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Competitive runs on Government debt☆

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ABSTRACT

We study how limiting Government bonds redemptions may precipitate a run. We consider an economy where infinitely-living Government bonds finance the public sector which contributes to output that moves according to a geometric Brownian motion. Agents are heterogeneous, some, Investors, holding bonds directly, others, Depositors, holding deposits in a bank that, in turn, hold bonds. When output faces a negative shock agents have the incentive to sell bonds. Bond sales continue gradually until a floor is reached and the Government stops buying them. The presence of a floor may trigger a run as competing agents attempt to sell before the others. Our model captures the interdependence between heterogenous agents' exits decisions when a negative shock propagates both within a group and from one group to the other and to the bank. We show how the level of uncertainty determines whether Depositors or Investors exit first, whether exit is sequential, which group runs, whether an economy with financial intermediation is more resilient than one without.

1. Introduction

In this paper we study the dynamic of the exit from Government bonds when there are limits to redemptions and agents are heterogenous.

The motivation for this study arises from stylized facts related to financial crises. First, there is evidence that runs, that is panic sales of financial instruments, do not happen suddenly but are preceded by an orderly exit process of variable length. For example, in the 2007 asset-backed commercial paper crisis the index of mortgage-related securities dropped by about 20 percentage points in the months before runs intensified (Schroth et al., 2014). Similarly, the exit by U.S. Money Markets Mutual Funds (MMFs) from European banks in the Summer of 2011 due to concerns by U.S. investors on some European sovereign bonds held by European banks, happened at slow motion and came to be known as the "quiet run" (Chernenko & Sunderam, 2014). In the 2023 U.S. regional banks crisis, First Republic Bank lost $100 billion of deposits in March, received $30 billion of deposits from 11 banks, but was closed only on May 1. Credit Suisse's demise was even slower: in the last three months of 2022, customers drained around $120 billion from bank assets under management and approximately another $70 billion came out in the first three months of 2023 along with around $75 billion in deposits, until the authorities orchestrated its take over by rival UBS on March 19.

Second there is empirical evidence that limits to redemptions are the drivers of recent financial crises and that whole-sale creditors are the first ones to exit. The run on MMFs in March 2020 offers a good illustration of both stylized facts. As we detail later in the paper, the presence of redemption limits on MMFs combined with passive retail subscribers and reactive institutional investors was responsible for the run on these instruments at the beginning of the pandemic. Institutional investors rushed to redeem their shares of some classes of MMFs for fear that the redemption limits would come into effect, while retails subscribers barely moved.

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The failure of Silicon Valley Bank (SVB) in 2023 offers a further illustration of the role of the heterogeneneity of the creditors in the dynamic of runs. SVB, like many U.S. banks, had invested the large inflow of deposits following the monetary expansion of the previous decade in U.S. Treasuries. Its depositors were both FDIC-insured small retail depositors and large depositors — venture-capitalists and their technology start ups with balances in excess of the insurance cap (approximately 90% of its deposits). The fast and large increase of interest rates in 2022–23 resulted both in a decline of U.S. Treasuries prices and in a decline of the value of the technology sectors, thus inducing the whole-sale depositors to withdraw from SVB to cover their losses. To finance these withdrawals SVB sold a fraction of its U.S. Treasuries holdings, hence booking losses, which, in turn, prompted further withdrawals of the large depositors, with the retail ones exiting last.

While each of these episodes has unique features the underlying economics shows similarities that we aim to capture in our paper.

In our model we consider a stylized economy where Government debt finances the public sector which contributes to output at a decreasing rate. Output is subject to shocks that follow a geometric Brownian motion. There are two groups of atomistics agents: Depositors and Investors. Depositors have preferences for "moneyness" and place their endowment in a bank in exchange for payment instruments backed by Government bonds. Investors have preferences for output and buy Government bonds directly with their endowment.

Through taxes the Government appropriates output to service its debt by paying either a coupon to bondholders (bank and Investors), or the current output itself when this is short of the coupon. When output faces a negative shock Investors may have the incentive to sell bonds, and Depositors to withdraw from the bank, which, in turn, sells an identical amount of bonds. The only buyer of bonds is the Government itself which pays both with the current output and with the liquidation of part of the investment. Crucially, the Government stops buying back bonds when their level reaches an exogenous floor. This floor arises because the Government does not want to downsize the public sector beyond a certain level and it cannot issue new debt.

Since the exit decision (selling bonds and withdrawing deposits) is irreversible each agent determines his optimal exit strategy taking into account that by waiting he obtains new information on what he can receive in the future, reducing the downside risk.

To preview our main results, we capture the interdependence between agents' exits decisions when a negative shock propagates both within a group and from one group to the other and to the bank. We analyze this interdependence when atomistics individuals compete both across groups and within their group. Agents do not sell bonds at once, rather, they do it gradually until the floor approaches. As the floor approaches runs start, that is bonds are sold in bulk. In proximity of the floor individuals intensify their trades in the attempt to sell their bonds before the others. We show that while the exit of both groups is triggered by fundamentals, the order of the exit is determined by the volatility of the output. In particular the size of the shock (the trigger) below which a bondholder sells his bonds differs across the two groups. Which of the two groups exits first (that is whose trigger is higher) depends, among other variables, on whether output volatility is "low" or "high", conditions that we will specify in detail below.

If output volatility is low Investors exit first and they do so even if the Government is able to pay the coupon. Their exit stabilizes the economy. To understand why, observe that when agents (Investors or Depositors) exit, these bonds disappear from the economy and cease to contribute to output. Since bonds' contribution to output exhibits declining marginal productivity, the marginal productivity of the remaining bonds increases. Thus exits help to maintain constant the value of the resources per unit of bond that each bondholder is entitled to obtain. When all Investors have sold their bonds the exit decision rests only with Depositors. They begin to exit only when the shock is such that the Government is no longer able to pay the coupon. Also Depositors exit sequentially, that is they are willing to continue to hold bonds even if the Government cannot pay the entire coupon. Their sales, however, do not stabilize the economy as with fewer bonds outstanding their preferences for payment instruments make them very sensitive to the increased probability that the Government will run out of resources, which lowers the value of the resources per unit of bond that each depositor is entitled to obtain. Their exit becomes a run as the floor approaches.

When output volatility is high, we show that depositors, who are most in need of liquidity in the short term, exit first because they are no longer willing to wait for a future recovery in fundamentals. They do it sequentially when Government bonds do not pay the full value of the coupon. Additionally, as there are fewer bonds remaining, their incentive to leave after a negative shock is reinforced, that is the value of the resources per unit of bond that each depositor is entitled to obtain declines until they have sold all their bonds.

Furthermore, we measure the resiliency of the economy as the expected time of reaching the floor after a negative shock triggered the sale of the first bonds. The expected time depends on the different reactions of Investors and Depositors to a depletion of the stock of bonds. If volatility is low a competitive economy with both agents is always more resilient than one with Investors only. That is financial intermediation stabilizes the economy. This is so because when the level of bonds is high and volatility is low, by exiting before the prorata resources fall short to pay the coupon level, Investors keep the prorata resources high which induces Depositors to delay exit. If, on the contrary, volatility is high the presence of Investors does not necessarily stabilize because, also for them, it may be optimal to exit in the hope to obtain the coupon.

Our contributions are many. First, we show that runs on Government bonds crucially depend on a limited amount of redeemable bonds smaller than the total bonds issued. Second, our model accounts for the stylized fact that runs do not happen suddenly but are preceded by an orderly exit process of variable length. Thus our model captures the highly non-linear nature of financial crises as documented by recent episodes. Third, under low volatility conditions the presence of a financial intermediary that holds Government bonds far from destabilizing the economy as the bank-sovereign nexus literature argues, makes the economy better equipped to withstand negative shocks versus an economy where all agents hold Government bonds directly.

We conclude this section by observing that our analysis is general enough to be applied to runs on any borrower who is indebted to different creditors. However, we choose to focus on runs on Government debt because its liabilities have reserve-like features that, in turn, can back an intermediary's liabilities.

The rest of the paper is organized as follows. Section 2 reviews the literature. Section 3 sets up the model, and introduces the preferences of Investors and Depositors. In Section 4 we show that runs may happen since competing agents attempt to sell before the others as the level of remaining bonds approaches the floor (Proposition 1). In Section 5 we show how the level of uncertainty determines whether Depositors or Investors exit first, whether exit is sequential, and which group runs (Proposition 2). Section 6 is devoted to comparative statics analysis with respect to some key parameters of the model. In Section 7 we analyze the resilience of the economy, that is its capacity to resist to adverse events, by calculating the expected time of reaching the floor after a negative shock triggered the sale of the first bonds (Proposition 3). The Conclusions are in Section 8 and the proofs are in 3 Appendices.

2. Related literature

This paper is related to several strands of literature. The work most closely related to ours is He and Xiong (2012) who analyze the coordination problem of the creditors of a firm with a time-varying fundamental. The firm finances its asset by rolling over short term debt held by a continuum of small creditors. As the firm debt expiration is spread over time, the creditors decide to roll over taking into account the risk that in the future other creditors refuse to roll over. He and Xiong derive the unique equilibrium in which each maturing creditor chooses to run if the fundamental falls below a certain threshold. Anticipating the future behavior of other agents, each creditor sets a roll over threshold higher (i.e. more prone to runs) than that absent the coordination problem. We share with He and Xiong the underlying nature of Government debt. In fact, a short term debt with roll over is the same as a infinitely-living debt where bondholders have the option to demand repayment from the Government at any time.

Our paper shares with He and Xiong an exit acceleration effect during the run, when agents panic anticipating that the economy will not recover. However, when exit is sequential our paper exhibits also an opposite effect as agents delay exit waiting to see if the economy improves. Our paper also shares with He and Xiong's the determination of the roll over threshold. We add to their work the study of the interactions between agents with heterogeneous preferences, the role of the ceiling to bonds redemptions in precipitating runs, the study of the interaction between agents holding bonds directly and through a financial intermediary.

Bartolini (1993) was the first to study the role of a capacity constraint that is akin to the ceiling on Government bonds redemptions. He shows that in a dynamic model with stochastic returns, capital ceiling, and exogenous fixed exit price, decentralized decisions by a continuum of agents give rise to competitive runs. As the capacity constraint approaches, agents intensify their trades in the attempt to capture the scarcity rent that the ceiling entails. This is, indeed, what happens in our model because of the Government-introduced constraint on the agents' ability to exit. Once the stabilization effect generated by the sale of bonds by a group is over the remaining individuals compete in the attempt to capture the last rents before the Government blocks redemptions.

Our paper is related to the classical literature on bank runs. In the model of Diamond and Dybvig (1983) banks transform maturities which increases welfare in the good equilibrium, but exposes them to the risk of runs. The more recent modeling approaches that use global games (e.g. Rochet & Vives, 2004) avoid the multiplicity of equilibria but fall short of identifying the dynamic of the withdrawals, which, instead is at the heart of our paper. Pedersen (2009) studies the run dynamic in a model with two otherwise identical agents, one of which (Ms 1) suffers a negative shock. To cover her losses Ms 1 consumes its available liquidity and liquidates its portfolio holdings, which, in turn, lowers the values of these assets. If the other agent (Mr 2), is aware of the upcoming assets sales, he anticipates that also his portfolio will depreciate. Thus also Mr 2, that would have some liquidity to buy some of Ms 1's assets at a discount, ends up selling. That is liquidity may evaporate precisely when it is most useful.

Calvo (1988) and Cole and Kehoe (2000) model runs on public debt as rollover panics in a model of self-fulfilling expectations. In our model we do not have herding behavior as runs occur because of shocks to fundamentals when the stock of bonds is low enough. More generally the literature shows that the absence of herds is due to two conditions, which are met in our model. First herds occur when the investment decision is discrete. Lee (1993) shows that herding disappears when investment decisions are continuous. Second, herds occur when there are no traded assets with market-determined prices. Avery and Zemsky (1988) and Glosten and Milgrom (1985) show that once we allow for trade at market prices herds disappear. In our model the agents that want to exit can trade their bonds against their "liquidation value".

As mentioned, our paper is also related to the redemption limits to prevent runs on MMFs whose whole sale investors are more reactive than the retail ones. Cipriani et al. (2014) extend the Diamond and Dybvig (1983) model and show how a constraint similar to the one in this paper – the 2014 U.S. SEC ruling imposing limits to redemption on MMFs – may cause runs. MMFs liabilities are cash-like instruments used by corporate treasurers. MMFs offer a slightly higher return than cash because they invest in short term securities whose values, however, fluctuate. This exposes MMFs to runs like after Lehman bankruptcy, and during the European sovereign debt crisis in 2011. To prevent a repeat of similar episodes in 2014 the SEC allowed the managers of the Prime MMFs to suspend redemptions requests and impose fees if the share of their liquid assets falls below a certain fraction of their overall portfolio.[1] A consequence of such a floor was that, during the market turmoil of March 2020 corporate treasurers redeemed their shares of Prime MMFs before that floor is crossed, while households subscribers were much less reactive. Since MMFs are crucial to the plumbing of the U.S. financial system the federal Government again stepped in to backstop them (Cipriani & La Spada, 2021; Cipriani et al., 2020). In a lab experiment Huber et al. (2022) document a similar finding, namely finance professionals react to changes in fundamentals while non-professionals react to the frequency of negative returns.

[1] MMFs including tax-exempt funds, those that invest in government securities, and Prime funds (those not investing in Government securities or tax exempt assets) hold about $4.4 trillion assets, according to the Investment Company Institute. With the 2014 SEC ruling also the NAV changed from fixed to floating to better reflect the underlying values of the assets and discourage runs.

Finally, our paper is related to the literature on the bank-sovereign nexus. When banks hold significant amounts of the bonds of their sovereign a vicious circle may arise as a shock to the value of the bonds propagate to bank assets and back to the Government. (See for example Acharya et al., 2014; Fahri & Tirole, 2018; Gennaioli et al., 2014, among others). On the contrary, we show that it is the not the bank-sovereign nexus as such that may generate instability.

Regarding methodology, we refer primarily to the papers by Leahy (1993), Grenadier (2002) and Back and Paulsen (2009). Leahy (1993) is the benchmark model for continuous-time analysis of infinitely divisible capacity expansion (and scrapping) under perfect competition. He shows that the optimal investment strategy of a competitive firm is equal to that of a single firm in isolation. That, is, the investment timing of a single firm in isolation is identical to that of a firm that correctly anticipates the other firm's strategies. Grenadier (2002) describes the investment strategy of an oligopoly with n symmetric firms producing a single, homogeneous, non-storable good where each firm can increase capacity incrementally at any time at a fixed cost. Grenadier derives the Nash equilibria in open-loop strategies and shows that, introducing strategic considerations into a real options framework, the option-holding firms may face the risk of preemption and the likely option value erosion.[2] Furthermore, Back and Paulsen (2009) show that when the number of firms is large, $(n \to \infty)$, myopic (open-loop) behaviors is optimal[3] and that the perfect competition outcome derived in Leahy (1993) is part of a closed-loop equilibrium.[4]

Our specific contribution consists of extending their methodology to the case where two groups of atomistics agents, homogeneous among themselves but asymmetrical to each other, jointly determine the optimal exit strategy. For the strategic interaction between the two groups we focus on open-loop strategies. In addition, since the two groups have different preferences, we derive sufficient conditions that guarantee the existence of sequential exits and we study the order in which the groups leave the bond market. We show that sequential exit and heterogeneity contributes to make the economy more resilient.

3. Model set up

3.1. The economy

The economy consists of a Government, a bank, and a large number of agents: Investors (group I) and of Depositors (group D). In the economy there is only one good which is both the investment input of the production process and its output, consumed continuously. Only the Government can invest, at $t = 0$.

At $t = 0$ each Investors and each Depositors have an endowment of the good, which is infinitesimally small with respect to the total B_0^I and B_0^D respectively. Because of their preference, that we will describe later, at $t = 0$ Investors devote all their endowments B_0^I to buy bonds directly from the Government, and Depositors place all their endowments B_0^D into the bank in exchange of payment instruments issued by the bank. The bank, in turn, buys Government bonds by the same amount. Thus the sum of the endowments $B_0 = B_0^I + B_0^D$ equals the amount of bonds that the Government issues and the number of agents.

Government bonds are infinitely-divisible and infinitely-living, and give the bondholder the option to exit at any time, in which case the Government will buy them back according to the rules that we will specify below.

We assume that the Government can issue bonds only at $t = 0$. We stress that the Government cannot issue bonds afterwards. A fortiori, at $t > 0$ when bondholders exit the Government would not be able to issue new bonds. This captures a situation where the Government has lost further access to the market and thus cannot expand its size.

As a result the stock of bonds B_t at time $t > 0$ cannot increase, i.e. $B_t \leq B_0$, and declines when the Government buys them back.

At $t = 0$ the Government invests B_0. At time $t > 0$ the economy produces the good X_t which is a function $X\left(B_t\right)$ of the stock of Government bonds and of a shock y_t according to the following production function:

$$X_t = X\left(B_t\right) y_t. \tag{1}$$

Government bonds are a proxy for the size of the Government sector. The larger the stock of Government bonds at time t, the larger the size of the Government sector at that time and the larger its contributions to current output. Thus we assume that $X'\left(B_t\right) > 0$. We also assume that $X''\left(B_t\right) < 0$, i.e. the marginal contribution of Government sector to output declines, and that if the stock of Government debt goes to zero output vanishes, i.e. $X(0) = 0$. Moreover we rule out increasing average productivity by focusing on a fairly concave production function, that is we assume $\varepsilon(B) \equiv \frac{X'(B)B}{X(B)} < 1$ and $\varepsilon'(B) \leq 0$.

The shock y_t is the only fundamental source of uncertainty and follows a trendless geometric diffusion process:

$$\frac{dy_t}{y_t} = \sigma dW_t \quad \text{with } y_0 = y \text{ and } \sigma > 0, \tag{2}$$

where $\sigma > 0$ is the instantaneous volatility and $W_t \sim N(0,t)$ is a standard Wiener process having distribution with zero mean and variance t. The assumption of a trendless random walk allows us to focus on the pure effect of uncertainty, namely, on the effect of σ on both the strategic relationship between the two groups and the optimal exit timing of each individual within his own group.

[2] Baldursson (1998) discusses both expansion and downsizing.

[3] For Back and Paulsen (2009), at each instant in time, the investment game can be viewed as one of Stackelberg competition, in which each firm chooses its investment with all other firms instantaneously following. Since each firm would like to be the Stackelberg leader, when the number of firms increases, a stable point of this joint Stackelberg leadership is perfect competition.

[4] While open-loop strategies condition investment only on the information concerning exogenous uncertainty, the dynamic capacity-expansion problem in closed-loop strategies poses severe conceptual problems on the proper definition of feedback strategies (Back & Paulsen, 2009, p. 4532).

However, by the Markov property of (2), our results would not be qualitatively altered by using a non-zero trend for y_t.[5] We also assume that every agent has the same information about the current value of y_t.

The Government appropriates all the output through taxes. Its ability to service its debt at time t is given only by its tax revenues, $X\left(B_t\right) y_t$. Hence each bond pays a coupon, $c > 0$, exogenously predetermined at $t = 0$ or, the prorata resources available at time t, i.e. $\frac{X(B_t)}{B_t} y_t$ if they fall short of c.[6]

We assume that output is non-storable. This assumption, which is not new in either the banking or the macro literature (see e.g. Lucas, 1978; Myers & Rajan, 1998; Parlour et al., 2012), implies that no agent (Investors, Depositors, Government, bank) can build reserves to offset output short fall. For the same reason we assume that the bank does not store the endowment received at $t = 0$; that is the bank spends in Government bonds all it raises from Depositors. Finally we assume that all variables are observable including bonds sales and deposits withdrawals.

When either Investors or the bank sell their bonds the Government is committed to buy them back until a minimum level of debt is reached. In particular we assume that the Government stops buying bonds if their stock reaches the critically low level θB_0 with $0 < \theta < 1$. That is, bonds in the interval $\left[0, \theta B_0\right]$ are not redeemable. Since B_t is the size of the public sector in a given year the ceiling θB_0 can be interpreted as a policy decision that the size of the public sector does not fall short of it. Recall that the Government cannot issue additional debt to make up for any tax revenues shortfall.

We assume that the Government continues to pay interests on the remaining bonds θB_0 with its tax revenues. We also assume that $\theta B_0 < B_0^I$, B_0^D, which implies that the exit of neither group can be stopped entirely. We assume that the bonds sold back to the Government disappear from the economy and cease to contribute to produce output. The idea is that as the bonds outstanding decline, so is the size of the Government sector and its contribution to output.

Here we observe that Collard et al. (2015) show that indeed Governments default on their debt before exhausting their ability to pay. As documented by Reinhart and Rogoff (2009) there are countless instances of Governments that consolidate or default their debt, or temporarily suspend the redemption of their bonds or interests payment. In our model we focus on the irreversible stop of debt repayment when the threshold θB_0 is reached. We treat the threshold as exogenous and explore how its presence affects the exit decisions of the agents when they act individually.[7]

3.2. Investors

Investors have a lifetime utility function defined over the consumption of the output good. The utility of each member of group I at date t depends on y_t, B_t^I and B_t^D. Recall that if the Investor remains (does not exit) he is entitled to receive at each time $t > 0$ output in the amount $\min[\frac{X(B_t)}{B_t} y_t,\ c]$. Hence his instantaneous utility is:

$$u^I(y_t, B_t) = \min[\frac{X(B_t)}{B_t} y_t,\ c], \tag{3}$$

for all $B_t \leq B_0$. If the Investor exits, the Government is required to pay both the prorata resources available at time t, and the liquidation value of a unit of investment. For simplicity we assume that the liquidation value of one unit of the investment is proportional to the prorata output. Therefore upon exiting the bondholder receives a total

$$k^I(y_t, B_t) = \xi \cdot \frac{X(B_t)}{B_t} y_t, \tag{4}$$

where $\xi \geq 1$ indicates the liquidation rate[8], provided the floor θB_0 is not crossed. For simplicity in what follows we will refer to $\xi \cdot \frac{X(B_t)}{B_t} y_t$ as the liquidation value of one unit of bonds.

Eq. (3) says that if the Investor does not sell his bond he accepts to obtain the coupon or the prorata amount of resources available to the Government at time t. In other words, Investors accept that if $X\left(B_t\right) y_t > c B_t$ the Government distributes c to the Investor, otherwise it gives Investor the amount $X\left(B_t\right) y_t$ prorata.[9] The assumption that investors are willing to accept the prorata resources implies that they are not running immediately if they receive less than c. They are willing to accept less than c for a certain amount of time while waiting for the economy to improve.

By the properties of (1), the utility $u^I(\cdot) = \frac{X(B_t)}{B_t} y_t$ is twice continuously differentiable in (y, B^I, B^D), increasing in y and decreasing in B^I and B^D, with $\frac{\partial^2 u^I}{\partial B^I \partial y} \leq 0$.[10] Eq. (4) shares the properties as (3).

[5] Alternatively, we can introduce a constant drift μ and a risk-adjusted discount rate $r > \mu$, to admit that the economy generates a positive rate-of-cash-flow $\delta = r - \mu$ (McDonald & Siegel, 1984). None of our results depend on this assumption.

[6] Of course nothing of importance would change if the Government could only appropriate a constant fraction of the output.

[7] As mentioned the assumption of a floor on Government bonds is reminiscent of Bartolini (1993). In our model, however, while the floor is exogenous as in Bartolini (1993), the exit price is linked to the fundamentals, that is the prorata resources available and the liquidation value of the investment. It is outside the objective of this paper to model why the Government imposes the threshold θB_0, that is why the Government stops buying back its debt when still it has resources to pay for all or part of it.

[8] Nothing of important would change if the liquidation value of the investment is a positive quantity that adds up to the prorata output.

[9] This corresponds to an equal treatment of all bondholders at time t.

[10] The assumption that $\varepsilon(B) < 1$ guarantees that the share of resources that each individual can appropriate is strictly decreasing in the amount of bonds outstanding.

3.3. Depositors and bank

We model financial intermediation in a stylized way to capture the notions that a bank is needed to provide payment services and that to back these services the bank must hold Government bonds. In particular we assume that Depositors have preferences for "moneyness" that is for the service flow that money offers as a medium of exchange. Hence they need a financial instrument to make their payments each period. To that end Depositors put all their endowment good B_0^D in the bank in exchange for demand deposits (i.e. bank money). Only bank money and not the output good is accepted to make payments. Each unit of endowment deposited in the bank entitles a depositor to receive a per period interest rate b, with $b < c$, in bank money. With b at any date $t > 0$ Depositors make their payments which we assume exogenous. At $t = 0$ the only assets of the bank are B_0^D Government bonds that it has acquired with the B_0^D endowment of the Depositors. At $t = 0$ its only liabilities are the B_0^D deposits which are therefore 100% backed by Government bonds. To put it differently the bank issues deposits up to the amount of Government bonds it holds. This was indeed the case in early banking systems around the world.[11] At any date $t > 0$ the bank is purely passive and sells an amount of Government bonds only to satisfy Depositors' withdrawals. Thus we will refer for short to Depositors selling Government bonds directly.

If the bank receives the promised coupon cB_t^D and if Depositors do not withdraw it pays b per unit of deposit in bank money and consumes the difference, hence building no reserves. However, as output and hence tax revenues are stochastic and may fall short of the Government obligations at any date t, $c(B_t^I + B_t^D)$, it may happen that the bank receives less than bB_t^D. In that case the bank cannot pay b per unit of deposits.[12]

With these premises, we describe the utility of each member of group D at date t as:

$$u^D\left(B_t\right) = \varphi(B_t)u(w|w \geq b) + (1 - \varphi(B_t)u(w|w < b), \tag{5}$$

where $u(w|w \geq b)$ and $u(w|w < b)$ are the utilities in case of receiving a payment w not less or less than b respectively, and $\varphi(B_t)$ is the probability assigned to these utilities. That is, each depositor minimizes the discomfort from the fear of negative outcomes, i.e. $w < b$, by assigning a subjective distribution function $\varphi(B_t) = \varphi(B_t|B_0)$ defined on $B_t \in [0, B_0]$. We assume that $\varphi(B_t)$ is monotone non-decreasing $\varphi'(B_t) \geq 0$, and $\varphi'(0) \geq 0$, $\varphi'(B_0) \geq 0$,[13] and that all depositors share the same subjective distribution function.

The intuition is that the larger the difference between the initial and the current stock of bonds and the lower is the Depositors' subjective assessment on the solvency of the Government. If there are bond sales (by the Depositors themselves or by the Investors) it means that the economy has received a negative shock. The reduction of the stock, $B_t < B_0$, lowers the ability of the economy to withstand additional negative shocks. In addition, the assumption that $\varphi'(B_t)$ is monotone non-decreasing indicates that the first exits have greater impact on the credibility of the Government in complying with the terms of the contract than the last ones.[14]

Finally, we simplify (5) by assuming that $u(w|w \geq b) = b$ and $u(w|w < b) = 0$. That is, if the service that bank money offers is to allow Depositors to respect the terms of payments they must make each period, it does not matter how much the bank will give them in case it cannot give b, as the utility is zero anyway. Thus, if a Depositor does not withdraw his utility at each time $t \geq 0$ is:

$$u^D\left(B_t\right) = \varphi(B_t)b, \tag{6}$$

which is increasing and convex in B^D and B^I, with $\frac{\partial^2 u^D}{\partial B^D \partial y} = 0$.[15]

If a Depositor withdraws at date t, the bank sells an equal amount of bonds and, above the floor θB_0, the Government pays the bank, that in turn pays the Depositors the liquidation value of one unit of bonds (as for the Investors):

$$k^D(y, B_t) = \xi \cdot \frac{X\left(B_t\right)}{B_t} y_t. \tag{7}$$

[16]

[11] For example, during the Free Banking Era before the U.S. Civil War banks had to back their notes one-for-one with designated bonds that were deposited with a state authority. Solvent banks were entitled to the interest on the bonds. The notes were redeemable at par in USD on demand. Should the banker fail to honor its notes, the state would sell the securities and reimburse note holders out of the proceeds. When bank assets were worth less than thought, runs and suspension of convertibility were frequent. (Gorton & Zhang, 2023; Rockoff, 1974).

[12] Since the bank is passive, the qualitative results would not change even if it could hold reserves. Infact if the bank could hold reserves two cases could happen. If the reserves are always sufficient to pay b when there are negative shocks the model reduces to one with Investors only as Depositors would have no exit strategies. If on the contrary, reserves could not always be sufficient to pay b (which is the most realistic case as the model is stochastic) there is always a positive probability that Depositors exit. Hence, assuming that the bank does not hold reserves is the most extreme case.

[13] By the assumption $\varphi(B_t) = \varphi(B_t|B_0)$ we exclude the possibility of learning.

[14] Since $1 - \varphi(B_t)$ is the subjective belief of not receiving b, and since $\varphi'(B_t)$ is monotone non-decreasing, then $\frac{d^2(1-\varphi)}{dB_t^2} = -\varphi''(B_t) \leq 0$. This indicates that a decrease in B_t increases the subjective belief of not receiving b at a decreasing rate.

[15] By introducing $\varphi(B_t)$ we modify the rational expectations decision process. That is, in addition to risk, represented by the Geometric Brownian motion (2), we assume that the Depositors' decision criterion is derived as if the state variable were governed by the worst-case probability measure among the measures considered. The distribution $\varphi(B_t)$ defines their worst-case probability measure as in Hey (1984).

[16] Note that the exit process plays an important role in the formation of Depositors' beliefs. However, although the probability of receiving the interest reflects interdependence as it depends also on bond sales of the other group, we do not exclude the possibility that Depositors withdraw all their deposits even if their subjective belief of receiving b does not drop to zero.

4. Equilibrium exit strategies

In this section we derive conditions for analyzing the strategic relationship between the two groups when each individual, within his own group, competes with the other individuals of the same group when selling bonds. Since the two groups are heterogeneous we focus on the case in which exit from Government bonds is sequential, i.e. one group exits before the other.[17]

We limit our attention to monotone equilibria, i.e. the individual exit strategy is monotonic with respect to the fundamentals. Following Leahy (1993), Grenadier (2002) and Back and Paulsen (2009), this equilibria can be obtained without resorting to the solution of a fixed point problem. In particular, it can be obtained as solution of a real option problem, where each individual behaves "myopically" in determining his exit strategy. That is, the divestment plan of each individual will be contingent on the exogenous shock y_t and the stock of bonds B_t only but not on the others' exit strategy. By the properties of y_t, $u^i(\cdot)$, and $k^i(\cdot)$, $i = I, D$, the optimal exit strategies will take the form of trigger strategies, where each agent disinvests the first moment that y_t hits, from above, a trigger value $y^{*i}(B_t)$, $i = I, D$.[18] Formally, each agent solves a stopping problem like:

$$v^i(y, B) = \max_{\tau} \mathbb{E}_0 \left[\int_0^{\tau} u^i(y_t, B_t) e^{-rt} dt + k^i(y_\tau, B_\tau) e^{-r\tau} \right], \tag{8}$$

where $v^i(y, B)$ is the maximum value of the individual's life-time expected utility normalized by the bond unit, r is the discount rate, $\tau = \inf\,(t \geq 0$, such that $y_t \leq y^{*i}(B_t))$ is the exit timing, and $y^{*i}(B_t)$ specifies the critical value of the shock y beyond which the individual exits as a function of the current stock of bonds.

However, we show that when agents within a group compete, bulk sales, that is runs, may happen since agents attempt to sell before the others as the level of remaining bonds approaches the floor. As individuals compete with all the others their behavior will approach that of a perfectly competitive market.

In addition, we show that the order in which the groups exit matters. In the presence of a floor nothing changes for the first group to exit, as agents continue to exit sequentially, differently, for the second group, for whom, at a certain point in time the presence of a floor gives rise to a "competitive exit run". Individuals exit sequentially only up to a certain point in time, and then a run starts, exhausting at once the residual stock of bonds that can be sold. Hence, the conditions that lead one group to leave before the other are also worth to be studied.

The following Proposition summarizes the optimal equilibrium strategies (hereafter we drop the time index when this does not cause confusion).

Proposition 1.

Let $v^i(y, B)$ and $v^{-i}(y, B)$, be the value of the intertemporal utility of each individual belonging to the first group and of each individual belonging to the second group respectively, conditional on the current state (y, B). There exists a unique monotone equilibrium in which each individual of group i, $(-i)$ exits if y is below the trigger $y^{*i}(B)(y^{*-i}(B))$. In particular we obtain that:

If $y^{*i}(B) > y^{*-i}(B)$ for all B (i.e. the group $i$exits before the group $-i$), the function $v^i(y, B)$and the trigger $y^{*i}(B)$ are jointly determined by the solution of the differential equation:

$$\frac{1}{2}\sigma^2 y^2 v^i_{yy} - r v^i + u^i(y, B) = 0 \tag{9}$$

subject to the regular matching value and smooth pasting conditions.

On the contrary, the function $v^{-i}(y, B)$ and the trigger $y^{*-i}(B)$ are determined by the solution of the differential equation:

$$\frac{1}{2}\sigma^2 y^2 v^{-i}_{yy} - r v^{-i} + u^{-i}(y, B) = 0 \tag{10}$$

with the following distinction:

i) For all individuals belonging to the range $B > \hat{B}$, where $\hat{B}$ is the level of bonds at which the remaining bondholders choose to run, the regular matching value and smooth pasting conditions hold.

ii) On the contrary, for all individuals in the range $B \in [\theta B_0, \hat{B}]$, the exit trigger is constant at $y^{*-i}(\theta B_0)$, while the marginal value satisfies:

$$v^{-i}(y^{*-i}(\theta B_0), B) = k^{-i}(y^{*-i}(\theta B_0), B) \tag{11}$$

where $y^{*-i}(\theta B_0)$ and $\hat{B}$ are given by:

$$v^{-i}(y^{*-i}(\theta B_0), \theta B_0) = k^{-i}(y^{*-i}(\theta B_0), \theta B_0) \tag{12}$$

$$y^{*-i}(\theta B_0) = y^{*-i}(\hat{B}). \tag{13}$$

Proof. See Appendix A.

[17] The case of simultaneous exits is treated in Appendix A as a special case when the two groups are symmetric.

[18] To understand why the optimal exit strategies will take the form of trigger strategies, observe that y_t is Markovian, and $u^i(.)$ and $k^i(.)$ are continuous and differentiable functions independent from the time dimension. These are the typical conditions that transform a stopping time problem into one that determines a threshold above which to exert one's option. For the proof, we have referred to dynamic optimization solutions extensively studied in the Operations Research literature (see Harrison, 1985; Harrison & Taksar, 1983; Karatzas & Shreve, 1984), and to some well-known applications to a competitive economy (Bartolini, 1993; Dixit & Pindyck, 1994; Leahy, 1993), network externalities (Moretto, 2008).

As already stated, the proof of Proposition 1 is mainly based on the results of Leahy (1993), subsequently confirmed by Grenadier (2002) and Back and Paulsen (2009) for the case of a finite number of agents possessing a non-negligible quantity of infinitely divisible assets. Both Grenadier (2002) and Back and Paulsen (2009), provide conditions for optimal behavior in terms of trigger functions. They derive a set of value matching and smooth pasting conditions to identify the value function and best reply trigger for each individual player as open-loop exit strategies, where each player assumes that the strategy of the other players does not change. Then, they use Leahy (1993) to claim that, when the individual are negligibly small, the myopic triggers where individuals' exit decision is not affected by the concurrent decision of exit by others, are optimal.

Although the agents are negligibly small so that an individual's exit does not affect the stock of bonds in the economy, the exiting agents collectively generate an aggregate disinvestment process. The sequential process of exit continues until the shock y hits the lower trigger $y^{*-i}(\hat{B})$ at which the remaining bondholders choose to run. Note, in fact, that the exit trigger of individuals belonging to group $-i$ is constant and equal to $y^{*-i}(\theta B_0)$ in the interval $[\theta B_0, \hat{B}]$, therefore, for continuity it will also be equal to the trigger $y^{*-i}(\hat{B})$ which starts the run. When $B = \hat{B}$, if $y$hits the trigger $y^{*-i}(\hat{B})$ exit occurs but, in contrast with what would happen in the interval $B > \hat{B}$, the trigger is not decreased by the reduction of B as $\frac{dy^{*}(B)}{dB} = 0$. Thus, y remains at the threshold and exit continues (in a run) until the lower level θB_0 is binding. Conditions (11) and (13) imply that if a run is started, it will be arrested only when all redeemable bonds are sold.

Thanks to the above Proposition, the condition (13) can also tell us how the disinvestment process works when $\theta \to 0$:

Corollary 1. *By condition (13) it is immediate to note that when $\theta \to 0$also $\hat{B} \to 0$, and the exit process continues without any run.*[19]

The following corollary allows us to highlight the role of financial intermediaries by comparing an investor-only economy with a depositor-cum-bank only economy.

Corollary 2. *As Proposition 1 indicates, the order of the exit is dictated by whether $y^{*i}(B) > y^{*-i}(B)$ or vice versa. In the case where $y^{*i}(B) = y^{*-i}(B)$ the distinction between the two groups vanishes. Both behave as a single group and the exit process will be described by the conditions (10)–(13) of the text. The same is true if $B_0^D = 0$ or $B_0^I = 0$, when the economy is run by only Investors or only Depositors.*

We are now ready to use Proposition 1 to analyze the behavior of Investors and Depositors.

5. Sequential exits and debt runs

Given the endowments of the two groups and the individuals' utility functions, we now analyze the equilibrium in exit strategies and the debt runs. In particular we show how the level of uncertainty determines whether Depositors or Investors exit first, whether exit is sequential, and which group runs. As noted in Proposition 1 we distinguish whether the exit threshold of the Investors is above or below that of the Depositors, that is whether Investors exit before Depositors or viceversa.[20] Investors exit before Depositors if $y^{*I}(B) > y^{*D}(B)$ for all $B \in [0, B_0]$, while if $y^{*D}(B) > y^{*I}(B)$ for all $B \in [0, B_0]$, we obtain the reverse. Before proceeding, to make our analysis interesting, we impose some parameter restrictions. In particular, we assume that:[21]

$$\frac{1}{r} > \xi \tag{14}$$

i.e. the expected rent produced by the economy is greater than what the government pays to those who decide to leave. This rules out the scenario where exit becomes the dominant strategy even when the fundamental y is close to zero. Using (3), (6) and (4) in Proposition 1 and the condition that $b < c$, we summarize the candidate policy for the optimal exit process in the following proposition:

Proposition 2.

Part A. If $\frac{\beta}{\beta-1} \geq \xi r$, where $\beta = \frac{1}{2} - \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$ Investors always exit before Depositors. Investors begin to exit sequentially when the shock y drops below the trigger value given by:

$$y^{*I}(B) = c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]\frac{B}{X(B)}. \tag{15}$$

Afterwards, Depositors in the range $B \in (\hat{B}, B_0^D]$, exit sequentially when the shock drops below the trigger value:

$$y^{*D}(B) = b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]\frac{B}{X(B)}, \tag{16}$$

and finally, Depositors in the range $B \in [\theta B_0, \hat{B}^D]$ run to exit the first time that the shock y hits the trigger value $y^{*D}(\hat{B}^D)$, which is given by:

$$y^{*D}(\hat{B}^D) = b\frac{1}{\xi r}\frac{\beta}{\beta-1}\varphi(\hat{B}^D)\frac{\hat{B}^D}{X(\hat{B}^D)}, \tag{17}$$

[19] The properties of (1) guarantee that the optimal trigger $y^{*-i}(B)$ is strictly monotone.

[20] Recall that we treat the Depositor group as selling Government bonds directly, since the bank from which they withdraw is purely passive.

[21] A similar assumption is made by He and Xiong (2012).

while $\hat{B}^D$ is given by direct application of (13).

Part B. If $\frac{\beta}{\beta-1} < \xi r$ and $\frac{c}{b} < \left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{-\frac{1}{\gamma-1}}$, where $\gamma = \frac{1}{2} + \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} > 1$, we get $y^{*D}(B) > y^{*I}(B)$ for all B. Depositors exit first and sequentially when the shock y drops below the trigger value given by:

$$y^{*D}(B) = b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]\frac{B}{X(B)}. \tag{18}$$

Afterward Investors in the range $B \in (\hat{B}^I, B_0^I]$, exit sequentially when the shock y drops below the trigger value:

$$y^{*I}(B) = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}\frac{B}{X(B)}, \tag{19}$$

and finally, Investors in the range $B \in [\theta B_0, \hat{B}^I]$ run to exit the first time that the shock y hits the trigger $y^{*I}(\hat{B}^I)$which, is given by:

$$y^{*I}(\hat{B}^I) = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}\frac{\hat{B}^I}{X(\hat{B}^I)}, \tag{20}$$

and $\hat{B}^I$ by (13). On the contrary, if $\frac{\beta}{\beta-1} < \xi r$ and $\frac{c}{b} > \left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{-\frac{1}{\gamma-1}}$, which guarantees that $y^{*I}(B) > y^{*D}(B)$ for all B, Investors exit first and sequentially when y drops below the trigger value given by (19), while Depositors exit following the strategy described in (16) and (17).

Proof. See Appendix B.

Several comments are in order. First, as is standard in the real option literature, the term $\frac{\beta}{\beta-1} = 1 + \frac{1}{\beta-1} < 1$ accounts for the presence of uncertainty and irreversibility. That is, by waiting to sell the bond, each individual obtains new information on what he can receive in the future, reducing the downside risk. The value of this option to wait is captured by the term $\frac{1}{\beta-1}$ (Dixit & Pindyck, 1994, Ch.5).

Second, as the production function is concave with decreasing marginal productivity the Investors' optimal exit trigger is increasing in B (i.e. $\frac{\partial y^{*I}(B)}{\partial B} > 0$). That is, the lower the stock of bonds outstanding, more serious must be the negative shock to induce additional exit.[22]

Third, unlike the Investors, the exit of Depositors generates two opposite effects. While on a one hand a decrease in B^D increases the marginal productivity of the remaining bonds, on the other hand Depositors' expected discounted value of the utility decreases because it lowers the probability that the Government, and thus the bank, honor their commitment. However, the productivity effect prevails and, even for Depositors, the trigger function $y^{*D}(B)$ is increasing in B (See Appendix B).

Fourth, the distinction between Part A and Part B in the proposition relies on the inequality $\frac{\beta}{\beta-1} < \xi r$. It is easy to see that this inequality depends on the level of uncertainty measured by σ. In particular, since $\beta = \frac{1}{2} - \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$ satisfies the standard property that $\frac{\partial\beta}{\partial\sigma} > 0$, it is immediate to show that $\frac{\partial\frac{\beta}{\beta-1}}{\partial\sigma} < 0$ (see Appendix B). That is, $\frac{\beta}{\beta-1} < \xi r$ is satisfied only if σ is very high. How high depends on the discount rate r and the liquidation rate ξ, according to the following formula:

$$\sigma \geq \bar{\sigma} \equiv (1-\xi r)\sqrt{\frac{2}{\xi}}.$$

In the rest of the paper we refer to high and low volatility when $\sigma > \bar{\sigma}$, and $\sigma \leq \bar{\sigma}$, respectively.[23] Finally, quite intuitively as the liquidation value ξincreases the exit triggers decrease, that is to induce bondholders to exit greater shocks are necessary. However, when the uncertainty is high (i.e. $\sigma > \bar{\sigma}$), the reaction of Depositors to change of ξ is weaker than that of the Investors.[24] Let us now discuss in more detail the results presented in Proposition 2.

5.1. Investors exit first and frantic run by depositors

Let assume that $\frac{\beta}{\beta-1} \geq \xi r$ (i.e. $\sigma \geq \bar{\sigma}$). We can have a better intuition of both Investors and Depositors equilibrium strategy by writing the above triggers in terms of the prorata $\frac{X(B)}{B}y$. By simple algebra we can write (15) and (16) as:

$$\frac{X(B)}{B}y^{*I}(B) = c\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right], \tag{21}$$

[22] With constant returns to scale, i.e. $X''(B) = 0$ the trigger would be constant and Investors would exit altogether.

[23] For example, if we assume $r = 1.5\%$ as in He and Xiong (2012), and $\xi = 2$, we obtain $\bar{\sigma} = 0.97$, and with $\xi = 1.5$ the cut off $\bar{\sigma}$ rises to 1.13. In this respect, Danielsson et al. (2018), report standard deviations for emerging countries in excess of 150% for short periods of time.

[24] The elasticity of triggers with respect to the liquidation rate is $\frac{1}{\gamma-1}\frac{\xi r}{1-\xi r}$ for Investors and 1 for the Depositors respectively. As $\gamma > 1$, and $\xi \geq 1$, it can be proved that $\frac{1}{\gamma-1}\frac{\xi r}{1-\xi r}$ is always greater than 1.

and:

$$\frac{X(B)}{B}y^{*D}(B) = b\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\varphi(B)\right]. \tag{22}$$

According to (21), starting with a stock $B = B^I + B^D$, each Investors will sell his bond well *before* the value $\frac{X(B)}{B}y$ drops below the coupon c. Once Investors are out, according to (22), starting with a stock of bonds $B = B^D$, each member of group D sells his bonds if the value of the prorata $\frac{X(B)}{B}y$ drops below the threshold given by $b\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\varphi(B)\right]$. This threshold is given by the discounted value of the individual utility normalized by the rent that the government has to pay to those who decide to leave

$$\frac{1}{\xi}\int_0^\infty b\varphi(B)be^{-rt}dt = \frac{b}{r\xi}\varphi(B), \tag{23}$$

multiplied by $\frac{\beta}{\beta-1} < 1$ to capture the value of the gain to delay the exit to obtain new information on the evolution of y. Thus, in the region below $b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]$ the expected discounted value of the utility is lower than the liquidation value and Depositors disinvest.

With reference to (21) and (22), the dynamic of the optimal exit strategy is the same and it can be described in the following way. Given any total stock B, if the shock y lowers the prorata $\frac{X(B)}{B}y$ below $c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]$, $(b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]$, respectively) the Investors (Depositors), finds it convenient to immediately sells their bonds. Bond sales are sufficient to bring the value of the prorata back to $c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]$, $(b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]$, respectively). Because of the concavity of the production function a reduction of the outstanding bonds increases the marginal productivity of the remaining bonds and thus also the value $\frac{X(B)}{B}y$ preventing it to fall below $c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]$, $(b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]$, respectively). In the region above the triggers the optimal policy is inaction; Investors (Depositors), wait until the shock y drops again and then a new mass of Investors (Depositors) will exit just enough to keep the value $\frac{X(B)}{B}y$ from crossing the threshold $c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]$, $(b\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\varphi(B)\right]$, respectively).[25]

Although the exit strategy is the same, there is one important difference between Investors and Depositors. While for the bonds in the segment $[B_0^D, \hat{B}^D)$ the exit is sequential following the trigger (22), the last bonds, those in the segment $[\theta B_0, \hat{B}^D]$, will be sold in bulk. Furthermore, during the run, bonds are sold at a higher price than it would be in case of no floor. This is because individuals fear that they will lose their exit option while postponing the exit decision beyond $y^{*D}(\hat{B}^D)$.

To appreciate the intuition behind this result let us consider an exit starting slightly higher than $\hat{B}^D$. By (16) the net payoff at exit is

$$\frac{X(\hat{B}^D)}{\hat{B}^D}y^{*D}(\hat{B}^D) - \frac{b}{r\xi}\varphi(\hat{B}^D) = \frac{1}{\beta-1}\frac{b}{r\xi}\varphi(\hat{B}^D) < 0, \tag{24}$$

where:

$$\frac{b}{r}\varphi(\hat{B}^D) = \int_0^\infty u^D(\hat{B}^D)e^{-rt}dt \tag{25}$$

is the discounted value of the individual's utility. The negative difference is covered by the option to postpone the exit decision that is worth $\frac{1}{\beta-1}\frac{b}{r\xi}\varphi(\hat{B}^D)$. For values lower than $\hat{B}^D$ competitive individuals do not have the option of postponing their exit decisions as other individuals would seize the opportunity and exhaust the residual stock of bonds. However, as (17) shows, the increase in the value of the prorata that accompanies the sale of bonds more than offsets the decrease in the expected utility, such that the individuals' net payoff at exit is zero. Note in fact that, once the floor becomes binding, their option to exit is worthless, i.e. $\frac{X(\theta B_0)}{\theta B_0}y^{*D}(\hat{B}^D) - \frac{b}{r\xi}\varphi(\theta B_0) = 0$. See Fig. 1 for the illustration of this case.

5.2. Exit strategies with high and low uncertainty

Although the exit strategy described in the previous section is the same for both Part A and B, there are two important differences between the cases of low and high uncertainty. First, if $\frac{\beta}{\beta-1} \geq \xi r$ holds (i.e. $\sigma \leq \bar{\sigma}$), Investors' sequential exit sustains the value of the prorata $\frac{X(B)}{B}y$ at a level above c. By lowering the remaining stock of bonds exits prevent the output from falling below $c\left[\frac{\beta}{\beta-1}\frac{1}{r\xi}\right]$ (see Fig. 1). This guarantees that the Investors who decide not to leave will continue to receive the coupon.

On the contrary, when $\frac{\beta}{\beta-1} < \xi r$(i.e. $\sigma > \bar{\sigma}$), whether the Investors exit first or second, at the exit the value of the prorata is well below c. That is, with high volatility, Investors are willing to accept less than $c$for long periods of time before exiting. This is especially true if Depositors exit first. This is an event which, as was underlined in the discussion of Proposition 2, can happen when the liquidation rate ξ is sufficiently high. See Fig. 2 for the illustration of the case where Depositors exit before Investors and both below c.

The intuition for this result lies on the Investors' continuation value $v^I(y,B)$ at exit. It is obvious that an Investor in making the decision not to sell his bond rationally anticipates that the exit of other individuals generates a loss, i.e. $v^I(y,B) < \frac{c}{r}$ (See

[25] In the technical parlance, $\frac{X(B)}{B}y$ behaves as regulated process with $\frac{X(B)}{B}y^{*I}(B)(\frac{X(B)}{B}y^{*D}(B))$ as lower reflecting barrier. A reflected process is like a process that has the same dynamics as the original process, but is required to stay above a given barrier whenever the original process tends to cross it (Harrison, 1985).

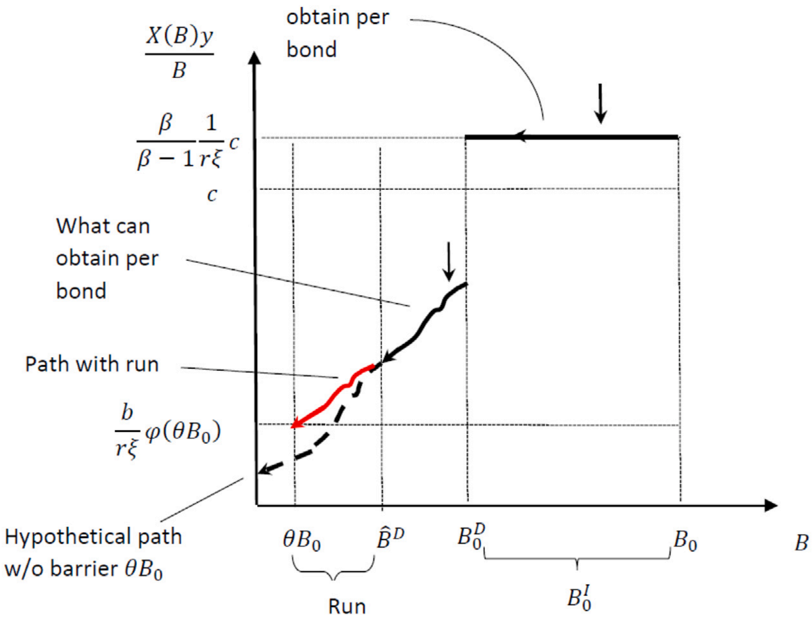

Fig. 1. Illustration of Proposition 2, part A. Investors exit sequentially and completely until B_0^D keeping the value of the prorata at $\frac{\beta}{\beta-1}\frac{1}{r\xi}c$. From B_0^D to $\hat{B}^D$ Depositors exit sequentially. Then, from $\hat{B}^D$ to θB_0 Depositors run.

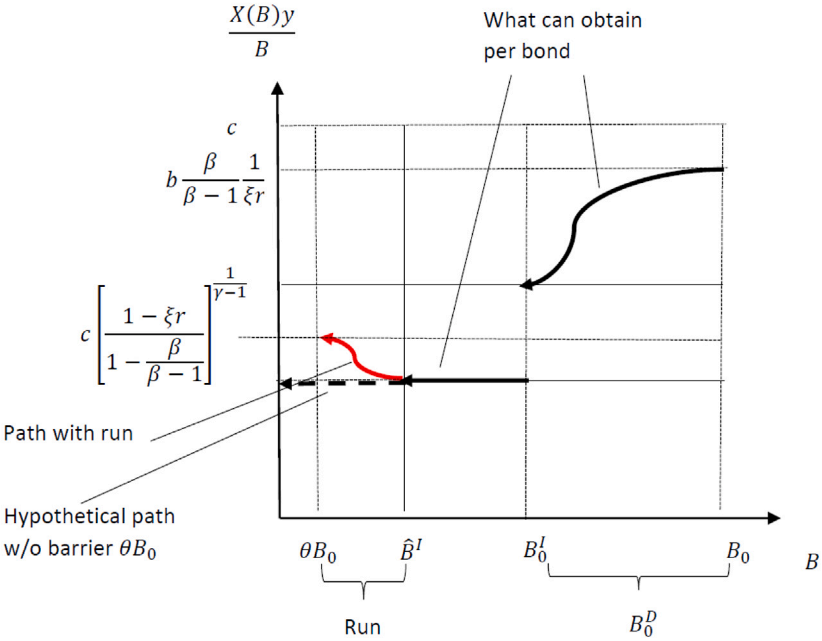

Fig. 2. Illustration of Proposition 2, part B with $\frac{c}{b} > \left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{-\frac{1}{\gamma-1}}$. Depositors exit first, completely and sequentially until B_0^I. Then Investors exit sequentially until $\hat{B}^I$. From $\hat{B}^I$ to θB_0 Investors run.

Appendix B). But how big is the loss he is willing to accept before exiting? By simple algebra, we can rewrite (15) and (19) as:

$$\xi \frac{X(B)}{B} y^{*I}(B) = \frac{c}{r} - \frac{c}{r}\frac{1}{1-\beta}, \tag{26}$$

and

$$\xi \frac{X(B)}{B} y^{*I}(B) = \frac{c}{r} - \frac{c}{r}\left\{1 + \frac{1}{\beta-1}\left[\frac{r}{1-r\xi}\right]\left[\frac{\frac{1}{r}-\xi}{-\frac{1}{\beta-1}\frac{1}{r}}\right]^{\frac{\gamma}{\gamma-1}}\right\}. \tag{27}$$

In both cases an Investor will agree to sell his bond for a liquidation value below the expected present value of the rent $\frac{c}{r}$ promised by the bond. However, when the uncertainty is low the loss he is willing to accept is also low. Thus the decision to sell the bond is justified when the liquidation value covers the difference between $\frac{c}{r}$ and the losses due to the new exit $-\frac{c}{r}\frac{1}{1-\beta}$. On the contrary, if the uncertainty is high, the amplitude of the shocks of y will also be high. An Investor will face long periods of time in which the payoff of the bond is, in fact, the prorata whose value is below c. This induces to sell the bond for a liquidation value far below $\frac{c}{r}$.

6. Comparative statics analysis

The following section is devoted to comparative statics analysis with respect to some key parameters of the model.

6.1. The effect of the floor

Regardless of whether Depositors or Investors run, the stock of bonds that starts the run depends positively on the floor introduced by the Government (see (16) and (20)**)**. However, adopting a Cobb–Douglas production function $X(B) = B^{\zeta}$, $\zeta \in (0,1)$ and an uniform distribution for the probability $\varphi(B) = \frac{B}{B_0}$ we are able to obtain close form solutions for both $\hat{B}^D$ and $\hat{B}^I$. In particular:

$$\hat{B}^D = \left(\frac{\beta-1}{\beta}\right)^{\frac{1}{2}-\zeta}\theta B_0, \tag{28}$$

and

$$\hat{B}^I = \left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{(1-\gamma)(\zeta-1)}}\theta B_0. \tag{29}$$

If to ensure greater stability to the economic system, the Government tightens the constraint, i.e. θ is increased, there will be, ceteris paribus, an acceleration of the run.

These arguments point to a possible trade-off faced by the Government between potential benefits of raising θ and the costs of run-acceleration. If the Government chooses to increase θ with the aim to maintain resources in the economy, it also lowers the resilience of the system, that is the time of reaching the floor after a negative shock triggered the sale of the first bond. In Section 7 we will analyze this aspect in detail.

Finally both (28) and (29) confirm what is expressed in Corollary 1. If $\theta \to 0$ also both $\hat{B}^D$ and $\hat{B}^I$ tend to zero and the optimal exit triggers reduce to (16) only for Depositors and to (19) only for Investors, respectively.

6.2. The effect of volatility σ on the equilibrium strategies

As already pointed out, the fundamental volatility σaffects the exit process of the two groups of agents in several ways. We now discuss in more detail its effect on the equilibrium exit strategies. There are three main channels on which we focus. First, it is easy to see that, when the fundamental volatility increases, both the triggers (15) and (16), decrease. The reason is that higher volatility increases the opportunity cost of exiting forever abandoning the possibility of gaining from a recovery of the economy, although higher volatility implies also a higher probability that the Government will not be able to pay the coupon c and/or the interest b. This opportunity cost increases even more than $\xi\frac{X(B)}{B}y$. Hence, instead of accelerating the exit rate, an increase in uncertainty implies more inertia.

Second, a higher volatility also increases the stock of bonds that starts the run, i.e. $\frac{\partial \hat{B}^i}{\partial \sigma} > 0$, $i = I, D$. That is, the same increase in the probability of Government insolvency also increases the fear of agents of not being able to recover their funds. This strategic effect of the uncertainty induces agents to intensify their trades in the attempt to sell their bonds before the others, hence causing an acceleration of the race.

Third, as volatility tends to spike during crises, a sudden increase in σ may make Investors to switch between the regime where the trigger is (15), towards the regime where the trigger is (19), hence strengthening the “wait and see” effect for both agents. That is, if avoiding losses is not possible the good strategy will be to try to minimize them and, then, taking corrective action before losses worsen is indeed the good strategy.

Finally, we stress that in our model runs rest on a mechanism – the floor on Government bonds – which is qualitatively similar to the ceiling of Bartolini (1993). The floor artificially creates resource scarcity which in turn generates rents and the incentive to capture them before the others.

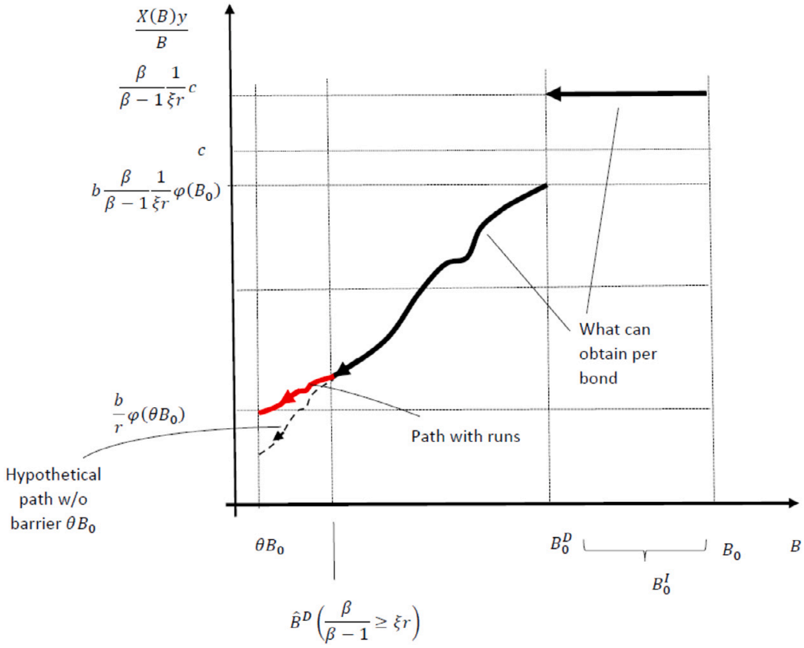

Fig. 3. Runs when uncertainty is low and investors exit first.

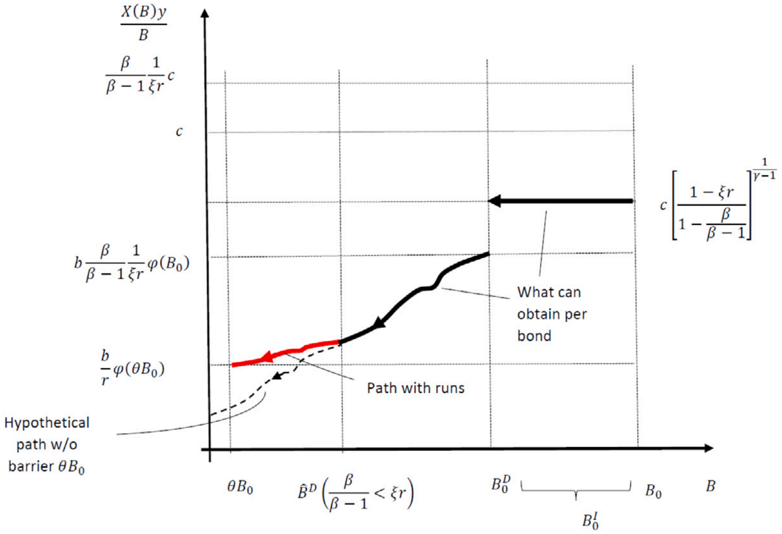

Fig. 4. Runs when uncertainty is high and investors exit first.

For a better comparison between the exit strategies under different uncertainty conditions, in Figs. 3 and 4 we illustrate the runs, described in Proposition 2, when Investors exit first, under low (i.e. $\frac{\beta}{\beta-1} \geq \xi r$; $\sigma \leq \bar{\sigma}$) and high (i.e. $\frac{\beta}{\beta-1} < \xi r$; $\sigma > \bar{\sigma}$) uncertainty conditions, respectively.

Figs. 3 and 4 show how the different exit paths between low and high uncertainty are important for the study of stability which we take on in Section 5.2.

6.3. Volatility: extreme cases

Related to what was highlighted in the previous section, it is interesting to study the two extreme cases of $\sigma \to 0$ and $\sigma \to \infty$.

If $\sigma \to 0$ we have $\beta \to -\infty$. Thus, absent uncertainty, we have $\frac{\beta}{\beta-1} \to 1$, from which it follows that the condition $\frac{\beta}{\beta-1} \geq \xi r$ is always satisfied and only Part A of Proposition 2 holds. This indicates that a reduction of the fundamental volatility anticipates the exit of both Investors and Depositors. Furthermore, for Investors exit happens as soon as the liquidation value equals the expected rent from the bond

$$\xi \frac{X(B)}{B} y^{*I}(B) = \frac{c}{r}, \tag{30}$$

that is Investors do not accept losses.

On the contrary, if $\sigma \to \infty$ we have $\beta \to 0$. With maximum uncertainty, we have $\frac{\beta}{\beta-1} \to 0$, and the condition $\frac{\beta}{\beta-1} < \xi r$ is always satisfied. However, as $\frac{b}{c} > 0$, only the case where Investors exit first of Part B holds. More formally we obtain

$$\xi \frac{X(B)}{B} y^{*I}(B) = c \left[\frac{1-\xi r}{1} \right]^{\frac{1}{0^+}} = 0 \tag{31}$$

which indicates that neither Investors nor Depositors will ever exit and they will accept the maximum level of losses. Hence in this model of real option, ceteris paribus, uncertainty is stabilizing because it makes postponing the exercise of the option optimal.

6.4. A tax on bond sales accelerates exit

Finally, from a policy standpoint it is interesting to investigate whether a tax on bond sales may deter exit.[26] We will illustrate the working of a tax in this model by focusing on Investors. It is easy to see that imposing a tax $0 < \tau < 1$ on bond sales the exit trigger (15) becomes:[27]

$$\hat{y}^{*I}(B^I, B^D) = \frac{c}{(1-\tau)\frac{\xi X(B)}{B}} \left[\frac{\beta}{\beta-1} \frac{1}{r} \right] > y^{*I}(B^I, B^D). \tag{32}$$

Quite intuitively, a tax lowers the amount received by Investors at exit, that will sell when the value of the bond is still high so that they can recover the loss imposed on the sale. The reason is that the shocks affect directly the value at which Investors can realize their bonds and only indirectly their payoffs if they do not sell. Hence a tax speeds exit up and may destabilize and, on the contrary, a retention bonus stabilizes. The opposite would happen if the redemption value is constant.

7. Resilience

We conclude analyzing whether an economy where a financial intermediary issues liabilities backed by the Government bonds is more resilient to shocks than an economy where agents hold those bonds directly. As noted there is a large literature that argues that the bank-sovereign nexus is a threat to financial stability as it amplifies and propagates shocks between banks and their sovereign and viceversa.

In particular we measure the capacity of an economy to be resistant to adverse events by calculating the expected time of reaching the floor after a negative shock triggered the sale of the first bonds. More precisely, as the range $B \in [\theta B_0, \hat{B}^i]$, $i = I, D$, is exhausted immediately in a run, this is equivalent to compute the expected time to reach $\hat{B}^i$, $i = I, D$. The higher is the expected time of reaching $\hat{B}^i$ the more resilient is the economy.

Continuing with the example of a Cobb–Douglas production function and an uniform distribution we distinguish two cases:

Case $\frac{\beta}{\beta-1} \geq \xi r$ (i.e. $\sigma \leq \bar{\sigma}$). Investors exit before Depositors. The expected time of reaching θB_0 is:

$$\mathbb{E}^{I,D}(T) = \psi^{-1} \ln \left(\frac{\beta-1}{\beta} \frac{b}{c} \theta^{2-\zeta} \right) > 0, \tag{33}$$

where $\psi = (-(1/2)\sigma^2)$.

Case $\frac{\beta}{\beta-1} < \xi r$ (i.e. $\sigma > \bar{\sigma}$). Investors exit before Depositors only if $y^{*I}(B) > y^{*D}(B)$. In this case the expected time of reaching θB_0 becomes:[28]

$$\mathbb{E}^{I,D}(T) = \psi^{-1} \ln \left(\frac{\frac{b}{\xi r}}{c \left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}} \right]^{\frac{1}{\gamma-1}}} \theta^{2-\zeta} \right) > 0. \tag{34}$$

[26] The SEC 2014 ruling on MMFs provides an example of such a tax. Besides suspension of reimbursements under stress conditions, that ruling allowed the managers of the prime MMFs to impose a fee on redemptions.

[27] The same effect applies also to (16).

[28] In theory there could also be the case $y^{*D}(B) > y^{*I}(B)$, where are the Depositors that exit before the Investors, and the expected time of reaching θB_0 is:

$$\mathbb{E}^{D,I}(T) = \psi^{-1} \ln \left(\frac{c \left[\frac{\gamma-\beta}{1-\beta} \right]^{\frac{1}{\gamma-1}} \left[\frac{1-r\xi}{1-\frac{\beta}{\beta-1}} \right]^{\frac{1}{\gamma-1}}}{b \left[\frac{\beta}{\beta-1} \frac{1}{r\xi} \right]} \theta^{1-\zeta} \right).$$

However if the uncertainty is high enough only the case where Investors exit before Depositors can happen.

Now, based on Proposition 2 and on (33) and (34), we are able to determine whether a competitive economy with both groups of agents is always more resilient than an economy with only one. This is:

Proposition 3. *If Investors exit before Depositors when uncertainty is low, a competitive economy with both agents is always more resilient than an economy with Investors only, i.e.:*

$$\mathbb{E}^{I,D}(T) - \mathbb{E}^{I}(T) = \psi^{-1} \ln\left(\frac{b}{c}\theta\right) \tag{35}$$

where $\mathbb{E}^{I}(T)$ indicates the mean time of reaching θB_0 when only Investors own the bonds.

When on the contrary, Investors exit before Depositors when uncertainty is high, the sign of the difference $E^{I,D}(T) - E^{I}(T)$ is not unique, except when $\frac{b}{c}$ or θ are sufficiently low which makes the difference positive and equal to:

$$\mathbb{E}^{I,D}(T) - \mathbb{E}^{I}(T) = \psi^{-1} \ln\left(\frac{b}{c}\theta\right) - \psi^{-1} \ln\left(\xi r\left[(\gamma-\beta)(1-r\xi)\right]^{\frac{1}{\gamma-1}}\right). \tag{36}$$

Proof. See Appendix C.

Under low volatility a financial intermediary that issues payment instruments backed by the bonds of its sovereign makes the economy better equipped to withstand negative shocks with respect to an economy where agents hold those bonds directly only. The intuition is that when the level of bonds is high and volatility is low, by exiting before the prorata resources fall short to pay the coupon level, Investors keep the prorata resources high which induces Depositors to delay exit.

Similarly, even when volatility is high financial intermediation makes the economy better equipped to withstand negative shocks if the claims of the Depositors with respect to those of the Investors, $\frac{b}{c}$, are sufficiently low, or the redemption floor θ is sufficiently low, to deter Depositor's exit.

However, when uncertainty is high the intermediary's ability to stabilize the economy is lowered. In fact, as when the uncertainty is very high (i.e. $\sigma \to \infty$), we have $\beta \to 0$ and $\gamma \to 1$, we may conclude that the second term on the R.H.S. of (36) is always negative, which makes the presence of two agents less important in terms of economic resilience than in the case of low volatility.

Finally, recall that one of our results is that Investors are willing to continue holding bonds for a while even if they do not receive the promised coupon c. This happens only when uncertainty is high as described in Fig. 2.

We then investigate the "patience" of the individual Investor, that is for how long on average an Investor is willing to hold his bond without receiving $c$before selling it. We can easily obtain a close form solution for the Investor's patience as (see Appendix C):

$$\mathbb{E}^{I}(\text{average time w/o coupon before exit}) = \psi^{-1} \ln\left(\left[(1-\beta)(1-\xi r)\right]^{\frac{1}{\gamma-1}}\right), \tag{37}$$

that we can calibrate. To this end we assume that $r = 1\%$ per year, that in period of high uncertainty the constant instantaneous volatility is $\sigma = 1.2$, and that upon exit the bondholder receives $\xi = 2$. With these parameters we obtain $\beta = -1.3701 \times 10^{-2}$ and $\gamma = 1.0137$. Recalling that $\psi = (-(1/2)\sigma^2)$, then (37) becomes:

$$\mathbb{E}^{I}(\text{average time w/o coupon before exit}) = 0.66856. \tag{38}$$

That is, under the maintained assumptions, the individual Investor exhibits a patience of about 8 months without coupon before selling, which is consistent with the highly non-linear nature of financial crises.

8. Conclusions

In this paper we study the dynamic of the exit from Government bonds when there are limits to bonds redemptions and agents are heterogeneous. We show that such limits may have the unintended consequence of inducing additional redemptions eventually precipitating a run.

The dynamic of the exit from Government bonds exhibits a non-linear pattern. After negative shocks individuals sell Government bonds gradually. Runs occur when atomistics individuals compete with each other and the remaining stock of Government bonds approaches the exogenous redemption floor.

Agents are heterogenous in the sense that a group, Depositors, have preferences for "moneyness". This motivates the presence of a financial intermediary where they place their endowment in exchange for payment instruments backed by Government bonds.

Our model captures the interdependence between heterogenous agents' exits decisions when a negative shock propagate both within a group and from one group to the other and to the bank. We show how the level of uncertainty determines whether Depositors or Investors exit first, whether exit is sequential, and which group runs.

The bank-sovereign nexus is affected by the contagion between the groups of individuals: as Investors sell, the stock of Government bonds declines, which increases the probability that Depositors withdraw from the bank out of fear that the Government does not honor its obligations. However, it is the not the bank-sovereign nexus *per se* that may generate instability, rather instability depends on the volatility of the underlying output and on the size of the Depositors' claims on the financial intermediary with respect to those of the Investors on the Government.

Finally our paper offers several testable implications. First, we show that in an economy with heterogenous agents the speed of the groups' reactions to negative shocks differs and depends on the output volatility. Under low volatility professional investors exit first, retail depositors late. Only under high volatility retail depositors exit first (Proposition 2). Second, a tax on bond sales to limit the exit from financial instruments, has the unintended consequence of speeding exit up, for the same reason of a redemption limit. Third, under low output volatility, the resiliency of the system is higher in an intermediated economy than in an economy where agents hold Government securities directly (Proposition 3).

Appendix A. Appendix 1

A.1. Proof of Proposition 1

We first derive the optimal myopic triggers when the agents, within each group, coordinate their actions (Lemmas 1 and 2). Then we show that the myopic property remains valid even when the agents are atomistics. We conclude by deriving the optimal individual's trigger when the agents face the floor.

We establish the following:

Lemma 1. *Let $V^i(y, B^I, B^D)$ be the value of the intertemporal utility of group $i = I, D$, conditional on the current state (y, B^I, B^D). A Nash equilibrium in open-loop exit strategies is characterized by two non-increasing processes such that:*

$$B^{i*} = \sup\left(B^i \mid y \geq y^{*i}(B^I, B^D)\right) \; for \; i = I, D, \tag{39}$$

where $y^{*i}(B^I, B^D)$ is the optimal exit trigger of group $i = I, D$.

In addition, if $y^{*i}(B^I, B^D) > y^{*-i}(B^I, B^D)$, the function $V^i(y, B^I, B^D)$ and the trigger $y^{*i}(B^I, B^D)$ are jointly determined by the solution of the following differential equation:

$$\frac{1}{2}\sigma^2 y^2 V^i_{yy} - rV^i + U^i(y, B^I, B^D) = 0, \tag{40}$$

subject to the value matching condition

$$\frac{\partial V^i(y^{*i}(B^I, B^D), B^I, B^D)}{\partial B^i} = k^i(y^{*i}(B^I, B^D), B^I, B^D), \tag{41}$$

and the smooth pasting condition

$$\frac{\partial^2 V^i(y^{*i}(B^I, B^D), B^I, B^D)}{\partial B^i \partial y} = \frac{\partial k^i(y^{*i}(B^I, B^D), B^I, B^D)}{\partial y}, \tag{42}$$

where $U^i(y, B^I, B^D) = u^i(y, B^I, B^D)B^i$.

Proof of Lemma 1. Open-loop exit strategies. For the sake of convenience the reader may consider $i = I$ (Investors) and $-i = D$ (Depositors), in the rest of the proof. Denoting with $U^i(y_t, B^i_t, B^{-i}_t)$ the total utility of group i, is the sum of the (same) utility of each individual of the group, i.e. $U^i(y_t, B^i_t, B^{-i}_t) = u^i(y_t, B^i_t, B^{-i}_t)B^i_t$. Consistently with the assumptions made on the utility functions of each individual, we assume that $U^i(\cdot)$ is twice continuously differentiable in (y, B^i, B^{-i}) non-decreasing in y, increasing in B^i and decreasing in B^{-i}, with $\frac{\partial^2 U^i}{\partial B^i \partial y} \geq 0$.

Let us consider first the case where each group coordinates the exit process. That is, at each time t each group can disinvest an infinitesimal amount $dB^i_t < 0$. Exit involves a payment by the Government, which depends on both B^i_t and B^{-i}_t, and the shock y_t. Denoting with $k^i(y_t, B^i_t, B^{-i}_t)$ this payment per unit of bond, we assume that $k^i(\cdot)$ is twice continuously differentiable in (y, B^i, B^{-i}), increasing in y and decreasing in B^i and B^{-i}, with $\frac{\partial^2 k^i}{\partial B^i \partial y} \leq 0$.

We consider the optimal exit by the two groups as part of a Nash equilibrium solution in open-loop exit strategies. Each group chooses its exit process B^i_t for $t \geq 0$ so as to maximize its intertemporal utility, conditional on the assumed exit strategies of the other group. The pair $(B^i_t, B^{-i}_t; t \geq 0)$ is a Nash equilibrium in open-loop strategies if, based on its initial stock B^i_0, the group i chooses a strategy B^i_t taking the strategies of the other group, i.e. B^{-i}_t, as given and viceversa.

Let $V^i(y, B^i, B^{-i}; B^i_t, B^{-i}_t)$ be the intertemporal utility of group i, with strategies $\left(B^i_t, B^{-i}_t; t \geq 0\right)$ and "generic" initial values (y, B^i, B^{-i}). This is given by:

$$V^i(y, B^i, B^{-i}; B^i_t, B^{-i}_t) = \mathbb{E}_0\left[\int_0^\infty U^i(y_t, B^i_t, B^{-i}_t)e^{-rt}dt - \int_0^\infty k^i(y_t, B^i_t, B^{-i}_t)e^{-rt}dB^i_t\right], \tag{43}$$

where the risk-neutral expectation operator is taken on the current state (y, B^i, B^{-i}). Given the above assumptions, determining agent's i optimal strategy against a given process B^{-i}_t corresponds to the solution of an optimal control problem of the type:

$$V^i(y, B^i, B^{-i}; B^{i*}_t, B^{-i}_t) = \max_{B^i_t} V^i(y, B^i, B^{-i}; B^i_t, B^{-i}_t), \quad \text{for all } t \geq 0. \tag{44}$$

In addition, given the properties of the utility functions of each individual, the optimal exit strategies will take the form of trigger strategies, where each group disinvests the first moment that y_t hits from above a threshold that is a function of the current stock of bonds B^i_t and of the exogenous process B^{-i}_t, i.e.:

$$B^{i*}_t = \sup\left(B^i \text{ such that } y_t \geq y^{*i}(B^i_t, B^{-i}_t)\right). \tag{45}$$

General solution. Consider the solution of (44) for the group i, the same procedure can be used for the group $-i$. In addition, in the following we will drop the time subscript for notational convenience and, without confusion, we always indicate dependence on B^I and B^D of the intertemporal utilities as well as of the trigger functions. The solution to (44) can be obtained starting within a time interval within which group i (Investors) has no new exits and takes the rival group's strategies as given. Furthermore, following Grenadier (2002), suppose that group $-i$ (Depositors) divests whenever the shock y drops to a given trigger function that we indicate with $y^{*-i} = y^{-i}(B^I, B^D)$.

Denote with $V^i(y, B^I, B^D; y^{*-i})$ the utility of group i contingent on the optimal exit strategy of group $-i$. Over the interval where group i has no exit the Government bonds pay a flow of utility U^i per unit of time, and experience a " capital" gain $\mathbb{E}[dV^i)]$ as y evolves stochastically. Assuming $V^i(y, B^I, B^D; y^{*-i})$ to be a twice-differentiable function with respect to y and using Itô's Lemma to expand dV^i we obtain:

$$\frac{1}{2}\sigma^2 y^2 V^i_{yy} - rV^i + U^i = 0. \tag{46}$$

The solution of (46) must satisfy four boundary conditions. The first one says that, without new Government bonds issues, to keep $V^i(y, B^I, B^D; y^{*-i})$ finite as y becomes high we need to impose that

$$\lim_{y\to\infty} V^i(y, B^I, B^D; y^{*-i}) = \mathbb{E}_0\left[\int_0^\infty U^i(y_t, B^I, B^D)e^{-rt}dt\right] < \infty \tag{47}$$

is satisfied for each fixed pair $(B^I, B^D; t \geq 0)$.

The second one is the matching value condition for i (Investors). If group i decides an infinitesimal exit dB^i at the optimal trigger y^{*i}, it must be the case that:

$$V^i(y^{*i}, B^I + dB, B^D; y^{*-i}) - k^i(y^{*i}, B^I, B^D)dB^i = V^i(y^{*i}, B^I, B^D; y^{*-i}).$$

Dividing by dB^i taking the limit for $dB^i \to 0$, the value matching condition becomes:

$$\frac{\partial V^i(y^{*i}, B^I, B^D; y^{*-i})}{\partial B^i} = k^i(y^{*i}, B^I, B^D). \tag{48}$$

The third one is the smooth pasting condition. If y^{*i} optimally triggers the exit of dB^i then by totally differentiating the matching value condition with respect to y we obtain:

$$V^i_y(y^{*i}, B^I + dB^i, B^D; y^{*-i}) - k^i_y(y^{*i}, B^I, B^D)dB^i = V^i_y(y^{*i}, B^I, B^D; y^{*-i}),$$

or in derivative way:

$$\frac{\partial^2 V^i(y^{*i}, B^I, B^D; y^{*-i})}{\partial B^i \partial y} = \frac{\partial k^i(y^{*i}, B^I, B^D)}{\partial y}. \tag{49}$$

Finally, we obtain the matching value condition at the group $-i$'s (Depositors) optimal exit trigger y^{*-i}. At the moment group $-i$ decreases the stock of bonds by the infinitesimal increment dB^{-i} it must be:

$$V^I(y^{*-i}, B^I, B^D + dB^{-i}; y^{*-i}) = V^i(y^{*-i}, B^I, B^D; y^{*-i}). \tag{50}$$

Dividing by dB^{-i} and taking the limit for $dB^{-i} \to 0$, we obtain:

$$\frac{\partial V^i(y^{*-i}, B^I, B^D; y^{*-i})}{\partial B^{-i}} = 0. \tag{51}$$

Myopic solution. We are able to reduce the above problem to a much simpler one in which group i's optimal exit strategies can ignore group $-i$'s exit strategies, and viceversa (Grenadier, 2002, Proposition 2; Back & Paulsen, 2009, Proposition 1). Assume that group i (Investors) decides to exit at y^{*i} by divesting $dB^i < 0$, on the assumption that the stock of the other group remains constant forever. In this case, the loss of utility for group i is the perpetual flow (the integral over time) of the marginal reduction $\frac{\partial U^i(y,B^I,B^D)}{\partial B^i}$. However, the trigger y^{*i} is the optimal trigger even if B^{-i} decreases by dB^{-i} as long as this occurs after y^{*i}, i.e. $y^{*i} > y^{*-i}$. Since, in this case, there is no fear of preemption prior to y^{*i}, this trigger turns out to be the optimal exit trigger for a group that behaves "myopically" as in Leahy (1993). At the time of group $-i$ exits, group i will receive the negative flow of utility $\frac{\partial U^i(y_t,B^I,B^D)}{\partial B^{-i}}$. However, this negative flow is beyond group i control and then future change in B^{-i} can be ignored in determining the optimal trigger y^{*i}.

Therefore, under the condition that $y^{*-i} < y^{*i}$, the solution of the intertemporal utility of group i and the associated trigger y^{*i}, reduces to the "myopic" optimal strategy given by the solution of the following differential equation (Grenadier, 2002, Proposition 2; Back & Paulsen, 2009, Proposition 1, Part A):[29]

$$\frac{1}{2}\sigma^2 y^2 V^i_{yy} - rV^i + U^i = 0, \tag{52}$$

subject to the value matching condition

$$\frac{\partial V^i(y^{*i}, B^I, B^D)}{\partial B^i} = k^i(y^{*i}, B^I, B^D), \tag{53}$$

[29] Note that this problem is very standard in the real option literature (Dixit & Pindyck, 1994, ch.5).

and the smooth pasting condition

$$\frac{\partial^2 V^i(y^{*i}, B^I, B^D)}{\partial B^i \partial y} = \frac{\partial k^i(y^{*i}, B^I, B^D)}{\partial y}. \tag{54}$$

End of Proof of Lemma 1.

Lemma 1 can be explained in this way. The game between the two groups can be seen as one of Stackelberg competition where each group chooses its exit strategy while the other one follows immediately. Suppose that group i is able to commit, at time zero, to an exit strategy that always makes it move first, and this commitment is announced to group $-i$. Subsequently, group $-i$ makes its decision to exit believing that group i will honor its commitment. Then, if group i will honor its commitment, both groups will carry out their exit strategies, which involves a sequential exit. With reference to our case, such a commitment implies that group i, in determining its exit trigger $y^{*i}(B^I, B^D)$ is able to behave myopically. That is, we assume that the other group will always sell its bonds after. However, since this strategy also applies to group $-i$, the sufficient condition that guarantees who can be considered the leader is $y^{*i}(B^I, B^D) > y^{*-i}(B^I, B^D)$.[30]

Making use of Lemma 1, we obtain a second important result:

Lemma 2. *Suppose that the two groups are symmetric, i.e.* $V^i = V^{-i}$, $k^i = k^{-i}$ *and* $B^i = B^{-i}$*, a symmetric Nash equilibrium in open-loop trigger strategies exists such that* $y^{*i} = y^{*-i} = y^*(B)$*.*

Proof of Lemma 2. Let us consider the intertemporal utility of group i, when group $-i$ decides to disinvest at $y^{*i} + \omega$ for an infinitesimally small number $\omega > 0$, and where y^{*i} is the "myopic" trigger given by the solution of (52)–(54). In this case the solution of (43) and the associated trigger, which we denote by $\tilde{y}^{*i}$ is given by:

$$\frac{1}{2}\sigma^2 y^2 V^i_{yy} - rV^i + U^i = 0, \tag{55}$$

subject to the matching value condition

$$\frac{\partial V^i(\tilde{y}^{*i}, B^I, B^D; y^{*i} + \omega)}{\partial B^i} = k^i(\tilde{y}^{*i}, B^I, B^D), \tag{56}$$

the smooth pasting condition

$$\frac{\partial^2 V^i(\tilde{y}^{*i}, B^I, B^D; y^{*i} + \omega)}{\partial B^i \partial y} = \frac{\partial k^i(\tilde{y}^{*i}, B^I, B^D)}{\partial y}, \tag{57}$$

and the matching value condition at the group's $-i$ exit trigger $y^{*i} + \omega$:

$$\frac{\partial V^i(y^{*i} + \omega, B^I, B^D; y^{*i} + \omega)}{\partial B^{-i}} = 0. \tag{58}$$

Since V^i is continuous with respect to (y, B^I, B^D) and bonds are infinitely divisible, letting $\omega \to 0$ the above problem becomes:

$$\frac{1}{2}\sigma^2 y^2 V^i_{yy} - rV^i + U^i = 0, \tag{59}$$

subject to:

$$\frac{\partial V^i(\tilde{y}^{*i}, B^I, B^D; y^{*i})}{\partial B^i} = k^i(y^{*i}, B^I, B^D), \tag{60}$$

$$\frac{\partial^2 V^i(\tilde{y}^{*i}, B^I, B^D; y^{*i})}{\partial B^i \partial y} = \frac{\partial k^i(y^{*i}, B^I, B^D)}{\partial y}, \tag{61}$$

$$\frac{\partial V^i(y^{*i}, B^I, B^D; y^{*i})}{\partial B^{-i}} = 0. \tag{62}$$

If the two groups are symmetric, the Nash equilibrium in open-loop trigger strategies exists that satisfies (59)–(62), and it is given by group i to choose $\tilde{y}^{*i} = y^{*i}$. Then, by the symmetry $y^{*i} = y^{*-i} = y^*(B)$ where $B = B^i + B^{-i}$ and $B^i = B^{-i}$. That is, the "myopic" trigger that satisfies (56)–(58) is also a symmetric Nash equilibrium in open-loop strategies (see Grenadier, 2002, Proposition 3; Back & Paulsen, 2009, Proposition 1, Part B).

End of Proof of Lemma 1.

Atomistics agents. Let us now assume that each individual, within his group, independently determines his exit strategy. Since within the group agents are perfectly symmetrical, by Lemma 2, they will adopt the same trigger. In addition, since they are negligibly small, denoting with $v^i(y, B) = \frac{\partial V^i(y,B)}{\partial B}$ the value of the marginal bond, the trigger $y^{*i}(B)$ takes the double meaning of both the optimal exit trigger for each individual within the group i in response to group $-i$ and the critical value of the shock y beyond which he decides to exit from the bond market.

30 This condition is, in fact, the essence of an open-loop Stackelberg equilibria (see Dockner et al., 2000, 2000 ch.5).

Now exit competition introduces, within the group, a fear of preemption. In equilibrium optimal exit timing yields zero expected net profits. The symmetric Nash equilibrium in open-loop strategies reduces to an equilibrium in symmetric "myopic" exit strategy as in Leahy (1993) and Dixit and Pindyck (1994), Ch.8), where each individual ignores the effect that other individuals exert on the utility level. In other words, the above stopping problem is equivalent to a problem of a bondholder that decides to hold the bond until he considers optimal to exit, under the myopic assumption that no one else will exit before and after him.

Thus, by Lemma 1, if $y^{*i}(B) > y^{*-i}(B)$, we can restate the problem (59)–(62) as:

$$\frac{1}{2}\sigma^2 y^2 v^i_{yy} - rv^i + u^i(y,B) = 0, \tag{63}$$

subject to the matching value condition

$$v^i(y^{*i}(B),B) = k^i(y^{*i}(B),B), \tag{64}$$

and the smooth pasting condition

$$\frac{\partial v^i(y^{*i}(B),B)}{\partial y} = \frac{\partial k^i(y^{*i}(B),B)}{\partial y}. \tag{65}$$

While for each individual in the group $-i$, the marginal value $v^{-i}(y^{*-i},B)$ as well as the exit trigger $y^{*-i}(B)$ are given by the solution of the following problem:

$$\frac{1}{2}\sigma^2 y^2 v^{-i}_{yy} - rv^{-i} + u^{-i}(y,B) = 0, \tag{66}$$

subject to the matching value condition

$$v^{-i}(y^{*-i}(B),B) = k^{-i}(y^{*-i}(B),B), \tag{67}$$

and the complementary slackness condition

$$\left(\frac{\partial v^{-i}(y^{*-i}(B),B)}{\partial y} - \frac{\partial k^{-i}(y^{*-i}(B),B)}{\partial y}\right)\frac{dy^{*-i}(B)}{dB} = 0. \tag{68}$$

The last condition implies that along the exit trigger $y^{*-i}(B)$, either the marginal value of the bond smooth-pastes the marginal payment, i.e. $\frac{\partial v^{-i}(y^{*-i}(B),B)}{\partial y} - \frac{\partial k^{-i}(y^{*-i}(B),B)}{\partial y} = 0$, or $y^{*-i}(B)$ does not change with B. In the former case, the smooth pasting condition holds and the exit trigger function $y^{*-i}(B)$ is as in the regular case without a floor. In the latter case, since $\frac{\partial v^{-i}(y^{*-i}(B),B)}{\partial y} - \frac{\partial k^{-i}(y^{*-i}(B),B)}{\partial y} \neq 0$, the same level of the state variable y triggers the exit of a mass of individuals in a run (Bartolini, 1993, Propositions 2 and 3 pp. 928–929).

To solve the problem (66)–(68) and to determine the range of B in which the run takes place, we start from the general solution of (66) (Dixit & Pindyck, 1994, Ch.8) :

$$v^{-i}(y,B) = a^{-i}(B)y^{\beta} + \frac{u^{-i}(y,B)}{r}, \tag{69}$$

where $a^{-i}(B)$ is a constant and $\beta = \frac{1}{2} - \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$ is the negative root of the characteristic equation $\frac{1}{2}\sigma^2 x(x-1) - r = 0$. The term $a^{-i}(B)y^{\beta}$ is negative and it accounts for how future changes in the stock of Government bonds due to further exits, affects the utility value of who remains. When B is at θB_0 no more individuals are able to exit, consequently $a^{-i}(\theta B_0) = 0$ and the marginal value of the θB_0th individual is simply given by $v^{-i}(y,\theta B_0) = \frac{u^{-i}(y,\theta B_0)}{r}$. Thus, by the matching value (67) we are able to obtain the trigger $y^{*-i}(\theta B_0)$:

$$\frac{u^{-i}(y^{*-i}(\theta B_0),\theta B_0)}{r} = k^{-i}(y^{*-i}(\theta B_0),\theta B_0). \tag{70}$$

Now since the smooth pasting condition does not hold at θB_0 then, by continuity, it also does not hold within the interval $[\theta B_0, \hat{B}]$ where, as $\frac{dy^{*-i}(B)}{dB} = 0$, $\hat{B}$ is the largest stock that satisfies:

$$y^{*-i}(\hat{B}) = y^{*-i}(\theta B_0). \tag{71}$$

End of Proof of Proposition 1

Appendix B. Appendix 2

B.1. Proof of Proposition 2

The road map towards the proof consists of four steps:

Step 1: Ignoring the order of exit we explicitly solve the optimal exit trigger for Investors and Depositors as coordinated groups (i.e. we apply Lemma 1).

Step 2: Ignoring the floor imposed by the Government, we show that the competitive exit triggers follows directly from the coordinated ones (i.e. we apply Lemma 2).

Step 3: We derive necessary and sufficient conditions for sequential exit: Investors to exit before Depositors and viceversa (i.e. we apply the first part of Proposition 1).

Step 4: Finally we impose the Government floor and derive the moment in which the run begins (i.e. we apply the second part of Proposition 1).

Step 1.

Let us consider first the Investors. By Lemma 1, the solution for the value of the intertemporal utility of Investors as coordinated group I, is given by:

$$\frac{1}{2}\sigma^2 y^2 V^I_{yy} - rV^I + \max[\frac{X(B)}{B}B^I y,\ cB^I] = 0 \tag{72}$$

subject to the value matching condition

$$\frac{\partial V^I(y^{*I}, B^I, B^D)}{\partial B^I} = \xi \frac{X(B)}{B} y^{*I} \tag{73}$$

and the smooth pasting condition

$$\frac{\partial^2 V^I(y^{*I}, B^I, B^D)}{\partial B^I \partial y} = \xi \frac{X(B)}{B}. \tag{74}$$

A general solution of (72) can be written as:

$$V^I(y, B^I, B^D) = \begin{cases} M_1(B)y^\gamma + N_1(B)y^\beta + \frac{cB^I}{r} & for \frac{X(B)y}{B} \geq c \\ M_2(B)y^\gamma + N_2(B)y^\beta + \frac{X(B)B^I}{B}\frac{y}{r} & for \frac{X(B)y}{B} < c \end{cases} \tag{75}$$

where $\gamma > 1,\ \beta < 0$ are, respectively, the positive and the negative roots of the characteristic equation

$$\frac{1}{2}\sigma^2 x(x-1) - r = 0, \tag{76}$$

$$\gamma = \frac{1}{2} + \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} > 1, \tag{77}$$

$$\beta = \frac{1}{2} - \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0, \tag{78}$$

and M_1, M_2, N_1, N_2 are constants to be determined together with the exit trigger y^{*I}.

Without new Government bond issues, to keep $V^I(y, B^I, B^D))$ finite as y becomes high we discard the term in the positive power of y, setting $M_1 = 0$, i.e.:[31]

$$\lim_{y\to\infty} V^I(y, B^I, B^D)) = \frac{cB^I}{r}. \tag{79}$$

Then, the general solution reduces to:

$$V^I(y, B^I, B^D) = \begin{cases} N_1(B)y^\beta + \frac{cB^I}{r} & \text{for } \frac{X(B)y}{B} \geq c \\ M_2(B)y^\gamma + N_2(B)y^\beta + \frac{X(B)B^I}{B}\frac{y}{r} & \text{for } \frac{X(B)y}{B} < c, \end{cases} \tag{80}$$

where the terms $N_1(B)y^\beta$ and $N_2(B)y^\beta$ is the value of the group $I's$ options to optimally dispose of the bonds in the future. Since group I is able to sell bonds both before and after the Government falls short of resources to pay c, i.e. both when $\frac{X(B)y}{B} \geq c$, and $\frac{X(B)y}{B} < c$, we need to analyze both cases separately.

Case $\frac{X(B)y}{B} < c$. It is easy to show that the constant $M_2(B)$ can be determined by applying the matching value condition and the smooth pasting condition at $\bar{y}(B) = c\frac{B}{X(B)}$. In particular we obtain:

$$N_1(B)\bar{y}^\beta + \frac{cB^I}{r} = M_2(B)\bar{y}^\gamma + N_2(B)\bar{y}^\beta + \frac{X(B)B^I}{B}\frac{\bar{y}}{r} \tag{81}$$

[31] Observe that when the value of y approaches infinity, the probability of not receiving the coupon goes to zero and the bond's value must approach its nominal value $\frac{cB^I}{r}$.

$$N_1(B)\beta\bar{y}^{\beta-1} = M_2(B)\gamma\bar{y}^{\gamma-1} + N_2(B)\beta\bar{y}^{\beta-1} + \frac{X(B)\,B^I}{B}\frac{1}{r}.$$

Substituting $N_1(B)\bar{y}^{\beta}$ in the first equation we obtain:

$$M_2(B) = -\frac{1}{\gamma-\beta}\frac{cB^I}{r}c^{-\gamma}(\frac{X(B)}{(B)}^{\gamma}, \tag{82}$$

which shows that $M_2(B) < 0$.

Substituting $M_2(B)$ in (80), we obtain the Investors' utility when $y < \bar{y}(B)$ as:

$$V^I(y, B^I, B^D) = N_2(B)y^{\beta} + \underbrace{\frac{1}{\gamma-\beta}\frac{cB^I}{r}c^{-\gamma}(\frac{X(B)}{B})^{\gamma}y^{\gamma}}_{M_2(B)y^{\gamma}} + \frac{X(B)\,B^I}{B}\frac{y}{r}. \tag{83}$$

The term $M_2(B)y^{\gamma} + \frac{X(B)B^I}{B}\frac{y}{r}$ represents the value of the life time utility of group I in the absence of new exit. This is composed by the expected discounted flow of payments $\frac{X(B)B^I}{B}\frac{y}{r}$ when $y < \bar{y}(B)$ plus the negative value of the contractual commitment to cash only cB^I when $y \geq \bar{y}(B)$. In other words, the term $M_2(B)y^{\gamma}$ takes into account that when $X(B)\,y$ goes above cB^I the bondholders cannot take advantage of it since they cash only the promised coupon c. Thus, this term measures the (negative) value of the implicit "cost" that the bondholders have accepted subscribing bonds with a coupon c. From (83) we also observe that $M_2(B)y^{\gamma}$ is increasing in c. This makes sense considering that an increase in the coupon implies that $\frac{X(B)}{B}y$ will reach the cap c less frequently. In addition, since the loss $M_2(B)y^{\gamma}$ is larger the larger is group's I share of Government bonds, the derivative of $M_2(B)y^{\gamma}$ with respect to B^I must be negative, i.e.:

$$\frac{\partial M_2(B)}{\partial B^I} = M_2(B)\left[1-\gamma(1-\varepsilon(B))\frac{B^I}{B}\right]\frac{1}{B^I} < 0. \tag{84}$$

To ensure that this occurs we need that $1-\gamma(1-\varepsilon(B))\frac{B^I}{B} > 0$, where $\varepsilon(B) < 1$. Finally, as said before, the term $N_2(B)y^{\beta}$ in (83) is the correction of the group's value due to new exits. It accounts for how future changes in the stock of Government bonds due to further exits affect the utility value and $N_2(B)$ must therefore be positive**.**

To determine $N_2(B)$ and y^{*I} we need to impose some boundary conditions to (83). In the specific the matching value condition:

$$N_{B^I,2}(B)(y^{*I})^{\beta} + M_{B^I,2}(B)(y^{*I})^{\gamma} + \frac{1}{r}\frac{\partial U^I(y^{*I}, B^I, B^D)}{\partial B^I} = k^I(y^{*I}, B^I, B^D), \tag{85}$$

and the smooth pasting condition:

$$N_{B^I2}(B)\beta(y^{*I})^{\beta-1} + M_{B^I2}(B)\gamma(y^{*I})^{\gamma-1} + \frac{1}{r}\frac{\partial^2 U^I(y^{*I}, B^I, B^D)}{\partial B^I\partial y} = \frac{\partial k^I(y^{*I}, B^I, B^D)}{\partial y}, \tag{86}$$

where $N_{B^I,2}(B)$ and $M_{B^I,2}(B)$ are partial derivatives with respect to B^I.

Substituting (85) into (86) to eliminate $N_{B^I,2}(B)$, we obtain:

$$M_{B^I2}(B)(y^{*I})^{\gamma-1} = \frac{(1-\beta)}{(\gamma-\beta)}\left(-\frac{1}{r}\frac{\partial^2 U^I(y^{*I}, B^I, B^D)}{\partial B^I\partial y} + \frac{\partial k^I(y^{*I}, B^I, B^D)}{\partial y}\right). \tag{87}$$

Note that condition $M_{B^I2}(B) < 0$ ensures that the second order condition for maximization holds, i.e.:

$$(\gamma-1)M_{B^I2}(B)(y^{*I})^{\gamma-2} < 0. \tag{88}$$

Multiplying both sides of (87) by y^{*I} and rearranging we obtain:

$$\frac{1}{r}\frac{\partial U^I(y^{*I}, B^I, B^D)}{\partial B^I} = k^I(y^{*I}, B^I, B^D) - \frac{(\gamma-\beta)}{(1-\beta)}M_{B^I2}(B)(y^{*I})^{\gamma}. \tag{89}$$

The interpretation of (89) is straightforward. Suppose that group I decides to sell a unit of bond when the current stock is B and the shock is y^{*I}. The loss in terms of expected present flow of utility is:

$$\mathbb{E}[\int_0^{\infty}\frac{\partial U^I}{\partial B^I}e^{-rt}dt] = \frac{1}{r}\frac{\partial U^I(y^{*I}, B^I, B^D)}{\partial B^I} \tag{90}$$

while the payoff from the sale of this unit is:

$$k^I(y^{*I}, B^I, B^D) - \frac{\gamma-\beta}{1-\beta}M_{B^I2}(B)(y^{*I})^{\gamma}, \tag{91}$$

i.e. the resources obtained by the Government plus the reduction of the loss $M_{B^I2}(B)(y^{*I})^{\gamma}$ that the group faces with a reduced stock. Thus (89) says that the decision to sell a marginal unit of bond is justified when the expected present value of the losses equals the expected present value of the benefits.

Now, from (86) we are able to obtain $N_{B^I2}(B)$ as:

$$N_{B^I2}(B)(y^{*I})^{\beta-1} = \frac{\gamma-1}{\gamma-\beta}\left(-\frac{1}{r}\left[\frac{X(B)}{B}\left(1-\xi r-(1-\varepsilon(B))\frac{B^I}{B}\right)\right]\right), \tag{92}$$

where the necessary and sufficient condition for the utility of group I to increase by selling some bonds, i.e. $N_{B^I 2}(B) < 0$, is given by $1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} > 0$. That is, when group I decides to exercise its option to sell some bonds, it gives up the marginal option value, which is why $N_{B^I 1}(B)$ is negative. Note also that this implies $1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} < 1$. Once determined $N_{B^I 2}(B)$, we can find $N_2(B)$ by integrating (92) on the interval $[B_0^I, 0]$ keeping B_0^D constant. Integrating both parties of (92) we obtain:

$$\int_{B_0^I}^{0} N_{B^I 2}(x + B^D)(y^{*I}(x, B^D))^{\beta-1} dx \tag{93}$$

$$= \int_0^{B_0^I} -N_{B^I 2}(x + B^D)(y^{*I}(x, B^D))^{\beta-1} dx$$

$$= \frac{(\gamma - 1)}{r(\gamma - \beta)} \int_0^{B_0^I} \left[\frac{X(x + B^D)}{x + B^D} \left(1 - \xi r - (1 - \varepsilon(x + B^D)) \frac{x}{x + B^D} \right) \right] dx$$

$$\leq \frac{(\gamma - 1)}{r(\gamma - \beta)} \int_0^{B_0^I} \frac{X(x + B^D)}{x + B^D} dx, \tag{94}$$

where $1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} < 1$ and the concavity of $X(B)$ guarantee the convergence of the integral.

Finally, by (89) we are able to isolate the optimal exit trigger as:

$$y^{*I}(B^I, B^D) = \frac{c}{\frac{X(B)}{B}} \left[\frac{1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B}}{\left[1 - \frac{\beta}{\beta-1}\right]\left[1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B}\right]} \right]^{\frac{1}{\gamma-1}}. \tag{95}$$

Note that the exit trigger is always positive if both $1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B} > 0$ and $1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} > 0$.

Summarizing, when $\frac{X(B)y}{B} < c$, in order for a decrease in B^I to result in an increase in the value of the expected utility (i.e. $N_{B^I 2}(B) < 0$) and to obtain a viable exit strategy (i.e. $y^{*I}(B^I, B^D) > 0$), we need two conditions. The first one from (84) is:

$$1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B} > 0, \tag{96}$$

and the second one from (92) is:

$$1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} > 0. \tag{97}$$

To conclude we need to check two other conditions. First, in order to have sequential exits it must be $\frac{\partial y^{*I}(B^I, B^D)}{\partial B^I} \geq 0$. Second, the optimal trigger must guarantee that exits occur when the liquidation value is below c. For the sign of $\frac{\partial y^{*I}(B^I, B^D)}{\partial B^I}$, let us define:

$$\Gamma \equiv [\frac{\frac{X(B)}{B} y^{*I}(B^I, B^D)}{c}]^{\gamma-1} = \frac{1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B}}{\left[1 - \frac{\beta}{\beta-1}\right]\left[1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B}\right]}. \tag{98}$$

Taking the derivative with respect to B^I, we obtain:

$$\frac{\partial \Gamma}{\partial B^I} = (\gamma - 1)[\frac{\frac{X(B)}{B} y^{*I}(B^I, B^D)}{c}]^{\gamma-2} \left[\frac{\partial \frac{X(B)}{B}}{\partial B^I} y^{*I}(B^I, B^D) + \frac{X(B)}{B} \frac{\partial y^{*I}}{\partial B^I} \right]$$

$$\propto -\varepsilon'(B)\left(1 - \frac{B^I}{B}\right)\left(1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B}\right) + ((1 - \xi r)\gamma - 1)\left(-\varepsilon'(B) + (1 - \varepsilon(B))\frac{B - B^I}{(B)^2}\right). \tag{99}$$

As $\frac{\partial \frac{X(B)}{B}}{\partial B^I} < 0$ and $(1 - \xi r)\gamma - 1 > 0$, if $1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B} > 0$ the R.H.S. of (99) is positive, then we may conclude that also $\frac{\partial y^{*I}}{\partial B^I} > 0$. Finally, if both $1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B} > 0$ and $1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B} > 0$, the condition $\frac{\beta}{\beta-1} - \xi r < 0$ is sufficient to guarantee that:

$$\frac{1 - \xi r - (1 - \varepsilon(B))\frac{B^I}{B}}{\left[1 - \frac{\beta}{\beta-1}\right]\left[1 - \gamma(1 - \varepsilon(B))\frac{B^I}{B}\right]} < 1. \tag{100}$$

Case $\frac{X(B)y}{B} \geq c$. By the matching value and smooth pasting conditions we obtain:

$$N_{B^I,1}(B)(y^{*I})^{\beta} + \frac{c}{r} = \xi \frac{X(B)\, y^{*I}}{B} \tag{101}$$

$$N_{B^I,1}(B)\beta(y^{*I})^{\beta-1} = \xi \frac{X(B)}{B}.$$

Solving for y^{*I}, the optimal exit trigger is:

$$y^{*I}(B^I, B^D) = \frac{\beta}{\beta - 1} \frac{c}{r} \frac{B}{\xi X(B)}, \tag{102}$$

which, under the assumption that $\frac{\beta}{\beta-1} > \xi r$, shows that the prorata at exit is greater than c. As before, the loss in terms of expected present flow of utility if one unit of bond is sold is $\frac{\beta}{\beta-1}\frac{c}{r}$, while the gain is $\xi\frac{X(B)}{B}y^{*I}(B^I, B^D)$.

In addition, the trigger is increasing in B^I because of the concavity of the utility function

$$\frac{\partial y^{*I}(B^I, B^D)}{\partial B^I} \geq 0. \tag{103}$$

Finally, note that, as expected,

$$N_{B^I,1}(B) = \frac{1}{\beta}k^I(y^{*I}, B^I, B^D)(y^{*I})^{-\beta} < 0. \tag{104}$$

This makes economic sense since $N_{B^I,1}(B)$ accounts for the marginal increase of the utility of group I by selling some bonds. As before, once determined $N_{B^I,1}(B)$, one may easily find $N_1(B)$ by integrating on the interval $[B^I, 0]$.

Let us move to the Depositors now. Again, by Lemma 1, the solution for the value of the intertemporal utility of group D as well as for the exit trigger y^{*D} is given by (Proposition 1):

$$\frac{1}{2}\sigma^2 y^2 V^D_{yy} - rV^D + [\varphi(B)b]B^D = 0, \tag{105}$$

subject to the value matching condition:

$$\frac{\partial V^D(y^{*D}, B^I, B^D)}{\partial B^D} = \xi\frac{X(B)}{B}y^{*D}, \tag{106}$$

and the smooth pasting condition:

$$\frac{\partial^2 V^D(y^{*D}, B^I, B^D)}{\partial B^D \partial y} = \xi\frac{X(B)}{B}. \tag{107}$$

A general solution of (105) can be written as:

$$V^D(y, B^I, B^D) = R(B)y^\gamma + O(B)y^\beta + \frac{\varphi(B)b}{r}B^D \tag{108}$$

where R and O are two constants to be determined. As in the previous case, to keep $V^D(y, B^I, B^D)$ finite as y increases, i.e.:

$$\lim_{y\to\infty} V^D(y, B^I, B^D) = \frac{\varphi(B)b}{r}B^D, \tag{109}$$

we set $R = 0$. Then, the general solution of (108) reduces to:

$$V^D(y, B^I, B^D) = O(B)y^\beta + \frac{\varphi(B)b}{r}B^D. \tag{110}$$

To determine both the optimal trigger y^{*D} and constant $O(B)$, we apply the matching value condition and the smooth pasting (106) and (107), respectively:

$$\begin{aligned} O_{B^D}(B)(y^{*D})^\beta + \frac{\varphi'(B)bB^D}{r} + \frac{\varphi(B)b}{r} &= \xi\frac{X(B)}{B}(y^{*D}) \\ O_{B^D}(B)(y^{*D})^\beta &= \xi\frac{X(B)}{\beta B}(y^{*D}). \end{aligned} \tag{111}$$

Recalling that $U^D(y^{*D}, B^I, B^D) = \varphi(B)bB^D$ from (111) we obtain:

$$\frac{1}{r}\frac{\partial U^D(y^{*D}, B^I, B^D)}{\partial B^D} = \xi\frac{X(B)}{B}y^{*D} - \frac{1}{\beta}\xi\frac{X(B)}{B}y^{*D}, \tag{112}$$

where:

$$\mathbb{E}[\int_0^\infty \frac{\partial U^D}{\partial B^D}e^{-rt}dt] = \frac{1}{r}\frac{\partial U^D(y^{*D}, B^I, B^D)}{\partial B^D} = \left[\varphi'(B)B^D + \varphi(B)\right]\frac{b}{r}. \tag{113}$$

Similarly to Eq. (89), suppose that group D decides to sell a unit of bond when the current stock is B and the market shock y^{*D}. The loss in terms of expected present flow of utility is now $\frac{1}{r}\frac{\partial U^D(y^{*D},B^I,B^D)}{\partial B^D}$, while the marginal benefit is given by the liquidation value available to the Government $\xi\frac{X(B)}{B}y^{*D}$ plus the term $-\frac{1}{\beta}\xi\frac{X(B)}{B}y^{*D} > 0$ which reflects the gain of information in waiting to exit the bond market. Thus (112) says that the decision to sell a marginal unit of bond is justified when the expected present value of the losses equals the expected present value of the benefits.

Solving for the optimal exit trigger we obtain:

$$y^{*D}(B^I, B^D) = \frac{\beta}{\beta-1}\left[\varphi'(B)B^D + \varphi(B)\right]\frac{b}{r}\frac{B}{\xi X(B)}, \tag{114}$$

whilst $O_{B^D}(B)$ is negative and given by:

$$O_{B^D}(B) = \xi\frac{X(B)}{\beta B}(y^{*D})^{1-\beta} < 0. \tag{115}$$

This makes economic sense since $O_{B^D}(B)$ accounts for the marginal increase of the utility of group D by selling bonds. Once we have determined $O_{B^D}(B)$, we can find $O(B)$ by integrating on the interval $[B^D, 0]$ as before.

Table 1

Investors, with $\frac{\beta}{\beta-1} > \xi r$	$y^{*I}(B^I, B^D) = \frac{\beta}{\beta-1}\frac{c}{r}\frac{B}{\xi X(B)}$
Investors, with $\frac{\beta}{\beta-1} < \xi r$	$y^{*I}(B^I, B^D) = \left[\frac{1-\xi r-(1-\varepsilon(B))\frac{B^I}{B}}{\left[1-\frac{\beta}{\beta-1}\right]\left[1-\gamma(1-\varepsilon(B))\frac{B^I}{B}\right]}\right]^{\frac{1}{\gamma-1}} c\frac{B}{X(B)}$
Depositors	$y^{*D}(B^I, B^D) = \frac{\beta}{\beta-1}\left[\varphi'(B)B^D + \varphi(B)\right]\frac{b}{r}\frac{B}{\xi X(B)}$

The trigger is increasing in B^D because of the concavity of the utility function and the properties of the cumulative distribution function, i.e.:

$$\frac{\partial y^{*D}(B^I, B^D)}{\partial B^D} = \left[\varphi''(B)B^D + 2\varphi'(B)\right]B + \left[\varphi'(B)B^D + \varphi(B)\right]\frac{(1-\varepsilon(B))}{\xi X(B)} > 0. \tag{116}$$

We summarize the optimal exit triggers for the coordinated case, in the following Table 1:

Step 2.

Denote with $v^I(y, B) = \frac{\partial V^I(y,B)}{\partial B}$ the Investors' marginal value of the bond. Solving (63) for the case $\frac{X(B)y}{B} < c$, and proceeding as in the step 1, we obtain:

$$v^I(B, y) = n_2(B)y^\beta + m_2(B)y^\gamma + \frac{X(B)}{B}\frac{y}{r} \text{ for } \frac{X(B)\,y}{B} < c, \tag{117}$$

where:

$$m_2(B) = \frac{1}{\beta-\gamma}\frac{1}{r}c^{1-\gamma}\left(\frac{X(B)}{B}\right)^\gamma < 0. \tag{118}$$

While $m_2(B)y^\gamma$ is the implicit cost that a bondholder accepts when holding a bond that pays c instead of $\frac{X(B)}{B}$ when y rises above c, the term $n_2(B)y^\beta$ is still the correction of the individual's value due to new exits. Applying the matching value and smooth pasting:

$$n_2(B)(y^{*I})^\beta + m_2(B)(y^{*I})^\gamma + \frac{X(B)}{B}\frac{(y^{*I})}{r} = \xi\frac{X(B)}{B}(y^{*I}) \tag{119}$$

$$n_2(B)\beta(y^{*I})^{\beta-1} + m_2(B)\gamma(y^{*I})^{\gamma-1} + \frac{X(B)}{B}\frac{1}{r} = \xi\frac{X(B)}{B}, \tag{120}$$

and solving for $y^{*I}(B)$ we get the optimal exit trigger function:

$$y^{*I}(B) = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{1/\gamma-1}\frac{B}{X(B)}, \tag{121}$$

which is increasing in B. The constant $n_2(B)$ is:

$$n_2(B) = -m_2(B)y^{*I}(B)^{\gamma-\beta}\left[\frac{(\gamma-1)}{(\beta-1)}\right] < 0. \tag{122}$$

Now, solving (63) for the case $\frac{X(B)y}{B} \geq c$, and proceeding as in step 1, we obtain:

$$v^I(y, B) = n_1(B)y^\beta + \frac{c}{r} \qquad \text{for } \frac{X(B)\,y}{B} \geq c, \tag{123}$$

where the term $n_1(B)y^\beta$ is the correction of the single investor's value due to the new exit. Applying the matching value and smooth pasting conditions:

$$n_1(B)(y^{*I})^\beta + \frac{c}{r} = \xi\frac{X(B)\,y^{*I}}{B} \tag{124}$$

$$n_1(B)\beta(y^{*I})^{\beta-1} = \xi\frac{X(B)}{B}, \tag{125}$$

we obtain the optimal exit trigger, which is equal to (102), and the constant $n_1(B)$, i.e.:

$$y^{*I}(B) = \frac{\beta}{\beta-1}\frac{c}{r}\frac{B}{\xi X(B)} > c, \text{ and } n_1(B) = \xi\frac{(y^{*I})^{1-\beta}}{\beta}\frac{X(B)}{B} < 0. \tag{126}$$

Denote now with $v^D(y, B) = \frac{\partial V^D(y,B)}{\partial B}$ the Depositors' marginal value of the bond. Along the same line, for the Depositors, solving (66), we obtain:

$$v^D(y, B) = o(B)y^\beta + \frac{\varphi(B)b}{r}. \tag{127}$$

Applying the matching value and smooth pasting conditions:

$$o(B)(y^{*D})^\beta + \frac{\varphi(B)b}{r} = \xi\frac{X(B)\,y^{*D}}{B} \tag{128}$$

Table 2

Investors, with $\frac{\beta}{\beta-1} \geq \xi r$	$y^{*I}(B) = \frac{\beta}{\beta-1}\frac{c}{r}\frac{B}{\xi X(B)}$
Investors, with $\frac{\beta}{\beta-1} < \xi r$	$y^{*I}(B) = \left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{1/\gamma-1} c\frac{B}{X(B)}$
Depositors	$y^{*D}(B) = \frac{\beta}{\beta-1}\frac{\varphi(B)b}{r}\frac{B}{\xi X(B)}$

$$o(B)\beta(y^{*D})^{\beta-1} = \xi\frac{X(B)}{B} \tag{129}$$

we obtain:

$$y^{*D}(B) = \frac{\beta}{\beta-1}\frac{\varphi(B)b}{r}\frac{B}{\xi X(B)}, \text{ and } o(B) = \xi\frac{(y^{*D})^{1-\beta}}{\beta}\frac{X(B)}{B} < 0. \tag{130}$$

Also in this case we are able to summarize the competitive exit triggers in Table 2:

Note now that in the absence of coordination $\frac{B^i}{B^i+B^{-i}} \to 0$ for $i = I, D$. Then, $y^{*I}(B^I, B^D) \to y^{*I}(B)$ and $y^{*D}(B^I, B^D) \to y^{*D}(B)$. In other words, the limits of coordinated action triggers $y^{*I}(B^I, B^D)$ and $y^{*D}(B^I, B^D)$, displayed in Table 1, become the perfectly competitive exit triggers defined by Leahy (1993) and reported in Table 2. In addition, it is easy to prove that $N_{B^I 2}(B) \to n_2(B)$, $N_{B^I,1}(B) \to n_1(B)$ and $O_{B^D}(B) \to o(B)$.

Step 3.

In case of the coordinated actions, sufficient conditions for having complete sequential exit requires $y^{*i}(B^I, B^D) > y^{*-i}(B^I, B^D)$ for all $B \in [0, B_0]$. In the specific, Investors exit before Depositors if $y^{*D}(B^I, B^D) > y^{*I}(B^I, B^D)$ and viceversa if $y^{*D}(B^I, B^D) > y^{*I}(B^I, B^D)$. Along the same lines as in step 2, we are able now to extend these conditions to the case where the behavior of each individual that competes with the other individuals of the same group when selling bonds. Although the "myopic" behavior continues to remain optimal, as competition in exit by infinitesimally small agents destroys any competitive advantage, the optimal strategies are also mutual best responses.

Recalling that both (102) and (114) are increasing functions of B^I and B^D respectively, if $\frac{\beta}{\beta-1} \geq \xi r$ and $b < c$, the sufficient conditions for all Investors to exit before Depositors is $\frac{c}{b} > \left[\varphi'(B)B^D + \varphi(B)\right]$ for $B \in [0, B_0]$. Since for an atomistics individual $\varphi'(B) = 0$, and $\varphi(B_0) = 1$, it is immediate to show that $c > b$ is the only necessary and sufficient condition for all Investors to exit before Depositors, i.e. $y^{*I}(B) > y^{*D}(B) \Leftrightarrow c > b\varphi(B)$ which is satisfied for all $B \in [0, B_0]$. Note that comparing (126) and (130), we obtain the same condition.

When $\frac{\beta}{\beta-1} < \xi r$, unlike the previous case we can have either that group I exits completely and then group D exits or the opposite. In the specific for Investors to exit before Depositors we need:

$$\frac{c}{b} > \frac{\beta}{\beta-1}\left[\varphi'(B)B^D + \varphi(B)\right]\frac{1}{\xi r}\left[\frac{1-\xi r-(1-\varepsilon(B))\frac{B^I}{B}}{\left[1-\frac{\beta}{\beta-1}\right]\left[1-\gamma(1-\varepsilon(B))\frac{B^I}{B}\right]}\right]^{-\frac{1}{\gamma-1}}, \tag{131}$$

while the inverse inequality brings Depositors to exit before Investors.

Also in this case it is possible to obtain conditions for the competitive case. Recalling that $\varphi(B_0) = 1$, and using the triggers in Table 2, the necessary and sufficient condition reduces to:

$$y^{*I} > y^{*D} \to \frac{c}{b} > \frac{\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]}{\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}, \tag{132}$$

while the opposite brings Depositors to exit before Investors:

$$y^{D} > y^{*I} \to \frac{c}{b} < \frac{\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]}{\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}. \tag{133}$$

Step 4.

Hence, when $\frac{\beta}{\beta-1} \geq \xi r$, are always the Depositors who experience a run. On the contrary, when $\frac{\beta}{\beta-1} < \xi r$, if (132) holds are the Depositors who experience a run while are the Investors to have a run if (133) holds.

Let us consider first the case $\frac{\beta}{\beta-1} \geq \xi r$. When $B = \theta B_0$ no more Depositors can exit and consequently the value associated to these variations is zero. Thus formally we obtain $o(\theta B_0) = 0$ and the matching value condition (128) reduces to:

$$\frac{\varphi(\theta B_0)b}{r} = \xi\frac{X(\theta B_0)\,y^{*D}}{\theta B_0}, \tag{134}$$

from which we obtain the trigger to leave at θB_0 as:

$$y^{*D}(\theta B_0) = \frac{\varphi(\theta B_0)b}{r}\frac{\theta B_0}{\xi X\left(\theta B_0\right)}. \tag{135}$$

Now since by Proposition 1 the smooth pasting condition does not hold at θB_0 then, by continuity, it also does not hold within the interval $[\theta B_0, \hat{B}^D]$ where, as $\frac{dy^{*D}(B)}{dB} = 0$, $\hat{B}^D$ is the largest stock that satisfies:

$$\begin{aligned} y^{*D}(\hat{B}^D) &= y^{*D}(\theta B_0) \\ \frac{\beta}{\beta-1}\varphi(\hat{B}^D)\frac{\hat{B}^D}{X(\hat{B}^D)} &= \varphi(\theta B_0)\frac{\theta B_0}{X\left(\theta B_0\right)}. \end{aligned} \tag{136}$$

Let us consider now the case $\frac{\beta}{\beta-1} < \xi r$. In particular let assume that (133) holds so that are the Investors to experience the run. In this case when $B = \theta B_0$ we get $n_2(\theta B_0) = 0$ and the zero-utility condition (119) reduces to:

$$m_2(\theta B_0)y^{*I}(\theta B_0)^{\gamma} + \frac{X\left(\theta B_0\right)}{\theta B_0}\frac{y^{*I}(\theta B_0)}{r} - \xi\frac{X\left(\theta B_0\right)}{\theta B_0}y^{*I}(\theta B_0) = 0, \tag{137}$$

from which we obtain the exit threshold at θB_0, i.e.:

$$y^{*I}(\theta B_0) = c\left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{\gamma-1}}\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{1/\gamma-1}\frac{\theta B_0}{X(\theta B_0)}. \tag{138}$$

As before, by continuity $y^{*I}(\theta B_0) = y^{*I}(\hat{B}^I)$ within the interval $[\theta B_0, \hat{B}^I]$, and then $\hat{B}^I$ is given by:

$$\frac{X(\hat{B}^I)}{\hat{B}^I} = \left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{1-\gamma}}\frac{X(\theta B_0)}{\theta B_0}. \tag{139}$$

End of Proof of Proposition 2.

Appendix C. Appendix 3

C.1. *Proof of Proposition 3*

By Ito's Lemma we can write (2) as a diffusion process defined in the real domain $\mathbb{R} = (-\infty, +\infty)$:

$$d\ln[y_t] = -(1/2)\sigma^2 dt + \sigma dW_t. \tag{140}$$

The mean time $\mathbb{E}(T)$ that y_t takes, starting from an initial point y_1, to reach a lower boundary $y_2 < y_1$ for the first time is given by (Dixit, 1989 pp. 54–56):

$$\mathbb{E}(T) = \psi^{-1}\log\left(\frac{y_2}{y_1}\right), \tag{141}$$

where ψ is the constant negative drift $(-(1/2)\sigma^2)$.

Case $\frac{\beta}{\beta-1} \geq \xi r$. We know from Proposition 2 that Investors exit first. From (126) the starting point at $t = 0$ is $y_1 = y^{*I}(B_0) = \frac{c}{\xi}\left[\frac{\beta}{\beta-1}\frac{1}{r}\right]B_0^{1-\zeta}$ and from (135) the end point is $y_2 = y^{*D}(\hat{B}^D) = \frac{b}{\xi}\frac{\theta}{r}(\theta B_0)^{1-\zeta}$. Replacing these values in (141) we obtain:

$$\mathbb{E}^{I,D}(T) = \psi^{-1}\ln\left(\frac{\beta-1}{\beta}\frac{b}{c}\theta^{2-\zeta}\right) > 0. \tag{142}$$

Let us now compare this economy with one of only Investors. The starting point at $t = 0$ remains $y_1 = y^{*I}(B_0) = \frac{c}{\xi}\left[\frac{\beta}{\beta-1}\frac{1}{r}\right]B_0^{1-\zeta}$ while the end point is now $y_2 = y^{*I}(\theta B_0) = c\frac{1}{\xi r}(\theta B_0)^{1-\zeta}$.[32] Thus:

$$\mathbb{E}^{I}(T) = \psi^{-1}\ln\left(\frac{\beta-1}{\beta}\theta^{1-\zeta}\right) > 0. \tag{143}$$

The difference shows:

$$\mathbb{E}^{I,D}(T) - \mathbb{E}^{I}(T) = \psi^{-1}\ln\left(\frac{b}{c}\theta\right) > 0. \tag{144}$$

[32] This is obtained by setting $n_1(B) = 0$ in (124).

Case $\frac{\beta}{\beta-1} < \xi r$. From (132) we know that Investors exit before Depositors if $y^{*I}(B) > y^{*D}(B)$, that is if:

$$\frac{c}{b} > \frac{\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]}{\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}. \tag{145}$$

From (121), the starting point at $t = 0$ is $y_1 = y^{*I}(B_0) = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}} B_0^{1-\zeta}$ and the end point is $y_2 = y^{*D}(\theta B_0) = \frac{b}{\xi}\frac{\theta}{r}(\theta B_0)^{1-\zeta}$. Replacing in (141) we obtain:

$$\mathbb{E}^{I,D}(T) = \psi^{-1}\ln\left(\frac{\frac{b}{\xi r}}{c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}\theta^{2-\zeta}\right) > 0. \tag{146}$$

As before if we consider only Investors, from (121) and (138) we are able to calculate $\mathbb{E}^I(T)$. The starting point is $y_1 = y^{*I}(B_0) = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}} B_0^{1-\zeta}$ and the end point is $y_2 = y^{*I}(\theta B_0) = c\left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{\gamma-1}}\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{1/\gamma-1}\theta^{1-\zeta}B_0^{1-\zeta}$. Then:

$$\mathbb{E}^I(T) = \psi^{-1}\ln\left(\left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{\gamma-1}}\theta^{1-\zeta}\right) \tag{147}$$

and the difference is:

$$\mathbb{E}^{I,D}(T) - \mathbb{E}^I(T) = \psi^{-1}\ln\left(\frac{b}{c}\frac{\theta}{\xi r\left[\frac{\gamma-\beta}{1-\beta}\right]^{\frac{1}{\gamma-1}}\left[\frac{1-r\xi}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}\right). \tag{148}$$

The sign of the difference in (148) is ambiguous except when $\frac{b}{c}$ or θ are sufficiently low in which case (148) > 0. In addition, by simple algebra, (148) can be written as:

$$\mathbb{E}^{I,D}(T) - \mathbb{E}^I(T) = \psi^{-1}\ln\left(\frac{b}{c}\theta\right) - \psi^{-1}\ln\left(\xi r\left[(\gamma-\beta)(1-r\xi)\right]^{\frac{1}{\gamma-1}}\right). \tag{149}$$

As, when $\sigma \to \infty$, we have $\beta \to 0$ and $\gamma \to 1$we may conclude that $\ln\left(\xi r\left[(\gamma-\beta)(1-r\xi)\right]^{\frac{1}{\gamma-1}}\right) < 0$.

Further, from (133) if $y^D(B) > y^{*I}(B)$ and:

$$\frac{c}{b} < \frac{\left[\frac{\beta}{\beta-1}\frac{1}{\xi r}\right]}{\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}, \tag{150}$$

are the Depositors to exit first. However, this case cannot happen as when volatility is sufficiently high $\beta \to 0$ which is inconsistent with $\frac{c}{b} > 0$.

Finally, in the same way, we are able to calculate the average time that an Investor is willing to hold the bond without receiving c. In this case the starting point is c and the ending point is $y^{*I}(B)B^{\zeta-1} = c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}$, i.e.:

$$\begin{aligned}\mathbb{E}^I(\text{average time w/o coupon before exit}) &= \psi^{-1}\ln\left(\frac{c\left[\frac{1-\xi r}{1-\frac{\beta}{\beta-1}}\right]^{\frac{1}{\gamma-1}}}{c}\right) \\ &= \psi^{-1}\ln\left(\left[(1-\beta)(1-\xi r)\right]^{\frac{1}{\gamma-1}}\right).\end{aligned} \tag{151}$$

End of Proof of Proposition 3.

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