Given the bounds on A and B (23) and combining (35)–(37) and (25), there exists a constant $C > 0$ such that

$$\begin{aligned} &C\,\|\psi_R\|_2\,\|v_0\|^2_{H^1} \\ &\quad\geqslant \int_{-T}^{T}\Big(3\frac{1}{R\,\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big) - C\,\|\psi_R\|_2\,\|\widetilde{V}\|_{T,1,N}\,\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}\,. \end{aligned}$$

Since $\|\psi_R\|_2$ is uniformly bounded in R, we get a constant $C_2 > 0$ such that

$$C_2\,\|v_0\|^2_{H^1} \geqslant \int_{-T}^{T}\Big(3\frac{1}{R\,\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big) - C_2\|\widetilde{V}\|_{T,1,N}\,\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}\,.$$

Let $0 < \varepsilon \leqslant 1$, and let us assume that $\|\widetilde{V}\|_{T,1,N} \leqslant \varepsilon$. Thus,

$$\int_{-T}^{T}\Big(3\frac{1}{R\,\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big) - C_2\varepsilon\,\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})} \leqslant C_2\,\|v_0\|^2_{H^1}\,.$$

We take the sup in R and we get

$$\sup_R \int_{-T}^{T} \Big(3\frac{1}{R\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big) - C_2\varepsilon \|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})} \leqslant C_2\|v_0\|^2_{H^1}. \tag{38}$$

As $N > \frac{3}{2}$, from Eq. (32), we have

$$\|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})} \lesssim \sup_R \int_{-T}^{T} \Big(3\frac{1}{R\langle R\rangle}\int_{S_R}|v|^2 + 4\frac{1}{\langle R\rangle}\int_{B_R}|\nabla v|^2\Big). \tag{39}$$

Plugging (39) into (38), and taking ε small enough, we get that there exists a constant $C(\varepsilon)$ dependent on ε such that

$$\|v\|_{L^2_T H^1(\langle x\rangle^{-N})} \leqslant C(\varepsilon)\|v_0\|_{H^1}.$$

We use the fact that $\widetilde{V} = H^{s-1}VH^{1-s}$ and $v = H^{s-1}u$ to conclude estimate (16), as indeed

$$\|V\|_{T,s,N} = \|\widetilde{V}\|_{T,1,N},$$

and

$$\|u\|^2_{L^2_T H^s(\langle x\rangle^{-N})} = \|v\|^2_{L^2_T H^1(\langle x\rangle^{-N})}, \quad \|u_0\|_{H^s} = \|v_0\|_{H^1}. \quad \square$$

2.3. Strichartz estimates

We are in a position for proving Strichartz estimates for solutions to (7).

Proof of Theorem 1.3. By Duhamel's formula, we know that

$$u(t) = S_0(t)u_0 - i\int_0^t S_0(t-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau.$$

We prove (10): we write

$$\|u\|_{L^p_T L^q} = \|S_V(t)u_0\|_{L^p_T L^q} \leqslant \|S_0(t)u_0\|_{L^p_T L^q} + \left\|\int_0^t S_0(t-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau\right\|_{L^p_T L^q}.$$

Thanks to the Christ–Kiselev Lemma (see [8]), since we are only interested in the non-endpoint case ($p > 2$) it is sufficient to estimate the untruncated integral

$$\int S_0(t-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau = S_0(t)\int S_0(-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau.$$

As (p,q,s) is Dirac admissible, according to Definition 1.2 we get

$$\left\|S_0(t)\int_0^T S_0(-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau\right\|_{L^p_T L^q} \lesssim \left\|\int_0^T S_0(-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau\right\|_{H^s}.$$

Now, we use the dual form of estimate (17) to obtain

$$\begin{aligned}\left\|\int_0^T S_0(-\tau)V(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau\right\|_{H^s} &\leqslant \left\|\int_0^T S_0(-\tau)H^sV(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\tau\right\|_{L^2} \\ &\lesssim \left\|\langle x\rangle^N H^s(Vu)\right\|_{L^2_T L^2_x}.\end{aligned}$$

Hence by Proposition 2.3 and the assumption

$$\|V\|_{T,s,N} \leqslant \varepsilon,$$

we finally get

$$\begin{aligned}\|S_V u_0\|_{L^p_T L^q} &\lesssim \|u_0\|_{H^s} + \left\| \langle x\rangle^N H^s V H^{-s} \langle x\rangle^N \langle x\rangle^{-N} H^s u \right\|_{L^2_T L^2_x} \\ &\lesssim \|u_0\|_{H^s} + \|V\|_{T,s,N} \|u\|_{L^2_T H^s(\langle x\rangle^{-N})} \lesssim \|u_0\|_{H^s}\end{aligned}$$

and this concludes the proof of (10). Estimate (9) can be proved in much the same way, using also the fact that $\|S_0(t)u\|_{H^s} = \|u\|_{H^s}$. □

We also have some form of continuity in the operator V in the sense of the following proposition.

Proposition 2.4. *Let (p, r, s) be a Dirac admissible triple as given by Definition 1.2 and $T \in (0, +\infty]$. Let $N > 3/2$ and let V_1, V_2 be two operators belonging to $C((0,T), H^s \to H^s)$ such that*

$$\|V_j\|_{T,s,N} \ll 1.$$

for $j = 1, 2$. Let $u_0 \in H^s$. Then the following bounds hold:

$$\|S_{V_1}(t)u_0 - S_{V_2}(t)u_0\|_{L^p_T L^q} \lesssim \|V_1 - V_2\|_{T,s,N} \|u_0\|_{H^s}, \tag{40}$$

$$\|S_{V_1}(t)u_0 - S_{V_2}(t)u_0\|_{L^\infty_T H^s} \lesssim \|V_1 - V_2\|_{T,s,N} \|u_0\|_{H^s}. \tag{41}$$

Proof. We prove (40). Setting $u_j(t) = S_{V_j}(t)u_0$ for $j = 1, 2$, from Duhamel's formula we get

$$\begin{aligned}u_1(t) - u_2(t) = &- i \int_0^t S_0(t-\tau)V_1(\tau)\Big(u_1(\tau) - u_2(\tau)\Big)d\tau \\ &- i \int_0^t S_0(t-\tau)\Big(V_1(\tau) - V_2(\tau)\Big)u_2(\tau)d\tau.\end{aligned}$$

By (17) we get

$$\|u_1 - u_2\|_{L^2_T H^s(\langle x\rangle^{-N})} \lesssim \|V_1\|_{T,s,N} \|u_1 - u_2\|_{L^2_T H^s(\langle x\rangle^{-N})} + \|V_1 - V_2\|_{T,s,N} \|u_2\|_{L^2_T H^s(\langle x\rangle^{-N})}.$$

where we have used the fact that $H^s S_0(t) = S_0(t)H^s$. Taking V_1 and V_2 small enough and using local smoothing on S_{V_2}, we get

$$\|u_1 - u_2\|_{L^2_T H^s(\langle x\rangle^{-N})} \lesssim \|V_1 - V_2\|_{T,s,N} \|u_0\|_{H^s}.$$

Finally, using the same strategy as in the previous proof, we get

$$\|u_1 - u_2\|_{L^p_T L^r} \lesssim \|V_1\|_{T,s,N} \|u_1 - u_2\|_{L^2_T H^s(\langle x\rangle^{-N})} + \|V_1 - V_2\|_{T,s,N} \|u_0\|_{H^s}$$

and we conclude using the first inequality we proved. The proof of (41) follows the same lines, the only difference being that we use estimate (9) instead of (10). □

3. Properties of the solution to the Klein–Gordon equation

This section is devoted to the study of the Klein–Gordon equation on W:

$$\begin{cases} \partial_{tt}W + W - \Delta W = \chi(x - q(t)), & W(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R} \\ \quad W(0,x) = w_0, \quad \partial_t W(0,x) = w_1 \end{cases} \tag{42}$$

Our aim is provide some suitable estimates on W in view of proving well-posedness for systems (1) and (11).

As mentioned in the introduction, the main idea here consists in splitting W into a "dispersive part" (that will be given by the sum of two terms, W_2 and W_3) and a "non-dispersive part" (that will be denoted by W_1). When dealing with the study of the well-posedness for system (1), the idea is that W_1 will be treated as a perturbation of the free Dirac equation, and we will be able to evoke Theorem 1.3 to deduce Strichartz estimates, while W_2 and W_3, which enjoy their own dispersive estimates, will be regarded as inhomogeneous terms. Therefore, to be more precise, on the one hand, we will need to ensure ourselves that the term W_1 satisfies condition (8): Proposition 3.5 goes in this direction (see also Section 4.1). On the other hand, for the remaining terms, it will be enough to prove that

$$\|W_2 + W_3\|_{L^1_T H^{s,\infty}} < +\infty, \tag{43}$$

and this will be done in Propositions 3.6 and 3.7. In addition, in view of the proof of Theorem 1.4, we also need the continuity of W with respect to q and its derivatives, which will appear in the same propositions.

More precisely, we prove the following proposition.

Proposition 3.1. *Let $T \in (0, +\infty]$ and $s \in [1,2]$. Assume that $q \in W^{1,+\infty}_T$ and $\|\ddot{q}\|_{L^1_T} \leqslant \frac{1}{2}$. Provided that*

$$\chi \in L^\infty \cap W^{s+2,1}, \quad w_0 \in W^{s+3,1}, \quad w_1 \in W^{s+2,1},$$

the unique solution to the Klein–Gordon equation

$$\partial_t^2 W + W - \Delta W = \chi(x - q(t)) \tag{44}$$

writes

$$W = W_1(q) + \tilde{W}_2(q)$$

such that $W_1(q) \in C^0_T L^\infty$ and the multiplication by the function $W_1(q)$ satisfies

$$\|W_1(q)\|_{T,s,N} \lesssim \left\langle \|q\|_{L^\infty_T} \right\rangle^{2N} \| \langle x \rangle^{2N} \chi\|_{L^\infty}$$

and $\tilde{W}_2(q) \in L^1_T H^{s,\infty}$ with

$$\|\tilde{W}_2(q)\|_{L^1_T H^{s,\infty}} \lesssim \|w_0\|_{W^{s+3,1}} + \|w_1\|_{W^{s+2,1}} + \|\chi\|_{W^{s+2,1}}.$$

What is more, under the same assumptions for q_1 and q_2 as for q, we have the continuity estimates :

$$\|W_1(q_1) - W_1(q_2)\|_{T,s,N} \lesssim \left(\left\langle \|q_1\|_{L^\infty_T} \right\rangle^{2N} + \left\langle \|q_2\|_{L^\infty_T} \right\rangle^{2N} \right) \| \langle x \rangle^{2N} \chi\|_{L^\infty} \|q_1 - q_2\|_{W^{1,\infty}_T}$$
$$\|\tilde{W}_2(q_1) - \tilde{W}_2(q_2)\|_{L^1_T H^{s,\infty}} \lesssim \|\chi\|_{W^{s,1}} \|q_1 - q_2\|_{W^{2,1}_T}.$$

3.1. Decomposition of W

It is well known indeed that, by Duhamel's formula, W can be written as

$$W(t,x)=\cos(\sqrt{1-\Delta}t)w_0+\frac{\sin(\sqrt{1-\Delta}t)}{\sqrt{1-\Delta}}w_1-\int_0^t\frac{\sin(\sqrt{1-\Delta}(t-\tau))}{\sqrt{1-\Delta}}\chi(x-q(\tau))\,\mathrm{d}\tau. \tag{45}$$

In our case, it is possible to provide a much more explicit representation of the solution:

Proposition 3.2. *Let W solve the Klein–Gordon equation*

$$\partial_{tt}W+W-\Delta W=\chi(x-q(t)).$$

Then it is possible to decompose W as follows

$$W(q,\dot q,\ddot q)(t,x)=W_1(q,\dot q)(t,x)+W_2(t,x)+W_3(q,\dot q,\ddot q)(t,x)$$

with

$$W_1(q,\dot q)(t,x):=\chi_1(\dot q,x-q(t)),$$
$$W_2(t,x):=\cos(\sqrt{1-\Delta}t)w_0+\frac{\sin(\sqrt{1-\Delta}t)}{\sqrt{1-\Delta}}w_1+\frac{\cos(\sqrt{1-\Delta}t)}{1-\Delta}\chi(x),$$

and

$$\begin{aligned}&W_3(q,\dot q,\ddot q)(t,x):=\\&\int_0^t\left(e^{i\sqrt{1-\Delta}(t-\tau)}\chi_2(q(\tau),\dot q(\tau),\ddot q(\tau))-e^{-i\sqrt{1-\Delta}(t-\tau)}\chi_3(q(\tau),\dot q(\tau),\ddot q(\tau))\right)\mathrm{d}\tau,\end{aligned}$$

where

$$\begin{aligned}\widehat{\chi}_1(\dot q,\xi)=&\frac{\widehat{\chi}(\xi)}{\langle\xi\rangle^2+(i\xi\cdot\dot q(t))^2},\\ \widehat{\chi}_2(q,\dot q,\ddot q,\xi)=&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{-i\xi\cdot q(t)}\frac{i\xi\cdot\ddot q(t)}{(-i\langle\xi\rangle-i\xi\cdot\dot q(t))^2},\\ \widehat{\chi}_3(q,\dot q,\ddot q,\xi)=&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{-i\xi\cdot q(t)}\frac{i\xi\cdot\ddot q(t)}{(i\langle\xi\rangle-i\xi\cdot\dot q(t))^2}.\end{aligned}$$

Proof. First of all, let

$$U=\int_0^t\frac{\sin(\sqrt{1-\Delta}(t-\tau))}{\sqrt{1-\Delta}}\chi(x-q(\tau))\,\mathrm{d}\tau.$$

We pass in Fourier variables to obtain

$$\begin{aligned}\widehat{U}=&\int_0^t\frac{\sin(\langle\xi\rangle(t-\tau))}{\langle\xi\rangle}\widehat{\chi}(\xi)e^{-i\xi\cdot q(\tau)}\,\mathrm{d}\tau\\=&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{i\langle\xi\rangle t}\int_0^t e^{-i\langle\xi\rangle\tau-i\xi\cdot q(\tau)}\,\mathrm{d}\tau-\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{-i\langle\xi\rangle t}\int_0^t e^{i\langle\xi\rangle\tau-i\xi\cdot q(\tau)}\,\mathrm{d}\tau.\end{aligned}$$

Let

$$I_\pm=\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{\pm i\langle\xi\rangle t}\int_0^t e^{\mp i\langle\xi\rangle\tau-i\xi\cdot q(\tau)}\,\mathrm{d}\tau;$$

then

$$\widehat{U}=I_+-I_-.$$

Integrating by parts, we get

$$\begin{aligned}I_\pm =&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{\pm i\langle\xi\rangle t}\int_0^t \frac{\mp i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau)}{\mp i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau)}e^{\mp i\langle\xi\rangle\tau - i\xi\cdot q(\tau)}\,\mathrm{d}\tau\\ =&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{\pm i\langle\xi\rangle t}\left(\frac{e^{\mp i\langle\xi\rangle\tau - i\xi\cdot q(t)}}{\mp i\langle\xi\rangle - i\xi\cdot\dot{q}(t)} \pm \frac{1}{\langle\xi\rangle} - \int_0^t e^{\mp i\langle\xi\rangle\tau - i\xi\cdot q(\tau)}\frac{-i\xi\cdot\ddot{q}(\tau)}{(\pm i\langle\xi\rangle - \xi\cdot\dot{q}(\tau))^2}\right)\\ =&\frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}\left(\frac{e^{-i\xi\cdot q(\tau)}}{\mp i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau)} \pm \frac{e^{\pm i\langle\xi\rangle t}}{i\langle\xi\rangle} - \int_0^t e^{\pm i\langle\xi\rangle(t-\tau) - i\xi\cdot q(\tau)}\frac{-i\xi\cdot\ddot{q}(\tau)}{(\mp i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau))^2}\right).\end{aligned}$$

Computing $I_+ - I_-$, we get

$$\frac{e^{-i\xi\cdot q(\tau)}}{-i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau)} - \frac{e^{i\xi\cdot q(\tau)}}{-i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau)} = e^{-i\xi\cdot q(\tau)}\frac{2i\langle\xi\rangle}{\langle\xi\rangle^2 - (\xi\cdot\dot{q}(\tau))^2}$$

and

$$\frac{e^{i\langle\xi\rangle t}}{i\langle\xi\rangle} + \frac{e^{-i\langle\xi\rangle t}}{i\langle\xi\rangle} = -2i\frac{\cos(\langle\xi\rangle t)}{\langle\xi\rangle}$$

from which we get

$$\begin{aligned}\widehat{U} =& \frac{\widehat{\chi}(\xi)e^{-i\xi\cdot q(\tau)}}{\langle\xi\rangle^2 + (i\xi\cdot\dot{q}(\tau))^2} - \frac{\widehat{\chi}(\xi)\cos(\langle\xi\rangle t)}{\langle\xi\rangle^2}\\ &+ \int_0^t \frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{-i\xi\cdot q(\tau)}\left(\frac{i\xi\cdot\ddot{q}(\tau)e^{i\langle\xi\rangle(t-\tau)}}{(-i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau))^2} - \frac{i\xi\cdot\ddot{q}(\tau)e^{-i\langle\xi\rangle(t-\tau)}}{(i\langle\xi\rangle - i\xi\cdot\dot{q}(\tau))^2}\right)\mathrm{d}\tau\end{aligned}$$

and this concludes the proof. □

3.2. Some auxiliary operators and technical preliminaries

We now introduce some operators based on translation and scaling that will play an important role in this section.

Let $v\in\mathbb{R}^3$ with $0<|v|<1$, we define the operator $L_v:\mathbb{R}^3\to\mathbb{R}^3$ as

$$L_v x := \frac{1}{\sqrt{1-|v|^2}}\frac{v\cdot x}{v\cdot v}v + \left(x - \frac{v\cdot x}{v\cdot v}v\right) = \frac{1}{\sqrt{1-|v|^2}}P_v x + P_v^\perp x.$$

This operator is clearly invertible, and

$$L_v^{-1}x := \sqrt{1-|v|^2}\frac{v\cdot x}{v\cdot v}v + \left(x - \frac{v\cdot x}{v\cdot v}v\right) = \sqrt{1-|v|^2}P_v x + P_v^\perp x.$$

In particular when $v=0$ we define $L_0 x = L_0^{-1}x = x$. Based on L_v, we also define the operator $\mathcal{L}_v$ and its inverse as follows:

$$\mathcal{L}_v f(x) = f(L_v x),\quad \mathcal{L}_v^{-1}f(x) = f(L_v^{-1}x).$$

Notice that

$$\mathcal{L}_v(fg) = (\mathcal{L}_v f)(\mathcal{L}_v g).$$

Finally, we define the operators

$$(-\Delta_v)^{s/2} = \mathcal{L}_v(-\Delta)^{s/2}\mathcal{L}_v^{-1} = \left((1-|v|^2)\left|\frac{v\cdot\nabla}{v\cdot v}v\right|^2 + \left|\nabla - \frac{v\cdot\nabla}{v\cdot v}\cdot v\right|^2\right)^{s/2}$$

and

$$H_v^s f = \mathcal{L}_v H^s\mathcal{L}_v^{-1}f = (1-\Delta_v)^{s/2}f. \tag{46}$$

Indeed, we have

$$\nabla = \frac{v \cdot \nabla}{v \cdot v} v + (\nabla - \frac{v \cdot \nabla}{v \cdot v} v),$$

so

$$\mathcal{L}_v \nabla \mathcal{L}_v^{-1} = \sqrt{1-|v|^2} \frac{v \cdot \nabla}{v \cdot v} v + (\nabla - \frac{v \cdot \nabla}{v \cdot v} v),$$

and

$$-\Delta_v = -(\mathcal{L}_v \nabla \mathcal{L}_v^{-1}) \cdot (\mathcal{L}_v \nabla \mathcal{L}_v^{-1}) = (1-|v|^2) \left| \frac{v \cdot \nabla}{v \cdot v} v \right|^2 + \left| \nabla - \frac{v \cdot \nabla}{v \cdot v} v \right|^2 .$$

For the fractional Laplacian operator $(-\Delta)^{s/2}$ and any function $f \in L^2$, we also have

$$(-\Delta_v)^{s/2} f = \left((1-|v|^2) \left| \frac{v \cdot \nabla}{v \cdot v} v \right|^2 + \left| \nabla - \frac{v \cdot \nabla}{v \cdot v} \cdot v \right|^2 \right)^{s/2} f$$

It is not difficult to see that for $0 < |v| \leqslant \frac{1}{2}$,

$$\|(-\Delta_v)^{s/2} u\|_{L^2} \lesssim \|(-\Delta)^{s/2} u\|_{L^2} \lesssim \|(-\Delta_v)^{s/2} u\|_{L^2} \tag{47}$$

and

$$\|H_v^s u\|_{L^2} \lesssim \|H^s u\|_{L^2} \lesssim \|H_v^s u\|_{L^2}. \tag{48}$$

Letting $y = L_v^{-1} x$ we get

$$\begin{aligned} \|\mathcal{L}_v^{-1} f\|_{L^p}^p &= \int |\mathcal{L}_v^{-1} f(x)|^p \,\mathrm{d}x = \int |\mathcal{L}_v^{-1} f(L_v y)|^p \,\mathrm{d}L_v y \\ &= \int |f(y)|^p \,\mathrm{d}L_v y = \frac{1}{\sqrt{1-|v|^2}} \|f\|_{L^p}^p. \end{aligned}$$

Thus, for any $1 \leqslant p \leqslant +\infty$ and $|v| < \frac{1}{2}$, we have

$$\|f\|_{L^p} \lesssim \|\mathcal{L}_v^{-1} f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{49}$$

We remark that the functions χ_1, χ_2 and χ_3 can be seen as convolution terms:

Lemma 3.3. *Let $Y(x) = \frac{e^{-|x|}}{4\pi|x|}$, $Z(x) = e^{-|x|}$ and let K_1 be the modified Bessel function of the second kind and order 1. For any $v \in \mathbb{R}^3$ with $|v| < 1$, up to some multiplicative constants we have:*

(1) $\mathscr{F}\big(\frac{1}{\langle\xi\rangle^2 - (\xi\cdot v)^2}\big) = \frac{1}{\sqrt{1-|v|^2}} Y(L_v x) \in W^{1,1}(\mathbb{R}^3)$
(2) $\mathscr{F}\big(\frac{1}{(\langle\xi\rangle^2 - (\xi\cdot v)^2)^2}\big) = \frac{1}{\sqrt{1-|v|^2}} Z(L_v x) \in W^{3,1}(\mathbb{R}^3)$,
(3) $\mathscr{F}\big(\frac{1}{\langle\xi\rangle}\big) = \frac{K_1(|x|)}{|x|} \in L^1(\mathbb{R}^3)$.

Proof. To compute the Fourier transforms we use the identity

$$\int_{\mathbb{R}^3} f(|x|^2) e^{ix\cdot p} dx = \frac{2\pi}{i|p|} \int_{-\infty}^{\infty} r f(r^2) e^{ir|p|} dr. \tag{50}$$

By Cauchy's residue formula, we easily find that the Fourier transform of $(1+|\xi|^2)^{-1}$ (resp. $(1+|\xi|^2)^{-2}$) is $e^{-|x|}/(4\pi|x|)$ (resp. $Ce^{-|x|}$ for some $C > 0$). Both functions are integrable. Furthermore, $Y(x) \in W^{1,1}$ and $Z(x) \in W^{3,1}$. We get the formula of $\mathscr{F}(\langle\xi\rangle^{-1})$ by showing that $|x|\mathscr{F}(\langle\xi\rangle^{-1})$ satisfies the same ODE as K_1's. We recall that $K_1(r)$ has exponential decay and diverges at $r = 0$ with singularity $\frac{1}{r}$ [16, Sections 3.71 and 7.23]. Thus $\frac{K_1(|x|)}{|x|}$ is in L^1.

Now, for any ξ, we have the decomposition $\xi = P_v\xi + P_v^{\perp}\xi$. Setting $z_1 = P_v\xi$, and $(z_2, z_3) = P_v^{\perp}\xi$, we have

$$\langle\xi\rangle^2 - (\xi\cdot v)^2 = 1 + (1-|v|^2)|P_v\xi|^2 + |P_v^{\perp}\xi|^2. \tag{51}$$

Changing the variables, we conclude by the dilation formula for the Fourier transform:

$$\begin{aligned}&\mathscr{F}\left(\frac{1}{\langle\xi\rangle^2 - (\xi\cdot v)^2}\right) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{\langle\xi\rangle^2 - (\xi\cdot v)^2}e^{i\xi\cdot x}\,\mathrm{d}\xi\\ =&\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{1+(1-|v|^2)z_1^2+z_2^2+z_3^2}e^{i(z_1\cdot P_v x + (z_2,z_3)\cdot P_v^{\perp}x)}\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\\ =&\frac{1}{\sqrt{1-|v|^2}}Y\Big(\frac{1}{\sqrt{1-|v|^2}}P_v x + P_v^{\perp}x\Big).\end{aligned}$$

We get the Fourier transform of $\frac{1}{(\langle\xi\rangle^2-(\xi\cdot v)^2)^2}$ in a similar fashion. □

The estimates on the W_1 part of the solution to the Klein–Gordon equation requires that we identify a function and the operator that consists in multiplying by this function. We have the following relationship between the norms of these two objects.

Lemma 3.4. *Let $N \geqslant 0$, $s \in \mathbb{R}$ and $v \in \mathbb{R}^3$, $|v| \leqslant 1/2$. Then for any function $V : \mathbb{R}^3 \to \mathbb{C}$, we have the following bound:*

$$\left\|\langle x\rangle^N H^s V H^{-s}\langle x\rangle^N\right\|_{L^2\to L^2} \lesssim \left\|H_v^s\langle x\rangle^{2N} V\right\|_{L^\infty},$$

where H_ν is defined in (46).

We postpone the proof to Appendix B.

3.3. Estimates on W

Now, in view of applying a contraction argument to prove the well-posedness for our differential systems, we need to provide some estimates on the terms W_j, $j = 1, 2, 3$. The idea is that to deal with the term W_1 we will make use of Theorem 1.3, and thus we will check that the potential W_1 satisfies the necessary conditions, while for the terms W_2 and W_3 we will exploit their own dispersive properties driven by the Klein–Gordon flow.

We estimate the terms W_j one by one.

Proposition 3.5 (*Estimates on W_1*)**.** *Let $N \geqslant 0$, $T \in (0, +\infty]$ and $s \in [0, 2]$. If $|\dot q| \leqslant \frac{1}{2}$, and $q \in L_T^\infty$, then*

$$\|W_1\|_{T,s,N} \lesssim \left\langle \|q\|_{L_T^\infty}\right\rangle^{2N}\left\|\langle x\rangle^{2N}\chi\right\|_{L^\infty}.$$

What is more, let q_1 and q_2 in $W_T^{1,\infty}$, and assume that $\|\dot q_1\|_{L_T^\infty}, \|\dot q_2\|_{L_T^\infty}$ are less than $\frac{1}{2}$. We have

$$\begin{aligned}&\|W_1(q_1,\dot q_1) - W_1(q_2,\dot q_2)\|_{T,s,N}\\ &\lesssim \left(\left\langle\|q_1\|_{L_T^\infty}\right\rangle^{2N} + \left\langle\|q_2\|_{L_T^\infty}\right\rangle^{2N}\right)\left\|\langle x\rangle^{2N}\nabla\chi\right\|_{L^\infty}\|q_1-q_2\|_{W_T^{1,\infty}}.\end{aligned} \tag{52}$$

Proof. Because of Lemma 3.4, it is sufficient to estimate

$$\left\|H_{\dot q}^s\langle x\rangle^{2N}W_1\right\|_{L_T^\infty L^\infty}.$$

If suffices to prove the case $s = 2$ and $s = 0$, and the conclusion follows from the standard interpolation.

From definition of W_1 and (51), we have $W_1 = H_{\dot q}^{-2}\tau_q\chi$ with $\tau_q(\chi)(x) = \chi(x-q)$. By definition of $H^s_{\dot q}$ and (49), it suffices to estimate $\|H^s \left\langle L_{\dot q}^{-1}x\right\rangle^{2N} H^{-2}\mathcal{L}_{\dot q}^{-1}\tau_q\chi\|_{L^\infty_T L^\infty}$ for $s\in[0,2]$. For the case $s=0$, thanks to Lemma 3.3, we have

$$\begin{aligned}
&\| \left\langle L_{\dot q}^{-1}x\right\rangle^{2N} H^{-2}\mathcal{L}_{\dot q}^{-1}\tau_q\chi\|_{L^\infty_T L^\infty}\\
&\quad= \left\|\left\langle L_{\dot q}^{-1}x\right\rangle^{2N}\int_{\mathbb{R}^3} Y(y)\mathcal{L}_{\dot q}^{-1}\tau_q\chi(x-y)\,\mathrm{d}y\right\|_{L^\infty_T L^\infty}\\
&\quad\lesssim \int_{\mathbb{R}^3}\left\|Y(y)\left\langle L_{\dot q}^{-1}(x-y)\right\rangle^{2N}\mathcal{L}_{\dot q}^{-1}\chi(x-y-q(t))\right\|_{L^\infty_T L^\infty}\mathrm{d}y\\
&\qquad+\int_{\mathbb{R}^3}\left\|\left\langle L_{\dot q}^{-1}(y)\right\rangle^{2N}Y(y)\mathcal{L}_{\dot q}^{-1}\chi(x-y-q(t))\right\|_{L^\infty_T L^\infty}\mathrm{d}y\\
&\quad\lesssim \|\left\langle x-q\right\rangle^{2N}\chi\|_{L^\infty}\lesssim\left\langle\|q\|_{L^\infty_T}\right\rangle^{2N}\|\left\langle x\right\rangle^{2N}\chi\|_{L^\infty}
\end{aligned}\tag{53}$$

where as $|x|\lesssim|\mathcal{L}_{\dot q}^{-1}x|\lesssim|x|$ the second inequality holds. On the other hand, by Leibniz rule, we have

$$\begin{aligned}
H^2\left\langle L_{\dot q}^{-1}x\right\rangle^{2N}H^{-2}\mathcal{L}_{\dot q}^{-1}\tau_q\chi &= (-\Delta\left\langle L_{\dot q}^{-1}x\right\rangle^{2N})H^{-2}\mathcal{L}_{\dot q}^{-1}\tau_q\chi\\
&\quad-2\nabla\left\langle L_{\dot q}^{-1}x\right\rangle^{2N}\cdot\int_{\mathbb{R}^3}\nabla Y(y)\mathcal{L}_{\dot q}^{-1}\tau_q\chi(x-y)\,\mathrm{d}y+\left\langle L_{\dot q}^{-1}x\right\rangle^{2N}\mathcal{L}_{\dot q}^{-1}\tau_q\chi.
\end{aligned}$$

Mimicking the estimate (53), and using the exponential decay properties of ∇Y, we get the result. This ends the proof of the first part of the proposition.

For the continuity estimate, we observe that

$$\begin{aligned}
W_1(q_1,\dot q_1)-W_1(q_2,\dot q_2) &= \int_0^1\nabla_q W_1(q_1+\tau(q_2-q_1),\dot q_1)(q_2-q_1)\,\mathrm{d}\tau\\
&\quad+\int_0^1\nabla_{\dot q}W_1(q_2,\dot q_1+\tau(\dot q_2-\dot q_1))(\dot q_2-\dot q_1)\,\mathrm{d}\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|W_1(q_1,\dot q_1)-W_1(q_2,\dot q_2)\|_{T,s,N}\\
&\quad\leqslant\|q_1-q_2\|_{L^\infty_T}\sup_{\tau\in[0,1]}\|\nabla_q W_1(q_1+\tau(q_2-q_1),\dot q_1)\|_{T,s,N}\\
&\qquad+\|\dot q_1-\dot q_2\|_{L^\infty_T}\sup_{\tau\in[0,1]}\|\nabla_{\dot q}W_1(q_2,\dot q_1+\tau(\dot q_2-\dot q_1))\|_{T,s,N}.
\end{aligned}\tag{54}$$

Let $v_\tau(t) = q_1(t)+\tau(q_2(t)-q_1(t))$. Thus, from Lemma 3.4

$$\|\nabla_q W_1(v_\tau(t),\dot q_1)\|_{T,s,N}\lesssim\|H^s_{v_\tau}\left\langle x\right\rangle^{2N}\nabla_x\chi_1(\dot q_1,x-v_\tau(t))\|_{L^\infty_T L^\infty}.$$

Notice that

$$\nabla_x\chi_1(\dot q,x-v_\tau(t)) = \int_{\mathbb{R}^3}\frac{1}{\sqrt{1-|\dot q_1|^2}}Y(L_{\dot q_1}y)\nabla_x\chi(x-v_\tau(t)-y)\,\mathrm{d}y;$$

as a consequence of the first part of the proposition we get

$$\begin{aligned}
\|\nabla_q W_1(v_\tau(t),\dot q_1)\|_{T,s,N} &\lesssim\sup_{\tau\in[0,1]}\left\langle\|v_\tau\|_{L^\infty_T}\right\rangle^{2N}\left\|\left\langle x\right\rangle^{2N}\nabla\chi\right\|_{L^\infty_T}\\
&\lesssim\left(\left\langle\|q_1\|_{L^\infty_T}\right\rangle^{2N}+\left\langle\|q_2\|_{L^\infty_T}\right\rangle^{2N}\right)\left\|\left\langle x\right\rangle^{2N}\nabla\chi\right\|_{L^\infty_T}.
\end{aligned}$$

Now, we consider the second term on the right-hand side of (54). Set $w_\tau = \dot{q}_1(t)+\tau(\dot{q}_2(t)-\dot{q}_1(t))$. Obviously, $\|w_\tau\|_{L_T^\infty} \leqslant (1-\tau)\|\dot{q}_1\|_{L_T^\infty} + \tau\|\dot{q}_2\|_{L_T^\infty} \leqslant \frac{1}{2}$. From Lemma 3.4,

$$\|\nabla_{\dot{q}} W_1(q_2, \dot{q}_1 + \tau(\dot{q}_2 - \dot{q}_1))\|_{T,s,N} \lesssim \left\| H^s_{w_\tau} \langle x\rangle^{2N} \nabla_{\dot{q}}\chi_1(w_\tau, x-q_2)\right\|_{L_T^\infty L^\infty}.$$

Notice that

$$\nabla_{\dot{q}}\widehat{\chi}_1(w_\tau, x-q_2)(t,\xi) = \frac{2\xi(\xi\cdot w_\tau)\widehat{\chi}(\xi)e^{-i\xi\cdot q(\tau)}}{(\langle\xi\rangle^2 + (i\xi\cdot w_\tau)^2)^2}.$$

Let $\widehat{G}_\tau(t,\xi) = \frac{2(\xi\cdot w_\tau)}{(\langle\xi\rangle^2+(i\xi\cdot w_\tau)^2)^2}$; according to Lemma 3.3, we find that $G_\tau(t,x) \in W^{2,1}(\mathbb{R}^3)$. Moreover,

$$\nabla_{\dot{q}}\chi(w_\tau, x-q_2) = \int_{\mathbb{R}^3} \nabla_y G_\tau(t,y)\chi(x-q_2-y)\,\mathrm{d}y = \int_{\mathbb{R}^3} G_\tau(t,y)\nabla\chi(x-q_2-y)\,\mathrm{d}y.$$

As for the proof of the first part of the proposition, we find

$$\left\|\langle x\rangle^{2N}\nabla_{\dot{q}}\chi(w_\tau, x-q_2)\right\|_{L^\infty L_T^\infty} \lesssim \left\langle \|q_2\|_{L_T^\infty}\right\rangle^{2N} \left\|\langle x\rangle^{2N}\nabla\chi\right\|_{L^\infty}.$$

Hence (52) follows. □

Proposition 3.6 (*Estimates on* W_2)**.** *Let* $T \in (0,+\infty]$ *and* $s \in [0,2]$*. We have*

$$\|W_2\|_{L_T^1 H^{s,\infty}} \lesssim \|w_0\|_{W^{s+3,1}} + \|w_1\|_{W^{s+2,1}} + \|\chi\|_{W^{s+1,1}},$$

Proof. It is easy to see that W_2 is the solution of the following linear Klein–Gordon equation:

$$\partial_{tt}W + W - \Delta W = 0; \quad W(0,x) = w_0 + (1-\Delta)^{-1}\chi(x), \quad \partial_t W(0,x) = w_1.$$

According to [13, Corollary 2.3] and (12),

$$\|W_2(t,\cdot)\|_{H^{s,\infty}} \lesssim (1+|t|)^{-3/2}\left(\|w_0\|_{W^{s+3,1}} + \|w_1\|_{W^{s+2,1}} + \|\chi\|_{W^{s+1,1}}\right). \tag{55}$$

The result follows immediately. □

Remark 3.1. Notice that the term W_2 does not depend on q, and thus we do not need to study its continuity with respect to q and its derivatives.

Now we turn to the estimates on W_3.

Proposition 3.7 (*Estimates on* W_3)**.** *Let* $T \in (0,+\infty]$*,* $s \in [1,2]$ *and* $\|\ddot{q}(t)\|_{L_T^1} \leqslant \frac{1}{2}$*. Then there exists* $C = C(\varepsilon)$ *such that:*

$$\|W_3\|_{L_T^1 H^{s,\infty}} \lesssim \|\chi\|_{W^{2+s,1}}.$$

What is more, if $q_j \in W_T^{2,1}$ *and* $\|\ddot{q}_j(t)\|_{L_T^1} \leqslant \frac{1}{2}$ *for* $j=1,2$*, then there exists* $C = C(\varepsilon)$ *such that:*

$$\|W_3(q_1,\dot{q}_1,\ddot{q}_1) - W_3(q_2,\dot{q}_2,\ddot{q}_2)\|_{L_T^1 H^{s,\infty}} \lesssim \|q_1-q_2\|_{W_T^{2,1}}\|\chi\|_{W^{s,1}}. \tag{56}$$

Proof. By symmetry of treatment, we only deal with the χ_2-term. We rewrite:

$$\begin{aligned}\widehat{\chi}_2(q,\dot q,\ddot q)(t,\xi) &= \frac{\widehat{\chi}(\xi)}{2i\langle\xi\rangle}e^{-i\xi\cdot q(t)}\frac{i\xi\cdot\ddot q(t)}{(-i\langle\xi\rangle - i\xi\cdot\dot q(t))^2},\\ &= -\frac{e^{-i\xi\cdot q(t)}}{(\langle\xi\rangle^2-[\xi\cdot\dot q(t)]^2)^2}\frac{(\langle\xi\rangle-\xi\cdot\dot q(t))^2}{2\langle\xi\rangle}\xi\cdot\ddot q(t)\widehat{\chi}(\xi).\end{aligned}$$

As $|\dot q(t)|\leqslant\|\ddot q\|_{L^1_T}\leqslant\frac12$ by assumption, according to Lemma 3.3 we have

$$\chi_2(q,\dot q,\ddot q)(t,x) = \frac{i}{2}H^{-1}(H+i\dot q\cdot\nabla)^2\nabla\cdot\ddot q\int_{\mathbb{R}^3}\frac{1}{\sqrt{1-|\dot q|^2}}Z(L_{\dot q}y)\chi(x-y-q(t))\,\mathrm{d}y. \tag{57}$$

Hence Young's convolution inequality gives, for all $s\geqslant 1$,

$$\|H^s\chi_2(q,\dot q)(t,x)\|_{L^1_x}\lesssim|\ddot q(t)|\sup_{t\in[0,T]}\|H^{s-1}\chi(x-q(t))\|_{L^1}\lesssim|\ddot q|\|\chi\|_{W^{s-1,1}}.$$

It follows from the decay estimate [13, Corollary 2.3] that

$$\begin{aligned}\|H^sW_3(q,\dot q,\ddot q)\|_{L^\infty_x} &\leqslant \sum_{j=2,3}\int_0^t\|e^{i(t-\tau)H}H^s\chi_j(q(\tau),\dot q(\tau),\ddot q(\tau))\|_{L^\infty_x}\,\mathrm{d}\tau,\\ &\lesssim\sum_{j=2,3}\int_0^t\frac{\|H^s\chi_j(q(\tau),\dot q(\tau),\ddot q(\tau))\|_{W^{3,1}}}{(1+|t-\tau|)^{3/2}}\,\mathrm{d}\tau,\\ &\lesssim\|\chi\|_{W^{s+2,1}}\int_0^t\frac{|\ddot q(\tau)|}{(1+|t-\tau|)^{3/2}}\,\mathrm{d}\tau.\end{aligned}$$

Integrating in t and using the fact that $(1+|t|)^{-3/2}\in L^1(\mathbb{R})$ we deduce

$$\begin{aligned}\|W_3\|_{L^1_TH^{s,\infty}} &\lesssim\|\chi\|_{W^{s+2,1}}\int_{[0,T]}\int_0^t\frac{|\ddot q(\tau)|}{(1+|t-\tau|)^{3/2}}\,\mathrm{d}\tau\,\mathrm{d}t\\ &=\|\chi\|_{W^{s+2,1}}\int_{[0,T]}\int_\tau^T\frac{|\ddot q(\tau)|}{(1+|t-\tau|)^{3/2}}\,\mathrm{d}t\,\mathrm{d}\tau\lesssim\|\chi\|_{W^{s+2,1}}\end{aligned}$$

and this concludes the proof of the first part of the proposition.

For the continuity estimates, we have

$$\begin{aligned}&\|H^s(W_3(q_1,\dot q_1,\ddot q_1)-W_3(q_2,\dot q_2,\ddot q_2))\|_{L^\infty_x}\\ &\leqslant\sum_{j=2,3}\int_0^t\|e^{i(t-t')H}H^s(\chi_j(q_1(t'),\dot q_1(t'),\ddot q_1(t'))-\chi_j(q_2(t'),\dot q_2(t'),\ddot q_2(t')))\|_{L^\infty_x}\,\mathrm{d}t',\\ &\lesssim\sum_{j=2,3}\int_0^t\frac{\|H^s(\chi_j(q_1(t'),\dot q_1(t'),\ddot q_1(t'))-\chi_j(q_2(t'),\dot q_2(t'),\ddot q_2(t')))\|_{W^{3,1}}}{(1+|t-t'|)^{3/2}}\,\mathrm{d}t'.\end{aligned}$$

Let $v_1=q_2-q_1$, $v_2=\dot q_2-\dot q_1$ and $v_3=\ddot q_2-\ddot q_1$; we have

$$\begin{aligned}&\|W_3\|_{L^1_TW^{s,\infty}}\lesssim\\ &\quad\sum_{j=2,3}\int_{[0,T]}\int_0^t\frac{\|H^s(\chi_j(q_1(t'),\dot q_1(t'),\ddot q_1(t'))-\chi_j(q_2(t'),\dot q_2(t'),\ddot q_2(t')))\|_{W^{3,1}}}{(1+|t-t'|)^{3/2}}\,\mathrm{d}t'\,\mathrm{d}t\end{aligned}$$

which yields

$$\|W_3\|_{L^1_TW^{s,\infty}}\lesssim\sum_{j=2,3}\int_{[0,T]}\|H^s(\chi_j(q_1(t'),\dot q_1(t'),\ddot q_1(t'))-\chi_j(q_2(t'),\dot q_2(t'),\ddot q_2(t')))\|_{W^{3,1}}\,\mathrm{d}t'.$$

We expand the right hand side and get

$$\begin{aligned}\|W_3\|_{L^1_T W^{s,\infty}} \lesssim \sum_{j=2,3} \sup_{\tau\in[0,1]} \int_{[0,T]} \|H^s \nabla_q \chi_j(q_1(t')+\tau v_1, \dot q_1(t'), \ddot q_1(t'))\|_{W^{3,1}} |v_1(t')|\, \mathrm{d}t' \\ + \sum_{j=2,3} \sup_{\tau\in[0,1]} \int_{[0,T]} \|H^s \nabla_{\dot q} \chi_j(q_1(t'), \dot q_1(t')+\tau v_2(t'), \ddot q_1(t'))\|_{W^{3,1}} |v_2(t')|\, \mathrm{d}t' \\ + \sum_{j=2,3} \sup_{\tau\in[0,1]} \int_{[0,T]} \|H^s \nabla_{\ddot q} \chi_j(q_1(t'), \dot q_1(t'), \ddot q_1(t')+\tau v_3(t'))\|_{W^{3,1}} |v_3(t')|\, \mathrm{d}t'.\end{aligned}$$

We deal with the χ_2-term, as the other one can be dealt with similarly by symmetry. According to (57) and Lemma 3.3, we get

$$\begin{aligned}&\|H^s \nabla_{\dot q} \chi_j(q_1(t'), \dot q_1(t')+\tau v_2(t'), \ddot q_1(t'))\|_{W^{3,1}} \\ &\quad = \|\nabla_x H^s \nabla_{\dot q} \chi_j(q_1(t'), \dot q_1(t')+\tau v_2(t'), \ddot q_1(t'))\|_{W^{3,1}} \lesssim |\ddot q(t')|\, \|\chi\|_{W^{s,1}}\,,\end{aligned}$$

and

$$\|H^s \nabla_{\dot q} \chi_j(q_1(t'), \dot q_1(t')+\tau v_2(t'), \ddot q_1(t'))\|_{W^{3,1}} \lesssim |\ddot q(t')|\, \|\chi\|_{W^{s-1,1}}$$

as well as

$$\|H^s \nabla_{\ddot q} \chi_j(q_1(t'), \dot q_1(t'), \ddot q_1(t')+\tau v_3(t'))\|_{W^{3,1}} \lesssim \|\chi\|_{W^{s-1,1}}\,.$$

Hence,

$$\begin{aligned}&\|W_3(q_1,\dot q_1,\ddot q_1) - W_3(q_2,\dot q_2,\ddot q_2)\|_{L^1_T H^{s,\infty}} \\ &\quad \lesssim \left(\|q_1-q_2\|_{L^1_T} + \|\dot q_1-\dot q_2\|_{L^1_T} + \|\ddot q_1-\ddot q_2\|_{L^1_T}\right) \|\chi\|_{W^{s,1}}\end{aligned}$$

and this concludes the proof. □

4. Proof of Theorems 1.1–1.4

This section is devoted to the proof of Theorems 1.1 and 1.4, that will be proved respectively in Sections 4.2 and 4.3. The strategy is very standard, and consists in the application of a fixed point theorem based on Strichartz estimates for the operator $\mathcal{D}+W_1$: therefore, in Section 4.1 we shall prove these estimates, essentially showing that the potential W_1 satisfies condition (8).

4.1. Strichartz estimates for the Dirac equation in the Dirac–Klein–Gordon system

We now show that solutions to the following equation

$$i\partial_t u = \mathcal{D}u + W_1 u \qquad \text{with} \quad W_1 = \chi_1(t, x-q(t)), \tag{58}$$

satisfy Strichartz estimates: we prove in fact the following

Proposition 4.1. *Let $T \in (0,+\infty]$, (p,r,s) any Dirac admissible triple with $s\in[0,2]$, u be a solution to (58) with initial condition $u(0,x)=u_0(x)$, $q=q(t)$ be such that $\|\ddot q(t)\|_{L^1_T} \leqslant \frac12$ and $q\in L^\infty_T$ and let χ be such that*

$$\left\| \langle x\rangle^{3+} \chi \right\|_{L^\infty} < C\varepsilon$$

for some constant C and ε small enough. Then u satisfies the Strichartz estimates (10) and (9) for the triple (p,r,s).

Proof. We need to check that the operator W_1 satisfies the conditions required in Theorem 1.3. To do that, we perform a change of variables, and consider the function $v(t,x) = u(t, x+q(t))$ which solves the equation

$$i\partial_t v = \mathcal{D}v + i\dot{q}(t)\cdot\nabla v + \chi_1(t,x)v. \tag{59}$$

In our assumption on $\ddot{q}$, we have that $\|\dot{q}\|_{L^\infty} \leqslant 1/2$ and this ensures that $H_1 := \mathcal{D} + i\dot{q}(t)\cdot\nabla + \chi_1(t,x)$ is a uniform (in t) perturbation of $\mathcal{D}$. Therefore the L^2 norm of $H_1 f$ is uniformly in time equivalent to the H^1 norm of f and H_1, which is symmetric, is also essentially self-adjoint. Notice that,

$$\partial_t H_1 = -i\dot{q}\cdot\nabla + \partial_t\chi_1,$$

and as

$$\partial_t\hat{\chi}_1(\xi) = \frac{\hat{\chi}(\xi)}{(\langle\xi\rangle^2 + (i\xi\cdot\dot{q}(t))^2)^2}2\xi\cdot\dot{q}\xi\cdot\ddot{q},$$

we get that $\partial_t H_1$ belongs to $L^1(\mathbb{R}, H^{s+1} \to H^s)$, and hence H_1 is of bounded variations in time as an operator from H^{s+1} to H^s. This means in particular that the equation

$$i\partial_t v = H_1 v$$

is well-posed in H^s for any $s \geqslant 0$ as long as $\ddot{q}$ and $\widehat{\chi}$ are small in L^1 norm: in other words, we have that there exists a constant $C > 0$ such that for any solution v of (59) with initial condition v_0 and for any time $t \in \mathbb{R}$ then

$$\|v\|_{H^s} \leqslant C\|v_0\|_{H^s}.$$

Now, we re-change variables to get back to the function u: as the translations in time do not alter the H^s norm in space, we get for any solution u to Eq. (58) the following bound

$$\|u\|_{H^s} \leqslant C\|u_0\|_{H^s}$$

and thus (58) is well-posed in H^s for any $s \geqslant 0$.

Now, thanks to Lemma 3.4 and Proposition 3.5 we get that for any $N \in \mathbb{R}^+$ and $s \in [0,2]$, there is a constant C' such that

$$\|W_1\|_{T,s,N} \leqslant C'\left\langle \|q\|_{L^\infty_T}\right\rangle^{2N} \left\|\langle x\rangle^{2N}\chi\right\|_{L^\infty}.$$

Let $\left\|\langle x\rangle^{3+}\chi\right\|_{L^\infty}$ sufficiently small such that

$$C'\left\langle \|q\|_{L^\infty_T}\right\rangle^{3+} \left\|\langle x\rangle^{3+}\chi\right\|_{L^\infty} < \epsilon.$$

Then, for $s \in [0,2]$

$$\|W_1\|_{T,s,3+} < \epsilon.$$

Applying Theorem 1.3, the conclusion follows. □

4.2. Proof of Theorem 1.1

We are now in a position for proving the global existence of solutions for the nonlinear Dirac equation

$$\begin{cases} i\partial_t u = \mathcal{D}u + Wu + \mathcal{N}(u), \\ u(0,x) = u_0(x). \end{cases} \tag{60}$$

Here $\mathcal{N}(u)$ is a generic nonlinearity.

According to Proposition 3.2, we write $W = W_1 + W_2 + W_3$; letting $V = W_1$, the above Dirac equation can be rewritten in integral form:

$$\begin{aligned} u =& S_0(t)u_0 + i\int_0^t S_0(t-\tau)(Wu)(\tau)\,\mathrm{d}\tau + i\int_0^t S_0(t-\tau)\mathcal{N}(u)(\tau)\,\mathrm{d}\tau \\ =& S_0(t)u_0 + i\int_0^t S_0(t-\tau)((W_1+W_2+W_3)u)(\tau)\,\mathrm{d}\tau + i\int_0^t S_0(t-\tau)\mathcal{N}(u)(\tau)\,\mathrm{d}\tau \\ =& S_{W_1}(t)u_0 + i\int_0^t S_0(t-\tau)((W_2+W_3)u)(\tau)\,\mathrm{d}\tau + i\int_0^t S_0(t-\tau)\mathcal{N}(u)(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{61}$$

The proof of the well-posedness is now very standard, and it consists in the application of the contraction mapping principle on the solution map above on the ball

$$X_K = \{\psi \in X \,|\, \|\psi\|_X = \|\psi\|_{L_T^\infty H^s} + \|\psi\|_{L_T^{p-1}L^\infty} \leqslant K\} \tag{62}$$

where $X = L_T^\infty H^s \cap L_T^{p-1}L^\infty$ and $s \in [s(p), 2]$ with $s(p) = \frac{3}{2} - \frac{1}{p-1}$.

The only additional tool that we need (with respect to the unperturbed case) is given by the following Lemma, that allows us to control the terms involving W_2 and W_3:

Lemma 4.2. *Let*

$$C_{w,\chi} := \|w_0\|_{W^{s+3,1}} + \|w_1\|_{W^{s+2,1}} + \|\chi\|_{W^{s+1,1}}\,,$$

and

$$C_{q,\chi} := \|H^s\chi\|_{W^{2,1}}\,.$$

Then,

$$\left\|\int_0^t S_0(t-\tau)((W_2+W_3)u)\,\mathrm{d}\tau\right\|_X \lesssim (C_{w,\chi} + C_{q,\chi})\,\|u\|_X\,. \tag{63}$$

Proof. Thanks to Strichartz estimates for the free flow, the left-hand side of (63) can be bounded by the term $\|(W_2+W_3)u\|_{L^1H^s}$. By the Kato–Ponce inequality (71), Propositions 3.6 and 3.7, as $s > 1$, we then get

$$\begin{aligned} \|(W_2+W_3)u\|_{L^1,H^s} &\lesssim \|W_2+W_3\|_{L^1H^{s,\infty}}\,\|u\|_{L^\infty H^s} \\ &\lesssim (C_{w,\chi}+C_{q,\chi})\,\|u\|_{L^\infty H^s} \lesssim (C_{w,\chi}+C_{q,\chi})\,\|u\|_X \end{aligned}$$

and this concludes the proof of the lemma. □

The rest of the proof is now completely standard (see [10]), and we thus omit it.

In what follows we will also need the continuity in q of the solution map. We thus prove the following

Proposition 4.3. *Let χ, w_0, w_1, q_1, q_2 be as in the assumptions of Theorem 1.1 with the additional assumption that $\|\langle x\rangle^{3+}\nabla\chi\|_{L^\infty}$ is sufficiently small. Let $T \in (0, +\infty)$ and let Ψ_q denote the global flow associated to system (60) with p and s as in the assumptions of Theorem 1.1, and let $\|u_0\|_{H^s}$ small enough. Then Ψ_q satisfies the following properties:*

$$\|\Psi_{q_j}(t)u_0\|_{L_T^\infty H^s} \leqslant C\|u_0\|_{H^s}, \qquad j = 1, 2, \tag{64}$$

$$\|\Psi_{q_1}(t)u_0 - \Psi_{q_2}(t)u_0\|_X \leqslant C\|u_0\|_{H^s}\left(\|q_1-q_2\|_{W_T^{1,\infty}} + \|\ddot{q}_1 - \ddot{q}_2\|_{L_T^1}\right) \tag{65}$$

where the norm X is given in (62) and the constant $C = C(s, w_0, w_1, \chi, \|q_1\|_{L_T^\infty}, \|q_2\|_{L_T^\infty})$.

Proof. We only need to prove (65) as indeed (64) is a consequence of the contraction argument. Let us take $q_1 \neq q_2$; we start from representation (61) that we rewrite as

$$\begin{aligned} u_j = \Psi_{q_j}(t)u_0 =& S_{W_1^{q_j}}(t)u_0 + i\int_0^t S_0(t-\tau)((W_2^{q_j} + W_3^{q_j})u_j)(\tau)\,\mathrm{d}\tau \\ &+ i\int_0^t S_0(t-\tau)\mathcal{N}(u_j)(\tau)\,\mathrm{d}\tau \end{aligned}$$

for $j = 1, 2$. Then we have that

$$\| \Psi_{q_1}(t)u_0 - \Psi_{q_2}(t)u_0\|_X \leqslant I + II + III,$$

with

$$I = \|S_{W_1^{q_1}}(t)u_0 - S_{W_1^{q_2}}(t)u_0\|_X \leqslant C^1_{q_1,q_2,\chi}\|q_1 - q_2\|_{W_T^{1,\infty}}$$

thanks to Propositions 2.4 and 3.5 (the constant $C_{q_1,q_2,\chi}$ is the one given by (52)). Then,

$$\begin{aligned} II &= \left\| i\int_0^t S_0(t-\tau)\big[(W_2^{q_1} + W_3^{q_1})u_1 - (W_2^{q_2} + W_3^{q_2})u_2\big](\tau)\,\mathrm{d}\tau\right\|_X \\ &\leqslant \|(W_2^{q_1} + W_3^{q_1})u_1 - (W_2^{q_2} + W_3^{q_2})u_2\|_{L^1_T H^s} \\ &\leqslant \|(W_2^{q_1} - W_2^{q_2})u_1\|_{L^1_T H^s} + \|(W_3^{q_1} - W_3^{q_2})u_2\|_{L^1_T H^s} + \|(W_3^{q_1} + W_2^{q_2})(u_1 - u_2)\|_{L^1_T H^s} \\ &= II_A + II_B + II_C. \end{aligned}$$

Notice now that $II_A = 0$ as indeed the term W_2 does not depend on q. We estimate the other terms as follows:

$$II_B \lesssim \|W_3^{q_1} - W_3^{q_2}\|_{L^1_T H^{s,\infty}}\|u_2\|_{L^\infty_T H^s} \leqslant C^2_{q_1,q_2,\chi}\|u_0\|_{H^s}\|q_1 - q_2\|_{W_T^{2,1}}$$

where we have used Proposition 3.7 with the constant given in (56) and estimate (64), and

$$\begin{aligned} II_C &\leqslant \|(W_3^{q_1} + W_2^{q_2})(u_1 - u_2)\|_{L^1 H^s} \\ &\leqslant \|W_3^{q_1} + W_2^{q_2}\|_{L^1_t H^{s,\infty}}\|u_1 - u_2\|_{L^\infty H^s} \\ &\leqslant (C_{w,\chi} + C_{q,\chi})\|u_1 - u_2\|_{L^\infty H^s} \end{aligned}$$

where the constants are given in Lemma 4.2. Finally, writing $\mathcal{N}(u) = |\langle u, \beta u\rangle|^{\frac{p-1}{2}}\beta u$, combining free Strichartz with classical nonlinear estimates yields

$$III = \left\| i\int_0^t S_0(t-\tau)\big[\mathcal{N}(u_1) - \mathcal{N}(u_2)\big](\tau)\,\mathrm{d}\tau\right\|_X \leqslant C(\|u_1\|_X^{p-1} + \|u_2\|_X^{p-1})\|u_1 - u_2\|_X.$$

As shown in the proof of Theorem 1.1, for $\|u_0\|$ small enough the solution map ψ is contracting, and thus absorbing the necessary terms on the LHS (notice that $T < +\infty$) yields (65). □

4.3. Proof of Theorem 1.4

We now deal with the proof of Theorem 1.4, that is we prove local well-posedness for system (11). To do this, we essentially follow the strategy developed in [6] (see also [2]).

First of all, we need to deal with the classical dynamics driven by q. Let us consider the following system

$$\begin{cases} \ddot{q} = F(q) = \dfrac{1}{M}\left\langle \Psi_q u_0 \left| \dfrac{x-q}{|x-q|^3} \right| \Psi_q u_0 \right\rangle, \\ q(0) = q_0, \qquad \dot{q}(0) = v_0. \end{cases} \tag{66}$$

We prove the following

Proposition 4.4. *Let $s \in \left(\frac{3}{2}, 2\right]$. There exists a constant C such that for all q_0, v_0 and $u_0 \in H^s$ system (66) admits a unique solution $C^2([0,T])$ for any $T \leqslant \frac{M}{C\|u_0\|_{H^s}^2}$.*

Proof. Let Z be the completion of $\mathcal{C}^2([0,T])$ induced by the norm

$$q \mapsto \|q\|_{L^\infty_T} + \|\ddot{q}\|_{L^1_T}.$$

We want to apply a contraction (Picard) argument onto the ball

$$B = B(T) = \left\{ q \in Z : \|\ddot{q}\|_{L^1_T} \leqslant \frac{1}{2}, \|q\|_{L^\infty} \leqslant 1, q(0) = 0, \dot{q}(0) = v_0 \right\}.$$

We denote with P the solution map, that is the map such that $\partial_t^2[P(q)] = F(q)$ with $P(0) = 0$ and $\partial_t P(0) = 1$. First of all, we prove that B is stable under the action of P:

Lemma 4.5. *Let $u_0 \in H^1$. There exists a constant C_1 such that if $T_1 \leqslant \frac{C_1\sqrt{M}}{\|u_0\|_{H^1}^2}$ then P maps B in B.*

Proof. Thanks to Hardy inequality we have

$$\|F(q)\|_{L^\infty} \leqslant \frac{C}{M} \| \Psi_q(u_0)\|_{H^1}^2.$$

This and (64) imply that

$$\|F(q)\|_{L^1_T} \leqslant \frac{CT}{M} \|u_0\|_{H^1}^2.$$

As a consequence we get

$$\|P(q)\|_{L^\infty} \leqslant |v_0|T + \frac{CT^2}{M} \|u_0\|_{H^1}^2$$

so that choosing $T_1 \leqslant K \frac{\sqrt{M}}{C\|u_0\|_{H^1}^2}$ and $T \leqslant K|v_0|^{-1}$ with K small enough, we get

$$\|F(q)\|_{L^1_{T_1}} \leqslant \frac{1}{2}, \quad \|P(q)\|_{L^\infty} \leqslant 1$$

that implies that $P(q) \in B$, and so P maps B in B. □

Then, we show that F is uniformly Lipschitz-continuous in q, that is the following

Lemma 4.6. *Let $u_0 \in H^s$ for some $s \in \left(\frac{3}{2}, 2\right]$ and let $q_1, q_2 \in B$. There exists a constant $C_2 > 0$ such that for any $T \leqslant T_1$ with T_1 as in Lemma 4.5 such that*

$$\|P(q_1) - P(q_2)\|_{C^2([0,T])} \leqslant C_2 T^2 \|u_0\|_{H^s}^2 \|q_1 - q_2\|_{C^2([0,T])} \tag{67}$$

Proof. We rewrite the difference

$$F(q_1) - F(q_2) = \left\langle \Psi_{q_1}(u_0) | \frac{x - q_1}{|x - q_1|^3} | \Psi_{q_1}(u_0) \right\rangle - \left\langle \Psi_{q_2}(u_0) | \frac{x - q_2}{|x - q_2|^3} | \Psi_{q_2}(u_0) \right\rangle$$

as follows

$$F(q_1) - F(q_2) = I + II + II$$

with

$$I = \left\langle (\Psi_{q_1}(u_0) - \Psi_{q_2}(u_0)) \left| \frac{x - q_1}{|x - q_1|^3} \right| \Psi_{q_1}(u_0) \right\rangle,$$

$$II = \left\langle \Psi_{q_2}(u_0) \right| \left(\frac{x-q_1}{|x-q_1|^3} - \frac{x-q_2}{|x-q_2|^3} \right) \left| \Psi_{q_1}(u_0) \right\rangle$$

$$III = \left\langle \Psi_{q_2}(u_0) \right| \frac{x-q_2}{|x-q_2|^3} \left| (\Psi_{q_1}(u_0) - \Psi_{q_2}(u_0)) \right\rangle$$

and we estimate the three terms one by one.

For I, we write

$$\begin{aligned} |I| &\leqslant \int_{\mathbb{R}^3} \frac{|\Psi_{q_1}(u_0) - \Psi_{q_2}(u_0)\,||\, \Psi_{q_1}(u_0)|}{|q_1 - x|^2} \\ &= \int_{\mathbb{R}^3} \frac{|\Psi_{q_1}(u_0) - \Psi_{q_2}(u_0)|}{|q_1 - x|^{s-1}} \frac{|\Psi_{q_1}(u_0)|}{|q_1 - x|^{3-s}} \\ &\leqslant C \left\| \frac{\Psi_{q_1}(u_0) - \Psi_{q_2}(u_0)}{|q_1 - x|^{s-1}} \right\|_{L^2} \left\| \frac{\Psi_{q_1}(u_0)}{|q_1 - x|^{3-s}} \right\|_{L^2} \\ &\leqslant C \| \Psi_{q_1}(u_0) - \Psi_{q_2}(u_0) \|_{H^{s-1}} \| \Psi_{q_1}(u_0) \|_{H^{3-s}} \end{aligned}$$

where we have made use of (68), and thanks to Proposition 4.3 we get (notice that $3-s<s$ since $s>\frac{3}{2}$)

$$|I| \leqslant C \|u_0\|_{H^s}^2 \|q_1 - q_2\|_{W_T^{2,1}} \leqslant C \|u_0\|_{H^s}^2 \|q_1 - q_2\|_{C^2([0,T_2])}.$$

The same strategy allows to control the term III. To deal with the term II, We consider the quantity

$$G(q) = \left\langle u \left| \frac{x-q}{|x-q|^3} \right| v \right\rangle.$$

Recall in the proof of Proposition 3.5:

$$\tau_q v(t,x) = v(t, x-q).$$

Then after a change of variable (the translation $y = x - q$), we have

$$G(q) = \left\langle \tau_{-q} u \left| \frac{x}{|x|^3} \right| \tau_{-q} v \right\rangle$$

where $u_q(x) = u(x+q)$. After differentiating in q, we get

$$\nabla_q G(q) = \left\langle \tau_{-q} \nabla u \left| \frac{x}{|x|^3} \right| \tau_{-q} v \right\rangle + \left\langle \tau_{-q} u_q \left| \frac{x}{|x|^3} \right| \tau_{-q} \nabla v \right\rangle$$

from which, by the use of (68), we obtain

$$\begin{aligned} |\nabla G(q)| &\leqslant C \|\tau_{-q} \nabla u\|_{H^{2-s}} \|\tau_{-q} v\|_{H^s} + C \|\tau_{-q} u\|_{H^s} \|\tau_{-q} \nabla v\|_{H^{2-s}} \\ &= C \|u\|_{H^{3-s}} \|v\|_{H^s} + C \|u\|_{H^s} \|v\|_{H^{3-s}}. \end{aligned}$$

We thus get

$$|G(q_1) - G(q_2)| \leqslant |q_1 - q_2| \Big(C \|u\|_{H^{3-s}} \|v\|_{H^s} + C \|u\|_{H^s} \|v\|_{H^{3-s}} \Big).$$

Thus by Proposition 4.3, we obtain, as $s > 3/2$,

$$|II| \leqslant C \|u_0\|_{H^s} \|u_0\|_{H^{3-s}} \|q_1 - q_2\|_{L_T^\infty} \leqslant C \|u_0\|_{H^s}^2 \|q_1 - q_2\|_{L_T^\infty}.$$

We integrate these bounds twice and get the result. □

Remark 4.1. Notice how we have used the fact that $s > 3/2$ twice, in the application of inequality (68): this therefore turns out to be a necessary condition in our proof above.

Now, the proof of Proposition 4.4 follows from the two Lemmas: it is a contraction argument for the map P in B for the topology of Z. □

Proof of Theorem 1.4. We only need to combine Theorem 1.1, Propositions 4.3 and 4.4. Let $u_0 \in H^s$, $w_0 \in W^{s+3,1}$, $w_1 \in W^{s+2,1}$ and $\chi \in W^{s+1,1}$ with $s > 3/2$. Let $q \in Z$ be the solution to (66) as given in Proposition 4.4, with any $T \leqslant T_1 := \min\{\frac{\sqrt{M}}{C\|u_0\|_{H^s}^2}, K|v_0|^{-1}\}$. Let $u = \Psi_q(t)u_0$ defined in Proposition 4.3. Then, the couple $(u, q) \in C([0, T], H^s) \times Z$ for $T \leqslant MC$ where the constant C depends on $\|u_0\|_{H^s}$, $\|w_0\|_{W^{s+3,1}}$, $\|w_1\|_{W^{s+2,1}}$, $\|\chi\|_{W^{s+1,1}}$ (follows from the proof of Theorem 1.1), and it satisfies system (11). The fact that q belongs to $\mathcal{C}^2$ is due to the fact that $\Psi_q(t)(u_0)$ belongs to $\mathcal{C}(\mathbb{R}, H^s)$ if $q \in Z$.

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Appendix A. Useful inequalities

We devote this small appendix to recalling some useful (and classical) inequalities and small variations of them that were needed during our proofs.

First, we recall the following generalized Hardy inequality

$$\| |x|^{-a}(-\Delta)^{-a/2}\|_{L^p\to L^p} \leqslant C \tag{68}$$

and by duality,

$$\|(-\Delta)^{-a/2}|x|^{-a}\|_{L^q\to L^q} \leqslant C \tag{69}$$

which holds for $a > 0$, and any $1 < p < \frac{3}{a}$ and $p^{-1} + q^{-1} = 1$ (see e.g. [12]).

Then, we recall the classical Kato–Ponce inequality:

Lemma A.1 (*Kato–Ponce Inequality [11]*). *For $r \geqslant 1$, $s \geqslant 0$ and $1 < p_1, q_1, p_2, q_2 \leqslant +\infty$ such that $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, we have*

$$\left\|(-\Delta)^{s/2}fg\right\|_{L^r} \lesssim \|f\|_{L^{p_1}} \left\|(-\Delta)^{s/2}g\right\|_{L^{q_1}} + \left\|(-\Delta)^{s/2}f\right\|_{L^{p_2}} \|g\|_{L^{q_2}}, \tag{70}$$

and

$$\|H^s fg\|_{L^r} \lesssim \|f\|_{L^{p_1}} \|H^s g\|_{L^{q_1}} + \|H^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}, \tag{71}$$

It is possible to prove an analogue for estimate (70) in the case of the operator H_v^s as defined in (46). By replacing fg with $\mathcal{L}_v^{-1}(fg)$, we get the following

$$\|H^s\mathcal{L}_v^{-1}(fg)\|_{L^r} \lesssim \|H^s\mathcal{L}_v^{-1}f\|_{L^{p_1}}\|\mathcal{L}_v^{-1}g\|_{L^{q_1}} + \|\mathcal{L}_v^{-1}f\|_{L^{p_2}}\|H^s\mathcal{L}_v^{-1}g\|_{L^{q_2}}.$$

For any $1 \leqslant p \leqslant +\infty$,

$$\|H^s \mathcal{L}_v^{-1} f\|_{L^p} = \|\mathcal{L}_v^{-1} H_v^s f\|_{L^p}.$$

Therefore, we get from (49) the following Kato–Ponce inequality:

$$\|H_v^s fg\|_{L^r} \lesssim \|f\|_{L^{p_1}} \|H_v^s g\|_{L^{q_1}} + \|H_v^s f\|_{L^{p_2}} \|g\|_{q_2}, \tag{72}$$

with $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ and $|v| \leqslant \frac{1}{2}$.

Appendix B. Proof of Lemma 3.4

Up to taking the dual operator, we can assume $s \geqslant 0$. We decompose

$$\langle\cdot\rangle^N H^s V H^{-s} \langle\cdot\rangle^N = \left(\langle\cdot\rangle^N H^s \langle\cdot\rangle^{-N} H^{-s}\right) \left(H^s \langle\cdot\rangle^N V \langle\cdot\rangle^N H^{-s}\right) \left(H^s \langle\cdot\rangle^{-N} H^{-s} \langle\cdot\rangle^N\right).$$

As $|v| \leqslant \frac{1}{2}$, from (48), for any $u \in L^2$, we have

$$\left\|H^s \langle x\rangle^N V \langle x\rangle^N u\right\|_{L^2} \lesssim \left\|H_v^s \langle x\rangle^N V \langle x\rangle^N u\right\|_{L^2}.$$

Then by Kato–Ponce inequality (72), we have

$$\begin{aligned}\left\|H^s \langle x\rangle^N V \langle x\rangle^N u\right\|_{L^2} &\lesssim \|H_v^s \langle x\rangle^{2N} V\|_{L^\infty} \|u\|_{L^2} + \|\langle x\rangle^{2N} V\|_{L^\infty} \|H_v^s u\|_{L^2} \\ &\leqslant \|H_v^s \langle x\rangle^{2N} V\|_{L^\infty} \|H^s u\|_{L^2}\end{aligned}$$

Here in the second inequality, we use (48) again. Then we get

$$\left\|H^s \langle x\rangle^N V \langle x\rangle^N H^{-s}\right\|_{L^2 \to L^2} \lesssim \left\|H_v^s \langle x\rangle^{2N} V\right\|_{L^\infty}.$$

To conclude the proof, we need to show that

$$F_1 := \langle x\rangle^N H^s \langle x\rangle^{-N} H^{-s}$$

and

$$F_2 := H^s \langle x\rangle^{-N} H^{-s} \langle x\rangle^N$$

are bounded L^2-operators.

Before turning to the proof, we introduce some notations: we write

$$N = 2p + r,\ p \in \mathbb{N},\ r \in [0, 2), \tag{73}$$

and $m_s(\xi) = (1 + |\xi|^2)^{s/2}$. Notice that by induction we have for any multi-index α

$$m_{s,\alpha}(\xi) := \partial^\alpha m_s(\xi) = w_\alpha(\xi) \langle\xi\rangle^{s-|\alpha|}, \tag{74}$$

where w_α is a smooth bounded function (rational function of ξ and $\langle\xi\rangle$).

Boundedness of F_1. By Leibniz rule (in Fourier space) we write F_1 as a linear combination of terms

$$\langle x\rangle^r m_{s,\alpha_1}(-i\nabla) \frac{x^{\alpha_2}}{\langle x\rangle^N} H^{-s},\ |\alpha_1| + |\alpha_2| \leqslant 2p.$$

We write $r = \lfloor r\rfloor + \varepsilon$. Using $\langle x\rangle \leqslant 1 + |x|$, we realize that we only need to show the boundedness of terms of type $\langle x\rangle^\varepsilon m_{s,\alpha_1}(-i\nabla) \frac{x^{\alpha_2}}{\langle x\rangle^N} H^{-s}$ with $|\alpha_1| + |\alpha_2| \leqslant 2p + \lfloor r\rfloor$ (recall (73)).

We now commute $\langle x\rangle^{\varepsilon}$ with $m_{s,\alpha_1}(-i\nabla)$. The second term $m_{s,\alpha_1}(-i\nabla)\frac{x^{\alpha_2}\langle x\rangle^{\varepsilon}}{\langle x\rangle^N}H^{-s}$ is bounded by Kato–Ponce inequality and the estimate $\left\|m_{s,\alpha_1}(-i\nabla)H^{|\alpha_1|-s}\right\|_{L^2\to L^2} < +\infty$. Let us now deal with the commutator term $\left[\langle\cdot\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)\right]\frac{x^{\alpha_2}}{\langle x\rangle^N}H^{-s}$: if $\epsilon=0$, then $\left[\langle\cdot\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)\right]\frac{x^{\alpha_2}}{\langle x\rangle^N}H^{-s}=0$; if $\epsilon\in(0,1)$, we have

$$\langle x\rangle^{\varepsilon}=C_{\varepsilon}\int_0^{+\infty}\frac{du}{u^{1-\varepsilon}}\frac{\langle x\rangle^2}{\langle x\rangle^2+u^2},\ C_{\varepsilon}<+\infty.$$

Using $\frac{\langle x\rangle^2}{\langle x\rangle^2+u^2}=1-\frac{u^2}{\langle x\rangle^2+u^2}$, we infer

$$[\langle x\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)]=C_{\varepsilon}\int_0^{\infty}u^{1+\varepsilon}du[m_{s,\alpha_1}(-i\nabla),\frac{1}{\langle x\rangle^2+u^2}]=:A.$$

We now use Plancherel and estimate the integral kernel $\widehat{A}(\xi,\eta)$ with the help of Lemma 3.3. Before going further, we rewrite the operator $(\langle u\rangle^2-\Delta)^{-1}$ on $\mathbb{R}^3$ (see [14, Eq. (8), Section 6.23]):

$$(\langle u\rangle^2-\Delta)^{-1}\phi=\frac{1}{4\pi}\int_{\mathbb{R}^3}|\xi-\eta|^{-1}e^{-\langle u\rangle|\xi-\eta|}\phi(\eta)d\eta.$$

Then for a test function ϕ we have

$$\begin{aligned}
&4\pi\int_0^{+\infty}u^{1+\varepsilon}du[m_{s,\alpha_1}(\xi),\frac{1}{\langle u\rangle^2-\Delta}]\phi\\
&\quad=\int_0^{+\infty}u^{1+\varepsilon}\int_{\mathbb{R}^3}\frac{m_{s,\alpha_1}(\xi)-m_{s,\alpha_1}(\eta)}{|\xi-\eta|}e^{-\langle u\rangle|\xi-\eta|}\phi(\eta)\,d\eta du\\
&\quad=\int_{\mathbb{R}^3}d\eta\underbrace{\left(\int_0^{+\infty}u^{1+\varepsilon}e^{-(\langle u\rangle-1/2)|\xi-\eta|}|\xi-\eta|^{2+\varepsilon}du\right)}_{\mathcal{I}(|\xi-\eta|)}\frac{e^{-|\xi-\eta|/2}}{|\xi-\eta|^{3+\varepsilon}}\left[m_{s,\alpha_1}(\xi)-m_{s,\alpha_1}(\eta)\right]\phi(\eta).
\end{aligned}$$

Using $\langle u\rangle-1/2\geqslant\frac{\sqrt{3}}{2}u$, we get $\sup_{\delta\geqslant 0}\mathcal{I}(\delta)<+\infty$. Using the Taylor expansion of $m_{s,\alpha_1}(\xi)$ with respect to η up to order $\lceil s\rceil-|\alpha_1|$, we get:

$$\frac{e^{-|\xi-\eta|/2}}{|\xi-\eta|^{3+\varepsilon}}(m_{s,\alpha_1}(\xi)-m_{s,\alpha_1}(\eta))=\frac{e^{-|\xi-\eta|/2}}{|\xi-\eta|^{3+\varepsilon}}\Big[\sum_{k=1}^{\lceil s\rceil-|\alpha_1|-1}d^k m_{s,\alpha_1}(\eta)(\xi-\eta)^k+R_{s,\alpha_1}(\xi,\eta)\Big],$$

where $|d^k m_{s,\alpha_1}(\eta)(\xi-\eta)^k|\lesssim|\xi-\eta|^k\langle\eta\rangle^{s-|\alpha_1|-k}$ and the remainder satisfies $|R_{s,\alpha_1}(\xi,\eta)|\lesssim|\xi-\eta|^{s-|\alpha_1|}$. Since the function $|\cdot|^{k-3-\varepsilon}e^{-|\cdot|/2}$ is integrable for any $k\geqslant 1$, it follows that the operator $\left[\langle\cdot\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)\right]$ $H^{-s+|\alpha_1|+1}$ is $\|\cdot\|_{L^2\to L^2}$-bounded. Then by Kato–Ponce inequality the operator $H^{s-|\alpha_1|-1}\frac{x^{\alpha_2}}{\langle x\rangle}H^{-s}$ is $\|\cdot\|_{L^2\to L^2}$-bounded. We have shown the boundedness of

$$\left[\langle\cdot\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)\right]\frac{x^{\alpha_2}}{\langle x\rangle^N}H^{-s}=\Big(\left[\langle\cdot\rangle^{\varepsilon}, m_{s,\alpha_1}(-i\nabla)\right]H^{-s+|\alpha_1|+1}\Big)\Big(H^{s-|\alpha_1|-1}\frac{x^{\alpha_2}}{\langle x\rangle^N}H^{-s}\Big).$$

Boundedness of F_2. Let us now write

$$s=2q+t,\quad q\in\mathbb{N},\quad t\in[0,2).$$

By Leibniz rule $[H^{2q},\langle x\rangle^{-N}]$ is a linear combination of terms of type $m_{-N,\alpha_1}(x)\partial^{\alpha_2}$ (we recall $m_{-N,\alpha_1}=\partial^{\alpha_1}(\langle\cdot\rangle^{-N})$). Let $\zeta=t-\lfloor t\rfloor\in[0,1)$, we only need to check the boundedness of $H^{\zeta}m_{-N,\alpha_1}\partial^{\alpha_2}H^{-s}\langle x\rangle^N$ with $|\alpha_1|+|\alpha_2|\leqslant 2q+\lfloor t\rfloor$.

As for F_1, we commute H^{ζ} with m_{-N,α_1}. We first deal with the second term $m_{-N,\alpha_1}(x)\partial^{\alpha_2}H^{-2q-\lfloor t\rfloor}\langle x\rangle^N$ $:=T_2$. By duality and by Proposition 2.2 we have

$$\|T_2\|_{L^2\to L^2}=\|\langle x\rangle^N\partial^{\alpha_2}H^{-2q-\lfloor t\rfloor}m_{-N,\alpha_1}\|_{L^2\to L^2}\lesssim\|\langle x\rangle^N m_{-N,\alpha_1}\|_{L^\infty}<+\infty.$$

Let us now consider the commutator term $[H^\zeta, m_{-N,\alpha_1}(x)]\partial^{\alpha_2} H^{-s} \langle x\rangle^N$ which we further decompose:

$$[H^\zeta, m_{-N,\alpha_1}(\cdot)]\partial^{\alpha_2} H^{-s} \langle\cdot\rangle^N = \big([H^\zeta, m_{-N,\alpha_1}(\cdot)]\langle\cdot\rangle^N\big)\big(\langle\cdot\rangle^{-N}\partial^{\alpha_2} H^{-s}\langle\cdot\rangle^N\big).$$

As for T_2, the operator $\langle x\rangle^{-N}\partial^{\alpha_2}H^{-s}\langle x\rangle^N$ is bounded. Then, proceeding as we did for F_1, given a test function ϕ, for $\zeta \neq 0$, we have

$$[H^\zeta, m_{-N,\alpha_1}(\cdot)]\langle\cdot\rangle^N \phi = \frac{C_\zeta}{4\pi}\int_y dy U^\zeta \frac{m_{-N,\alpha_1}(x) - m_{-N,\alpha_1}(y)}{|x-y|^{4\zeta}} e^{-|x-y|/2}\langle y\rangle^N \phi(y)$$

with

$$U^\zeta = \Big(\int_0^{+\infty} u^{1+\zeta} e^{-(\langle u\rangle -1/2)|x-y|}|x-y|^{2+\zeta}du\Big).$$

By the mean-value theorem, we have:

$$|m_{-N,\alpha_1}(x) - m_{-N,\alpha_1}(y)|\langle y\rangle^N \lesssim |x-y|\big(|\partial^{\alpha_1}\nabla m_{-N}(x)| + |\partial^{\alpha_1}\nabla m_{-N}(y)|\big)\langle y\rangle^N.$$

We have $\sup_y |\partial^{\alpha_1}\nabla m_{-N}(y)|\langle y\rangle^N < +\infty$. Then using $\langle y\rangle^N \lesssim \langle x\rangle^N + \langle x-y\rangle^N$ we get

$$\frac{e^{-|x-y|/2}}{|x-y|^{2+\zeta}}|\partial^{\alpha_1}\nabla m_{-N}(x)|\langle y\rangle^N \lesssim \frac{e^{-|x-y|/2}}{|x-y|^{2+\zeta}}|\partial^{\alpha_1}\nabla m_{-N}(x)|(\langle x\rangle^N + \langle x-y\rangle^N).$$

Since $\frac{e^{-|y|/2}}{|y|^{2+\zeta}}[1+|y|^N]$ is integrable, we get that $[H^\zeta, m_{-N,\alpha_1}(x)]\langle\cdot\rangle^N$ is bounded.

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