Maximally symmetric nonlinear extension of electrodynamics and charged particles

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We consider couplings of electrically and magnetically charged sources to the maximally symmetric nonlinear extension of Maxwell's theory called ModMax. The aim is to reveal physical effects which distinguish ModMax from Maxwell's electrodynamics. We find that, in contrast to generic models of nonlinear electrodynamics, Lienard-Wiechert fields induced by a moving electric or magnetic particle, or a dyon, are exact solutions of the ModMax equations of motion. We then study whether and how ModMax nonlinearity affects properties of electromagnetic interactions of charged objects, in particular the Lorentz force, the Coulomb law, the Lienard-Wiechert fields, Dirac's and Schwinger's quantization of electric and magnetic charges, and the Compton Effect. In passing we also present an alternative form of the ModMax Lagrangian in terms of the coupling of Maxwell's theory to axion-dilaton-like auxiliary scalar fields which may be relevant for revealing the effective field theory origin of ModMax.

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I. INTRODUCTION

A nonlinear extension of free $D = 4$ Maxwell's electrodynamics, called ModMax, which preserves all the symmetries of the latter, namely the four-dimensional conformal symmetry and electric-magnetic duality, was discovered in Ref. [1]. This is the unique specimen with such properties among the variety of models of nonlinear electrodynamics (see [2] for a review). This theory exhibits interesting features, such as a peculiar form of birefringence [1,3] and, in spite of its intrinsic nonanalyticity, has plane waves [1] and topologically nontrivial knotted null electromagnetic field configurations [4] as exact solutions in its Hamiltonian formulation. Properties of the ModMax Lagrangian formulation were studied in more detail in [5,6] and its Hamiltonian formulation in [7].

ModMax arises as a weak field limit of a generalized two-parameter Born-Infeld theory [1,8]. Further generalizations of these theories were discussed in [9,10]. Quite remarkably, as was shown recently [11–13], both, ModMax and the generalized Born-Infeld theory arise as different $T\bar{T}$-like deformations of Maxwell's and the Born-Infeld theory, and the generalized Born-Infeld theory is a $T\bar{T}$ deformation of ModMax. Their supersymmetric extensions were constructed in [14,15], and in [16] (super)conformal higher-spin generalizations of ModMax were derived. A possibility of linking the generalized Born-Infeld theory to string theory by uncovering a stringlike nature of the former was discussed in [17].

Effects of ModMax and its generalizations on properties and thermodynamics of charged black holes (e.g., Taub-NUT, Reissner-Nordström ones, and others) have been studied in a number of papers [3,18–27]. The aim of this article is to study how the ModMax nonlinearity affects properties of interactions of electrically and magnetically charged point particles, in particular the Lorentz force, the Coulomb law, the Lienard-Wiechert fields, Dirac's and Schwinger's quantization of electric and magnetic charges, and the Compton effect. We will show that the Lienard-Wiechert fields created by a moving electric or magnetic particle, or a dyon, are exact solutions of the ModMax equations of motion, while this is not the case for most of nonlinear electrodynamics models. In passing we will also present an alternative form of the ModMax Lagrangian in terms of the coupling of Maxwell's theory to axion-dilaton-like auxiliary scalar fields which may be relevant for revealing the effective field theory origin of ModMax.

We will show that for a certain choice of the definition of physical electric and magnetic charges, which is associated with an appropriate rescaling of the source-free ModMax Lagrangian, and the standard minimal electromagnetic

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coupling of the charges, there is no difference in the Coulomb law and Lorentz forces describing interactions of two electric particles (one of which is a test particle) in ModMax and Maxwell's theory. The difference appears if magnetic monopoles or dyons are present. On the other hand, as is the case of vacuum birefringence of ModMax [1,3], the Compton scattering differs from that in Maxwell's theory for any scaling of the ModMax Lagrangian.

Notation and conventions.—We use the almost minus signature of the Lorentz metric $(+,-,-,-)$ and natural units in which the speed of light c and the Planck constant $\hbar$ are set to one.

II. MODMAX ELECTRODYNAMICS

The Lagrangian density of ModMax has the following form [1]

$$\begin{aligned}\mathcal{L} &= \cosh\gamma S + \sinh\gamma\sqrt{S^2+P^2}\\ &= \frac{\cosh\gamma}{2}(\mathbf{E}^2-\mathbf{B}^2) + \frac{\sinh\gamma}{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}\cdot\mathbf{B})^2},\end{aligned} \tag{2.1}$$

where

$$S=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2), \quad P=-\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}=\mathbf{E}\cdot\mathbf{B} \tag{2.2}$$

are the two independent $D=4$ Lorentz invariants constructed from the electromagnetic field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and its Hodge dual $\tilde{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$. $E_i=F_{0i}$ is the electric three-vector field $(i=1,2,3)$, $B^i=\tilde{F}^{0i}$ is the magnetic vector field, and γ is a dimensionless coupling constant. The conditions of causality and unitarity require this constant to be non-negative $\gamma\geq 0$ [1]. These values of γ also ensure that the Lagrangian density is a convex function of the electric field E_i [1] and that its energy-momentum tensor satisfies the weak, strong, and dominant energy conditions [28]. Note that Maxwell's electrodynamics is not a weak field limit of ModMax because of conformal invariance, but is recovered when the ModMax coupling constant tends to zero $\gamma\to 0$.

The Lagrangian field equations of the theory, accompanied by the Bianchi identities, are

$$\begin{aligned}&\partial_\mu G^{\mu\nu} = \cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma\partial_\mu\left(\frac{SF^{\mu\nu}+P\tilde{F}^{\mu\nu}}{\sqrt{S^2+P^2}}\right)=0,\\ &\partial_\mu\tilde{F}^{\mu\nu}=0,\end{aligned} \tag{2.3}$$

where

$$G^{\mu\nu}:=-2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}=\cosh\gamma F^{\mu\nu}+\sinh\gamma\left(\frac{SF^{\mu\nu}+P\tilde{F}^{\mu\nu}}{\sqrt{S^2+P^2}}\right). \tag{2.4}$$

The equations are nonlinear, but they linearize for field configurations for which $P=cS$ with c being a constant. So all the solutions of Maxwell's equations with $P=cS$ are solutions of ModMax theory.

One can notice that the equations of motion (2.3) are nonanalytic and are not well defined when the electromagnetic fields are null, i.e., the fields for which the Lorentz scalar and pseudoscalar (2.2) are zero

$$S=0, \qquad P=0. \tag{2.5}$$

In the null-field limit the ambiguity of the values of the scalar factors $\frac{S}{\sqrt{S^2+P^2}}$ and $\frac{P}{\sqrt{S^2+P^2}}$ in (2.3) range from -1 to $+1$. This might be an issue, since the class of solutions for which the electromagnetic fields are null, such as the plane waves, are not well defined in the ModMax Lagrangian formulation.[1] However, somewhat surprisingly, the Hamiltonian formulation of ModMax comes to the rescue [1]. In the ModMax Hamiltonian formulation the null electromagnatic fields are well defined. Among these configurations, the plane waves [1] and topologically nontrivial knotted electromagnetic fields [4] (generalizing those of Maxwell's theory [29]) are exact solutions of the ModMax Hamiltonian equations (see [1,2,4] for more details).

A. Conformal and duality invariance

The ModMax action $I=\int d^4x\mathcal{L}(S,P)$ is invariant under the $D=4$ conformal transformations [1,8], which can be easily checked for the rescaling of the coordinates and the fields with a constant parameter b

$$x^\mu\to bx^\mu, \qquad A_\mu\to b^{-1}A_\mu, \qquad F_{\mu\nu}\to b^{-2}F_{\mu\nu}. \tag{2.6}$$

The ModMax field equations and the Bianchi identities (2.3) are invariant under electric-magnetic duality $SO(2)$ rotations of $G^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ [1,5]

$$\begin{pmatrix} G^{\mu\nu}(F') \\ \tilde{F}'^{\mu\nu}\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix} G^{\mu\nu}(F) \\ \tilde{F}^{\mu\nu}\end{pmatrix}. \tag{2.7}$$

The duality invariance is ensured by the fact that the ModMax Lagrangian density (2.1) satisfies a condition which must hold for any duality-invariant nonlinear electrodynamics [30], namely

[1]Note, though, that the vacuum (in which $F_{\mu\nu}=0$) is a well-defined solution of the ModMax Lagrangian field equations (2.3) with zero energy-momentum tensor.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} - G_{\mu\nu}\tilde{G}^{\mu\nu} = 0 \Rightarrow \mathcal{L}_S^2 - \frac{2S}{P}\mathcal{L}_S\mathcal{L}_P - \mathcal{L}_P^2 = 1, \quad (2.8)$$

where $\mathcal{L}_S = \frac{\partial\mathcal{L}}{\partial S}$ and $\mathcal{L}_P = \frac{\partial\mathcal{L}}{\partial P}$.

III. ALTERNATIVE FORMS OF THE MODMAX LAGRANGIAN

The ModMax Lagrangian density (2.1) contains the square root function of S and P, which may hinder the quantization of this theory. Using three auxiliary scalar fields ψ_1, ψ_2, and ρ one can construct a Lagrangian density which is classically equivalent to ModMax and has the following analytic (polynomial) form[2]

$$\mathcal{L} = \cosh\gamma S + \sinh\gamma(S\psi_1 + P\psi_2) - \frac{1}{2}\rho^2(\psi_1^2 + \psi_2^2 - 1). \quad (3.1)$$

The equations of motion for the scalar fields are

$$\sinh\gamma S = \rho^2\psi_1, \qquad \sinh\gamma P = \rho^2\psi_2 \quad (3.2)$$

and

$$\rho(\psi_1^2 + \psi_2^2 - 1) = 0 \quad (3.3)$$

which for $\rho \neq 0$ reduces to

$$(\psi_1^2 + \psi_2^2 - 1) = 0. \quad (3.4)$$

For $\rho = 0$, the values of ψ_1 and ψ_2 are not defined from (3.3), while from (3.2) we get $S = 0 = P$, i.e., the electromagnetic fields are null. In this case the Lagrangian density (3.1) reduces to

$$\mathcal{L} = (\cosh\gamma + \psi_1 \sinh\gamma)S + (\sinh\gamma\psi_2)P \quad (3.5)$$

which is nothing but the Lagrangian density of the Bialynicki-Birula theory describing all possible configurations of the null electromagnetic fields (see [8,30,32] for details about this theory).

Let us now proceed with the case of finite ρ. Substituting the solution of equations (3.2) for ψ_1 and ψ_2 into (3.1) one gets

$$\mathcal{L} = \cosh\gamma S + \frac{1}{2}(\rho^{-2}\sinh^2\gamma(S^2 + P^2) + \rho^2). \quad (3.6)$$

Now the equation of motion of ρ gives

$$\rho^4 = \sinh^2\gamma(S^2 + P^2). \quad (3.7)$$

[2]This is somewhat similar to the case of the Born-Infeld theory in which the square root can be removed by introducing four real auxiliary scalar fields [31].

Substituting this back into (3.6) we get the original ModMax Lagrangian density (2.1).

Note that the choice of the square of the field ρ in the Lagrangian densities (3.1) and (3.6) ensured that (3.7) has the unique solution with the positive sign on its right-hand side (provided that $\gamma > 0$). If instead of ρ^2 we chose a generic auxiliary field $\hat{\rho}$, the corresponding field equation would have two solutions $\hat{\rho} = \pm\sinh\gamma\sqrt{S^2 + P^2}$, which would correspond to the ModMax Lagrangian densities with the plus and minus sign of $\sinh\gamma$ in (2.1), respectively.

Alternatively, the equations (3.3) and (3.2) can be solved as follows:

$$\psi_1 = \cos\varphi, \qquad \psi_2 = \sin\varphi, \quad (3.8)$$

$$\rho^2 = \frac{S\sinh\gamma}{\cos\varphi} = \frac{P\sinh\gamma}{\sin\varphi} > 0. \quad (3.9)$$

Substituting the expressions for ψ_1 and ψ_2 as the functions of the single auxiliary field φ into (3.1), one gets the Lagrangian density of the following form:

$$\mathcal{L} = \cosh\gamma S + \sinh\gamma(S\cos\varphi + P\sin\varphi). \quad (3.10)$$

To get back the original ModMax Lagrangian density (2.1) one eliminates the auxiliary field φ by solving its equation of motion and substitutes the solution back into Eq. (3.10)

$$S\sin\varphi - P\cos\varphi = 0 \Rightarrow \tan\varphi = \frac{P}{S} \Rightarrow \sin\varphi = \pm\frac{P}{\sqrt{S^2+P^2}},$$
$$\cos\varphi = \pm\frac{S}{\sqrt{S^2+P^2}}. \quad (3.11)$$

Substituting into (3.10) the solution for φ in (3.11) with the upper sign we recover again the ModMax Lagrangian density (2.1). In the formulation with the three auxiliary fields, the solution of (3.11) with the minus sign is excluded by the positive definiteness of the expressions in (3.9).

If, however, one considers the Lagrangian density (3.10) as an independent starting point without imposing the conditions (3.9), then *a priori* the second solution in (3.11) cannot be excluded and, upon substitution into (3.10), results in the Lagrangian density

$$\mathcal{L} = \cosh\gamma S - \sinh\gamma\sqrt{S^2 + P^2}. \quad (3.12)$$

For $\gamma > 0$ this Lagrangian density has problems with causality and unitarity, but is "healthy" for $\gamma \leq 0$ and coincides with (2.1) after the replacement $\gamma \to -\gamma$. So, starting with the Lagrangian density (3.10) without requiring (3.9), upon elimination of the auxiliary field φ one gets the ModMax Lagrangian with positive or negative coupling constant γ.

In spite of the above issue which requires further study, it is interesting to have a closer look at the Lagrangian density

(3.10). Let us note that the coupling between the scalar and electromagnetic fields described by the term $S\cos\varphi + P\sin\varphi$ was introduced in [33,34] as a building block of a model which the authors called nonlinear axion-dilaton electrodynamics. Here we would like to discuss a similarity of the Lagrangian density (3.10) to that of Maxwell's theory coupled to an axion $a(x)$ and a dilaton $\phi(x)$

$$\mathcal{L} = -\frac{e^{-\phi}}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = e^{-\phi}S - aP. \qquad (3.13)$$

If in (3.10) we formally promote the coupling constant γ to a scalar field $\gamma(x)$ we see that the Lagrangians (3.10) and (3.13) are related to each other by the field redefinitions

$$a = -\sinh\gamma\sin\varphi, \quad e^{-\phi} = \cosh\gamma + \sinh\gamma\cos\varphi \qquad (3.14)$$

whose inverse are

$$\cosh\gamma = \frac{e^{\phi}}{2}(e^{-2\phi} + a^2 + 1),$$
$$\sin\varphi = \mp\frac{2ae^{-\phi}}{\sqrt{(e^{-2\phi} + a^2 + 1)^2 - 4e^{-2\phi}}}. \qquad (3.15)$$

As is well known, a and ϕ parametrize a hyperbolic space $SL(2,R)/SO(2)$. So Eqs. (3.14) and (3.15) are just the change of coordinates on $SL(2,R)/SO(2)$ from (γ,φ) to (a,ϕ) and vice versa. More precisely the coordinates (γ,φ) parametrize two copies of $SL(2,R)/SO(2)$ since the relations (3.14) are invariant under the map $(\gamma,\varphi) \to (-\gamma, \varphi + \pi)$, which is a discrete symmetry of the Lagrangian density (3.10).

Maxwell's theory coupled to the axion and the dilaton (3.13) is well known to possess the electric-magnetic duality symmetry which is enhanced from $SO(2)$ to $SL(2,R)$ [35,36]. This symmetry is of course preserved by the field redefinitions (3.14). If γ is constant [as in (3.10)], then $SL(2,R)$ gets broken to $SO(2)$. The infinitesimal $SO(2)$ duality transformation parametrized by α acts on the scalar field ϕ as follows:

$$\delta\varphi = 2\alpha(\cosh\gamma + \sinh\gamma\cos\varphi) = 2\alpha\mathcal{L}_S. \qquad (3.16)$$

It is obtained from an $SL(2,R)$ transformation of a and ϕ which can be found e.g., in [35]. One can notice that φ transforms under $SO(2)$ nonlinearly and with a constant shift which resembles a Goldstone behavior of this field.

The axion and the dilaton are dynamical fields whose propagation is described by the $SL(2,R)$ invariant Lagrangian density

$$\mathcal{L}_{a,\phi} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{e^{2\phi}}{2}\partial_\mu a\partial^\mu a. \qquad (3.17)$$

Substituting the expressions (3.14) into (3.17) one gets the kinetic terms for the fields γ and φ which have the following form:

$$L_{\gamma,\varphi} = \frac{(\cosh 2\gamma + \sinh 2\gamma\cos\varphi)\partial_\mu\gamma\partial^\mu\gamma - 2\sinh^2\gamma\sin\varphi\partial_\mu\gamma\partial^\mu\varphi + \sinh^2\gamma\partial_\mu\varphi\partial^\mu\varphi}{2(\cosh\gamma + \sinh\gamma\cos\varphi)^2}. \qquad (3.18)$$

It would be of interest to figure out whether the Lagrangian density (3.10) with constant γ and nondynamical $\varphi(x)$ can be regarded as a certain conformal effective field theory limit of an axion-dilaton-coupled Maxwell's theory. We hope to address this problem elsewhere.

IV. COUPLING OF MODMAX TO ELECTRIC AND MAGNETIC CHARGES

In this section we will consider effects of the ModMax nonlinearity on the electromagnetic interactions of charged particles.

A. Electrically charged point particles

Consider first the minimal coupling of ModMax to a point particle of mass m_e carrying an electric charge e. The corresponding action has the following form:

$$S[A,y] = \int d^4x\,\mathcal{L} - \int d^4x\, j_e^\nu A_\nu - m_e\int d\tau\sqrt{\frac{dy^\mu(\tau)}{d\tau}\frac{dy_\mu(\tau)}{d\tau}} \qquad (4.1)$$

where $\mathcal{L}$ is the ModMax Lagrangian density (2.1), $y^\mu(\tau)$ is the particle worldline parametrized by the parameter τ, and

$$j_e^\mu = e\int d\tau\,\delta^{(4)}(x - y(\tau))\frac{dy^\mu}{d\tau}. \qquad (4.2)$$

The electromagnetic field equations (2.3) now acquire a source and take the following form:

$$\partial_\mu G^{\mu\nu} = \cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma\partial_\mu\left(\frac{SF^{\mu\nu} + P\tilde{F}^{\mu\nu}}{\sqrt{S^2 + P^2}}\right) = j_e^\nu,$$
$$\partial_\mu\tilde{F}^{\mu\nu} = 0. \qquad (4.3)$$

The equations of motion obtained by the variation of (4.1) with respect to the particle worldline $y^\mu(\tau)$ describe the conventional Lorentz force acting on the electrically charged particle

$$\frac{dp^\mu}{d\tau} = eF^{\mu\nu}(y)v_\nu, \tag{4.4}$$

where

$$v^\mu = \frac{dy^\mu(\tau)}{d\tau}, \qquad p^\mu = \frac{mv^\mu}{\sqrt{v^\nu v_\nu}}$$

are respectively the relativistic particle velocity and momentum.

As was mentioned in Sec. II, the Eqs. (4.3) are nonlinear in general; however, for some classes of fields they reduce to Maxwell's equations (modulo a rescaling). For instance, in the case in which the electric and magnetic fields are orthogonal i.e., satisfy $P = 0$ (but $S \neq 0$), the equations become linear,

$$e^{\gamma\frac{S}{|S|}}\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{4.5}$$

If electric and magnetic fields have the same strength, i.e., $S = 0$ (but $P \neq 0$) the equations also linearize

$$\cosh\gamma\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{4.6}$$

More generically we can study the case in which both S and P are nonzero, but proportional to each other, such that $P = cS$ with c being a constant. Then the equations of motion again reduce to linear ones,

$$\left(\cosh\gamma + \frac{S}{|S|}\frac{\sinh\gamma}{\sqrt{1+c^2}}\right)\partial_\mu F^{\mu\nu} = j_e^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{4.7}$$

The possibility that the equations of motion are linearized for certain configurations of electromagnetic fields is due to the conformal invariance of the theory. This does not happen in the nonconformal nonlinear electrodynamics such as the Born-Infeld theory, unless S and P are constant. For latest developments in the construction of solutions of duality-symmetric nonlinear electrodynamics models see [37].

B. Lienard-Wiechert fields

Above we have seen that particular classes of electromagnetic fields satisfy equations of motion which are similar to Maxwell ones, but with different effective coupling constants between the fields and the charges. Therefore, the solutions of Maxwell's equations which describe the fields satisfying $P = cS$ with c being a constant, are also the solutions of ModMax theory. For instance, the solutions of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{4.8}$$

describing the fields generated by a moving electrically charged point particle are called Lienard-Wiechert fields (see e.g., [38]). Their 4-vector potential is

$$A^\mu_{\mathrm{LW}} = \frac{e}{4\pi}\frac{v^\mu}{v^\nu l_\nu}\bigg|_{s=s_0} \tag{4.9}$$

where s is the particle proper time parameter defined by the relation $ds^2 = \eta_{\mu\nu}dy^\mu(\tau)dy^\nu(\tau)$, $l^\nu = x^\nu - y^\nu(s)$, $v^\nu(s) = \frac{dy^\nu(s)}{ds}$, and s_0 is the solution of $l^2 = 0$ with the condition $x^0 > y^0(s_0)$. The corresponding Lienard-Wiechert field strength has the following form:

$$F^{\mu\nu}_{\mathrm{LW}} = \frac{e}{4\pi}\frac{1}{(l_\rho v^\rho)^3}[l^\mu v^\nu + l^\mu l_\lambda(v^\lambda a^\nu - a^\lambda v^\nu) - (\mu \leftrightarrow \nu)]|_{s=s_0} \equiv ef^{\mu\nu}, \qquad \partial_\mu \tilde{f}^{\mu\nu} = 0, \tag{4.10}$$

where $a^\nu(s) = \frac{d^2y^\nu(s)}{ds^2}$.

One can see that the Lienard-Wiechert fields satisfy the condition $P = 0$, since $f_{\mu\nu}\tilde{f}^{\mu\nu} = 0$. Note also that

$$f^{\mu\nu}f_{\mu\nu} = -\frac{1}{8\pi^2(v^\mu l_\mu)^4} < 0 \Rightarrow S > 0. \tag{4.11}$$

Therefore, we can easily adopt (4.10) to be a solution of the ModMax equations (4.5) by making the following rescaling:

$$F^{\mu\nu}_{\mathrm{MMLW}} = e^{-\gamma}ef^{\mu\nu}. \tag{4.12}$$

So a difference of ModMax and Maxwell theory is that the electric charge in the Lienard-Wiechert fields gets effectively rescaled by the ModMax coupling constant. Let us elaborate on this difference in more detail by considering an electric particle at rest. In Maxwell's theory the fields produced by this particle are the Coulomb ones. In the ModMax theory the Coulomb fields have the same form, but with the rescaled electric charge

$$\vec{E} = e^{-\gamma}\frac{e}{4\pi}\frac{\vec{r}}{|\vec{r}|^3}, \qquad \vec{B} = \vec{0}, \tag{4.13}$$

where $\vec{r} = (x, y, z)$ is the position vector in space with the particle placed at $\vec{r} = 0$. Therefore, from (4.4) it follows that the Coulomb force acted on a test point particle of charge q at the position $\vec{r}_0$ is given by

$$\vec{F} = e^{-\gamma}\frac{eq}{4\pi}\frac{\vec{r}_0}{|\vec{r}_0|^3}, \tag{4.14}$$

so that the permittivity of the vacuum differs from Maxwell's theory by the factor e^γ.

The Coulomb force is tested in experiments with a very high precision, so for ModMax to be a theory consistent with the experiment for electromagnetic fields in the vacuum its coupling constant γ should be very small.[3] On the other hand, if one assumes that ModMax may serve as an effective theory for the description of electromagnetic properties of certain materials, there is *a priori* no restrictions on γ. For instance, in some materials the variation of the permittivity ϵ and the permeability μ from their vacuum values can be presumably modeled by a function of γ.

In Sec. V we will moreover show that the Coulomb law in the ModMax theory can be made exactly the same as in Maxwell's electrodynamcs, i.e., the scale factor e^{γ} can be removed from the field equations (4.5) by rescaling the source-free ModMax Lagrangian but keeping the minimal coupling to the electric charges intact. This is equivalent to an appropriate rescaling of the electric charge in the minimal coupling term of (4.1). However, e^{γ} factors will reappear in the magnetic sector of the theory, as we will see.

C. Magnetic monopoles

Let us now add to the action (4.1) terms that describe the coupling of the electromagnetic field to a point particle carrying a magnetic charge g (monopole) and having the mass m_g. To this end we will use the Dirac formulation in which the electromagnetic potential of the monopole is defined everywhere except for a line going from the monopole to infinity (the so-called "Dirac's string") (see e.g., [38,40] for a review). The presence of the magnetic current along the particle trajectory $z^{\mu}(\tau)$

$$j_g^{\mu} = g \int d\tau\, \delta^{(4)}(x - z(\tau)) \frac{dz^{\mu}}{d\tau} \tag{4.15}$$

modifies the Bianchi identity in (4.8) as follows:

$$\partial_{\mu} \tilde{F}^{\mu\nu} = j_g^{\nu}. \tag{4.16}$$

The solution of this equation is

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - \tilde{C}_{\mu\nu}, \tag{4.17}$$

where $\tilde{C}_{\mu\nu}$ is the Hodge dual of the antisymmetric tensor distribution

$$C^{\mu\nu}(x) = -g \iint d\tau d\sigma \left(\frac{\partial w^{\mu}}{\partial \tau} \frac{\partial w^{\nu}}{\partial \sigma} - \frac{\partial w^{\nu}}{\partial \tau} \frac{\partial w^{\mu}}{\partial \sigma} \right) \times \delta^{(4)}(x - w(\tau, \sigma)) \tag{4.18}$$

that has its support on the Dirac string worldsheet $\mathcal{N}$ stemming from the monopole

$$w^{\mu}(\tau, \sigma) = z^{\mu}(\tau) + u^{\mu}(\tau, \sigma), \qquad u^{\mu}(\tau, 0) = 0. \tag{4.19}$$

Using Stokes's theorem one can directly check that

$$\partial_{\mu} C^{\mu\nu} = j_g^{\nu} \tag{4.20}$$

and hence Eq. (4.16) holds. The action describing the coupling of ModMax to the electric and magnetic charges has the following form:

$$\begin{aligned} I[A, y, z, u] = \int d^4x\, \mathcal{L} - \int d^4x\, j_e^{\mu} A_{\mu} \\ - m_e \int d\tau \sqrt{\frac{dy^{\mu}(\tau)}{d\tau} \frac{dy_{\mu}(\tau)}{d\tau}} \\ - m_g \int d\tau \sqrt{\frac{dz^{\mu}(\tau)}{d\tau} \frac{dz_{\mu}(\tau)}{d\tau}} \end{aligned} \tag{4.21}$$

where $\mathcal{L}$ is as in (2.1) but with $F_{\mu\nu}$ defined in (4.17).

The action depends on the electromagnetic vector potential, the trajectories of the charges, and the worldsheet of the Dirac string. The latter must however be invisible, i.e., the physical effects should not depend on the string position in space.

We will now show, following [41], that the action (4.21) changes under a different choice of the string worldsheet by a term proportional to eg. The requirement of the independence of the path integral of the quantum theory of the Dirac string results in Dirac's quantization condition for the electric and magnetic charge. First, let us notice that Eq. (4.20) holds for any string which stems from the monopole. Let us take two strings whose worldsheets $\mathcal{N}_1$ and $\mathcal{N}_2$ described by $C_1^{\mu\nu}$ and $C_2^{\mu\nu}$ do not coincide. Since $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same boundary given by the worldline of the monopole, the boundaryless worldsheet $\mathcal{N}_2 - \mathcal{N}_1$ is the boundary of a three-dimensional manifold $\mathcal{S}$. Poincaré duality [42] then implies for the associated tensorial δ functions the relations

$$C_2^{\mu\nu} - C_1^{\mu\nu} = g \partial_{\rho} D^{\rho\mu\nu} \leftrightarrow \mathcal{N}_2 - \mathcal{N}_1 = \partial \mathcal{S}, \tag{4.22}$$

where $D^{\rho\mu\nu}$ is the tensorial δ function with support on $\mathcal{S}$ [analogous to (4.18)]. We thus have

$$\tilde{C}_2^{\mu\nu} - \tilde{C}_1^{\mu\nu} = g(\partial^{\mu} \tilde{D}^{\nu} - \partial^{\nu} \tilde{D}^{\mu}), \tag{4.23}$$

where $\tilde{D}^{\mu} = \frac{1}{6} \varepsilon^{\nu\rho\sigma\mu} D_{\nu\rho\sigma}$, i.e., the Hodge dual of the three-tensor $D^{\rho\mu\nu}$ defined in (4.22). Since the field strength (4.17) must be invariant under a change of Dirac strings, we require the four-potential to transform as follows when passing from one choice of the string to another:

$$A^{\mu} \to A^{\mu} + g \tilde{D}^{\mu}. \tag{4.24}$$

[3] A bound on the value of γ which one extracts from data of the experiment PVLAS [39] on the vacuum birefringence (discussed in Sec. VI) is $\gamma \leq 3 \times 10^{-22}$ [2].

Therefore, a finite change of the Dirac string only affects the minimal-interaction term of the action (4.21) which changes as follows (setting $j_e^\mu = e j^\mu$):

$$\Delta I = \Delta\left(\int d^4x\, j_e^\mu A_\mu\right) = eg\int j^\mu \tilde{D}_\mu d^4x. \quad (4.25)$$

According to the theory of "integer forms" [42], the integral on the right-hand side is an integer n, counting the number of intersections of the particle worldline with the three-volume $\mathcal{S}$.[4] We thus have

$$\Delta I = egn. \quad (4.26)$$

In the quantum theory the functional integrals always contain a factor e^{iI}. From (4.26) we see that e^{iI} is Dirac string independent if and only if

$$eg = 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots \quad (4.27)$$

which is the famous Dirac quantization condition.

Let us now derive the equations of motion of $A_\mu(x)$, $z^\mu(\tau)$ and $y^\mu(\tau)$ from the variation of (4.21). Varying the action with respect to A^μ we get the electromagnetic field equations

$$\partial_\mu G^{\mu\nu} = j_e^\nu \quad (4.28)$$

accompanied by "Bianchi's identities" sourced by the magnetic current (4.16).

The monopole equation of motion is obtained by varying the action with respect to $z^\mu(\tau)$, taking into account the dependence of w^μ on z^μ in (5.5)

$$\frac{dp_g^\mu}{d\tau} = g\tilde{G}^{\mu\nu}\frac{dz_\nu}{d\tau}. \quad (4.29)$$

Finally, let us show that the equation of motion of the electrically charged particle derived from the action (4.21) describes the Lorentz force (4.34) acting on the particle with $F^{\mu\nu}$ given in (4.17). To this end we perform the variation of the relevant terms in (4.21) with respect to $y_\mu(\tau)$ in the following way:

$$\begin{aligned}\delta_y\left(\int j_e^\nu A_\nu d^4x - m_e\int d\tau\right) &= \int\left(\frac{dp_e^\mu}{d\tau} - e(\partial^\mu A^\nu - \partial^\nu A^\mu)v_\nu\right)\delta y_\mu d\tau \\ &= \int\left(\frac{dp_e^\mu}{d\tau} - e(\partial^\mu A^\nu - \partial^\nu A^\mu - \tilde{C}^{\mu\nu})v_\nu\right)\delta y_\mu d\tau - e\int \tilde{C}^{\mu\nu}v_\nu\delta y_\mu d\tau \\ &= \int\left(\frac{dp_e^\mu}{d\tau} - eF^{\mu\nu}v_\nu\right)\delta y_\mu d\tau - \delta_y\left(\frac{1}{2}\int d^4x\,\tilde{C}_{\mu\nu}C_e^{\mu\nu}\right) \end{aligned} \quad (4.30)$$

where $C_e^{\mu\nu}$ is associated with a formal Dirac string worldsheet attached to the electric current, namely

$$C_e^{\mu\nu}(x) = -e\iint d\tau d\sigma\left(\frac{\partial w_e^\mu}{\partial\tau}\frac{\partial w_e^\nu}{\partial\sigma} - \frac{\partial w_e^\nu}{\partial\tau}\frac{\partial w_e^\mu}{\partial\sigma}\right) \times \delta^{(4)}(x - w_e(\tau,\sigma)) \quad (4.31)$$

and

$$w^\mu(\tau,\sigma) = y^\mu(\tau) + u_e^\mu(\tau,\sigma), \qquad u_e^\mu(\tau,0) = 0, \quad (4.32)$$

so that

$$\partial_\mu C^{\mu\nu} = j_e^\mu. \quad (4.33)$$

Note that the last term in (4.30), namely

$$\frac{1}{2}\int d^4x\,\tilde{C}_{\mu\nu}C_e^{\mu\nu}$$

is eg times the number of intersections between the Dirac string worldsheets of the electric particle and the monopole. Under infinitesimal variations of the electric particle worldline it is zero.[5]

The equation of motion of the electrically charged particle is thus

$$\frac{dp^\mu}{d\tau} = eF^{\mu\nu}(y)v_\nu, \quad (4.34)$$

which is the standard Lorentz force (4.4) induced by the field strength (4.17).

[4]In this derivation we put aside the issue of the *Dirac veto* discussed in detail in [41].

[5]In the derivation described in (4.30) we have added and subtracted the term with $\tilde{C}^{\mu\nu}$ and argued that one of the two terms drops out from the final result. For the explanation of this peculiarity see the more rigorous treatment of [41], tackling also the issue of the Dirac veto.

D. ModMax coupled to dyons

In this section we consider an action that describes the ModMax theory coupled to an arbitrary system of dyons with masses m_r, world lines y_r, and (electric and magnetic) charges (e_r, g_r), $r = 1, 2\ldots, N$. This action is

$$I[A,y] = \int \left(-\frac{1}{4}\cosh\gamma F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}\sinh\gamma\sqrt{(F^{\mu\nu}F_{\mu\nu})^2 + (F^{\mu\nu}\tilde{F}_{\mu\nu})^2} - A_\mu J_e^\mu\right)d^4x - \sum_r m_r \int ds_r. \tag{4.35}$$

In this case the field strength $F^{\mu\nu}$ is still given by (4.17), but the δ functions associated with the r Dirac strings are now given by

$$C^{\mu\nu} = \sum_r g_r C_r^{\mu\nu}, \qquad C_r^{\mu\nu} = -\iint d\tau d\sigma\left(\frac{\partial w_r^\mu}{\partial\tau}\frac{\partial w_r^\nu}{\partial\sigma} - \frac{\partial w_r^\nu}{\partial\tau}\frac{\partial w_r^\mu}{\partial\sigma}\right)\delta^{(4)}(x - w_r(\tau,\sigma)). \tag{4.36}$$

The total electric current is

$$J_e^\mu = \sum_r e_r j_r^\mu, \qquad j_r^\mu = \int d\tau\, \delta^{(4)}(x - z_r(\tau))\frac{dz_r^\mu}{d\tau}. \tag{4.37}$$

The electromagnetic field equations become

$$\partial_\mu G^{\mu\nu} = J_e^\nu \qquad \partial_\mu \tilde{F}^{\mu\nu} = J_g^\nu \tag{4.38}$$

instead of (4.16) and (4.28), where $J_g^\mu = \sum_r g_r j_r^\mu$ and $G^{\mu\nu}$ was defined in (2.4). And the equations of motion of the dyons are

$$\frac{dp_r^\mu}{d\tau_r} = (e_r F^{\mu\nu} + g_r\tilde{G}^{\mu\nu})\frac{dy_{r\nu}}{d\tau_r}. \tag{4.39}$$

Note that the equations of motion (4.38) and (4.39) are formally invariant under the duality rotations (2.7) if we assume that the electric and magnetic charges of each dyon get transformed accordingly

$$\begin{pmatrix} e' \\ g' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} e \\ g \end{pmatrix}. \tag{4.40}$$

Under a different choice of the Dirac strings [see (4.23)] accompanied by the shift of the four-vector potential

$$\begin{aligned} C_1^{\mu\nu} \to C_2^{\mu\nu} &= C_1^{\mu\nu} + \sum_r g_r(\partial^\mu\tilde{D}_r^\nu - \partial^\nu\tilde{D}_r^\mu), \\ A^\mu &\to A^\mu + \sum_r g_r\tilde{D}_r^\mu, \end{aligned} \tag{4.41}$$

the action changes as follows:

$$\begin{aligned} \Delta I &= \Delta\left(\int d^4x\, J_e^\mu A_\mu\right) = \sum_{r,s} e_r g_s \int j_{r\mu}\tilde{D}_s^\mu d^4x \\ &= \sum_{r,s} e_r g_s N_{rs}, \qquad N_{rs} \in \mathbb{N}, \end{aligned} \tag{4.42}$$

which leads to the Dirac-quantization conditions

$$e_r g_s = 2\pi n_{rs}, \qquad n_{rs} = 0, \pm 1, \pm 2, \ldots, \tag{4.43}$$

for each pair (r, s), that generalize the condition (4.27).

Notice that the conditions (4.43) violate the $SO(2)$ duality invariance (4.40). They are only invariant under the discrete Z_4 transformations generated by $e_r \to g_r$, $g_r \to -e_r$ (and the same for the s dyon) and $n_{rs} \to -n_{sr}$. To get an $SO(2)$-invariant charge quantization condition, following [41] we add to the action (4.35) a term involving only the Dirac strings, which does not change the equations of motion of the electromagnetic field, nor those of the dyons. The modified action takes the form

$$I' = I + \frac{1}{4}\int \sum_{r,s} e_r\tilde{C}_r^{\mu\nu} g_s C_{s\mu\nu}. \tag{4.44}$$

The dependence of the second term in (4.44) on the particle trajectories is proportional to an integer multiple of π, and hence does not contribute to infinitesimal variations of the classical action with respect to $z_r^\mu(\tau)$.

Under the change of the choice of the Dirac strings (4.41), with the use of (4.20) and (4.42), one finds that the variation of the modified action is

$$\begin{aligned} \Delta I' &= \Delta I - \frac{1}{2}\sum_{r,s}(e_r g_s + e_s g_r)\int j_\mu^r\tilde{D}_s^\mu d^4x \\ &= \frac{1}{2}\sum_{r,s}(e_r g_s - e_s g_r)\int j_\mu^r\tilde{D}_s^\mu d^4x \\ &= \frac{1}{2}\sum_{r,s}(e_r g_s - e_s g_r)N_{rs}. \end{aligned} \tag{4.45}$$

Requiring that this variation does not change e^{iI} in the functional integrals we get the Schwinger quantization condition

$$\frac{1}{2}(e_r g_s - e_s g_r) = 2\pi n_{rs}, \qquad n_{rs} = 0, \pm 1, \pm 2, \ldots, \tag{4.46}$$

which is invariant under the $SO(2)$ rotation (4.40) of the charges.

But there is even more, as was shown in [41] for Maxwell's theory, by adding the same second term as in (4.44) the resulting effective action for dyons obtained by performing the functional integral over the gauge field A^μ is invariant under $SO(2)$ duality. We assume that also the ModMax theory coupled to dyons and based on the modified action (4.44), will eventually be invariant under $SO(2)$ duality.

E. The Lienard-Wiechert field of a single dyon

Let us now consider the coupling to ModMax of a single dyon carrying the charges (e, g), i.e., we take $r = 1$ in (4.35)–(4.39). The electromagnetic field equations have the same form as in (4.38). We will now show that for the single dyon the equations (4.38) are satisfied by the Lienard-Wiechert fields induced by the moving dyon. As the ansatz we take the following generalization of the Lienard-Wiechert field strength (4.12)

$$F^{\mu\nu}_{\mathrm{dLW}} = e^{-\gamma} e f^{\mu\nu} - g\tilde{f}^{\mu\nu}, \tag{4.47}$$

where $f^{\mu\nu}$ has been defined in (4.10) and is such that

$$\partial_\mu f^{\mu\nu} = \int d\tau \delta^{(4)}(x - y(\tau)) \frac{dy^\nu}{d\tau}, \quad \partial_\mu \tilde{f}^{\mu\nu} = 0, \quad f_{\mu\nu}\tilde{f}^{\mu\nu} = 0. \tag{4.48}$$

So the field strength (4.47) is such that

$$\partial_\mu F^{\mu\nu}_{\mathrm{dLW}} = e^{-\gamma} j_e, \qquad \partial_\mu \tilde{F}^{\mu\nu}_{\mathrm{dLW}} = j_g. \tag{4.49}$$

So the second equation in (4.38) is also satisfied and it remains to show that the above relations imply that the first equation is satisfied as well. To this end we should compute the form of $G^{\mu\nu}$ in (2.4) for the ansatz (4.47). Using the last equation in (4.48) we find that[6]

$$\begin{aligned} S &= -\frac{1}{4}(e^{-2\gamma}e^2 - g^2) f_{\mu\nu} f^{\mu\nu}, \\ P &= -\frac{e^{-\gamma}}{2} eg f_{\mu\nu} f^{\mu\nu} \Rightarrow P = \frac{2e^{-\gamma} eg}{e^{-2\gamma} e^2 - g^2} S \end{aligned} \tag{4.50}$$

and hence

$$G^{\mu\nu} = e f^{\mu\nu} - e^{-\gamma} g \tilde{f}^{\mu\nu}, \tag{4.51}$$

which satisfies the first equation in (4.38). Thus we have proved that the Lienard-Wiechert fields created by the dyon are indeed exact solutions of the ModMax equations of motion. However, in general, linear combinations of Lienard-Wiechert fields created by several dyons will not be solutions of the ModMax nonlinear field equations.

The Lorentz force of the Lienard-Wiechert fields acting on a test dyon of an electric charge q and a magnetic charge r has the form [see (4.39) with $r = 1$, and use (4.51) and (4.47)]

$$\frac{dp^\mu}{d\tau} = [e^{-\gamma}(qe + rg) f^{\mu\nu} + (re - qg)\tilde{f}^{\mu\nu}] v_\nu. \tag{4.52}$$

As has already been observed for the Coulomb force (4.14), we see that in the "electric" part of the ModMax theory [first term on the l.h.s. of (4.52)], the Lorentz force differs from that in Maxwell's theory by a factor of $e^{-\gamma}$, while the "magnetic" parts are the same [the second term in (4.52)]. We will now show that the electric parts of the Lorentz forces in the two theories can be made equal by a suitable rescaling of the ModMax Lagrangian. But this will lead to a difference in the magnetic part of the Lorentz forces.

V. DIFFERENT SCALINGS OF THE MODMAX LAGRANGIAN

Let us consider the ModMax Lagrangian rescaled with the $e^{-\gamma}$ factor[7]

$$\mathcal{L} = e^{-\gamma}(\cosh\gamma S + \sinh\gamma\sqrt{S^2 + P^2}). \tag{5.1}$$

Then

$$\begin{aligned} G^{\mu\nu} = -\frac{2\partial\mathcal{L}}{\partial F_{\mu\nu}} = e^{-\gamma}\Bigg[\left(\cosh\gamma + \sinh\gamma\frac{S}{\sqrt{P^2 + S^2}}\right) F^{\mu\nu} \\ + \left(\sinh\gamma\frac{P}{\sqrt{P^2 + S^2}}\right)\tilde{F}^{\mu\nu}\Bigg] \end{aligned} \tag{5.2}$$

and now for $P = 0$ and $S > 0$ we have $G^{\mu\nu} = F^{\mu\nu}$ and the field equations reduce to the linear Maxwell equations. These have the standard Lienard-Wiechert fields (4.10) as solutions, while the Lorentz force remains the same as in (4.4). Therefore now the Coulomb law takes exactly the same form as in Maxwell's theory.

Let us note that the rescaling of the ModMax Lagrangian results in the following modification of the duality-invariance condition (2.8):

$$G_{\mu\nu}\tilde{G}^{\mu\nu} = e^{-2\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{5.3}$$

and so the duality transformation (2.7) now involves the rotation of $(G^{\mu\nu}, e^{-\gamma}\tilde{F}^{\mu\nu})$ instead of $(G^{\mu\nu}, \tilde{F}^{\mu\nu})$. Hence, if

[6]A recent result obtained in ModMax for a charged black hole in a Rindler frame [27] is a particular case of the general solution considered here.

[7]As "Note added" of [8], a different form of this Lagrangian was somewhat implicitly present in [43] among a more general class of conformally invariant models of nonlinear electrodynamics.

we want to couple the Lagrangian density (5.1) to a dyon in such a way that the formal $SO(2)$ duality invariance of the equations of motion holds for the electric and magnetic charges transformed as in (4.40), the field equations (4.38) should be modified as follows:

$$\partial_\mu G^{\mu\nu} = j_e^\nu, \qquad e^{-\gamma}\partial_\mu \tilde{F}^{\mu\nu} = j_g^\nu. \tag{5.4}$$

With this choice of the form of the second equation above, the tensor distribution $C^{\mu\nu}$ in (4.17) acquires the factor e^γ in comparison with (5.5), namely

$$C^{\mu\nu}(x) = -e^\gamma g \iint d\tau d\sigma \left(\frac{\partial w^\mu}{\partial \tau}\frac{\partial w^\nu}{\partial \sigma} - \frac{\partial w^\nu}{\partial \tau}\frac{\partial w^\mu}{\partial \sigma}\right) \times \delta^{(4)}(x - w(\tau,\sigma)). \tag{5.5}$$

The above rescalings then imply that the Lienard-Wiechert fields of the dyon that satisfy (5.4) have the following form:

$$F^{\mu\nu} = ef^{\mu\nu} - e^\gamma g\tilde{f}^{\mu\nu}, \qquad G^{\mu\nu} = ef^{\mu\nu} - e^{-\gamma} g\tilde{f}^{\mu\nu}. \tag{5.6}$$

These fields produce the following Lorentz force on a test dyon carrying the charges (q, r):

$$\frac{dp^\mu}{d\tau} = [(qe + rg)f^{\mu\nu} + e^\gamma(re - qg)\tilde{f}^{\mu\nu}]v_\nu. \tag{5.7}$$

The “electric” part of the force is the same as in Maxwell’s theory, while the “magnetic” part differs by the factor of e^γ. Note also that the Dirac quantization condition (4.27) and the Schwinger quantization condition (4.46) acquire now the factor of $e^{-\gamma}$

$$qg = 2\pi n e^{-\gamma}, \qquad \frac{1}{2}(re - qg) = 2\pi n e^{-\gamma}. \tag{5.8}$$

One can remove the factors of $e^{-\gamma}$ from the equations (5.4), the magnetic part of (5.7) and the quantization conditions (5.8) by redefining the physical values of the magnetic charges $g \to e^{-\gamma} g$ and $r \to e^{-\gamma} r$. Then the Lorentz force takes the form

$$\frac{dp^\mu}{d\tau} = [(qe + e^{-2\gamma} rg)f^{\mu\nu} + (re - qg)\tilde{f}^{\mu\nu}]v_\nu, \tag{5.9}$$

which preserves Schwinger’s quantization condition (4.46).

In any case, from (5.7) and (5.9) it follows that via the Lorentz force one can distinguish ModMax described by the Lagrangian (5.1) from Maxwell’s theory only in the presence of magnetic charges.[8] This suggests that one looks for other physical effects in which ModMax nonlinearity is manifested independently of the rescaling of its Lagrangian and charges. One of these effects is the birefringence of light propagating in a uniform strong electromagnetic background computed for ModMax in [1,3]. Another one is the Compton effect which we will analyze in the next section.

[8]Note that the rescaling of the ModMax Lagrangian and magnetic charges considered in this section also affects the contribution of charges to the black hole solutions in general relativity coupled to ModMax considered in [3,18–27].

VI. COMPTON EFFECT IN MODMAX

The Compton effect is a well-known photon-electron scattering process. It consists in the change of the wavelength of the photon when scattered by a free electron and manifests the corpuscular nature of light. In [44] properties of Compton scattering were derived in a generic theory of nonlinear electrodynamics (analytic in S and P) in a uniform magnetic background and used to study features of this effect in some concrete models, such as the Heisenberg-Euler, Born-Infeld, and a logarithmic electrodynamics. Like in the case of birefringence, switching on a strong uniform electromagnetic background allows one to distinguish the Compton effect in a given nonlinear electrodynamics model from that in Maxwell’s theory. In this section we will compute the Compton effect in ModMax in the presence of a magnetic background.

To this end we take the dispersion relations for the 4d wave vector $k^\mu = (\omega, \mathbf{k})$ of an electromagnetic wave propagating in a uniform magnetic background $\mathbf{B}$ computed for ModMax in [1]. The wave undergoes birefringence and splits in two rays. One of them propagates along the standard relativistic light cone

$$\omega^2 = |\mathbf{k}|^2, \tag{6.1}$$

while the other one propagates along a path characterized by the following dispersion relation:

$$\omega^2 = |\mathbf{k}|^2(\cos^2\beta + e^{-2\gamma}\sin^2\beta), \tag{6.2}$$

where

$$\cos\beta = \frac{\mathbf{k}\cdot\mathbf{B}}{|\mathbf{k}||\mathbf{B}|} \tag{6.3}$$

and β is the angle between the direction of the light ray and the background magnetic field. In this case (for $\beta \neq 0$ and $\gamma > 0$) the velocity of propagation $v = \sqrt{(\cos^2\beta + e^{-2\gamma}\sin^2\beta)}$ is less than the speed of light in the vacuum ($c = 1$).

We are interested in the Compton effect of the photon obeying Eq. (6.2), because the Compton scattering of the relativistic photon (6.1) will be the same as in Maxwell’s theory. Since we are using natural units in which $c = 1$ and $\hbar = 1$, the dispersion relation (6.2) is straightforwardly promoted to the condition on the four-momentum $p^\mu = (E, \mathbf{p})$ of the photon

$$E^2 = |\mathbf{p}|^2(\cos^2\beta + e^{-2\gamma}\sin^2\beta) \quad \text{with} \quad \cos\beta = \frac{\mathbf{p}\cdot\mathbf{B}}{|\mathbf{p}||\mathbf{B}|}. \tag{6.4}$$

A. Compton effect in the magnetic field orthogonal to the incoming photon momentum

As in [44] we first consider a magnetic field orthogonal to the direction of the incoming photon. We assume that an experimental setup is such that the electron can be considered at rest before scattering. Hence, the initial four-momentum of the electron is $p_e^\mu = (m_e, 0)$, where m_e and $\lambda_e = m_e^{-1}$ are the electron mass and the Compton wavelength, respectively. Upon scattering with the photon the electron acquires the four-momentum $p_e^\mu = (E_e, \mathbf{p}_e)$. The four-momentum of the incoming photon is $p^\mu = (E, \mathbf{p})$ and that of the outgoing one is $p'^\mu = (E', \mathbf{p}')$. Because of our choice of the orientation of $\mathbf{B}$ ($\mathbf{p}\cdot\mathbf{B} = 0$), from (6.4) we have

$$E^2 = e^{-2\gamma}|\mathbf{p}|^2,$$
$$E'^2 = |\mathbf{p}'|^2(\cos^2\beta' + e^{-2\gamma}\sin^2\beta') \equiv |\mathbf{p}'|^2 f^2(\beta', \gamma), \tag{6.5}$$

where β' is the angle between $\mathbf{B}$ and the momentum $\mathbf{p}'$ of the outgoing photon.

Conservation of the four-momentum implies that

$$E_e^2 = (E + m - E')^2,$$
$$\mathbf{p}_e^2 = (\mathbf{p} - \mathbf{p}')^2 = |\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2\mathbf{p}\cdot\mathbf{p}'. \tag{6.6}$$

Remembering that $E_e^2 - \mathbf{p}_e^2 = m_e^2$ and using (6.5) we get the following relation:

$$p^2(e^{-2\gamma} - 1) + p'^2(f^2 - 1) - 2pp'(e^{-\gamma}f - \cos\theta) + 2m(e^{-\gamma}p - p'f) = 0, \tag{6.7}$$

where $p = |\mathbf{p}|$, $p' = |\mathbf{p}'|$ and θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$.

For the wavelength $\lambda = 1/p$ and $\lambda' = 1/p'$ of the incoming and outgoing photon the above equation takes the following form:

$$\lambda'^2(e^{-2\gamma} - 1) + \lambda^2(f^2 - 1) - 2\lambda\lambda'(e^{-\gamma}f - \cos\theta) + 2\frac{\lambda\lambda'}{\lambda_e}(e^{-\gamma}\lambda' - \lambda f) = 0. \tag{6.8}$$

Solving this equation for positive λ' we find

$$\lambda' = \frac{1}{2 - 2\sinh\gamma(\lambda_e/\lambda)}\{f\lambda + \lambda_e(f - \cos\theta) + \sqrt{[e^\gamma f\lambda + \lambda_e(f - e^\gamma\cos\theta)]^2 + 2e^\gamma\lambda\lambda_e(1 - f^2)(1 - \sinh\gamma\lambda_e/\lambda)}\}. \tag{6.9}$$

One can see that for $\gamma = 0$ we have $f = 1$ and (6.9) reduces to the usual Compton scattering formula, as expected.

The above expression is rather complicated, but assuming that γ is small one can perform the Taylor expansion of (6.9) up to the first order in γ which results in the following simpler expression:

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta) + \gamma\lambda\left[1 - \frac{\sin^2\beta'}{1 + \lambda_e/\lambda(1 - \cos\theta)}\right]\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] + O(\gamma^2). \tag{6.10}$$

In ModMax the effect differs from that in Maxwell's theory by

$$\begin{aligned}\Delta\lambda \equiv \lambda' - \lambda - \lambda_e(1 - \cos\theta) &= \gamma\lambda\left[1 - \frac{\sin^2\beta'}{1 + \lambda_e/\lambda(1 - \cos\theta)}\right]\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] + O(\gamma^2)\\ &= \gamma\left[\frac{\lambda\cos^2\beta'}{1 + \lambda_e/\lambda(1 - \cos\theta)} + \frac{\lambda_e(1 - \cos\theta)}{1 + \lambda_e/\lambda(1 - \cos\theta)}\right]\left[1 + \left(\frac{\lambda_e}{\lambda} + \frac{\lambda_e^2}{\lambda^2}\right)(1 - \cos\theta)\right] + O(\gamma^2).\end{aligned} \tag{6.11}$$

We see that, with the chosen direction of the background magnetic field, in ModMax the difference of the wavelengths of the incoming and outgoing photons is a bit greater than that in Maxwell's electrodynamics. This can be attributed to two contributions. The first one is proportional to $\cos^2\beta'$ in (6.11) and is due to the refraction in the chosen magnetic background which is such that the effective velocity of the outgoing photon $v = \sqrt{\cos^2\beta' + e^{-2\gamma}\sin^2\beta'} \geq e^{-\gamma}$ [read from (6.5)] is in general greater than that of the incoming one. This contribution increases for larger wavelengths of the incoming photon. And the second contribution is nonzero even in the case in which $\beta' = \pi/2$ and the velocity does not change but is still less than the vacuum speed of light ($v = e^{-\gamma}$)

$$\Delta\lambda = \gamma \frac{\lambda_e(1-\cos\theta)}{1+\lambda_e/\lambda(1-\cos\theta)} \times \left[1+\left(\frac{\lambda_e}{\lambda}+\frac{\lambda_e^2}{\lambda^2}\right)(1-\cos\theta)\right] + O(\gamma^2).$$

When the trajectory of the scattered photon is along the magnetic field ($\beta' = 0$) the relation (6.10) simplifies to

$$\Delta\lambda = \gamma\lambda\left(1+\left(\frac{\lambda_e}{\lambda}+\frac{\lambda_e^2}{\lambda^2}\right)(1-\cos\theta)\right) + O(\gamma^2). \qquad (6.12)$$

B. Compton effect in the magnetic field parallel to the incoming photon momentum

One can consider a different setup for studying the Compton scattering by orienting the uniform magnetic field $\mathbf{B}$ along the momentum $\mathbf{p}$ of the incoming photon. With this configuration, indicating as before with θ the angle between $\mathbf{p}$ and $\mathbf{p}'$ [which is now the same as the angle between $\mathbf{B}$ and $\mathbf{p}'$ $(\theta = \beta')$], we see that the energy-momentum relations (6.4) take the following form:

$$E^2 = \mathbf{p}^2, \qquad E'^2 = \mathbf{p}'^2 f^2(\gamma,\theta) \quad \text{where}$$
$$f^2(\gamma,\theta) = \cos^2\theta + e^{-2\gamma}\sin^2\theta. \qquad (6.13)$$

Imposing the conservation of the four-momentum and proceeding as above we find the relation

$$\lambda' - f\lambda - \frac{\lambda\lambda_e}{2\lambda'}(1-f^2) - \lambda_e f + \lambda_e\cos\theta = 0. \qquad (6.14)$$

Solving this equation for positive λ' we get

$$\lambda' = \frac{1}{2}f\lambda + \frac{1}{2}\lambda_e(f-\cos\theta) + \frac{1}{2}\sqrt{\lambda_e^2(f-\cos\theta)^2 + \lambda^2 f^2 + 2\lambda\lambda_e(1-f\cos\theta)}. \qquad (6.15)$$

The Taylor expansion of this expression up to the first order in γ gives

$$\Delta\lambda = \lambda' - \lambda - \lambda_e(1-\cos\theta) = -\gamma\frac{\lambda\sin^2\theta}{1+\lambda_e/\lambda(1-\cos\theta)} \times \left[1+\left(\frac{\lambda_e}{\lambda}+\frac{\lambda_e^2}{\lambda^2}\right)(1-\cos\theta)\right] + O(\gamma^2). \qquad (6.16)$$

In contrast to the case discussed in Sec. VI A we see that the difference of the wavelengths of outgoing and incoming photons is smaller than in Maxwell's electrodynamics. This can be attributed to the fact that, as can be deduced from (6.13), the effective velocity of the outgoing photon is less than that of the incoming one due to the refraction in the magnetic background. In general, the difference between the effects in the two theories increases for larger wavelengths, but when the photon is recoiled, i.e., $\theta = \pi$, there is no difference with Maxwell's theory, because in this case the incoming and the recoiled photons travel with the speed of light in the vacuum.

VII. CONCLUSION

In this paper we have studied the interaction of electrically and magnetically charged particles in the ModMax theory. We have found that the Lienard-Wiechert fields induced by a moving electric or magnetic particle, or a dyon, are exact solutions of the ModMax equations of motion (with appropriately rescaled charges). The Lorentz force, and in particular the Coulomb law between two electric particles, can be made the same as in Maxwell's theory by choosing a suitable definition of the physical electric charges (which corresponds to a certain rescaling of the ModMax Lagrangian). However, in the presence of magnetic charges the Lorentz force in ModMax always differs from that in Maxwell's theory independently of the rescaling of the charges. Different rescalings with functions of the ModMax parameter γ result in different values of the permittivity and the permeability in ModMax in comparison with their vacuum values in Maxwell's theory. This can presumably be used for mimicking properties of some materials. We have then considered physical effects, namely birefringence and the Compton scattering, which manifest the discrepancy of ModMax from Maxwell's theory independently of the scaling of the ModMax Lagrangian and the definition of the physical charges. It will be of interest to study other physical effects, e.g., light-by-light scattering and the Hall effect as a measure of the fine structure constant in this theory. We have also presented alternative forms of the ModMax Lagrangian constructed with the use of auxiliary scalar fields. These forms may be useful for understanding the origin of ModMax as an effective field theory and for its quantization.

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