Domain perturbation for the solution of a periodic Dirichlet problem

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Abstract. We prove that the solution of the periodic Dirichlet problem for the Laplace equation depends real analytically on a suitable parametrization of the shape of the domain, on the periodicity parameters, and on the Dirichlet datum.

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1. Introduction

In this paper we study the dependence of the solution of the periodic Dirichlet problem for the Laplace equation in $\mathbb{R}^n$ upon joint perturbation of the shape of the domain, of the periodicity structure, and of the Dirichlet datum. The shape of the domain is determined by the image of a fixed domain through a map ϕ in a suitable class of diffeomorphisms and the periodicity cell is a box of edges of length $q_{11}, \dots, q_{nn}$. As a main result, we prove that the solution of the problem depends real analytically on the 'periodicity-domain-Dirichlet datum' triple $((q_{11}, \dots, q_{nn}), \phi, g)$. Our method is based on a periodic version of potential theory which has already revealed to be a powerful tool to analyze boundary value problems for elliptic differential equations in periodic domains.

Many authors have exploited potential theory to analyze perturbation problems. In the non-periodic setting, Potthast [20] and Potthast and Stratis [21] have proved a Fréchet differentiability result for layer potentials associated to the Helmholtz operator, with an application to inverse problems in scattering theory. Lanza de Cristoforis and Preciso [15] have shown that the Cauchy integral depends real analytically on domain perturbations. Lanza de Cristoforis and Rossi [16] have considered the case of layer potentials associated to the Laplace operator and have obtained real analyticity results.

Later on, Lanza de Cristoforis [11, 12] has exploited these results to prove that the solutions of boundary value problems for the Laplace and Poisson equations depend real analytically upon domain perturbation. Then, these results have been extended to singular perturbation problems and to systems of partial differential equations (see, *e.g.*, Dalla Riva and Lanza de Cristoforis [4] for the Lamé equations and Dalla Riva [3] for the Stokes' system). Moreover, analyticity results for domain perturbation problems for eigenvalues and eigenfunctions have been obtained for example for the Laplace equation by Lanza de Cristoforis and Lamberti [10] and for the biharmonic operator by Buoso [2]. We mention also Keldysh [9], Henry [8] and Sokolowski and Zolésio [22] for elliptic domain perturbation problems.

Now, we introduce our problem. We fix once for all $n \in \mathbb{N} \setminus \{0,1\}$. If $(q_{11},\dots,q_{nn}) \in]0,+\infty[^n$ we introduce a periodicity cell Q and a matrix $q \in \mathbb{D}_n^+(\mathbb{R})$ by setting

$$Q \equiv \prod_{j=1}^{n}]0,q_{jj}[, \qquad q \equiv \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix},$$

where $\mathbb{D}_n(\mathbb{R})$ is the space of $n \times n$ diagonal matrices with real entries and $\mathbb{D}_n^+(\mathbb{R})$ is the set of elements of $\mathbb{D}_n(\mathbb{R})$ with diagonal entries in $]0,+\infty[$. We also denote by $|Q|_n$ the n-dimensional measure of the cell Q, by ν_Q the outward unit normal to ∂Q, where it exists, and by q^{-1} the inverse matrix of q. Clearly $q\mathbb{Z}^n$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q. Moreover, we find convenient to set

$$\widetilde{Q} \equiv]0,1[^n, \qquad \tilde{q} \equiv I_n \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we take

$$\begin{aligned} &\alpha \in]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^n \\ &\text{of class } C^{1,\alpha} \text{ such that } \mathbb{R}^n \setminus \overline{\Omega} \text{ is connected.} \end{aligned} \tag{1.1}$$

The symbol '$\overline{\cdot}$' denotes the closure of a set. For the definition of sets and functions of the Schauder class $C^{1,\alpha}$ we refer, *e.g.*, to Gilbarg and Trudinger [7]. Then we consider a class of diffeomorphisms $\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$ from $\partial\Omega$ into their images contained in $\widetilde{Q}$ (see (2.1) below). If $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, *e.g*, Deimling [6, Thm. 5.2, p. 26]), and we denote by $\mathbb{I}[\phi]$ the bounded open connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega)$. Since $\phi(\partial\Omega) \subseteq \widetilde{Q}$, a simple topological argument shows that $\widetilde{Q} \setminus \overline{\mathbb{I}[\phi]}$ is also connected. Then we consider the following two periodic domains:

$$\mathbb{S}_q[q\mathbb{I}[\phi]] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + q\mathbb{I}[\phi]) , \qquad \mathbb{S}_q[q\mathbb{I}[\phi]]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]} .$$

Now, we take $g \in C^{1,\alpha}(\partial\Omega)$ and we consider the following periodic Dirichlet problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^- , \\ u(x+qz) = u(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-} , \forall z \in \mathbb{Z}^n , \\ u(x) = g\circ\phi^{(-1)}(q^{-1}x) & \forall x \in \partial q\mathbb{I}[\phi] . \end{cases} \tag{1.2}$$

If $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, then the solution of problem (1.2) in the space $C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$ of q-periodic functions in $\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}$ of class $C^{1,\alpha}$ exists and is unique and we denote it by $u[q,\phi,g]$. Then we pose the following question:

$$\text{What can be said on the regularity of the map } (q,\phi,g) \mapsto u[q,\phi,g]? \tag{1.3}$$

Our work stems from Lanza de Cristoforis [11, 12] where the author proved the real analytic dependence of the solution of the Dirichlet problem for the Laplace and Poisson equations upon domain perturbations. Moreover, it can be thought as a continuation of [17] where the authors proved a real analyticity result for the periodic layer potentials upon variation of the periodicity, of the shape of the support of integration, and of the density. We note that this paper generalizes a part of [18] where the authors proved an analyticity result for the longitudinal flow along a periodic array of cylinders.

In this work, we answer to the question in (1.3) by proving that the map $(q,\phi,g) \mapsto u[q,\phi,g]$ is real analytic between suitable Banach spaces (see Theorem 3.6). Such a result implies that if $\delta_0 > 0$ and we have a family of triples $\{(q_\delta,\phi_\delta,g_\delta)\}_{\delta\in]-\delta_0,\delta_0[}$ in a suitable Banach space such that the map $\delta \mapsto (q_\delta,\phi_\delta,g_\delta)$ is real analytic, then, if x belongs to the domain of $u[q_\delta,\phi_\delta,g_\delta]$ for all $\delta \in]-\delta_0,\delta_0[$, we can deduce the possibility to expand $u[q_\delta,\phi_\delta,g_\delta](x)$ as a power series in δ, *i.e.*,

$$u[q_\delta,\phi_\delta,g_\delta](x) = \sum_{k=0}^{\infty} c_k\delta^k \tag{1.4}$$

for δ close to zero. Moreover, the coefficients $(c_k)_{k\in\mathbb{N}}$ in (1.4) can be constructively determined by exploiting the method developed in [5].

2. Preliminary results

In order to consider shape perturbations, we introduce a class of diffeomorphisms. Let Ω be as in (1.1). We denote by $\mathcal{A}_{\partial\Omega}$ the set of functions of class $C^1(\partial\Omega,\mathbb{R}^n)$ which are injective and whose differential is injective at all points of $\partial\Omega$. One can verify that $\mathcal{A}_{\partial\Omega}$ is open in $C^1(\partial\Omega,\mathbb{R}^n)$ (see, *e.g.*, Lanza de Cristoforis and Rossi [16, Lem. 2.5, p. 143]). Then we set

$$\mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \equiv \{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq \widetilde{Q}\}. \tag{2.1}$$

Our method is based on a periodic version of classical potential theory. Therefore, to introduce layer potentials, we replace the fundamental solution of the Laplace operator by a q-periodic tempered distribution $S_{q,n}$ such that $\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n}$, where δ_{qz} is the Dirac measure with mass in qz (see *e.g.*, [13, p. 84]). The distribution $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) = - \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|Q|_n 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x}$$

in the sense of distributions in $\mathbb{R}^n$ (see *e.g.*, Ammari and Kang [1, p. 53], [13, §3]). Moreover, $S_{q,n}$ is even, real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (see *e.g.*, [13, §3]). We now introduce the periodic double layer potential. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ such that $\overline{\Omega_Q} \subseteq Q$. Then we consider the following two periodic domains:

$$\mathbb{S}_q[\Omega_Q] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_Q)\,, \qquad \mathbb{S}_q[\Omega_Q]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}_q[\Omega_Q]}\,.$$

We set

$$w_q[\partial\Omega_Q, \mu](x) \equiv - \int_{\partial\Omega_Q} \nu_{\Omega_Q}(y) \cdot DS_{q,n}(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

for all $\mu \in L^2(\partial\Omega_Q)$. The symbol ν_{Ω_Q} denotes the outward unit normal field to $\partial\Omega_Q$, $d\sigma$ denotes the area element on $\partial\Omega_Q$, and $DS_{q,n}(\xi)$ denotes the gradient of $S_{q,n}$ computed at the point $\xi \in \mathbb{R}^n \setminus q\mathbb{Z}^n$. The function $w_q[\partial\Omega_Q, \mu]$ is called the q-periodic double layer potential. As is well known, if $\mu \in C^0(\partial\Omega_Q)$ then $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}_q[\Omega_Q]}$ admits a continuous extension to $\overline{\mathbb{S}_q[\Omega_Q]}$, which we denote by $w_q^+[\partial\Omega_Q, \mu]$ and $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}_q[\Omega_Q]^-}$ admits a continuous extension to $\overline{\mathbb{S}_q[\Omega_Q]^-}$, which we denote by $w_q^-[\partial\Omega_Q, \mu]$ (cf. *e.g.*, [13, §3]). We also need the following lemma about the real analyticity upon the diffeomorphism ϕ of some maps related to the change of variables in the integrals and to the outer normal field (for a proof, see Lanza de Cristoforis and Rossi [16, p. 166]).

Lemma 2.1. *Let α, Ω be as in* (1.1). *Then the following statements hold.*

(i) *For each $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, there exists a unique $\tilde{\sigma}[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\phi] > 0$ and*

$$\int_{\phi(\partial\Omega)} w(s)\, d\sigma_s = \int_{\partial\Omega} w \circ \phi(y) \tilde{\sigma}[\phi](y)\, d\sigma_y, \qquad \forall\omega \in L^1(\phi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ from $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) *The map from $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ which takes ϕ to $\nu_{\mathbb{I}[\phi]} \circ \phi$ is real analytic.*

3. Analyticity of the solution

As we shall see, we will reduce the analysis of the solution $u[q,\phi,g]$ of problem (1.2) to that of a related integral equation. To do so, we start with a result on a boundary integral operator, which is proved in [19, Prop. A.3].

Lemma 3.1. *Let $q \in \mathbb{D}_n^+(\mathbb{R})$. Let α, Ω be as in (1.1). Let $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\tilde{Q}}$. Let N be the map from $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$ to itself, defined by*

$$N[\mu] \equiv -\frac{1}{2}\mu + w_q[\partial q\mathbb{I}[\phi],\mu] \qquad \forall \mu \in C^{1,\alpha}(\partial q\mathbb{I}[\phi]).$$

Then N is a linear homeomorphism from $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$ to $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$.

Now we are able to establish a correspondence between the solution of our Dirichlet problem and the solution of an integral equation in the proposition below, whose proof follows from a straightforward modification of the proof of [17, Prop. 5.2].

Proposition 3.2. *Let $q \in \mathbb{D}_n^+(\mathbb{R})$. Let α, Ω be as in (1.1). Let $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\tilde{Q}}$. Let $g \in C^{1,\alpha}(\partial\Omega)$. Then the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-, \\ u(x+qz) = u(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}, \forall z \in \mathbb{Z}^n, \\ u(x) = g \circ \phi^{(-1)}(q^{-1}x) & \forall x \in \partial q\mathbb{I}[\phi] \end{cases}$$

has a unique solution $u[q,\phi,g]$ in $C_q^{1,\alpha}(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$. Moreover,

$$u[q,\phi,g](x) = w_q^-[\partial q\mathbb{I}[\phi],\mu](x) \qquad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-},$$

where μ is the unique solution in $C^{1,\alpha}(\partial q\mathbb{I}[\phi])$ of the integral equation

$$-\frac{1}{2}\mu(x) + w_q[\partial q\mathbb{I}[\phi],\mu](x) = g \circ \phi^{(-1)}(q^{-1}x) \qquad \forall x \in \partial q\mathbb{I}[\phi]\,. \tag{3.1}$$

Next, we analyze the dependence of the solution of (3.1) upon (q,ϕ,g). Since equation (3.1) is defined on the (q,ϕ)-dependent domain $\partial q\mathbb{I}[\phi]$, the first step is to provide a reformulation on a fixed domain. More precisely, we have the following lemma. The proof follows by a change of variable and by Lemma 3.1 (cf. [19, Lem. 3.4]).

Lemma 3.3. *Let $q \in \mathbb{D}_n^+(\mathbb{R})$. Let α, Ω be as in (1.1). Let $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\tilde{Q}}$. Let $g \in C^{1,\alpha}(\partial\Omega)$. Then the function $\theta \in C^{1,\alpha}(\partial\Omega)$ solves the equation*

$$-\frac{1}{2}\theta(t) - \int_{q\phi(\partial\Omega)} \nu_{q\mathbb{I}[\phi]}(s) \cdot DS_{q,n}(q\phi(t)-s)(\theta\circ\phi^{(-1)})(q^{-1}s)\,d\sigma_s = g(t) \qquad \forall t \in \partial\Omega\,, \tag{3.2}$$

if and only if the function $\mu \in C^{1,\alpha}(\partial q\mathbb{I}[\phi])$, with μ delivered by

$$\mu(x) = (\theta\circ\phi^{(-1)})(q^{-1}x) \qquad \forall x \in \partial q\mathbb{I}[\phi],$$

solves the equation

$$-\frac{1}{2}\mu(x) + w_q[\partial q\mathbb{I}[\phi], \mu](x) = g \circ \phi^{(-1)}(q^{-1}x) \qquad \forall x \in \partial q\mathbb{I}[\phi]\,.$$

Moreover, equation (3.2) *has a unique solution* θ *in* $C^{1,\alpha}(\partial\Omega)$.

Now, our aim is to prove the analyticity upon (q, ϕ, g) of the function θ which solves equation (3.2). Inspired by Lemma 3.3, we introduce the map Λ from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times \left(C^{1,\alpha}(\partial\Omega)\right)^2$ to $C^{1,\alpha}(\partial\Omega)$ by setting

$$\begin{aligned}\Lambda[q, \phi, g, \theta](t) &\equiv -\frac{1}{2}\theta(t)\\ &- \int_{q\phi(\partial\Omega)} \nu_{q\mathbb{I}[\phi]}(s) \cdot DS_{q,n}(q\phi(t) - s)(\theta \circ \phi^{(-1)})(q^{-1}s)d\sigma_s - g(t) \quad \forall t \in \partial\Omega,\end{aligned}$$

for all $(q, \phi, g, \theta) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times \left(C^{1,\alpha}(\partial\Omega)\right)^2$. Next, we apply the implicit function theorem to the equation $\Lambda[q, \phi, g, \theta] = 0$.

Proposition 3.4. *Let* α, Ω *be as in* (1.1). *Then the following statements hold.*

(i) Λ *is real analytic.*

(ii) *For each* $(q, \phi, g) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times C^{1,\alpha}(\partial\Omega)$, *there exists a unique* θ *in* $C^{1,\alpha}(\partial\Omega)$ *such that*

$$\Lambda[q, \phi, g, \theta] = 0 \qquad \text{on } \partial\Omega,$$

and we denote such a function by $\theta[q, \phi, g]$.

(iii) *The map* $\theta[\cdot, \cdot, \cdot]$ *from* $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times C^{1,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes* (q, ϕ, g) *to* $\theta[q, \phi, g]$ *is real analytic.*

Proof. Statement (i) follows from [17, Thm. 3.2 (ii)], while (ii) is a consequence of Lemma 3.3. Next we consider (iii). Since the analyticity is a local property, we fix (q_0, ϕ_0, g_0) in $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times C^{1,\alpha}(\partial\Omega)$ and we show that $\theta[\cdot, \cdot, \cdot]$ is real analytic in a neighborhood of (q_0, ϕ_0, g_0) in the product space $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\right) \times C^{1,\alpha}(\partial\Omega)$. By standard calculus in normed spaces, the partial differential $\partial_\theta\Lambda[q_0, \phi_0, g_0, \theta[q_0, \phi_0, g_0]]$ of Λ at $(q_0, \phi_0, g_0, \theta[q_0, \phi_0, g_0])$ with respect to the variable θ is delivered by

$$\begin{aligned}&\partial_\theta\Lambda[q_0,\phi_0, g_0, \theta[q_0, \phi_0, g_0]](\psi)(t)\\ &= -\frac{1}{2}\psi(t) - \int_{q_0\phi_0(\partial\Omega)} \nu_{q_0\mathbb{I}[\phi_0]}(s) \cdot DS_{q_0,n}(q_0\phi_0(t) - s)(\psi \circ \phi_0^{(-1)})(q_0^{-1}s)\, d\sigma_s\\ &\qquad \forall t \in \partial\Omega,\end{aligned}$$

for all $\psi \in C^{1,\alpha}(\partial\Omega)$. By Lemma 3.1, $\partial_\theta\Lambda[q_0, \phi_0, g_0, \theta[q_0, \phi_0, g_0]]$ is a linear homeomorphism from $C^{1,\alpha}(\partial\Omega)$ onto $C^{1,\alpha}(\partial\Omega)$. Then the implicit function theorem for real analytic maps in Banach spaces (see, *e.g.*, Deimling [6, Thm.

15.3]) implies that $\theta[\cdot,\cdot,\cdot]$ is real analytic in a neighborhood of (q_0,ϕ_0,g_0) in $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times C^{1,\alpha}(\partial\Omega)$. □

Remark 3.5. By Lemma 2.1, Proposition 3.2 and Proposition 3.4, the solution $u[q,\phi,g]$ of problem (1.2) can be written as

$$u[q,\phi,g](x) = -\int_{\partial\Omega} \nu_{q\mathbb{I}[\phi]}(q\phi(s)) \cdot DS_{q,n}(x - q\phi(s))\theta[q,\phi,g](s)\tilde{\sigma}[q\phi]\, d\sigma_s \qquad \forall x \in \mathbb{S}_q[q\mathbb{I}[\phi]]^-,$$

for all $(q,\phi,g) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times C^{1,\alpha}(\partial\Omega)$.

We are now able to deduce our main result, which answers to (1.3).

Theorem 3.6. *Let α, Ω be as in* (1.1). *Let*

$$(q_0,\phi_0,g_0) \in \mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times C^{1,\alpha}(\partial\Omega).$$

Let U be a bounded open subset of $\mathbb{R}^n$ such that $\overline{U} \subseteq \mathbb{S}_{q_0}[q_0\mathbb{I}[\phi_0]]^-$. Then there exists an open neighborhood $\mathcal{U}$ of (q_0,ϕ_0,g_0) in

$$\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right) \times C^{1,\alpha}(\partial\Omega)$$

such that the following statements hold.

(i) $\overline{U} \subseteq \mathbb{S}_q[q\mathbb{I}[\phi]]^-$ *for all* $(q,\phi,g) \in \mathcal{U}$

(ii) *Let $k \in \mathbb{N}$. Then the map of $\mathcal{U}$ to $C^k(\overline{U})$ which takes (q,ϕ,g) to the restriction $u[q,\phi,g]_{|\overline{U}}$ of $u[q,\phi,g]$ to $\overline{U}$ is real analytic.*

Proof. By taking $\mathcal{U}$ small enough, we can deduce the validity of statement (i). Statement (ii) follows from the representation formula of Remark 3.5 together with Lemma 2.1, Proposition 3.4 and standard properties of integral operators with real analytic kernels and with no singularity (cf. [14]). □

Remark 3.7. We considered the periodic Dirichlet problem for the Laplace equation. Our method can be used for other periodic problems. For example, one can consider the Dirichlet problem

$$\begin{cases} \Delta v = 1 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\ v(x+qz) = v(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}\,, \forall z \in \mathbb{Z}^n\,, \\ v(x) = 0 & \forall x \in \partial q\mathbb{I}[\phi]\,, \end{cases} \tag{3.3}$$

which generalizes the one considered in [18]. Then, if we denote by $v[q,\phi]$ the solution to problem (3.3), by exploiting the periodic volume potential we can prove that the map from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\right)$ to $\mathbb{R}$

$$(q,\phi) \mapsto \int_{Q\setminus q\mathbb{I}[\phi]} v[q,\phi](x)\, dx$$

is real analytic. Moreover, one can replace the right-hand side in the first equation of problem (3.3) by a more general sufficiently regular periodic function.

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