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Eigenvalue Bounds for Symmetric, Multiple Saddle-Point Matrices with SPD Preconditioners

Luca Bergamaschi * and Michele Bergamaschi

Department of Civil Environmental and Architectural Engineering, University of Padua, Via Marzolo, 9, 35100 Padua, Italy; michele15456@gmail.com

* Correspondence: luca.bergamaschi@unipd.it

Abstract

We derive the eigenvalue bounds for symmetric block-tridiagonal multiple saddle-point systems preconditioned with the symmetric positive definite (SPD) preconditioner proposed by J. Pearson and A. Potschka in 2024 and further studied by L. Bergamaschi and coauthors, and for double saddle-point problems with inexact Schur complement matrices. The analysis applies to an arbitrary number of blocks. We validate the proposed estimates with both synthetic and realistic test problems, and show the good performance of the proposed preconditioner under the condition that the Schur complements are accurately approximated.

Keywords: multiple saddle-point systems; preconditioned iterative methods; symmetric positive definite preconditioner; eigenvalue bounds

1. Introduction

We consider the iterative solution of a block-tridiagonal multiple saddle-point linear system $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathcal{A} = \begin{bmatrix} A_0 & B_1^\top & 0 & \dots & 0 \\ B_1 & -A_1 & B_2^\top & \ddots & \vdots \\ 0 & B_2 & A_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B_N^\top \\ 0 & \dots & 0 & B_N & (-1)^N A_N \end{bmatrix}. \tag{1}$$

We assume that $A_0 \in \mathbb{R}^{n_0 \times n_0}$ is symmetric positive definite, all other square block matrices $A_k \in \mathbb{R}^{n_k \times n_k}$ are symmetric positive semi-definite and $B_k \in \mathbb{R}^{n_k \times n_{k-1}}$ have full rank (for $k = 1, \dots, N$). Denoting with $S_1 = A_1 + B_1 A_0^{-1} B_1^\top$, and $S_k = A_k + B_k S_{k-1}^{-1} B_k^\top, k = 2, \dots, N$ the Schur complements, we finally assume that

$$\lambda_{\min}(A_k) \cdot \lambda_{\min}(B_k S_{k-1}^{-1} B_k^\top) \neq 0, \quad k = 1, \dots, N. \tag{2}$$

This condition ensures that $S_k, k = 1, \dots, N$ are invertible, which also guarantees the invertibility of $\mathcal{A}$, see e.g., the discussion in [1–3].

The name *saddle-point*, and, by generalization, multiple saddle-point, linear system, comes from the fact that the the solution of $\begin{bmatrix} A_0 & B_1^\top \\ B_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$ is a saddle-point for the Lagrangian [4]

$$\frac{1}{2} x^T A_0 x - f^\top x + (B_1 x - g)^\top y.$$

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With the notation *double saddle-point linear system* we refer to (1) in the case that $N = 2$, providing a 3×3 block system. In general, we will refer to multiple saddle-point systems when the system matrix is of the form (1) and $N > 1$.

Linear systems involving matrix $\mathcal{A}$ arise mostly with $N = 2$ (double saddle-point systems) in many scientific applications including magma–mantle dynamics [5], liquid crystal director modeling [6] or in the coupled Stokes–Darcy problem [7–10], and the preconditioning of such linear systems has been considered in [11–13]. In particular, block-diagonal preconditioners for matrix $\mathcal{A}$ have been thoroughly examined in [1,14–18]. Augmented Lagrangian preconditioners have also been investigated in [19].

Multiple saddle-point linear systems with $N > 2$ have recently attracted the attention of a number of researchers. Such systems often stem from modeling multiphysics processes, i.e., the simultaneous simulation of different aspects of physical systems and the interactions among them. A recent example of this is the so-called four-field Thermo-Poroelasticity model which gives rise to a 4×4 block linear system. Discretization and preconditioning issues are considered e.g., in [20] and, more recently, in [21,22]. Other examples of multiple saddle-point linear systems can be found in [23–25].

The accurate and efficient solution of linear systems in which $\mathcal{A}$ takes the form described in (1) is required to obtain a sound numerical solution of the previously mentioned applications. Due to its size and sparsity, direct (factorization) methods, like the $LDL^\top$ factorization of $\mathcal{A}$ (see e.g., [26] for more details) are not recommended, since they usually produce an excessively dense triangular factor which can be hardly stored when the size of the problems is large, say more than 10^5.

A common way to overcome this issue is to consider iterative methods for the solution to $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$. Among those methods, the most popular ones are in the class of the Krylov subspace methods [27], which exploit the sparsity of the matrices arising from the previously described discretizations. Matrix $\mathcal{A}$ in (1) is highly indefinite. This means that it has a number of both negative and positive eigenvalues, posing some problems for the convergence of such iterative methods, see e.g., [4,28] and the references therein. In particular the spectrum of $\mathcal{A}$ is the union of a negative (I_-) and a positive (I_+) real interval, which we will denote in the following sequel:

$$I_- = [\lambda_-^{LB}, \lambda_-^{UB}] \qquad \text{and} \qquad I_+ = [\lambda_+^{LB}, \lambda_+^{UB},] \tag{3}$$

with symbol $-(+)$ standing for negative (positive) eigenvalues, and LB and UB for lower bounds and upper bounds, respectively. The large length of such intervals as well as the proximity of λ_-^{UB} and λ_+^{LB} to zero are detrimental for the convergence.

In any case, the use of iterative methods cannot come without preconditioning, namely the action of (1) devising a suitable matrix (the preconditioner, $\mathcal{P}$), approximating in some sense the original matrix $\mathcal{A}$, (2) transforming the original system $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$ into an equivalent one $\mathcal{P}^{-1}\mathcal{A}\boldsymbol{x} = \mathcal{P}^{-1}\boldsymbol{b}$, with, hopefully, more convenient spectral properties. To efficiently precondition $\mathcal{A}$, two different strategies can be followed, which take into account the block structure of the matrix. In the first, $\mathcal{P}$ is chosen to be symmetric and indefinite, with a block structure, and spectral properties, mimicking those of A. In this case, the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ are usually clustered around one, in a subset of the complex right half-plane. This second strategy is devoted to find an SPD approximation, $\mathcal{P}$, of $\mathcal{A}$, with the aim of using short-term recurrence iterative solvers, such as the MINRES Krylov subspace method. In this case, the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ again lie in two, negative and positive, intervals, with the action of the preconditioner consisting of shrinking both I_- and I_+. In both strategies, the convergence of the iterative solvers is expected to speed up.

An example of the first class is given by the block-triangular preconditioner [29], while to the second class belongs the block diagonal preconditioner [4,30], with both of them

based on the efficient approximation of the Schur complements of $\mathcal{A}$. In this work we will consider a recent preconditioner, belonging to the second class, which combines the strengths of both the block-triangular and the block-diagonal preconditioners.

Another preconditioning strategy, usually denoted as the *monolithic* approach, consists of developing a multigrid preconditioner (AMG) for the whole system matrix. Examples in this direction can be found e.g., in [31,32].

Arguably the most prominent Krylov subspace methods for solving the linear system $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$ are preconditioned variants of the MINimal RESidual method, MINRES [33], and the Generalized Minimal RESidual method, GMRES [34,35]. As opposed to GMRES, the MINRES algorithm can explicitly exploit the symmetry of $\mathcal{A}$. As a consequence, MINRES features a three-term recurrence relation, which is beneficial for its implementation (low memory requirements because subspace bases do not need to be stored) and its purely eigenvalue-based convergence analysis (via the famous connection to orthogonal polynomials; see [36,37]). Specifically, if the eigenvalues of the preconditioned matrix (referring to (3)) are contained within $I_- \cup I_+$, and I_- and I_+ have the same length, then at iteration k the Euclidean norm of the preconditioned residual r_k satisfies the bound

$$\frac{\|\boldsymbol{r}_k\|}{\|\boldsymbol{r}_0\|} \leq 2\left(\frac{\sqrt{|\lambda_-^{LB}\lambda_+^{UB}|}-\sqrt{|\lambda_-^{UB}\lambda_+^{LB}|}}{\sqrt{|\lambda_-^{LB}\lambda_+^{UB}|}+\sqrt{|\lambda_-^{UB}\lambda_+^{LB}|}}\right)^{\lfloor k/2\rfloor} = 2\left(\frac{1-\sqrt{\frac{\lambda_-^{UB}}{\lambda_-^{LB}}\frac{\lambda_+^{LB}}{\lambda_+^{UB}}}}{1+\sqrt{\frac{\lambda_-^{UB}}{\lambda_-^{LB}}\frac{\lambda_+^{LB}}{\lambda_+^{UB}}}}\right)^{\lfloor k/2\rfloor} \equiv 2\left(\frac{1-\zeta}{1+\zeta}\right)^{\lfloor k/2\rfloor}. \tag{4}$$

To guarantee fast convergence of the MINRES method, the quantity $\zeta = \sqrt{\frac{\lambda_-^{UB}}{\lambda_-^{LB}}\frac{\lambda_+^{LB}}{\lambda_+^{UB}}}$ should be as close as possible to 1, which occurs when boths intervals I_- and I_+ have small lengths and are bounded away from zero. The relation (4) allows us to predict with a good level of accuracy the number of iterations needed by MINRES to converge to a fixed accuracy, exclusively in terms of the bounds on the extremal eigenvalues $(\lambda_-^{LB}, \lambda_-^{UB}; \lambda_+^{LB}, \lambda_+^{UB})$ of the (preconditioned) matrix.

In this work we consider the preconditioner proposed in [3] (PP preconditioner in short) for multiple saddle-point linear systems, which show its good performances ont PDE-constrained optimization problems, in comparison with other SPD preconditioners. We develop a spectral analysis of the preconditioned matrix for an arbitrary number of blocks and in the presence of inexact Schur complements. The study we conduct in this paper, which will provide tight bounds for the endpoints of the intervals I_- and I_+, containing all the eigenvalues of the preconditioned matrix, is therefore paramount to predict the performance of the MINRES solver equipped with the considered PP preconditioner.

The novelty of this work consists of extending to an arbitrary number of blocks (hence allowing $N > 2$) the analysis carried out in [38] which holds only for double saddle-point systems. Motivated by the large amount of models that do require the solution of multiple ($N > 2$)-saddle-point linear systems, we develop a sequence of polynomials defined by recurrence whose extremal roots are connected with the endpoints of the intervals containing the eigenvalues of the preconditioned matrix. These polynomials are constructed using a similar idea as in [17] for the block-diagonal preconditioner. The present analysis is, however, complicated by the expression of the considered preconditioner (see e.g., the generalized eigenvalue problem in (7)). Moreover, differently from [17], we allow here an approximate evaluation of the Schur complement matrices, which makes this analysis useful for realistic settings.

This paper is structured as follows: In Section 2 we describe the preconditioner and write the eigenvalue problem for the preconditioned matrix. In Section 3, a sequence of polynomials is defined by recurrence, and the connection of these with the eigenvalues of the preconditioned matrix is stated. Then, in Section 4, we characterize the extremal roots

of such polynomials and the dependence of such roots with two vectors of parameters, containing information on the inexact Schur complements. Section 5 is devoted to comparing the bounds with the exactly computed eigenvalues on a large set of synthetic experiments for the number of blocks ranging from 3 to 5. In Section 6, we present as a realistic test case, the 3D mixed finite element discretization of the Biot model, which gives rise to a double saddle-point linear system, to again validate the bounds and to show the good performance of the PP preconditioner, also in comparison with the most known family of block-diagonal preconditioners, see e.g., [4,30], which we will describe in Section 2. Section 7 presents some conclusions.

2. The Preconditioner

Consider the exact block factorization of matrix $\mathcal{A} = U^\top D^{-1} U$, where

$$D = \begin{bmatrix} S_0 & & & & \\ & -S_1 & & & \\ & & S_2 & & \\ & & & \ddots & \\ & & & & (-1)^N S_N \end{bmatrix}, \qquad U = \begin{bmatrix} S_0 & B_1^\top & & & \\ & -S_1 & B_2^\top & & \\ & & S_2 & \ddots & \\ & & & \ddots & B_N^\top \\ & & & & (-1)^N S_N \end{bmatrix}, \tag{5}$$

with $S_0 = A_0, S_k = A_k + B_k S_{k-1}^{-1} B_k^\top, k = 1, \ldots, N$. The preconditioner is initially obtained by removing the minus signs in D, thus yielding a symmetric positive definite matrix $P = U^\top |D^{-1}| U$, and then, in view of its practical application, by approximating the Schur complements with their inexact counterparts. The final expression of the preconditioner is therefore $\mathcal{P} = \mathcal{P}_U^\top \mathcal{P}_D^{-1} P_U$, where

$$\mathcal{P}_D = \begin{bmatrix} \widehat{S}_0 & & & & \\ & \widehat{S}_1 & & & \\ & & \widehat{S}_2 & & \\ & & & \ddots & \\ & & & & \widehat{S}_N \end{bmatrix}, \qquad \mathcal{P}_U = \begin{bmatrix} \widehat{S}_0 & B_1^\top & & & \\ & -\widehat{S}_1 & B_2^\top & & \\ & & \widehat{S}_2 & \ddots & \\ & & & \ddots & B_N^\top \\ & & & & (-1)^N \widehat{S}_N \end{bmatrix}. \tag{6}$$

For our analysis, we denote the approximated Schur complements as $\widehat{S}_i$ for $i = 0, \ldots, N$. It is important to point out that each of these should approximate a perturbed Schur complement $\widetilde{S}_i$ that takes the previous Schur complement approximations into account according to

$$\begin{aligned} & & \widehat{S}_0 &\approx A_0 \\ \widetilde{S}_k &= A_k + B_k \widehat{S}_{k-1}^{-1} B_k^\top & \widehat{S}_k &\approx \widetilde{S}_k. \end{aligned}$$

The preconditioner (6) has been proposed by Pearson and Potschka in [3], in the framework of PDE-constrained optimization. Since $\mathcal{P}$ is symmetric and positive definite, this allows the use of the MINRES iterative method.

Finding the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ is equivalent to solving the generalized eigenproblem $\mathcal{P}_D^{-1/2} \mathcal{A} \mathcal{P}_D^{-1/2} \boldsymbol{u} = \lambda \mathcal{P}_D^{-1/2} \mathcal{P} \mathcal{P}_D^{-1/2} \boldsymbol{u}$, or $\mathcal{Q}\boldsymbol{u} = \lambda \mathcal{L}\mathcal{L}^\top \boldsymbol{u}$:

$$\underbrace{\mathcal{P}_D^{-1/2} \mathcal{A} \mathcal{P}_D^{-1/2}}_{\mathcal{Q}} \boldsymbol{u} = \lambda \underbrace{\mathcal{P}_D^{-1/2} P_U^\top \mathcal{P}_D^{-1/2}}_{\mathcal{L}} \underbrace{\mathcal{P}_D^{-1/2} P_U \mathcal{P}_D^{-1/2}}_{\mathcal{L}^\top} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \ldots \\ \boldsymbol{u}_{N+1} \end{bmatrix}.$$

Exploiting the blocks of the generalized eigenvalue problem

$$\mathcal{Q}\boldsymbol{u} = \lambda \mathcal{L}\mathcal{L}^\top \boldsymbol{u}, \tag{7}$$

we get

$$\mathcal{Q} = \mathcal{P}_D^{-1/2}\mathcal{A}\mathcal{P}_D^{-1/2} = \begin{bmatrix} E_0 & R_1^\top & & & \\ R_1 & -E_1 & R_2^\top & & \\ 0 & R_2 & E_2 & R_3^\top & \\ & \cdots & \cdots & \cdots & \\ & & R_{N-1} & (-1)^{N-1}E_{N-1} & R_N^\top \\ & & & R_N & (-1)^N E_N \end{bmatrix} \tag{8}$$

$$\begin{aligned} \mathcal{L} &= \mathcal{P}_D^{-1/2}\mathcal{P}_U^\top \mathcal{P}_D^{-1/2} = \\ &= \begin{bmatrix} \widehat{S}_0^{-1/2} & & & & \\ & \widehat{S}_1^{-1/2} & & & \\ & & \widehat{S}_2^{-1/2} & & \\ & & & \ddots & \\ & & & & \widehat{S}_N^{-1/2} \end{bmatrix} \begin{bmatrix} \widehat{S}_0 & & & & \\ B_1 & -\widehat{S}_1 & & & \\ & B_2 & \widehat{S}_2 & & \\ & & & \ddots & \\ & & & B_N & (-1)^N \widehat{S}_N \end{bmatrix} \begin{bmatrix} \widehat{S}_0^{-1/2} & & & & \\ & \widehat{S}_1^{-1/2} & & & \\ & & \widehat{S}_2^{-1/2} & & \\ & & & \ddots & \\ & & & & \widehat{S}_N^{-1/2} \end{bmatrix} = \\ &= \begin{bmatrix} I & & & & \\ R_1 & -I & & & \\ & R_2 & I & & \\ & \cdots & \cdots & & \\ & & R_{N-1} & (-1)^{N-1}I & \\ & & & R_N & (-1)^N I \end{bmatrix}, \quad \text{and, finally} \end{aligned}$$

$$\mathcal{L}\mathcal{L}^\top = \begin{bmatrix} I & R_1^\top & & & \\ R_1 & I + R_1 R_1^\top & -R_2^\top & & \\ 0 & -R_2 & I + R_2 R_2^\top & R_3^\top & \\ & \cdots & \cdots & \cdots & \\ & & (-1)^N R_{N-1} & I + R_{N-1}R_{N-1}^\top & (-1)^{N+1}R_N^\top \\ & & & (-1)^{N+1}R_N & I + R_N R_N^\top \end{bmatrix} \tag{9}$$

where

$$\begin{aligned} R_k &= \widehat{S}_k^{-1/2} B_k \widehat{S}_{k-1}^{-1/2}, \ k = 1, \ldots, N \\ E_k &= \widehat{S}_k^{-1/2} A_k \widehat{S}_k^{1/2}, \ k = 0, \ldots, N \\ R_k^\top R_k + E_k &= \widehat{S}_k^{-1/2} \widetilde{S}_k \widehat{S}_k^{-1/2} \equiv \overline{S}_k. \end{aligned}$$

With $\overline{S}_k$ we denote the k-th perturbed Schur complement $\widetilde{S}_k$, preconditioned with its SPD approximation $\widehat{S}_k$. Componentwise, we rewrite (7), by using (8) and (9), as

$$\begin{aligned} \mathbf{0} &= (E_0 - \lambda I)\boldsymbol{u}_1 + (1-\lambda)R_1^\top \boldsymbol{u}_2, \\ \mathbf{0} &= (1-\lambda)R_1\boldsymbol{u}_1 + (-E_1 - \lambda(I + R_1 R_1^\top))\boldsymbol{u}_2 + (1+\lambda)R_2^\top \boldsymbol{u}_3, \\ \mathbf{0} &= (1+\lambda)R_2\boldsymbol{u}_2 + (E_2 - \lambda(I + R_2 R_2^\top))\boldsymbol{u}_3 + (1-\lambda)R_3^\top \boldsymbol{u}_4, \\ & \cdots \quad \cdots \quad \cdots \\ \mathbf{0} &= (1 + (-1)^{N-1}\lambda)R_{N-1}\boldsymbol{u}_{N-1} + ((-1)^{N-1}E_{N-1} - \lambda(I + R_{N-1}R_{N-1}^\top))\boldsymbol{u}_N \\ &\quad + (1 + (-1)^N \lambda)R_N^\top \boldsymbol{u}_{N+1}, \\ \mathbf{0} &= (1 + (-1)^N \lambda)R_N \boldsymbol{u}_N + ((-1)^N E_N - \lambda(I + R_N R_N^\top))\boldsymbol{u}_{N+1}. \end{aligned} \tag{10}$$

The matrices $R_k R_k^\top$ are all symmetric positive semi-definite. We define two indicators $\gamma_E^{(k)}$ and $\gamma_R^{(k)}$ using the Rayleigh quotient

$$\begin{aligned} &\alpha_E^{(k)} \equiv \lambda_{\min}(E_k), \qquad \beta_E^{(k)} \equiv \lambda_{\max}(E_k), \qquad \gamma_E^{(k)}(\boldsymbol{w}) = \frac{\boldsymbol{w}^\top E_k \boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}} \in \left[\alpha_E^{(k)}, \beta_E^{(k)}\right] \equiv \mathcal{I}_{E_k}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, \ldots, N \\ &\alpha_R^{(k)} \equiv \lambda_{\min}\left(R_k R_k^\top\right), \quad \beta_R^{(k)} \equiv \lambda_{\max}\left(R_k R_k^\top\right), \quad \gamma_R^{(k)}(\boldsymbol{w}) = \frac{\boldsymbol{w}^\top R_k R_k^\top \boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}} \in \left[\alpha_R^{(k)}, \beta_R^{(k)}\right] \equiv \mathcal{I}_{R_k}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, N. \end{aligned} \tag{11}$$

The role of these indicators will be clear when considering the eigenvalue problem (10). They represent a generic value between the minimum and the maximum eigenvalues of the corresponding matrix. The vector $\boldsymbol{w}$ is a generic nonzero vector of the suitable dimension.

Since A_k is positive semi-definite and $R_k R_K^\top$ could not have full rank, we allow that $\alpha_E^{(k)} = 0$ for some $k \geq 1$ as well as $\alpha_R^{(k)} = 0$ for some $k \geq 1$. However, according to condition (2), this situation cannot occur simultaneously, for the same k.

A third set of indicators, $\gamma_S^{(k)}$, is linked to the previous ones as

$$\gamma_S^{(k)}(\boldsymbol{w}) = \gamma_E^{(k)}(\boldsymbol{w}) + \gamma_R^{(k)}(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \mathbb{R}^{n_k}, \tag{12}$$

which describes the field of values of each preconditioned Schur complement $\overline{S}_k$:

$$\alpha_S^{(k)} \equiv \lambda_{\min}\left(\overline{S}_k\right), \ \beta_S^{(k)} \equiv \lambda_{\max}\left(\overline{S}_k\right), \ \gamma_S^{(k)}(\boldsymbol{w}) = \frac{\boldsymbol{w}^\top \overline{S} \boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}} \in \left[\alpha_S^{(k)}, \beta_S^{(k)}\right] \equiv \mathcal{I}_{S_k}, \ i = 1, \ldots, N. \tag{13}$$

If the Schur complements are exactly inverted, then $E_0 = I$ and $R_k R_k^\top + E_k = I$. In such a case, denoting $\gamma_R^{(0)} = 0$, we have $\gamma_S^{(k)}(\boldsymbol{w}) = \gamma_E^{(k)}(\boldsymbol{w}) + \gamma_R^{(k)}(\boldsymbol{w}) \equiv 1$, $j = 0, \ldots, N, \forall \boldsymbol{w}$, and the eigenvalues of (10) are either -1 or 1, with multiplicity, respectively, $n_1 + n_3 + \ldots$ and $n_0 + n_2 + \ldots$ (see [3], Theorem 2.1). In the following, we will often remove the argument $\boldsymbol{w}$ from the previously defined indicators. We define the following two vectors of parameters:

$$\boldsymbol{\gamma}_E = \left[\gamma_E^{(0)}, \ldots, \gamma_E^{(N)}\right], \qquad \boldsymbol{\gamma}_R = \left[\gamma_R^{(1)}, \ldots, \gamma_R^{(N)}\right].$$

We will now make some very mild assumptions on two of these indicators:

Assumptions 1. *The value 1 is strictly included in both $\mathcal{I}_{E_0}$ and $\mathcal{I}_{S_1}$, namely*

1. $\alpha_E^{(0)} < 1 < \beta_E^{(0)}$,
2. $\alpha_S^{(1)} < 1 < \beta_S^{(1)}$.

These assumptions are very commonly satisfied in practice, meaning that 1 is in both the spectra of the preconditioned $(1,1)$ block and of the preconditioned Schur complement $\overline{S}_1$. In fact usually preconditioners *symmetrically* cluster the eigenvalues around one, with the number one strictly belonging to the spectral interval. As an example, the Jacobi preconditioner always satisfies this property since, by construction, the arithmetic mean of the eigenvalues of the preconditioned matrix is exactly 1.

3. Characterization of the Eigenvalues of the Preconditioned Matrix

We now recursively define a sequence of polynomials, with $\boldsymbol{\gamma}_E$ and $\boldsymbol{\gamma}_R$ as parameters, whose roots are strictly related with eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$.

Definition 1.

$$\begin{aligned} U_0(\lambda,\gamma_R,\gamma_E) &= 1, \\ U_1(\lambda,\gamma_R,\gamma_E) &= \lambda-\gamma_E^{(0)}, \\ U_{k+1}(\lambda,\gamma_R,\gamma_E) &= (\lambda(1+\gamma_R^{(k)})+(-1)^{k+1}\gamma_E^{(k)})U_k(\lambda,\gamma_R,\gamma_E) \\ &\quad -\gamma_R^{(k)}(\lambda+(-1)^k)^2U_{k-1}(\lambda,\gamma_R,\gamma_E), \qquad k\geq 1. \end{aligned} \tag{14}$$

These monic polynomials will play an important role in the evaluation of the bounds on the eigenvalues of the preconditioned matrix, as will stand out from Lemma 2 and from Theorem 1. Notice that, in the case of exact Schur complements, and hence for $\gamma_E^{(k)}+\gamma_R^{(k)}\equiv 1$, $k=0,\ldots,N$, this sequence of polynomials reduces to

$$U_k(\lambda)=(\lambda-1)^{\lfloor\frac{k+1}{2}\rfloor}(\lambda+1)^{\lfloor\frac{k}{2}\rfloor}, \qquad k=1,\ldots,N+1.$$

We now define a sequence of matrix-valued function, again by recurrence:

Definition 2.

$$\begin{aligned} &Y_1(\lambda)=\lambda I-E_0, \\ &Y_{k+1}(\lambda)=(-1)^{k+1}E_k+\lambda\left(I+R_kR_k^\top\right)-(\lambda+(-1)^k)^2R_kY_k(\lambda)^{-1}R_k^\top, \quad k\geq 1, \\ &\quad \textit{and } \lambda \textit{ s.t. } 0\notin\sigma(Y_k(\lambda)). \end{aligned}$$

Notation. We use the notation $\mathcal{I}_k$ to denote the union of intervals bounding the roots of polynomials of the form U_k over the valid range of γ_E,γ_R.

We recall a technical lemma, based on an idea in [39]; its proof can be found in [38].

Lemma 1. *Let Y be a symmetric matrix valued function defined in $F\subset\mathbb{R}$ and*

$$0\notin[\min\{\sigma(Y(\xi))\},\max\{\sigma(Y(\xi))\}] \qquad \textit{for all } \xi\in F.$$

Then, for arbitrary $\boldsymbol{s}\neq\mathbf{0}$, there exists a vector $\boldsymbol{v}\neq\mathbf{0}$ such that

$$\frac{\boldsymbol{s}^\top Y(\xi)^{-1}\boldsymbol{s}}{\boldsymbol{s}^\top\boldsymbol{s}}=\frac{1}{\gamma_Z} \qquad \textit{with } \gamma_Z=\frac{\boldsymbol{v}^\top Y(\xi)\boldsymbol{v}}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

The next lemma, which will be used in the proof of the subsequent Theorem 1, links together the two sequences previously defined.

Lemma 2. *For every $\boldsymbol{u}\neq\mathbf{0}$, there is a choice of $\boldsymbol{\gamma}$ for which*

$$\frac{\boldsymbol{u}^\top Y_{k+1}(\lambda)\boldsymbol{u}}{\boldsymbol{u}^\top\boldsymbol{u}}=\frac{U_{k+1}(\lambda)}{U_k(\lambda)} \qquad \textit{for all } \lambda\notin\bigcup_{j=1}^{k}\mathcal{I}_j.$$

Proof. This is shown by induction. We first define

$$\eta_k(\lambda)=(-1)^{k+1}\gamma_E^{(k)}+\lambda(1+\gamma_R^{(k)}), \quad \overline{\mu}_k(\lambda)=(\lambda+(-1)^k)^2. \tag{15}$$

For $k=0$ we have $\dfrac{\boldsymbol{u}^\top Y_1(\lambda)\boldsymbol{u}}{\boldsymbol{u}^\top\boldsymbol{u}}=\lambda-\gamma_E^{(0)}=\dfrac{U_1(\lambda)}{U_0(\lambda)}$ for all $\lambda\in\mathbb{R}$. If $k\geq 1$, the condition $\lambda\notin\mathcal{I}_k$, together with the inductive hypothesis $\dfrac{\boldsymbol{u}^\top Y_k(\lambda)\boldsymbol{u}}{\boldsymbol{u}^\top\boldsymbol{u}}=\dfrac{U_k(\lambda)}{U_{k-1}(\lambda)}$, implies invertibility

of $Y_k(\lambda)$. Moreover, this is equivalent to the condition $0 \notin [\min\{\sigma(Y(\xi))\}, \max\{\sigma(Y(\xi))\}]$ that guarantees the applicability of Lemma 1. Therefore, we can write

$$\frac{\boldsymbol{u}^\top Y_{k+1}(\lambda)\boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}} = \eta_k(\lambda) - \overline{\mu}_k(\lambda)\frac{\boldsymbol{u}^\top R_k Y_k(\lambda)^{-1} R_k^\top \boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}} \underset{\boldsymbol{w}=R_k^\top \boldsymbol{u}}{=} \eta_k(\lambda) - \overline{\mu}_k(\lambda)\frac{\boldsymbol{w}^\top Y_k(\lambda)^{-1}\boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}}\gamma_R^{(k)} \tag{16}$$

We then apply Lemma 1 and the inductive hypothesis to write

$$\frac{\boldsymbol{w}^\top Y_k(\lambda)^{-1}\boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}} = \frac{U_{k-1}(\lambda)}{U_k(\lambda)}.$$

Substituting into (16) and using relation (14) we obtain

$$\eta_k(\lambda) - \overline{\mu}_k(\lambda)\frac{\boldsymbol{w}^\top Y_k(\lambda)^{-1}\boldsymbol{w}}{\boldsymbol{w}^\top \boldsymbol{w}}\gamma_R^{(k)} = \frac{\eta_k(\lambda)U_k(\lambda) - \overline{\mu}_k(\lambda)\gamma_R^{(k)}U_{k-1}(\lambda)}{U_k(\lambda)} = \frac{U_{k+1}(\lambda)}{U_k(\lambda)}.$$

□

Remark 1. *An important consequence of Lemma 2 is that* $\boldsymbol{u}^\top Y_k(\lambda)\boldsymbol{u} = 0 \iff U_k(\lambda) = 0$, *for a particular choice of the parameters. Therefore if* $\lambda \notin \mathcal{I}_k$ *(satisfying* $U_k(\lambda) \neq 0$ *for any choice of the parameters) then* $\boldsymbol{u}^\top Y_k(\lambda)\boldsymbol{u} \neq 0$ *for any* $\boldsymbol{u} \neq 0$. *This also implies that* $Y_k(\lambda)$ *is definite, and consequently also invertible.*

We now state the main results of this section: in Theorem 1 we will show that all the eigenvalues of the preconditioned matrix are located in the union of the intervals containing the zeros of the previously defined sequence of polynomials.

Theorem 1. *The eigenvalues of* $\mathcal{P}^{-1}\mathcal{A}$ *are located in* $\mathcal{I} = \bigcup_{k=1}^{N+1} \mathcal{I}_k$.

Proof. The proof is carried out through induction on k, that is for every $k \leq N+1$ either

$$\text{(i) } \lambda \in \mathcal{I}_k \qquad \text{or} \qquad \text{(ii) } \boldsymbol{u}_k = (1 + (-1^k)\lambda)Y_k(\lambda)^{-1}R_k^\top \boldsymbol{u}_{k+1},$$

and for $k = N+1$ only condition (i) can hold. Let $\boldsymbol{u} = \left[\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_{k+1}^\top\right]^\top$ be an eigenvector of (10). Assume that $\lambda \notin \mathcal{I}_{E_0}$ (for $k = 0$), then $Y_1(\lambda)$ is invertible. From the first equation of (10) we get

$$(\lambda I - E_0)\boldsymbol{u}_1 = (1-\lambda)R_1^\top \boldsymbol{u}_2 \quad \Rightarrow \quad Y_1(\lambda)\boldsymbol{u}_1 = (1-\lambda)R_1^\top \boldsymbol{u}_2, \tag{17}$$

whereupon inserting (17) into the second row of (10) yields

$$\underbrace{\left(E_1 + \lambda(I + R_1R_1^\top) - (\lambda-1)^2 R_1(\lambda I - E_0)^{-1}R_1^\top\right)}_{Y_2(\lambda)}\boldsymbol{u}_2 = (1+\lambda)R_2^\top \boldsymbol{u}_3. \tag{18}$$

Pre-multiplying the left hand side of (18) by $\frac{\boldsymbol{u}_2^\top}{\boldsymbol{u}_2^\top \boldsymbol{u}_2}$, we obtain that

$$\frac{\boldsymbol{u}_2^\top Y_2(\lambda)\boldsymbol{u}_2}{\boldsymbol{u}_2^\top \boldsymbol{u}_2} = \frac{U_2(\lambda)}{U_1(\lambda)}$$

If λ is a zero of U_2 then $\lambda \in \mathcal{I}_2$, otherwise $Y_2(\lambda)$ is invertible and definite (see Remark 1), and we can write $\boldsymbol{u}_2 = (1+\lambda)Y_2(\lambda)^{-1}\boldsymbol{u}_3$. Assume now the inductive hypothesis holds for $k-1$. If $\lambda \notin \mathcal{I}_{k-1}$, then $Y_{k-1}(\lambda)$ is definite and invertible. We can write

$$\begin{aligned} Y_k(\lambda)\boldsymbol{u}_k &\equiv \left((-1)^k E_{k-1} + \lambda(I + R_{k-1}R_{k-1}^\top) - (\lambda + (-1)^{k-1})^2 R_{k-1}Y_{k-1}^{-1}R_{k-1}^\top\right)\boldsymbol{u}_k \\ &= (1+(-1)^k\lambda)R_k^\top \boldsymbol{u}_{k+1}. \end{aligned}$$

Since $\lambda \notin \mathcal{I}_{k-1}$, then Y_{k-1} is definite and therefore, by Lemma 2,

$$\frac{\boldsymbol{u}_k^\top Y_k(\lambda)\boldsymbol{u}_k}{\boldsymbol{u}_k^\top \boldsymbol{u}_k} = \frac{U_k(\lambda)}{U_{k-1}(\lambda)}.$$

We have that either λ is a zero of $U_k(\lambda)$, and hence $\lambda \in \mathcal{I}_k$, or $Y_k(\lambda)$ is definite, in such a case we can write $\boldsymbol{u}_k = (1+(-1)^k\lambda)Y_k(\lambda)^{-1}\boldsymbol{u}_{k+1}$. The induction process ends for $k = N+1$. In this case we have that

$$Y_{N+1}(\lambda)\boldsymbol{u}_{N+1} = \mathbf{0},$$

and the condition $\lambda \in \mathcal{I}_{N+1}$ must hold, noticing that $\boldsymbol{u}_{N+1} = \mathbf{0}$ would imply that also $\boldsymbol{u}_N = \cdots = \boldsymbol{u}_1 = \mathbf{0}$ contradicting the definition of an eigenvector. □

4. Bounds on the Roots of $\{U_k\}$

In this section we will first characterize the set $\mathcal{I} = \bigcup \mathcal{I}_k$. Then we will specify the dependence of the zeros $U_k(\lambda)$ on the parameters γ_E, γ_R.

Proposition 1. *Since the polynomials of the sequence* (14) *are monic*

$$\lim_{\lambda\to+\infty} U_k(\lambda) = +\infty \qquad \lim_{\lambda\to-\infty} U_k(\lambda) = (-1)^k \cdot \infty$$

Lemma 3. *Given the sequence of polynomials* (14)*, for all* $k \geq 1$

$$\operatorname{sgn}(U_{2k-1}(0)) = \operatorname{sgn}(U_{2k}(0)) = (-1)^k.$$

In other words, the signs of $U_k(0)$ *follow the repeating pattern* $--, ++, --, ++, \ldots$ *starting from* $k = 1$*. Equivalently,* $\operatorname{sgn}(U_k(0)) = (-1)^{\lceil \frac{k}{2} \rceil}$.

Proof. We prove this by induction of k. The claim holds for $k = 1$:

$$U_1(0) = -\gamma_E^{(0)} < 0, \qquad U_2(0) = -\gamma_E^{(1)}\gamma_E^{(0)} - \gamma_R^{(1)} < 0.$$

Assume that the claim holds for some $k \geq 1$, that is the sign of $U_{2k-1}(0)$ and $U_{2k}(0)$ is $s = (-1)^k$, then

$$\begin{aligned} \operatorname{sgn}(U_{2k+1}(0)) &= (-1)^{2k+1}\operatorname{sgn}(U_{2k}(0)) - \operatorname{sgn}(U_{2k-1}(0)) = -s = (-1)^{k+1}. \\ \operatorname{sgn}(U_{2k+2}(0)) &= (-1)^{2k+2}\operatorname{sgn}(U_{2k+1}(0)) - \operatorname{sgn}(U_{2k}(0)) = -s = (-1)^{k+1}. \end{aligned}$$

□

We now show that, based on Assumption 1, both -1 and 1 are strictly included in $\mathcal{I}$, in fact $\alpha_E^{(0)} < 1 < \beta_E^{(0)}$ implies 1 is strictly included in $\mathcal{I}_1$, and $\alpha_S^{(1)} < 1 < \beta_S^{(1)}$ implies that -1 is strictly included in $\mathcal{I}_2$. In fact

$$U_2(-1; \gamma_E^{(0)}, \gamma_E^{(1)}, \gamma_R^{(1)}) = 1 + \gamma_E^{(0)} - 3\gamma_R^{(1)} + \gamma_E^{(0)}(\gamma_R^{(1)} - \gamma_E^{(1)}) - \gamma_E^{(1)}.$$

Using $\gamma_E^{(0)} = 1$, we have $U_2(-1;1,\gamma_E^{(1)},\gamma_R^{(1)}) = 2(1-\gamma_E^{(1)}-\gamma_R^{(1)}) = 2(1-\gamma_S^{(1)}) \equiv U_2(-1;1,\gamma_S^{(1)})$. That -1 strictly belongs to $\mathcal{I}_2$ comes from $U_2(-1;1,\alpha_S^{(1)}) > 0$ and $U_2(-1;1,\beta_S^{(1)}) < 0$.

We now state the results regarding the extremal roots of the polynomials $\{U_k\}$. In particular we prove that the smallest negative root is decreasing, for increasing degree, and the largest positive root is increasing, with k increasing. The assumptions $\xi_1^{(1)} < -1$ and $\xi_{s_k}^{(k)} > 1$ are made with the aim of finding the smallest negative and the largest positive roots, that must be less than -1 and larger than 1, respectively, in view of the above observation.

Theorem 2. *Assume that the polynomial $U_k(\lambda)$ has s_k distinct roots, and let us denote as $\xi_1^{(k)} < \xi_2^{(k)} < \ldots \xi_{s_k}^{(k)}$ the roots of $U_k(\lambda)$. If $\xi_1^{(k)} < -1$ and $\xi_{s_k}^{(k)} > 1$, $\forall k$, then the extremal roots of the polynomials U_k satisfy*

$$\xi_1^{(k)} \le \xi_1^{(k-1)} \le \ldots \le \xi_1^{(2)}, \qquad \textit{and} \qquad \xi_1^{(1)} \le \ldots \le \xi_{s_k-1}^{(k-1)} \le \xi_{s_k}^{(k)}.$$

Proof. We prove the claim by induction, starting from $\ell = 2$. Consider first the positive roots. The basis of the induction is

$$1 \le \xi_1^{(1)} \le \xi_2^{(2)},$$

which is readily proved by taking $\xi_1^{(1)} = \gamma_E^{(0)} > 1$ and observing that $U_2(\gamma_E^{(0)}) = -\gamma_R^{(0)}(\gamma_E^{(0)}-1)^2 \le 0$. Assume now that the claim holds for $\ell-1$, that is, $\xi_{s_{\ell-1}}^{(\ell-1)} \le \xi_{s_\ell}^{(\ell)}$. This implies that $U_{\ell-1}(\xi_{s_\ell}^{(\ell)}) \ge 0$. Then, from the recursion (14)

$$U_{\ell+1}(\xi_{s_\ell}^{(\ell)}) = -\gamma_R(\xi_{s_\ell}^{(\ell)} + (-1)^\ell)^2 U_{\ell-1}(\xi_{s_\ell}^{(\ell)}) \le 0,$$

which implies that $\xi_{s_\ell}^{(\ell)} \le \xi_{s_{\ell+1}}^{(\ell+1)}$.

Consider now the negative roots. If $\gamma_E^{(0)} < 1$ and $\gamma_S^{(1)} > 1$, direct computation shows that $U_2(-1) < 0$, providing $\xi_1^{(2)} < -1$. Moreover, $U_3(\xi_1^{(2)}) = -\gamma_R^{(2)}(\xi_1^{(2)}+1)^2(\xi_1^{(2)} - \gamma_E^{(0)}) \ge 0$, which implies $\xi_1^{(3)} \le \xi_1^{(2)}$. Assume now that the claim holds for $l-1$, that is $\xi_1^{(\ell-1)} \ge \xi_1^{(\ell)}$. This can occur if either $U_{\ell-1}(\xi_1^{(\ell)}) = 0$ (from which also $U_{\ell+1}(\xi_1^{(\ell)}) = 0$) or $\mathrm{sgn}(U_{\ell-1}(\xi_1^{(\ell)})) = (-1)^{\ell-1}$. In this case, from the recursion (14)

$$\mathrm{sgn}(U_{\ell+1}(\xi_1^{(\ell)})) = \begin{cases} 0 & \text{if } \gamma_R^{(k)} = 0 \\ -\mathrm{sgn}\Big(U_{\ell-1}(\xi_1^{(\ell)})\Big) = -(-1)^{\ell-1} = (-1)^\ell & \text{if } \gamma_R^{(k)} > 0, \end{cases}$$

which finally shows that $\xi_1^{(\ell+1)} \le \xi_1^{(\ell)}$, as desired. □

The results of this section allow us to characterize the set $\mathcal{I}$ in Theorem 1 like

$$\mathcal{I} = [\lambda_-^{LB}, \lambda_-^{UB}] \cup [\lambda_+^{LB}, \lambda_+^{UB}],$$

where

$$\begin{aligned} \lambda_-^{LB} &= \xi_1^{(N+1)} \\ \lambda_-^{UB} &= \max_j\{\max_\ell\{\xi_\ell^{(j)}, \text{s.t. } \xi_\ell^{(j)} < 0\}\} \\ \lambda_+^{LB} &= \min_j\{\min_\ell\{\xi_\ell^{(j)}, \text{s.t. } \xi_\ell^{(j)} > 0\}\} \\ \lambda_+^{UB} &= \xi_{s_{N+1}}^{(N+1)}. \end{aligned}$$

4.1. How Zeros of $\{U_k\}$ Move Depending on $\gamma_E^{(j)}, \gamma_R^{(j)}$

Now the question is for which values of the parameters $\gamma_E^{(i)}, \gamma_R^{(i)}$ is the extremal values of the roots attained? The following results will allow us to state that the bounds on the zeros of $U_{k+1}(\lambda, \gamma_E, \gamma_R)$ are obtained at the extremal values of γ_E and γ_R, namely $\{\alpha_E^{(i)}, \beta_E^{(i)}; \alpha_R^{(i)}, \beta_R^{(i)}\}$.

Let ξ an extremal zero of the polynomial U_{k+1} (smallest/largest negative, smallest/largest positive) for a particular combination of the parameters. Then taking separately one of the parameters, γ_* we can write γ_* as a convex combination of its extremal values, $\gamma_* = \alpha\gamma_*^{\min} + (1-\alpha)\gamma_*^{\max}$ and write

$$\begin{aligned} U_{k+1}(\lambda, \gamma_*) &= s_1(\lambda) + \gamma_* s_2(\lambda) = s_1(\lambda) + (\alpha\gamma_*^{\min} + (1-\alpha)\gamma_*^{\max}) s_2(\lambda) \\ &= \alpha(s_1(\lambda) + \gamma_*^{\min} s_2(\lambda)) + (1-\alpha)(s_1(\lambda) + \gamma_*^{\max} s_2(\lambda)) \\ &= \alpha U_{k+1}(\lambda, \gamma_*^{\min}) + (1-\alpha) U_{k+1}(\lambda, \gamma_*^{\max}). \end{aligned}$$

Hence,

$$0 = U_{k+1}(\xi, \gamma_*) = \alpha U_{k+1}(\xi, \gamma_*^{\min}) + (1-\alpha) U_{k+1}(\xi, \gamma_*^{\max}),$$

with $\rho_1 = U_{k+1}(\xi, \gamma_*^{\min})$ and $\rho_2 = U_{k+1}(\xi, \gamma_*^{\max})$ therefore either taking opposite signs, or being both zero. In the first case it is clear that one between ρ_1 and ρ_2 improves the sought root, in particular we select $\gamma \in \{\gamma_*^{\min}, \gamma_*^{\max}\}$ corresponding to the $\rho_* \in \{\rho_1, \rho_2\}$ such that

1. **(Smallest negative root)**: The sign of ρ_* is opposite to the sign of U_{k+1} at $-\infty$.
2. **(Largest negative root)**: The sign of ρ_* is opposite to the sign of U_{k+1} at 0.
3. **(Smallest positive root)**: The sign of ρ_* is opposite to the sign of U_{k+1} at 0.
4. **(Largest positive root)**: The sign of ρ_* is opposite to the sign of U_{k+1} at $+\infty$.

If, finally, both ρ_1 and ρ_2 are zero, for linearity, it means that the root is independent of this parameter and we can choose one of its extremal values.

We will now prove (in Theorem 3, and in Section 4.2) that we can predict whether $\gamma_E^{(i)} = \alpha_E^{(i)}$ or $\gamma_E^{(i)} = \beta_E^{(i)}$ in order to obtain the intervals bounding the roots of U_k.

Lemma 4. *Given a fixed k, there exist polynomials W_ℓ for any $0 \le \ell \le k$ such that*

$$U_{k+1} = U_j W_{k-j+1} - U_{j-1} W_{k-j} \mu_j,$$

with $\mu_j = \gamma_R^{(j)}(\lambda + (-1)^j)^2$.

Proof. The statement holds for $j = k$, by setting $W_1 = \eta_k$ and $W_0 = 1$. Let the statement be true for $j \ge 2$, then,

$$\begin{aligned} U_{k+1} &= U_j W_{k-j+1} - U_{j-1} W_{k-j} \mu_j = \\ &= W_{k-j+1}(\eta_{j-1} U_{j-1} - \mu_{j-1} U_{j-2}) - U_{j-1} W_{k-j} = \\ &= \underbrace{(W_{k-j+1}\eta_{j-1} - W_{k-j})}_{W_{k-j+2}} U_{j-1} - W_{k-j+1}\mu_{j-1} U_{j-2} \end{aligned}$$

It is also clear by induction that W_{k-j} depends only on the parameters with index greater than j. □

We denote with $e_j \in \{\alpha_E^{(j)}, \beta_E^{(j)}\}$ one choice of the extremal values of $\gamma_E^{(j)}$ and, analogously, $r_j \in \{\alpha_R^{(j)}, \beta_R^{(j)}\}$; moreover we denote by e_j^* and r_j^* the other extremal value of the parameter. We now consider the polynomial U_k with $\gamma_E^{(j)} \equiv e_j$, $\gamma_R^{(j)} \equiv r_j$ and also define

${}^{E}U_k^j$ as the polynomial U_k with $\gamma_E^{(j)} \equiv e_j^*$. Similarly we define ${}^{R}U_k^j$ as the polynomial with $\gamma_R^{(j)} = r_j^*$. Then it is immediate from the recursion of U_k that

$$
\begin{aligned}
{}^{E}U_{j+1}^j &= U_{j+1} + (-1)^{j+1}(e_j^* - e_j)U_j && (19)\\
{}^{R}U_{j+1}^j &= U_{j+1} + [\lambda U_j - (\lambda + (-1)^j)^2 U_{j-1}](r_j^* - r_j). && (20)
\end{aligned}
$$

Setting $z_j(\lambda) = \lambda U_j - (\lambda + (-1)^j)^2 U_{j-1}$, (20) can be rewritten simply as

$$
{}^{R}U_{j+1}^j = U_{j+1} + z_j(r_j^* - r_j).
$$

Before stating the main result of this section, we premise two Lemmas.

Lemma 5. *We have the following relations*

$$
\begin{aligned}
{}^{E}U_{k+1}^j &= U_{k+1} + (-1)^{j+1} W_{k-j} U_j (e_j^* - e_j) && (21)\\
{}^{R}U_{k+1}^j &= U_{k+1} + W_{k-j} z_j (r_j^* - r_j). && (22)
\end{aligned}
$$

Proof. We write

$$
U_{k+1} = U_{j+1} W_{k-j} - U_j W_{k-j-1} \mu_{j+1},
$$

and notice that the only term in the right hand side that depends on $\gamma_E^{(j)}$ is U_{j+1}. Thus, using (19),

$$
{}^{E}U_{k+1}^j - U_{k+1} = W_{k-j}({}^{E}U_{j+1}^j - U_{j+1}) = (-1)^{j+1} W_{k-j} U_j (e_j^* - e_j).
$$

This proves the first formula. The second one is proved in the same way, using (20). □

Lemma 6. *The following statements hold:*

(a) *Let ξ be a root of $U_{\ell+1}$. If $W_{\ell-j}(\xi) = 0$ then ξ is a root for all values of the parameter $\gamma_E^{(j)}$.*

(b) *Let ξ as above and suppose further $U_j(\xi) \neq 0$ for $j \leq \ell$. If $W_{\ell-j}(\xi) = 0$ then $W_{\ell-s}(\xi) = 0$ for all $s \leq j$.*

(c) *If ξ is as in (b) and $r_j = 0$, then $W_{\ell-s}(\xi) = 0$ for all $s < j$.*

Proof.

(a) Using (21) we have

$$
{}^{E}U_{\ell+1}^j(\xi) = U_{\ell+1}(\xi) + (-1)^{j+1} W_{\ell-j}(\xi) U_j(\xi)(e_j^* - e_j) = 0.
$$

$U_{\ell+1}(\xi)$ can be interpreted as a linear polynomial in the variable $\gamma_E^{(j)}$. Since this has two zeros, it must be identically zero. This also means that ξ is a root of $U_{\ell+1}$ independently of $\gamma_E^{(j)}$.

(b) We start from the decomposition

$$
U_{k+1} = U_j W_{\ell-(j-1)} - U_{j-1} W_{\ell-j} \mu_j,
$$

and we evaluate at ξ obtaining

$$
0 = U_j(\xi) W_{\ell-(j-1)}(\xi) - 0.
$$

Since $U_j(\xi)$ is different from zero, it follows that $W_{\ell-(j-1)}(\xi) = 0$. Then we simply conclude by induction.

(c) Since $r_j = 0$ implies $\mu_j = 0$, we follow the same reasoning as above to conclude that $W_{\ell-(j-1)}(\xi) = 0$. Then we apply part (b) to finish the proof.

□

The result in Theorem 3, together with the discussion in Section 4.2, will drive the selection of which of the extremal values of $\gamma_E^{(k)}$ is to be chosen to determine the extremal roots of the polynomials. This will be further summarized in Section 4.4.

Theorem 3. *Let ξ be the largest (resp. smallest) positive or negative root of $U_{\ell+1}$ and let e_j, r_j be the parameters that realize this root. Suppose further that ξ is not in $\mathcal{I}_l$, that is $U_j(\xi) \neq 0$ for $j \leq \ell$. Then ξ assumes the largest (resp. smallest) value when the $\gamma_E^{(j)}$ take alternatively their maximum or minimum value.*

Proof. We prove equivalently that $(e_j^* - e_j)(e_{j-1}^* - e_{j-1}) < 0$. First we assume $r_j \neq 0$ and $W_{\ell-j}(\xi) \neq 0$ for all $j \leq \ell$. We know from the hypothesis that $U_{j+1}(\xi), U_j(\xi) \neq 0$. We begin with the decomposition

$$U_{\ell+1} = U_j W_{\ell+1-j} - U_{j-1} W_{\ell-j} \mu_j.$$

Now since $r_j \neq 0$ implies $\mu_j > 0$ all the quantities at the right hand side are nonzero. Evaluating at ξ we get

$$\operatorname{sgn}(U_j W_{\ell-j}) = \operatorname{sgn}(U_{j-1} W_{\ell-j+1}).$$

Now we observe that ${}^E U_{k+1}^j(\xi)$ must have a fixed sign as j varies to guarantee that the root ξ is either maximal or minimal (this sign depends on the type of of the root—smallest/largest, positive/negative—and on k but cannot depend on j). We call this sign s. Next we have, using (21) and recalling that $U_{k+1}(\xi) = 0$,

$$\begin{aligned}
&\operatorname{sgn}(e_j^* - e_j)(e_{j-1}^* - e_{j-1}) \\
&= \operatorname{sgn}({}^E U_{\ell+1}^j(\xi)({}^E U_{\ell+1}^{j-1}(\xi))(-1)^{j+1}(-1)^j \operatorname{sgn}(W_{\ell-j+1} U_j)\operatorname{sgn}(W_j U_{j-1}) \\
&= s^2(-1)^{2j+1} = -1.
\end{aligned}$$

Thus in this case all the e_j are alternating. Notice in particular that to show that e_{j-1} and e_j are alternating it is enough to assume that $r_j, W_{k-j}(\xi), W_{k-(j-1)}(\xi) \neq 0$.

We now drop our assumptions that r_j and $W_{\ell-j}(\xi)$ are nonzero. Let j_* be the largest j such that $W_{\ell-j_*}(\xi) = 0$, which implies that $W_{\ell-j}(\xi) \neq 0$ for all $j > j_*$. From Lemma 6 (c) we also obtain $r_j \neq 0$ for $j \geq j_* + 2$. But then the first part of the proof implies that the parameters $e_\ell, e_{\ell-1}, \ldots, e_{j_*+1}$ are alternating. Since $W_{\ell-j}(\xi) = 0$, from Lemma 6 (b) we get that $W_{\ell-j}(\xi) = 0$ for all $j \leq j_*$ and so Lemma 6 (a) implies that ξ does not depend on e_j for $j \leq j_*$. Thus we can choose the parameter $\gamma_E^{(j)}$ such that $\gamma_E^{(j)}, \gamma_E^{(j+1)}$ are alternating, then choose $\gamma_E^{(j-1)}$ so that $\gamma_E^{(j-1)}, \gamma_E^{(j)}$ are alternating and so on. This shows that we can always find a choice of parameters where the $\gamma_E^{(j)}$ are alternating that gives the desired extremal root. □

4.2. Choice of $\gamma_E^{(k)}$

In the previous section we have shown that the $\gamma_E^{(k)}$ assume alternatively the maximum or minimum value. It is enough to fix the value of $\gamma_E^{(k)}$ to determine all the $\gamma_E^{(j)}$. We know, from (19) that $\operatorname{sign}(e_k^* - e_k)$ is equal to $(-1)^{k+1}\operatorname{sign}(U_k(\xi)^E U_{k+1}^k(\xi))$. We will consider four cases:

- Let $\xi = \lambda_+^{UB} = \xi_{s_{k+1}}^{(k+1)}$. We know that ξ is larger than all the roots of U_k and ${}^E U_{k+1}^k$ and so $U_k(\xi), {}^E U_{k+1}^k(\xi) > 0$. Thus

$$\operatorname{sign}(e_k^* - e_k) = (-1)^{k+1}$$

and so

$$\operatorname{sign}(e_j^* - e_j) = (e_k^* - e_k)(-1)^{k-j} = (-1)^{k+1}(-1)^{k-j} = (-1)^{j+1}. \tag{23}$$

- Let $\xi = \lambda_-^{LB} = \xi_1^{(k+1)}$. The root ξ is negative and smaller than all the roots of U_k and ${}^E U_{k+1}^k$ and so

$$\operatorname{sign}(U_k(\xi)) = (-1)^k \operatorname{sign}({}^E U_{k+1}^k(\xi)) = (-1)^{k+1}.$$

Thus

$$\operatorname{sign}(e_k^* - e_k) = (-1)^{k+1}(-1)^k(-1)^{k+1} = (-1)^k$$

and we have

$$\operatorname{sign}(e_j^* - e_j) = \operatorname{sign}(e_k^* - e_k)(-1)^{k-j} = (-1)^k(-1)^{k-j} = (-1)^j. \tag{24}$$

- Now let $\xi = \lambda_+^{LB}$ and let m be the (smallest) index such that $U_{m+1}(\xi) = 0$. Then the sign of $U_m(\xi)$ and ${}^E U_{m+1}^m$ must be the sign that this polynomial assumes on 0. Thus

$$\begin{aligned}\operatorname{sign}(e_m^* - e_m) &= (-1)^{m+1}\operatorname{sign}(U_m(0))\operatorname{sign}({}^E U_{m+1}^m(0)) \\ &= (-1)^{m+1}(-1)^{\lceil \frac{m+1}{2} \rceil}(-1)^{\lceil \frac{m}{2} \rceil} \\ &= (-1)^{m+1}(-1)^{m+1} = +\end{aligned}$$

and

$$\operatorname{sign}(e_j^* - e_j) = \operatorname{sign}(e_m^* - e_m)(-1)^{m-j} = (-1)^{m-j}. \tag{25}$$

- Finally let $\xi = \lambda_-^{UB}$ and let m be the (smallest) index such that $U_{m+1}(\xi) = 0$. Reasoning as above shows that

$$\operatorname{sign}(e_m^* - e_m) = + \qquad \text{and} \qquad \operatorname{sign}(e_j^* - e_j) = (-1)^{m-j}. \tag{26}$$

4.3. Sign of $r_j^* - r_j$ in Terms of the Sign of $z_j(\xi)$

Recall that we have

$$\begin{aligned}{}^E U_{k+1}^j(\xi) &= (-1)^{j+1} W_{j+1}(\xi) U_j(\xi)(e_j^* - e_j), \\ {}^R U_{k+1}^j(\xi) &= W_{j+1}(\xi) z_j(\xi)(r_j^* - r_j).\end{aligned}$$

If $W_{j+1}(\xi) = 0$, then the root is independent of $\gamma_R^{(j)}$, so we can assume $W_{j+1}(\xi) \neq 0$. We also assume, without loss of generality, that $U_j(\xi) \neq 0$. Then we have

$$\begin{aligned}\operatorname{sign}(r_j^* - r_j) &= \operatorname{sign}(z_j(\xi)\,{}^R U_{k+1}^j(\xi))\operatorname{sign}(W_{j+1}) \\ &= \operatorname{sign}(z_j(\xi))\operatorname{sign}({}^R U_{k+1}^j(\xi))(-1)^{j+1}\operatorname{sign}({}^E U_{k+1}^j(\xi) U_j(\xi))\operatorname{sign}(e_j^* - e_j) \\ &= (-1)^{j+1}\operatorname{sign}(U_j(\xi))\operatorname{sign}(e_j^* - e_j)\operatorname{sign}(z_j(\xi)),\end{aligned}$$

where the last step follows from the fact that ${}^E U_{k+1}^j(\xi)$ and ${}^R U_{k+1}^j(\xi)$ must have the same sign. Again we distinguish four cases depending on the type of root we are considering.

- Let $\xi = \lambda_+^{UB} = \xi_{s_{k+1}}^{(k+1)}$. Then $U_j(\xi) > 0$ and $\text{sign}(e_j^* - e_j) = (-1)^{j+1}$ so

$$\text{sign}(r_j^* - r_j) = (-1)^{j+1}(-1)^{j+1}\text{sign}(z_j(\xi)) = \text{sign}(z_j(\xi)). \tag{27}$$

- Let $\xi = \lambda_-^{UB} = \xi_1^{(k+1)}$. Then

$$\text{sign}(U_j(\xi)) = (-1)^j \quad \text{and} \quad \text{sign}(e_j^* - e_j) = (-1)^j$$

so

$$\text{sign}(r_j^* - r_j) = (-1)^{j+1}(-1)^j(-1)^j\text{sign}(z_j(\xi)) = (-1)^{j+1}\text{sign}(z_j(\xi)). \tag{28}$$

- Now let $\xi = \lambda_+^{LB}$ and $m \leq k$ be the (smallest) index such that $U_{m+1}(\xi) = 0$. Then

$$\text{sign}(U_j(\xi)) = (-1)^{\left\lceil \frac{j}{2} \right\rceil} \quad \text{and} \quad \text{sign}(e_j^* - e_j) = (-1)^{m-j}$$

so

$$\begin{aligned}\text{sign}(r_j^* - r_j) &= (-1)^{j+1}(-1)^{\left\lceil \frac{j}{2} \right\rceil}(-1)^{m-j}\text{sign}(z_j(\xi)) \\ &= (-1)^{m+1}(-1)^{\left\lceil \frac{j}{2} \right\rceil}\text{sign}(z_j(\xi)).\end{aligned}$$

- Finally let $\xi = \lambda_-^{UB}$ and $m \leq k$ be the (smallest) index such that $U_{m+1}(\xi) = 0$. Reasoning as above shows that

$$\text{sign}(r_j^* - r_j) = (-1)^{m+1}(-1)^{\left\lceil \frac{j}{2} \right\rceil}\text{sign}(z_j(\xi)).$$

The sign of $z_j(\xi)$ cannot be computed in general, since this depends on the signs of polynomials U_j and U_{j-1} at a particular root of another polynomial in the sequence. As a consequence, we must try all the extremal values of these indicators to find the roots of the corresponding polynomials, as further described in the following Section 4.4.

4.4. Bounds for the Eigenvalues of the Preconditioned Matrix

A procedure to compute the intervals bounding the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ with $N+1$ blocks can be described as follows:

- **Upper bound for the positive eigenvalues**. In view of Theorem 2, this is the largest root of the polynomial $U_{N+1}(\lambda)$. The sign of (23) is $(-1)^{j+1}$. When the sign of $e_j^* - e_j$ is positive, this means that the *wrong* indicator is larger than the *correct* one, and therefore we must set $\gamma_E^{(j)} = \alpha_E^{(j)}$. Hence, as $\text{sgn}(e_0^* - e_0) = -1$, we should start with $\gamma_E^{(0)} = \beta_E^{(0)}$, and then alternating between the two extremal values: $\beta_E^{(0)}, \alpha_E^{(1)}, \beta_E^{(2)}, \ldots$.
- **Lower bound for the negative eigenvalues**. In view of Theorem 2, this is given by the smallest zero of the polynomial $U_{N+1}(\lambda)$. The sign of (24) is $(-1)^j$. This means that we should start with $\gamma_E^{(0)} = \alpha_E^{(0)}$ (since $e_0^* - e_0 > 0$) and then alternating between the two extremal values: $\alpha_E^{(0)}, \beta_E^{(1)}, \alpha_E^{(2)}, \ldots$.
- **Upper bounds for the negative eigenvalues**. We consider evaluating the largest negative root $\xi_-^{(m)}$ of all polynomials $U_m, 2 \leq m \leq N+1$. The sign in (25) is $(-1)^{m+j}$ suggesting that we have to start with $\alpha_E^{(0)}$ if m is even and with $\beta_E^{(0)}$, if m is odd, and proceed as before. Finally we compute

$$\lambda_-^{UB} = \max_{m \leq N+1} \{\xi_-^{(m)}\}.$$

- **Lower bounds for the positive eigenvalues**. We consider evaluating the smallest positive root $\xi_+^{(m)}$ of all polynomials $U_m, 1 \le m \le N+1$. The sign in (26) is again $(-1)^{m+j}$ suggesting that we have to start with $\alpha_E^{(0)}$ if m is even and with $\beta_E^{(0)}$, if m is odd, and proceed as before. Finally we compute

$$\lambda_+^{LB} = \min_{m \le N+1} \{\xi_+^{(m)}\}.$$

In each of the four cases, the choice of $\gamma_R^{(j)} \in [\alpha_R^{(j)}, \beta_R^{(j)}]$ must be performed by taking all the combinations of these values. In summary, to evaluate the bounds for the extremal eigenvalues of the preconditioned matrix with $N+1$ blocks, we have to compute the roots of a number of polynomials. e.g., by means of the `MATLAB`'s `roots` function. To add more detail, for $m = 1, \dots N$ we compute the roots of U_m with 2^m different combinations of parameters and the roots of U_{N+1} for 4×2^N combinations, totalling $\sum_{m=1}^{N} 2^m + 2^{N+2} \le 6 \times 2^N$ calls to the function `roots`. This preprocessing stage takes less than 0.1 s for $N \le 8$.

Some insights on how to select the extremal values of the γ_R indicators, upon additional assumptions, are given in Lemma 7, whose proof is deferred to Appendix A, and in Theorem 4.

Lemma 7. *Let $\alpha_R^{(k)} > 0$, $k \ge 1$, and $\rho \in (0,2)$. Assume that, for every $j = 0, \dots, k$, $\alpha_E^{(j)} \le 2 - \rho$ and $\beta_E^{(j)} \ge 2 + \rho$ and that $|\xi| > \frac{1}{\rho}$, with $\xi \in \left\{\xi_1^{(k+1)}, \xi_{s_{k+1}}^{(k+1)}\right\}$. Then $z_j(\xi_{s_{k+1}}^{(k+1)}) < 0$, and $\operatorname{sgn}\left(z_j(\xi_1^{(k+1)})\right) = (-1)^j$.*

Under the assumptions of Lemma 7 we are now able to determine the monotonicity of the extremal roots of U_{k+1} depending on $\gamma_R^{(j)}$.

Theorem 4. *Under the assumptions of Lemma 7, the largest (respectively, the smallest) root of U_{k+1} assumes its largest (respectively, smallest) value, in combination with $\gamma_R^{(j)} = \beta_R^{(j)}$, $j = 1, \dots, k$.*

Proof. Let ξ be the largest positive root. Then the previous lemma shows that $\operatorname{sign}(z_j(\xi)) = (-1)$. But then (27) implies

$$\operatorname{sign}(r_j^* - r_j) = -1$$

and so $r_j = \beta_R^{(j)}$. Now let ξ be the smallest negative root. The previous lemma shows that $\operatorname{sign}(z_j(\xi)) = (-1)^j$ but then (28) implies

$$\operatorname{sign}(r_j^* - r_j) = (-1)^{j+1}(-1)^j = -1$$

and so again $r_j = \beta_R^{(j)}$. □

5. Numerical Results: Randomly Generated Matrices

We now undertake numerical tests to validate the theoretical bounds of Theorem 1, and subsequent characterizations. We determine the extremal eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ on randomly generated linear systems. Specifically, we considered a simplified case with $E_i \equiv 0, i > 0$ and ran a number of different test cases , combining the values of the extremal eigenvalues (Table 1) of the symmetric positive definite matrices involved.

In particular

- Case $N = 2$. We have three parameters with six different endpoints of the corresponding intervals. Overall we run $3^6 = 729$ test cases.
- Case $N = 3$. We have four parameters with four different endpoints of the corresponding intervals. Overall we run $4^4 = 256$ test cases.

- Case $N = 4$. We have five parameters with four different endpoints of the corresponding intervals. Overall we run $5^4 = 1024$ test cases.

Table 1. Extremal eigenvalues of the relevant symmetric positive definite matrices used in the verification of the bounds.

$\alpha_E^{(0)}$, $\alpha_R^{(1)}$, $\alpha_R^{(2)}$	0.1	0.3	0.9	$\alpha_E^{(0)}$, $\alpha_R^{(1)}$, $\ldots$ $\alpha_R^{(N)}$	0.1	0.8
$\beta_E^{(0)}$, $\beta_R^{(1)}$, $\beta_R^{(2)}$	1.2	1.8	3	$\beta_E^{(0)}$, $\beta_R^{(1)}$, $\ldots$ $\beta_R^{(N)}$	1.2	2
Case $N = 2$				Cases $N > 2$		

Each test case has been run 10 times, generating random matrices that satisfy the relevant spectral properties, and we record the most extreme eigenvalues for each test case.

In more detail, the dimensions $n_0, n_1, \ldots$ are computed using the closest integer to $50 + 10 * \texttt{rand}$, using MATLAB's `rand` function, recomputing as necessary to ensure that $n_k \geq n_{k+1}, \forall k \leq N - 1$, to guarantee that all matrices $R_k R_k^\top$ are invertible. Matrices A_k and B_k are computed using MATLAB's `randn` function, whereupon we take the symmetric part of A_k and then add an identity matrix times the absolute value of the smallest eigenvalue to ensure symmetric positive semi-definiteness (definiteness if $k = 0$). We then choose $\widehat{S}_0$ as a linear combination of A and the identity matrix, such that the eigenvalues of E_0 are contained in $[\alpha_E^{(0)}, \beta_E^{(0)}]$, and similarly to construct $\widehat{S}_k$ for any k.

From Figures 1–3 we notice that three out of four bounds perfectly capture the eigenvalues of the preconditioned matrix, while the upper bounds for negative eigenvalues are not as tight for $N = 2, 4$ and lower bounds for positive eigenvalues are less effective for $N = 3$. All in all, our technique is able to predict the behavior of the MINRES iterative solver applied to the preconditioned saddle-point linear system.

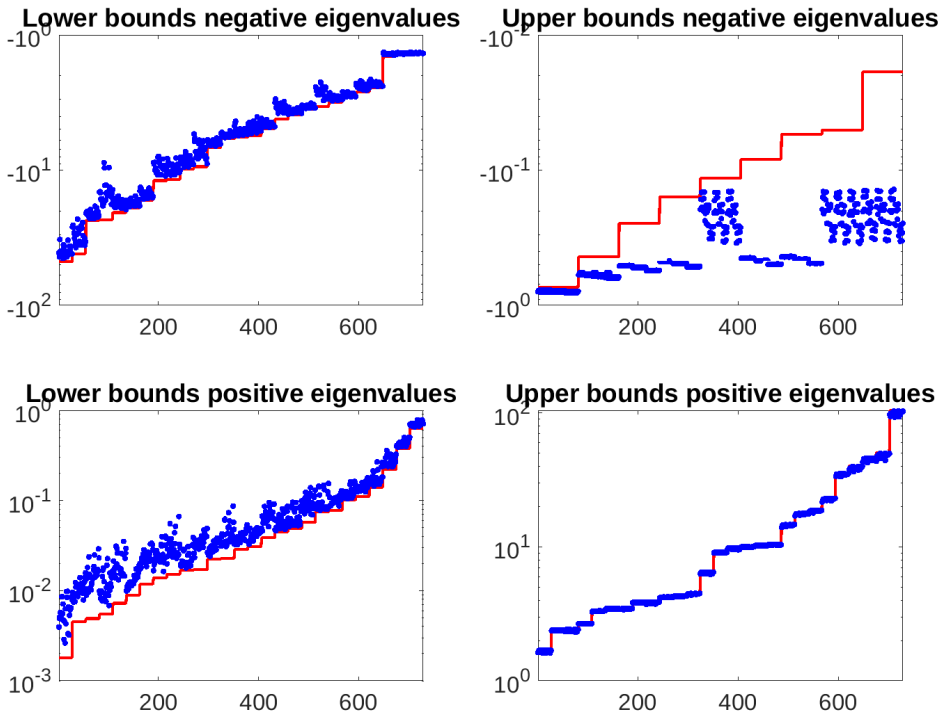

Figure 1. Double saddle-point linear system. Extremal eigenvalues of the preconditioned matrix (blue dots) and bounds (red line) after 10 runs with each combination of the parameters from Table 1.

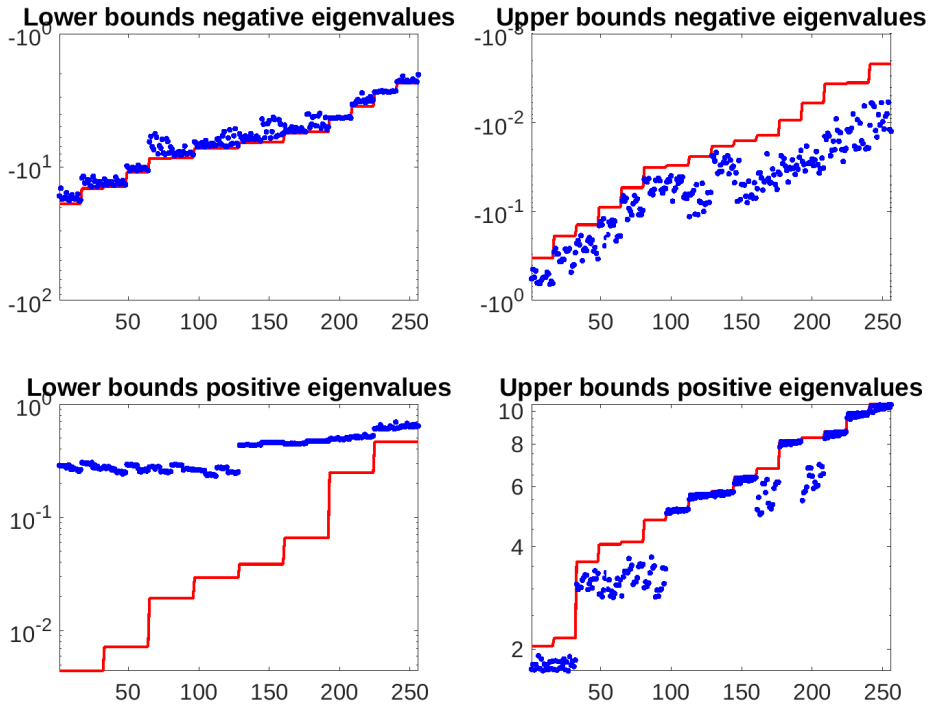

Figure 2. Multiple saddle-point linear system with $N = 3$. Extremal eigenvalues of the preconditioned matrix (blue dots) and bounds (red line) after 10 runs with each combination of the parameters from Table 1.

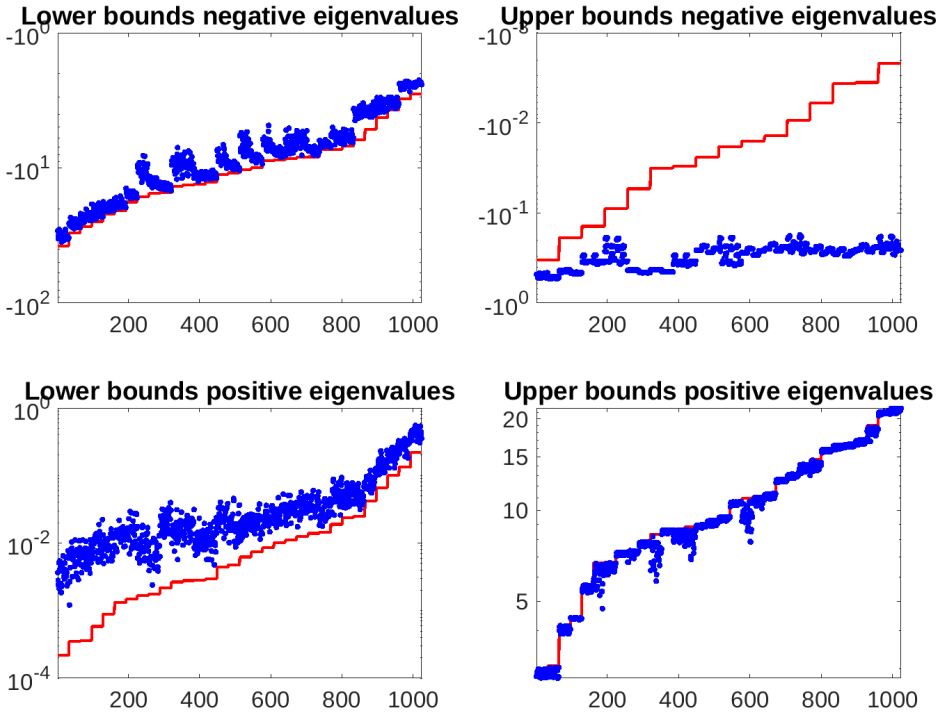

Figure 3. Multiple saddle-point linear system with $N = 4$. Extremal eigenvalues of the preconditioned matrix (blue dots) and bounds (red line) after 10 runs with each combination of the parameters from Table 1.

Comparisons Against the Block-Diagonal Preconditioner

The preconditioner we have studied so far, being SPD, allows the use of the MINRES iterative solver for the multiple saddle-point linear systems. Another, most known, preconditioner sharing the same properties is the block-diagonal preconditioner $\mathcal{P}_D$, based on

the same inexact Schur complements as in the PP preconditioner, defined in (6), which we recall here below

$$\mathcal{P}_D = \begin{bmatrix} \widehat{S}_0 & & & & \\ & \widehat{S}_1 & & & \\ & & \widehat{S}_2 & & \\ & & & \ddots & \\ & & & & \widehat{S}_N \end{bmatrix}.$$

We compare the MINRES number of iterations in all the previously described test cases for $N = 3$. In Figure 4 we plot the number of iterations with the PP preconditioner and the block-diagonal preconditioner, together with the square root of κ, an indicator of the ill-conditioning of the overall saddle-point system, namely

$$\kappa = \frac{\beta_E^{(0)}}{\alpha_E^{(0)}} \frac{\beta_R^{(1)} \beta_R^{(2)} \beta_R^{(3)}}{\alpha_R^{(1)} \alpha_R^{(2)} \alpha_R^{(3)}}.$$

From the figure, we notice that the two preconditioners are both influenced by the κ indicator, with the PP preconditioner more effective than $\mathcal{P}_D$ for small to modest values of κ. When the Schur complements are instead not optimally approximated, the block-diagonal preconditioner is to be preferred.

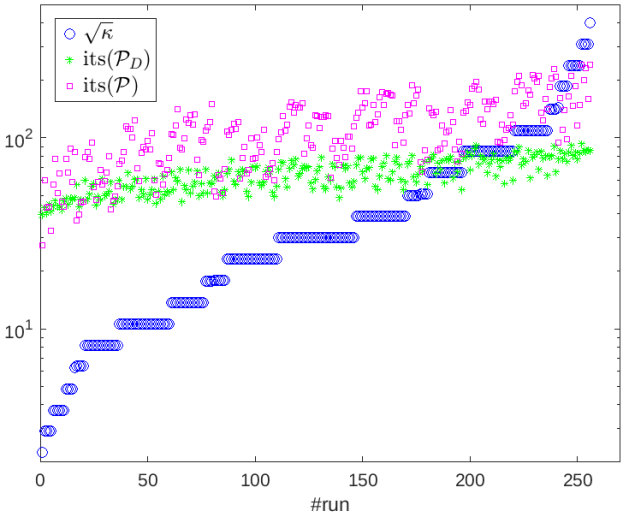

Figure 4. Iteration numbers of $\mathcal{P}$ and $\mathcal{P}_D$ for all the tests with $N = 3$. The condition number indicator $\sqrt{\kappa}$ is also displayed.

6. A Realistic Example: The 3D Biot's Consolidation Model

A numerical experiment, concerning the mixed form of Biot's poroelasticity equations, is conducted to investigate the quality of the bounds for the eigenvalues of the preconditioned matrix and the effectiveness of the proposed triangular preconditioners on a realistic problem. For the PDEs and boundary conditions governing this model, we refer to the description in [25]. As a reference test case for the experiments, we considered the porous cantilever beam problem originally introduced in [40] and already used in [14]. The domain is the unit cube with no-flow boundary conditions along all sides, zero displacements along the left edge, and a uniform load applied on top. The material properties are described in ([25], Table 5.1). We considered the mixed-hybrid finite element discretization of the coupled Biot equations, which generates a double saddle-point linear system. The test case refers to a unitary cubic domain, uniformly discretized using a $h = 0.05$ mesh size in all the

spatial dimensions. The size of the block matrices and the overall number of nonzeros in the double saddle-point system are reported in Table 2.

Table 2. Results for 3D cantilever beam problem on a unit cube: size and number of nonzeros of the test matrices.

$1/h$	n_0	n_1	n_2	$n_0 + n_1 + n_2$	Nonzeros
20	27,783	8000	25,200	60,983	2.4×10^6

6.1. Verification of the Bounds

To approximate the $(1,1)$ block, we employed the classical incomplete Cholesky factorization (IC) with fill-in based on a drop tolerance δ. The approximations $\widehat{S}_1$ and $\widehat{S}_2$ of the Schur complements are as in [25], which prove to be completely scalable with the problem size. Even if the IC preconditioner does not scale with the discretization parameter, it is helpful to conduct a spectral analysis as a function of the drop tolerance.

In Table 3 we report the extremal values of the five indicators for this problem. We notice that $\alpha_R^{(2)} = 0$, which is consistent with the sizes of the blocks. In fact, $n_1 > n_2$, see Table 2, implies that the matrix $R_2 R_2^\top$, of dimension $n_1 = 25{,}200$, has rank $n_1 = 8000$.

Table 3. Extremal values of the indicators for different values of the threshold parameter δ. The endpoints of the intervals $\mathcal{I}_{E_j}$ and $\mathcal{I}_{R_j}, j = 1, 2$ do not change with δ.

	$\delta = 10^{-4}$	$\delta = 10^{-5}$	$\delta = 10^{-6}$	$\delta = 10^{-7}$
$\mathcal{I}_{E_0}$	$[0.0422, \quad 1.2036]$	$[0.2563, \quad 1.2232]$	$[0.9600, \quad 1.0118]$	$[0.9976, \quad 1.0001]$
$\mathcal{I}_{E_1}$	$[5 \times 10^{-4}, \quad 0.0889]$			
$\mathcal{I}_{E_2}$	$[0.9976, \quad 1.0001]$			
$\mathcal{I}_{R_1}$	$[2 \times 10^{-4}, \quad 0.7790]$			
$\mathcal{I}_{R_2}$	$[0, \quad 0.0023]$			

We computed the relevant eigenvalues of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$, using the `MATLAB` function `eigs` with a function handle for the application of the preconditioned matrix (which we could not compute explicitly, due to its size and density) as well as the four endpoints of $\mathcal{I}$, as predicted by the theory, and reported all of them in Table 4.

Table 4. Results for 3D cantilever beam problem on a unit cube: eigenvalue bounds vs real eigenvalues, for each value of the threshold parameter δ.

		λ_-^{LB}	λ_-^{UB}	λ_+^{LB}	λ_+^{UB}
$\delta = 10^{-4}$	Bounds	-2.6786	-0.0012	0.0424	1.2216
	Exact Eigvs	-1.2408	-0.3192	0.0453	1.2055
$\delta = 10^{-5}$	Bounds	-2.4104	-0.0012	0.2564	1.2447
	Exact Eigvs	-1.2408	-0.3192	0.2895	1.2252
$\delta = 10^{-6}$	Bounds	-1.7001	-0.0012	0.9600	1.0119
	Exact Eigvs	-1.2408	-0.3192	0.9600	1.0118
$\delta = 10^{-7}$	Bounds	-1.6701	-0.0012	0.9976	1.0070
	Exact Eigvs	-1.2408	-0.3190	0.9979	1.0013

6.2. Improving the Upper Bounds for the Negative Eigenvalues

From Table 4 we see that the bounds perfectly capture the extremal eigenvalues, except for the upper bound for the negative eigenvalues, which is still loose. However, this can be

improved by taking into account the indicator $\gamma_S^{(k)}(\boldsymbol{w})$, defined in (13). The endpoints of this indicator are, for this test case,

$$\mathcal{I}_{S_1} = [0.3187, 1.2408], \quad \mathcal{I}_{S_2} = [0.9998, 1.0002],$$

for every value of δ. In order to use these indicators in the eigenvalue analysis, we perform a change in variable, using (12), in the polynomial recurrence that reads as, for $k \geq 1$,

$$U_{k+1}(\lambda) = (\lambda(1 + \gamma_S^{(k)} - \gamma_E^{(k)}) + (-1)^{k+1}\gamma_E^{(k)})U_k(\lambda) + (\gamma_E^{(k)} - \gamma_S^{(k)})(\lambda + (-1)^k)^2 U_{k-1}(\lambda), \tag{29}$$

for which all the results obtained in Section 3 still hold, as well as the observation at the beginning of Section 4.1, stating that the bounds for the roots of U_{k+1} are taken at the endpoints of the parameters. On the contrary, in this case, we cannot select which of the extremal values of $\gamma_E^{(k)}$ and $\gamma_S^{(k)}$ provides the bound. However, in view of the low degree of the polynomial we can compute its roots for every combination of the endpoints of the indicators, and take the minimum/maximum values of the roots, depending on the bounds sought. To do this we also should consider the following:

Remark 2. *The definition of $\gamma_S^{(k)}(\boldsymbol{w})$ constrains this indicator to be larger than the corresponding $\gamma_E^{(k)}(\boldsymbol{w})$. Hence in correspondence with $\gamma_E^{(k)} = \beta_E^{(k)}$, the lower bound for $\gamma_S^{(k)}$ is $\min\{\beta_E^{(k)}, \alpha_S^{(k)}\}$.*

Taking this into account, we compute the roots of U_1, U_2 and U_3 for each combination of the extremal values of the parameters $\gamma_E^{(j)}$, $j = 0, 1, 2$ and $\gamma_S^{(j)}$, $j = 1, 2$ and take the extremal values of the obtained roots. which we report in Table 5, together with the exact eigenvalues.

Table 5. Refined eigenvalue bounds, obtained with the new indicators γ_S, vs real eigenvalues, for each value of the threshold parameter δ.

		$\lambda_-^{\mathbf{LB}}$	$\lambda_-^{\mathbf{UB}}$	$\lambda_+^{\mathbf{LB}}$	$\lambda_+^{\mathbf{UB}}$
$\delta = 10^{-4}$	Bounds	-2.8268	-0.2630	0.0422	1.2274
	Exact Eigvs	-1.2408	-0.3192	0.0453	1.2055
$\delta = 10^{-5}$	Bounds	-2.4204	-0.2585	0.1809	1.2517
	Exact Eigvs	-1.2408	-0.3192	0.2895	1.2252
$\delta = 10^{-6}$	Bounds	-1.2913	-0.2952	0.9593	1.0119
	Exact Eigvs	-1.2408	-0.3192	0.9600	1.0118
$\delta = 10^{-7}$	Bounds	-1.2434	-0.3175	0.9979	1.0029
	Exact Eigvs	-1.2408	-0.3190	0.9979	1.0013

The adherence of the bounds to the extremal eigenvalues is now impressive.

The bounds in Table 5 allow us to predict with some accuracy the number of iterations we expect from the MINRES solver, by taking into account (4). Considering for instance the case $\delta = 10^{-5}$, inserting the computed bounds into (4), we obtain that, after k iterations, the expected reduction of the residual norm is $(0.883)^k$, thus predicting an average residual norm reduction of 0.883. Using instead the *true* eigenvalue bound, we obtain, as the right-hand-side of (4), $(0.778)^k$, which accounts for a 0.778 reduction factor of the residual norm.

6.3. Comparisons with the Block-Diagonal Preconditioner for the Biot Problem

We finally compare the block-diagonal preconditioner, $\mathcal{P}_D$, with the PP preconditioner in terms of number of iterations and CPU time, for different values of the parameter δ, and report the obtained results in Table 6, showing that the preconditioner analyzed in this work represents a valid alternative to the block-diagonal preconditioner, especially when the Schur complements are all well approximated. Figure 5 compares the convergence

profile of MINRES with both preconditioners for $\delta = 10^{-5}, 10^{-6}$. All tests are carried out in MATLAB on an Intel(R) Core(TM) Ultra 7 165U, CPU @ 3.80 GHz processor with 12 cores.

Table 6. Number of MINRES iterations and CPU times for the block-diagonal and the PP preconditioner, for different values of the threshold parameter δ. With $\delta = 0$ we indicate that the exact Cholesky factor of A_0 is used for preconditioning.

δ	10^{-4}		10^{-5}		10^{-6}		10^{-7}		0	
	Its	**CPU**	**Its**	**CPU**	**Its**	**CPU**	**Its**	**CPU**	**Its**	**CPU**
$\mathcal{P}_D$ preconditioner	132	1.46	83	2.08	64	2.97	63	3.21	63	3.31
PP preconditioner	99	1.92	61	2.86	31	2.88	23	2.28	10	1.10

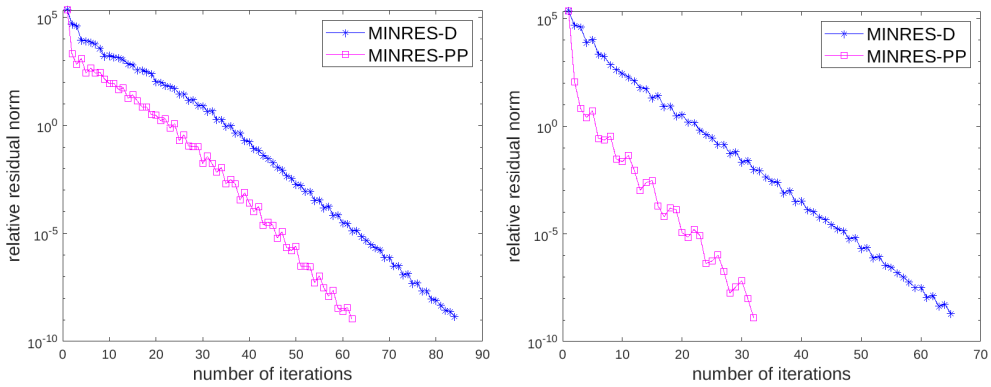

Figure 5. The 3D cantilever beam problem. Convergence profiles of MINRES preconditioner, i.e., logarithm of the norm of the residual vs iteration number, with $\mathcal{P}_D$ and $\mathcal{P}$. Drop tolerance of the Cholesky parameter: $\delta = 10^{-5}$ (**left**), $\delta = 10^{-6}$ (**right**).

A final remark is in order: in this study we have not been concerned with the cost of the application of the two preconditioners, at each MINRES iteration, which is crucial to evaluate the performance of both. This comparison is outside the scope of this paper, for it is mainly concerned with eigenvalue distribution and the number of iterations. However, even taking into account that the application of the PP preconditioner is from 1.5 to (slightly less than) 2 times more expensive than that of the block-diagonal preconditioner, our results show that the PP preconditioner can be a viable alternative to accelerate the MINRES solution of multiple saddle-point linear systems, see e.g., the CPU times in Table 6, which are progressively favorable to the PP preconditioner when δ is decreasing.

To stress this behavior, on purpose we let δ go to zero. Of course, this is not necessarily the most efficient strategy for the Biot's consolidation problem, as, the smaller δ is, the more costly the computation of the approximation of the $(1,1)$ block. Other applications, such as PDE constrained optimization, requires less effort to approximate the A_0 matrix, making the PP preconditioner more appealing than the block-diagonal preconditioner.

7. Conclusions

We have developed an eigenvalue analysis of preconditioned multiple saddle-point linear systems, with the SPD preconditioner (PP in short) proposed in [3]. This analysis is based on the (approximate) knowledge of the smallest and the largest eigenvalues of a number of SPD matrices, related to the preconditioned blocks of the original system. The obtained bounds turn out to be very close to the exact extremal eigenvalues of the preconditioned matrix, and therefore, this study constitutes a valuable tool to predict with some accuracy the number of MINRES iterations needed to solve the multiple saddle-point linear system. In a number of both synthetic and realistic test cases, the PP precondi-

tioner is shown to display a comparable performance as the block-diagonal preconditioner, especially when the Schur complement matrices are well approximated.

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Data Availability Statement: The MATLAB script for obtaining the bounds, given the γ parameters, is available at www.dmsa.unipd.it/~berga/Code (accessed on 10 March 2026).

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Proof of Lemma 7

Proof. We start with the definition of $z_j(\lambda)$ and apply the following recursion:

$$\begin{aligned} z_j(\lambda) &= \lambda U_j(\lambda) - (\lambda + (-1)^j)^2 U_{j-1}(\lambda) \\ &= \lambda\Big(U_{j-1}(\lambda)(\lambda(1+\gamma_R^{(j-1)}) + (-1)^j\gamma_E^{(j-1)}) - \gamma_R^{(j-1)}(\lambda + (-1)^{j-1})^2 U_{j-2}(\lambda)\Big) \\ &\quad - (\lambda + (-1)^j)^2 U_{j-1}(\lambda). \end{aligned}$$

We now apply the definition to $z_{j-1}(\lambda)$:

$$z_{j-1}(\lambda) = \lambda U_{j-1}(\lambda) - (\lambda + (-1)^{j-1})^2 U_{j-2}(\lambda).$$

Combining these two expressions we get

$$\begin{aligned} z_j(\lambda) - \gamma_R^{(j-1)}\lambda z_{j-1}(\lambda) &= U_{j-1}(\lambda)\Big(\lambda^2(1+\gamma_R^{(j-1)}) + (-1)^j\lambda\gamma_E^{(j-1)} \\ &\quad - (\lambda + (-1)^j)^2 - \lambda^2\gamma_R^{(j-1)}\Big) \\ &= U_{j-1}(\lambda)\Big((-1)^j\lambda\gamma_E^{(j-1)} + 2(-1)^{j+1}\lambda - 1\Big) \\ &= U_{j-1}(\lambda)(-1)^{j+1}\Big(\lambda(2-\gamma_E^{(j-1)}) + (-1)^j\Big) \\ &\equiv U_{j-1}(\lambda)(-1)^{j+1}t_j(\lambda). \end{aligned}$$

We know that $z_1(\lambda) = t_1(\lambda)$ and we have just proved that

$$z_j(\lambda) = \lambda\gamma_R^{(j-1)} z_{j-1}(\lambda) + (-1)^{j+1}U_{j-1}(\lambda)t_j(\lambda).$$

Sign of $z_j(\xi)$ for the largest positive root. Now let $\xi \equiv \xi_{s_{k+1}}^{(k+1)}$ be the largest positive root. In such a case the previous analysis shows that $\gamma_E^{(j-1)} = \begin{cases} \alpha_E^{(j-1)} & \text{if } j \text{ is even} \\ \beta_E^{(j-1)} & \text{if } j \text{ is odd} \end{cases}$.

Now using the hypotheses, we have that

$$\operatorname{sgn}(t_j(\xi)) = \operatorname{sgn}(2 - \gamma_E^{(j-1)}) = (-1)^j.$$

Now we prove by induction that $\operatorname{sgn}(z_j(\xi)) = -1$ for all j. The base step is as follows:

$$\operatorname{sgn}(z_1(\xi)) = \operatorname{sgn}(t_1(\xi)) = (-1)^1 = -1.$$

We know that

$$z_j(\xi) = \lambda \gamma_R^{(j-1)} z_{j-1}(\xi) + (-1)^{j+1} U_{j-1}(\xi) t_j(\xi).$$

The first term is negative according to induction hypothesis. As for the second term

$$\operatorname{sgn}((-1)^{j+1} U_{j-1}(\xi) t_j(\xi) = (-1)^{j+1}(+1)(-1)^j = -1.$$

Thus z_j is negative.

Sign of $z_j(\xi)$ for the smallest negative root. Let $\xi \equiv \xi_1^{(k+1)}$ be the smallest negative root. According to assumptions, $-\xi > \dfrac{1}{\rho}$. Moreover, we know that $\gamma_E^{(j-1)} = \begin{cases} \beta_E^{(j-1)} & \text{if } j \text{ is even} \\ \alpha_E^{(j-1)} & \text{if } j \text{ is odd} \end{cases}$.

Then $\operatorname{sgn}(t_j(\xi)) = -\operatorname{sgn}(2 - \gamma_E^{(j-1)}) = (-1)^j$.

We now prove by induction that $\operatorname{sgn}(z_j(\xi)) = (-1)^j$. The base step is as follows:

$$\operatorname{sgn}(z_1(\xi)) = \operatorname{sgn}(t_1(\xi)) = -1.$$

Now we assume that $z_{j-1}(\xi)$ has sign $(-1)^{j-1}$. We know that

$$z_j = \xi \gamma_R^{(j-1)} z_{j-1}(\xi) + (-1)^{j+1} U_{j-1}(\xi) t_j(\xi).$$

The first term has sign $(-1)^j$ according to induction hypothesis (since ξ is negative). As for the second term

$$\operatorname{sgn}((-1)^{j+1} U_{j-1}(\xi) t_j(\xi)) = (-1)^{j+1}(-1)^{j-1}(-1)^j = (-1)^j.$$

Thus $z_j(\xi)$ has sign $(-1)^j$.

□

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