# Università degli Studi di Padova


Dipartimento di Fisica e Astronomia "Galileo Galilei"

*Corso di Dottorato in Fisica*


# Cosmic Birefringence as a Probe of Fundamental Parity-Violation


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“In the beginning there was nothing, which exploded.”
— Terry Pratchett (1948 - 2015)

# Abstract

The fact that nature exhibits a parity-violating behavior has been mostly accepted since parity is maximally broken in the electroweak sector of the standard model of particle physics. Hence, a question naturally arises: is that possible that besides weak interactions also electromagnetism encode parity-breaking signatures? The purpose of this thesis is to describe how cosmology can provide the natural setting for testing this kind of hypothesis about fundamental physics. In particular, we will show how parity-violating extensions of Maxwell electromagnetism can induce a rotation of the linear polarization plane of photons during propagation, causing the so-called cosmic birefringence. This effect impacts on the cosmic microwave background observations producing a mixing of $E$ and $B$ polarization modes which is otherwise null in the standard scenario. In the literature, several models of cosmic birefringence have been proposed, and most of them involve a Chern-Simons coupling of an axion-like field with the photons, yielding the possibility to exploit such a phenomenon to investigate this kind of exotic cosmic species beyond the standard cosmological paradigm. The structure of this thesis is organized as follows. In Chap. I, we provide all the necessary mathematical tools for understanding the physics of CMB polarization. In Chap. II we discuss the theoretical formalism underlying the theory of cosmological birefringence, and how it is possible ot use it to probe axion-like field as candidates for some components of the Universe's dark sector. In Chap. III, we show how anisotropic cosmic birefringence is able to induce some promising non-Gaussian signatures and we estimate their signal-to-noise ratio for a future CMB experiment. Chap. IV is dedicated to the conclusion.

# List of Papers

- Greco A., Bartolo N., Gruppuso A. (2022),
  *Cosmic Birefringence: Cross-Spectra and Cross-Bispectra with CMB Anisotropies*,
  Journ. of Cosmol. and Astrop. Phys., 03, 050 (arXiv), (JCAP).

- Greco A., Bartolo N., Gruppuso A. (2023),
  *Probing Axions through Tomography of Anisotropic Cosmic Birefringence*,
  Journ. of Cosmol. and Astrop. Phys., 05, 026 (arXiv), (JCAP).

# Contents

# List of Acronyms

**CMB** .......... Cosmic Microwave Background

**SR** ............ Special Relativity

**EOM** .......... Equation Of Motion

**GR** ............ General Relativity

**LIF** ............ Local Inertial Frame

**FLRW** ......... Friedmann-Lemaître-Robertson-Walker

**EOS** ........... Equation of State

**CDM** .......... Cold Dark Matter

**SM** ............ Standard Model

**QFT** ........... Quantum Field Theory

**GOA** .......... Geometric Optics Approximation

**ACB** ........... Anisotropic Cosmic Birefringence

**EDE** ........... Early Dark Energy

**SNR** ........... Signal-to-Noise Ratio

# 1
# Introduction

In the last decades, the investigation of parity-violating signatures in cosmology has become one of the most ambitious goals (see e.g. Refs. [1–3]). Many efforts have been done in order to constrain parity-breaking effects coming, e.g., from non-standard inflationary models, not only at the level of the cosmic microwave background (CMB) angular power spectra (see e.g. Refs. [4–12]), but also by looking at higher-order correlation functions, such as bispectra and trispectra (see e.g. Refs. [13–27]). Furthermore, besides CMB observables, recently the research on parity-breaking signals in large scale structures (see e.g. Refs. [28–36]) and from astrophysical gravitational waves (see e.g. Refs. [37–44]) has known an increasing interest. However, one of the most intriguing source of cosmological parity violation seems to come from cosmic birefringence, which is nothing but the rotation of the linear polarization plane of CMB photons of a angle $\alpha$, when free-streaming as a consequence of an electromagnetic Chern-Simons coupling with a pseudoscalar field. Indeed, this extension of the Maxwell theory induces a rotation of the observed Stokes parameters describing the linear CMB polarization (see e.g. Refs. [45, 46]). An observational consequence of such a rotation is, e.g., the switching-on of a parity-breaking angular cross-correlation between the $E$ and $B$ modes of CMB polarization. In fact, cosmic birefringence can be seen as a probe for the existence of such a pseudoscalar field, which could be a candidate for early and late dark energy (see e.g. Refs. [47–54]) or dark matter (see e.g. Refs. [55–61]), in the form of an axion-like field (see e.g. Refs. [62–73]). The tantalizing idea of succeeding in unveiling the nature of the dark sector of the Universe by investigating cosmological parity-violation has also brought with it the necessity to break the degeneracies between different theoretical models able to induce the birefringence effect, and according to this purpose, ven a tomographic approach has been recently proposed (see e.g. Refs. [74–78]). Moreover, a precise treatment of cosmic birefringence should consider the possibility that the pseudoscalar field in general may not be homogeneous, implying the presence of a non-zero anisotropic component in the birefringence angle (see e.g. Refs. [79–88]): such anisotropies in the birefringence angle can provide by themselves a further and complementary observational tests of models for birefringence. An increasing number of observational constraints on both isotropic and anisotropic cosmological birefringence are present in literature, as results of several CMB experiments: WMAP (see Refs. [89–92]), POLARBEAR (see Refs. [93, 94]), ACTPol (see Ref. [95]), SPTpol (see Ref. [96]), BICEP/Keck (see Refs. [97, 98]), and the *Planck* satellite (see Refs. [99–105]). In particular, the authors of [106], exploiting the latest *Planck* data release, have found an hint of detection of the isotropic birefringence angle $\alpha = (0.30 \pm 0.11)^\circ$ [106]. However, a more detailed analysis is required in order to be sure that such a rotation has effectively a cosmological origin, and it is not instead caused by galactic dust or miscalibration angles [107–116]. Nevertheless, let us just mention that if the new physics hypothesis for the existence of a non-zero $EB$ cross-correlation would be confirmed, most probably this could only be explained by cosmic birefringence, since any observed $EB$ correlation sourced by primordial chiral gravitational waves does not work due to the overproduction of the $B$ modes with respect to the current constraints on the tensor-to-scalar ratio [117]. Therefore, as one could infer from the previous lines, there is a number of strong motivations for studying the phenomenon of cosmic birefringence. However, in order to fully understand how the mechanism of birefringence works, to which Chap. 2 is devoted, we have firstly to review the standard theory of CMB polarization. Hence, the structure of this chapter is organized as follows. In Sec. 1.1 we review the basics of the Maxwell's description of photons' polarization. In Sec. 1.2 we review some elements of general relativity. In Sec. 1.3 we provide all the mathematical formalism needed to understand the anisotropies of CMB polarization.

## 1.1 Elements of Classical Electromagnetism

In this section, we are going to give an overview of the Maxwell's theory of electromagnetism, since the formalism exposed here is necessary to understand the nature of the birefringence effect, which is the main topic of this thesis. The content of this section is a partial review of the standard notions that can be found in many famous textbooks, such as Refs. [118–123]).

### 1.1.1 Covariant Formulation of Electrodynamics

It is well known that the behavior of electric and magnetic fields and their interactions with charges and currents can be described by a theory which unifies electric and magnetic phenomena into a single coherent theory: the Maxwell's theory of electromagnetism. The modern description of the classical[1] electromagnetic phenomena is based on the Einstein's theory of **special relativity** (see Ref. [124]). When applying special relativity[2] to electromagnetism, it is possible to show that Maxwell's fundamental equations describing classical electromagnetism, maintain the same form and retain their validity when we switch from one inertial reference frame to another using the mathematical transformations of special relativity, known as **Lorentz transformations**. Indeed, special relativity provides the mathematical framework to describe electromagnetism accurately for all observers, regardless of their relative motion and has been confirmed by numerous experimental tests. The key contribution of the SR formulation of Maxwell's electromagnetism is encapsulated in his Lagrangian density,

$$\boxed{\mathcal{L}_{\rm EM} \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},} \tag{1.1}$$

where $F_{\mu\nu}$ is the **covariant electromagnetic field tensor**, which is defined as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.2}$$

$A_\mu$ and $\partial_\mu$ being the **covariant electromagnetic field** (or **covariant 4-potential**) and the **covariant 4-gradient**, respectively. These two quantities in special relativity have the following expressions

$$A_\mu \equiv \left(-\frac{1}{c}\phi,\, A_1,\, A_2,\, A_3\right) \tag{1.3}$$

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t},\, \frac{\partial}{\partial x_1},\, \frac{\partial}{\partial x_2},\, \frac{\partial}{\partial x_3}\right), \tag{1.4}$$

where $c$ is the **speed of light in vacuum**, $\phi(t, \mathbf{x})$ is the **Coulomb potential** and $A_1$, $A_2$, $A_3$ are the components of the **vector potential** $\mathbf{A}(t, \mathbf{x})$, whereas $t$ is the time coordinate, and $\mathbf{x} \equiv (x_1, x_2, x_3)$ denotes the vector of the spatial ones. For sake of completeness, we also give the definition of the **contravariant electromagnetic field tensor** which appears in Eq. (1.1) as

$$F^{\mu\nu} \equiv \eta^{\mu\sigma} F_{\sigma\nu}, \tag{1.5}$$

[1]With the attribute "classical" here we mean "non-quantum".

[2]Of course, here we are not going to review special relativity (SR), but we just recall the basic notions and definitions to set the conventions used in this thesis right on the start.

where $\eta_{\mu\nu}$ is the **covariant metric tensor** which in special relativity is simply the **Minkowski tensor**:

$$\eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\mu\nu}. \tag{1.6}$$

The physical meaning of the metric tensor cannot be underestimated since it defines the spacetime's geometry. Moreover, as can be seen by looking at Eq. (1.5), it can be used to transform covariant tensors to contravariant ones, and vice versa. In fact, the mathematical operation on the right-hand side of Eq. (1.5) is called **Lorentz contraction**, and for practical reasons we can think of it as a matrix multiplication between the two tensors, where the summation between repeated indices has to be understood, according to the Einstein's summation convention, i.e.

$$\eta^{\mu\sigma}F_{\sigma\nu} \equiv \sum_{\sigma=0}^{3} \eta^{\mu\sigma}F_{\sigma\nu} = \eta^{\mu 0}F_{0\nu} + \eta^{\mu 1}F_{1\nu} + \eta^{\mu 2}F_{2\nu} + \eta^{\mu 3}F_{3\nu} = \eta^{\mu 0}F_{0\nu} + \sum_{i=1}^{3} \eta^{\mu i}F_{i\nu} = \eta^{\mu 0}F_{0\nu} + \eta^{\mu i}F_{i\nu}. \tag{1.7}$$

Indeed, it is customary to use Greek indices when running from 0 to 4, and Latin ones when running just over the spatial indices, i.e. from 1 to 3. Before to proceed, let us just clarify that, starting from now, we are going to work in the so called "God-given" units, so that some the following fundamental constants of the Universe (see e.g. Ref. [125])

| | | |
|---|---|---|
| speed of light in vacuum | $c \simeq 2.998 \times 10^{8}$ m/s, | (1.8) |
| reduced Planck mass | $\hbar \simeq 1.055 \times 10^{-34}$ J · s, | (1.9) |
| boltzmann constant | $k_B \simeq 1.381 \times 10^{-23}$ J/K, | (1.10) |
| vacuum dielectric permittivity | $\varepsilon_0 \simeq 8.854 \times 10^{-12}$ F/m, | (1.11) |
| vacuum magnetic permeability | $\mu_0 \simeq 1.257 \times 10^{-6}$ N/A$^2$, | (1.12) |

are taken as dimensionless and set equal to 1. This is done in order to make our equations more compact and removing the need for unit conversions.

### 1.1.2 Maxwell's Equations

The Maxwell Lagrangian density defined in Eq. (1.1) completely describes the classical electromagnetic field theory, and this becomes evident if we compute the associated equation of motion (EOM). In fact, let us recall that, according to classical field theory, the dynamics of a generic field $A_\mu$, defined in the spacetime continuum, is described by the following 4-dimensional integral of the Lagrangian density, which is called **action**:

$$S \equiv \int \mathrm{d}^4x\, \mathcal{L}\left[A_\mu(x^\lambda), \partial_\mu A_\nu(x^\lambda)\right], \tag{1.13}$$

where $x^\lambda = (t, x, y, z)^\mathsf{T}$ is the **contravariant 4-position**, and $\mathrm{d}^4x$ is the spacetime volume element, which in the Minkowski's case simply reads

$$\mathrm{d}^4x = \mathrm{d}t\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 = \mathrm{d}t\, \mathrm{d}^3x, \tag{1.14}$$

since it is defined in order to be invariant under Lorentz transformations. Now, the **Hamilton's principle** states that the "true" field configuration is such that the action is stationary, namely the EOM of the field $A_\mu$ is obtained by imposing that the **functional derivative** of the action $S$ with respect to the field itself equals zero, i.e.

$$\frac{\delta S}{\delta A_\mu(x^\lambda)} \equiv \int \mathrm{d}^4\tilde{x} \left\{ \frac{\partial \mathcal{L}}{\partial\left[A_\sigma(\tilde{x}^\lambda)\right]} \frac{\delta A_\sigma(\tilde{x}^\lambda)}{\delta A_\mu(x^\lambda)} + \frac{\partial \mathcal{L}}{\partial\left[\partial_\rho A_\sigma(\tilde{x}^\lambda)\right]} \frac{\delta\left[\partial_\varrho A_\sigma(\tilde{x}^\sigma)\right]}{\delta A_\mu(x^\sigma)} \right\} = 0, \tag{1.15}$$

where we have renamed some dummy variables and indices. Eq. (1.15) is just the definition of the functional derivative of an action, and then it can be used for any kind of covariant tensor field of whatever Lorentz-invariant theory. The functional derivative of the field simply reads

$$\frac{\delta A_\sigma(\tilde{x}^\lambda)}{\delta A_\mu(x^\lambda)} \equiv \delta^\mu_\sigma\, \delta^{(4)}\left(\tilde{x}^\lambda - x^\lambda\right), \tag{1.16}$$

where $\delta^{(4)}$ is a 4-dimensional Dirac delta, and $\delta^\mu_\sigma$ is a 4-dimensional Kronecker delta, $\delta^\mu_\sigma \equiv \eta^{\mu\nu}\eta_{\nu\sigma}$. By means of Eq. (1.16), we can rewrite Eq. (1.15) for $\mathcal{L} = \mathcal{L}_{\mathrm{EM}}$ as

$$0 = \int \mathrm{d}^4\tilde{x} \left\{ \frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left[\partial_\varrho A_\sigma(\tilde{x}^\lambda)\right]} \partial_\varrho \right\} \delta^\mu_\sigma\, \delta^{(4)}\left(\tilde{x}^\lambda - x^\lambda\right), \tag{1.17}$$

where we have used that the Maxwell's Lagrangian density just depends on the derivatives of the electromagnetic field, and that the functional derivatives commute with the ordinary ones. We now integrate by parts, so that Eq. (1.17) further simplifies once we recall the properties of the Dirac delta:

$$
\begin{aligned}
0 &= -\int \mathrm{d}^4\tilde{x}\, \delta^{\mu}_{\sigma}\, \delta^{(4)}\left(\tilde{x}^{\lambda} - x^{\lambda}\right) \partial_{\varrho} \left\{ \frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial \left[\partial_{\varrho} A_{\sigma}(\tilde{x}^{\lambda})\right]} \right\} \\
&= \frac{1}{4} \delta^{\mu}_{\sigma} \partial_{\varrho} \frac{\partial}{\partial \left(\partial_{\varrho} A_{\sigma}\right)} \left[\eta^{\alpha\gamma}\eta^{\beta\kappa} \left(\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}\right) \left(\partial_{\gamma} A_{\kappa} - \partial_{\kappa} A_{\gamma}\right)\right] \\
&= \frac{1}{4} \delta^{\mu}_{\sigma} \partial_{\varrho}\, \eta^{\alpha\gamma}\eta^{\beta\kappa} \left[\left(\delta^{\varrho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\varrho}_{\beta}\delta^{\sigma}_{\alpha}\right)\left(\partial_{\gamma} A_{\kappa} - \partial_{\kappa} A_{\gamma}\right) + \left(\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}\right)\left(\delta^{\varrho}_{\gamma}\delta^{\sigma}_{\kappa} - \delta^{\varrho}_{\kappa}\delta^{\sigma}_{\gamma}\right)\right] \\
&= \partial_{\varrho} \left(\partial^{\varrho} A^{\mu} - \partial^{\mu} A^{\varrho}\right)
\end{aligned}
\tag{1.18}
$$

where we have explicitly substituted Eq. (1.1). Therefore, Eq. (1.18), after a renomination of the indices, simply reduces to

$$
\boxed{\partial_{\mu} F^{\mu\nu} = 0.} \tag{1.19}
$$

Eq. (1.19), as well as of the Maxwell's Lagrangian density, is invariant under U(1) **gauge transformations**, which means that the theory remains unchanged if the 4-gradient of a scalar function $h\left(x^{\lambda}\right)$ is added to the electromagnetic field,

$$
A_{\mu}(x^{\lambda}) \mapsto A_{\mu}(x^{\lambda}) + \partial_{\mu}\, h\left(x^{\lambda}\right), \tag{1.20}
$$

since, as it can be verified by direct substitution,

$$
F_{\mu\nu} \mapsto \left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right) + \left(\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu}\right) h = F_{\mu\nu}. \tag{1.21}
$$

Eq. (1.19) is the EOM for the electromagnetic field written in an explicit covariant way, but if we recall the definition of $A_{\mu}$ we can easily recover the standard Maxwell equations. Indeed, for $\nu = 0$ we have

$$
0 = \partial_{\mu} F^{\mu 0} = \partial_{i} F^{i0} = \eta^{00}\eta^{ij}\partial_{i}\left(\partial_{j} A_{0} - \partial_{0} A_{j}\right) = \nabla^2 \phi(t, \mathbf{x}) + \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A}(t, \mathbf{x}), \tag{1.22}
$$

where, after recognizing the definition of the **electric field**, we can easily understand that what we have found is nothing but the **Gauss' law** in absence of sources:

$$
\mathbf{E}(t, \mathbf{x}) \equiv -\boldsymbol{\nabla}\phi(t, \mathbf{x}) - \frac{\partial \mathbf{A}(t, \mathbf{x})}{\partial t} \quad \Longrightarrow \quad \boxed{\boldsymbol{\nabla} \cdot \mathbf{E}(t, \mathbf{x}) = 0.} \tag{1.23}
$$

Similarly, the $\nu = k$ component of Eq. (1.19) reads[3]

$$\begin{aligned} 0 &= \partial_\mu F^{\mu k} \\ &= \partial_0 F^{0k} + \partial_i F^{ik} = \left\{ -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\mathbf{A}(t,\mathbf{x}) + \boldsymbol{\nabla}\phi(t,\mathbf{x})\right] + \boldsymbol{\nabla}^2\mathbf{A}(t,\mathbf{x}) - \boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\mathbf{A}(t,\mathbf{x})\right]\right\}, \end{aligned} \tag{1.24}$$

which, by means of a vector identity,

$$\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x})\right] = \boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\mathbf{A}(t,\mathbf{x})\right] - \boldsymbol{\nabla}^2\mathbf{A}(t,\mathbf{x}), \tag{1.25}$$

and of the definition of the **magnetic field**, just leads to the **Ampère's circuital's law** in absence of sources:

$$\mathbf{B}(t,\mathbf{x}) \equiv \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) \qquad \Longrightarrow \qquad \boxed{\boldsymbol{\nabla}\times\mathbf{B}(t,\mathbf{x}) = \frac{\partial\mathbf{E}(t,\mathbf{x})}{\partial t}.} \tag{1.26}$$

Up to now, we have found two of the original four Maxwell's equations: the remaining ones can be obtained by exploiting another field equation that cannot be derived by means of the Hamilton's principle, but just exploiting geometrical properties,

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0,} \tag{1.27}$$

where $\tilde{F}$ is the **dual Maxwell tensor**,

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\frac{\varepsilon^{\mu\nu\varrho\sigma}}{\sqrt{-\eta}}F_{\varrho\sigma}, \tag{1.28}$$

with $\eta = -1$ being the determinant of $\eta_{\mu\nu}$, and $\varepsilon^{\mu\nu\varrho\sigma}$ being the **Levi-Civita antisymmetric symbol**,

$$\varepsilon^{\mu\nu\varrho\sigma} \equiv \begin{cases} +1 & \text{for even permutations of (1,2,3,4)}, \\ -1 & \text{for odd permutations of (1,2,3,4)}, \\ 0 & \text{if two or more indices are equal.} \end{cases} \tag{1.29}$$

The validity of Eq. (1.27) can be easily proved by exploiting the symmetry properties of the tensor indices:

$$\partial_\mu \tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\varrho\sigma}\partial_\mu\partial_\varrho A_\sigma = 0, \tag{1.30}$$

[3] The vector Laplacian $\boldsymbol{\nabla}^2\mathbf{A}$ is a vector quantity whose components are $\boldsymbol{\nabla}^2\mathbf{A} \equiv (\nabla^2 A_1,\ \nabla^2 A_2,\ \nabla^2 A_3)$.

since the Levi-Civita tensor is antisymmetric under the exchange of the indices labeling the two derivatives, that instead commute. Therefore, the $\nu = 0$ component of Eq. (1.27) simply leads to the **Gauss' law for magnetism**,

$$0 = \partial_\mu \tilde{F}^{\mu 0} = -\varepsilon^{0ijk}\partial_i \partial_k A_k = -\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x})] \quad \Longrightarrow \quad \boxed{\boldsymbol{\nabla} \cdot \mathbf{B}(t, \mathbf{x}) = 0,} \tag{1.31}$$

from which we can understand why it is possible to define the magnetic field as the curl of a vector field without loss of generality, since the divergence of a curl identically vanishes. Similarly the $\nu = k$ component of Eq. (1.27),

$$\begin{aligned} 0 = \partial_\mu \tilde{F}^{\mu k} &= \varepsilon^{0ijk}\left[\partial_0\left(\partial_i A_j - \partial_j A_i\right) - \partial_i \partial_j A_0\right] \\ &= -\frac{\partial}{\partial t}\left[\boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x})\right] + \boldsymbol{\nabla} \times \left[\frac{\partial \mathbf{A}(t, \mathbf{x})}{\partial t} + \boldsymbol{\nabla}\phi(t, \mathbf{x})\right], \end{aligned} \tag{1.32}$$

yields the **Faraday-Neumann-Lenz law**,

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial \mathbf{B}(t, \mathbf{x})}{\partial t},} \tag{1.33}$$

and also here we can appreciate why it is possible to define the electric field as the sum of the gradient of a scalar field plus a vector one, since the curl of gradient identically vanishes. Now that we have derived all the four Maxwell's equations, we are in the position to understand why when talking about electric and magnetic fields, we also use the expression **electromagnetic waves**. Indeed, if we take the curl of Eq. (1.33), and we substitute Eq. (1.26) in the result, we get

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \mathbf{E}(t, \mathbf{x})] = -\frac{\partial}{\partial t}[\boldsymbol{\nabla} \times \mathbf{B}(t, \mathbf{x})] \implies \boldsymbol{\nabla}[\boldsymbol{\nabla} \cdot \mathbf{E}(t, \mathbf{x})] - \boldsymbol{\nabla}^2 \mathbf{E}(t, \mathbf{x}) = -\frac{\partial^2 \mathbf{E}(t, \mathbf{x})}{\partial t^2}, \tag{1.34}$$

where we have exploited the vector identity of Eq. (1.25). Finally, thanks to Eq. (1.23), the equation above reduces to a **wave equation**,

$$\boxed{\boldsymbol{\nabla}^2 \mathbf{E}(t, \mathbf{x}) - \frac{\partial^2 \mathbf{E}(t, \mathbf{x})}{\partial t^2} = 0,} \tag{1.35}$$

implying that the electric field (as well as the magnetic one[4]) can be treated as a wave propagating at the speed of light in the vacuum. As well known, in order to solve Eq. (1.35), the electric and magnetic fields must depends on space and time in the following way:

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 f_E\left(\hat{\mathbf{p}}_E \cdot \mathbf{x} - t\right), \qquad \mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0 f_B\left(\hat{\mathbf{p}}_B \cdot \mathbf{x} - t\right), \tag{1.36}$$

as can be verified by direct substitution. Here $\hat{\mathbf{p}}_E$ and $\hat{\mathbf{p}}_B$ are in principle distinct directions of propagation, $f_E$ and $f_B$ two distinct functions, $\mathbf{E}_0$ and $\mathbf{B}_0$ are the amplitudes of the wave describing the electric field and of the one describing the magnetic field, respectively. By means of the Maxwell's equations, we can constrain all these degrees of freedom. In fact, we know that the divergence of the electric and magnetic fields vanishes because of Eq. (1.23) and Eq. (1.31), so that

$$\nabla \cdot \mathbf{E}(t,\mathbf{x}) = 0 \quad \Longrightarrow \quad \hat{\mathbf{p}}_E \cdot \mathbf{E}_0 = 0 \quad \Longrightarrow \quad \boxed{\hat{\mathbf{p}}_E \perp \mathbf{E}_0,} \tag{1.37}$$

$$\nabla \cdot \mathbf{B}(t,\mathbf{x}) = 0 \quad \Longrightarrow \quad \hat{\mathbf{p}}_B \cdot \mathbf{B}_0 = 0 \quad \Longrightarrow \quad \boxed{\hat{\mathbf{p}}_B \perp \mathbf{B}_0,} \tag{1.38}$$

whereas from Eq. (1.33) and Eq. (1.26) we can infer that

$$\nabla \times \mathbf{E}(t,\mathbf{x}) = -\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \quad \Longrightarrow \quad \hat{\mathbf{p}}_E \times \mathbf{E}_0 = \left(\dot{f}_B/\dot{f}_E\right)\mathbf{B}_0, \tag{1.39}$$

$$\nabla \times \mathbf{B}(t,\mathbf{x}) = \frac{\partial}{\partial t}\mathbf{E}(t,\mathbf{x}) \quad \Longrightarrow \quad \hat{\mathbf{p}}_B \times \mathbf{B}_0 = -\left(\dot{f}_E/\dot{f}_B\right)\mathbf{E}_0, \tag{1.40}$$

where the dot $\dot{} \equiv \partial/\partial t$ denotes differentiation with respect to time. Therefore, we have learned that both the electric and the magnetic fields oscillate perpendicularly to the direction of their propagation. Since, by definition the dot product of two vectors orthogonal each to the other is zero, we can write

$$0 = \hat{\mathbf{p}}_E \cdot \left(\hat{\mathbf{p}}_E \times \mathbf{E}_0\right) = \left(\dot{f}_B/\dot{f}_E\right)\left(\hat{\mathbf{p}}_E \cdot \mathbf{B}_0\right) \quad \Longrightarrow \quad \boxed{\mathbf{B}_0 \perp \hat{\mathbf{p}}_E,} \tag{1.41}$$

and therefore the magnetic field is perpendicular to both the directions of propagation. On the other hand, we have

$$\hat{\mathbf{p}}_B \times \left(\hat{\mathbf{p}}_E \times \mathbf{E}_0\right) = \left(\dot{f}_B/\dot{f}_E\right)\left(\hat{\mathbf{p}}_B \times \mathbf{B}_0\right) = -\mathbf{E}_0, \tag{1.42}$$

[4]It is trivial to see that it is possible to obtain exactly the same equation also for **B** by taking the curl of Eq. (1.26) and plugging the result with the other Maxwell's equations.

so that, by means of the rule for the double cross product[5], we then obtain

$$\begin{aligned} \mathbf{E}_0 &= -\hat{\mathbf{p}}_B \times (\hat{\mathbf{p}}_E \times \mathbf{E}_0) \\ &= (\hat{\mathbf{p}}_E \cdot \hat{\mathbf{p}}_B)\, \mathbf{E}_0 - (\hat{\mathbf{p}}_B \cdot \mathbf{E}_0)\, \hat{\mathbf{p}}_E \\ &= (\hat{\mathbf{p}}_E \cdot \hat{\mathbf{p}}_B)\, \mathbf{E}_0 + \left(\dot{f}_B/\dot{f}_E\right) [\hat{\mathbf{p}}_B \cdot (\hat{\mathbf{p}}_B \times \mathbf{B}_0)]\, \hat{\mathbf{p}}_E \\ &= (\hat{\mathbf{p}}_E \cdot \hat{\mathbf{p}}_B)\, \mathbf{E}_0 \qquad \Longrightarrow \qquad \boxed{\hat{\mathbf{p}}_E = \hat{\mathbf{p}}_B \equiv \hat{\mathbf{p}}.} \end{aligned} \tag{1.43}$$

Thus, we have proven that electric and magnetic field propagate in the same direction, which is the propagation's direction of the electromagnetic wave itself. Without loss of generality we can also take $f_E = f_B$, and still we have a solution of the Maxwell's equations. Hence, thanks to this information we can use Eq. (1.39) to find

$$\boxed{\mathbf{B}(t, \mathbf{x}) = \hat{\mathbf{p}} \times \mathbf{E}(t, \mathbf{x}),} \tag{1.44}$$

which is the standard expression for the magnetic field, written in such a way to make evident its orthogonality with respect to the electric field.

### 1.1.3 Stokes Parameters

As we have shown in Sec. 1.1.2, an electromagnetic wave consists of an oscillating electric field coupled with an oscillating magnetic one, which are always perpendicular to each other; by convention, we call **polarization** of electromagnetic waves the direction of the electric field. According to Eq. (1.36), we can write an electromagnetic field propagating along the $\hat{\mathbf{p}} = \hat{\mathbf{x}}_3$ axis as

$$\mathbf{E}(t, \mathbf{x}) = \begin{bmatrix} E_1(t, \mathbf{x}) \\ E_2(t, \mathbf{x}) \\ 0 \end{bmatrix} = \begin{bmatrix} E_{0,1} \cos(x_3 - t) \\ E_{0,2} \cos(x_3 - t + \beta) \\ 0 \end{bmatrix}, \tag{1.45}$$

where $\beta$ is the relative phase between the two directions of oscillations. If we take the square modulus of Eq. (1.45), we obtain

$$\begin{aligned} \left(\frac{E_1}{E_{0,1}}\right)^2 + \left(\frac{E_2}{E_{0,2}}\right)^2 &= \cos^2(x_3 - t) + [\cos(x_3 - t)\cos(\beta) - \sin(x_3 - t)\sin(\beta)]^2 \\ &= 2\cos^2(x_3 - t)\cos^2(\beta) + \sin(\beta)\,[1 - 2\cos(x_3 - t)\cos(\beta)\sin(x_3 - t)] \end{aligned} \tag{1.46}$$

[5] i.e. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$.

where we have used the following trigonometric identities:

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta), \tag{1.47}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta). \tag{1.48}$$

We can easily recast Eq. (1.46) to take a more suitable form,

$$\left(\frac{E_1}{E_{0,1}}\right)^2 + \left(\frac{E_2}{E_{0,2}}\right)^2 - \frac{2E_1E_2}{E_{0,1}E_{0,2}}\cos(\beta) = \sin^2(\beta), \tag{1.49}$$

which, again, by defining

$$E_{0,1} \equiv \mathcal{C}\cos(\theta) \qquad\qquad E_{0,2} \equiv \mathcal{C}\sin(\theta), \tag{1.50}$$

allows us to rewrite Eq. (1.49) as

$$\frac{E_1^2}{\cos^2(\theta)} + \frac{E_2^2}{\sin^2(\theta)} - \frac{4E_1E_2\cos(2\theta)}{\sin^2(2\theta)}\frac{\sin(2\alpha)}{\cos(2\alpha)} + \mathcal{C}^2\cos^2(\beta) = \mathcal{C}^2, \tag{1.51}$$

where we have defined the angle $\alpha$ as

$$\tan(2\alpha) \equiv \tan(2\theta)\cos(\beta). \tag{1.52}$$

Finally, we now define other two parameters, namely

$$a^2 \equiv \frac{\mathcal{C}^2}{2}\left[1 + \sqrt{1 - \sin^2(2\theta)\sin^2(\beta)}\right], \qquad\qquad b^2 \equiv \frac{\mathcal{C}^2}{2}\left[1 - \sqrt{1 - \sin^2(2\theta)\sin^2(\beta)}\right], \tag{1.53}$$

so that Eq. (1.51) takes the form of an ellipse rotated by an angle $\alpha$ in the $E_1 - E_2$ plane:

$$\boxed{\frac{[E_1\cos(\alpha) + E_2\sin(\alpha)]^2}{a^2} + \frac{[E_1\sin(\alpha) - E_2\cos(\alpha)]^2}{b^2} = 1,} \tag{1.54}$$

which is known as the **polarization ellipse** (see Fig. 1.1). Let use notice that when $\beta = 0$ we have $b = 0$, i.e. the ellipse degenerates in a straight line, tilted by an angle $\alpha$ with respect to the $E_x$−axis: in this case the electromagnetic wave is said to be purely **linearly polarized**. On the other hand, when $\beta = \pm\pi/2$ and $\theta = \pi/4$ we have $b = a$, which means that the ellipse degenerates in a circle: in this case the electromagnetic wave is said to be purely **circularly polarized**. In order to deal with the interplay of all the quantities used

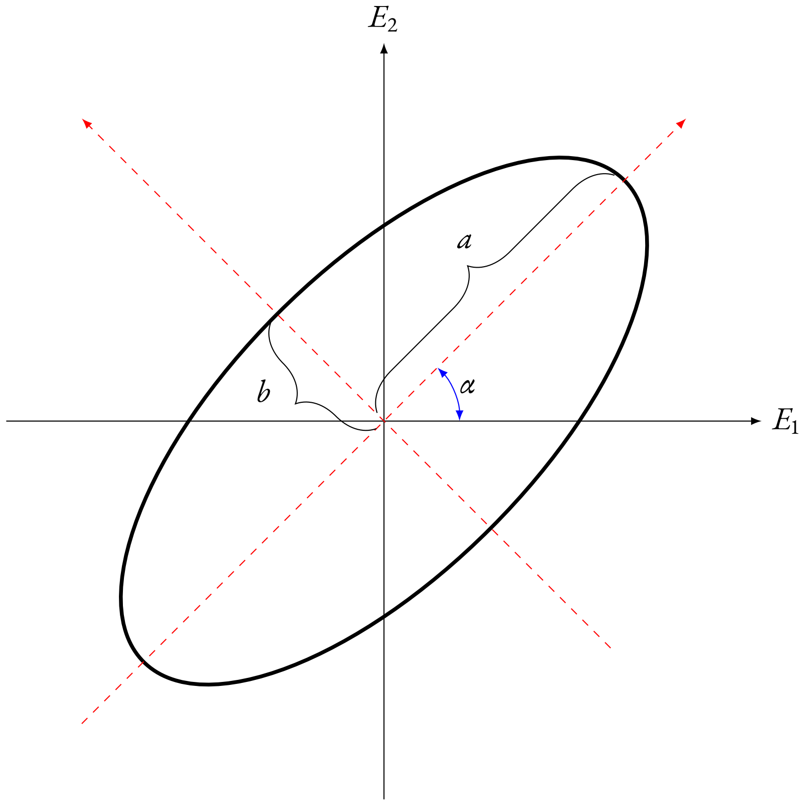


**Figure 1.1:** Example of polarization ellipse in the $a > b$ case; when $\alpha = n\pi/2$ (with $n \in \mathbb{Z}$), the dashed axes coincide with the $E_1, E_2$ ones, and the ellipse takes an horizontal or vertical orientation.

insofar, it is customary to define the so called **Stokes parameters**:

$$I \equiv a^2 + b^2, \tag{1.55}$$

$$Q \pm iU \equiv \left(a^2 - b^2\right) e^{\pm 2i\alpha}, \tag{1.56}$$

$$V \equiv 2ab. \tag{1.57}$$

Let us notice, that already from Eqs. (1.55)-(1.57), we can infer a fundamental property of the Stokes parameters, i.e. their behaviour under a spatial rotation of the polarization plane. In fact, it is clear that $I$ and $V$ are left unchanged by a rotation, whereas the linear combination $Q \pm iU$ behaves as a spin-2 field. This means that under a rotation R of an angle $\alpha$, we have

$$I \xrightarrow{\mathsf{R}(\alpha)} I, \qquad (Q \pm iU) \xrightarrow{\mathsf{R}(\alpha)} (Q \pm iU)\, e^{\pm 2i\alpha}, \qquad V \xrightarrow{\mathsf{R}(\alpha)} V. \tag{1.58}$$

The Stokes parameter $I$ is then a scalar, and it is nothing but the squared amplitude of the electric field

$$I = a^2 + b^2 = \mathcal{C}^2 = E_{0,1}^2 + E_{0,2} = |\mathbf{E}(t,\mathbf{x})|^2 , \tag{1.59}$$

whereas the Stokes parameter $V$ describes the circular polarization, because when the linear combination $Q \pm iU$ is vanishing (which occurs when $a = b$), the ellipse reduces to a circle and we have a pure circular polarization parameterized by

$$V = 2ab = \mathcal{C}^2 \sin(2\theta)\sin(\beta) = \mathcal{C}^2 = I, \tag{1.60}$$

Analogously, when $V = 0$ (which occurs when $a$ or $b$ equal zero), only $I$ and $Q \pm iU$ survives, where the latter completely describes linear polarization. For instance, if $b = 0$ we get

$$Q \pm iU = a^2 e^{\pm 2i\alpha} = \mathcal{C}^2 \cos(2\theta) = I. \tag{1.61}$$

By the way, when using the definition given in Eq. (1.45) we have implicitly considered a monochromatic wave, eventually made up of many waves but all coherent, since we have assumed $E_{0,1}$, $E_{0,2}$ and $\beta$ as constant in time. On the other hand, if we consider the more general case of a superposition of incoherent waves, then we have to account for the fact that the phases of individual wave components are no more correlated, meaning they have random phase relationships. In order to address this lack of generality in Eqs. (1.55)-(1.57), we now write down the following redefinition of the Stokes parameters:

$$\boxed{I\,(t,\mathbf{x},\hat{\mathbf{p}} = \hat{\mathbf{x}}_3) \equiv \langle E_1^2(\tilde{t},\mathbf{x})\rangle_t + \langle E_2^2(\tilde{t},\mathbf{x})\rangle_t,} \tag{1.62}$$

$$\boxed{[\,Q \pm iU]\,(t,\mathbf{x},\hat{\mathbf{p}} = \hat{\mathbf{x}}_3) \equiv \langle E_1^2(\tilde{t},\mathbf{x})\rangle_t - \langle E_2^2(\tilde{t},\mathbf{x})\rangle_t \pm 2i\langle E_1(\tilde{t},\mathbf{x})E_2(\tilde{t},\mathbf{x})\cos[\beta(\tilde{t})\,]\rangle_t,} \tag{1.63}$$

$$\boxed{V\,(\tilde{t},\mathbf{x},\hat{\mathbf{p}} = \hat{\mathbf{x}}_3) \equiv 2\langle E_1(\tilde{t},\mathbf{x})E_2(\tilde{t},\mathbf{x})\sin[\beta(\tilde{t})\,]\rangle_t,} \tag{1.64}$$

where $\langle \dots \rangle_t \equiv \int_0^t (\dots)\, \mathrm{d}\tilde{t}/t$ indicates a time average on a time variable $\tilde{t}$, such that the time $t$ over which the integration is performed is long enough to capture the relevant information about the electromagnetic wave, while filtering out rapid fluctuations. Indeed, in the monochromatic case when the amplitudes and the phase are time-independent, one need only to drop the angular brackets in the expressions above to get the applicable Stokes parameters, since, as can be verified by direct substitution, one should recover Eqs. (1.55)-(1.57). Moreover, the parameters describing the polarization are just three ($\theta$, $\beta$ and $\mathcal{C}$), which means that, as one could have inferred by looking e.g. at Eqs. (1.61), the four Stokes parameters are not independent but obey the following constraint:

$$Q^2 + U^2 + V^2 \leq I^2. \tag{1.65}$$

### 1.1.4 Electric Field of a Moving Electron

Up to now, we have been able just to describe electric and magnetic fields just in vacuum, i.e. without considering the presence of anything else but the electromagnetic field itself. However, adding a source term to the Maxwell's equations is essential to account for the presence of electric charges and currents in the surrounding space, so that Maxwell's equations become a complete description of how electric and magnetic fields interact with charges and currents, allowing us to analyze and predict electromagnetic phenomena in the presence of matter. This extension can be done by adding by hand a source term to the Maxwell's Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^{\mu}A_{\mu}, \tag{1.66}$$

where $J^{\mu} \equiv (\rho, J_1, J_2, J_3)^{\mathsf{T}}$ is the **contravariant 4-current**, whose entries are the **charge volumetric density** $\rho(t, \mathbf{x})$, and the **current surface density** $\mathbf{J}(t, \mathbf{x}) \equiv \rho(t, \mathbf{x})\mathbf{v}(t)$ of a charge distribution moving at speed $\mathbf{v}(t)$, respectively. By applying again the Hamilton's principle, it is finally possible to derive the two Maxwell's equation in presence of sources, i.e.

$$\partial_{\mu}F^{\mu\nu} + \mu_0 J^{\nu} = 0 \qquad \Longrightarrow \qquad \begin{cases} \nabla \cdot \mathbf{E}(t, \mathbf{x}) = \rho(t, \mathbf{x}), \\ \nabla \times \mathbf{B}(t, \mathbf{x}) = \mathbf{J}(t, \mathbf{x}) + \dfrac{\partial \mathbf{E}(t, \mathbf{x})}{\partial t}. \end{cases} \tag{1.67}$$

As shown in Eq. (1.21), the Maxwell's equations are invariant under the gauge transformation defined in Eq. (1.20), and this means that the electromagnetic theory is invariant also under the following transformation,

$$\partial_{\mu}A^{\mu}(x^{\lambda}) \mapsto \partial_{\mu}A^{\mu}(x^{\lambda}) + \partial_{\mu}\partial^{\mu}h\,(x^{\lambda}) \tag{1.68}$$

which implies that we have the freedom of choosing the value of $\partial_{\mu}A^{\mu}$ as we like. Therefore, we now adopt the **Lorenz gauge** and we simply set

$$\partial_{\mu}A^{\mu} = 0 \qquad \Longrightarrow \qquad \nabla \cdot \mathbf{A}(t, \mathbf{x}) = -\frac{\partial \phi(t\,\mathbf{x})}{\partial t}, \tag{1.69}$$

so that, by recalling the definition of the electric and magnetic fields in terms of $A^\mu$, Eqs. (1.67) yields

$$\nabla^2\phi(t,\mathbf{x}) - \frac{\partial^2}{\partial t^2}\phi(t,\mathbf{x}) = -\rho(t,\mathbf{x}), \tag{1.70}$$

$$\boldsymbol{\nabla}^2\mathbf{A}(t,\mathbf{x}) - \frac{\partial^2}{\partial t^2}\mathbf{A}(t,\mathbf{x}) = -\mathbf{J}(t,\mathbf{x}), \tag{1.71}$$

where we have exploited Eq. (1.25). Therefore, we have found that both the scalar $\phi$ and the vector potential $\mathbf{A}$ satisfy a wave equation. Thanks to this fact, we can easily write down the formal solutions of Eqs. (1.70)-(1.71) as

$$\phi(t,\mathbf{x}) = \int d^3\tilde{x}\,\frac{\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}, \qquad \mathbf{A}(t,\mathbf{x}) = \int d^3\tilde{x}\,\frac{\mathbf{J}(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}, \tag{1.72}$$

that identically solves the differential equation above. In fact, for instance, if we take the gradient of $\phi$,

$$\begin{aligned}
\boldsymbol{\nabla}\phi(t,\mathbf{x}) &= \int \frac{d^3\tilde{x}}{4\pi}\left[\frac{1}{|\mathbf{x}-\tilde{\mathbf{x}}|}\boldsymbol{\nabla}\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}}) + \rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\tilde{\mathbf{x}}|}\right)\right]\\
&= -\int \frac{d^3\tilde{x}}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}\left\{\left[\frac{\partial}{\partial t}\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})\right]\boldsymbol{\nabla}\left(|\mathbf{x}-\tilde{\mathbf{x}}|\right) + \rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})\frac{\hat{\mathbf{x}}-\hat{\tilde{\mathbf{x}}}}{|\mathbf{x}-\tilde{\mathbf{x}}|}\right\}\\
&= -\int \frac{d^3\tilde{x}}{4\pi}\frac{\mathbf{x}-\tilde{\mathbf{x}}}{|\mathbf{x}-\tilde{\mathbf{x}}|^3}\left[1+|\mathbf{x}-\tilde{\mathbf{x}}|\frac{\partial}{\partial t}\right]\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}}),
\end{aligned} \tag{1.73}$$

and we then take the divergence of such a result, we can find a suitable expression for $\nabla^2\phi$. In order, to do this, let us exploit a vector identity,

$$\boldsymbol{\nabla}\cdot(\lambda\mathbf{V}) = \mathbf{V}\cdot\boldsymbol{\nabla}\lambda + \lambda\boldsymbol{\nabla}\cdot\mathbf{V}, \tag{1.74}$$

which holds true for any scalar function $\lambda$ and vector function $\mathbf{V}$, so that we get

$$\begin{aligned}
\nabla^2\phi(t,\mathbf{x}) &= -\int \frac{d^3\tilde{x}}{4\pi}\left[\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{x}-\tilde{\mathbf{x}}}{|\mathbf{x}-\tilde{\mathbf{x}}|^3}\right)\right]\left[1+|\mathbf{x}-\tilde{\mathbf{x}}|\frac{\partial}{\partial t}\right]\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})\\
&\qquad -\int \frac{d^3\tilde{x}}{4\pi}\frac{\mathbf{x}-\tilde{\mathbf{x}}}{|\mathbf{x}-\tilde{\mathbf{x}}|^3}\cdot\boldsymbol{\nabla}\left[1+|\mathbf{x}-\tilde{\mathbf{x}}|\frac{\partial}{\partial t}\right]\rho(t-|\mathbf{x}-\tilde{\mathbf{x}}|,\tilde{\mathbf{x}})\\
&= -\rho(t,\mathbf{x}) + \frac{\partial^2}{\partial t^2}\phi(t,\mathbf{x}),
\end{aligned} \tag{1.75}$$

where we have used the following vector identity,

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{x}-\tilde{\mathbf{x}}}{|\mathbf{x}-\tilde{\mathbf{x}}|^3}\right)=-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\tilde{\mathbf{x}}|}\right)=-\nabla^2\left(\frac{1}{|\mathbf{x}-\tilde{\mathbf{x}}|}\right)=4\pi\delta^{(3)}\left(\mathbf{x}-\tilde{\mathbf{x}}\right). \tag{1.76}$$

Let us notice that Eq. (1.75) is exactly Eq. (1.70), and in the very same fashion it can be shown that the expression we provided for $\mathbf{A}$ in Eq. (1.71) identically solves Eq. (1.71). Yes, indeed we have found some formal solutions for the electromagnetic potentials, but now we want to compute them for the specific case of a moving electron of charge $e \simeq 1.602\times 10^{-19}$ C, whose trajectory is given as a function of time by $\mathbf{s}(t)$, at speed $\mathbf{v}(t)=\mathrm{d}\mathbf{s}(t)/\mathrm{d}t$, so that its charge density reads

$$\rho(t,\mathbf{x})=e\,\delta^{(3)}\left[\mathbf{x}-\mathbf{s}(t)\right] \tag{1.77}$$

Therefore, Eq. (1.72) yields

$$\phi(t,\mathbf{x})=e\int \mathrm{d}^3\tilde{x}\;\frac{\delta^{(3)}\left[\tilde{\mathbf{x}}-\mathbf{s}(t-|\mathbf{x}-\tilde{\mathbf{x}}|)\right]}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}, \tag{1.78}$$

$$\mathbf{A}(t,\mathbf{x})=e\int \mathrm{d}^3\tilde{x}\;\frac{\delta^{(3)}\left[\tilde{\mathbf{x}}-\mathbf{s}(t-|\mathbf{x}-\tilde{\mathbf{x}}|)\right]}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}\mathbf{v}(t-|\mathbf{x}-\tilde{\mathbf{x}}|). \tag{1.79}$$

After defining a new time variable $\tilde{t}$, we can rewrite the two potentials as

$$\phi(t,\mathbf{x})=e\int \mathrm{d}\tilde{t}\int \mathrm{d}^3\tilde{x}\;\frac{\delta^{(3)}\left[\tilde{\mathbf{x}}-\mathbf{s}(\tilde{t})\right]}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}\delta(\tilde{t}-t+|\mathbf{x}-\tilde{\mathbf{x}}|), \tag{1.80}$$

$$\mathbf{A}(t,\mathbf{x})=e\int \mathrm{d}\tilde{t}\int \mathrm{d}^3\tilde{x}\;\frac{\delta^{(3)}\left[\tilde{\mathbf{x}}-\mathbf{s}(\tilde{t})\right]}{4\pi|\mathbf{x}-\tilde{\mathbf{x}}|}\mathbf{v}(\tilde{t})\delta(\tilde{t}-t+|\mathbf{x}-\tilde{\mathbf{x}}|), \tag{1.81}$$

and, by integrating over $\tilde{\mathbf{x}}$, these expression further reduce to

$$\phi(t,\mathbf{x})=e\int \mathrm{d}\tilde{t}\,\frac{\delta[\tilde{t}-t+|\mathbf{x}-\mathbf{s}(\tilde{t})|]}{4\pi|\mathbf{x}-\mathbf{s}(\tilde{t})|},\qquad \mathbf{A}(t,\mathbf{x})=e\int \mathrm{d}\tilde{t}\,\mathbf{v}(\tilde{t})\frac{\delta[\tilde{t}-t+|\mathbf{x}-\mathbf{s}(\tilde{t})|]}{4\pi|\mathbf{x}-\mathbf{s}(\tilde{t})|}. \tag{1.82}$$

To evaluate the time integral, we can use the well known formula which allows for relating the Dirac delta of a function $F(x)$ to the Dirac deltas of the function's roots $F(x_i)=0$,

$$\delta\left[F(x)\right]=\sum_i\left[\frac{\mathrm{d}F(x)}{\mathrm{d}x}\Big|_{x=x_i}\right]^{-1}\delta(x-x_i), \tag{1.83}$$

so that, in our case, we have

$$
\begin{aligned}
\delta[\tilde{t} - t + |\mathbf{x} - \mathbf{s}(\tilde{t})|] &= \left\{ \frac{\partial}{\partial \tilde{t}} [\tilde{t} - t + |\mathbf{x} - \mathbf{s}(\tilde{t})|] \Big|_{\tilde{t} = t - |\mathbf{x} - \mathbf{s}(t)|} \right\}^{-1} \delta[\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|] \\
&= \frac{\delta[\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|]}{1 - [\hat{\mathbf{x}} - \hat{\mathbf{s}}(\tilde{t})] \cdot \mathbf{v}(\tilde{t})}.
\end{aligned}
\tag{1.84}
$$

Therefore, if substitute this result in the equations above, we find the **Liénard–Wiechert potentials**,

$$
\begin{aligned}
\phi(t, \mathbf{x}) &= \int \frac{\mathrm{d}\tilde{t}}{4\pi} \frac{e\, \delta[\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|]}{|\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})] \cdot \mathbf{v}(\tilde{t})}, \\
\mathbf{A}(t, \mathbf{x}) &= \int \frac{\mathrm{d}\tilde{t}}{4\pi} \frac{e\, \mathbf{v}(\tilde{t})\, \delta[\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|]}{|\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})] \cdot \mathbf{v}(\tilde{t})}.
\end{aligned}
\tag{1.85}
$$

We are now in the position to evaluate the electric field of a point electron in arbitrary motion, since the only thing we have to do is to substitute the Liénard–Wiechert potentials in the electric field's definition,

$$
\mathbf{E}(t, \mathbf{x}) = -\boldsymbol{\nabla}\phi(t, \mathbf{x}) - \frac{\partial \mathbf{A}(t, \mathbf{x})}{\partial t}. \tag{1.86}
$$

By defining $t_\star \equiv t - |\mathbf{x} - \mathbf{s}(t)|$, the gradient of the scalar potential can be expressed as

$$
\boldsymbol{\nabla}\phi(t, \mathbf{x}) = -\frac{e}{4\pi} \frac{\boldsymbol{\nabla}\left[\, |\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)] \cdot \mathbf{v}(t_\star) \right]}{\left\{ |\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)] \cdot \mathbf{v}(t_\star) \right\}^2}. \tag{1.87}
$$

The first contribution comes from the following gradient

$$
\boldsymbol{\nabla}\left[\, |\mathbf{x} - \mathbf{s}(t_\star)| \,\right] = \boldsymbol{\nabla}\left(t - t_\star\right) = -\boldsymbol{\nabla} t_\star, \tag{1.88}
$$

whereas the second one can be worked out by exploiting the following vector identity,

$$
\boldsymbol{\nabla}\left\{ [\mathbf{x} - \mathbf{s}(t_\star)] \cdot \mathbf{v}(t_\star) \right\} = \mathbf{v}(t_\star) + \Big\{ [\mathbf{x} - \mathbf{s}(t_\star)] \cdot \mathbf{a}(t_\star) - v^2(t_\star) \Big\}\, \boldsymbol{\nabla} t_\star \tag{1.89}
$$

where we have recognized $\mathbf{a}(t) \equiv \mathrm{d}\mathbf{v}(t)/\mathrm{d}t$ as the electron's acceleration, and we have used the following vector identities:

$$\boldsymbol{\nabla}(\mathbf{U}\cdot\mathbf{V}) = (\mathbf{U}\cdot\boldsymbol{\nabla})\,\mathbf{V} + (\mathbf{V}\cdot\boldsymbol{\nabla})\,\mathbf{U} + \mathbf{U}\times(\boldsymbol{\nabla}\times\mathbf{V}) + \mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{U}) \tag{1.90}$$

$$\mathbf{U}\times(\mathbf{V}\times\mathbf{W}) = \mathbf{V}\,(\mathbf{U}\cdot\mathbf{W}) - \mathbf{W}\cdot(\mathbf{V}\cdot\mathbf{V})\,, \tag{1.91}$$

that hold true for any set of vector functions $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$. To complete the calculation we need to know $\boldsymbol{\nabla} t_\star$:

$$\begin{aligned}\boldsymbol{\nabla} t_\star &= \{[\hat{\mathbf{x}} - \hat{\mathbf{s}}(t_\star)]\cdot\boldsymbol{\nabla}\}\,[\mathbf{x} - \mathbf{s}(t_\star)] + [\hat{\mathbf{x}} - \hat{\mathbf{s}}(t_\star)]\times\{\boldsymbol{\nabla}\times[\mathbf{x} - \mathbf{s}(t_\star)]\}\\ &= [\hat{\mathbf{x}} - \hat{\mathbf{s}}(t_\star)] - \{[\hat{\mathbf{x}} - \hat{\mathbf{s}}(t_\star)]\cdot\mathbf{v}(t_\star)\}\,\boldsymbol{\nabla} t_\star,\end{aligned} \tag{1.92}$$

so that, by inverting such a result, we finally find

$$\begin{aligned}\boldsymbol{\nabla}\phi(t,\mathbf{x}) = \frac{e}{4\pi}&\frac{\mathbf{v}(t_\star)}{\{\,|\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{v}(t_\star)\,\}^2}\\ &- \frac{e}{4\pi}\frac{1 - v^2(t_\star) + [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{a}(t_\star)}{\{\,|\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{v}(t_\star)\,\}^3}\,[\mathbf{x} - \mathbf{s}(t_\star)]\,,\end{aligned} \tag{1.93}$$

and, in the very same fashion, it is easy to show that

$$\begin{aligned}\frac{\partial\mathbf{A}(t,\mathbf{x})}{\partial t} = \frac{e}{4\pi}&\frac{|\mathbf{x} - \mathbf{s}(t_\star)|\,\mathbf{a}(t_\star) - \mathbf{v}(t_\star)}{\{\,|\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{v}(t_\star)\,\}^2}\\ &+ \frac{e}{4\pi}\frac{|\mathbf{x} - \mathbf{s}(t_\star)|\,\{1 - v^2(t_\star) + [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{a}(t_\star)\}\,\mathbf{v}(t_\star)}{\{\,|\mathbf{x} - \mathbf{s}(t_\star)| - [\mathbf{x} - \mathbf{s}(t_\star)]\cdot\mathbf{v}(t_\star)\,\}^3}\end{aligned} \tag{1.94}$$

Therefore, by recalling Eq. (1.86), the electric field sourced by a moving electron reads

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) = e\int\frac{\mathrm{d}\tilde{t}}{4\pi}&\frac{[1 - v^2(\tilde{t})]\,[\mathbf{x} - \mathbf{s}(\tilde{t}) - \mathbf{v}(\tilde{t})]}{\{\,|\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})]\cdot\mathbf{v}(\tilde{t})\,\}^3}\delta[\,\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|\,]\\ &+ e\int\frac{\mathrm{d}\tilde{t}}{4\pi}\frac{[\mathbf{x} - \mathbf{s}(\tilde{t})]\times\{[\mathbf{x} - \mathbf{s}(\tilde{t}) - \mathbf{v}(\tilde{t})]\times\mathbf{a}(\tilde{t})\}}{\{\,|\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})]\cdot\mathbf{v}(\tilde{t})\,\}^3}\delta[\,\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|\,].\end{aligned} \tag{1.95}$$

## 1.2 From Flat to Curved Spacetime

Up to now, we have reviewed how classical electromagnetism is described in the context of a flat spacetime, i.e. when the metric tensor $\eta_{\mu\nu}$ is just the Minkowski's one. This is correct when assuming that special relativity is a theory general enough to describe the Universe. However, SR is limited to inertial frames and does not account for accelerated motion or gravity. For instance, it cannot explain how gravity affects the trajectory of objects or how massive bodies like planets, stars, and galaxies interact with each other. Since in this thesis we are going to discuss cosmological phenomena, it is clear that we have to move to **general relativity** (GR), which is a more comprehensive theory that includes special relativity as a special case (see Ref. [126]). The content of this section is a partial review of the standard notions that can be found in many famous textbooks, such as Refs. [127–130]).

### 1.2.1 Classical Field Theory in General Relativity

Although we are not going to completely review GR, let us just mention that it is a theory of gravity and spacetime, describing how mass and energy curve the fabric of spacetime and how objects move in response to this curvature. This is the reason for that GR provides a unified framework for understanding both inertial and accelerated motion, as well as the effects of gravity on the behavior of celestial bodies. In this thesis, we are moving from SR to GR just now, simply because even if the conceptual foundations of these two frameworks are so different, the formalism describing a field theory (such as electromagnetism) in SR spacetime is extremely similar to that adopted in GR. In fact, the Einstein's principle of (special) relativity is replaced by a principle of general invariance, which states that all laws of physics must be invariant under **general coordinate transformations**. However, it is clear that such a GR principle demands that the laws have a symmetry that goes beyond the simple invariance under Lorentz transformations. In fact, Lorentz transformations are "just" a set of equations that describe how coordinates and physical quantities change when transitioning between two inertial reference frames in special relativity, whereas general coordinate transformations are mathematical tools used in GR to express the same physical laws in different coordinate systems, taking into account the curved nature of spacetime due to mass and energy. Indeed, the key-point of general relativity is that gravitation is a manifestation of geometry, and this fact is well described by the Einstein's field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{T_{\mu\nu}}{m_{Pl}^2}, \tag{1.96}$$

where $m_{Pl} \simeq 2.4353 \times 10^{27}$ eV is the **reduced Planck mass**, $\Lambda$ is the **cosmological constant**, $g_{\mu\nu}$ is the metric tensor[6], $R \equiv g_{\mu\nu} R^{\mu\nu}$ is the **Ricci scalar**, whereas

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\alpha\nu} \tag{1.97}$$

is the **Ricci tensor**, with $\Gamma^\lambda_{\alpha\beta}$ being the **Christoffel symbols**:

$$\Gamma^\lambda_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\kappa} \left( \partial_\alpha g_{\kappa\beta} + \partial_\beta g_{\alpha\kappa} - \partial_\kappa g_{\alpha\beta} \right) . \tag{1.98}$$

When the metric tensor is just the Minkowski one, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$, it is straightforward to see that the Christoffel symbols vanish, and so also all the left-hand side of Eq. (1.96), implying that also the right-hand side must vanish to satisfy the equation. Indeed, $\mathcal{T}_{\mu\nu}$ is nothing but the **energy-momentum tensor**, i.e. a quantity encoding all the information about masses and energy, and that plays here the role of a source term in the field equation. Therefore, in absence of sources, a solution of Eq. (1.96) is exactly the Minkowski tensor, which defines a **flat spacetime** geometry and we recover SR, whereas in more general situations the solution of the Einstein's field equations is a metric tensor defining a **curved spacetime** geometry. Hence, we have understood that the presence of a source curves the spacetime geometry, but another crucial point of GR is that the metric tensor $g_{\mu\nu}$ is also the mathematical object describing what we call **gravitational field**, which according to GR, is not described by a separate force or field like in classical physics. Instead, it is represented by the curvature of spacetime itself. Indeed, massive objects, such as stars, planets, and galaxies, curve the spacetime around them, and this curvature affects the motion of other objects in their vicinity. As well expressed by Eq. (1.96), the metric tensor is a mathematical tool that quantifies this curvature. As already said, when a mass or energy is present, it causes spacetime to curve around it. The metric tensor defines the curvature of spacetime at every point and tells us how distances and time intervals are affected in the presence of gravitational fields. The motion of objects, including light rays, follows the curved paths determined by this curvature. Now a question naturally arises: how can we generalize the formulas describing electromagnetism we derived in Sec. 1.1 to the case of a curved spacetime? Intriguing enough we have just to recast our expression in order to satisfy the GR principle for that laws of physics must be expressed in a generally covariant form. However, we have already provided a covariant formulation of electromagnetism in Sec. 1.1.1, so that we have simply to replace the covariance under Lorentz transformation with that one under general coordinate transformations. This can be achieved by taking our theory valid in SR

[6] In the context of GR, it is customary to define the metric tensor as $g_{\mu\nu}$, in order to differentiate it from the Minkowski one $\eta_{\mu\nu}$, which is able to describe geometry only in SR. In fact, $g_{\mu\nu}$ could in general be non-diagonal and can vary in space and time, so that usually $g_{\mu\nu} \neq g^{\mu\nu}$.

and making the following replacements:

$$\eta_{\mu\nu} \mapsto g_{\mu\nu}, \qquad \mathrm{d}^4x \mapsto \mathrm{d}^4x\,\sqrt{-g}, \qquad \partial_\mu \mapsto \nabla_\mu, \tag{1.99}$$

where $g$ is the determinant of the metric tensor and

$$\begin{aligned}\nabla_\sigma M^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \equiv \partial_\sigma M^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} M^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} M^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ - \Gamma^{\lambda}_{\sigma\nu_1} M^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} M^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots,\end{aligned} \tag{1.100}$$

is the **covariant derivative** of a given tensor, which is defined in order to vanish when applied to the metric tensor and reduce to an ordinary 4-gradient when applied to a scalar field. Thanks to the recipe of Eq. (1.99), we can now see how the Maxwell's equations change in a GR context. The Maxwell's field tensor is not affected by the presence of the covariance derivatives, as it can be seen by substituting the definition of Eq. (1.100),

$$F_{\mu\nu} = \nabla_\mu A_\mu - \nabla_\nu A_\mu = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \left(\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu}\right) A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.101}$$

where the Christoffel symbols cancel out because they are symmetric in their lower indices, as can be deduced by inspecting Eq. (1.98). As a consequence, the property of gauge-invariance we pointed out in Eq. (1.21) holds true also in curved spacetime. Moreover, the equation of motion involving the Maxwell dual tensor in Eq. (1.27),

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \tag{1.102}$$

is also preserved[7],

$$\begin{aligned}\frac{1}{2}\nabla_\mu \varepsilon^{\mu\nu\varrho\sigma} \frac{F_{\varrho\sigma}}{\sqrt{-g}} &= \frac{1}{2}\frac{\varepsilon^{\mu\nu\varrho\sigma}}{\sqrt{-g}} \left(\nabla_\mu F_{\varrho\sigma} + \nabla_\varrho F_{\sigma\mu} + \nabla_\sigma F_{\mu\varrho}\right) \\ &= \frac{1}{2}\frac{\varepsilon^{\mu\nu\varrho\sigma}}{\sqrt{-g}} \left(\left[\nabla_\mu, \nabla_\varrho\right] A_\sigma + \left[\nabla_\varrho, \nabla_\sigma\right] A_\mu + \left[\nabla_\sigma, \nabla_\mu\right] A_\varrho\right) \\ &= -\frac{1}{2}\frac{\varepsilon^{\mu\nu\varrho\sigma}}{\sqrt{-g}} \left(R^{\nu}{}_{\sigma\mu\varrho} + R^{\nu}{}_{\mu\varrho\sigma} + R^{\nu}{}_{\varrho\sigma\mu}\right) A_\nu = 0,\end{aligned} \tag{1.103}$$

[7] Let us recall that the Levi-Civita symbol is not a tensor but a number, and hence its covariant derivative, as well its ordinary one, simply vanishes.

where $R_{\mu\nu\varrho\sigma}$ is the **Riemann tensor**,

$$R^{\mu}_{\ \nu\varrho\sigma} \equiv \partial_\varrho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\varrho\nu} + \Gamma^{\mu}_{\varrho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\varrho\nu} \qquad \Longrightarrow \qquad R^{\mu}_{\ \nu\mu\sigma} = R_{\nu\sigma}, \tag{1.104}$$

which obeys the **first Bianchi identity**,

$$R_{\mu\nu\varrho\sigma} + R_{\mu\varrho\sigma\nu} + R_{\mu\sigma\nu\varrho} = 0, \tag{1.105}$$

with $R_{\mu\nu\varrho\sigma} = g_{\lambda\mu} R^{\lambda}_{\ \nu\varrho\sigma}$, and it is related to the **commutator** of covariant derivatives applied on a tensor:

$$\begin{aligned} \left[\nabla_\varrho, \nabla_\sigma\right] M^{\mu_1\cdots\mu_k}_{\nu_1\ldots\nu_l} &= \left[\nabla_\varrho\nabla_\sigma - \nabla_\sigma\nabla_\varrho\right] M^{\mu_1\cdots\mu_k}_{\nu_1\ldots\nu_l} \\ &= R^{\mu_1}_{\ \lambda\varrho\sigma} M^{\lambda\mu_2\cdots\mu_k}_{\nu_1\ldots\nu_l} + R^{\mu_2}_{\ \lambda\varrho\sigma} M^{\mu_1\lambda\ldots\mu_k}_{\nu_1\ldots\nu_l} - R^{\lambda}_{\ \nu_1\varrho\sigma} M^{\mu_1\cdots\mu_k}_{\lambda\nu_2\ldots\nu_l} - R^{\lambda}_{\ \nu_2\varrho\sigma} M^{\mu_1\cdots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots . \end{aligned} \tag{1.106}$$

Let us notice that in proving the validity of Eq. (1.103) we have used that the covariant derivative of $\sqrt{-g}$ gives no contribution, since

$$\begin{aligned} \nabla_\varrho \sqrt{-g} &= -\frac{1}{2\sqrt{-g}} \nabla_\varrho g = -\frac{1}{2\sqrt{-g}} \nabla_\varrho \exp\left[g^{\mu\nu} \ln\left(g^{\mu}_{\ \nu}\right)\right] \\ &= \frac{1}{2}\sqrt{-g}\,\nabla_\varrho \left[g^{\mu}_{\ \nu} \ln\left(g_{\mu\nu}\right)\right] = \frac{1}{2}\sqrt{-g}\, g^{\mu}_{\ \nu} g^{\lambda\nu} \nabla_\varrho g_{\mu\lambda} \\ &= \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \nabla_\varrho g_{\mu\nu} = 0, \end{aligned} \tag{1.107}$$

where we have used the relation between determinant of a matrix $M$ and its trace, i.e.

$$|M| = \exp\left\{\mathrm{Tr}\left[\ln\left(M\right)\right]\right\}. \tag{1.108}$$

Analogously, we are now going to show that also the other Maxwell's equations remain unchanged moving from SR to GR. Indeed, by applying the rules collected in Eq. (1.99) to the Hamilton's principle, we find that Eq. (1.15) for the Maxwell's Lagrangian density becomes

$$0 = -\int \mathrm{d}^4\tilde{x}\, \sqrt{-g(\tilde{x}^\lambda)}\, \delta^{\mu}_{\sigma}\, \delta^{(4)}\left(\tilde{x}^\lambda - x^\lambda\right) \nabla_\varrho \left\{\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left[\nabla_\varrho A_\sigma(\tilde{x}^\lambda)\right]}\right\} = g^{\varrho\alpha} g^{\mu\beta} \nabla_\varrho \left(\sqrt{-g}\, F_{\alpha\beta}\right). \tag{1.109}$$

However, as shown in Eq. (1.101) the Maxwell's field tensor is the same both in flat and curved spacetime, so that Eq. (1.109) simply reduces to

$$\boxed{\nabla_\mu F^{\mu\nu} = 0,} \tag{1.110}$$

as expected. Similarly, also Eq. (1.96), which is the equation of motion for the gravitational field $g_{\mu\nu}$, can be derived with a Lagrangian approach. Indeed, the field theory describing Einstein's relativistic gravity is defined by the **Einstein-Hilbert action**:

$$S_{\rm EH} \equiv \frac{m_{Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda)\,, \tag{1.111}$$

and if we take the functional derivative of $S_{\rm EH}$ with respect to $g^{\mu\nu}$, we get[8]

$$\begin{aligned} \frac{\delta S_{\rm EH}}{\delta g^{\mu\nu}} &= \frac{m_{Pl}^2}{2} \int \mathrm{d}^4\tilde{x} \left\{ \left[ (R - 2\Lambda) \frac{\partial \sqrt{-g}}{\partial g^{\varrho\sigma}} + \sqrt{-g} R_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial g^{\varrho\sigma}} + \sqrt{-g}\, g^{\alpha\beta} \frac{\partial R_{\alpha\beta}}{\partial g^{\varrho\sigma}} \right] \frac{\delta g^{\varrho\sigma}(\tilde{x}^\lambda)}{\delta g^{\mu\nu}(x^\lambda)} \right\} \\ &= \frac{m_{Pl}^2}{2} \int \mathrm{d}^4\tilde{x} \sqrt{-g} \left[ R_{\varrho\sigma} - \frac{1}{2} g_{\varrho\sigma} R + g_{\varrho\sigma}\Lambda + g^{\alpha\beta} \frac{\partial R_{\alpha\beta}}{\partial g^{\varrho\sigma}} \right] \delta^{(4)}(\tilde{x}^\lambda - x^\lambda)\, \delta^\varrho_\mu \delta^\sigma_\nu, \end{aligned} \tag{1.112}$$

where we have used that, since

$$\frac{\partial \delta^\alpha_\beta}{\partial g^{\varrho\sigma}} = 0 \quad \Longrightarrow \quad g^{\alpha\lambda} \frac{\partial g_{\lambda\beta}}{\partial g_{\varrho\sigma}} = -g_{\beta\lambda} \frac{\partial g^{\lambda\alpha}}{\partial g^{\varrho\sigma}} \quad \Longrightarrow \quad g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial g^{\varrho\sigma}} = -g_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial g^{\varrho\sigma}}, \tag{1.113}$$

we can partially inherit the result of Eq. (1.107), so that we get

$$\frac{\partial \sqrt{-g}}{\partial g^{\varrho\sigma}} = \frac{1}{2} \sqrt{-g}\, g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial g^{\varrho\sigma}} = -\frac{1}{2} \sqrt{-g}\, g_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial g^{\varrho\sigma}}. \tag{1.114}$$

Moreover, we can easily work out the functional derivative of the Ricci tensor as

$$\begin{aligned} \frac{\partial R_{\alpha\beta}}{\partial g^{\varrho\sigma}} &= \partial_\mu \frac{\partial \Gamma^\mu_{\alpha\beta}}{\partial g^{\varrho\sigma}} - \partial_\beta \frac{\partial \Gamma^\mu_{\alpha\mu}}{\partial g^{\varrho\sigma}} + \Gamma^\nu_{\alpha\beta} \frac{\partial \Gamma^\mu_{\mu\nu}}{\partial g^{\varrho\sigma}} + \Gamma^\mu_{\mu\nu} \frac{\partial \Gamma^\nu_{\alpha\beta}}{\partial g^{\varrho\sigma}} - \Gamma^\nu_{\mu\beta} \frac{\partial \Gamma^\mu_{\alpha\nu}}{\partial g^{\rho\sigma}} - \Gamma^\mu_{\alpha\nu} \frac{\partial \Gamma^\nu_{\mu\beta}}{\partial g^{\varrho\sigma}} \\ &= \nabla_\mu \left( \frac{\partial \Gamma^\mu_{\alpha\beta}}{\partial g^{\varrho\sigma}} \right) - \nabla_\beta \left( \frac{\partial \Gamma^\mu_{\alpha\mu}}{\partial g^{\varrho\sigma}} \right), \end{aligned} \tag{1.115}$$

[8] Of course, there is no term depending on $\nabla_\varrho g^{\mu\nu} = 0$.

where we have used that the functional derivative between two Christoffel symbols is a well defined tensor in GR. Hence, by substituting Eq. (1.115) in Eq. (1.112)

$$\begin{aligned}\frac{\delta S_{\rm EH}}{\partial g^{\mu\nu}} &= \frac{m_{Pl}^2}{2}\sqrt{-g}\left\{\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda\right)+g^{\alpha\beta}\left[\nabla_\lambda\left(\frac{\partial\Gamma^\lambda_{\alpha\beta}}{\partial g^{\mu\nu}}\right)-\nabla_\beta\left(\frac{\partial\Gamma^\lambda_{\alpha\lambda}}{\partial g^{\mu\nu}}\right)\right]\right\}\\ &= \frac{m_{Pl}^2}{2}\sqrt{-g}\left\{\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda\right)+\nabla_\lambda\left[g^{\alpha\beta}\left(\frac{\partial\Gamma^\lambda_{\alpha\beta}}{\partial g^{\mu\nu}}\right)-g^{\alpha\lambda}\left(\frac{\partial\Gamma^\beta_{\alpha\beta}}{\partial g^{\mu\nu}}\right)\right]\right\}.\end{aligned} \tag{1.116}$$

Thanks to Eq. (1.113), the functional derivative of the Christoffel symbols can be expressed as

$$\begin{aligned}\frac{\partial\Gamma^\lambda_{\alpha\beta}}{\partial g^{\mu\nu}} = \frac{1}{2}\Bigg\{ & g^{\lambda\kappa}\left[\partial_\alpha\left(\frac{\partial g_{\kappa\beta}}{\partial g^{\mu\nu}}\right)+\partial_\beta\left(\frac{\partial g_{\kappa\alpha}}{\partial g^{\mu\nu}}\right)-\partial_\kappa\left(\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}\right)\right]\\ & -g^{\lambda\varrho}g^{\kappa\sigma}\frac{\partial g_{\varrho\sigma}}{\partial g^{\mu\nu}}\left(\partial_\alpha g_{\kappa\beta}+\partial_\beta g_{\kappa\alpha}-\partial_\kappa g_{\alpha\beta}\right)\Bigg\},\end{aligned} \tag{1.117}$$

which simply reduces to

$$\begin{aligned}\frac{\partial\Gamma^\lambda_{\alpha\beta}}{\partial g^{\mu\nu}} &= \frac{1}{2}g^{\lambda\kappa}\left[\partial_\alpha\left(\frac{\partial g_{\kappa\beta}}{\partial g^{\mu\nu}}\right)+\partial_\beta\left(\frac{\partial g_{\kappa\alpha}}{\partial g^{\mu\nu}}\right)-\partial_\kappa\left(\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}\right)-2\Gamma^\sigma_{\alpha\beta}\frac{\partial g_{\kappa\sigma}}{\partial g^{\mu\nu}}\right]\\ &= \frac{1}{2}\left[\nabla_\alpha\left(g^{\lambda\kappa}\frac{\partial g_{\kappa\beta}}{\partial g^{\mu\nu}}\right)+\nabla_\beta\left(g^{\lambda\kappa}\frac{\partial g_{\kappa\alpha}}{\partial g^{\mu\nu}}\right)-\nabla_\kappa\left(g^{\lambda\kappa}\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}\right)\right]\\ &= \frac{1}{2}\left[g^{\lambda\kappa}\nabla_\kappa\left(g_{\alpha\nu}g_{\beta\mu}\right)-\delta^\lambda_\nu\nabla_\alpha g_{\beta\mu}-\delta^\lambda_\nu\nabla_\beta g_{\alpha\mu}\right],\\ &= 0,\end{aligned} \tag{1.118}$$

and similarly

$$\frac{\partial\Gamma^\beta_{\alpha\beta}}{\partial g^{\mu\nu}} = \frac{1}{2}\left(\nabla_\mu g_{\alpha\nu}-\nabla_\alpha g_{\mu\nu}-\nabla_\nu g_{\alpha\mu}\right)=0, \tag{1.119}$$

so that by setting Eq. (1.116) equal to zero we finally obtain

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda=0 \tag{1.120}$$

which is Eq. (1.96) in absence of sources. Indeed to have a non-vanishing energy-momentum tensor we should include an extra contribution in the Einstein-Hilbert action, representing the source term due to the presence of other fields in the Universe with respect to the gravitational one:

$$S_{\text{GR}} = S_{\text{EH}} + S_{\text{fields}}. \tag{1.121}$$

Therefore, by repeating all the procedure with this addition, we find the following equation of motion

$$\frac{m_{Pl}^2}{2}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\right) + \frac{\delta S_{\text{fields}}}{\delta g^{\mu\nu}} = 0, \tag{1.122}$$

which means that the energy-momentum tensor can be defined as

$$\boxed{\mathcal{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{fields}}}{\delta g^{\mu\nu}}.} \tag{1.123}$$

Let us notice that the law of energy-momentum conservation in curved spacetime is guaranteed by the **second Bianchi identity**:

$$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0 \qquad \Longrightarrow \qquad \nabla_\mu T^{\mu\nu} = 0. \tag{1.124}$$

In fact, the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ is defined as the flux of four-momentum $p^\mu$ across a surface of constant $x^\nu$. According to this definition, it is possible e.g. to identify $\mathcal{T}^0_{\ 0} = -\varrho$, with $\varrho$ being the **energy density**, whereas $\mathcal{T}^i_{\ j}$ as the **pressure** $\mathcal{P}$ when $i = j$, since such elements represents the $i$-component of the force being exerted (per unit area) in the $j$-direction.

### 1.2.2 The Geodesic Equation

There is another fundamental equation in GR together with Eq. (1.96), which is that characterizing the motion of a particle in curved spacetime, e.g. a **photon**. Now, a ray of light can be thought as a path along which photons travel, and according to the **Fermat's principle**, the path taken by a ray between two given points $A$ and $B$ is the path that can be traveled in the least time. This generalizes in general relativity by assuming that the length of the path connecting two spacetime points $A$ and $B$,

$$\boxed{ds \equiv \sqrt{g_{\mu\nu}\,dx^\mu dx^\nu}} \qquad \Longrightarrow \qquad s_{AB} = \int_A^B \sqrt{g_{\mu\nu}\,dx^\mu dx^\nu}, \tag{1.125}$$

is stationary under infinitesimal variation of $x^\mu \mapsto x^\mu + \delta x^\mu$, i.e.

$$\delta s_{AB} = \delta \int_A^B \mathrm{d}s = \delta \int_A^B \sqrt{g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu}, \tag{1.126}$$

with $\delta x_A^\mu = \delta x_B^\mu = 0$. To describe the path of the particle as it moves along a curve in spacetime we use $s$ itself, i.e. $x^\mu = x^\mu(s)$, so that we have

$$\begin{aligned}
\delta \int_A^B \sqrt{g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu} &= \int_A^B \frac{1}{\sqrt{g_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta}} \left[g_{\mu\nu}\, \mathrm{d}x^\mu \delta x^\nu + \frac{1}{2} \mathrm{d}x^\mu \mathrm{d}x^\nu \delta x^\lambda \partial_\lambda g_{\mu\nu}\right] \\
&= \int_A^B \mathrm{d}s \left[g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}\delta x^\nu}{\mathrm{d}s} + \frac{1}{2} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \delta x^\lambda \partial_\lambda g_{\mu\nu}\right] \\
&= -\int_A^B \mathrm{d}s \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\mu\lambda} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) - \frac{1}{2} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \partial_\lambda g_{\mu\nu}\right] \delta x^\lambda \\
&= -\int_A^B \mathrm{d}s \left[g_{\mu\lambda} \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \partial_\nu g_{\mu\lambda} - \frac{1}{2} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \partial_\lambda g_{\mu\nu}\right] \delta x^\lambda \\
&= -\int_A^B \mathrm{d}s \left[g_{\mu\lambda} \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \frac{1}{2} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right)\right] \delta x^\lambda.
\end{aligned} \tag{1.127}$$

Since $\delta x^\lambda$ is arbitrary, after imposing the condition $\delta \int \mathrm{d}s = 0$ we finally get the **geodesic equation**,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0. \tag{1.128}$$

By direct inspection of Eq. (1.128), we can see that if on a given point the spacetime is locally flat, then locally in that point $g_{\mu\nu} = \eta_{\mu\nu}$, making the Christoffel symbols vanishing, so that the geodesic equation is simply a straight line,

$$\frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}s^2} = 0. \tag{1.129}$$

The coordinates $x'^\mu$ that locally make the Christoffel symbols zero are called **local inertial coordinates** (at that point), because the geodesic equation in these coordinates implies motion at constant velocity, like the inertial motion of a free particle in flat spacetime. Strictly speaking, geometrically the local inertial coordinates correspond to replacing the curved space by a small flat patch tangent to the former at that point. Moreover, it is possible to introduce also new coordinates, called **Fermi coordinates**, such that the Christoffel symbols are zero at every point on a given geodesic curve. Such coordinates are built by defining coordinate axes at the initial point where the Christoffel symbols were zero. The $x'^0$-axis is assumed to lie along the geodesic, whereas the other three spacetime axes lie in the spatial slice within spacetime of constant $x'^0$ for the given initial value of $x'^0$. Then, the unit vectors associated with the coordinate axes are then carried forward along the geodesic by a procedure called **parallel transport**, which allows for transporting the vectors along the curved path while keeping their orientation unchanged with respect to the local geometry, compensating for the curvature of spacetime, so that it is possible to define the coordinate axes at any later point on the geodesic. The set of four unit vectors needed to define the directions of the local coordinate axes are called a **tetrad**. Since these local coordinate axes are constructed by parallel transport, it is obvious that whenever we parallel-transport some vector along the geodesic, its components cannot change, and this immediately implies that all the Christoffel symbols involved in parallel transport along the geodesic must be zero on the geodesic. Therefore, physically speaking, the Fermi coordinates along a geodesic represent a freely falling reference frame, whose spatial orientation is defined by parallel transport. One of the reason for that we have spent some lines talking about tetrads is that the Stokes parameters we introduced in Sec. 1.1.3 are well defined in Minkowski spacetime, but in general relativity, these definitions should be generalized, and this can be done by using the tetrad formalism. As said, a tetrad is a set of four orthogonal unit basis vectors $e^{\mu}_{(a)}$, with $a = \overline{0,3}$, and at each point we can attach a tetrad which transforms between the coordinate frame and the **local inertial frame** (LIF) at that point: for a vector field $A^{\mu}(x)$, its components in the local inertial frame are then given as

$$A_a|_{\text{LIF}} = e^{\mu}_{(a)} A_{\mu}. \tag{1.130}$$

The Latin indices are lowered and raised by the Minkowski metric $\eta^{ab}$, the Greek indices instead by the coordinate metric $g^{\mu\nu}$. The tetrad has the following properties:

$$g_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)} = \eta_{ab}, \qquad\qquad \eta^{ab} e^{\mu}_{(a)} e^{\nu}_{(b)} = g^{\mu\nu}. \tag{1.131}$$

We will see in the Sec. 2.1.2 how it is possible to exploit this formalism in order to obtain suitable expressions for the Stokes parameters in curved spacetime. By the way, let us notice that we have derived the geodesic equation by implicitly considering a **test particle**, i.e. a body which does not itself influence the geometry through which it moves, and we have found Eq. (1.128) by parametrizing the particle's path with the spacetime interval itself $s$. Of course, it is clear that the geodesic equation is left invariant by the following transformation:

$$s \mapsto \lambda = As + B, \tag{1.132}$$

for some constants $A$ and $B$. Any parameter related to the spacetime interval in this way is said to be an **affine parameter**, and is just as good as the spacetime interval for parametrizing a geodesic. In particular, it is often convenient to choose the normalization of the affine parameter $\lambda$ such that

$$\boxed{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \equiv p^2 - \mathcal{E}^2 = -m^2,} \tag{1.133}$$

where $p^2 = \delta^{ij}p_ip_j$ is the square modulus of the physical **three-momentum** of the test particle, $\mathcal{E}$ is its **energy**, and $m$ is its **mass**. In fact, by reordering the terms in Eq. (1.133) we recover the **dispersion relation**:

$$\mathcal{E}(p) = \sqrt{m^2 + p^2}. \tag{1.134}$$

This can be done because the coefficients $A$ and $B$ in Eq. (1.132) are not necessarily dimensionless, but at the contrary their normalization can be fixed by the choice we are freely making in Eq. (1.133). Now, let us try to understand how we can determine actual physical quantities we are able to measure in cosmological surveys, in terms of the more abstract GR tensor. For instance, let us try to determine the energy of a test particle. This can be done by considering a particle which remains fixed in one and the same point in space at two different times, so that we can set $\mathrm{d}x^i/\mathrm{d}\lambda = 0$ and find

$$\boxed{\mathcal{E} = \sqrt{-g_{00}}\,\frac{\mathrm{d}x^0}{\mathrm{d}\lambda},} \tag{1.135}$$

since the particle is not moving and hence it has no three-momentum. Indeed, Eq. (1.135) provides a relation which determines the actual energy of a test particle for a parameterized change of the coordinate $x^0$ Similarly, we now determine the physical **three-momentum** by considering this time a moving particle, so that Eq. (1.133) reads

$$g_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\right)^2 + 2g_{0i}\frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + m^2 = 0, \tag{1.136}$$

which is solved with respect to $\mathrm{d}x^0/\mathrm{d}\lambda$ by two roots:

$$\sqrt{-g_{00}}\,\frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\bigg|_\pm = \frac{g_{0i}}{\sqrt{-g_{00}}}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \pm \sqrt{m^2\left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}, \tag{1.137}$$

corresponding to the particle's propagation in the two directions between a point with spatial coordinates $x^i$ and a point with spatial coordinates $x^i + \mathrm{d}x^i$, as shown in Fig. 1.2. In fact, if $x^0$ is the moment of arrival of

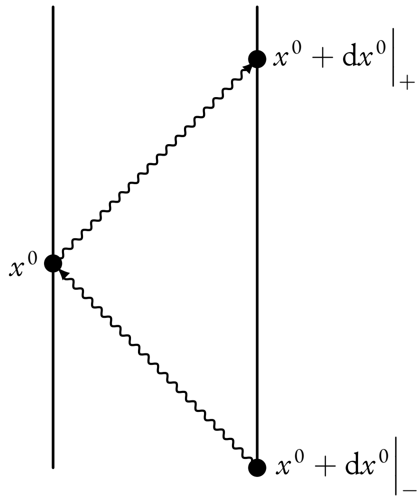


**Figure 1.2:** A particle directed from some point in space with coordinates $x^i + \mathrm{d}x^i$ to a point having coordinated $x^i$ infinitely near to it and then back over the same path.

the particle at the point with coordinates $x^i$, the times when it left the point with coordinates $x^i + \mathrm{d}x^i$ and when it will come back to the same point are, respectively, $\mathrm{d}x^0 + \mathrm{d}x^0\big|_-$ and $\mathrm{d}x^0 + \mathrm{d}x^0\big|_+$, so that, by recalling Eq. (1.135), we can infer that the total energy of the photon due to the motion between its departure and its return to the original point will be given by the following semi-difference

$$\mathcal{E} = \frac{\sqrt{-g_{00}}}{2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\bigg|_+ - \frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\bigg|_-\right) = \sqrt{m^2 + \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}, \tag{1.138}$$

so that, by recalling Eq. (1.134), we finally get

$$\boxed{p^2 = \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}.} \tag{1.139}$$

Armed with the result of Eq. (1.135) and Eq. (1.139), in the next sections, we will be able to characterize the dynamics of the particles populating the Universe.

## 1.3 The Cosmic Microwave Background Radiation

In the previous sections we have reviewed some fundamental notions of electromagnetism and general relativity. This has been done because the main goal of this thesis is to show how it is possible to probe parity-violating extensions of a fundamental theory, such as electrodynamics, by means of cosmological observables. Hence, it is now the moment to review what kind of cosmic phenomenon allows us for doing that. In fact, as well known, cosmology can be seen as the study of the composition and the evolution of the Universe as a whole. The Hot Big Bang model explains extraordinarily well the evolution of our Universe from its early stages to its current state today. As we are going to see, such a cosmological model is set within a theoretical framework based on GR, while from the observational side one of its supporting pillars is the detection of the **cosmic microwave background** radiation (see Ref. [131]). This is an extremely almost isotropic radiation emitted when the Universe, while cooling down, reached a temperature low enough to allow the formation of neutral atoms. After that, photons decoupled from matter and free-streamed to us. Therefore, CMB can be seen as a sort of primordial witness of an early stage of the Universe. Since CMB is the main source of cosmological electromagnetic waves, it is clear that it also provides a natural laboratory for testing deviations from the standard theory of electrodynamics taking place in unique regimes. Now, the purpose of this section is to review the formalism describing the physics of CMB. The content of this section is a partial review of the standard notions that can be found in many famous textbooks, such as Refs. [132–142].

### 1.3.1 The ΛCDM Model

Einstein's theory of general relativity explains how the geometry of spacetime is related to the energy content of the Universe, and this is expressed by means of Eq. (1.96). In general, it is not so simple to solve a priori the Einstein equation, and the strategy usually adopted is to simplify the problem by making strong physical assumptions on $g_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$. For instance, this is done in standard cosmology, in order to describe the Universe on large scales. Experimental observations tell us that the Universe is expanding. Indeed, in 1929, Edwin Hubble (see Ref. [143]) discovered that the farther a galaxy is the faster it recedes from us. This phenomenon is mathematically expressed by the **Hubble's law**[9],

$$v = H_0\, d, \tag{1.140}$$

[9] Despite its name, the Belgian astronomer Georges Lemaître was the first to publish research deriving what is now known as Hubble's law.

where $v$ us the recessional velocity, $d$ is the distance and $H_0 = (67.4 \pm 0.5)$ km/s/MPc is a constant (see Ref. [144]), named after Hubble[10]. The expansion of the Universe is not the only assumption we can make about the metric of the Universe at large scales. Indeed, a very high symmetry for the Universe, called the **cosmological principle**, is assumed. Such a principle is minimally stated as follows: *"the Universe is isotropic and homogeneous"*, i.e. there is no preferred direction or preferred position. The cosmological principle seems to be compatible with observations at very large scales, according to Ref. [145]. We now briefly summarize the procedure for building a metric tensor which satisfies the cosmological principle: one has to consider a generic spacetime, and suppose that it is possible to define a reference frame in which spacetime can be sliced in hyper-surfaces of constant time. On each of those slices, the time will have a different, fixed value. Two generic hyper-surfaces will be then separated by a constant time distance $\delta t$, since they are all parallel on the time axis. If we take a spatial point $\mathbf{x} = (x, y, z)$ on one of those hyper-surfaces and then associate it to the constant time $t$ of the hyper-surface, we define the so called **synchronous frame**. The synchronous frame is nothing but the reference frame in which the metric of the Universe shows the property we are looking for. First of all, we consider a geodesic where only the time varies, and such a geodesic must be orthogonal to all the hyper-surfaces. Then, those geodesics will have the same shape in every point of space, and the space-time interval between two hyper-surfaces will be the same for all the geodesics, and equal to $\delta t$. Observers on the same hyper-surface will measure the same time, which we call **cosmic time**. The cosmic time will be uniquely defined for every observer, given a precise hyper-surface. Since the spacetime must be equal to the time distance, then it follows that $g_{00}$ must be necessarily equal to $-1$, and it can be proved that in this reference frame $g_{0j} = 0$. Then, it can be shown that the metric which satisfies these assumptions is the **Friedmann-Lemaître-Robertson-Walker** (FLRW) one, which, in spherical coordinates, reads

$$g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}x^2}{1-Kr^2} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)\right], \tag{1.141}$$

where $t$ is the cosmic time, $r$ is the radial coordinate, $\vartheta$ and $\varphi$ denote the angular ones, $K$ is the curvature of space (see Fig. 1.3), and $a$ is the **scale factor**, since it tells us how the distance between two points in the expanding Universe scales with time. For this reason, $a(t)$ is related to the **Hubble parameter**, which is defined as

$$H \equiv \frac{1}{a(t)}\frac{\mathrm{d}\,a(t)}{\mathrm{d}t}, \tag{1.142}$$

and takes the value of $H_0$ for $t = t_0$, i.e. at the present time, when $a(t_0)$ is conventionally set equal to 1. As a consequence, the spatial coordinates are called **comoving coordinates**. Having fixed the form of $g_{\mu\nu}$, a consistent expression of the energy-momentum tensor, which satisfies the cosmological principle, is

[10] However, let us remark that such an expression holds only for galaxies that are not so away from us.

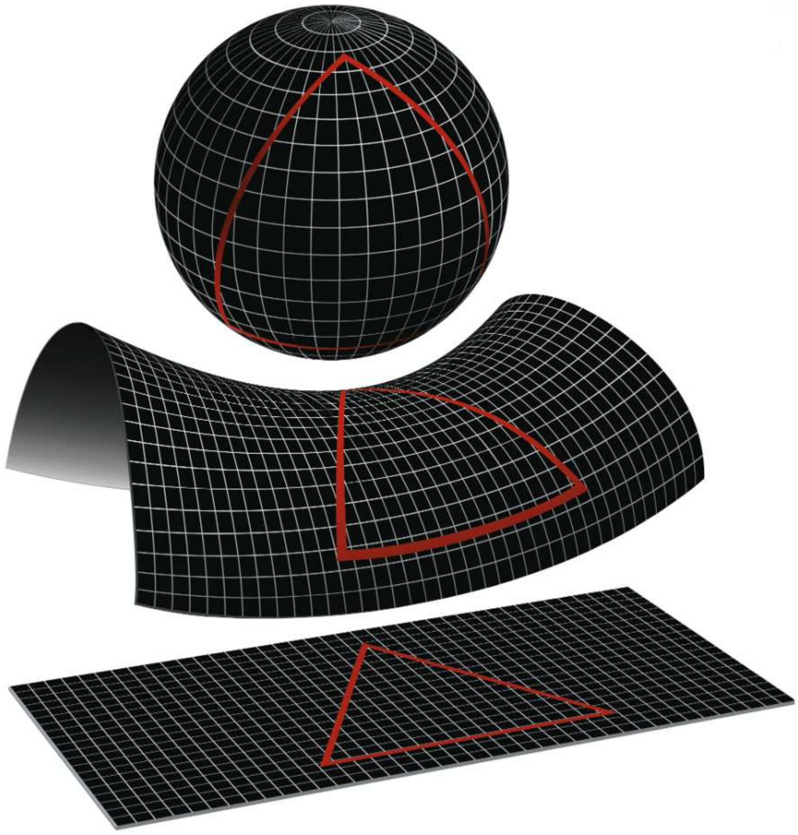

**Figure 1.3:** The three possible geometries of our Universe in a two-dimensional analogy (picture borrowed from the NASA official website). From up to down: $K = 1$ denotes a positively curved (closed) Universe, $K = -1$ denotes a negatively curved (open) one, whereas $K = 0$ denotes a spatially flat one.

achieved by thinking the Universe as a perfect isotropic fluid, whose energy-momentum tensor is given as

$$\mathcal{T}^{\mu}_{\ \nu}(t) = \begin{bmatrix} -\varrho(t) & 0 & 0 & 0 \\ 0 & \mathcal{P}(t) & 0 & 0 \\ 0 & 0 & \mathcal{P}(t) & 0 \\ 0 & 0 & 0 & \mathcal{P}(t) \end{bmatrix}, \tag{1.143}$$

where, as we mentioned in Sec. 1.2.1, $\varrho$ is the energy density, and $\mathcal{P}$ is the pressure of such a fluid. Eq. (1.143) has a reasonable form: thanks to the assumption of isotropy, there is no preferred direction, so that the pressure is equal in all spatial directions. Moreover, such a tensor is diagonal because it represents a fluid with no shear stress or viscosity, and so without internal forces causing tangential stresses. Indeed, the diagonal elements correspond to the energy density and the principal pressures along the coordinate axes, reflecting the fluid's isotropic nature without internal friction or shearing effects. According to Eq. (1.124), let us compute the $\nu = 0$ component of the conservation law for the energy-momentum tensor,

$$\nabla_\mu \mathcal{T}^\mu_{\ 0} = \partial_\mu \mathcal{T}^\mu_{\ 0} + \Gamma^\mu_{\mu\sigma} \mathcal{T}^\sigma_{\ 0} - \Gamma^\sigma_{\mu 0} \mathcal{T}^\mu_{\ \sigma} = 0, \tag{1.144}$$

where we have used Eq. (1.100). Now, we can easily work out the Christoffel symbols for the metric defined in Eq. (1.141) by recalling their definition presented in Eq. (1.98), so that we get

$$\Gamma^0_{00} = 0 \qquad \Gamma^0_{11} = \frac{a^2 H}{1 - Kr^2} \qquad \Gamma^0_{22} = a^2 H r^2 \qquad \Gamma^0_{33} = a^2 H r^2 \sin^2 \vartheta \tag{1.145}$$

$$\Gamma^1_{01} = H \qquad \Gamma^1_{11} = \frac{Kr}{1 - Kr^2} \qquad \Gamma^1_{22} = -r\left(1 - kr^2\right) \qquad \Gamma^1_{33} = -r\left(1 - Kr^2 \sin^2 \vartheta\right) \tag{1.146}$$

$$\Gamma^2_{02} = H \qquad \Gamma^2_{12} = \frac{1}{r} \qquad \Gamma^2_{22} = 0 \qquad \Gamma^2_{33} = -\sin\vartheta \cos\vartheta \tag{1.147}$$

$$\Gamma^3_{03} = H \qquad \Gamma^3_{13} = \frac{1}{r} \qquad \Gamma^3_{23} = \frac{1}{\tan\vartheta} \qquad \Gamma^3_{33} = 0, \tag{1.148}$$

whereas all the other ones are vanishing or can deduced by means of the symmetry in the lower indices. By substituting Eq. (1.143) and Eqs. (1.145)-(1.148) in Eq. (1.144), we obtain

$$\boxed{\frac{\mathrm{d}\varrho(t)}{\mathrm{d}t} + 3H(t)\left[\varrho(t) + \mathcal{P}(t)\right] = 0,} \tag{1.149}$$

which is known as the **continuity equation**. Analogously, we can use the definition of the Ricci tensor we wrote in Eq. (1.97), to compute its component for the FLRW metric, finding

$$R_{00} = -\frac{3}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2}, \tag{1.150}$$

$$R_{11} = \frac{a^2(t)}{1 - Kr^2} \left[\frac{1}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} + 2H^2(t) + \frac{2K}{a^2(t)}\right], \tag{1.151}$$

$$R_{22} = a^2(t) r^2 \left[\frac{1}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} + 2H^2(t) + \frac{2K}{a^2(t)}\right], \tag{1.152}$$

$$R_{33} = a^2(t) r^2 \left[\frac{1}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} + 2H^2(t) + \frac{2K}{a^2(t)}\right] \sin^2 \vartheta. \tag{1.153}$$

In fact, by direct inspection of Eq. (1.96), it is clear that the Ricci tensor is non-vanishing only for $\mu = \nu$ since both $g_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ are diagonal. Therefore, the Ricci scalar reads

$$R = g^{\mu\nu} R_{\mu\nu} = 6 \left[ \frac{1}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} + H^2(t) + \frac{K}{a^2(t)} \right], \tag{1.154}$$

so that the $\mu = \nu = 0$ component of the Einstein equation yields the **first Friedmann equation**,

$$\boxed{H^2(t) = \frac{\varrho(t)}{3m_{Pl}^2} - \frac{K}{a^2(t)} + \frac{\Lambda}{3}} \tag{1.155}$$

we can see that here $m_{Pl}^2 \Lambda$ plays the role of an energy density associated with empty space (since it survives even when $\varrho = 0$), and this is why the cosmological constant is also called **vacuum energy**. Moreover, if we differentiate with respect to the cosmic time Eq. (1.155), and we substitute the first Friedmann equation and the continuity one within the result, we find the **second Friedmann equation**,

$$\boxed{\frac{1}{a(t)} \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} = -\frac{1}{6m_{Pl}^2} [\varrho(t) + 3\mathcal{P}(t)] + \frac{\Lambda}{3},} \tag{1.156}$$

which describes the acceleration of the cosmic scale factor as a function of various components contributing to the energy content of the universe. Therefore, Eq. (1.149) and the two Friedmann equations, together describe the dynamics of an homogeneous and isotropic Universe. By the way, to solve the continuity equation, a further condition is needed: it is provided by an **equation of state** (EOS), i.e. a mathematical relationship between density and pressure, $\mathcal{P} = \mathcal{P}(\varrho)$. The explicit dependence of $\mathcal{P}$ on $\varrho$ usually takes the following form:

$$\mathcal{P} = w(\varrho)\varrho, \tag{1.157}$$

where $w(\varrho)$ is a parameter which depends on what type of content we consider for the Universe. In particular, let us now focus on the simplified situation in which $w$ is just a number not depending on $\varrho$. Then, we can separate the variables $\rho$ and $a$, and after integrating each side separately, we find a relation $\varrho = \varrho(a)$.

$$\int_{\text{past}}^{\text{today}} \frac{\mathrm{d}\varrho}{\varrho} = -3(1+w) \int_{\text{past}}^{\text{today}} \frac{\mathrm{d}a}{a} \quad \Longrightarrow \quad \varrho(a) = \frac{\varrho_0}{a^{3(1+w)}}, \tag{1.158}$$

where we have recalled that today $a = a_0 = 1$ and we have defined $\varrho_0 \equiv \varrho(a_0)$. There exist three well known examples in literature:

- non-relativistic particles (or **matter**). They are characterized by $w = 0$, and so by $\varrho \propto a^{-3}$ since the typical velocities and energies of non-relativistic particles are much smaller than the speed of light. As a result, the kinetic energy contributions to pressure are minimal, and the particles' motion does not exert significant pressure on the system;
- ultra-relativistic particles (or **radiation**), characterized by $w = 1/3$ (we will see the reason in Sec. 1.3.2) and so by $\varrho \propto a^{-4}$;
- cosmological constant, characterized by $w = -1$, since it leads to a constant in time energy density.

The reason we are underlining this classification is that it is roughly possibly to include all the relevant particles populating the Universe in these categories: neutrinos and photons, being approximately and exactly massless, respectively, are ultra-relativistic particles, so that they contribute to radiation, whereas baryons and leptons (such as protons and electrons) contribute to matter (although it is not always true that these particles behave as non-relativistic bodies). Therefore, the energy density entering in the two Friedmann equation has to be understood as the sum of the contributions coming from the different cosmic species populating the Universe:

$$\varrho(t) = \frac{\varrho_{r,0}}{a^4(t)} + \frac{\varrho_{m,0}}{a^3(t)}, \tag{1.159}$$

$\varrho_{r,0}$, $\varrho_{m,0}$ being the energy density evaluated today of radiation and matter, respectively. Now, if we divide the first Friedmann equation by $H_0^2$, we obtain a more suitable expression,

$$\frac{H(t)}{H_0} = \sqrt{\frac{\Omega_{r,0}}{a^4(t)} + \frac{\Omega_{m,0}}{a^3(t)} + \frac{\Omega_{K,0}}{a^2(t)} + \Omega_\Lambda}, \tag{1.160}$$

where we have defined the following **density parameters**:

$$\Omega_{r,0} \equiv \frac{\varrho_{r,0}}{3H_0^2 m_{Pl}^2}, \qquad \Omega_{m,0} \equiv \frac{\varrho_{m,0}}{3H_0^2 m_{Pl}^2} \qquad \Omega_{K,0} \equiv -\frac{K}{H_0^2} \qquad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}. \tag{1.161}$$

The reason for that we have put the Friedmann equation in the form of Eq. (1.160) is twofold: first of all, we can notice that today the sum of all the $\Omega$'s must equal 1, which means that they are not independent, but they have to satisfy a closure relation; secondly, since in the past the scale factor was smaller than today because of the expansion of the Universe, it is clear that different cosmic species were dominant in different epochs. In particular, in its early life, the Universe experienced a radiation-dominate epoch, which was followed by a matter-dominated one, and then by a curvature-dominate one, and finally today the dominant contribution is given by the cosmological constant. According to the most recent results coming from the *Planck* satellite (see Ref. [144])

$$\Omega_{m,0} = 0.3081 \pm 0.0065, \quad \Omega_{K,0} = 0.0007 \pm 0.0019, \quad \Omega_{\Lambda} = 0.6911 \pm 0.0062, \tag{1.162}$$

whereas $\Omega_{r,0} \sim 10^{-4}$. These results seem to suggest that our Universe is consistent with a spatially flat one, and for this reason from now we will set $K = 0$, since this assumption will simplify a lot our calculations. Moreover, it seems that today almost all the energy content of the Universe is given by matter and cosmological constant, but let us now stress that such matter is mainly not made by particles of the standard model. In fact, observations are telling us that baryons and leptons in the Universe are not enough to yield the 30 % of matter in the Universe, but instead the large majority of this must be made by a unknown kind of matter. Apart from cosmology, many astrophysical observations from different sources at different distance scales point out the existence of this mysterious dark component, such as the dynamics of galaxies in clusters and the rotation curves of spiral galaxies (see Refs. [146, 147], respectively). If this sort of dark matter has the same equation of state, then it could be made by an unknown type of non-relativistic particles, and this hypothesis is called **cold dark matter** (CDM), and several candidates have been taken into account to unveil its nature (see e.g. Ref. [148]). However, the main part of the current energy budget of the Universe is provided by $\Lambda$, whose equation of state defines a fluid of negative pressure, and let us notice that if the cosmological constant is the dominant term in Eq. (1.156), we can directly see that the Universe experiences a phase of accelerated expansion, since $\mathrm{d}^2a/\mathrm{d}t^2 \geq 0$. Of course, this could happen even if $\Lambda = 0$ but with an extra mysterious fluid having equation of state $w \leq 1/3$. Since in the last decades observations have shown the Universe is effectively in a state of accelerated expansion (see e.g. Ref. [149]), this means that there must exist a **dark energy** able to drive this acceleration, and at the moment the main candidate is $\Lambda$, even if, as said, the same result can be achieved by some cosmic species with the adequate equation of state. Hence, the current Universe is mainly made up by dark energy (that our paradigm for the moment identifies with the cosmological constant itself) and cold dark matter, and for this reason the standard cosmological model is said to be the **$\Lambda$CDM model**. As we will see in the course of this thesis, the phenomenon of cosmic birefringence provides an indirect tool for testing candidates for dark matter and dark energy.

### 1.3.2 Cosmological Perturbation Theory

The assumption of an homogeneous and isotropic Universe makes us able to solve the Einstein field equation for such an Universe, but it is reliable only on very large scales. For instance, its shortcomings become evident when one starts to investigate how galaxies and their clusters form. Indeed they seem to be huge deviations from the cosmological principle. In cosmology, small deviations from the cosmological principle are addressed, by considering perturbations in the FLRW metric,

$$\bar{g}_{\mu\nu}(\tau) = a^2(\tau)\eta_{\mu\nu}, \tag{1.163}$$

where we have rewritten in Cartesian coordinates the metric shown in Eq. (1.141) for $K = 0$ in terms of the Minkowski one, after moving from the cosmic time $t$ to the **conformal time** $\tau$,

$$\mathrm{d}\tau \equiv \frac{\mathrm{d}t}{a(t)}, \tag{1.164}$$

whose physical meaning is that, if we integrate Eq. (1.164) from $\tau_A$ to $\tau_B$, the difference $(\tau_B - \tau_A)$ is nothing but the comoving distance traveled by a photon in a time interval equal to $(t_B - t_A)$. Indeed, in the general theory of relativity, the choice of a coordinate system is not limited in any way; the triplet of space coordinates $(x^1, x^2, x^3)$ can be any quantities defining the position of bodies in space, and the time coordinate $x^0$ can be defined by an arbitrarily running clock. Indeed, we have taken advantage of this fact by moving from the $(t, r, \theta, \varphi)$ coordinates used instead in Eq. (1.141) to the $(\tau, x, y, z)$ ones used in Eq. (1.163). As said before, Eq. (1.163) mathematically describes a Universe which satisfies the cosmological principle. In other words, the four-dimensional surface over which $\bar{g}_{\mu\nu}$ is defined can be seen as a background spacetime, that does not include deviations from homogeneity and isotropy, since $\bar{g}_{\mu\nu}$ is diagonal and does not depend on the spatial coordinates. Therefore, the "real" spacetime is something different described by the "real" metric $g_{\mu\nu}$, such that

$$g_{\mu\nu}(\tau, \mathbf{x}) \equiv \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, \mathbf{x}). \tag{1.165}$$

Anyway, this means that $\delta g_{\mu\nu}$, i.e. the "difference" between the two metrics, reveals to be ill-defined, since $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ are tensors defined on different four-dimensional spaces, and consequently $(\tau,\ \mathbf{x})$ are a set of spacetime coordinates defined through different ways. In order to take care of such a problem, we need a rule, called **gauge**, which identifies points of the first spacetime with those of the second one. The gauge is arbitrary and allows us to still use $\tau$ (or $t$) plus $\mathbf{x}$ defined in the background, also for the points in the physical one, i.e. even when working with the perturbed metric (see e.g. Refs. [150, 151]).

$$g_{00}(\tau, \mathbf{x}) = -a^2(\tau)\left[1 + \sum_{n=1}^{\infty} \frac{2}{n!}\Psi^{(n)}(\tau, \mathbf{x})\right], \tag{1.166}$$

$$g_{0i}(\tau, \mathbf{x}) = +a^2(\tau)\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{\omega}_i^{(n)}(\tau, \mathbf{x})\right], \tag{1.167}$$

$$g_{ij}(\tau, \mathbf{x}) = +a^2(\tau)\left\{\left[1 + \sum_{n=1}^{\infty} \frac{2}{n!}\Phi^{(n)}(\tau, \mathbf{x})\right]\delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{\gamma}_{ij}^{(n)}(\tau, \mathbf{x})\right\}, \tag{1.168}$$

where we have denoted by $^{(n)}$ the perturbative order. The key concept of the perturbative approach is that, by denoting with $M$ any GR tensor representing whatever cosmological quantity, we impose

$$\left|\left[\delta M^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}(\tau, \mathbf{x})\right]^{(n+1)}\right| \ll \left|\left[\delta M^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}(\tau, \mathbf{x})\right]^{(n)}\right| \ll \left|\overline{M}^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}(\tau)\right| \qquad \forall\, n \in \mathbb{N}_0 \tag{1.169}$$

and also that for any combination of tensors $A$, $B$, $C$, the product of two perturbations has the same perturbative order of the sum of their own perturbative orders, i.e.

$$\left|\left[\delta A^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}(\tau, \mathbf{x})\right]^{(n)}\left[\delta B^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}(\tau, \mathbf{x})\right]^{(m)}\right| = \left|\left[\delta C^{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_k}_{\nu_1\ldots\nu_l\beta_1\ldots\beta_l}(\tau, \mathbf{x})\right]^{(n+m)}\right| \qquad \forall\, n, m \in \mathbb{N}_0 \tag{1.170}$$

It is standard to split the perturbations into their scalar, vector and tensor parts according to their transformation properties with respect to the three-dimensional space with metric $\delta_{ij}$, where, thanks to the **Helmholtz decomposition**, scalar parts are related to a scalar potential, vector parts to transverse (i.e. divergence-free) vectors and tensor parts to transverse trace-free tensors. Thus, in our case

$$\hat{\omega}_i^{(n)}(\tau, \mathbf{x}) = \partial_i \omega^{(n)}(\tau, \mathbf{x}) + \omega_i^{(n)}(\tau, \mathbf{x}) \tag{1.171}$$

$$\hat{\gamma}_{ij}^{(n)}(\tau, \mathbf{x}) = \left[\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right]\gamma^{(n)}(\tau, \mathbf{x}) + \partial_i\gamma_j^{(n)}(\tau, \mathbf{x}) + \partial_j\gamma_i^{(n)}(\tau, \mathbf{x}) + \gamma_{ij}^{(n)}(\tau, \mathbf{x}), \tag{1.172}$$

where $\omega_i$ and $\gamma_i$ are divergenceless vectors, $\gamma_{ij}^{(n)}$ is a symmetric divergenceless and trace-free tensor, and the operator acting on $\gamma^{(n)}$ is a trace-free operator, i.e.

$$\delta^{ij}\partial_i\omega_j^{(n)}(\tau, \mathbf{x}) = 0 \qquad \delta^{ij}\partial_i\gamma_j^{(n)}(\tau, \mathbf{x}) = 0 \tag{1.173}$$

$$\delta^{ij}\partial_i\gamma_{jk}^{(n)}(\tau, \mathbf{x}) = 0 \qquad \delta^{ij}\gamma_{ij}^{(n)}(\tau, \mathbf{x}) = 0 \qquad \delta^{ij}\left[\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right]\gamma^{(n)}(\tau, \mathbf{x}) = 0. \tag{1.174}$$

Such a splitting has been introduced because, at least at first order, these different modes decouple from each other in the perturbed evolution equations so that they can be studied separately. At the contrary, this property does not hold anymore beyond the linear regime, where e.g. second-order perturbations are coupled by first-order perturbations. For our purposes, it will be convenient to work in a specific gauge which will allow us to simplify the main computations. By the way, let us just say that different gauges might lead to different interpretations of the same physical phenomenon, and it can be challenging to ensure that the chosen gauge accurately represents the underlying physics without introducing unnecessary complications, such as the usage of explicitly gauge-invariant quantities. In fact, we have seen that the metric tensor (as well as any other symmetric tensor of the same rank) has 10 independent elements, and this is why we can decompose such a tensor in 4 scalar functions ($\Psi$, $\omega$, $\Phi$, and $\gamma$), plus 2 divergenceless three-vectors ($\omega_i$, $\gamma_i$), for a total of $2 \times 2 = 4$ independent components[11], plus a divergenceless trace-free spatial tensor of rank, having 2 independent components[12]. Therefore, we know that this decomposition is the most general one, as the functions we have introduced represent $4 + 4 + 2$ independent components corresponding to the 10 components of the most general metric $g_{\mu\nu}$. However, since the Einstein equation is manifestly covariant under general coordinate transformations, we are also allowed to choose our 4 spacetime coordinates $(\tau, \mathbf{x})$ in any way we want without changing any physical quantities, so that only 6 of these fields represent physical degrees of freedom. Hence, from this moment we are going to adopt the **Poisson gauge**[13],

$$g_{\mu\nu} \equiv a^2 \begin{pmatrix} -\left[1 + \sum_{n=1}^{\infty} \frac{2}{n!}\Psi^{(n)}\right] & \sum_{n=1}^{\infty} \frac{1}{n!}\omega_j^{(n)} \\ \sum_{n=1}^{\infty} \frac{1}{n!}\omega_i^{(n)} & \left[1 + \sum_{n=1}^{\infty} \frac{2}{n!}\Phi^{(n)}\right]\delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!}\gamma_{ij}^{(n)} \end{pmatrix}, \tag{1.175}$$

or equivalently we have set $\omega$, $\gamma$ and $\gamma_i$ equal to zero. In the very same fashion we adopted for the metric, similar considerations follow for the components of the energy-momentum tensor. Indeed, one can introduce small deviations from the perfect isotropic fluid, such that

$$\mathcal{T}_{\mu\nu}(\tau, \mathbf{x}) = \overline{\mathcal{T}}_{\mu\nu}(\tau) + \sum_{n=1}^{\infty} \delta\mathcal{T}_{\mu\nu}^{(n)}(\tau, \mathbf{x}). \tag{1.176}$$

[11]The 3 original independent elements of each three-vector reduce to just 2, because of the divergenceless condition.

[12]The 6 original independent elements of the spatial symmetric tensor reduce to just 5 because of the trace-free condition, and they further reduce to just 2 because the divergenceless conditions provides a vector equation imposing 3 extra constraints.

[13]To be rigorous, the metric in Eq. (1.175) is written in the **longitudinal gauge**, which, when it involves only scalar perturbations of the metric, is also called **Newtonian conformal gauge**, but instead, when including also vector and tensor perturbations, it is called Poisson gauge (see e.g. Ref. [152]).

## Perturbed Einstein Equations

Armed with Eqs. 1.175-(1.176), we are now able to compute the perturbed Einstein equation by means of the definition given in Eq. (1.96),

$$\frac{\mathcal{T}^{\mu}{}_{\nu}(\tau, \mathbf{x})}{m_{Pl}^2} = R^{\mu}{}_{\nu}(\tau, \mathbf{x}) - \frac{1}{2}\left[R(\tau, \mathbf{x}) - 2\Lambda\right]\delta^{\mu}{}_{\nu}. \tag{1.177}$$

However, doing this would involve evaluating the Ricci tensor and the Ricci scalar, and consequently the Christoffel symbols, for the metric defined in Eq. (1.175). Although this is conceptually straightforward, since we have has just to substitute the perturbed metric in Eq. (1.98), it could lead to a very long calculation. In order to avoid this, we take advantage of the `xPand` package (see Ref. [153]) for `Mathematica`, which is able to that for us. Therefore, we write down here the time-time, time-space and space-space components, respectively, of the perturbed Einstein equation obtained in the just mentioned way. At zero-th order, we then recover the first Friedmann equation written in terms of the conformal time assuming a spatially flat Universe,

$$\overline{\mathcal{T}}^{0}_{\ 0}(\tau) = \frac{m_{Pl}^2}{a^2(\tau)}\left[\Lambda a^2(\tau) - 3\mathcal{H}^2(\tau)\right], \tag{1.178}$$

$$\overline{\mathcal{T}}^{0}_{\ i}(\tau) = 0, \tag{1.179}$$

$$\overline{\mathcal{T}}^{i}_{\ j}(\tau) = \frac{m_{Pl}^2}{a^2(\tau)}\left[\Lambda a^2(\tau) - \mathcal{H}^2(\tau) - 2\frac{\mathrm{d}\mathcal{H}(\tau)}{\mathrm{d}\tau}\right]\delta^{i}_{\ j}, \tag{1.180}$$

where $\mathcal{H} \equiv (\mathrm{d}a/\mathrm{d}\tau)/a$ is the **conformal Hubble parameter**. Analogously at first order we have

$$\frac{a^2(\tau)}{m_{Pl}^2}\delta\mathcal{T}^{0\,(1)}_{\ 0}(\tau,\mathbf{x}) = 6\mathcal{H}(\tau)\left[\mathcal{H}(\tau)\Psi^{(1)} - \frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})\right] + 2\nabla^2\Phi^{(1)}(\tau,\mathbf{x}), \tag{1.181}$$

$$\frac{a^2(\tau)}{m_{Pl}^2}\delta\mathcal{T}^{0\,(1)}_{\ i}(\tau,\mathbf{x}) = \frac{1}{2}\nabla^2\omega_i^{(1)}(\tau,\mathbf{x}) + 2\partial_i\left[\frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x}) - \mathcal{H}(\tau)\Psi^{(1)}(\tau,\mathbf{x})\right], \tag{1.182}$$

$$\begin{aligned}\frac{a^2(\tau)}{m_{Pl}^2}\delta\mathcal{T}^{i\,(1)}_{\ j}(\tau,\mathbf{x}) = {}& \frac{1}{2}\left[\frac{\partial^2}{\partial\tau^2} + 2\mathcal{H}(\tau)\frac{\partial}{\partial\tau} - \nabla^2\right]\gamma^{i\,(1)}_{j}(\tau,\mathbf{x}) \\ &+ \frac{2}{\mathcal{H}(\tau)}\left[\mathcal{H}(\tau) + \frac{\partial}{\partial\tau}\right]\left[\mathcal{H}^2(\tau)\Psi^{(1)}(\tau,\mathbf{x})\right]\delta^i_j \\ &+ \left(\delta^i_j\nabla^2 - \delta^{ik}\partial_k\partial_j\right)\left[\Psi^{(1)}(\tau,\mathbf{x}) + \Phi^{(1)}(\tau,\mathbf{x})\right] \\ &- \left[\mathcal{H}(\tau) + \frac{1}{2}\frac{\partial}{\partial\tau}\right]\left[\delta^{ik}\partial_k\omega_j(\tau,\mathbf{x}) + \delta^{ik}\partial_j\omega_k^{(1)}(\tau,\mathbf{x})\right. \\ &\left. + 2\delta^i_j\frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})\right],\end{aligned} \tag{1.183}$$

and we do not write here the very long expressions valid at second order in perturbation theory, which can be found e.g. in Ref. [154], whereas going at third order is beyond the purpose of this thesis.

### Perturbed Energy-Momentum Tensor

Now, in order to further proceed, we have understand how we can link the components of the perturbed energy-momentum tensor arising in Eqs. (1.181)-(1.183) to the kinetic properties of cosmic species. However, the Universe is extremely vast and the number of particles within it is enormous, so that it is impractical to track the exact behavior of each individual particle. Instead, we will use statistical methods to describe the collective behavior of particles: this can be done by means of the **distribution function**. It is nothing but a scalar function $f(\tau,\mathbf{x},\mathbf{p})$ which represents the probability density of finding a particle at a specific position $\mathbf{x}$ at the conformal time $\tau$ with a specific three-momentum $\mathbf{p}$ in a multi-particle system. By simply following our definition, it is then clear that we can make the following identification:

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}f(\tau,\mathbf{x},\mathbf{p}) = \text{density of particles located at } \mathbf{x} \text{ at time } \tau \equiv n(\tau,\mathbf{x}). \tag{1.184}$$

The normalization $(2\pi)^3$ is due to the fact that, according to the **Heisenberg's principle**, no quantum particle can be exactly localized at the point $(\mathbf{x}, \mathbf{p})$ in the position-momentum space, but at most in a small volume of size equal to the Planck constant $2\pi\hbar$ about that point, representing the fundamental cell. However, since we are working in God-given units, $\hbar$ is set equal to 1. As shown in Eq. (1.134), we can denote by $\mathcal{E}(p)$ the energy of a particle with three-momentum $\mathbf{p}$, so that, by slightly modifying Eq. (1.184), we obtain the energy density,

$$\varrho(\tau, \mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \mathcal{E}(p) f(\tau, \mathbf{x}, \mathbf{p}). \tag{1.185}$$

By further proceeding in this way, we can make another identification,

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^i p_j}{E(p)} f(\tau, \mathbf{x}, \mathbf{p}) = \text{the flux of } p^i \text{ due to motion in the } j\text{-direction.} \tag{1.186}$$

or, in other words, $i$-component of the force being exerted per unit area in the $j$-direction[14] Hence, by recalling the discussion made in Sec. 1.2.1, Eq. (1.186) represents nothing but the spatial components of the energy-momentum tensor, $\mathcal{T}^i{}_j$, as well as, by definition, Eq. (1.185) represents the $\mathcal{T}^0{}_0$ one (times a minus sign). Therefore, by collecting all together, we see that, with a bit of imagination, it is possible to define the energy-momentum tensor in terms of the distribution function by exploiting Eq. (1.133) as

$$\mathcal{T}^\mu_\nu(\tau, \mathbf{x}) \equiv \sqrt{\frac{g_{00}(\tau, \mathbf{x})}{g(\tau, \mathbf{x})}} \det \left| g_{ij}(\tau, \mathbf{x}) - \frac{g_{0i}(\tau, \mathbf{x}) g_{0j}(\tau, \mathbf{x})}{g_{00}(\tau, \mathbf{x})} \right| \int \frac{d^3p}{(2\pi)^3} \frac{dx^\mu}{d\lambda} \frac{dx_\nu}{d\lambda} \frac{f(\tau, \mathbf{x}, \mathbf{p})}{\mathcal{E}(p)}, \tag{1.187}$$

as it can be verified by direct substitution. Let us notice that in a FLRW spacetime, according to Eq. (1.143), the homogeneous pressure of a perfect fluid can be evaluated as

$$\overline{\mathcal{P}}(\tau) = \frac{1}{3}\delta^j{}_i \overline{\mathcal{T}}^i{}_j(\tau) = \delta^j{}_i \bar{g}_{jk}(\tau) \int \frac{d^3p}{(2\pi)^3} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} \frac{\bar{f}(\tau, p)}{3E(p)} = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\mathcal{E}(p)} \bar{f}(\tau, p), \tag{1.188}$$

where, as we did for the other cosmological quantities, we have split the distribution function in a homogeneous background contribution plus a perturbation,

$$f(\tau, \mathbf{x}, \mathbf{p}) = \bar{f}(\tau, p) + \sum_{n=1}^{\infty} f^{(n)}(\tau, \mathbf{x}, \mathbf{p}). \tag{1.189}$$

[14] A missing $c^2$ factor due to the fact we are working in God-given units is necessary to achieve the standard dimensionality.

Therefore, Eq. (1.188) for an ultra-relativistic particle with $\mathcal{E} \simeq p$ simply reduces to

$$\overline{\mathcal{P}}(\tau) = \frac{1}{3}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathcal{E}(p)\bar{f}(\tau,p) = \frac{1}{3}\bar{\varrho}(\tau), \tag{1.190}$$

which is exactly what we have previously stated in Sec. 1.3.1. Armed with Eq. (1.187) we can now compute the left-hand sides of the perturbed Einstein equation. For instance, at first order in perturbation theory, we have

$$\sqrt{\frac{g_{00}^{(1)}(\tau,\mathbf{x})}{g^{(1)}(\tau,\mathbf{x})}\det\left| g_{ij}^{(1)}(\tau,\mathbf{x}) - \frac{g_{0i}^{(1)}(\tau,\mathbf{x})g_{0j}^{(1)}(\tau,\mathbf{x})}{g_{00}^{(1)}(\tau,\mathbf{x})}\right|} = 1, \tag{1.191}$$

so that we obtain

$$\delta\mathcal{T}^{\mu(1)}_{\ \nu}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x_\nu}{\mathrm{d}\lambda}\frac{\delta f^{(1)}(\tau,\mathbf{x},\mathbf{p})}{\mathcal{E}(p)}. \tag{1.192}$$

It is then clear that the deviations from isotropy and homogeneity is encoded in the perturbative term of the distribution function: this is why in Sec. 1.3.1 we previously considered the energy density and pressure as just functions of time, since the background distribution function is just $\bar{f}(\tau,p)$ without any dependence on $\mathbf{x}$ and $\hat{\mathbf{p}}$. However, for our purposes is not now necessary to explicitly evaluate Eq. (1.192), but we just say that since its components appear at the left-hand side of Eqs. (1.181)-(1.183), in order to completely track the evolution of cosmological perturbation we also need to know that one for the distribution function, and this is exactly what we are going to focus on in the next sections. By the way, it is clear that since the time-time component of the energy-momentum tensor is by definition associated with the energy density, we can without loss of generality define the perturbative counterpart of the energy density as

$$\varrho(\tau,\mathbf{x}) = \bar{\varrho}(\tau) + \sum_{n=1}\delta\varrho^{(n)}(\tau,\mathbf{x}) \qquad \Longrightarrow \qquad \delta\mathcal{T}^{0\,(n)}_{\ 0}(\tau,\mathbf{x}) = -\bar{\varrho}(\tau)\left[1+\sum_{n=1}^{\infty}\delta^{(n)}(\tau,\mathbf{x})\right], \tag{1.193}$$

with $\delta^{(n)} \equiv \delta\varrho^{(n)}/\bar{\varrho}$ being the so called **density contrast**.

### 1.3.3 Boltzmann's Description of Cosmic Species

As shown in Sec. 1.3.2, the source terms for the perturbed Einstein equations are given by the components of the energy-momentum tensor that are in turn related to the distribution function of the cosmic species populating the Universe. Therefore, the distribution function at the left-hand side appearing in the Einstein equations we have just derived in Sec. 1.3.2 must be seen as the sum of distribution functions of each cosmic species, which according to the **Standard Model** (SM) of particle physics can be divided in **bosons** and

**fermions**. In particular, bosons are quantum particles with an integer spin number that are described by symmetric wave functions, whereas fermions have an half-integer spin number and they are described by antisymmetric wave functions. This implies that the statistics of a systen of bosons is different from that of a fermions' one: bosons obey a **Bose-Einstein** statistics and fermions a **Fermi-Dirac** one, so that the full distribution function of the Universe can be written as

$$f(\tau, \mathbf{x}, \mathbf{p}) = \sum_{\text{particles}} f_s(\tau, \mathbf{x}, \mathbf{p}) = \sum_{\text{bosons}} f_{\text{BE}}(\tau, \mathbf{x}, \mathbf{p}) + \sum_{\text{fermions}} f_{\text{FD}}(\tau, \mathbf{x}, \mathbf{p}). \tag{1.194}$$

In particular, it can be shown that if we consider a system containing a number of identical particles of the cosmic species $s$ with energy $\mathcal{E}_s$ in thermodynamic equilibrium at temperature $T$, then such distribution function read[15]

$$\begin{aligned} f_{\text{BE}}(\tau, \mathbf{x}, \mathbf{p}) &= g_s \left\{ \exp\left[ \frac{\mathcal{E}_s(p)}{T_s(\tau, \mathbf{x}, \mathbf{p})} \right] - 1 \right\}^{-1}, \\ f_{\text{FD}}(\tau, \mathbf{x}, \mathbf{p}) &= g_s \left\{ \exp\left[ \frac{\mathcal{E}_s(p)}{T_s(\tau, \mathbf{x}, \mathbf{p})} \right] + 1 \right\}^{-1}, \end{aligned} \tag{1.195}$$

where $g_s$ is the particles' **degeneracy** of the specific species, i.e. the number of possible spin states (e.g. it equals 2 for photons, that are bosons, and electrons, that are fermions). If we adopt the usual perturbative decomposition also for the temperature of the cosmic species $s$,

$$T_s(\tau, \mathbf{x}, \mathbf{p}) = \overline{T}_s(\tau) + \sum_{n=1}^{\infty} \delta T_s^{(n)}(\tau, \mathbf{x}, \mathbf{p}), \tag{1.196}$$

we see that we can Taylor-expand Eq. (1.195), so that we find

$$f_s(\tau, \mathbf{x}, \mathbf{p}) = \bar{f}(\tau, p) - \mathcal{E}_s(p) \frac{\partial \bar{f}_s(\tau, p)}{\partial E_s(p)} \frac{\delta T_s^{(1)}(\tau, \mathbf{x}, \mathbf{p})}{\overline{T}_s(\tau)} + \sum_{n=2}^{\infty} \delta f_s^{(n)}(\tau, \mathbf{x}, \mathbf{p}), \tag{1.197}$$

which makes evident that the perturbed distribution function for cosmological particles can be physically interpreted as its relative temperature fluctuations. In order to study the evolution of the distribution function, we need some kind of differential equation for $f_s(\tau, \mathbf{x}, \mathbf{p})$. What we are looking for is provided by the

[15] The careful reader expert in statistical mechanics should notice that we are setting the chemical potential equal to zero. This can be done by exploiting the fact, as shown e.g. in Ref. [155], such a contribution is subdominant.


**Figure 1.4:** A collision of a particle 1 with initial three-momentum $\mathbf{q}_1$ and energy $\mathcal{E}_1(q_1)$ with a particle 2 with initial three-momentum $\mathbf{q}_2$ and energy $\mathcal{E}_2(q_2)$. Here, the effect of the collision is just that of modifying the three-momentum of the particles (i.e. it is not an annihilation or a creation process), so that, after the collision, the particles have three-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, and their energies are $\mathcal{E}_1(p_1)$ and $\mathcal{E}_2(p_2)$, respectively.

**Boltzmann equation**,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_s} f_s(\tau, \mathbf{x}, \mathbf{p}) = \mathcal{C}[f_s(\tau, \mathbf{x}, \mathbf{p})], \tag{1.198}$$

which associates in a very general way the variation of the distribution function of the species $s$ with respect to an affine parameter $\lambda_s$ with a **collision term** $\mathcal{C}$, a functional of the distribution function itself describing the interactions among the particles constituting the system under investigation. We can rewrite Eq. (1.198) as

$$\frac{\partial}{\partial\tau} f_s(\tau, \mathbf{x}, \mathbf{p}) = \frac{\sqrt{-g_{00}(\tau, \mathbf{x})}}{E(p)} \left\{ \mathcal{C}[f_s(\tau, \mathbf{x}, \mathbf{p})] - \frac{\mathrm{d}x^i}{\mathrm{d}\lambda_s} \partial_i f_s(\tau, \mathbf{x}, \mathbf{p}) - \frac{\mathrm{d}p^i}{\mathrm{d}\lambda_s} \frac{\partial}{\partial p^i} f_s(\tau, \mathbf{x}, \mathbf{p}) \right\} \tag{1.199}$$

where we have substituted Eq. (1.135).

### Collision Term of the Boltzmann Equation

As just mentioned, $\mathcal{C}$ is a scalar function which sources the modification of the distribution function of some particles of the species $s$ as a consequence of some scattering processes or particles decays. Therefore, we can write the more general collision term as the sum of the terms coming from any kind of process involving the particles of the species $s$,

$$\mathcal{C}[f_s(\tau, \mathbf{x}, \mathbf{p})] = \sum_{\text{processes}} \mathcal{C}_{\text{process}}[f_s(\tau, \mathbf{x}, \mathbf{p})]\,. \tag{1.200}$$

In order to understand what is the form of the terms at the right-hand side of Eq. (1.200), let us consider a very standard process where a particle of a species $s = 1$ collides with another particle of a species 2, as shown in Fig. 1.4. In a completely general way, we can reasonably define the collision term for such a $(1,\ 2) \mapsto (1, 2)$

process as

$$
\begin{aligned}
\mathcal{C}_{(1,2)\mapsto(1,2)}[f_1(\tau,\mathbf{x},\mathbf{p}_1)] \equiv \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \int \mathrm{d}\,\mathcal{E}_2(p_2) \int \mathrm{d}\,\mathcal{E}_1(q_1) \int \mathrm{d}\,\mathcal{E}_2(q_2) \\
\Big\{ f_1(\tau,\mathbf{x},\mathbf{q}_1) f_2(\tau,\mathbf{x},\mathbf{q}_2) W[\mathcal{E}_1(q_1),\mathcal{E}_2(q_2);\mathbf{q}_1,\mathbf{q}_2|\mathcal{E}_1(p_1),\mathcal{E}_2(p_2);\mathbf{p}_1,\mathbf{p}_2] \\
- f_1(\tau,\mathbf{x},\mathbf{p}_1) f_2(\tau,\mathbf{x},\mathbf{p}_2) W[\mathcal{E}_1(p_1),\mathcal{E}_2(p_2);\mathbf{p}_1,\mathbf{p}_2|\mathcal{E}_1(q_1),\mathcal{E}_2(q_2);\mathbf{q}_1,\mathbf{q}_2] \Big\},
\end{aligned}
\tag{1.201}
$$

where $W$ is a scalar function encoding the information about the probability that such a collision process could occur. Let us notice that the dependence of the distribution functions on $\tau$ and $\mathbf{x}$ is the same for any of the distribution functions appearing in Eq. (1.201), because we can reasonably assume that the difference in spacetime coordinates of the colliding particles before and after the collision may be neglected. The presence of the distribution functions $f_1$ and $f_2$ is of course due to the fact that the probability that such a process could occur must be proportional to the probability to have particles of the species 1 and 2 interacting in the point with coordinates $\mathbf{x}$ at the conformal time $\tau$. The integration over the initial three-momenta and energies, as well over the particle 2's final ones, is performed in order to average along our "ignorance" about the initial states of the system and the final one of the particle 2. The difference between the braces is instead due to the fact that the net change of number of particles of the species 1 with three-momentum $\mathbf{p}_1$ is given by the gain of particles due to the collision process minus the loss of particles due to the reverse process. Now, it is clear that, for any kind of process, function $W$ must be of the form

$$
\begin{aligned}
& W[\mathcal{E}_1(q_1),\mathcal{E}_2(q_2);\mathbf{q}_1,\mathbf{q}_2|\mathcal{E}_1(p_1),\mathcal{E}_2(p_2);\mathbf{p}_1,\mathbf{p}_2] = \\
&\quad = (2\pi)^4 |\mathcal{M}_{(1,2)\mapsto(1,2)}(\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2)|^2 \\
&\qquad \delta[\mathcal{E}_1(p_1)+\mathcal{E}_2(p_2)-\mathcal{E}_1(q_1)-\mathcal{E}_2(q_2)]\,\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{q}_1-\mathbf{q}_2) \\
&\qquad \delta\left[\mathcal{E}_1^2(p_1)-m_1^2-p_1^2\right]\Theta[\mathcal{E}_1(p_1)]\,\delta\left[\mathcal{E}_2^2(p_2)-m_2^2-p_2^2\right]\Theta[\mathcal{E}_2(p_2)] \\
&\qquad \delta\left[\mathcal{E}_1^2(q_1)-m_1^2-q_1^2\right]\Theta[\mathcal{E}_1(q_1)]\,\delta\left[\mathcal{E}_2^2(q_2)-m_2^2-q_2^2\right]\Theta[\mathcal{E}_2(q_2)],
\end{aligned}
\tag{1.202}
$$

i.e. it has to be equal to another function $(2\pi)^4|\mathcal{M}^2_{(1,2)\mapsto(1,2)}|$ with the same physical meaning multiplied by several Dirac deltas enforcing the dispersion relations of Eq. (1.134) for all the cosmic species, as well as the conservation of the system's energy and three-momentum. The reason we are defining a squared quantity instead of just $\mathcal{M}$ is to ensure the positiveness, given its probabilistic nature. Moreover, such a function has to be multiplied by some Heaviside functions guaranteeing that the particles' energy is positive. Now, if we substitute Eq. (1.201), we see that we can easily evaluate the integrals over the particles' energies. For instance, we have

$$\int \mathrm{d}\,\mathcal{E}_2(p_2)\,\delta\left[\mathcal{E}_2^2(p_2) - m_2^2 - p_2^2\right]\Theta\left[\mathcal{E}_2(p_2)\right] =$$
$$= \int_0^{+\infty} \mathrm{d}\,\mathcal{E}_2(p_2)\left\{\frac{\delta\left[\mathcal{E}_2(p_2) - \sqrt{m_2^2+p_2^2}\right]}{2\sqrt{p_2^2+m_2^2}} - \frac{\delta\left[\mathcal{E}_2(p_2) + \sqrt{m_2^2+p_2^2}\right]}{2\sqrt{p_2^2+m_2^2}}\right\} = \frac{1}{2\mathcal{E}_2(p_2)}, \tag{1.203}$$

where we have used Eq. (1.83). Therefore, we find

$$\begin{aligned}\mathcal{C}_{(1,2)\mapsto(1,2)}[f_1(\tau,\mathbf{x},\mathbf{p}_1)] \equiv \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2\mathcal{E}_2(p_2)} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\mathcal{E}_1(q_1)} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\mathcal{E}_2(q_2)} \\ \Big[f_1(\tau,\mathbf{x},\mathbf{q}_1) f_2(\tau,\mathbf{x},\mathbf{q}_2) |\mathcal{M}_{(1,2)\mapsto(1,2)}(\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2)|^2 \\ - f_1(\tau,\mathbf{x},\mathbf{p}_1) f_2(\tau,\mathbf{x},\mathbf{p}_2) |\mathcal{M}_{(1,2)\mapsto(1,2)}(\mathbf{p}_1,\mathbf{p}_2|\mathbf{q}_1,\mathbf{q}_2)|^2\Big] \\ (2\pi)^4 \delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{q}_1-\mathbf{q}_2)\,\delta\left[\mathcal{E}_1(p_1)+\mathcal{E}_2(p_2)-\mathcal{E}_1(q_1)-\mathcal{E}_2(q_2)\right].\end{aligned} \tag{1.204}$$

Now, let us take a moment to understand what the quantity $|\mathcal{M}|^2$ is: as said, this function encodes the information about the probability that the particles' collision can occur. Hence, it must be related to the fundamental interactions between the particles themselves, that are quantum objects whose collision could take place at very high energy (and so relativistic) regimes. The theoretical framework combining quantum mechanics with special relativity, allowing us to understand the behaviour of particles at both small scales and high energies is **quantum field theory** (QFT), which treats particles as excitations of underlying fields[16], and interactions as mediated by the exchange of virtual particles. This formalism is crucial for accurately modeling and predicting the behavior of particles in the subatomic world, making it a fundamental theory in modern physics. According to QFT, the quantity $|\mathcal{M}|^2$ is the **Feynman amplitude** of the process, and it is directly related to the process' **differential cross-section** as (see e.g. Ref. [156])

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)\Big|_{(1,2)\mapsto(1,2)} \equiv \\ \equiv \frac{|\mathbf{p}_1|^2\,\left|\mathcal{M}_{(1,2)\mapsto(1,2)}(\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2)\right|^2}{32\pi^2 v_{\mathrm{rel}}\mathcal{E}_1(q_1)\mathcal{E}_2(q_2)\mathcal{E}_1(p_1)\mathcal{E}_2(p_2)}\left\{\frac{\partial\left[\mathcal{E}_1(p_1)+\mathcal{E}_2(p_2)\right]}{\partial|\mathbf{p}_1|}\right\}^{-1},\end{aligned} \tag{1.205}$$

[16]For instance, photons are excitations of the electromagnetic field.

where the conservation of energy and three-momentum has to be understood. To clarify our notation, let us just mention that here $\mathrm{d}\Omega_1 \equiv \mathrm{d}^2\hat{p}_1 = \mathrm{d}\cos\vartheta_{p_1}\mathrm{d}\varphi_{p_1}$ denotes the infinitesimal solid angle of the scattered particle 1 and $v_{\mathrm{rel}}$ is the relative velocity between the particle 1 and the particle 2 before the collision. In fact, the cross-section of a process represents the probability per unit area for particles to undergo a specific interaction or scattering when they interact through the fields described by the theory. It plays the role of a fundamental quantity used to calculate the likelihood of particle interactions in high-energy particle physics experiments. However, let us just mention that Eq. (1.205) is valid only if after the process we have the same two particles we had before it: the generalization to the $(1,\,2)\,\mapsto\,(3,\,4,\,5,\,\dots)$ case is straightforward and it can be found e.g. in Refs. [120, 156]. Now, under the reasonable assumption that the amplitude for forward and reverse reactions is the same, we can rewrite Eq. (1.204) as

$$
\begin{aligned}
&\mathcal{C}_{(1,2)\mapsto(1,2)}[f_1(\tau,\mathbf{x},\mathbf{p}_1)] \equiv \\
&\quad\equiv \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2\mathcal{E}_2(p_2)} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\mathcal{E}_1(q_1)} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\mathcal{E}_2(q_2)} \\
&\qquad (2\pi)^4 |\mathcal{M}_{(1,2)\mapsto(1,2)}(\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2)|^2 \delta\left[\mathcal{E}_1(p_1)+\mathcal{E}_2(p_2)-\mathcal{E}_1(q_1)-\mathcal{E}_2(q_2)\right] \\
&\qquad \Big[f_1(\tau,\mathbf{x},\mathbf{q}_1)f_2(\tau,\mathbf{x},\mathbf{q}_2) - f_1(\tau,\mathbf{x},\mathbf{p}_1)f_2(\tau,\mathbf{x},\mathbf{p}_2)\Big]\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{q}_1-\mathbf{q}_2)\,.
\end{aligned}
\tag{1.206}
$$

Our heuristic derivation of the collision term for the Boltzmann equation is correct, but it has lead to a result which is good enough for our purposes but it is not the most general one. First of all, we have assumed that a process of the type $(1,2)\,\mapsto\,(1,2)$ but, as mentioned before, other more complicated situations can occur, as well as decay process of the type $1\,\mapsto\,(2,3,\dots)$, and also we have neglected the effects due to the so called **Bose enhancement** and **Pauli blocking**, which should modify a bit Eq. (1.206) to take into account the fact that it is easier to produce a boson rather than a fermion. This is due to the **Pauli exclusion principle**, stating that no two fermions can occupy the same quantum state simultaneously, so that there are more states available to bosons than fermions. However, as discussed e.g. in Ref. [142], we can safely neglect such corrective terms for our purposes.

### General Perturbed Boltzmann Equation

Armed with Eq. (1.206) we are now able to approach the resolution of Eq. (1.199), but a brute force approach would be extremely challenging, and so we are going to take advantage of the perturbative approached used insofar. Indeed, we are going to derive the Boltzmann equation at zero-th and first order in perturbation theory for the particles of a species $s$, and obviously to do that we are going to stop the perturbative expansion of all the cosmological quantities before the second order

$$\frac{\partial}{\partial \tau}\left[\bar{f}(\tau,p)+\delta f_s^{(1)}(\tau,\mathbf{x},\mathbf{p})\right]+\frac{a(\tau)}{\mathcal{E}_s(p)}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda_s}\partial_i\,\delta f_s^{(1)}(\tau,\mathbf{x},\mathbf{p})= \\ =\frac{a(\tau)}{\mathcal{E}_s(p)}\mathcal{C}[f_s(\tau,\mathbf{x},\mathbf{p})]-\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\partial}{\partial p}\left[\bar{f}_s(\tau,p)+\delta f_s^{(1)}(\tau,\mathbf{x},\mathbf{p})\right]+\dots, \tag{1.207}$$

with the dots $(\cdots)$ denoting terms beyond the linear order. Let us notice that in Eq. (1.207) we have decomposed $p^i = p\hat{p}^i$ and used that $\mathrm{d}\hat{p}^i/\mathrm{d}\lambda_s$, as well as the collision term, are vanishing at zero-th order in perturbation theory, because there is nothing that could deviate the path of a particle in a homogeneous and isotropic space nor produce collisions. In order to further proceed, we have to understand what is the perturbative order of $\mathrm{d}p/\mathrm{d}\tau$. This can be done by taking the derivative of the energy with respect to the conformal time in the Poisson gauge, so that, by using Eq. (1.135), we get

$$\frac{\mathrm{d}\mathcal{E}_s(p)}{\mathrm{d}\tau}=\mathcal{H}(\tau)\mathcal{E}_s(p)+\mathcal{E}_s(p)\frac{\partial}{\partial\tau}\Psi^{(1)}(\tau,\mathbf{x})+\frac{a^2(\tau)}{\mathcal{E}_s(p)}\left[1+2\Psi^{(1)}(\tau,\mathbf{x})\right]\frac{\mathrm{d}^2x^0}{\mathrm{d}\lambda_s^2}+\dots, \tag{1.208}$$

However, by looking at Eq. (1.128), we see that the time component of the geodesic equation yields

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}\lambda_s^2}=-\Gamma^0_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\lambda_s}\frac{\mathrm{d}x^0}{\mathrm{d}\lambda_s}-2\Gamma^0_{0i}\frac{\mathrm{d}x^0}{\mathrm{d}\lambda_s}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda_s}-\Gamma^0_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda_s}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda_s}, \tag{1.209}$$

where we can evaluate the perturbed Christoffel symbols in the Poisson gauge by means of `xPand`,

$$\Gamma^0_{00}(\tau,\mathbf{x})=\mathcal{H}(\tau)+\frac{\partial}{\partial\tau}\Psi^{(1)}(\tau,\mathbf{x})+\dots \tag{1.210}$$

$$\Gamma^0_{0i}(\tau,\mathbf{x})=\mathcal{H}(\tau)\,\omega_i^{(1)}(\tau,\mathbf{x})+\partial_i\Psi^{(1)}(\tau,\mathbf{x})+\dots \tag{1.211}$$

$$\Gamma^0_{ij}(\tau,\mathbf{x})=\left\{\mathcal{H}(\tau)\left[1+2\Phi^{(1)}(\tau,\mathbf{x})-2\Psi^{(1)}(\tau,\mathbf{x})\right]+\frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})\right\}\delta_{ij} \\ +\frac{1}{2}\left[\partial_i\omega_j^{(1)}(\tau,\mathbf{x})+\partial_j\omega_i^{(1)}(\tau,\mathbf{x})\right]+\left[\mathcal{H}(\tau)+\frac{1}{2}\frac{\partial}{\partial\tau}\right]\gamma_{ij}^{(1)}(\tau,\mathbf{x})+\dots. \tag{1.212}$$

Now, by recalling the definition of three-momentum derived in Eq. (1.139), we can easily see that for the perturbed FLRW metric we have

$$p^2=\left[a^2(\tau)\delta_{ij}+\delta g_{ij}^{(1)}(\tau,\mathbf{x})\right]\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}+\dots, \tag{1.213}$$

because any contribution coming from $g_{0i}g_{0j}/g_{00}$ is beyond the linear order, so that Eq. (1.209) reduces to

$$\begin{aligned}\frac{\mathrm{d}^2x^0}{\mathrm{d}\lambda_s^2} = &-\frac{\mathcal{E}_s^2(p)}{a^2(\tau)}\left[\mathcal{H}(\tau) - 2\mathcal{H}(\tau)\Psi^{(1)}(\tau,\mathbf{x}) + \frac{\partial}{\partial\tau}\Psi^{(1)}(\tau,\mathbf{x})\right] \\ &- \frac{2\mathcal{E}_s(p)}{a^2(\tau)}\left[\mathcal{H}(\tau)\,\omega_i^{(1)}(\tau,\mathbf{x}) + \partial_i\Psi^{(1)}(\tau,\mathbf{x})\right]p^i - \mathcal{H}(\tau)\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda_s}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda_s} \\ &- \frac{p^ip^j}{a^2(\tau)}\left\{\frac{1}{2}\left[\partial_i\omega_j^{(1)}(\tau,\mathbf{x}) + \partial_j\omega_i^{(1)}(\tau,\mathbf{x})\right] + \left[\mathcal{H}(\tau) + \frac{1}{2}\frac{\partial}{\partial\tau}\right]\gamma_{ij}^{(1)}(\tau,\mathbf{x})\right\} \\ &- \frac{p^2}{a^2(\tau)}\left\{\mathcal{H}(\tau)\left[2\Phi^{(1)}(\tau,\mathbf{x}) - 2\Psi^{(1)}(\tau,\mathbf{x})\right] + \frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})\right\} + \ldots .\end{aligned} \tag{1.214}$$

Therefore, if we substitute such a result within Eq. (1.208), we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}_s(p)}{\mathrm{d}\tau} = &-\frac{p^2}{\mathcal{E}_s(p)}\left[\mathcal{H}(\tau) + \frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})\right] - 2p^i\left[\mathcal{H}(\tau)\,\omega_i^{(1)}(\tau,\mathbf{x}) + \partial_i\Psi^{(1)}(\tau,\mathbf{x})\right] \\ &- \frac{1}{2}\frac{p^ip^j}{\mathcal{E}_s(p)}\left[\partial_i\omega_j^{(1)}(\tau,\mathbf{x}) + \partial_j\omega_i^{(1)}(\tau,\mathbf{x}) + \frac{\partial}{\partial\tau}\gamma_{ij}^{(1)}(\tau,\mathbf{x})\right] + \ldots\end{aligned} \tag{1.215}$$

By means of the dispersion relation given in Eq. (1.134), we can easily evaluate the derivative of $p$ with respect to the conformal time as

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \frac{\mathrm{d}\mathcal{E}_s(p)}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\mathcal{E}(p)} = \frac{\mathrm{d}\mathcal{E}_s(p)}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}_s(p)}\left(\sqrt{\mathcal{E}_s^2(p) - m_s^2}\right) = \frac{\mathcal{E}_s(p)}{p}\frac{\mathrm{d}\mathcal{E}_s(p)}{\mathrm{d}\tau}, \tag{1.216}$$

which is what we originally wanted. Finally, we can use the last results in such a way that Eq. (1.207) leads to the zero-th,

$$\boxed{\left[\frac{\partial}{\partial\tau} - \mathcal{H}(\tau)p\frac{\partial}{\partial p}\right]\bar{f}_s(\tau,p) = 0,} \tag{1.217}$$

and the first order perturbed Boltzmann equation for the species $s$, respectively,

$$\begin{aligned}&\left[\frac{\partial}{\partial\tau}+\frac{p^i}{\mathcal{E}_s(p)}\partial_i-\mathcal{H}(\tau)p\frac{\partial}{\partial p}\right]\delta f_s^{(1)}(\tau,\mathbf{x},\mathbf{p})-\frac{a(\tau)}{\mathcal{E}_s(p)}\mathcal{C}[f_s(\tau,\mathbf{x},\mathbf{p})]=\\&\quad=\left\{\frac{p^2}{\mathcal{E}_s(p)}\frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x})+2p^i\left[\mathcal{H}(\tau)\,\omega_i^{(1)}(\tau,\mathbf{x})+\partial_i\Psi^{(1)}(\tau,\mathbf{x})\right]\right.\\&\qquad\left.+\frac{p^ip^j}{2\mathcal{E}_s(p)}\left[\partial_i\omega_j^{(1)}(\tau,\mathbf{x})+\partial_j\omega_i^{(1)}(\tau,\mathbf{x})+\frac{\partial}{\partial\tau}\gamma_{ij}^{(1)}(\tau,\mathbf{x})\right]\right\}\frac{\partial}{\partial p}\bar{f}_s(\tau,p).\end{aligned}\tag{1.218}$$

### 1.3.4 Compton Scattering

We now have all the necessary ingredients for fully understanding the origin of the cosmic microwave background radiation. As shown in Eq. (1.198), the evolution of the distribution function of a cosmic species, which is fundamental for solving the perturbed Einstein equations, is sourced by all the collision processes involving such a species, as we wrote in Eq. (1.200). In particular, we are now going to focus on photons, since their dynamics encodes fundamental information about the electromagnetic theory. For this reason, cosmological photons provide a "natural laboratory" for testing the Maxwell's theory and any eventual deviations from it. Now, among all the possible processes where cosmological photons play a role, the most significant one is the **Compton Scattering**, i.e. the collision between a photon $\gamma$ with three-momentum $\mathbf{q}_\gamma$ and an electron $e^-$ with three-momentum $\mathbf{q}_e$,

$$\gamma(\mathbf{q}_\gamma)+e^-(\mathbf{q}_e)\leftrightarrows\gamma(\mathbf{p}_\gamma)+e^-(\mathbf{p}_e).\tag{1.219}$$

However, before to enter in the mathematical details, it is better to understand why such a process is so important in cosmology. Indeed, the key point is that at some point, the Universe has expanded to such an extent that it makes it less and less likely that photons and electrons will collide, so that the process defined in Eq. (1.219) becomes more and more inefficient. Because of that, cosmological photons start to decouple from free electrons, reaching us today. Analogously, these electrons were able to take part in the hydrogen atoms' **recombination** ($\mathsf{p}^+ + e^- \leftrightarrows \mathsf{H}+\gamma$). However, because of the expansion of the Universe, the overall temperature cooled down, so that cosmological photons were not able to break Hydrogen atoms anymore. In fact, the efficient way to form hydrogen is to form it in a excited state, but when it relaxes to the ground state, the photons emitted have not enough energy to ionize other hydrogen atoms. For example, an electron captured in the $n=2$ state generates a 3.4 eV photon. Subsequently, when the electron falls in the ground state, the hydrogen releases another 10.2 eV photon. Neither of the two photons has sufficient energy for ionizing another hydrogen atom in the ground state. Therefore, almost at the same time of recombination the decoupling of photons from electron takes place. This happens because very few free electrons remain after hydrogen formation and therefore photons are free to propagate undisturbed and seen by us as the CMB. Moreover, well after decoupling, ultraviolet light emitted by stars and gas is able to ionize again hydrogen atoms, causing the so-called **reionization**, which is another fundamental source of cosmological photons, as we will see. Therefore, Compton scattering is the main character in the history of cosmic photons, and for this reason, we have now to understand its impact on their Boltzmann equation. In the laboratory reference frame, where an incoming photons collides with an electron at rest (i.e. $\mathbf{q}_e = 0$), this process can be represented as shown in Fig. 1.5. The differential cross-section of the Compton scattering

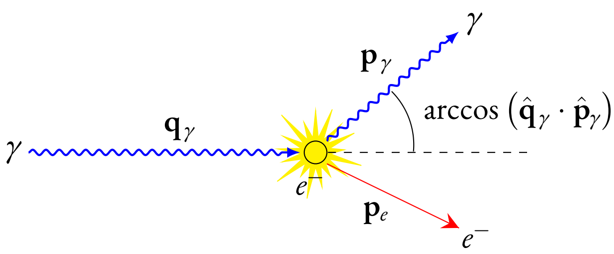


**Figure 1.5:** Representation of the Compton Scattering in a reference frame where the electron is initially at rest.

is provided by the **Klein-Nishina formula**, which has been derived for the first time in Ref. [157] as

$$\frac{d\sigma}{d\Omega}(\mathbf{q}_\gamma, \mathbf{q}_e, \mathbf{p}_\gamma, \mathbf{p}_e)\Big|_{(\gamma, e^-)\mapsto(\gamma, e^-)} = \frac{3\sigma_T}{32\pi}\left(\frac{p_\gamma}{q_\gamma}\right)^2 \left\{\frac{q_\gamma}{p_\gamma} + \frac{p_\gamma}{q_\gamma} + 4\left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 - 2\right\}, \quad (1.220)$$

where

$$\sigma_T \equiv \frac{8\pi}{3}\left(\frac{e^2}{4\pi m_e}\right)^2 \simeq 4.329 \times 10^{-5}\, 1/\mathrm{MeV}^2, \quad (1.221)$$

is the **Thomson's cross section**, with $m_e = 0.511$ MeV being the electron's mass, and $\varepsilon(\hat{\mathbf{p}})$ denotes the polarization vector associated with a photons with three-momentum $\mathbf{p}$, i.e., as we discussed in Sec. 1.1.3, the electric field's direction, so that by denoting with $\mathbf{E}_{\mathrm{in}}$ and $\mathbf{E}_{\mathrm{sc}}$ the electric fields of the incoming photon and the scattered one, respectively, we have

$$\hat{\mathbf{q}}_\gamma \cdot \varepsilon_{\lambda q}(\hat{\mathbf{q}}_\gamma) = \hat{\mathbf{q}}_\gamma \cdot \mathbf{E}_{\mathrm{in}}(t, \mathbf{x}) = 0, \qquad \hat{\mathbf{p}}_\gamma \cdot \varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma) = \hat{\mathbf{p}}_\gamma \cdot \mathbf{E}_{\mathrm{sc}}(t, \mathbf{x}) = 0, \quad (1.222)$$

and the labels $\lambda_q$, $\lambda_p$ denote the two possible polarization states of the incoming photon and the scattered one[17], respectively. By inverting Eq. (1.205), one could obtain the Feynman amplitude associated with the Compton's scattering differential cross-section. Before to do that, let us put ourselves in the reference frame defined by Fig. 1.5, in which the electron is at rest ($\mathbf{q}_e = 0$): this means that the conservation of energy and three-momentum lead to

$$\mathbf{p}_e = \mathbf{q}_\gamma - \mathbf{p}_\gamma \qquad \Longrightarrow \qquad p_\gamma = q_\gamma + m_e - \sqrt{m_e^2 + |\mathbf{q}_\gamma - \mathbf{p}_\gamma|^2}, \tag{1.223}$$

where we have substituted Eq. (1.134). By solving Eq. (1.223) with respect to $p_\gamma$, we find the famous **Compton shift formula**:

$$p_\gamma = \frac{q_\gamma m_e}{m_e + q_\gamma(1 - \hat{\mathbf{q}}_\gamma \cdot \hat{\mathbf{p}}_\gamma)}. \tag{1.224}$$

Let us now take one of the main approximations of this section, i.e. that $m_e \gg q_\gamma$: this is done by noting that the electron mass ($m_e \simeq 0.51\,\mathrm{MeV}$) is usually much larger than that of the average cosmological photon ($\mathcal{E}_\gamma \lesssim 13.6\,\mathrm{eV}$), so that the latter is almost unable to deviate the former from its path. Therefore, Eq. (1.224) simply reduces to $p_\gamma \simeq q_\gamma$ and in turn the Klein-Nishina differential cross-section simplifies to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\mathbf{q}_\gamma, \mathbf{q}_e, \mathbf{p}_\gamma, \mathbf{p}_e)\Big|_{(\gamma, e^-)\mapsto(\gamma, e^-)} \simeq \frac{3\sigma_T}{8\pi}\left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma) \cdot \varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2, \tag{1.225}$$

which is the differential cross-section of the **Thomson scattering**, i.e. the low-energy version of the Compton scattering. Now a question naturally arise: what is the Feynman amplitude which once substituted in Eq. (1.205) will yield the Thomson cross-section? Well, to avoid any computation in quantum electrodynamics, we are simply going to invert Eq. (1.205). Indeed, because of Eq. (1.223), we can write

$$\frac{\partial}{\partial p_\gamma}\left[p_\gamma + \mathcal{E}_e(q_e)\right] \simeq \frac{m_e}{\mathcal{E}_e(p_e)}, \tag{1.226}$$

and, since the relative velocity between the electron at rest and the incoming photon is 1, we then find

$$|\mathcal{M}_{(\gamma,e)\mapsto(\gamma,e)}\left(\mathbf{q}_\gamma, \mathbf{q}_e|\mathbf{p}_\gamma, \mathbf{p}_e\right)|^2 \simeq 12\pi m_e^2 \sigma_T \left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma) \cdot \varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 \tag{1.227}$$

where we highlight that such an amplitude only depends on $\hat{\mathbf{q}}_\gamma$ and $\hat{\mathbf{p}}_\gamma$.

[17] Indeed, in a three-dimensional space there are two unit vectors that are orthogonal to a third one.

### Boltzmann Equation for CMB Intensity

We now want to compute the intensity $I$ of the electromagnetic radiation for cosmological photons. By definition, intensity is nothing but a measure of the power per unit area carried by a wave, and the power is the amount of energy carried by the wave per unit time, so that the following general relation holds true:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \mathrm{d}^3\mathbf{x}\,\varrho_\gamma(t,\mathbf{x})\right] \equiv \int \mathrm{d}\mathbf{S}\cdot\mathbf{I}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) \tag{1.228}$$

where $\varrho_\gamma(t,\mathbf{x})$ is the photons' energy density, whereas $\mathbf{I}(t,\mathbf{x},\mathbf{p}_\gamma)$ is the **intensity vector** of an electromagnetic wave propagating in the $\hat{\mathbf{p}}_\gamma$-direction, i.e. a vector whose magnitude is given by the properly said intensity and whose direction is that indicating the path along which energy is being transported. It is then clear in the case of an electromagnetic wave, the intensity vector points in the direction of the wave's motion, i.e. $\mathbf{I} = I\,\hat{\mathbf{p}}_\gamma$. Without loss of generality, let us now consider a spherical volume of radius $|\mathbf{x}| = r$, so that we can easily imagine that $\mathbf{I}$ is "piercing" such a sphere, i.e. $\hat{\mathbf{p}}_\gamma\cdot\mathrm{d}\mathbf{S} = r^2\,\mathrm{d}^2\hat{p}_\gamma$, and we can rewrite Eq. (1.228) as

$$4\pi\frac{\mathrm{d}r}{\mathrm{d}t}r^2\varrho(t,\mathbf{x}) = \int \mathrm{d}^2\hat{p}_\gamma\,r^2\,I(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) \quad\Longrightarrow\quad \varrho(t,\mathbf{x}) = \int\frac{\mathrm{d}^2\hat{p}_\gamma}{4\pi}\,I(t,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.229}$$

where we have implicitly assumed that the energy density is spherically symmetric, and used that for an electromagnetic wave $\mathrm{d}r/\mathrm{d}t = 1$, since the wave-front propagates at the speed of light. Moreover, by comparing Eq. (1.185) with Eq. (1.229), we easily notice that we must have

$$I(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \int_0^\infty \frac{\mathrm{d}p_\gamma\,p_\gamma^2}{2\pi^2}\mathcal{E}_\gamma(p_\gamma)f_\gamma(\tau,\mathbf{x},\mathbf{p}_\gamma) = \frac{\pi^2}{15}\overline{T}_\gamma^4(\tau) + \sum_{n=1}^\infty \delta I^{(n)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.230}$$

where we have moved from the cosmic time $t$ to the conformal time $\tau$, as usually, and defined

$$\boxed{\delta I^{(n)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv \int\frac{\mathrm{d}p\,p^2}{2\pi^2}\,\mathcal{E}_\gamma(p_\gamma)\,\delta f_\gamma^{(n)}(\tau,\mathbf{x},\mathbf{p}_\gamma).} \tag{1.231}$$

Let us notice that, as a further evidence of the validity of our approach, we have derived the **Stefan-Boltzmann law** expressing the intensity of the power radiated by a photons' distribution, i.e. $\bar{I}(\tau) = \pi^2\overline{T}_\gamma^4(\tau)/15$. We now want to find the study the evolution of the photons' intensity at first order in perturbation theory, and for this reason we are going to provide a new definition,

$$\delta I^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv 4\varrho_\gamma(\tau)\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.232}$$

and we apply the $\int \left[\mathrm{d}^3 p_\gamma\, \mathcal{E}_\gamma(p_\gamma)\right] /(2\pi)^3$ operator to Eq. (1.218) for photons ($\mathcal{E}_\gamma = p_\gamma$), so that we find

$$\left[\frac{\partial}{\partial \tau} + \hat{p}_\gamma^i \partial_i\right] \Theta_T^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) + \mathcal{G}^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \frac{a(\tau)}{4\bar{\varrho}_\gamma(\tau)} \int \frac{\mathrm{d}p_\gamma p_\gamma^2}{2\pi^2} \mathcal{C}[f_\gamma(\tau, \mathbf{x}, \mathbf{p}_\gamma)], \tag{1.233}$$

where we have used the definition of energy density we gave in Eq. (1.185) together with the continuity equation, i.e. Eq. (1.149), and defined

$$\begin{aligned} \mathcal{G}^{(1)}[\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma] \equiv \frac{\partial}{\partial \tau}\Phi^{(1)}(\tau, \mathbf{x}) + 2\left[\mathcal{H}(\tau)\,\omega_i^{(1)}(\tau, \mathbf{x}) + \partial_i \Psi^{(1)}(\tau, \mathbf{x})\right] \hat{p}_\gamma^i \\ + \left[\partial_i \omega_j^{(1)}(\tau, \mathbf{x}) + \partial_j \omega_i^{(1)}(\tau, \mathbf{x}) + \frac{\partial}{\partial \tau}\gamma_{ij}^{(1)}(\tau, \mathbf{x})\right] \frac{\hat{p}_\gamma^i \hat{p}_\gamma^j}{2}. \end{aligned} \tag{1.234}$$

According to Eq. (1.206), the collision term for the Compton scattering at first order reads

$$\begin{aligned} &\mathcal{C}_{(\gamma, e^-) \mapsto (\gamma, e^-)}[f_\gamma(\tau, \mathbf{x}, \mathbf{p}_\gamma)] = \\ &\equiv \int \frac{\mathrm{d}^3 p_e}{(2\pi)^3 2\mathcal{E}_e(p_e)} \int \frac{\mathrm{d}^3 q_\gamma}{(2\pi)^3 2\mathcal{E}_\gamma(q_\gamma)} \int \frac{\mathrm{d}^3 q_e}{(2\pi)^3 2\mathcal{E}_e(q_e)} \delta^{(3)}\left(\mathbf{p}_\gamma + \mathbf{p}_e - \mathbf{q}_\gamma - \mathbf{q}_e\right) \\ &\left[\bar{f}_\gamma(\tau, q_\gamma)\delta f_e^{(1)}(\tau, \mathbf{x}, \mathbf{q}_e) + \delta f_\gamma^{(1)}(\tau, \mathbf{x}, \mathbf{q}_\gamma)\bar{f}_e(\tau, q_e)\right. \\ &\left. - \bar{f}_\gamma(\tau, p_\gamma)\delta f_e^{(1)}(\tau, \mathbf{x}, \mathbf{p}_e) - \delta f_\gamma^{(1)}(\tau, \mathbf{x}, \mathbf{p}_\gamma)\bar{f}_e(\tau, p_e)\right] \\ &(2\pi)^4 |\mathcal{M}_{(\gamma, e) \mapsto (\gamma, e)}\left(\mathbf{q}_\gamma, \mathbf{q}_e | \mathbf{p}_\gamma, \mathbf{p}_e\right)|^2 \delta\left[\mathcal{E}_\gamma(p_\gamma) + \mathcal{E}_e(p_e) - \mathcal{E}_\gamma(q_\gamma) - \mathcal{E}_e(q_e)\right], \end{aligned} \tag{1.235}$$

which can be further simplified by integrating over $\mathbf{p}_e$ with the help of the Dirac delta,

$$\begin{aligned} &\mathcal{C}_{(\gamma, e^-) \mapsto (\gamma, e^-)}[f_\gamma(\tau, \mathbf{x}, \mathbf{p}_\gamma)] \simeq \\ &\simeq \int \frac{\mathrm{d}^3 q_\gamma}{2q_\gamma} \int \frac{\mathrm{d}^3 q_e}{(2\pi)^5 4m_e^2} \delta\left[p_\gamma + \frac{|\mathbf{q}_\gamma + \mathbf{q}_e - \mathbf{p}_\gamma|^2}{2m_e} - q_\gamma - \frac{q_e^2}{2m_e}\right] \\ &\left[\bar{f}_\gamma(\tau, q_\gamma)\delta f_e^{(1)}(\tau, \mathbf{x}, \mathbf{q}_e) + \delta f_\gamma^{(1)}(\tau, \mathbf{x}, \mathbf{q}_\gamma)\bar{f}_e(\tau, q_e)\right. \\ &\left. - \bar{f}_\gamma(\tau, p_\gamma)\delta f_e^{(1)}(\tau, \mathbf{x}, \mathbf{q}_\gamma + \mathbf{q}_e - \mathbf{p}_\gamma) - \delta f_\gamma^{(1)}(\tau, \mathbf{x}, \mathbf{p}_\gamma)\bar{f}_e(\tau, |\mathbf{q}_\gamma + \mathbf{q}_e - \mathbf{p}_\gamma|)\right] \\ &|\mathcal{M}_{(\gamma, e) \mapsto (\gamma, e)}\left(\mathbf{q}_\gamma, \mathbf{q}_e | \mathbf{p}_\gamma, \mathbf{p}_\gamma - \mathbf{q}_\gamma - \mathbf{q}_e\right)|^2. \end{aligned} \tag{1.236}$$

where we have also reasonably assumed that the electron's velocity is negligible with respect to that of the photon, so that it behaves as a non-relativistic particle and its energy is dominated by the mass:

$$\mathcal{E}_e(q_e) = \sqrt{q_e^2 + m_e^2} = m_e + \frac{q_e^2}{2m_e} + \mathcal{O}(q_e^4/m_e^3). \tag{1.237}$$

In order to further proceed we can make another approximation: since the electron mass-energy is usually so much larger than that of the average photon, we can assume that the latter is almost unable to deviate the former from its path. Mathematically speaking, this implies $|\mathbf{p}_\gamma| \simeq |\mathbf{q}_\gamma| \ll |\mathbf{q}_e|$, so that

$$|\mathbf{q}_\gamma + \mathbf{q}_e - \mathbf{p}_\gamma|^2 = \frac{q_e^2}{2m_e} + \frac{\mathbf{q}_e \cdot (\mathbf{q}_\gamma - \mathbf{p}_\gamma)}{m_e} + \mathcal{O}\left(|\mathbf{q}_\gamma - \mathbf{p}_\gamma|^2\right), \tag{1.238}$$

and hence the Dirac delta over the energies can be rewritten as

$$\begin{aligned} \delta\left[p_\gamma + \frac{|\mathbf{q}_\gamma + \mathbf{q}_e - \mathbf{p}_\gamma|^2}{2m_e} - q_\gamma - \frac{q_e^2}{m_e}\right] &\simeq \delta\left[p_\gamma + \frac{\mathbf{q}_e \cdot (\mathbf{q}_\gamma - \mathbf{p}_\gamma)}{m_e} - q_\gamma\right] \\ &\simeq \delta(p_\gamma - q_\gamma) + \frac{\mathbf{q}_e \cdot (\mathbf{p}_\gamma - \mathbf{q}_\gamma)}{m_e}\frac{\partial\delta(p_\gamma - q_\gamma)}{\partial q_\gamma}. \end{aligned} \tag{1.239}$$

We are now going to substitute these expression, as well as the Feynman amplitude, within Eq. (1.236), but when doing that we also include "by hand" a summation over the photon's initial polarization states. This is done for exactly the same reason we are integrating over the three-momenta of the initial particles, that is for averaging over our ignorance about the situation before the scattering process. Therefore, after some trivial algebra, Eq. (1.236) reduces to

$$\begin{aligned} &\mathcal{C}_{(\gamma,e^-)\mapsto(\gamma,e^-)}[f_\gamma(\tau,\mathbf{x},\mathbf{p}_\gamma)] \simeq \\ &\simeq \frac{3\sigma_T}{2m_e}\int\frac{\mathrm{d}^3q_\gamma}{4\pi q_\gamma}\int\frac{\mathrm{d}^3q_e}{(2\pi)^3}\sum_{\lambda_q}\left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 \\ &\qquad \Big\{\mathbf{q}_e\cdot(\mathbf{p}_\gamma - \mathbf{q}_\gamma)\left[\bar{f}_\gamma(\tau,q_\gamma) - \bar{f}_\gamma(\tau,p_\gamma)\right]\delta f_e^{(1)}(\tau,\mathbf{x},\mathbf{q}_e)\frac{\partial\delta(p_\gamma - q_\gamma)}{\partial q_\gamma} \\ &\qquad\qquad + m_e\bar{f}_e(\tau,q_e)\left[\delta f_\gamma^{(1)}(\tau,\mathbf{x},\mathbf{q}_\gamma) - \delta f_\gamma^{(1)}(\tau,\mathbf{x},\mathbf{p}_\gamma)\right]\delta(p_\gamma - q_\gamma)\Big\}, \end{aligned} \tag{1.240}$$

where we have exploited the Dirac deltas and used that

$$\int \frac{d^3 q_e}{(2\pi)^3}\, \mathbf{q}_e\, \bar{f}_e(\tau, q_e) = \bar{\varrho}_e(\tau) \int_0^{2\pi} d\varphi_{q_e} \int_{-1}^{+1} \frac{d\vartheta_{q_e}}{4\pi} \begin{pmatrix} \cos\varphi_{q_e} \sin\vartheta_{q_e} \\ \sin\varphi_{q_e} \sin\vartheta_{q_e} \\ \cos\vartheta_{q_e} \end{pmatrix} = 0. \tag{1.241}$$

By recalling the definition of Eq. (1.184), we see that

$$\int \frac{d^3 q_e}{(2\pi)^3}\, \bar{f}_e(\tau, q_e) = \bar{n}_e(\tau), \qquad \int \frac{d^3 q_e}{(2\pi)^3}\, \mathbf{q}_e\, \delta f_e^{(1)}(\tau, \mathbf{x}, \mathbf{q}_e) = m_e \bar{n}_e(\tau) \mathbf{v}_e(\tau, \mathbf{x}), \tag{1.242}$$

where $m_e \mathbf{v}_e(\tau, \mathbf{x})$ is, by definition, the non-relativistic three-momentum of an electron. Thanks to these intermediate results, we are now able to rewrite Eq. (1.240) as

$$\begin{aligned} \mathcal{C}_{(\gamma, e^-) \mapsto (\gamma, e^-)}[f_\gamma(\tau, \mathbf{x}, \mathbf{p}_\gamma)] = \frac{3 p_\gamma}{2 a(\tau)} \frac{d\kappa(\tau)}{d\tau} \int \frac{d^2 \hat{q}_\gamma}{4\pi} \sum_{\lambda_q} \left[\boldsymbol{\varepsilon}_{\lambda_q}(\hat{\mathbf{q}}_\gamma) \cdot \boldsymbol{\varepsilon}_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 \\ \left\{ \delta f_\gamma^{(1)}(\tau, \mathbf{x}, \mathbf{p}_\gamma) - \delta f_\gamma^{(1)}(\tau, \mathbf{x}, p_\gamma \hat{\mathbf{q}}_\gamma) + \mathbf{v}_e(\tau, \mathbf{x}) \cdot \left(\hat{\mathbf{p}}_\gamma - \hat{\mathbf{q}}_\gamma\right) p_\gamma \frac{\partial}{\partial p_\gamma} \bar{f}_\gamma(\tau, p_\gamma) \right\}, \end{aligned} \tag{1.243}$$

where we have defined the so called **optical depth** $\kappa(\tau)$ as

$$\frac{d\kappa(\tau)}{d\tau} \equiv -\sigma_T a(\tau) \bar{n}_e(\tau). \tag{1.244}$$

By substituting Eq. (1.243) within Eq. (1.233) we finally find

$$\begin{aligned} \left[\frac{\partial}{\partial \tau} + \hat{\mathbf{p}}_\gamma \cdot \boldsymbol{\nabla}\right] \Theta_T^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) + \mathcal{G}^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \frac{3}{2} \frac{d\kappa(\tau)}{d\tau} \int \frac{d^2 \hat{q}_\gamma}{4\pi} \sum_{\lambda_q} \left[\boldsymbol{\varepsilon}_{\lambda_q}(\hat{\mathbf{q}}_\gamma) \cdot \boldsymbol{\varepsilon}_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 \\ \left\{ \Theta_T^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) - \Theta_T^{(1)}(\tau, \mathbf{x}, \hat{\mathbf{q}}_\gamma) + \mathbf{v}_e(\tau, \mathbf{x}) \cdot \left(\hat{\mathbf{q}}_\gamma - \hat{\mathbf{p}}_\gamma\right) \right\}. \end{aligned} \tag{1.245}$$

In order to further proceed, we must understand how to deal with the polarization vectors, that are defined by the property in Eq. (1.222). For instance, if we write down here the most general expression of $\hat{\mathbf{q}}_\gamma$ in spherical coordinates, it follows that the polarization vector can be written as

$$\hat{\mathbf{q}}_\gamma \equiv \begin{pmatrix} \cos\varphi_{q_\gamma}\sin\vartheta_{q_\gamma} \\ \sin\varphi_{q_\gamma}\sin\vartheta_{q_\gamma} \\ \cos\vartheta_{q_\gamma} \end{pmatrix} \implies \varepsilon_{\pm 1}(\hat{\mathbf{q}}_\gamma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\vartheta_{q_\gamma}\cos\varphi_{q_\gamma} \pm \sin\varphi_{q_\gamma} \\ \cos\vartheta_{q_\gamma}\sin\varphi_{q_\gamma} \mp \cos\varphi_{q_\gamma} \\ -\sin\vartheta_{q_\gamma} \end{pmatrix}, \tag{1.246}$$

since this is the most general unit vector satisfying the required properties, and, as just mentioned before, the label $\lambda_q = \pm 1$ defines the two possible polarization states. Therefore it is possible to evaluate the following integrals e.g. with `Mathematica`,

$$\int \frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi} \sum_{\lambda_q=\pm 1} \left[\varepsilon_{\lambda_p}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_q}(\hat{\mathbf{p}}_\gamma)\right]^2 = \frac{2}{3}, \tag{1.247}$$

$$\int \frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi}\, \hat{\mathbf{q}}_\gamma \sum_{\lambda_q=\pm 1} \left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 = 0, \tag{1.248}$$

so that we finally find

$$\begin{aligned} &\left[\frac{\partial}{\partial\tau} + \hat{\mathbf{p}}_\gamma\cdot\boldsymbol{\nabla} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right]\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) + \mathcal{G}^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \\ &= -\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\left\{\mathbf{v}_e(\tau,\mathbf{x})\cdot\hat{\mathbf{p}}_\gamma + \frac{3}{2}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi}\sum_{\lambda_q}\left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right\}. \end{aligned} \tag{1.249}$$

## Boltzmann Equation for CMB Polarization

After a bit of calculations we have finally found Eq. (1.249), i.e. the Boltzmann equation for the intensity of cosmological photons at first order in perturbation theory. However, as we discussed in Sec. 1.1.3, this is just part of the story, since also the other three Stokes parameter $Q$, $U$ and $V$ are necessary to completely characterize any kind of electromagnetic radiation. In Eq. (1.229) we have seen that the intensity of an electromagnetic wave is directly associated with its energy density, whose general expression can be derived by means of the energy-momentum tensor,

$$\begin{aligned} \varrho_\gamma(t,\mathbf{x}) &= -\mathcal{T}^0{}_0 = g^{0\mu}\mathcal{T}_{\mu 0}(t,\mathbf{x}) = -\frac{g^{0\mu}(t,\mathbf{x})\,\delta^\nu{}_0}{2\sqrt{-g}(t,\mathbf{x})}\frac{\delta}{\delta g^{\mu\nu}(t,\mathbf{x})}\int \mathrm{d}^4\tilde{x}\sqrt{-g(\tilde{t},\tilde{\mathbf{x}})}\,F_{\alpha\beta}(\tilde{t},\tilde{\mathbf{x}})F^{\alpha\beta}(\tilde{t},\tilde{\mathbf{x}}) \\ &= -\mathcal{L}_{\mathrm{EM}}(t,\mathbf{x}) - F^{0\mu}(t,\mathbf{x})F_{0\mu}(t,\mathbf{x}) = \frac{1}{2}\left[E^2(t,\mathbf{x}) + B^2(t,\mathbf{x})\right] = E^2(t,\mathbf{x}), \end{aligned} \tag{1.250}$$

where we used Eq. (1.123) and Eq. (1.44). In fact, let us notice that, in the case $\hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3$, it is easy to see that Eq. (1.62) yields

$$\rho_\gamma(\tau, \mathbf{x}) = \int \frac{d^2\hat{p}_\gamma}{4\pi} I(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3) = E_1^2(t, \mathbf{x}) + E_2^2(t, \mathbf{x}) = |\mathbf{E}(\mathbf{x}, t)|^2, \tag{1.251}$$

as expected. Hence our characterization of the Stokes parameter $I$ as the intensity of the electromagnetic wave is consistent with the definition of intensity we derived in Eq. (1.229). This should not surprise us, since it is a manifestation of the intrinsic wave-particle duality of light, which can be described in terms of of photons, as well as electromagnetic waves, but the two prescriptions must be consistent. Now, from a phenomenological point of view, the Compton scattering is nothing but the collision between a photon and an electron: when the electric field associated with such a photon interacts with the electron, this starts to oscillate and to emit electromagnetic waves in almost all directions. In Sec. 1.1.4 we have found that the expression of the electric field generated by a moving electron is provided by Eq. (1.95),

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) = e \int \frac{d\tilde{t}}{4\pi} \frac{[1 - v^2(\tilde{t})]\,[\mathbf{x} - \mathbf{s}(\tilde{t}) - \mathbf{v}(\tilde{t})]}{\{\, |\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})] \cdot \mathbf{v}(\tilde{t}) \,\}^3} \delta[\,\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|\,] \\ + e \int \frac{d\tilde{t}}{4\pi} \frac{[\mathbf{x} - \mathbf{s}(\tilde{t})] \times \{[\mathbf{x} - \mathbf{s}(\tilde{t}) - \mathbf{v}(\tilde{t})] \times \mathbf{a}(\tilde{t})\}}{\{\, |\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})] \cdot \mathbf{v}(\tilde{t}) \,\}^3} \delta[\,\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|\,].\end{aligned} \tag{1.252}$$

However, our purpose here is to discuss about the cosmic microwave background radiation, and not all the terms on the right-hand side of the equation above contribute to the electromagnetic energy flux as "radiation". Indeed, when talking about radiation, we have to consider only those contributions that give rise to a radiated energy, i.e. something which detaches itself from the source and propagates off to infinity: this is why we are able to see CMB photons today on Earth with our telescopes, even if the last Compton scattering occurred billions of years ago at the borders of the observable Universe. Therefore, it is clear that the first term of Eq. (1.252) is completely negligible when $|\mathbf{x} - \mathbf{s}(\tilde{t})| \to +\infty$, with respect to the second one, so that we can write

$$\mathbf{E}_{\rm sc}(t, \mathbf{x}) \simeq e \int \frac{d\tilde{t}}{4\pi} \frac{[\mathbf{x} - \mathbf{s}(\tilde{t})] \times \{[\mathbf{x} - \mathbf{s}(\tilde{t}) - |\mathbf{x} - \mathbf{s}(\tilde{t})|\mathbf{v}(\tilde{t})] \times \mathbf{a}(\tilde{t})\}}{\{\, |\mathbf{x} - \mathbf{s}(\tilde{t})| - [\mathbf{x} - \mathbf{s}(\tilde{t})] \cdot \mathbf{v}(\tilde{t}) \,\}^3} \delta[\,\tilde{t} - t + |\mathbf{x} - \mathbf{s}(t)|\,]. \tag{1.253}$$

Now, let us assume that, after the collision, the electron is moving with a non-relativistic velocity $v \ll 1$ at a distance from us which is large compared to the movements performed by the electron with respect to the

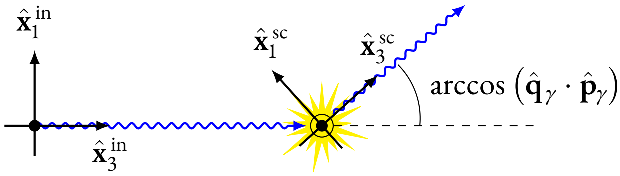


**Figure 1.6:** Plane where the Compton scattering occurs. The unit vectors $\hat{\mathbf{x}}_2^{\text{in}}$ and $\hat{\mathbf{x}}_2^{\text{sc}}$ are equal and going out from the page orthogonally.

origin of the reference frame, i.e. $|\mathbf{s}(t)| \ll |\mathbf{x}|$, so that we can write

$$\mathbf{E}_{\text{sc}}(t, \mathbf{x}) \simeq \frac{e}{4\pi|\mathbf{x}|} \{\hat{\mathbf{x}} \times [\hat{\mathbf{x}} \times \mathbf{a}(t)]\} \,. \tag{1.254}$$

Now, since we are assuming a non-relativistic regime for the electron, it is clear that its acceleration can be computed by considering the electrostatic force acting on it, i.e. that one sourced by the electric field of the incoming photon,

$$m_e\,\mathbf{a}(t) \simeq e\,\mathbf{E}_{\text{in}}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma) \implies \boxed{\mathbf{E}_{\text{sc}}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma) \simeq \sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} \left\{\hat{\mathbf{p}}_\gamma \times \left[\hat{\mathbf{p}}_\gamma \times \mathbf{E}_{\text{in}}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma)\right]\right\} ,} \tag{1.255}$$

where, by exploiting Eq. (1.41), we have also noticed that

$$0 = \hat{\mathbf{p}}_\gamma \cdot \mathbf{E}_{\text{sc}}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} \left\{\left(\hat{\mathbf{p}}_\gamma \cdot \hat{\mathbf{x}}\right) \left[\hat{\mathbf{x}} \cdot \mathbf{E}_{\text{in}}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma)\right] - \hat{\mathbf{p}}_\gamma \cdot \mathbf{E}_{\text{in}}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma)\right\} , \tag{1.256}$$

which is satisfied by $\hat{\mathbf{x}} = \pm\hat{\mathbf{p}}_\gamma$, as it can be verified by direct substitution. Without loss of generality, in order to make advantage of the expressions for the Stokes parameters we provided in Eqs. (1.62)-(1.57), let us now define our reference frame as shown in Fig. 1.6, in such a way that $\hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\text{in}}$ and $\hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\text{sc}}$. Therefore, by looking at Fig. 1.6, it is clear that with this simple geometry we can decompose the incoming and the scattered electric field as

$$\mathbf{E}_{\text{in}}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\text{in}}) = E_1^{\text{in}}(t, \mathbf{x})\hat{\mathbf{x}}_1^{\text{in}} + E_2^{\text{in}}(t, \mathbf{x})\hat{\mathbf{x}}_2^{\text{in}} \tag{1.257}$$

$$\mathbf{E}_{\text{sc}}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\text{sc}}) = E_1^{\text{sc}}(t, \mathbf{x})\hat{\mathbf{x}}_1^{\text{sc}} + E_2^{\text{sc}}(t, \mathbf{x})\hat{\mathbf{x}}_2^{\text{sc}}, \tag{1.258}$$

where the two frames are related by a rotation about an angle $\psi \equiv \arccos\left(\hat{\mathbf{q}}_\gamma \cdot \hat{\mathbf{p}}_\gamma\right)$ around the $\hat{\mathbf{x}}_2^{\rm in}$-axis,

$$\begin{pmatrix} \hat{\mathbf{x}}_1^{\rm sc} \\ \hat{\mathbf{x}}_2^{\rm sc} \\ \hat{\mathbf{x}}_3^{\rm sc} \end{pmatrix} = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1^{\rm in} \\ \hat{\mathbf{x}}_2^{\rm in} \\ \hat{\mathbf{x}}_3^{\rm in} \end{pmatrix}, \tag{1.259}$$

which means that, by recalling again Eq. (1.91), we can rewrite Eq. (1.255) as

$$\begin{aligned} \mathbf{E}_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) &= \sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} \left\{ \left[\hat{\mathbf{x}}_3^{\rm sc} \cdot \mathbf{E}_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in})\right] \hat{\mathbf{x}}_3^{\rm sc} - \mathbf{E}_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}) \right\}, \\ &= -\sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} \Big[ E_1^{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}) \cos^2\psi \, \hat{\mathbf{x}}_1^{\rm in} + E_2^{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}) \, \hat{\mathbf{x}}_2^{\rm in} \\ &\qquad - E_1^{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}) \sin\psi \cos\psi \, \hat{\mathbf{x}}_3^{\rm in} \Big]. \end{aligned} \tag{1.260}$$

We can then derive the components of $\mathbf{E}_{\rm sc}$ through the standard procedure,

$$E_1^{\rm sc}(t, \mathbf{x}) = \hat{\mathbf{x}}_1^{\rm sc} \cdot \mathbf{E}_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = -\sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} \cos\psi E_1^{\rm in}(t, \mathbf{x}), \tag{1.261}$$

$$E_2^{\rm sc}(t, \mathbf{x}) = \hat{\mathbf{x}}_2^{\rm sc} \cdot \mathbf{E}_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = -\sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}} E_2^{\rm in}(t, \mathbf{x}), \tag{1.262}$$

so that, by means of Eqs. (1.62)-(1.57), we are now in the position of relating the Stokes parameter of the scattered electromagnetic wave to those of the incoming one as

$$I_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = \frac{3\sigma_T}{8\pi|\mathbf{x}|^2} \left\{ \cos^2\psi \langle \left[E_1^{\rm in}(t, \mathbf{x})\right]^2 \rangle_t + \langle \left[E_2^{\rm in}(t, \mathbf{x})\right]^2 \rangle_t \right\}, \tag{1.263}$$

$$\begin{aligned} \left[Q \pm iU\right]_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = \frac{3\sigma_T}{8\pi|\mathbf{x}|^2} \Big\{ & \cos^2\psi \langle \left[E_1^{\rm in}(t, \mathbf{x})\right]^2 \rangle_t - \langle \left[E_2^{\rm in}(t, \mathbf{x})\right]^2 \rangle_t \\ & \pm i \cos\psi \, U_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}) \Big\}, \end{aligned} \tag{1.264}$$

$$V_{\rm sc}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = \frac{3\sigma_T}{8\pi|\mathbf{x}|^2} \cos\psi \, V_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}). \tag{1.265}$$

By rewriting the incoming electric field in terms of its Stokes parameter, after a bit of manipulation we then find the following relation, which holds true, as it can be verified by direct substitution:

$$
\begin{aligned}
&\lambda^{-1}(|\mathbf{x}|) \begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix} (t, \mathbf{x}, \hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3^{\rm sc}) = \\
&= \begin{bmatrix} \frac{1+\cos^2\psi}{2} & -\frac{\sin^2\psi}{4} & -\frac{\sin^2\psi}{4} & 0 \\ -\frac{\sin^2\psi}{2} & \frac{(1+\cos\psi)^2}{4} & \frac{(1-\cos\psi)^2}{4} & 0 \\ -\frac{\sin^2\psi}{2} & \frac{(1-\cos\psi)^2}{4} & \frac{(1+\cos\psi)^2}{4} & 0 \\ 0 & 0 & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} I_{\rm in} \\ (Q+iU)_{\rm in} \\ (Q-iU)_{\rm in} \\ V_{\rm in} \end{bmatrix} (t, \mathbf{x}, \hat{\mathbf{q}}_\gamma = \hat{\mathbf{x}}_3^{\rm in}),
\end{aligned}
\tag{1.266}
$$

where we have defined $\lambda(|\mathbf{x}|) \equiv 3\sigma_T/(8\pi|\mathbf{x}|^2)$. However, Eq. (1.266) is valid only in the scattering plane we have defined in Fig. 1.6, and it must be generalized to any reference frame. To do that, we have simply to apply a rotation of an angle (say $-\varsigma$) which will bring $\hat{\mathbf{q}}_\gamma$ from being along the $\hat{\mathbf{x}}_3^{\rm in}$ axes on a generic direction, and similarly a rotation of angle (say $\xi$) which will do the same for $\hat{\mathbf{p}}_\gamma$. Luckily, we already know from Sec. 1.1.3 how the Stokes parameters transform under a rotation, so that, by denoting with $\mathsf{S}(\psi)$ the **scattering matrix**, i.e. the $4\times 4$ matrix on the right-hand side of Eq. (1.266), and by $\mathsf{R}(-\varsigma)$, $\mathsf{R}(\xi)$ the rotation matrices parameterized by the angles $-\varsigma$, $\xi$, respectively, we have

$$
\mathsf{R}(\xi) \begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix} (t, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \lambda(|\mathbf{x}|)\mathsf{S}(\psi)\mathsf{R}(-\varsigma) \begin{bmatrix} I_{\rm in} \\ (Q+iU)_{\rm in} \\ (Q-iU)_{\rm in} \\ V_{\rm in} \end{bmatrix} (t, \mathbf{x}, \hat{\mathbf{q}}_\gamma),
\tag{1.267}
$$

which means the vector containing the Stokes parameters of the scattered electric field is obtained by multiplying the matrix $\lambda(|\mathbf{x}|)\mathsf{R}^{-1}(\xi)\mathsf{S}(\psi)\mathsf{R}(-\varsigma)$ by the vector containing the Stokes parameters of the incoming

**Table 1.1:** Explicit functional form of some of the spin-0 and spin-2 spherical harmonics as computed from Eq. (1.269). The complex conjugated functions are ${}_sY^*_{\ell m}(\psi,\varsigma) = (-1)^{s+m}\,{}_{-s}Y_{\ell-m}(\psi,\varsigma)$.

| $m$ | ${}_0Y_{2m}(\psi,\varsigma)$ | ${}_2Y_{2m}(\psi,\varsigma)$ |
|---|---|---|
| $0$ | $\frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\psi - 1)$ | $\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\psi$ |
| $\pm1$ | $\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\psi\cos\psi\, e^{\pm i\varsigma}$ | $\frac{1}{4}\sqrt{\frac{5}{\pi}}\sin\psi\,(1\mp\cos\psi)e^{\pm i\varsigma}$ |
| $\pm2$ | $\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\psi\, e^{\pm 2i\varsigma}$ | $\frac{1}{8}\sqrt{\frac{5}{\pi}}(1\mp\cos\psi)^2 e^{\pm 2i\varsigma}$ |

one[18]. After some trivial algebra, we then find

$$\mathsf{R}^{-1}(\xi)\mathsf{S}(\psi)\mathsf{R}(-\varsigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2i\xi} & 0 & 0 \\ 0 & 0 & e^{2i\xi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathsf{S}(\psi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2i\varsigma} & 0 & 0 \\ 0 & 0 & e^{2i\varsigma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos^2\psi & -e^{-2i\varsigma}\frac{\sin^2\psi}{2} & -e^{2i\varsigma}\frac{\sin^2\psi}{2} & 0 \\ -e^{-2i\xi}\sin^2\psi & e^{-2i(\xi+\varsigma)}\frac{(1+\cos\psi)^2}{2} & e^{-2i(\xi-\varsigma)}\frac{(1-\cos\psi)^2}{2} & 0 \\ -e^{2i\xi}\sin^2\psi & e^{2i(\xi-\varsigma)}\frac{(1-\cos\psi)^2}{2} & e^{2i(\xi+\varsigma)}\frac{(1+\cos\psi)^2}{2} & 0 \\ 0 & 0 & 0 & 2\cos\psi \end{bmatrix}. \tag{1.268}$$

In order to further proceed, it is convenient to collect the whole matrix's functional dependence on the angles $\psi,\varsigma$ in some special functions called **spin-weighted spherical harmonics** (see Refs. [158, 159]),

$${}_sY_{\ell m}(\psi,\varsigma) = \left(-\frac{s}{|s|}\right)^s \sqrt{\frac{(l-|s|)!}{(l+|s|)!}} \sum_{j=0}^{|s|} \binom{|s|}{j} \left(\frac{is}{|s|}\right)^j \frac{\partial^{|s|-j}}{\partial\psi^{|s|-j}} \frac{1}{\sin^j(\psi)} \frac{\partial^j}{\partial\varsigma^j}\, {}_0Y_{\ell m}(\psi,\varsigma), \tag{1.269}$$

where a non-vanishing result is obtained only if $|m|, |s| < \ell$, and the ${}_0Y_{\ell m}(\psi,\varsigma)$ can be found e.g. in the tables of the Particle Data Group. In fact, for sake of simplicity, we have computed the spin-weighted spherical harmonics relevant for us, and collected them in Tab. 1.1, so that it is possible to verify that the following

[18] Indeed, the function $\lambda(|\mathbf{x}|)$ only depends on the modulus of $\mathbf{x}$, and hence it is not affected by a rotation.

expression, derived in Ref. [160] starting from the results present in Ref. [161], holds true:

$$\frac{3}{2}\sqrt{\frac{5}{\pi}}\mathsf{R}^{-1}(\xi)\mathsf{S}(\psi)\mathsf{R}(-\varsigma) = \\ = \begin{bmatrix} {}_0Y_{20}(\psi,\varsigma)+2\sqrt{5}{}_0Y_{00}(\psi,\varsigma) & -\sqrt{\frac{3}{2}}{}_0Y_{2-2}(\psi,\varsigma) & -\sqrt{\frac{3}{2}}{}_0Y_{22}(\psi,\varsigma) & 0 \\ -\sqrt{6}e^{-2i\xi}{}_2Y_{20} & 3e^{-2i\xi}{}_2Y_{2-2}(\psi,\varsigma) & 3e^{-2i\xi}{}_2Y_{22}(\psi,\varsigma) & 0 \\ -\sqrt{6}e^{2i\xi}{}_{-2}Y_{20}(\psi,\varsigma) & 3e^{2i\xi}{}_{-2}Y_{2-2}(\psi,\varsigma) & 3e^{2i\xi}{}_{-2}Y_{22}(\psi,\varsigma) & 0 \\ 0 & 0 & 0 & \sqrt{15}\,{}_0Y_{10}(\psi,\varsigma) \end{bmatrix}. \tag{1.270}$$

We can further simplify such a result, by exploiting the **addition theorem of spherical harmonics**,

$$\sum_{m=-\ell}^{\ell} {}_{s_1}Y^*_{\ell m}(\vartheta_{q_\gamma},\varphi_{q_\gamma})\,{}_{s_2}Y_{\ell m}(\vartheta_{p_\gamma},\varphi_{p_\gamma}) = \sqrt{\frac{2\ell+1}{4\pi}}\,{}_{s_2}Y_{\ell -s_1}(\psi,\varsigma)e^{-is_2\xi}, \tag{1.271}$$

which relates the set of angles defining $\hat{\mathbf{p}}_\gamma$ and $\hat{\mathbf{q}}_\gamma$,

$$\hat{\mathbf{p}}_\gamma \equiv \left(\cos\varphi_{p_\gamma}\,\sin\vartheta_{p_\gamma},\ \sin\varphi_{p_\gamma}\,\sin\vartheta_{p_\gamma},\ \cos\vartheta_{p_\gamma}\right), \tag{1.272}$$

$$\hat{\mathbf{q}}_\gamma \equiv \left(\cos\varphi_{q_\gamma}\,\sin\vartheta_{q_\gamma},\ \sin\varphi_{q_\gamma}\,\sin\vartheta_{q_\gamma},\ \cos\vartheta_{q_\gamma}\right), \tag{1.273}$$

to the so called **Euler angles** $\varsigma$, $\psi$, $\xi$ as shown in Fig. 1.7. Therefore, we find

$$\mathsf{R}^{-1}(\xi)\mathsf{S}(\psi)\mathsf{R}(-\varsigma) - \frac{4\pi}{\sqrt{15}}\sum_{m=-1}^{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & {}_0Y^*_{1m}\,{}_0Y_{1m} \end{bmatrix}(\hat{\mathbf{q}}_\gamma,\hat{\mathbf{p}}_\gamma) = \\ = \frac{4\pi}{15}\sum_{m=-2}^{2}\begin{bmatrix} \frac{5}{2\pi}\delta_{m0}+{}_0Y^*_{2m}\,{}_0Y_{2m} & -\sqrt{\frac{3}{2}}\,{}_2Y^*_{2m}\,{}_0Y_{2m} & -\sqrt{\frac{3}{2}}{}_{-2}Y^*_{2m}\,{}_0Y_{2m} & 0 \\ -\sqrt{6}\,{}_0Y^*_{2m}\,{}_2Y_{2m} & 3\,{}_2Y^*_{2m}\,{}_2Y_{2m} & 3\,{}_{-2}Y^*_{2m}\,{}_2Y_{2m} & 0 \\ -\sqrt{6}\,{}_0Y^*_{2m}\,{}_{-2}Y_{2m} & 3\,{}_2Y^*_{2m}\,{}_{-2}Y_{2m} & 3\,{}_{-2}Y^*_{2m}\,{}_{-2}Y_{2m} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}(\hat{\mathbf{q}}_\gamma,\hat{\mathbf{p}}_\gamma). \tag{1.274}$$

where the complex conjugates spherical harmonics depend on $\hat{\mathbf{q}}_\gamma$ whereas the others depend on $\hat{\mathbf{p}}_\gamma$. Now we substitute such a result within Eq. (1.266), so that we finally find the expression of the Stokes parameters

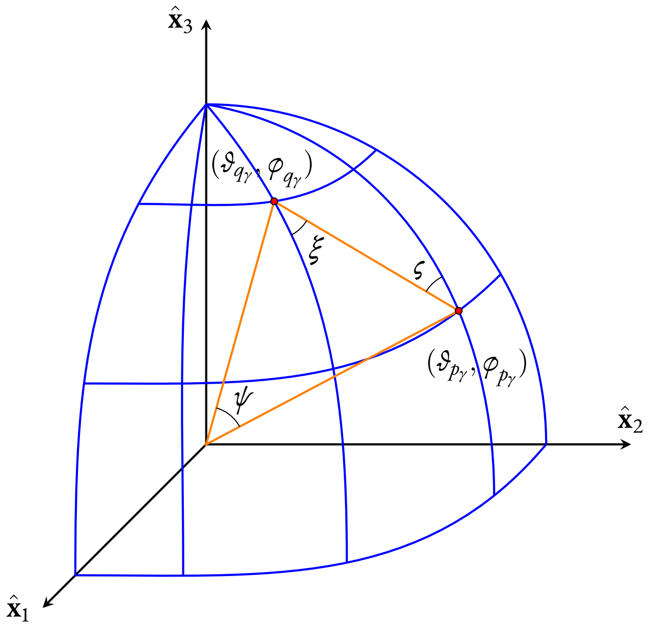


**Figure 1.7:** A rotation from $(\vartheta_{p_\gamma}, \varphi_{p_\gamma})$ through the origin to $(\vartheta_{q_\gamma}, \varphi_{q_\gamma})$ is equivalent to a direct rotation by the Euler angles $(\varsigma, \psi, \xi)$. In fact, these angles represent the rotation by $\varsigma$ from the $\hat{\mathbf{p}}_\gamma = \hat{\mathbf{x}}_3$ frame to the scattering frame, by the scattering angle $\psi$, and by $\xi$ back into the $\hat{\mathbf{p}}_\gamma$ frame.

for the scattered electric field in terms of incoming one's those,

$$\begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix}(t, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \frac{\sigma_T}{4\pi|\mathbf{x}|^2}\begin{bmatrix} I_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$+ \frac{\sigma_T}{10|\mathbf{x}|^2}\sum_{m=-2}^{2} \mathsf{P}_m(\hat{\mathbf{q}}_\gamma, \hat{\mathbf{p}}_\gamma)\begin{bmatrix} I_{\rm in} \\ (Q+iU)_{\rm in} \\ (Q-iU)_{\rm in} \\ 0 \end{bmatrix}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma) \tag{1.275}$$
$$+ \frac{3\sigma_T}{2\sqrt{15}|\mathbf{x}|^2}\sum_{m=-1}^{1} {}_0Y^*_{1m}(\hat{\mathbf{q}}_\gamma)\, {}_0Y_{1m}(\hat{\mathbf{p}}_\gamma)\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_{\rm in}(t, \mathbf{x}, \hat{\mathbf{q}}_\gamma) \end{bmatrix},$$

where to lighten up the notation we have defined the following matrix:

$$\mathsf{P}_m(\hat{\mathbf{q}}_\gamma,\hat{\mathbf{p}}_\gamma) \equiv \begin{bmatrix} {}_0Y^*_{2m}\,{}_0Y_{2m} & -\sqrt{\frac{3}{2}}\,{}_2Y^*_{2m}\,{}_0Y_{2m} & -\sqrt{\frac{3}{2}}\,{}_{-2}Y^*_{2m}\,{}_0Y_{2m} & 0 \\ -\sqrt{6}\,{}_0Y^*_{2m}\,{}_2Y_{2m} & 3\,{}_2Y^*_{2m}\,{}_2Y_{2m} & 3\,{}_{-2}Y^*_{2m}\,{}_2Y_{2m} & 0 \\ -\sqrt{6}\,{}_0Y^*_{2m}\,{}_{-2}Y_{2m} & 3\,{}_2Y^*_{2m}\,{}_{-2}Y_{2m} & 3\,{}_{-2}Y^*_{2m}\,{}_{-2}Y_{2m} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}(\hat{\mathbf{q}}_\gamma,\hat{\mathbf{p}}_\gamma). \tag{1.276}$$

If we multiply both sides of Eq. (1.275) by $\overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2$ and then apply the $\int \mathrm{d}^2\hat{q}_\gamma$ operator, we get

$$\begin{aligned} \int \mathrm{d}^2\hat{q}_\gamma \overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2 \begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) &= -\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi}\begin{bmatrix} I_{\rm in}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma) \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\quad -\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\frac{4\pi}{10}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi}\sum_{m=-2}^{2}\mathsf{P}_m(\hat{\mathbf{q}}_\gamma,\hat{\mathbf{p}}_\gamma)\begin{bmatrix} I_{\rm in} \\ (Q+iU)_{\rm in} \\ (Q-iU)_{\rm in} \\ 0 \end{bmatrix}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma) \\ &\quad +\frac{6\pi}{\sqrt{15}}\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi}\sum_{m=-1}^{1}{}_0Y^*_{1m}(\hat{\mathbf{q}}_\gamma)\,{}_0Y_{1m}(\hat{\mathbf{p}}_\gamma)\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_{\rm in}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma) \end{bmatrix}. \end{aligned} \tag{1.277}$$

Let us now focus on $I_{\rm sc}$: we know that thanks to Eq. (1.255), we can write

$$\begin{aligned} |\mathbf{E}_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma)|^2 &= \sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}}\left\{[\hat{\mathbf{p}}_\gamma\cdot\mathbf{E}_{\rm in}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma)]\hat{\mathbf{p}}_\gamma - \mathbf{E}_{\rm int}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right\}\cdot\mathbf{E}_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) \\ &= -\sqrt{\frac{3\sigma_T}{8\pi|\mathbf{x}|^2}}|\mathbf{E}_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma)||\mathbf{E}_{\rm in}(t,\mathbf{x},\hat{\mathbf{q}}_\gamma)|^2\hat{\mathbf{E}}_{\rm in}(\hat{\mathbf{q}}_\gamma)\cdot\hat{\mathbf{E}}_{\rm sc}(\hat{\mathbf{p}}_\gamma) \end{aligned} \tag{1.278}$$

so that, after dividing by $|\mathbf{E}_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma)|$ and squaring the result, it becomes evident that if Eq. (1.278) holds true, then can write

$$I_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \frac{3\sigma_T}{8\pi|\mathbf{x}|^2}\sum_{\lambda_q}\left[\varepsilon_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\varepsilon_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 I_{\rm in}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma), \tag{1.279}$$

It is then clear that the right-hand side of Eq. (1.277) can be rewritten as

$$
\begin{aligned}
4\pi\,\overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2 \begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) &= \\
= \int \frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi} & \begin{bmatrix} -\frac{3}{2}\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\sum_{\lambda_q}\left[\boldsymbol{\varepsilon}_{\lambda_q}(\hat{\mathbf{q}}_\gamma)\cdot\boldsymbol{\varepsilon}_{\lambda_p}(\hat{\mathbf{p}}_\gamma)\right]^2 4\overline{\varrho}_\gamma(\tau)\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma) \\ 4\pi\,\overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2\left[Q+iU\right]_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) \\ 4\pi\,\overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2\left[Q-iU\right]_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma) \\ 4\pi\,\overline{n}_e(\tau)a(\tau)|\mathbf{x}|^2 V_{\rm sc}(t,\mathbf{x},\hat{\mathbf{p}}_\gamma), \end{bmatrix},
\end{aligned}
\tag{1.280}
$$

where, after having recalled that at first order in perturbation theory $I \simeq 4\overline{\varrho}_\gamma\Theta_T^{(1)}$, we see that the first row of such a matrix equation is exactly the right-hand side of Eq. (1.249) times $4\overline{\varrho}_\gamma$. Hence, this implies that the right-hand side of Eq. (1.249) must be equal to the right-hand side of Eq. (1.277) divided by $4\overline{\varrho}_\gamma$. It then follows that if this equality holds true for the Stokes parameter associated with the intensity, the same will occur also for the other Stokes parameter, whose left-hand side can be easily built in analogy with $I$. Therefore, by defining some new quantities in the very same fashion of $\Theta_T^{(1)}$,

$$
\begin{bmatrix} \Theta_T^{(1)} \\ {}_{+}\Theta_P^{(1)} \\ {}_{-}\Theta_P^{(1)} \\ \Theta_V^{(1)} \end{bmatrix}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv \frac{1}{4\overline{\varrho}_\gamma(\tau)}\begin{bmatrix} I_{\rm sc} \\ (Q+iU)_{\rm sc} \\ (Q-iU)_{\rm sc} \\ V_{\rm sc} \end{bmatrix}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.281}
$$

we find the final form of the Boltzmann equations for cosmic photons at first order in perturbation theory,

$$
\left[\frac{\partial}{\partial\tau}+\hat{\mathbf{p}}_\gamma\cdot\boldsymbol{\nabla}-\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right]\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \mathcal{S}_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.282}
$$

$$
\left[\frac{\partial}{\partial\tau}+\hat{\mathbf{p}}_\gamma\cdot\boldsymbol{\nabla}-\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right]{}_{\pm}\Theta_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = {}_{\pm}\mathcal{S}_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.283}
$$

$$
\left[\frac{\partial}{\partial\tau}+\hat{\mathbf{p}}_\gamma\cdot\boldsymbol{\nabla}-\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right]\Theta_V^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \mathcal{S}_V^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma), \tag{1.284}
$$

where we have defined the **source terms** of the Boltzmann equation as

$$\begin{aligned}\mathcal{S}_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv & -\mathcal{G}^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\left[\frac{1}{4}\delta_\gamma^{(1)}(\tau,\mathbf{x}) + \mathbf{v}_e(\tau,\mathbf{x})\cdot\hat{\mathbf{p}}_\gamma\right] \\ & - \sqrt{\frac{3}{2}}\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{10}\sum_{m=-2}^{2}{}_0Y_{2m}(\hat{\mathbf{p}}_\gamma)\left[\sqrt{\frac{2}{3}}{}_0Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right. \\ & \left. - {}_2Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\,{}_+\Theta_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma) - {}_{-2}Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\,{}_-\Theta_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right],\end{aligned} \tag{1.285}$$

$$\begin{aligned}{}_\pm\mathcal{S}_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv & \frac{3}{10}\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\int\mathrm{d}^2\hat{q}_\gamma\sum_{m=-2}^{2}{}_{\pm2}Y_{2m}(\hat{\mathbf{p}}_\gamma)\left[\sqrt{\frac{2}{3}}{}_0Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\Theta_T^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right. \\ & \left. - {}_2Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\,{}_+\Theta_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma) - {}_{-2}Y_{2m}^*(\hat{\mathbf{q}}_\gamma)\,{}_-\Theta_P^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma)\right],\end{aligned} \tag{1.286}$$

where we have used Eq. (1.229) and the definition of density contrast we gave in Eq. (1.193), whereas

$$\mathcal{S}_V^{(1)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) \equiv \sqrt{\frac{3}{5}}\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\int\frac{\mathrm{d}^2\hat{q}_\gamma}{2}\sum_{m=-1}^{1}{}_0Y_{1m}(\hat{\mathbf{p}}_\gamma)\,{}_0Y_{1m}^*(\hat{\mathbf{q}}_\gamma)\Theta_V^{(1)}(\tau,\mathbf{x},\hat{\mathbf{q}}_\gamma). \tag{1.287}$$

These source terms have been obtained by simply collecting together all terms on the right-hand side of the Boltzmann equations, and by writing explicitly the matrix product between the matrix $\mathsf{P}_m$ and the vector containing the Stokes parameter of the incoming photon, as it follows from Eq. (1.277). Let us notice that $V$ is not coupled with the other Stokes parameter, and this is a direct consequence of the fact that the Compton scattering is unable to produce circular polarization, but just linear one. Moreover, Eqs. (1.282)-(1.284) have been derived by working at first order in perturbation theory, but by repeating all the procedure we have described in this section for any order, it is possible to generalize such results,

$$\left[\frac{\partial}{\partial\tau} + \hat{\mathbf{p}}_\gamma \cdot \boldsymbol{\nabla} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right] \Theta^{(n)}_T(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \mathcal{S}^{(n)}_T(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma), \tag{1.288}$$

$$\left[\frac{\partial}{\partial\tau} + \hat{\mathbf{p}}_\gamma \cdot \boldsymbol{\nabla} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right] {}_{\pm}\Theta^{(n)}_P(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = {}_{\pm}\mathcal{S}^{(n)}_P(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma), \tag{1.289}$$

$$\left[\frac{\partial}{\partial\tau} + \hat{\mathbf{p}}_\gamma \cdot \boldsymbol{\nabla} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right] \Theta^{(1)}_V(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) = \mathcal{S}^{(n)}_V(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma), \tag{1.290}$$

where the conversion factor connecting the Stokes parameters to the $\Theta's$ is still just a function of conformal time but not necessary $4\bar{\rho}_\gamma$, and where the $n$-th order source functions have more complicated expressions which can be found e.g. in Refs. [162–168]. However, what is true at any perturbative order is that we can decompose such source functions as

$${}_{\pm}\mathcal{S}^{(n)}_P(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma) \equiv -\frac{\mathrm{d}\kappa}{\mathrm{d}\tau}\sqrt{\frac{6\pi}{5}} \sum_{m=-2}^{2} {}_{\pm 2}Y_{2m}(\hat{\mathbf{p}}_\gamma)\Pi^{(n)}_m(\tau, \mathbf{x}), \tag{1.291}$$

so that, for instance, by comparing Eq. (1.291) with Eq. (1.286), we can easily infer that

$$\begin{aligned}\Pi^{(1)}_m(\tau, \mathbf{x}) = \sqrt{\frac{6\pi}{5}} \int \frac{\mathrm{d}^2\hat{q}_\gamma}{4\pi} \Bigg[ {}_{2}Y^*_{2m}(\hat{\mathbf{q}}_\gamma)\, {}_{+}\Theta^{(1)}_P(\tau, \mathbf{x}, \hat{\mathbf{q}}_\gamma) + {}_{-2}Y^*_{2m}(\hat{\mathbf{q}}_\gamma)\, {}_{-}\Theta^{(1)}_P(\tau, \mathbf{x}, \hat{\mathbf{q}}_\gamma) \\ - \sqrt{\frac{2}{3}}\, {}_{0}Y^*_{2m}(\hat{\mathbf{q}}_\gamma)\Theta^{(1)}_T(\tau, \mathbf{x}, \hat{\mathbf{q}}_\gamma)\Bigg].\end{aligned} \tag{1.292}$$

However, we are not going to solve here the perturbed Boltzmann equations, since we prefer to directly do that in Sec. 2.2 when also the contribution coming from cosmic birefringence are taken into account. Finally, let us mention here that our derivation is correct but not the most rigorous one, which can be found instead in Ref. [169].

# 2

# Theory of Cosmological Birefringence

In the previous chapter we have reviewed the basics of the CMB polarization theory, and here we are going to introduce the main topic of this thesis: cosmic birefringence. In fact, Chap. 1 had two purposes: providing the standard mathematical formalism and definitions to the reader, and explaining what is the description of the CMB polarization in the context of the standard cosmological paradigm. However, what if such a model can be extended? Indeed, as discussed in Sec. 1.3.1, the ΛCDM model is literally made up by Λ as a candidate for dark energy, and dark matter, and we do not know what is the exact nature of none of them. Therefore, it becomes quite natural trying to stress the paradigm e.g. by introducing extra cosmic species in order to explain the dark sector of the Universe, or investigate the phenomenological consequences of the modification of the fundamental theories ruling the Universe, such as gravity or electromagnetism. In this thesis, we are particularly interested in the possibility that, as well as the weak interactions (see Ref. [170]) also the electromagnetic theory could encode some parity-breaking signatures. In order to address this intriguing question, extensions of Maxwell's electromagnetism have been proposed in the literature, e.g. in the form of extra couplings between photons and new cosmic fields. As we will see in the next sections, a phenomenological consequence of this extension is the rotation of the photons' linear polarization plane of an angle $\alpha$, which could have left measurable imprints in the CMB polarization signal. In fact, cosmic birefringence yields a modification of the observed CMB power spectra and in particular provides a non-vanishing value for the parity-breaking cross-correlations *TB* and *EB* (see e.g. Refs. [2, 89]). As we will show in the course of this chapter, birefringence is a propagation effect and therefore larger is the path of the photon, larger will be the probability to produce an appreciable value for $\alpha$. This is the reason why the most promising observations of such a phenomenon come from cosmology: since CMB photons represent the oldest source of electromagnetic radiation in the Universe, they have traveled the longest possible path. Constraining parity-violation from CMB data is a well known historical effort for cosmologists, but in the last years there has been an increasing amount of important observational constraints on cosmic birefringence: these results. In particular, the authors of Ref. [101], by using *Planck* maps from the third public release (PR3), have been able to extract a promising measurement of a non-vanishing birefringence angle, and then such a treatment has been also extended to PR4 in Ref. [106], yielding the tantalizing result of $\alpha = (0.30 \pm 0.11)^\circ$. Similar results come by a joint analysis of polarization data from the space missions WMAP and *Planck*, as shown in Ref. [92]. Although there exists the possibility that this effect is just caused by interstellar dust emission, as discussed e.g. in Refs. [104, 110, 112–116], confirming these detections by observations with higher statistical significance in the future might have a profound implication for fundamental physics. The *Planck* result mentioned before has been possible thanks to a new technique which takes also into account information present in the Galactic foreground emission. Indeed, if one relied only on the CMB power spectra, it would not be possible to distinguish between $\alpha$ and a miscalibration of the instrumental polarization angle (see Refs. [90, 107, 108, 111]). However, since, as we will show, the birefringence angle is proportional to the difference between the value of the scalar field inducing birefringence today and that one at the emission time of the photon, it is reasonable to expect that the polarized emission from our Galaxy is only negligibly affected by cosmic birefringence. Therefore, it is possible to use this property to isolate the two different rotation angles (see Ref. [109]). However, as we will see in the next sections, the fact that the whole birefringence effect can be parameterized by a single number, i.e. the birefringence angle, is possible only if we assume that photons of the CMB were all emitted at the recombination time. However, they are instead statistically distributed over the photon visibility function, and hence $\alpha$ should be a function not a number. This is a crucial point that we are briefly mentioning right now, but that will follows in a very natural way after having developed the mathematical formalism needed for understanding cosmic birefringence. The structure of the chapter is organized as follows. In Sec. 2.1 we mathematically review the phenomenon of cosmic birefringence, mainly following Ref. [79]. In Sec. 2.2 we show the impact of cosmic birefringence on the Boltzmann equation of CMB polarization, mainly following Ref. [171]. In Secs. 2.3-2.4 we discuss how it is possible to exploit a tomographic description of cosmic birefringence to probe axion-like fields as candidates for dark matter or dark energy, mainly following Ref. [77].

## 2.1 A Parity-Violating Extension of Electrodynamics

In Sec. 1.1 we have partially reviewed the standard formulation of electromagnetism: here our purpose is instead to investigate the phenomenological consequences of a parity-violating deviation from such a theory. In particular, the modification we are going to consider involves adding the so-called **Chern-Simons term** to the Maxwell's action [172],

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \chi_\mu A_\nu \tilde{F}^{\mu\nu} \right]. \tag{2.1}$$

where $\tilde{F}^{\mu\nu}$ is the dual electromagnetic tensor as defined in Eq. (1.28), and $\chi_\mu$ is a yet unspecified four-vector. As we are going to see, this simple extension of standard electromagnetism has an important impact on cosmological observables, since it is responsible for a parity-breaking phenomenon affecting the CMB polarization, named cosmic birefringence (see e.g. Refs. [1, 2, 79]). We start by determining under which conditions the theory defined in Eq. (2.1) is gauge-invariant, in the same sense we expressed in Eq. (1.21). Indeed, this is due to the fact we want to consider a "new" theory of electrodynamics able to produce the parity-violating signatures we are interested in, but preserving the electromagnetic field's gauge-invariance. In fact, gauge-transforming Eq. (2.1) yields

$$\begin{aligned} S \mapsto S + \int \mathrm{d}^4x \sqrt{-g} \chi_\mu \nabla_\nu h \tilde{F}^{\mu\nu} &= S - \int \mathrm{d}^4x \sqrt{-g}\, h \left( \nabla_\nu \chi_\mu \right) \tilde{F}^{\mu\nu} \\ = S + \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g}\, h \tilde{F}^{\mu\nu} \left( \nabla_\mu \chi_\nu - \nabla_\nu \chi_\mu \right), \end{aligned} \tag{2.2}$$

where we have integrated by parts and used that we already know $F_{\mu\nu}$ is gauge-invariant, whereas, by definition, the dual Maxwell tensor satisfies Eq. (1.102). Gauge invariance requires that $S \mapsto S$ for an arbitrary scalar function $h$, and the non-trivial way this can happen is for $\chi_\mu = -(\beta/2)\nabla_\mu \chi$, i.e. if $\chi_\mu$ is the covariant derivative of a scalar field $\chi$ times a coupling constant $-\beta/2$, since, according to Eq. (1.100), we have

$$\nabla_\mu \chi_\nu - \nabla_\nu \chi_\mu = \frac{\beta}{2} \left[ \partial_\nu \partial_\mu \chi - \Gamma^\sigma_{\nu\mu} \partial_\sigma \chi - \partial_\mu \partial_\nu \chi + \Gamma^\sigma_{\mu\nu} \partial_\sigma \chi \right] = 0. \tag{2.3}$$

Therefore, because of gauge-invariance, after integrating by parts, the theory defined in Eq. (2.1) simply reduces to

$$\boxed{S = \int \mathrm{d}^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\beta}{4} \chi F_{\mu\nu} \tilde{F}^{\mu\nu} \right].} \tag{2.4}$$

We can easily find the equation of motion associated with the action above by applying the Hamilton's principle, as stated in Eq. (1.15), but with the rules collected in Eq. (1.99),

$$\frac{\delta S}{\delta A_\mu} = 0 \qquad \Longrightarrow \qquad \boxed{\nabla_\mu F^{\mu\nu} = \beta \tilde{F}^{\mu\nu} \nabla_\mu \chi.} \tag{2.5}$$

Now, by recalling the expression of the commutator for covariant derivatives we provided in Eq. (1.106), it is not difficult to show that the following identity holds,

$$\begin{aligned}\nabla_\mu F_{\rho\sigma} + \nabla_\rho F_{\sigma\mu} + \nabla_\sigma F_{\mu\rho} &= \left[\nabla_\rho, \nabla_\sigma\right] A_\mu + \left[\nabla_\sigma, \nabla_\mu\right] A_\rho + \left[\nabla_\mu, \nabla_\rho\right] A_\sigma \\ &= -g^{\alpha\beta} A_\alpha \left(R_{\beta\mu\rho\sigma} + R_{\beta\rho\sigma\mu} + R_{\beta\sigma\mu\rho}\right) \\ &= 0,\end{aligned} \tag{2.6}$$

because of the first Bianchi identity, as shown in Eq. (1.105). Therefore, by applying the $\nabla^\mu$ operator to Eq. (2.6) we get

$$\begin{aligned}\nabla^\mu \nabla_\mu F_{\rho\sigma} &= -g^{\mu\nu} \left\{\left[\nabla_\nu, \nabla_\rho\right] + \nabla_\rho \nabla_\nu\right\} F_{\sigma\mu} - g^{\mu\nu} \left\{\left[\nabla_\nu, \nabla_\sigma\right] + \nabla_\sigma \nabla_\nu\right\} F_{\mu\rho} \\ &= \beta \left[\nabla_\rho \left(\tilde{F}_{\mu\sigma} \nabla^\mu \chi\right) - \nabla_\sigma \left(\tilde{F}_{\mu\rho} \nabla^\mu \chi\right)\right] + g^{\mu\nu} \left(F_{\mu\rho} R_{\nu\sigma} - F_{\mu\sigma} R_{\nu\rho}\right) - F^{\mu\nu} R_{\mu\nu\rho\sigma},\end{aligned} \tag{2.7}$$

where we have used again Eq. (1.106) together with Eq. (1.105) to invert the order of differentiation, and substituted Eq. (2.5).

### 2.1.1 Solution of the Modified Maxwell Equations

Up to now, we have derived three differential equations involving $F_{\mu\nu}$, i.e. Eqs. (2.5)-(2.7), but solving them by brute force will be extremely challenging and the risk is that of losing the physical understanding of what is the impact of the Chern-Simons coupling on the theory of electromagnetism. In fact, we are going to approach the problem by following the original derivation shown in Ref. [79], and to adopt the **geometric optics approximation** (GOA), which is valid whenever the wavelength of the electromagnetic waves we are considering is very short compared to the length scales associated with the curvature of spacetime (see e.g. Ref. [127]), so that the waves can be regarded locally as plane waves propagating through a spacetime of negligible curvature. In such a case we can then disregard the terms depending on the Riemann and Ricci tensors in Eq. (2.7) and also assume the following ansatz for the electromagnetic tensor:

$$F_{\mu\nu} = \left[\sum_{n=0}^{\infty} \varepsilon^n \mathcal{F}^{(n)}_{\mu\nu}\right] \exp\left[-\frac{\Xi}{i\varepsilon}\right], \tag{2.8}$$

being $\Xi$ is a real scalar function and $\varepsilon$ a real small parameter (i.e. $|\varepsilon| \ll 1$), whereas the $\mathcal{F}^{\mu\nu}_{(n)}$'s form a set of tensors defining such a perturbative expansion, so that all the "post-GOA" corrections are put into them and none are put into $\Xi$. We substitute now Eq. (2.8) in Eqs. (2.6), finding

$$0 = \sum_{n=0}^{\infty} \exp\left[-\frac{\Xi}{i\varepsilon}\right]\Big\{\varepsilon^n \left[\nabla_\mu \mathcal{F}^{(n)}_{\rho\sigma} + \nabla_\rho \mathcal{F}^{(n)}_{\sigma\mu} + \nabla_\sigma \mathcal{F}^{(n)}_{\rho\sigma}\right] \\ + i\varepsilon^{n-1}\left[\mathcal{F}^{(n)}_{\rho\sigma}\nabla_\mu + \mathcal{F}^{(n)}_{\sigma\mu}\nabla_\rho + \mathcal{F}^{(n)}_{\rho\sigma}\nabla_\sigma\right]\Xi\Big\}, \tag{2.9}$$

which at dominant order (i.e. $n = 0$, since $\varepsilon$ is a small quantity) reduces to

$$\left[\mathcal{F}^{(0)}_{\rho\sigma}\nabla_\mu + \mathcal{F}^{(0)}_{\sigma\mu}\nabla_\rho + \mathcal{F}^{(0)}_{\rho\sigma}\nabla_\sigma\right]\Xi = 0. \tag{2.10}$$

Let us notice that Eq. (2.10) is telling us that $\mathcal{F}^{(0)}_{\mu\nu}$ must be an antisymmetric tensor like $F_{\mu\nu}$, which means that we can define it as

$$\mathcal{F}^{(0)}_{\mu\nu} \equiv \mathcal{A}^{(0)}_\nu \nabla_\mu \Xi - \mathcal{A}^{(0)}_\mu \nabla_\nu \Xi, \tag{2.11}$$

since this identically solves Eq. (2.10). Similarly, by substituting the GOA expansion in Eq. (2.7) and dropping out the terms proportional to the Ricci and Riemann tensors, we get

$$\sum_{n=0}^{\infty}\Big\{\varepsilon^n \nabla^\mu \nabla_\mu \mathcal{F}^{(n)}_{\rho\sigma} + i\varepsilon^{n-1}\left[\mathcal{F}^{(n)}_{\rho\sigma}\nabla^\mu + 2\nabla^\mu \mathcal{F}^{(n)}_{\rho\sigma}\right]\nabla_\mu \Xi - \varepsilon^{n-2}\mathcal{F}^{(n)}_{\rho\sigma}\nabla_\mu \Xi\,\nabla^\mu \Xi\Big\} = \\ = -\beta \sum_{n=0}^{\infty}\Big\{\varepsilon^n\left[\nabla_\sigma \tilde{\mathcal{F}}^{(n)}_{\mu\rho} - \nabla_\rho \tilde{\mathcal{F}}^{(n)}_{\mu\sigma} + \tilde{\mathcal{F}}^{(n)}_{\mu\rho}\nabla_\sigma - \tilde{\mathcal{F}}^{(n)}_{\mu\sigma}\nabla_\rho\right] \\ + i\varepsilon^{n-1}\left[\tilde{\mathcal{F}}^{(n)}_{\mu\rho}\nabla_\sigma - \tilde{\mathcal{F}}^{(n)}_{\mu\sigma}\nabla_\rho\right]\Xi\Big\}\nabla^\mu \chi, \tag{2.12}$$

whose dominant contribution is again the $n = 0$ one, yielding the following system of equations

$$\begin{cases} \boxed{\nabla_\mu \Xi\,\nabla^\mu \Xi = 0} & \text{terms} \propto \varepsilon^{-2}, \\ \left[\mathcal{F}^{(n)}_{\rho\sigma}\nabla^\mu + 2\nabla^\mu \mathcal{F}^{(0)}_{\rho\sigma}\right]\nabla_\mu \Xi = -\beta \nabla^\mu \chi \left[\tilde{\mathcal{F}}^{(n)}_{\mu\rho}\nabla_\sigma - \tilde{\mathcal{F}}^{(n)}_{\mu\sigma}\nabla_\rho\right]\Xi & \text{terms} \propto \varepsilon^{-1}. \end{cases} \tag{2.13}$$

The first equation implies the following orthogonality relation,

$$\nabla_\nu\left(\nabla_\mu \Xi\,\nabla^\mu \Xi\right) = 0 \qquad \Longrightarrow \qquad \boxed{\nabla^\mu \Xi \nabla_\mu \nabla_\nu \Xi = 0,} \tag{2.14}$$

since $\nabla_\mu \Xi = \partial_\mu \Xi$ because $\Xi$ is scalar function, whereas the second one can be rewritten with the help of Eq. (2.11) as

$$\begin{aligned}\nabla_\rho \Xi \nabla^\mu \Xi \nabla_\mu \mathcal{A}^{(0)}_\sigma - \nabla_\sigma \Xi \nabla^\mu \Xi \nabla_\mu \mathcal{A}^{(0)}_\rho + \frac{1}{2} \nabla_\mu \nabla^\mu \Xi \nabla_\rho \Xi \mathcal{A}^{(0)}_\sigma = \\ = \frac{1}{2} \nabla_\mu \nabla^\mu \Xi \nabla_\sigma \Xi \mathcal{A}^{(0)}_\rho + \frac{\beta\, \varepsilon^{\mu\beta\nu\kappa}}{2\sqrt{-g}} \nabla_\mu \chi \mathcal{A}^{(0)}_\kappa \nabla_\nu \Xi \left( g_{\beta\sigma} \nabla_\rho \Xi - g_{\beta\rho} \nabla_\sigma \Xi \right),\end{aligned} \tag{2.15}$$

where some terms vanish because of Eq. (2.14). By defining some new differential operators,

$$\mathcal{D} \equiv \nabla^\mu \Xi \nabla_\mu, \qquad \omega \equiv \nabla_\mu \nabla^\mu \Xi, \tag{2.16}$$

it is then possible to express the product between $\nabla^\sigma \Xi$ and Eq. (2.15) as

$$\mathcal{D} \mathcal{A}^{(0)}_\mu + \frac{\omega}{2} \mathcal{A}^{(0)}_\mu = \frac{\beta}{2} \frac{\varepsilon^{\alpha\beta\rho\sigma}}{\sqrt{-g}} \nabla_\alpha \chi g_{\mu\beta} \nabla_\rho \Xi \mathcal{A}^{(0)}_\sigma. \tag{2.17}$$

In the very same fashion, we can now apply the GOA also to Eq. (2.5), finding

$$\sum_{n=0} \left[ \varepsilon^n \nabla^\mu \mathcal{F}^{(n)}_{\mu\nu} + i \varepsilon^{n-1} \mathcal{F}^{(n)}_{\mu\nu} \nabla^\mu \Xi \right] = \beta \sum_{n=0}^{\infty} \tilde{\mathcal{F}}^{(n)}_{\mu\nu} \nabla^\mu \chi, \tag{2.18}$$

whose dominant contribution for $n = 0$ yields

$$\mathcal{A}^{(0)}_\mu \nabla^\mu \Xi = 0. \tag{2.19}$$

### 2.1.2 Impact on the Stokes Parameters

In order to investigate how the Chern-Simons extension of Maxwell's theory affects the propagation of the electromagnetic waves, we must study its impact on the Stokes parameters. In Sec, 1.1.3 we have provided some definitions of them, but they are valid only in the context of flat spacetime, and we can generalize this to GR by means of the tetrad formalism we briefly mentioned in Sec. 1.2. In fact, we are going to consider the local inertial reference frame in which an observer is at rest (so its spatial velocity is zero), and which sees an electromagnetic wave traveling along the $\hat{\mathbf{x}}_3$-direction. We know from Sec. 1.1.1 that the electric and magnetic fields in special relativity are defined as

$$E_i|_{\rm SR} = \partial_i A_0 - \partial_0 A_i = F_{i0} = F_{i\nu}\delta^{\nu}{}_0, \tag{2.20}$$
$$B_i|_{\rm SR} = \delta_{il}\varepsilon^{0ljk}\partial_j A_k = \tilde{F}^{0j}\delta_{ij} = \tilde{F}_{\mu i}\delta^{\mu}{}_0, \tag{2.21}$$

so that, according to Eq. (1.130), they should generalize in general relativity as

$$E_i|_{\rm LIF} = E_\mu e^{\mu}_{(i)}, \qquad B_i|_{\rm LIF} = B_\mu e^{\mu}_{(i)}. \tag{2.22}$$

In order to understand how to perform such an operation, it is convenient to define a "new" quantity, i.e. the **4-velocity** $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$, i.e. a four-vector satisfying Eq. (1.128) with respect to an affine parameter $\lambda = is$, called **proper time**. It then follows that

$$u^\mu u_\mu = g_{\mu\nu}u^\mu u^\nu = -g^{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = -\frac{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{\mathrm{d}s^2} = -1. \tag{2.23}$$

In the local inertial frame, the free fall observer's four-velocity is given by $u_i|_{\rm LIF} = \delta^0{}_i$, so that in the coordinate frame it reads

$$u^\mu = u^i|_{\rm LIF}\, e^{\mu}_{(i)} = \delta^i{}_0 e^{\mu}_{(i)} = e^{\mu}_{(0)}. \tag{2.24}$$

One of the reasons we introduced the concept of 4-velocity is that, with a bit of imagination, it allows us to generalize Eqs. (2.20)-(2.21) to curved spacetime as (see e.g. Ref. [173])

$$E_\mu \equiv F_{\mu\nu}u^\nu \qquad B_\mu \equiv \tilde{F}_{\mu\nu}u^\nu. \tag{2.25}$$

Indeed, if we choose a reference frame such that the observer's four-velocity matches the 4-velocity of an inertial observer in special relativity (i.e. the LIF), the effects of gravity become negligible, and we should recover the same expressions as in special relativity. Moreover, let us recall that the quantity $\Xi$ satisfies Eq. (2.14), so that we can rewrite it with the help of Eq. (1.100) as

$$0 = \nabla^\mu\Xi\nabla_\mu\nabla^\nu\Xi = \nabla^\mu\Xi\left(\partial_\mu\nabla^\nu\Xi + \Gamma^\nu_{\mu\sigma}\nabla^\sigma\Xi\right) = \frac{\nabla^\mu\Xi}{\mathrm{d}x^\mu/\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\nabla^\nu\Xi + \Gamma^\nu_{\mu\sigma}\nabla^\mu\Xi\nabla^\sigma\Xi, \tag{2.26}$$

which yields the geodesic equation, by setting $\nabla^\mu\Xi = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for a proper affine parameter. Without loss of generality, we can choose $\lambda$ to be the same which appears in Eq. (1.133), so that the vanishing of the inner product $\nabla^\mu\Xi\nabla_\mu\Xi$ fits perfectly for a massless particle. Therefore, $\nabla^\mu\Xi$ is related to the 4-momentum of the photons associated with the electromagnetic wave, and in the local inertial frame, we can then write

$$\nabla^i \Xi|_{\rm LIF} = y\left(\delta^i{}_0 + \delta^i{}_3\right), \tag{2.27}$$

since in the LIM we have chosen the wave propagates along the $\hat{\mathbf{x}}_3$-direction, with $y$ being a parameter with the right dimensionality. Therefore, by moving to the coordinate frame, we find

$$\nabla^\mu \Xi = \nabla^i \Xi|_{\rm LIF}\, e^\mu_{(i)} = y\left(\delta^i{}_0 + \delta^i{}_3\right) e^\mu_{(i)} = y\left[u^\mu + e^\mu_{(3)}\right] \quad \Longrightarrow \quad e^\mu_{(3)} = \frac{\nabla^\mu \Xi}{y} - u^\mu, \tag{2.28}$$

where we have substituted Eq. (2.24). Indeed, the other tetrad vectors $e^\mu_{(1)}$ and $e^\mu_{(2)}$ are orthogonal to each other and to $e^\mu_{(0)}$, as well as to $e^\mu_{(3)}$. Thanks to Eq. (2.22), we are finally in the position to compute the components of the electric field,

$$E_i|_{\rm LIF} = \sum_{n=0}\left[\varepsilon^n \mathcal{F}^{(n)}_{\mu\nu}\right] \exp\left(-\frac{\Xi}{i\varepsilon}\right) u^\nu e^\mu_{(i)} \simeq \left[\mathcal{A}^{(0)}_\nu \nabla_\mu \Xi - \mathcal{A}^{(0)}_\mu \nabla_\nu \Xi\right] \exp\left(-\frac{\Xi}{i\varepsilon}\right) u^\nu e^\mu_{(i)} \tag{2.29}$$

where, as previously done, we have adopted again the GOA and truncated at dominant order $n = 0$. Let us notice that all this procedure has been performed in order to use the definitions of Stokes parameter we provided in Sec. 1.1.3, that are valid in special relativity and so they can also be used in GR when considering the local inertial frame. For instance, the intensity of the electromagnetic wave is given as

$$\begin{aligned} I &= \langle E_1|_{\rm LIF} E_1^*|_{\rm LIF}\rangle_t + \langle E_2|_{\rm LIF} E_2^*|_{\rm LIF}\rangle_t \\ &= y^2 \left\langle \left| \left\{ g_{\mu\sigma} \mathcal{A}^{(0)}_\nu \left(e^\sigma_{(0)} + e^\sigma_{(3)}\right) - g_{\nu\varrho} \mathcal{A}^{(0)}_\mu \left[e^\varrho_{(0)} + e^\varrho_{(3)}\right]\right\} e^\nu_{(0)} e^\mu_{(1)} \right|^2 \right\rangle_t \\ &\qquad + y^2 \left\langle \left| \left\{ g_{\mu\sigma} \mathcal{A}^{(0)}_\nu \left(e^\sigma_{(0)} + e^\sigma_{(3)}\right) - g_{\nu\varrho} \mathcal{A}^{(0)}_\mu \left[e^\varrho_{(0)} + e^\varrho_{(3)}\right]\right\} e^\nu_{(0)} e^\mu_{(2)} \right|^2 \right\rangle_t \\ &= y^2 \left\langle \left| \mathcal{A}^{(0)}_\mu e^\mu_{(1)} \right|^2 \right\rangle_t + y^2 \left\langle \left| \mathcal{A}^{(0)}_\mu e^\mu_{(2)} \right|^2 \right\rangle_t , \end{aligned} \tag{2.30}$$

where we have exploited Eq. (1.131). In the very same fashion, we can easily work out also the other Stokes parameter, so that we find[1]

$$I = y^2 \langle \mathcal{A}_\mu^{(0)} \mathcal{A}_\nu^{(0)*} \rangle_t \left[ e_{(1)}^\mu e_{(1)}^\nu + e_{(2)}^\mu e_{(2)}^\nu \right], \tag{2.31}$$

$$Q \pm iU = y^2 \langle \mathcal{A}_\mu^{(0)} \mathcal{A}_\nu^{(0)*} \rangle_t \left\{ e_{(1)}^\mu e_{(1)}^\nu - e_{(2)}^\mu e_{(2)}^\nu \pm i \left[ e_{(1)}^\mu e_{(2)}^\nu + e_{(2)}^\mu e_{(1)}^\nu \right] \right\}, \tag{2.32}$$

$$V = iy^2 \langle \mathcal{A}_\mu^{(0)} \mathcal{A}_\nu^{(0)*} \rangle_t \left[ e_{(1)}^\mu e_{(2)}^\nu - e_{(2)}^\mu e_{(1)}^\nu \right]. \tag{2.33}$$

Let us now apply the operator $\mathcal{D}$ defined in Eq. (2.16) to the tensor $L_{\mu\nu}^{(0)} \equiv \langle \mathcal{A}_\mu^{(0)} \mathcal{A}_\nu^{(0)*} \rangle_t$,

$$\begin{aligned} \mathcal{D} L_{\mu\nu}^{(0)} &= \left\langle \mathcal{D} \left[ \mathcal{A}_\mu^{(0)} \right] \mathcal{A}_\nu^{(0)*} \right\rangle + \left\langle \mathcal{A}_\mu^{(0)} \mathcal{D} \left[ \mathcal{A}_\nu^{(0)*} \right] \right\rangle_t \\ &= -\omega L_{\mu\nu}^{(0)} + \frac{\beta \varepsilon^{\kappa\lambda\varrho\sigma}}{\sqrt{-g}} \nabla_\kappa \chi \nabla_\lambda \Xi \, L_{\mu\varrho}^{(0)} g_{\sigma\nu} \end{aligned} \tag{2.34}$$

where we have used Eq. (2.17). We now reasonably require the tetrad frame not to be physically rotating, i.e. $\mathcal{D} e_{(i)}^\mu = 0$, so that by applying the $\mathcal{D}$ operator also to Eqs. (2.31)-(2.33), we get

$$[\mathcal{D} + \omega] \, I = 0, \tag{2.35}$$

$$[\mathcal{D} + \omega] \, (Q \pm iU) = \pm i\beta \, (Q \pm iU) \, \mathcal{D}\chi, \tag{2.36}$$

$$[\mathcal{D} + \omega] \, V = 0. \tag{2.37}$$

Let us recall that we can evaluate the angle defining the orientation of the polarization ellipse shown in Fig. 1.1 by simply inverting Eq. (1.56),

$$\alpha \equiv \frac{1}{2} \arctan \left[ -i \frac{(Q + iU) - (Q - iU)}{(Q + iU) + (Q - iU)} \right] \quad \Longrightarrow \quad \mathcal{D}\alpha = \frac{\beta}{2} \mathcal{D}\chi. \tag{2.38}$$

Since $\alpha$ and $\chi$ are scalars, such a differential equation is identically solved for

$$\alpha_{\rm fin} - \alpha_{\rm ini} = \frac{\beta}{2} \int_{\rm ini}^{\rm fin} {\rm d}x^\mu \partial_\mu \chi = \frac{\beta}{2} \left( \chi_{\rm fin} - \chi_{\rm ini} \right), \tag{2.39}$$

with "fin" and "ini" denoting the final and initial spacetime points of the electromagnetic wave's path, respectively. Moreover, by recalling the definitions given in Eq. (2.16) and exploiting the fact that $\omega = \nabla^\mu \nabla_\mu \Xi$ is

[1] Let us notice that here the dependence on the difference in phase $\beta$ between the components $E_1$ and $E_2$ of the electric field has been reabsorbed in the definition of $\mathcal{A}_\mu^{(0)}$.

a scalar, we get

$$\mathcal{D}\omega = \nabla^{\mu}\Xi\nabla_{\mu}\omega = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\partial_{\mu}\omega, \tag{2.40}$$

where we have used that $\nabla^{\mu}\Xi$ solves the geodesic equation. Therefore, since also the Stokes parameters are scalars[2], we are now able to rewrite Eq. (2.36) as

$$\mathcal{D}\left[\ln\left(Q \pm iU\right)\right] = -\left(\omega \mp i\beta\mathcal{D}\chi\right) \implies \frac{(Q \pm iU)_{\text{fin}}}{(Q \pm iU)_{\text{ini}}} = \exp\left[-\int_{\text{ini}}^{\text{fin}} \mathrm{d}\,x^{\mu}\partial_{\mu}\omega\right] e^{\pm 2i\left(\chi_{\text{fin}} - \chi_{\text{ini}}\right)}, \tag{2.41}$$

where we have used that $\omega = \nabla^{\mu}\nabla_{\mu}\Xi$ is a scalar. Similarly, it is trivial to show that the Stokes parameters $I$ and $V$ are instead unaffected by the presence of the Chern-Simons term,

$$\frac{I_{\text{fin}}}{I_{\text{ini}}} = \exp\left[-\int_{\text{ini}}^{\text{fin}} \mathrm{d}\,x^{\mu}\partial_{\mu}\omega\right] = \frac{V_{\text{fin}}}{V_{\text{ini}}}. \tag{2.42}$$

By looking at Eq. (2.41), we then infer that the Chern-Simons term modifies the standard propagation of electromagnetic wave by merely adding a rotation of the linear polarization plane (see Fig. 1.1) parameterized by the angle $\alpha$ defined in Eq. (2.38), which is called **birefringence angle**,

$$\left[Q \pm iU\right]_{\text{obs}}(\tau, \mathbf{x}, \hat{\mathbf{p}}_{\gamma}) = \left[Q \pm iU\right]_{\text{EM}}(\tau_{\star}, \mathbf{x}_{\star}, \hat{\mathbf{p}}_{\gamma}) \exp\left\{\pm 2i\left[\chi(\tau, \mathbf{x}) - \chi(\tau_{\star}, \mathbf{x}_{\star})\right]\right\}, \tag{2.43}$$

for $\tau \geq \tau_{\star}$, where the underscript "EM" labels the Stokes parameters predicted by the standard Maxwell's theory, whereas "obs" the real observed ones in the extended model. By the way, as pointed out in Ref. [174], the term "birefringence" refers generically to the property exhibited by certain materials in which electromagnetic waves split into two distinct rays with different velocities when passing through the material. This occurs because the material has different refractive indices for waves polarized in different directions, resulting in optical effects like double refraction, i.e. a bi-refringence. However, in the cosmological literature, the term cosmic birefringence instead describes the specific case of the linear polarization plane's rotation induced by the Chern-Simons coupling, and in crystal optics, the same effect is instead defined as **optical activity**. Furthermore, let us notice that we can rewrite the argument of the complex exponential in Eq. (2.43) in terms of the birefringence angle of a photon observed today ($\tau = \tau_0$) on Earth ($\mathbf{x} = \mathbf{x}_0$) as

$$\begin{aligned}\pm 2i\left[\chi(\tau, \mathbf{x}) - \chi(\tau_{\star}, \mathbf{x}_{\star})\right] &= \pm 2i\left\{\left[\chi(\tau_0, \mathbf{x}_0) - \chi(\tau_{\star}, \mathbf{x}_{\star})\right] - \left[\chi(\tau_0, \mathbf{x}_0) - \chi(\tau, \mathbf{x})\right]\right\} \\ &= \pm 2i\left[\alpha(\tau_{\star}, \mathbf{x}_{\star}) - \alpha(\tau, \mathbf{x})\right],\end{aligned} \tag{2.44}$$

[2] In fact, although the Stokes parameters are not in general invariant under Lorentz transformations, e.g. under spatial rotations, and so they are not scalars in SR, they are coordinate scalars in GR, as shown e.g. in Ref. [79].

where we have used Eq. (2.39) to define the **physical birefringence angle** of a photon emitted at the time $\tau$ in the point with spatial coordinates $\mathbf{x}$ as

$$\boxed{\alpha(\tau,\mathbf{x}) \equiv \frac{\beta}{2}\left[\chi(\tau_0,\mathbf{x}) - \chi(\tau,\mathbf{x})\right],} \tag{2.45}$$

so that Eq. (2.43) reduces to[3]

$$\boxed{\left[Q \pm iU\right]_{\rm obs}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) = \left[Q \pm iU\right]_{\rm EM}(\tau_\star,\mathbf{x}_\star,\hat{\mathbf{p}}_\gamma)\exp\left\{\pm 2i\left[\alpha(\tau_\star,\mathbf{x}_\star) - \alpha(\tau,\mathbf{x})\right]\right\}.} \tag{2.46}$$

## 2.2 Modified Boltzmann Equations for CMB Polarization

In the previous section we have proved that if we add a Chern-Simons term to the Maxwell's action, the Stokes parameters associated with the linear polarization experience a rotation. It is now time to investigate how this phenomenon affects cosmological photons, i.e. to consider the case of **cosmic birefringence**. Indeed Eq. (2.39) shows that birefringence is a propagation effect and therefore larger is the path of the photon, larger will be the probability for the field $\chi$ to change enough to produce an appreciable value for $\alpha$. In other words, since in the model described by Eq. (2.4) the birefringence angle is proportional to the distance traveled by photons, a tiny coupling to the Chern-Simons term can become observable if the source of linearly polarized photons is the farthest possible, which is exactly the case of CMB. In order to see what is the impact of birefringence on cosmological photons, let us see how the standard polarized Boltzmann equation for CMB photons is modified because of cosmic birefringence. Let us rewrite here the Fourier transform of Eq. (1.289),

$$\left[\frac{\partial}{\partial\tau} + i\,\hat{\mathbf{p}}_\gamma\cdot\mathbf{k} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau}\right]{}_{\pm}\Theta_P^{(n)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma) = {}_{\pm}\mathcal{S}_P^{(n)}(\tau,\mathbf{k},\hat{\mathbf{p}}), \tag{2.47}$$

where we have used

$$\begin{cases} {}_{\pm}\Theta_P^{(n)}(\tau,\mathbf{x},\hat{\mathbf{p}}_\gamma) & \equiv \displaystyle\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,{}_{\pm}\Theta_P^{(n)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma)\,e^{i\mathbf{k}\cdot\mathbf{x}}, \\ {}_{\pm}\mathcal{S}_P^{(n)}(\tau,\mathbf{x},\hat{\mathbf{p}}) & \equiv \displaystyle\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,{}_{\pm}\mathcal{S}_P^{(n)}(\tau,\mathbf{k},\hat{\mathbf{p}})\,e^{i\mathbf{k}\cdot\mathbf{x}}. \end{cases} \tag{2.48}$$

[3] Indeed what can be physicaly observed is the not the angle itself, but the variation in the orientation's angle of the linear polarization plane.

Since, our goal here is to find how cosmic birefringence impacts on the Boltzmann equation, let us assume just for now that CMB polarization is only affected by the presence of the Chern-Simons term, so that we will include the contribution from the source function ${}_{\pm}\mathcal{S}_P^{(n)}$ only later. Let us now rewrite Eq. (2.46) in terms of ${}_{\pm}\Theta_P^{(n)}$,

$$
{}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma)|_{\text{obs}} = {}_{\pm}\Theta_P^{(n)}(\tau_\star, \mathbf{x}_\star, \hat{\mathbf{p}}_\gamma)|_{\text{EM}} \exp\left\{\pm 2i\left[\alpha(\tau_\star, \mathbf{x}_\star) - \alpha(\tau, \mathbf{x})\right]\right\}, \tag{2.49}
$$

and let us differentiate its left-hand side with respect to the conformal time,

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\left[{}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma)|_{\text{obs}}\right] = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{\partial}{\partial\tau} + i\,\mathbf{p}_\gamma \cdot \hat{\mathbf{k}}\right] {}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma)|_{\text{obs}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}. \tag{2.50}
$$

However, according to the right-hand side of Eq. (2.49), we should also have

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\left[{}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma)|_{\text{obs}}\right] = \mp 2i\,{}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{x}, \hat{\mathbf{p}}_\gamma)|_{\text{obs}}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau, \mathbf{x}), \tag{2.51}
$$

where we have omitted the underscript "obs" to lighten up the notation. We now adopt the standard perturbative approach also for the birefringence angle,

$$
\alpha(\tau, \mathbf{x}) = \overline{\alpha}(\tau) + \sum_{n=1}^{\infty} \delta\alpha^{(n)}(\tau, \mathbf{x}), \tag{2.52}
$$

as we are going to see in the course of this thesis $\overline{\alpha}$ is responsible for an isotropic cosmic birefringence, whereas its perturbations $\delta\alpha$ are associated with **anisotropic cosmic birefringence** (ACB). By Fourier-transforming the right-hand side of Eq. (2.50) we obtain

$$
\begin{aligned}
\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{\partial}{\partial\tau} + i\,\hat{\mathbf{p}}_\gamma \cdot \mathbf{k}\right] {}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{k}, \hat{\mathbf{p}}_\gamma) e^{i\,\mathbf{k}\cdot\mathbf{x}} = \mp 2i \Bigg\{ & \frac{\mathrm{d}\,\overline{\alpha}(\tau)}{\mathrm{d}\tau} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_{\pm}\Theta_P^{(n)}(\tau, \mathbf{k}, \hat{\mathbf{p}}_\gamma) e^{i\,\mathbf{k}\cdot\mathbf{x}} \\
& + \int \frac{\mathrm{d}^3k_a\, \mathrm{d}^3k_b}{(2\pi)^6}\, {}_{\pm}\Theta_P^{(n)}(\eta, \mathbf{k}_a, \hat{\mathbf{p}}_\gamma)\left[\frac{\partial}{\partial\tau} + i\,\hat{\mathbf{p}}_\gamma \cdot \mathbf{k}_b\right] \delta\alpha^{(n)}(\tau, \mathbf{k}_2)\, e^{i(\mathbf{k}_a + \mathbf{k}_b)\cdot\mathbf{x}} \Bigg\}.
\end{aligned} \tag{2.53}
$$

In order to add now the contribution from the Thomson scattering, we take the inverse Fourier transform of Eq. (2.53) and we include this result within Eq. (2.47). We now find two generalized Boltzmann equations:

the former is valid at first-order in perturbation theory,

$$\left[\frac{\partial}{\partial\tau} + i\,\hat{\mathbf{p}}_\gamma\cdot\mathbf{k} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau} \pm 2i\frac{\mathrm{d}\bar{\alpha}(\tau)}{\mathrm{d}\tau}\right] {}_{\pm}\Theta_P^{(1)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma) = {}_{\pm}\mathcal{S}_P^{(1)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma), \tag{2.54}$$

and the latter at second-order,

$$\begin{aligned}\left[\frac{\partial}{\partial\tau} + i\,\hat{\mathbf{p}}_\gamma\cdot\mathbf{k} - \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau} \pm 2i\frac{\mathrm{d}\bar{\alpha}(\tau)}{\mathrm{d}\tau}\right] {}_{\pm}\Theta_P^{(2)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma) &= {}_{\pm}\mathcal{S}_P^{(2)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma) \\ \mp 2i\int\frac{\mathrm{d}^3k_a\,\mathrm{d}^3k_b}{(2\pi)^3}\delta^{(3)}(\mathbf{k}-\mathbf{k}_a-\mathbf{k}_b)\,{}_{\pm}\Theta_P^{(1)}(\tau,\mathbf{k}_a,\hat{\mathbf{p}}_\gamma)&\left[\frac{\partial}{\partial\tau} + i\,\mathbf{k}_b\cdot\hat{\mathbf{p}}_\gamma\right]\delta\alpha^{(1)}(\tau,\mathbf{k}_b),\end{aligned} \tag{2.55}$$

whereas further perturbative orders are beyond the purpose of this thesis. Now it is clear why we have adopted a perturbative expansion of the relevant quantities: as it can be seen by looking at Eqs. (2.54)-(2.55), isotropic cosmic birefringence affects CMB polarization at any order in perturbation theory, whereas anisotropic cosmic birefringence does it starting from the second-order. This is obvious, since the inhomogeneous perturbation of the birefringence angle is, in fact, an extra cosmological perturbation. In order to solve the two differential equations, we firstly integrate over the conformal time both the sides of Eq. (2.54),

$$\begin{aligned}{}_{\pm}\Theta_P^{(1)}(\tau,\mathbf{k},\hat{\mathbf{p}}_\gamma) =\ & {}_{\pm}\Theta_P^{(1)}(0,\mathbf{k},\hat{\mathbf{p}}_\gamma)e^{-i\,\mathbf{k}\cdot\hat{\mathbf{p}}_\gamma\tau_0}e^{-[\kappa(0)-\kappa(\tau)]}e^{\pm 2i[\bar{\alpha}(0)-\bar{\alpha}(\tau)]} \\ & + \int_0^\tau \mathrm{d}\tilde{\tau}\,{}_{\pm}\mathcal{S}_P^{(1)}(\tilde{\tau},\mathbf{k},\hat{\mathbf{p}}_\gamma)e^{i\,\mathbf{k}\cdot\hat{\mathbf{p}}_\gamma(\tilde{\tau}-\tau)}e^{-[\kappa(\tilde{\tau})-\kappa(\tau)]}e^{\pm 2i[\bar{\alpha}(\tilde{\tau})-\bar{\alpha}(\tau)]}.\end{aligned} \tag{2.56}$$

This procedure in standard in cosmological perturbation theory, and it is done because what we are looking for is the expression of the ${}_{\pm}\Theta_P$'s for $\tau = \tau_0$ and $\mathbf{x} = \mathbf{x}_0$, since the observation of CMB anisotropies is made today on Earth, by looking at the photons propagating along the $\hat{\mathbf{p}}_\gamma$-direction. This means that our direction of observation, say $\hat{\mathbf{n}}$, is given by

$$\boxed{\hat{\mathbf{n}} = -\hat{\mathbf{p}}_\gamma,} \tag{2.57}$$

and from this moment we will use it instead of $\hat{\mathbf{p}}_\gamma$. However, in our case we do not need just ${}_{\pm}\Theta_P^{(1)}(\tau_0,\mathbf{k},\hat{\mathbf{n}})$, but also ${}_{\pm}\Theta_{P,\pm}^{(1)}(\tau,\mathbf{k},\hat{\mathbf{n}})$, since, Eq. (2.55) depends on the first-order transfer function. By the way, if we recall the definition of the optical depth we provided in Eq. (1.244), we see that it can be written as

$$\kappa(\tau) = \sigma_T\int_\tau^{\tau_0}\mathrm{d}\tilde{\tau}\,a(\tilde{\tau})\bar{n}_e(\tilde{\tau}), \tag{2.58}$$

so that, after solving it numerically, it is possible to notice that $\kappa(\tau)$ becomes extremely large in the radiation-dominated epoch (see e.g. Ref. [175]), i.e. for $\tau \to 0$, so that the first term on the right-hand side becomes exponentially suppressed and then completely negligible. Therefore, we find the same result obtained e.g. in Refs. [51, 176–180],

$$
{}_{\pm}\Theta_P^{(1)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) = \int_0^{\tau_0} \mathrm{d}\tau \, {}_{\pm}\mathcal{S}_P^{(1)}(\tau, \mathbf{k}, \hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} e^{-\kappa(\tau)} e^{\pm 2i\bar{\alpha}(\tau)}, \tag{2.59}
$$

where we have used that the optical depth's value is zero today because of Eq. (2.58), and that the isotropic birefringence angle for a photon emitted today is identically vanishing, according to the definition given in Eq. (2.45). Similarly, the second-order transfer function, after integrating by parts, reads

$$
\begin{aligned}
{}_{\pm}\Theta_P^{(2)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) = \int_0^{\eta_0} \mathrm{d}\eta \, e^{i\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} e^{-\kappa(\tau)} e^{\pm 2i\bar{\alpha}(\tau)} \Big[ & {}_{\pm}\mathcal{S}_P^{(2)}(\tau, \mathbf{k}, \hat{\mathbf{n}}) \\
& \pm 2i \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} \, \delta\alpha^{(1)}(\tau, \mathbf{k} - \tilde{\mathbf{k}}) \, {}_{\pm}\mathcal{S}_P^{(1)}(\tau, \tilde{\mathbf{k}}, \hat{\mathbf{n}}) \Big].
\end{aligned} \tag{2.60}
$$

Eqs. (2.59)-(2.60) are the core of our generalized treatment of CMB polarization. Armed with these expressions, we can investigate what is the impact on the main CMB observables. In order to do that, it is convenient to define the following quantity, just valid for the perturbative orders $n = 1, 2$,

$$
{}_{\pm}\Delta_\lambda^{(n)}(\tau, \mathbf{k}) \equiv e^{\pm 2i\bar{\alpha}(\tau)} \left[ \Pi_\lambda^{(n)}(\tau, \mathbf{k}) \pm 2i(n-1) \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} \, \delta\alpha^{(1)}(\tau, \mathbf{k} - \tilde{\mathbf{k}}) \Pi_\lambda^{(1)}(\tau, \tilde{\mathbf{k}}) \right]. \tag{2.61}
$$

Indeed, if we substitute Eq. (1.291) within Eqs. (2.59)-(2.60), we can then write a compact expression valid for the perturbative order $n = 1, 2$,

$$
{}_{\pm}\Theta_P^{(n)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) = \sqrt{\frac{6\pi}{5}} \int_0^{\tau_0} \mathrm{d}\tau \, e^{i\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} g(\tau) \sum_{\lambda=-2}^{2} {}_{\pm 2}Y_{2\lambda}(\hat{\mathbf{n}}) \, {}_{\pm}\Delta_\lambda^{(n)}(\tau, \mathbf{k}), \tag{2.62}
$$

where we have defined the **photons' visibility function** as

$$
g(\tau) \equiv -\left[ \frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau} \right] \exp\left[ -\kappa(\tau) \right], \tag{2.63}
$$

which physically represents the Poissonian probability that a photon is last scattered at a conformal time $\tau$. Let us highlight here that Eq. (2.62) is the main result of this section, and we have put it in such a specific form because now the mathematical computation becomes less challenging, since it has exactly the same form of the standard transfer function of CMB polarization, such as e.g. Eq. (14) of Ref. [181]. Let us now move to the harmonic space, and in order to do that we notice that the dependence of the ${}_{\pm}\Theta_P^{(n)}$ on $\hat{\mathbf{n}}$ is encoded in ${}_{\pm}\Delta_{\lambda}^{(n)}$ but also in the complex exponential, which can be rewritten via the plane wave expansion (see e.g. Ref. [182]),

$$e^{i\,\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} = 4\pi \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^L j_L[k(\tau_0-\tau)]\, {}_0Y^*_{LM}(\hat{\mathbf{k}})\, {}_0Y_{LM}(\hat{\mathbf{n}}), \tag{2.64}$$

where $j_L$ is the $L$-th spherical Bessel function. Therefore, let us evaluate the following harmonic transform:

$${}_{\pm2}P^{(n)}_{\ell m}(\tau_0, \mathbf{x}_0) \equiv \int \frac{\mathrm{d}^2\hat{n}}{4\pi}\, {}_{\pm2}Y^*_{\ell m}(\hat{\mathbf{n}}) \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_{\pm}\Theta_P^{(n)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) e^{i\,\mathbf{k}\cdot\mathbf{x}_0}. \tag{2.65}$$

By substituting Eq. (2.64) in Eq. (2.62), we can easily see that now the dependence on $\hat{\mathbf{n}}$ is encoded in the product of two spin-weighted spherical harmonics, which can rewritten as a single one by means of the composition of angular momenta (see e.g. Ref. [183]),

$$\begin{aligned} {}_{s_1}Y_{\ell_1 m_1}(\hat{\mathbf{n}})\, {}_{s_2}Y_{\ell_2 m_2}(\hat{\mathbf{n}}) = \sum_{\ell_3}\sum_{m_3}\sum_{s_3} &\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \\ &\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} {}_{s_3}Y^*_{\ell_3 m_3}(\hat{\mathbf{n}}), \end{aligned} \tag{2.66}$$

where the "matrix" is a **Wigner 3-j symbol**, which satisfies the following selection rules:

$$\begin{cases} |\ell_1-\ell_2| & \le \ell_3 \le |\ell_1+\ell_2|, \\ m_1+m_2+m_3 & = 0, \\ s_1+s_2+s_3 & = 0. \end{cases} \tag{2.67}$$

However, in order to better digest such a long computation, it is better to adopt a standard trick in CMB calculations: indeed, instead of directly evaluating Eq. (2.65). we apply a sort of "fake" rotation of the $\hat{\mathbf{k}}$

unit vector, that is we firstly compute

$$ {}_{\pm}\Theta_P^{(n)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) = \mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \left[ {}_{\pm}\Theta_P^{(n)}(\tau_0, k\,\hat{\mathbf{x}}_3, \hat{\mathbf{n}}) \right] . \tag{2.68} $$

In other words, we choose to work in the coordinate system where $\mathbf{k} \parallel \hat{\mathbf{x}}_3$, and then, before to perform the angular integration, we apply a rotation $\mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}}$ which brings $\mathbf{k}$ in a generic direction. Nevertheless, it is clear that rotating the reference system implies that also $\hat{\mathbf{n}}$ rotates and we have to take this into account. We then substitute Eq. (2.64) and Eq. (2.66) within Eq. (2.62) and we plug all together in Eq. (2.68), so that we obtain

$$ \begin{aligned} {}_{\pm}\Theta_P^{(n)}(\tau_0, \mathbf{k}, \hat{\mathbf{n}}) = \sqrt{\frac{3}{2}} \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) \sum_{LM} i^L j_L[k(\tau_0 - \tau)] \sum_{L'M'} \sqrt{(2L+1)(2L'+1)} \\ \begin{pmatrix} L & 2 & L' \\ 0 & \mp 2 & \pm 2 \end{pmatrix} \sum_{\lambda=-2}^{2} \begin{pmatrix} L & 2 & L' \\ M & \lambda & M' \end{pmatrix} \mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \left[ {}_0 Y^*_{LM}(\hat{\mathbf{x}}_3)\, {}_{\pm}\Delta_\lambda^{(n)}(\eta, k\,\hat{\mathbf{x}}_3)\, {}_{\mp 2} Y^*_{L'M'}(\hat{\mathbf{n}}) \right] . \end{aligned} \tag{2.69} $$

Thanks to our choice of $\hat{\mathbf{k}} = \hat{\mathbf{x}}_3$, the associated spherical harmonics is simply given as (see e.g. Ref. [184])

$$ {}_0 Y^*_{LM}(\hat{\mathbf{x}}_3) = \delta_{M0} \sqrt{\frac{2L+1}{4\pi}} , \tag{2.70} $$

so that in the last line of Eq. (2.69) we have

$$ \mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \left[ Y^*_{LM}(\hat{\mathbf{x}}_3)\, {}_{\pm}\Delta_\lambda^{(n)}(\tau, k\,\hat{\mathbf{x}}_3)\, {}_{\mp 2} Y^*_{L'M'}(\hat{\mathbf{n}}) \right] = \delta_{M0} \sqrt{\frac{2L+1}{4\pi}}\, {}_{\pm}\Delta_\lambda^{(n)}(\tau, \mathbf{k})\, \mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \left[ {}_{\mp 2} Y^*_{L'M'}(\hat{\mathbf{n}}) \right] , \tag{2.71} $$

where we have exploited that the rotation operator is unitary, and so applying it to a product of quantities is equivalent to multiply the rotated quantities themselves. The action of the rotation operator on the spin-weighted spherical harmonics is given as (see e.g. Ref. [184]),

$$ \begin{aligned} \mathsf{R}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \left[ {}_{\mp 2} Y^*_{L'M'}(\hat{\mathbf{n}}) \right] &= \sum_{m'=-L'}^{L'} \mathsf{D}^{(L')}_{m'M'} \left[ \mathsf{R}^{-1}_{k\,\hat{\mathbf{x}}_3 \mapsto \mathbf{k}} \right] \, {}_{\mp 2} Y^*_{L'm'}(\hat{\mathbf{n}}) \\ &= \sum_{m'=-L'}^{L'} \sqrt{\frac{4\pi}{2L'+1}}\, {}_{M'} Y^*_{L'm'}(\hat{\mathbf{k}})\, {}_{\pm 2} Y_{L'm'}(\hat{\mathbf{n}}) , \end{aligned} \tag{2.72} $$

where the $\mathrm{D}^{(L)}_{m'M'}$'s are elements of the **Wigner D-matrix**. We now substitute the results of Eqs. (2.71)-(2.72) in Eq. (2.69). By exploiting the orthonormality of spin-weighted spherical harmonics (see e.g. Ref. [159]),

$$\int \mathrm{d}^2\hat{n}\, {}_sY^*_{\ell_1 m_1}(\hat{\mathbf{n}})\, {}_sY_{\ell_2 m_2}(\hat{\mathbf{n}}) = \delta_{\ell_1\ell_2}\delta_{m_1m_2}, \tag{2.73}$$

$\forall\, s$ we can finally evaluate the right-hand side of Eq. (2.65),

$$\begin{aligned} {}_{\pm 2}P^{(n)}_{\ell m}(\tau_0, \mathbf{x}_0) = \sqrt{\frac{3}{2}} \sum_{L=|\ell-2|}^{\ell+2} i^L (2L+1) \begin{pmatrix} L & 2 & \ell \\ 0 & \mp 2 & \pm 2 \end{pmatrix} \sum_{\lambda=-2}^{2} \begin{pmatrix} L & 2 & \ell \\ 0 & \lambda & -\lambda \end{pmatrix} \\ \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_{-\lambda}Y^*_{\ell m}(\hat{\mathbf{k}}) \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau)\, {}_{\pm}\Delta^{(n)}_{\lambda}(\tau, \mathbf{k})\, j_L[k(\tau_0 - \tau)], \end{aligned} \tag{2.74}$$

where we have used that $\mathbf{x}_0 = 0$ by putting us at the center of the reference frame. At this point it becomes crucial to introduce that the CMB polarization field can be decomposed into "electric" and "magnetic" components that are signatures of distinct physical processes, and that behave differently under parity transformations (see e.g. Ref. [185]),

$$ {}_{\pm 2}P^{(n)}_{\ell m}(\tau_0, \mathbf{x}_0) \equiv -\left[E^{(n)}_{\ell m} \pm i B^{(n)}_{\ell m}\right](\tau_0, \mathbf{x}_0). \tag{2.75}$$

Therefore, we are now in the position to give the most general expression for the harmonic coefficients of the CMB polarization. By recalling all the procedure that we have made, it can be easily understood that the results of Eq. (2.76) and Eq. (2.77) are valid for any kind of cosmological perturbations (scalar, vector or tensor) up to the second-order in perturbation theory ($x = 1, 2$), and for any kind of initial conditions:

$$\begin{aligned} E^{(n)}_{\ell m}(\tau_0, \mathbf{x}_0) = \sqrt{\frac{3}{8}} \sum_{L=|\ell-2|}^{\ell+2} i^{L+2} (2L+1) \begin{pmatrix} L & 2 & \ell \\ 0 & -2 & 2 \end{pmatrix} \sum_{\lambda=-2}^{2} \begin{pmatrix} L & 2 & \ell \\ 0 & \lambda & -\lambda \end{pmatrix} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_{-\lambda}Y^*_{\ell m}(\hat{\mathbf{k}}) \\ \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) \left[ {}_{+}\Delta^{(n)}_{\lambda}(\tau, \mathbf{k}) + (-1)^{\ell+L}\, {}_{-}\Delta^{(n)}_{\lambda}(\tau, \mathbf{k}) \right] j_L[k(\tau_0 - \tau)], \end{aligned} \tag{2.76}$$

$$\begin{aligned} B^{(n)}_{\ell m}(\tau_0, \mathbf{x}_0) = \sqrt{\frac{3}{8}} \sum_{L=|\ell-2|}^{\ell+2} i^{L+1} (2L+1) \begin{pmatrix} L & 2 & \ell \\ 0 & -2 & 2 \end{pmatrix} \sum_{\lambda=-2}^{2} \begin{pmatrix} L & 2 & \ell \\ 0 & \lambda & -\lambda \end{pmatrix} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_{-\lambda}Y^*_{\ell m}(\hat{\mathbf{k}}) \\ \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) \left[ {}_{+}\Delta^{(n)}_{\lambda}(\tau, \mathbf{k}) - (-1)^{\ell+L}\, {}_{-}\Delta^{(n)}_{\lambda}(\tau, \mathbf{k}) \right] j_L[k(\tau_0 - \tau)]. \end{aligned} \tag{2.77}$$

Let us now focus on $n = 1$ case, for which we remark that if we stop our summation by just considering the $\lambda = 0$ contribution, we get

$$E^{(1)}_{\ell m}(\tau_0, \mathbf{x}_0)\big|_{\lambda=0} = \sqrt{\frac{3}{2}} \sum_{L=|\ell-2|}^{\ell+2} i^{L+2}(2L+1) \begin{pmatrix} L & 2 & \ell \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} L & 2 & \ell \\ 0 & 0 & 0 \end{pmatrix} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, Y^*_{\ell m}(\hat{\mathbf{k}}) \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) \cos[2\,\bar{\alpha}(\tau)]\Pi^{(1)}_0(\tau, \mathbf{k}) j_L[k(\tau_0 - \tau)], \tag{2.78}$$

$$B^{(1)}_{\ell m}(\tau_0, \mathbf{x}_0)\big|_{\lambda=0} = \sqrt{\frac{3}{2}} \sum_{L=|\ell-2|}^{\ell+2} i^{L+2}(2L+1) \begin{pmatrix} L & 2 & \ell \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} L & 2 & \ell \\ 0 & 0 & 0 \end{pmatrix} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, Y^*_{\ell m}(\hat{\mathbf{k}}) \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) \sin[2\,\bar{\alpha}(\tau)]\Pi^{(1)}_0(\tau, \mathbf{k}) j_L[k(\tau_0 - \tau)]. \tag{2.79}$$

By inspecting the first-order expressions, we can see that, in such a case, if the isotropic birefringence angle equals zero, we have no $B$-modes, and this is something completely expected, since in the standard $\Lambda$CDM model they are sourced just by inflationary tensor perturbations, i.e. **primordial gravitational waves**, or by the **gravitational lensing** of the $E$ modes. In fact, by recalling Eq. (1.292), it is possible to show that setting $\lambda = 0$ has the meaning of selecting just scalar perturbations, because of the axis-symmetry of the radiation field around the mode axis[4] for this case, as discussed e.g. in Ref. [186]. However, directly evaluating Eqs. (2.76)-(2.77) could be extremely challenging, since it involves the interplay between second-order perturbations. For this reason, we are going to motivate the usage of a really suitable approximation that will be exploited a lot in the course of this thesis. Let us start by substituting Eq. (1.291) within Eq. (2.60) expressed in the real space,

$${}_{\pm}\Theta^{(2)}_P(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}}) = \sqrt{\frac{6\pi}{5}} \sum_{\lambda=-2}^{2} {}_{\pm 2}Y_{2\lambda}(\hat{\mathbf{n}}) \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) e^{\pm 2i\,\bar{\alpha}(\tau)} \left\{ \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\,\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} \Pi^{(2)}_\lambda(\tau, \mathbf{k}) \pm 2i \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\,\mathbf{k}\cdot\hat{\mathbf{n}}(\tau_0-\tau)} \left[\delta\alpha^{(1)}(\tau) * \Pi^{(1)}_\lambda(\tau)\right](\mathbf{k}) \right\}, \tag{2.80}$$

[4] However, let us point out that this is no more true when going at second order in perturbation theory, because such a rotational symmetry around the wave-vector is broken by coupling to other modes (see e.g. Refs. [162, 181]).

where we have recognized the definition of the convolution product, and used again that $\mathbf{x}_0 = 0$. We can now simplify Eq. (2.80), by exploiting the convolution theorem, which allows us to deal with the Fourier transform of the convolution in the last line, so that we can write

$$
\begin{aligned}
{}_{\pm}\Theta_P^{(2)}(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}}) = \sqrt{\frac{6\pi}{5}} \sum_{\lambda=-2}^{2} {}_{\pm 2}Y_{2\lambda}(\hat{\mathbf{n}}) \int_0^{\tau_0} \mathrm{d}\tau\, g(\tau) e^{\pm 2i\bar{\alpha}(\tau)} \\
\left\{ \Pi_\lambda^{(2)}[\tau, (\tau_0 - \tau)\hat{\mathbf{n}}] \pm 2i\delta\alpha^{(1)}[\tau, (\tau_0 - \tau)\hat{\mathbf{n}}] \Pi_\lambda^{(1)}[\tau, (\tau_0 - \tau)\hat{\mathbf{n}}] \right\}.
\end{aligned} \tag{2.81}
$$

As said before, evaluating the time-integral involving second-order perturbations is not trivial at all, but we can reduce a lot the problem by adopting a simple but reasonable approximation. In fact, if in Eq. (2.81) we substitute the photons' visibility function with a series of Dirac deltas associated with the peaks of the original $g(\tau)$, as shown in Fig. 2.1, i.e., according to the discussion made in Sec. 1.3.4, the recombination and reionization epochs[5], respectively,

$$
g(\tau) \simeq g_{\mathrm{rec}} \delta(\tau - \tau_{\mathrm{rec}}) + g_{\mathrm{rei}} \delta(\tau - \tau_{\mathrm{rei}}), \tag{2.82}
$$

with $g_{\mathrm{rec}} \gg g_{\mathrm{rei}}$, then the time integral can be trivially computed leading to the following result:

$$
{}_{\pm}\Theta_P(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\mathrm{obs}} = \sum_{i=\mathrm{rec,\,rei}} \exp\left\{ \pm 2i\bar{\alpha}(\tau_i) \pm 2i\delta\alpha[\tau_i, (\tau_0 - \tau_i)\hat{\mathbf{n}}] \right\} {}_{\pm}\Theta_P(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\mathrm{EM}}, \tag{2.83}
$$

where we have resummed up all the perturbative orders. Let us underline here that Eq. (2.83) is the "master equation" used in literature to deal also with the anisotropic birefringence contribution (see e.g. Refs. [74–77, 79–85, 85, 86]). However, let us mention here, that from this moment, we will stop considering the second-order contributions to CMB polarization except for those coming from anisotropic cosmic birefringence. This is done, in order to be consistent with the main literature on the topic, and for this reason, we will not use the superscript $^{(2)}$ when dealing with CMB anisotropies, anymore.

[5] Although, for sake of simplicity, here we write some Dirac deltas, the numerical code we have used in this thesis to simulate cosmic birefringence, does not evaluate the relevant quantities only at $\tau_{\mathrm{rec}}$ or $\tau_{\mathrm{rei}}$ but in practice convolves it with the photon visibility function $g(\tau)$ in the neighborhood of the recombination and reionization peak, so that we can isolate the recombination contribution from the one from reionization epoch. This approach is not only numerically more accurate but also more physically correct since of course, we have an important amount of photons emitted also in a finite range of $\tau$ close to the peaks.

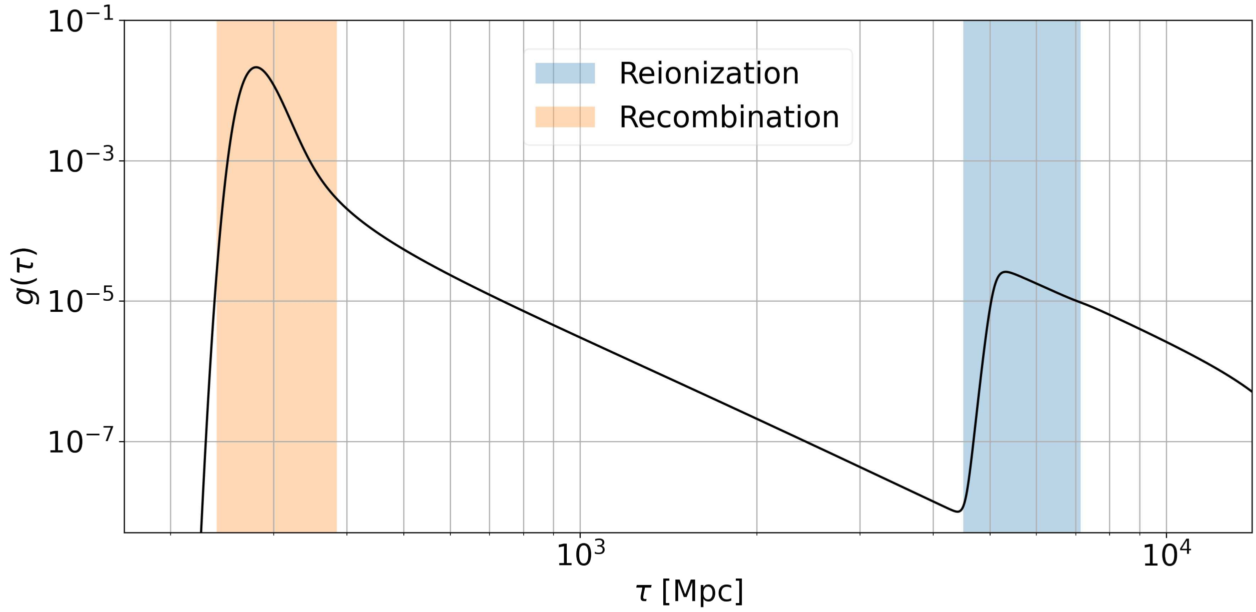


**Figure 2.1:** The photons visibility function $g$ as function of the conformal time $\tau$ from the numerical calculation performed with CLASS for the ΛCDM model.

## 2.3 The Axion Hypothesis

We have seen in the previous sections how the presence of a Chern-Simons coupling between a scalar field $\chi$ with the electromagnetic one can alter the photons' standard polarization. Therefore, it is now time to discuss about the nature of this field $\chi$, and this is strictly related to its property under **parity transformations**. In fact, a parity transformation is nothing but a spatial inversion, i.e. the flip of the signs of each spatial coordinates in the reference frame, which is something different from a standard rotation, since there is no rotation operator R able to induce $\mathbf{x} \mapsto -\mathbf{x}$. At the contrary, the only operator which can do that is the **parity operator**. Hence, if parity is not a rotation, there is no guaranty that the field $\chi$ will be left unchanged by this kind of coordinate transformations. For instance, although the quantity $F^{\mu\nu}F_{\mu\nu}$ is **parity-even**, the opposite occurs for $\tilde{F}^{\mu\nu}F_{\mu\nu}$, which is indeed **parity-odd**:

$$F^{\mu\nu}F_{\mu\nu}(t,\mathbf{x}) \xrightarrow{\mathbf{x}\mapsto-\mathbf{x}} +F^{\mu\nu}F_{\mu\nu}(t,-\mathbf{x}), \qquad \tilde{F}^{\mu\nu}F_{\mu\nu}(t,\mathbf{x}) \xrightarrow{\mathbf{x}\mapsto-\mathbf{x}} -\tilde{F}^{\mu\nu}F_{\mu\nu}(t,-\mathbf{x}), \tag{2.84}$$

because of the antisymmetric Levi-Civita symbol appearing in the definition of $\tilde{F}^{\mu\nu}$. By direct inspection of Eq. (2.5), we then easily see that since $\tilde{F}^{\mu\nu}F_{\mu\nu}$ changes sign under inversion of spatial coordinates, it follows that also $\chi$ needs to be a parity-odd quantity, such that the whole Chern-Simons term remains invariant.

However, let us point that such a requirement is not mandatory at all: the only fundamental rules a field theory in curved spacetime must obey are that its associated action has to be real and invariant under the GR version of **Poincarè group**'s transformations, i.e. spatial rotations, boosts and spacetime translations, and parity is none of them. However, the fact that Eq. (2.4) is parity-invariant under the assumption of a parity-odd field $\chi$ does not mean that there is not parity-violation: indeed, as we are going to see in the next sections, cosmic birefringence is effectively a parity-breaking phenomenon, since it allows us to probe parity-breaking cosmological observable. Now, if $\chi$ is parity-odd, then it is said to be a **pseudoscalar field**, which we are introducing in the jargon of cosmic species to switching-on cosmic birefringence. A question then naturally arises: who is it? The intriguing point is that $\chi$ can be seen as a candidate for a component of the Universe's dark sector, i.e. dark matter or dark energy, in the form of a pseudoscalar field. Indeed, the current cosmological paradigm is not able to explain the exact nature of none of them, and, as we briefly mentioned in Sec. 1.3.1, several candidates have been proposed in the literature. Therefore, CMB provides us a powerful way of testing the pseudoscalar field hypothesis for dark matter or dark energy thanks to its observable polarization, which is in principle sensitive to eventual birefringence effects sourced by $\chi$. Even before the advent of studies about cosmic birefringence, the possibility that dark matter (see e.g. Refs. [55–61]) or dark energy (see e.g. Refs. [47–54]) are present in the Universe in the form of pseudoscalar field, has been deeply investigated [62–73, 187–189], and the hypothetical particle associated with such a field $\chi$ is said to be an **axion**, in analogy with the elementary particle first proposed in Ref. [190] to solve the strong CP problem in quantum chromodynamics. The action of the axion-like field $\chi$ si then given by the standard Klein-Gordon one plus the Chern-Simons addition,

$$S_\chi = \int \mathrm{d}^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi) + \frac{\beta}{2}\chi\tilde{F}^{\mu\nu}F_{\mu\nu}\right], \tag{2.85}$$

where $V(\chi)$ is the axion potential, whose expression defines the model for the pseudoscalar field itself. As shown in Eq. (2.45), the birefringence angle is associated with the difference in value for $\chi$ between the moments of emission and observation of the photons. Therefore, since we have adopted a perturbative expansion for $\alpha$ in Eq. (2.52), it is clear that we can do the same also for the the axion-like field,

$$\chi(\tau,\mathbf{x}) = \bar{\chi}(\tau) + \sum_{n=1}^{\infty}\delta\chi^{(n)}(\tau,\mathbf{x}). \tag{2.86}$$

We are now going to investigate separately the homogeneous $\bar{\chi}$ and its perturbations, because, by recalling Eq. (2.45), it is clear that they are responsible for the isotropic and anisotropic birefringence, respectively:

$$\bar{\alpha}(\tau) = \frac{\beta}{2}\left[\bar{\chi}(\tau_0) - \bar{\chi}(\tau)\right], \qquad \delta\alpha^{(n)}(\tau, \mathbf{x}_0) = \frac{\beta}{2}\left[\delta\chi^{(n)}(\tau_0, \mathbf{x}_0) - \delta\chi^{(n)}(\tau, \mathbf{x})\right]. \tag{2.87}$$

### 2.3.1 Background Evolution of a Pseudoscalar Field

The evolution of $\bar{\chi}(\tau)$ is governed by its equation of motion, which is found by applying the Hamilton's principle we stated in Eq. (1.15) to the action defined Eq. (2.85) with respect to $\bar{\chi}$ in the FLRW metric,

$$\frac{\mathrm{d}^2\bar{\chi}(\tau)}{\mathrm{d}\tau^2} + 2\mathcal{H}(\tau)\frac{\mathrm{d}\bar{\chi}(\tau)}{\mathrm{d}\tau} + a^2(\tau)\frac{\mathrm{d}V(\bar{\chi})}{\mathrm{d}\bar{\chi}} = 0. \tag{2.88}$$

In order to track the evolution of such a pseudoscalar field, in this thesis we specialize our analysis to the case in which $\chi$ is a quintessence-like field playing the role of early dark energy, characterized by the following potential (see e.g. Refs. [49, 51, 81–83]:

$$V[\bar{\chi}(\tau)] = m_\chi^2 M_{Pl}^2 \left\{1 - \cos\left[\frac{\bar{\chi}(\tau)}{M_{Pl}}\right]\right\}^2, \tag{2.89}$$

where $M_{Pl} \equiv \sqrt{8\pi} m_{Pl}$ is the Planck mass and $m_\chi$ is a parameter which defines the mass of the field $\chi$. We have modified the Boltzmann code `CLASS` [191] in order to solve Eq. (2.88) and get the plot shown in Fig. 2.2, whose upper panel shows the evolution of $\bar{\chi}$ with respect to $\bar{\chi}(\tau_{\text{ini}})$ as a function of the **cosmic redshift**,

$$z(\tau) \equiv \frac{1}{a(\tau)} - 1, \tag{2.90}$$

for different values of the mass parameter $m_\chi$: by inspecting such a plot, we can observe that smaller the field mass is, slower its evolution is (see e.g. Refs. [75, 76]). It is possible to qualitatively identify five different phenomenological regimes:

- if $m_\chi \gg 10^{-27}$ eV, then $\bar{\chi}(\tau)$ fluctuates around zero at recombination, at reionization and today, so that, by recalling Eq. (2.87), there is no isotropic cosmic birefringence.
- if $10^{-29}$ eV $\ll m_\chi \lesssim 10^{-27}$ eV, then only recombination contributes to isotropic cosmic birefringence, since $\bar{\chi}(\tau_{\text{rec}}) \neq 0 = \bar{\chi}(\tau_0)$ but $\bar{\chi}(\tau_{\text{rei}}) = 0 = \bar{\chi}(\tau_0)$;

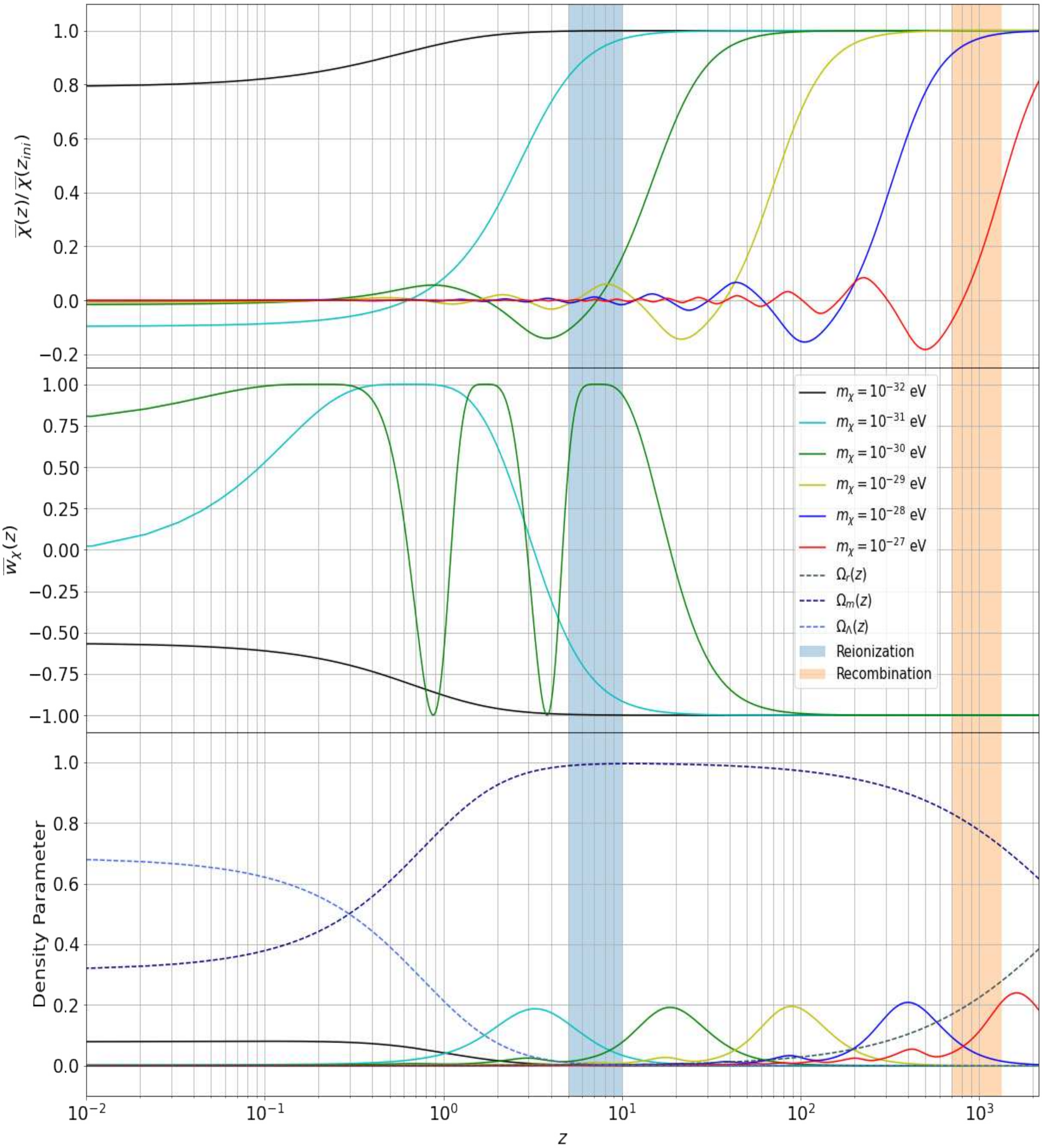


**Figure 2.2:** Background axion quantities as functions of $z$ for the model defined by Eq. (2.89). The numerical computation has been performed for several values of $m_\chi$ with $\bar{\chi}(\tau_{\rm ini}) = m_{pl}$, $\mathrm{d}\bar{\chi}/\mathrm{d}\tau(\tau_{\rm ini}) = 0$, and the fiducial ones of the $\Lambda$CDM parameters given in Ref. [144]. The colored regions have been numerically evaluated by using the `HyRec` algorithm for recombination and the `tanh()` model for reionization.

- if $10^{-32}$ eV $\ll m_\chi \lesssim 10^{-29}$ eV, then both recombination and reionization contribute to isotropic cosmic birefringence, with different rotation angles;
- if $m_\chi \lesssim 10^{-32}$ eV, then both recombination and reionization contribute to isotropic cosmic birefringence, but the rotation angle is the same, since $\overline{\chi}(\tau_{\rm rec}) = \overline{\chi}(\tau_{\rm rei})$;
- if $m_\chi \ll 10^{-32}$ eV, there is again no isotropic cosmic birefringence, since $\overline{\chi}(\tau_{\rm rec}) = \overline{\chi}(\tau_{\rm rei}) = \overline{\chi}(\tau_0)$.

From this analysis we can infer that an appreciable birefringence effect occurs only within a finite window of masses ($m_\chi \in [10^{-32}{\rm eV},\ 10^{-27}{\rm eV}]$). As we will see in the next sections, the anisotropic contribution to cosmic birefringence allows one to probe higher values for the axion mass. Note that independently on the axion mass, the field experiences a slow-roll phase at early times (and so at high redshifts): this is due to the fact that we have taken $\mathrm{d}\overline{\chi}/\mathrm{d}\tau(\tau_{\rm ini}) = 0$, which is a natural requirement for $\overline{\chi}$ if we want it to behave as **early dark energy** (EDE). To understand this, let us compute the energy-momentum tensor for the scalar field: this can be done by evaluating Eq. (1.123) for the action defined in Eq. (2.85)

$$\mathcal{T}_{\mu\nu}(\tau, \mathbf{x}) \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\chi}{\delta g^{\mu\nu}} = \partial_\mu \chi \partial_\nu \chi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha \chi\, \partial_\beta \chi + V(\chi)\right], \tag{2.91}$$

so that we can easily get the background energy density and pressure of the scalar field as

$$\overline{\varrho}_\chi(\tau) = -\overline{\mathcal{T}}^0{}_0(\tau) = \frac{1}{2a^2(\tau)}\left[\frac{\mathrm{d}\overline{\chi}(\tau)}{\mathrm{d}\tau}\right]^2 + V[\overline{\chi}(\tau)], \tag{2.92}$$

$$\overline{\mathcal{P}}_\chi(\tau) = \frac{1}{3}\delta^i{}_j \overline{\mathcal{T}}^j{}_i(\tau) = \frac{1}{2a^2(\tau)}\left[\frac{\mathrm{d}\overline{\chi}(\tau)}{\mathrm{d}\tau}\right]^2 - V[\overline{\chi}(\tau)]. \tag{2.93}$$

Therefore the equation of state of the homogeneous scalar field $\overline{\chi}(\tau)$ is given by

$$\overline{w}_\chi(\tau) = \frac{\overline{\mathcal{P}}_\chi(\tau)}{\overline{\varrho}_\chi(\tau)} = \frac{[\mathrm{d}\overline{\chi}(\tau)/\mathrm{d}\tau]^2 - 2a^2(\tau)V[\overline{\chi}(\tau)]}{[\mathrm{d}\overline{\chi}(\tau)/\mathrm{d}\tau]^2 + 2a^2(\tau)V[\overline{\chi}(\tau)]}, \tag{2.94}$$

and it is equal to $\overline{w}_\chi = -1$, i.e., as discussed in Sec. 1.3.1, it mimics the behavior of the cosmological constant, when $\overline{\chi}$ is frozen, i.e. for $\mathrm{d}\overline{\chi}/\mathrm{d}\tau = 0$. Hence, the requirement $\overline{w}_\chi(\tau_{\rm ini}) = -1$ implies $\mathrm{d}\overline{\chi}/\mathrm{d}\tau = 0$ for $\tau = \tau_{\rm ini}$. In the central panel of Fig. 2.2 we have used again our modified version of `CLASS` to track the evolution of the equation of state for the scalar field $\chi$. Let us notice that this axion-like field has a rich phenomenology, since different masses imply a different nature of the field: e.g., if $m_\chi \simeq 10^{-31}$ eV, although the field may have been dark energy in the early Universe, today it may be contributing to dark matter, since $\overline{w}_\chi(\tau_0) \simeq 0$. However, let us just mention that in this thesis, as well as in Refs. [75–77], for sake of simplicity, we have not added the contribution due to the axion field in the conformal Hubble parameter $\mathcal{H}$ appearing in Eq. (2.88): this is a sort of **spectator field approximation** for the axion field, which is valid only if $\bar{\varrho}_\chi$ makes $\bar{\chi}$ a subdominant cosmic species, and the field initials conditions are chosen to satisfy this condition, as shown in the lower panel of Fig. 2.2.

### 2.3.2 First-Order Axion Perturbations

We now derive the equation of motion of the axion valid at first order in perturbation theory. By applying again the Hamilton's principle, we then vary the action defined in Eq. (2.85) in the usual Poisson gauge we introduced in Eq. (1.175), getting

$$\begin{aligned}\frac{\partial^2}{\partial\tau^2}\delta\chi^{(1)}(\tau,\mathbf{x}) + 2\mathcal{H}(\tau)\frac{\partial}{\partial\tau}\delta\chi^{(1)}(\tau,\mathbf{x}) + \left[k^2 + a^2(\tau)\frac{\mathrm{d}^2V(\bar{\chi})}{\mathrm{d}\bar{\chi}^2}\right]\delta\chi^{(1)}(\tau,\mathbf{x}) = \\ = \left[\frac{\mathrm{d}^2\bar{\chi}(\tau)}{\mathrm{d}\tau^2} + 2\mathcal{H}(\tau)\frac{\mathrm{d}\bar{\chi}(\tau)}{\mathrm{d}\tau} + \frac{\mathrm{d}\bar{\chi}(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial\tau}\right]\left[3\Phi^{(1)}(\tau,\mathbf{x}) + \Psi^{(1)}(\tau,\mathbf{x})\right] \\ + a^2(\tau)\frac{\mathrm{d}V(\bar{\chi})}{\mathrm{d}\bar{\chi}}\left[3\Phi^{(1)}(\tau,\mathbf{x}) - \Psi^{(1)}(\tau,\mathbf{x})\right],\end{aligned} \tag{2.95}$$

where we have set the background component of the electromagnetic field equal to zero for ensuring statistical isotropy, and we have moved to the Fourier space. By substituting Eq. (2.88) in Eq. (2.95), we find

$$\begin{aligned}\frac{\partial^2}{\partial\tau^2}\delta\chi^{(1)}(\tau,\mathbf{x}) + 2\mathcal{H}(\tau)\frac{\partial}{\partial\tau}\delta\chi^{(1)}(\tau,\mathbf{x}) + \left[k^2 + a^2(\tau)\frac{\mathrm{d}^2V(\bar{\chi})}{\mathrm{d}\bar{\chi}}\right]\delta\chi^{(1)}(\tau,\mathbf{x}) = \\ = \frac{\mathrm{d}\bar{\chi}(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial\tau}\left[3\Phi^{(1)}(\tau,\mathbf{x}) + \Psi^{(1)}(\tau,\mathbf{x})\right] - 2a^2(\tau)\frac{\mathrm{d}V(\bar{\chi})}{\mathrm{d}\bar{\chi}}\Psi^{(1)}(\tau,\mathbf{x}).\end{aligned} \tag{2.96}$$

The form of Eq. (2.96) is telling us that the first-order anisotropic cosmic birefringence is sourced just by scalar perturbations, explaining how the field fluctuation $\delta\chi^{(1)}$ is related to the metric perturbations. Now, it is time to understand how the axion inhomogeneities can effectively induce the anisotropic signature in the birefringence angle: in fact, from Eq. (2.83) we see that the spatial dependence of $\delta\alpha$ enters as $(\tau - \tau_0)\hat{\mathbf{n}}$, so that, by recalling Eq. (2.87), we can infer

$$\delta\alpha^{(1)}\left[\tau,(\tau_0 - \tau)\mathbf{n}\right] = -\frac{\beta}{2}\delta\chi^{(1)}\left[\tau,(\tau_0 - \tau)\right], \tag{2.97}$$

since $\delta\chi^{(1)}(\tau_0, 0)$ only gives rise to a redefinition of $\bar{\chi}(\tau_0)$. The crucial point is that for each fixed $\tau$ it is now possible to expand the anisotropic birefringence angle over the sky through a standard spherical harmonics decomposition,

$$\delta\alpha^{(1)}\left[\tau, (\tau_0 - \tau)\hat{\mathbf{n}}\right] \equiv \sum_{\ell m} \alpha_{\ell m}^{(1)}(\tau)\, {}_0Y_{\ell m}(\hat{\mathbf{n}}), \tag{2.98}$$

since $\delta\alpha$ is a scalar quantity, whose harmonic coefficients $\alpha_{\ell m}$ are given at any emission time as

$$\alpha_{\ell m}^{(1)}(\tau) = -\frac{\beta}{2} \int \mathrm{d}^2\hat{n}\, {}_0Y_{\ell m}^{*}(\hat{\mathbf{n}})\, \delta\chi^{(1)}(\tau, \Delta\tau\,\hat{\mathbf{n}}), \tag{2.99}$$

with $\Delta\tau \equiv \tau_0 - \tau$. We now move to the Fourier space,

$$\delta\chi^{(1)}(\tau, \Delta\tau\,\hat{\mathbf{n}}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\Delta\tau\,\mathbf{k}\cdot\hat{\mathbf{n}}}\delta\chi^{(1)}(\tau, \mathbf{k}), \tag{2.100}$$

and we adopt the plane wave-expansion as we did in Eq. (2.64), so that we can rewrite Eq. (2.99) as

$$\boxed{\alpha_{\ell m}^{(1)}(\tau) = -4\pi i^{\ell}\frac{\beta}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, {}_0Y_{\ell m}^{*}(\hat{\mathbf{k}})\, j_\ell(k\Delta\tau)\delta\chi^{(1)}(\tau, \mathbf{k}),} \tag{2.101}$$

where we have exploited Eq. (2.73) to perform the angular integration. We are now in the position to introduce another fundamental concept in cosmology: if we statistically average over multiple realizations or observations of the CMB sky, i.e. we perform the **ensemble average** $\langle\cdots\rangle$, we can extract the underlying significant signal from random fluctuations of the CMB fields. In particular, it gives us the possibility to determine the **angular power spectrum** $C_\ell$ of the given observable, which tells us how much variation there is in such an observable across different angular sizes of regions in the sky:

$$\langle M_{\ell m}^{*} N_{\ell' m'}\rangle = C_\ell^{MN}\delta_{\ell\ell'}\delta_{mm'}, \tag{2.102}$$

for any couple of observable $M(\tau, \mathbf{x}, \hat{\mathbf{n}})$ and $N(\tau, \mathbf{x}, \hat{\mathbf{n}})$. The fact that the angular power spectrum just depends on $\ell$ is a consequence of the assumption that any kind of two-point correlation function, such as the left-hand side of Eq. (2.102), preserves statistical isotropy (see e.g. Refs. [137, 192]). The reason we are treating cosmological perturbations as random variables can be intuitively understood by thinking to the fact that, observationally, we are not in general interested in predicting e.g. the position of a certain galaxy at a certain time. Though this might be interesting to some extent, we are rather more concerned with averaged quantities, such as the average distance among galaxies, because these contain information on gravity and the expanding universe. Armed with the expression given in Eq. (2.101), we can then compute the first-order angular power spectrum of anisotropic cosmic birefringence:

$$\langle \alpha_{\ell m}^{(1)*}(\tau)\, \alpha_{\ell' m'}^{(1)}(\tau') \rangle = 4\pi^2 \beta^2 i^{\ell_2-\ell_1} \int \frac{d^3k\, d^3k'}{(2\pi)^6}\, {}_0Y_{\ell m}(\hat{\mathbf{k}})\, {}_0Y^*_{\ell' m'}(\hat{\mathbf{k}}') \\ j_\ell(k\Delta\tau)\, j_{\ell'}(k'\Delta\tau') \left\langle \delta\chi^{(1)*}(\tau,\mathbf{k})\delta\chi^{(1)}(\tau',\mathbf{k}') \right\rangle . \tag{2.103}$$

If we assume now statistically isotropic and **adiabatic initial conditions**[6] for the cosmological perturbations[7], it is possible to define the two-point correlation function for the first-order field fluctuations as

$$\left\langle \delta\chi^{(1)*}(\tau,\mathbf{k})\delta\chi^{(1)}(\tau',\mathbf{k}') \right\rangle = \frac{16\pi^5}{k^3} P_{\mathcal{R}}(k)\delta\chi^{(1)}(\tau,k)\delta\chi^{(1)}(\tau',k)\delta^{(3)}(\mathbf{k}_1-\mathbf{k}_2), \tag{2.104}$$

where, since, as shown in Eq. (2.96) the first-order axion fluctuations are sourced just by scalar perturbations, $P_{\mathcal{R}}(k)$ is the dimensionless power spectrum (in the Fourier space) of the **comoving curvature perturbation**, a gauge-invariant linear combination of many scalar perturbations, which in the Poisson gauge reads

$$\mathcal{R}^{(1)}(\tau,\mathbf{x}) \equiv \Phi^{(1)}(\tau,\mathbf{x}) + \frac{2m_{Pl}^2\mathcal{H}(\tau)}{a^2(\tau)[\bar{\rho}(\tau)+\overline{\mathcal{P}}(\tau)]}\left[\frac{\partial}{\partial\tau}\Phi^{(1)}(\tau,\mathbf{x}) - \mathcal{H}(\tau)\Psi^{(1)}(\tau,\mathbf{x})\right], \tag{2.105}$$

where $\bar{\rho}$ and $\overline{\mathcal{P}}$ are the total background energy density and pressure of the Universe. This quantity is particularly relevant in cosmology, because, when assuming the above mentioned adiabatic initial conditions, it becomes constant in time on large scales (i.e. for $k\tau \ll 1$), and any initial condition of the other scalar perturbations can be easily related to that of $\mathcal{R}$, making it the only real scalar perturbation whose initial conditions we have to deal with. This can be easily understood also because, by looking Eq. (2.96), we can see that nothing in such an EOM depends on $\hat{\mathbf{k}}$, which means that all the dependence of $\delta\chi^{(1)}$ on it is encoded in its initial condition, and so on $\mathcal{R}$, i.e.

$$\begin{cases} \delta\chi^{(1)}(\tau,\mathbf{k}) = \delta\chi^{(1)}(\tau,k)\mathcal{R}^{(1)}(0,\mathbf{k}) \\ \Phi^{(1)}(\tau,\mathbf{k}) = \Phi^{(1)}(\tau,k)\mathcal{R}^{(1)}(0,\mathbf{k}) \\ \Psi^{(1)}(\tau,\mathbf{k}) = \Psi^{(1)}(\tau,k)\mathcal{R}^{(1)}(0,\mathbf{k}). \end{cases} \tag{2.106}$$

[6]When talking about "adiabatic" cosmological perturbations, we refer to the fact that it is possible to show that for instance the scalar ones produce density variations in all forms of matter and energy with equal density contrasts in the number density. In such a way it is possible to prove, through thermodynamic relations, that there is no variation of entropy (see e.g. Ref. [141]).

[7]See e.g. Refs. [193, 194] for a discussion about isocurvature modes as initial conditions instead.

The power spectrum of the comoving scalar perturbation, in the scale-invariant case, has been observationally estimated as $P_{\mathcal{R}} \simeq 2.1 \times 10^{-9}$ (see Ref. [195]), whereas, as just mentioned, $\delta\chi^{(1)}(\tau, k)$ is the solution of Eq. (2.96), which here plays the role of a transfer function for the axion first-order perturbations, evolving it from early primordial epoch to the given time $\tau$. Therefore, we obtain (see e.g Refs. [79–83]):

$$\left\langle \alpha_{\ell m}^{(1)*}(\tau)\, \alpha_{\ell' m'}^{(1)}(\tau') \right\rangle = 4\pi \left(\frac{\beta}{2}\right)^2 \int \frac{\mathrm{d}k}{k}\, P_{\mathcal{R}}(k) j_\ell(k\Delta\tau) j_\ell(k\Delta\tau') \delta\chi^{(1)}(\tau, k) \delta\chi^{(2)}(\tau', k) \delta_{\ell\ell'} \delta_{mm'}, \quad (2.107)$$

so that, by recalling Eq. (2.102), we can parameterize the amplitude of the angular power-spectrum in Eq. (2.107) as

$$\left\langle \alpha_{\ell m}^{(1)*}(\tau_x)\, \alpha_{\ell' m'}^{(1)}(\tau_z) \right\rangle = C_\ell^{\alpha\alpha|(1)}\Big|_{xz} \delta_{\ell\ell'} \delta_{mm'}, \quad (2.108)$$

where now $x$ and $z$ are labels for the different epochs[8]. The angular power spectrum of first-order anisotropic birefringence is then given as

$$\boxed{C_\ell^{\alpha\alpha|(1)}\Big|_{xz} = 4\pi \left(\frac{\beta}{2}\right)^2 \int \frac{\mathrm{d}k}{k}\, P_{\mathcal{R}}(k) j_\ell(k\Delta\tau_x) j_\ell(k\Delta\tau_z) \delta\chi^{(1)}(\tau_x, k) \delta\chi^{(1)}(\tau_z, k).} \quad (2.109)$$

Similarly, it is possible to consider also cross-correlations between cosmic birefringence with CMB intensity and polarization [77, 80–83, 85]:

$$\left\langle \alpha_{\ell m}^{(1)*}(\tau_x)\, T_{\ell' m'}^{\mathrm{EM}|(1)}(\tau_z) \right\rangle = C_{\ell,\,\mathrm{EM}}^{\alpha T|(1)}\Big|_{xz} \delta_{\ell\ell'} \delta_{mm'}, \quad (2.110)$$

$$\left\langle \alpha_{\ell m}^{(1)*}(\tau_x)\, E_{\ell' m'}^{\mathrm{EM}|(1)}(\tau_z) \right\rangle = C_{\ell,\,\mathrm{EM}}^{\alpha E|(1)}\Big|_{xz} \delta_{\ell\ell'} \delta_{mm'}, \quad (2.111)$$

whereas the $C_{\ell,\,\mathrm{EM}}^{\alpha B|(1)}$ cross-correlation is predicted to be identically zero at first-order, since the $B$ modes of CMB polarization are instead sourced just by tensor perturbations. The harmonic coefficients of CMB intensity,

$$\Theta_T^{(1)}(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}}) = \sum_{\ell m} T_{\ell m}^{(1)}(\tau_0, \mathbf{x}_0)\, {}_0Y_{\ell m}(\hat{\mathbf{n}}) \quad (2.112)$$

are denoted by $T_{\ell m}$ because they encode the information about the **CMB temperature anisotropies**. This is simply due to the fact that by substituting Eq. (1.197) in the definition of $\Theta_T$, it becomes clear that all the dependence on the photons' direction of propagation is encoded in $\delta T^{(1)}$. In Ref. [77], we have modified the Boltzmann code CLASS, in order to implement anisotropic cosmic birefringence: our code evaluates the

[8] In our case they refer to recombination or reionization, as shown in Eq. (2.83).

first-order angular power spectra defined in Eqs. (2.109)-(2.111) by solving the perturbed EOM we derived in Eq. (2.96) together with the Einstein equations for the two scalar potentials $\Psi$ and $\Phi$. In particular, $C_\ell^{\alpha\alpha|(1)}$, $C_{\ell,\,\mathrm{EM}}^{\alpha T|(1)}$ and $C_{\ell,\,\mathrm{EM}}^{\alpha E|(1)}$ given in Eq.(2.109) and Eqs. (2.110)-(2.111) are plotted in Fig. 2.3 for $\tau_x = \tau_z = \tau_{\mathrm{rec}}$, where for now we have not considered the signal coming from reionization in order to be consistent with the current observational constraints. By looking at Fig. 2.3, we can remark that for the potential defined in Eq. (2.89), larger the scalar field mass is, larger the spectra's amplitudes are[9]: this is a peculiar behavior of anisotropic cosmic birefringence. Note that, as we discussed in Sec. 2.3.1, an heavy axion field would implies no isotropic birefringence, since the background field $\bar{\chi}(\tau)$ starts to oscillate before the recombination time; on the contrary Fig. 2.3 is telling us an interesting aspect: the more massive the axion field, the more amplitude is enhanced, allowing us to investigate a wider range of masses. There is in fact a clear physical explanation for this intriguing phenomenon. Indeed, since anisotropic birefringence is sourced by the perturbations of the axion field, the fact that the larger the axion mass is, larger the field fluctuations are, can seem counter-intuitive, since we expect a heavy field to fluctuate less than a light one. In order to clarify this aspect, let now us focus on Eq. (2.101): the birefringence angle is related to the value of the axion fluctuation at the recombination or at the reionization time, whose precise dynamics is ruled by Eq. (2.96). We can see that coupling with the metric perturbations, which is what is able to turn-on the correlation functions defined in Eqs. (2.109)-(2.111) because of the adiabatic initial conditions, enters in the EOM for $\delta\chi^{(1)}$ as multiplied by the time derivative of $\bar{\chi}$ or by the functional derivative of the axion potential. On the one hand, because of Eq. (2.89), we can then see that the strength of this coupling is proportional to $\mathrm{d}V/\mathrm{d}\bar{\chi}$, which is in turn proportional to $m_\chi^2$, and on the other hand we have already noticed in Fig. 2.2 that the larger the axion mass is, the faster its background time-evolution is. Therefore, this explains why increasing the axion mass implies an enhancement of the spectra's amplitude for anisotropic cosmic birefringence. Of course, a vanishing potential prevents the axion-like field perturbations from having any correlation with perturbations in the matter/radiation density (see e.g. Ref. [80]), so that a trivial consequence of this is that, if the axion mass is exactly zero, we would have $C_{\ell,\,\mathrm{EM}}^{\alpha T|(1)} = C_{\ell,\,\mathrm{EM}}^{\alpha E|(1)} = 0$. Finally, let us just mention that the theoretical results shown in Fig. 2.3 are consistent with those derived in e.g. in Refs. [82, 83], and are compared with the most recent measurements, in particular with the analysis of *Planck* PR3 data performed in Ref. [103], which gives the observational constraints on the scale-invariant angular correlations of anisotropic birefringence using the `Commander` component separation method. Other important constraints on anisotropic cosmic birefringence come by former analysis of the *Planck* mission (see Refs. [100, 102]), and by other experiments, such as ACTPol , SPTpol, Bicep-Keck, Polarbear and WMAP (see Refs. [91, 93, 95–98], respectively). Although a full comparison of theory with observations is beyond the purpose of this thesis, we

[9] We have checked that using a quadratic potential, $V(\chi) = m_\chi^2\chi^2/2$, considered e.g. in Refs. [75, 76], does not qualitatively affect our plots in a significant way.

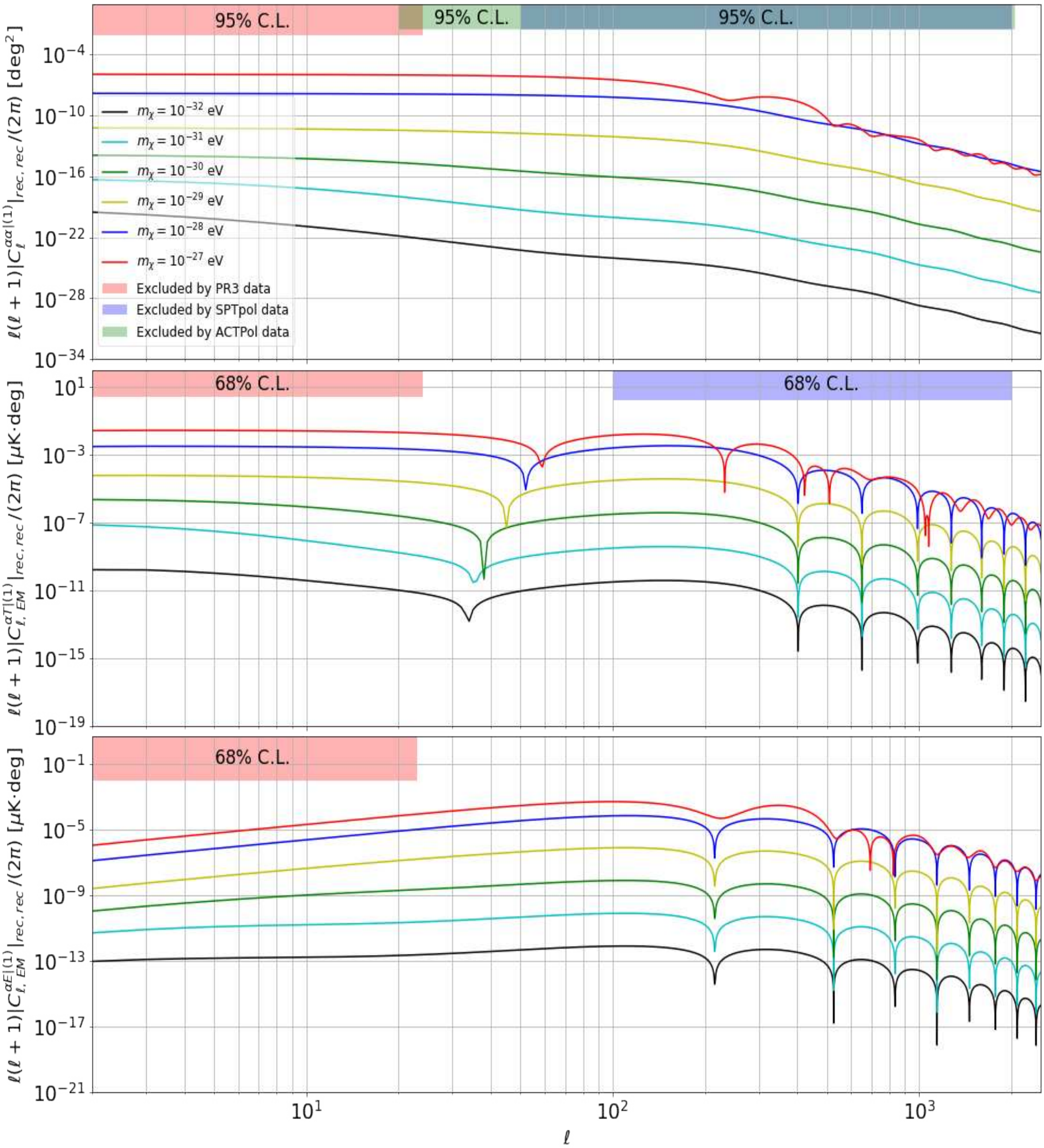


**Figure 2.3:** Absolute value of the angular power spectra involving anisotropic cosmic birefringence for the model defined by Eq. (2.89), with $\beta = 10^{-18}\,\text{GeV}^{-1}$ and fir the same set of parameters of Fig. 2.2. The shaded regions are excluded by the present constraints from *Planck* PR3 with the `Commander` component separation method, ACTPol and SPTpol (see Refs. [95, 96, 103], respectively).

nevertheless just point out that a joint investigation of all the cross-spectra of cosmic birefringence can allow us to extract fundamental information. For example, already from these results from anisotropic birefringence, one can conclude that, within the context of these models and for the fixed value of the axion-photon coupling parameter $\beta = 10^{-18}\,\mathrm{GeV}^{-1}$ that we have chosen in Ref. [77], spectra with higher masses than those we considered in this thesis are generally excluded by the current observational constraints, as shown in Fig. 2.3. A direct implication of that is that for sure a non-vanishing isotropic birefringence should be produced, since that the range of masses that are excluded by the anisotropic signal forces the background axion field to evolve in time as shown in Fig. 2.2.

## 2.4 Tomographic Approach to Cosmic Birefringence

It is now time to compute the angular power spectra of the cosmic microwave background, and for this purpose we are now going to directly use Eq. (2.83): in practice, the fact that CMB photons were mainly emitted at two different epochs, i.e. recombination and reionization, allows us for adopting a **tomographic approach**, in which we study the impact of birefringence on CMB observables by separating the contributions coming from reionization from those coming from reionization. As discussed in Sec. 1.1.3, since the linear combination of Stokes parameters $(Q \pm iU)$ behaves as a spin-2 field, it can be projected on the celestial sky via a proper set of spin-weighted spherical harmonics, as we implicitly did in Eq. (2.65),

$$
{}_{\pm}\Theta_P(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\mathrm{obs}} = 4\pi \sum_{\ell m} {}_{\pm}P^{\mathrm{obs}}_{\ell m}(\tau_0)\, {}_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}}), \tag{2.113}
$$

where we have suppressed the dependence of ${}_{\pm}P^{\mathrm{obs}}_{\ell m}$ on $\mathbf{x}_0$ to lighten-up the notation. Therefore, the related harmonic coefficients are then obtained by inverting Eq. (2.83),

$$
{}_{\pm}P^{\mathrm{obs}}_{\ell m}(\tau_0) = \sum_{i=\mathrm{rec,\,rei}} \sum_{LM} \int \mathrm{d}^2\hat{n}\, {}_{\pm 2}Y^*_{\ell m}(\hat{\mathbf{n}})\, {}_{\pm 2}Y_{LM}(\hat{\mathbf{n}})\, {}_{\pm}P^{\mathrm{EM}}_{LM}(\tau_i) e^{\pm 2i\left\{\bar{\alpha}_0(\tau_i) + \delta\alpha^{(1)}[\tau_i, (\tau_0 - \tau_i)\hat{\mathbf{n}}]\right\}}, \tag{2.114}
$$

where the ${}_{\pm}P^{\mathrm{EM}}_{\ell m}$'s are the unrotated harmonic coefficients for the Stokes parameters, i.e. those that we would have in absence of cosmic birefringence,

$$
{}_{\pm}P^{\mathrm{EM}}_{\ell m}(\tau_i) = \int \frac{\mathrm{d}^2\hat{n}}{4\pi}\, {}_{\pm 2}Y^*_{\ell m}(\hat{\mathbf{n}})\, {}_{\pm}\Theta_P(\tau_i, \mathbf{x}_0, \hat{\mathbf{n}})|_{\mathrm{EM}}. \tag{2.115}
$$

The harmonic coefficients of the $E$ and $B$ polarization modes are then obtained by simply recalling Eq. (2.75), so that, after some trivial calculations, it is easy to show that the following formula holds true :

$$\begin{bmatrix} E^{\rm obs}_{\ell m}(\tau_0) \\ B^{\rm obs}_{\ell m}(\tau_0) \end{bmatrix} = \frac{1}{2} \sum_{i\,=\,{\rm rec,\,rei}} \sum_{s\,=\,\pm 2} e^{is\bar{\alpha}(\tau_i)} \sum_{LM} \int {\rm d}\hat{n}\, {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{LM}(\hat{\mathbf{n}}) \begin{pmatrix} 1 & is/2 \\ -is/2 & 1 \end{pmatrix} \begin{bmatrix} E^{\rm EM}_{LM}(\tau_i) \\ B^{\rm EM}_{LM}(\tau_i) \end{bmatrix} \exp\left\{ is\delta\alpha^{(1)}[\tau_i, (\tau_0 - \tau_i)\hat{\mathbf{n}}] \right\}, \tag{2.116}$$

whereas the harmonic coefficients of CMB temperature anisotropies instead are not affected by cosmic birefringence:

$$\Theta_T(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\rm obs} = \sum_{\ell m} T^{\rm EM}_{\ell m}(\tau_0)\, {}_0Y_{\ell m}(\hat{\mathbf{n}}) = \Theta_T(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\rm EM}, \tag{2.117}$$

whence

$$T^{\rm obs}_{\ell m}(\tau_0) = \int {\rm d}^2\hat{n}\, {}_0Y^*_{\ell m}(\hat{\mathbf{n}})\Theta_T(\tau_0, \mathbf{x}_0, \hat{\mathbf{n}})|_{\rm obs} = \sum_{i\,=\,{\rm rec,rei}} T^{\rm EM}_{\ell m}(\tau_i). \tag{2.118}$$

The CMB power spectra affected by cosmic birefringence are then given as

$$C^{EE}_{\ell,\rm obs} = C^{EE}_{\ell,\rm obs}\Big|_{\rm rec,rec} + 2C^{EE}_{\ell,\rm obs}\Big|_{\rm rec,rei} + C^{EE}_{\ell,\rm obs}\Big|_{\rm rei,rei}, \tag{2.119}$$

$$C^{BB}_{\ell,\rm obs} = C^{BB}_{\ell,\rm obs}\Big|_{\rm rec,rec} + 2C^{BB}_{\ell,\rm obs}\Big|_{\rm rec,rei} + C^{BB}_{\ell,\rm obs}\Big|_{\rm rei,rei}, \tag{2.120}$$

$$C^{EB}_{\ell,\rm obs} = C^{EB}_{\ell,\rm obs}\Big|_{\rm rec,rec} + C^{EB}_{\ell,\rm obs}\Big|_{\rm rec,rei} + C^{EB}_{\ell,\rm obs}\Big|_{\rm rei,rec} + C^{EB}_{\ell,\rm obs}\Big|_{\rm rei,rei}, \tag{2.121}$$

$$C^{TE}_{\ell,\rm obs} = C^{TE}_{\ell,\rm obs}\Big|_{\rm rec,rec} + C^{TE}_{\ell,\rm obs}\Big|_{\rm rec,rei} + C^{TE}_{\ell,\rm obs}\Big|_{\rm rei,rec} + C^{TE}_{\ell,\rm obs}\Big|_{\rm rei,rei}, \tag{2.122}$$

$$C^{TB}_{\ell,\rm obs} = C^{TB}_{\ell,\rm obs}\Big|_{\rm rec,rec} + C^{TB}_{\ell,\rm obs}\Big|_{\rm rec,rei} + C^{TB}_{\ell,\rm obs}\Big|_{\rm rei,rec} + C^{TB}_{\ell,\rm obs}\Big|_{\rm rei,rei}. \tag{2.123}$$

### 2.4.1 Derivation of Birefringent CMB Angular Power Spectra

we are now going to compute all the terms collected in Eqs. (2.119)-(2.123). In order to do this, it is convenient to rewrite Eq. (2.116) as

$$a^{\rm obs}_{j,\ell m}(\tau_x) = \sum_{s=\pm 2} \frac{e^{is\bar{\alpha}(\tau_x)}}{2} \sum_{LM} \int {\rm d}^2\hat{n}\, {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{LM}(\hat{\mathbf{n}}) \mathscr{R}^{(s)}_{jk}\, a^{\rm EM}_{k,LM}(\tau_x) e^{is\delta\alpha^{(1)}[\tau_x, (\tau_0 - \tau_x)\hat{\mathbf{n}}]}, \tag{2.124}$$

where for $j, k = 1, 2$ we have defined

$$a_{\ell m} = \begin{pmatrix} E_{\ell m} \\ B_{\ell m} \end{pmatrix}, \qquad \mathscr{R}^{(s)} \equiv \begin{pmatrix} 1 & is/2 \\ -is/2 & 1 \end{pmatrix}, \tag{2.125}$$

and the sum over $k$ has to be understood. To calculate the components of the observed correlation functions of CMB anisotropies we now assume that the first-order EM CMB fields, as well as the first-order anisotropic birefringence angle can all be treated as **Gaussian random fields**, and that the underlying inflationary model is parity-conserving, so that $C^{EB}_{\ell,\,\mathrm{EM}} = 0 = C^{TB}_{\ell,\,\mathrm{EM}}$ for primordial unrotated modes. The simplest way to proceed is to evaluate the following general cross-correlator of CMB polarization:

$$\begin{aligned} \langle a^{\mathrm{obs}}_{j,\ell_1 m_1}(\tau_x) a^{\mathrm{obs}}_{i,\ell_2 m_2}(\tau_z)\rangle = \sum_{s_1 s_2} \frac{e^{is_1\bar{\alpha}(\tau_x)}e^{is_2\bar{\alpha}(\tau_z)}}{4} \sum_{L_1 M_1}\sum_{L_2 M_2} \int \mathrm{d}^2\hat{n}_1\,\mathrm{d}^2\hat{n}_2\, \mathscr{R}^{(s_1)}_{jk}\mathscr{R}^{(s_2)}_{il}\; {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1) \\ {}_{s_1}Y_{L_1 M_1}(\hat{\mathbf{n}}_1)\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2 M_2}(\hat{\mathbf{n}}_2) \langle a^{\mathrm{EM}}_{k,L_1 M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) e^{is_1\delta\alpha[\tau_x,(\tau_0-\tau_x)\hat{\mathbf{n}}_1]} e^{is_2\delta\alpha[\tau_z,(\tau_0-\tau_z)\hat{\mathbf{n}}_2]}\rangle . \end{aligned} \tag{2.126}$$

We then Taylor-expand the exponential containing the anisotropic angle at the quadratic order:

$$e^{is\delta\alpha^{(1)}[\tau,(\tau_0-\tau)\hat{\mathbf{n}}]} \simeq 1 + is\sum_{pq}\alpha^{(1)}_{pq}(\tau)\,{}_0Y_{pq}(\hat{\mathbf{n}}) - 2\sum_{pq}\sum_{uv}\alpha^{(1)}_{pq}(\tau)\alpha^{(1)}_{uv}(\tau)\,{}_0Y_{pq}(\hat{\mathbf{n}})\,{}_0Y_{uv}(\hat{\mathbf{n}}), \tag{2.127}$$

so that the ensemble average in the last line of Eq. (2.126) can be rewritten as follows

$$\begin{aligned} \langle a^{\mathrm{EM}}_{k,L_1M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2M_2}(\tau_z) e^{is_1\delta\alpha^{(1)}[\tau_x,(\tau_0-\tau_x)\hat{\mathbf{n}}_1]} e^{is_2\delta\alpha^{(1)}[\tau_z,(\tau_0-\tau_z)\hat{\mathbf{n}}_2]}\rangle \simeq \langle a^{\mathrm{EM}}_{k,L_1M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2M_2}(\tau_z)\rangle \\ - 2\sum_{p_1q_1}\sum_{u_1v_1} {}_0Y_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y_{u_1v_1}(\hat{\mathbf{n}}_1) \left\langle a^{\mathrm{EM}}_{k,L_1M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2M_2}(\tau_z)\alpha^{(1)}_{p_1q_1}(\tau_x)\alpha^{(1)}_{u_1v_1}(\tau_z)\right\rangle \\ - 2\sum_{p_2q_2}\sum_{u_2v_2} {}_0Y_{p_2q_2}(\hat{\mathbf{n}}_2)\,{}_0Y_{u_2v_2}(\hat{\mathbf{n}}_2) \left\langle a^{\mathrm{EM}}_{k,L_1M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2M_2}(\tau_z)\alpha^{(1)}_{p_2q_2}(\tau_z)\alpha^{(1)}_{u_2v_2}(\tau_z)\right\rangle \\ - s_1s_2\sum_{p_1q_1}\sum_{p_2q_2} {}_0Y_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y_{p_2q_2}(\hat{\mathbf{n}}_2) \left\langle a^{\mathrm{EM}}_{k,L_1M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2M_2}(\tau_z)\alpha^{(1)}_{p_1q_1}(\tau_x)\alpha^{(1)}_{p_2q_2}(\tau_z)\right\rangle . \end{aligned} \tag{2.128}$$

Each one of the four terms on the right-hand side of the previous equation can be in turn decomposed by means of the **Isserlis' theorem** (see Ref. [196]):

$$\langle G_1 \prod_{j=2}^{n} G_j \rangle = \begin{cases} \sum_{i=2}^{n} \langle G_1 G_i \rangle \langle \partial_{G_i} \prod_{j=2}^{n} G_j \rangle & \text{if } n \text{ is an even integer,} \\ 0 & \text{if } n \text{ is an odd integer,} \end{cases} \tag{2.129}$$

valid for any combination of Gaussian random fields $G_i$'s. For instance, the term in the second line at the right-hand side of Eq. (2.128) yields

$$\begin{aligned} -2 \sum_{p_1 q_1} \sum_{u_1 v_1} {}_0Y_{p_1 q_1}(\hat{\mathbf{n}}_1) {}_0Y_{u_1 v_1}(\hat{\mathbf{n}}_1) \left\langle a^{\mathrm{EM}}_{k,L_1 M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) \alpha^{(1)}_{p_1 q_1}(\tau_x) \alpha^{(1)}_{u_1 v_1}(\tau_x) \right\rangle = \\ = -2 \sum_{p_1 q_1} \sum_{u_1 v_1} {}_0Y_{p_1 q_1}(\hat{\mathbf{n}}_1) {}_0Y_{u_1 v_1}(\hat{\mathbf{n}}_1) \Bigg[ \left\langle a^{\mathrm{EM}}_{k,L_1 M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) \right\rangle \langle \alpha^{(1)}_{p_1 q_1}(\tau_x) \alpha^{(1)}_{u_1 v_1}(\tau_z) \rangle \\ + \langle a^{\mathrm{EM}}_{k,L_1 M_1}(\tau_x) \alpha^{(1)}_{p_1 q_1}(\tau_x) \rangle \left\langle a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) \alpha^{(1)}_{u_1 v_1}(\tau_x) \right\rangle \\ + \left\langle a^{\mathrm{EM}}_{k,L_1 M_1} \alpha^{(1)}_{u_1 v_1}(\tau_x) \right\rangle \langle a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) \alpha^{(1)}_{p_1 q_1}(\tau_x) \rangle \Bigg], \end{aligned} \tag{2.130}$$

which can be simplified with the definition of angular power spectrum, shown e.g. in Eq. (2.109) as

$$\begin{aligned} -2 \sum_{p_1 q_1} \sum_{u_1 v_1} {}_0Y_{p_1 q_1}(\hat{\mathbf{n}}_1) {}_0Y_{u_1 v_1}(\hat{\mathbf{n}}_1) \left\langle a^{\mathrm{EM}}_{k,L_1 M_1}(\tau_x) a^{\mathrm{EM}}_{l,L_2 M_2}(\tau_z) \alpha^{(1)}_{p_1 q_1}(\tau_x) \alpha^{(1)}_{u_1 v_1}(\tau_x) \right\rangle = \\ = -2 \sum_{p_1 q_1} \sum_{u_1 v_1} {}_0Y_{p_1 q_1}(\hat{\mathbf{n}}_1) {}_0Y_{u_1 v_1}(\hat{\mathbf{n}}_1) \Bigg[ C^{kl|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xz} C^{\alpha\alpha|(1)}_{p_1}\Big|_{xx} \delta_{L_1 L_2} \delta_{M_1,-M_2} \delta_{p_1 u_1} \delta_{q_1,-v_1} \\ + C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} \delta_{L_1 p_1} \delta_{M_1,-q_1} \delta_{L_2,u_1} \delta_{M_2,-v_1} \\ + C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} \delta_{L_1 u_1} \delta_{M_1,-v_1} \delta_{L_2 p_1} \delta_{M_2,-q_1} \Bigg], \end{aligned} \tag{2.131}$$

so that it further reduces to

$$-2\sum_{p_1q_1}\sum_{u_1v_1}{}_0Y_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y_{u_1v_1}(\hat{\mathbf{n}}_1)\left\langle a^{\rm EM}_{k,L_1M_1}(\tau_x)a^{\rm EM}_{l,L_2M_2}(\tau_z)\alpha^{(1)}_{p_1q_1}(\tau_x)\alpha^{(1)}_{u_1v_1}(\tau_x)\right\rangle =$$
$$= -2\Bigg[C^{kl|(1)}_{L_1,\,\rm EM}\Big|_{xz}\delta_{L_1L_2}\delta_{M_1,-M_2}\sum_{p_1q_1}{}_0Y_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{p_1q_1}(\hat{\mathbf{n}}_1)C^{\alpha\alpha|(1)}_{p_1}\Big|_{xx} + \left(C^{\alpha k|(1)}_{L_1\,\rm EM}\Big|_{xx}C^{\alpha l|(1)}_{L_2,\,\rm EM}\Big|_{xz} + C^{\alpha k|(1)}_{L_2,\,\rm EM}\Big|_{xx}C^{\alpha l|(1)}_{L_1,\,\rm EM}\Big|_{xz}\right){}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_1)\Bigg]. \tag{2.132}$$

We now define the first-order **variance** of the anisotropic birefringence angle as the

$$V^{(1)}_\alpha\Big|_{xx} \equiv \sum_{p_1q_1}{}_0Y_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{p_1q_1}(\hat{\mathbf{n}}_1)C^{\alpha\alpha|(1)}_{p_1}\Big|_{xx} = \sum_{p_1}\left(\frac{2p_1+1}{4\pi}\right)C^{\alpha\alpha|(1)}_{p_1}\Big|_{xx}, \tag{2.133}$$

where we have used the Unsöld's theorem (see Ref. [197]). By repeating the same procedure for all the terms on the right-hand side of Eq. (2.128), we find after lengthy calculations

$$\langle a^{\rm EM}_{k,L_1M_1}(\tau_x)a^{\rm EM}_{l,L_2M_2}(\tau_z)e^{is_1\delta\alpha^{(1)}[\tau_x,(\tau_0-\tau_x)\hat{\mathbf{n}}_1]}e^{is_2\delta\alpha^{(1)}[\tau_z,(\tau_0-\tau_z)\hat{\mathbf{n}}_2]}\rangle =$$
$$\begin{aligned}
&= C^{kl}_{L_1}\Big|_{xz}\left(1-2V^{(1)}_\alpha\Big|_{xx}-2V^{(1)}_\alpha\Big|_{zz}\right)\delta_{L_1L_2}\delta_{M_1,-M_2}\\
&\quad - 4C^{\alpha k|(1)}_{L_1,\,\rm EM}\Big|_{xx}C^{\alpha l|(1)}_{L_2,\,\rm EM}\Big|_{xz}{}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_1)\\
&\quad - 4C^{\alpha k|(1)}_{L_1,\,\rm EM}\Big|_{xz}C^{\alpha l|(1)}_{L_2,\,\rm EM}\Big|_{zz}{}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_2)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_2)\\
&\quad - s_1s_2\sum_{p_1q_1}C^{kl|(1)}_{L_1,\,\rm EM}\Big|_{xz}C^{\alpha\alpha|(1)}_{p_1}\Big|_{xz}{}_0Y^*_{p_1q_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{p_1q_1}(\hat{\mathbf{n}}_2)\delta_{L_1L_2}\delta_{M_1,-M_2}\\
&\quad - s_1s_2C^{\alpha k|(1)}_{L_1,\,\rm EM}\Big|_{xx}C^{\alpha l|(1)}_{L_2,\,\rm EM}\Big|_{zz}{}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_2)\\
&\quad - s_1s_2C^{\alpha k|(1)}_{L_1,\,\rm EM}\Big|_{xz}C^{\alpha l|(1)}_{L_2,\,\rm EM}\Big|_{xz}{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_1)\,{}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_2).
\end{aligned} \tag{2.134}$$

We now substitute Eq. (2.134) within Eq. (2.126), so that at the very end we have just to compute the following object:

$$\langle a^{\rm obs}_{j,\ell_1m_1}(\tau_x)a^{\rm obs}_{i,\ell_2m_2}(\tau_z)\rangle =$$
$$= \frac{1}{4}\sum_{s_1s_2}e^{is_1\bar{\alpha}(\tau_x)}e^{is_2\bar{\alpha}(\tau_z)}\mathscr{R}^{(s_1)}_{jk}\mathscr{R}^{(s_2)}_{il}\left[\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}\right]^{kl|(1)}_{\ell_1\ell_2m_1m_2s_1s_2}(\tau_x,\tau_z), \tag{2.135}$$

with

$$
\begin{aligned}
\mathrm{I}^{kl|(1)}_{\ell_1\ell_2m_1m_2s_1s_2}(\tau_x,\tau_z) \equiv& \left[1-2V^{(1)}_{\alpha}\big|_{xx}-2V^{(1)}_{\alpha}\big|_{zz}\right]\sum_{L_1M_1}\sum_{L_2M_2} C^{kl|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\delta_{L_1L_2}\delta_{M_1,-M_2} \\
&\int \mathrm{d}^2\hat{n}_1\,{}_{s_1}Y^*_{\ell_1m_1}(\hat{\mathbf{n}}_1)\,{}_{s_1}Y_{L_1M_1}(\hat{\mathbf{n}}_1)\int \mathrm{d}^2\hat{n}_2\,{}_{s_2}Y^*_{\ell_2m_2}(\hat{\mathbf{n}}_2)\,{}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2) \\
=& \left[1-2V^{(1)}_{\alpha}\big|_{xx}-2V^{(1)}_{\alpha}\big|_{zz}\right] C^{kl|(1)}_{L_1,\mathrm{EM}}\big|_{xz}\delta_{\ell_1\ell_2}\delta_{m_1,-m_2},
\end{aligned}
\tag{2.136}
$$

where we have used Eq. (2.73) to perform the angular integration. Let us proceed in the evaluation of all the pieces in Eq. (2.135): the second term is

$$
\begin{aligned}
\mathrm{II}^{kl|(1)}_{\ell_1\ell_2m_1m_2s_1s_2}(\tau_x,\tau_z) \equiv& -4\sum_{L_1M_1}\sum_{L_2M_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\big|_{zx}C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\big|_{zz}\int \mathrm{d}^2\hat{n}_1\,{}_{s_1}Y^*_{\ell_1m_1}(\hat{\mathbf{n}}_1)\,{}_{s_1}Y_{L_1M_1}(\hat{\mathbf{n}}_1) \\
&\int \mathrm{d}^2\hat{n}_2\,{}_{s_2}Y^*_{\ell_2m_2}(\hat{\mathbf{n}}_2)\,{}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2)\,{}_0Y^*_{L_1M_1}(\hat{\mathbf{n}}_2)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_2) \\
=& -4\sum_{L_2M_2} C^{\alpha k|(1)}_{\ell_1,\,\mathrm{EM}}\big|_{zx}\sum_{L_2} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\big|_{zz}\int \mathrm{d}^2\hat{n}_2\,{}_{s_2}Y^*_{\ell_2m_2}(\hat{\mathbf{n}}_2)\,{}_0Y^*_{\ell_1m_1}(\hat{\mathbf{n}}_2) \\
&\sum_{M_2=-L_2}^{L_2}\,{}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2)\,{}_0Y^*_{L_2M_2}(\hat{\mathbf{n}}_2) \\
=& \, 0,
\end{aligned}
\tag{2.137}
$$

because of the following identity involving spin-weighted spherical harmonics (see e.g. Ref.[159]):

$$
\sum_{m=-\ell}^{\ell} {}_sY_{\ell m}(\hat{\mathbf{n}})\,{}_{s'}Y^*_{\ell m}(\hat{\mathbf{n}}) = \frac{2\ell+1}{4\pi}\delta_{ss'}. \tag{2.138}
$$

For the same reason, also the third term identically vanishes

$$
\begin{aligned}
\mathrm{III}^{kl|(1)}_{\ell_1\ell_2 m_1 m_2 s_1 s_2}(\tau_x,\tau_z) &\equiv -4\sum_{L_1M_1}\sum_{L_2M_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz}\int \mathrm{d}^2\hat{n}_2\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2) \\
&\qquad \int \mathrm{d}^2\hat{n}_1\, {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1)\, {}_{s_1}Y_{L_1M_1}(\hat{\mathbf{n}}_1) Y^*_{L_1M_1}(\hat{\mathbf{n}}_1) Y^*_{L_2M_2}(\hat{\mathbf{n}}_1) \\
&= -4\sum_{L_1} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz}\int \mathrm{d}^2\hat{n}_1\, {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1) Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_1) \qquad (2.139) \\
&\qquad \sum_{M_2=-L_2}^{L_2} {}_{s_1}Y_{L_1M_1}(\hat{\mathbf{n}}_1) Y^*_{L_1M_1}(\hat{\mathbf{n}}_1) \\
&= 0.
\end{aligned}
$$

Evaluating the fourth term is more tricky,

$$
\begin{aligned}
\mathrm{IV}^{kl|(1)}_{\ell_1\ell_2 m_1 m_2 s_1 s_2}(\tau_x,\tau_z) &\equiv -s_1 s_2\sum_{L_1M_1}\sum_{L_2M_2}\sum_{p_1q_1} C^{kl|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xz} C^{\alpha\alpha|(1)}_{p_1}\Big|_{xz}\delta_{L_1L_2}\delta_{M_1,-M_2} \\
&\int \mathrm{d}^2\hat{n}_1\, {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1)\, {}_{s_1}Y_{L_1M_1}(\hat{\mathbf{n}}_2) Y_{p_1q_1}(\hat{\mathbf{n}}_1)\int \mathrm{d}^2\hat{n}_2\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2) Y^*_{p_1q_1}(\hat{\mathbf{n}}_2),
\end{aligned}
\qquad (2.140)
$$

since, in order perform the angular integration, we have to exploit a formula for the triple integral:

$$
\int \mathrm{d}^2\hat{n}\, {}_{s_1}Y_{\ell_1,m_1}(\hat{\mathbf{n}})\, {}_{s_2}Y_{\ell_2,m_2}(\hat{\mathbf{n}})\, {}_{s_3}Y_{\ell_3,m_3}(\hat{\mathbf{n}}) = I^{-s_1,-s_2,-s_3}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}, \qquad (2.141)
$$

where we have defined

$$
I^{-s_1,-s_2,-s_3}_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -s_1 & -s_2 & -s_3\end{pmatrix}, \qquad (2.142)
$$

and where we have made use of the Wigner 3$j$-symbols we have already mentioned in Sec. 2.2, that obey the following symmetries:

$$\begin{aligned}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} &= \begin{pmatrix}\ell_2 & \ell_3 & \ell_1\\ m_2 & m_3 & m_1\end{pmatrix} = \begin{pmatrix}\ell_3 & \ell_1 & \ell_2\\ m_3 & m_1 & m_2\end{pmatrix} = (-1)^{\ell_T}\begin{pmatrix}\ell_1 & \ell_3 & \ell_2\\ m_1 & m_3 & m_2\end{pmatrix}\\ &= (-1)^{\ell_T}\begin{pmatrix}\ell_3 & \ell_2 & \ell_1\\ m_3 & m_2 & m_1\end{pmatrix} = (-1)^{\ell_T}\begin{pmatrix}\ell_2 & \ell_1 & \ell_3\\ m_2 & m_1 & m_3\end{pmatrix}\\ &= (-1)^{\ell_T}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -m_1 & -m_2 & -m_3\end{pmatrix},\end{aligned} \tag{2.143}$$

with $\ell_T \equiv \ell_1 + \ell_2 + \ell_3$ being the total multipole number. Therefore, Eq. (2.140) can be rewritten as

$$\begin{aligned}\mathrm{IV}^{kl|(1)}_{\ell_1\ell_2 m_1 m_2 s_1 s_2}(\tau_x,\tau_z) &= -s_1 s_2 \sum_{L_1 M_1}\sum_{p_1 q_1} C^{kl|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xz} C^{\alpha\alpha|(1)}_{p_1}\Big|_{xz} I^{s_1,-s_1,0}_{\ell_1 L_1 p_1} I^{s_2,-s_2,0}_{\ell_2 L_1 p_1}\\ &\qquad \begin{pmatrix}\ell_1 & L_1 & p_1\\ -m_1 & M_1 & q_1\end{pmatrix}\begin{pmatrix}\ell_2 & L_1 & p_1\\ -m_2 & -M_2 & -q_1\end{pmatrix}\\ &= -\frac{s_1 s_2}{2\ell_1+1}\delta_{\ell_1\ell_2}\delta_{m_1,-m_2}\sum_{L_1 p_1}(-1)^{\ell_1+L_1+p_1} C^{kl|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xz} C^{\alpha\alpha|(1)}_{p_1}\Big|_{xz} I^{s_1,s_1,0}_{\ell_1 L_1 p_1} I^{s_2,-s_2,0}_{\ell_2 L_1 p_1},\end{aligned} \tag{2.144}$$

where we have exploited the orthogonality relation of the Wigner 3$j$-symbol (see e.g. Ref. [183]):

$$\sum_{m_1 m_2}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3'\\ m_1 & m_2 & m_3'\end{pmatrix} = \frac{\delta_{\ell_3\ell_3'}\delta_{m_3 m_3'}}{2\ell_3+1}. \tag{2.145}$$

At the contrary, we now show that the fifth term vanishes:

$$\begin{aligned}\mathrm{V}^{kl|(1)}_{\ell_1\ell_2 m_1 m_2 s_1 s_2}(\tau_x,\tau_z) &\equiv -s_1 s_2 \sum_{L_1 M_1}\sum_{L_2 M_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{zz}\\ &\quad \int d^2\hat{n}_1\, {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1)\, {}_{s_1}Y_{L_1 M_1}(\hat{\mathbf{n}}_1)\, {}_0 Y^*_{L_1 M_1}(\hat{\mathbf{n}}_1)\int d^2\hat{n}_2\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2 M_2}(\hat{\mathbf{n}}_2) Y^*_{L_2 M_2}(\hat{\mathbf{n}}_2)\\ &= -s_1 s_2 \sum_{L_1 M_1}\sum_{L_2 M_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{zz} I^{-s_1,s_1,0}_{\ell_1 L_1 L_1} I^{-s_2,s_2,0}_{\ell_2 L_2 L_2}\\ &\qquad \begin{pmatrix}\ell_1 & L_1 & L_1\\ -m_1 & M_1 & -M_1\end{pmatrix}\begin{pmatrix}\ell_2 & L_2 & L_2\\ -m_2 & M_2 & -M_2\end{pmatrix}\\ &= 0.\end{aligned} \tag{2.146}$$

Indeed, the previous expression can be set equal to zero because the selection rule of the Wigner $3j$-symbol forces $m_1 = m_2 = 0$, but this implies e.g. (see e.g. Ref. [184])

$$\begin{pmatrix} \ell_1 & L_1 & L_1 \\ 0 & M_1 & -M_1 \end{pmatrix} \propto \delta_{\ell_1 0} = 0, \tag{2.147}$$

since the monopole $\ell_1 = 0$ is not observable. Finally the last term is given as

$$\begin{aligned} \mathrm{VI}^{kl|(1)}_{\ell_1\ell_2 m_1 m_2 s_1 s_2}(\tau_x,\tau_z) &\equiv -s_1 s_2 \sum_{L_1 M_1}\sum_{L_2 M_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{zx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} \\ &\int \mathrm{d}^2\hat{n}_1\, {}_{s_1}Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_1)\, {}_{s_1}Y_{L_1 M_1}(\hat{\mathbf{n}}_1)\, {}_0Y^*_{L_2 M_2}(\hat{\mathbf{n}}_1) \int \mathrm{d}^2\hat{n}_2\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2 M_2}(\hat{\mathbf{n}}_2)\, {}_0Y^*_{L_1 M_1}(\hat{\mathbf{n}}_2) \\ &= -s_1 s_2 \sum_{L_1 M_1}\sum_{L_2 M_2} (-1)^{\ell_2 + L_1 + L_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{zx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} I^{s_1,-s_1,0}_{\ell_1 L_1 L_2} I^{s_2,0,-s_2}_{\ell_2 L_1 L_2} \\ &\qquad \begin{pmatrix} \ell_1 & L_1 & L_2 \\ -m_1 & M_1 & -M_2 \end{pmatrix} \begin{pmatrix} \ell_2 & L_1 & L_2 \\ m_2 & M_1 & -M_2 \end{pmatrix} \\ &= -\frac{s_1 s_2}{2\ell_1 + 1}\delta_{\ell_1\ell_2}\delta_{m_1,-m_2} \sum_{L_1 L_2} C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{zx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} I^{s_1,-s_1,0}_{\ell_1 L_1 L_2} I^{s_2,0,-s_2}_{\ell_1 L_1 L_2}. \end{aligned} \tag{2.148}$$

We are now in the position to plug all the terms in the right-hand side of Eq. (2.135) together and find the following general formula:

$$\begin{aligned} C^{ji|(1)}_{\ell,\mathrm{obs}}\Big|_{xz} &= \frac{1}{4}\sum_{s_1 s_2} e^{i s_1 \bar{\alpha}(\tau_x)} e^{i s_2 \bar{\alpha}(\tau_z)} \mathscr{R}^{(s_1)}_{jk}\mathscr{R}^{(s_2)}_{il} \Bigg\{ \left[1 - 2V^{(1)}_{\alpha}\Big|_{xx} - 2V^{(1)}_{\alpha}\Big|_{zz}\right] C^{kl|(1)}_{\ell,\,\mathrm{EM}}\Big|_{xz} \\ &- \frac{s_1 s_2}{2\ell+1}\sum_{L_1 L_2} (-1)^{\ell + L_1 + L_2} I^{s_1,-s_1,0}_{\ell L_1 L_2} \left[ C^{kl|(1)}_{L_1,\,\mathrm{EM}}\Big|_{xz} C^{\alpha\alpha|(1)}_{L_2}\Big|_{xz} I^{s_2,-s_2,0}_{\ell L_1 L_2} + C^{\alpha k|(1)}_{L_1,\,\mathrm{EM}}\Big|_{zx} C^{\alpha l|(1)}_{L_2,\,\mathrm{EM}}\Big|_{xz} I^{s_2,0,-s_2}_{\ell L_1 L_2} \right] \Bigg\}. \end{aligned} \tag{2.149}$$

Such an expression has to be specialized for the cases of interest: this is done by performing the summation over $s_1, s_2 = \pm 2$ for the elements of the vector $a_{\ell m}$ defined in Eq. (2.125):

$$
\begin{aligned}
C^{EE|(1)}_{\ell,\mathrm{obs}}\big|_{xz} = &\left[1 - 2V^{(1)}_{\alpha}\big|_{xx} - 2V^{(1)}_{\alpha}\big|_{zz}\right]\Big\{C^{EE|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\cos[2\overline{\alpha}(\tau_x)]\cos[2\overline{\alpha}(\tau_z)] \\
&\qquad + C^{BB|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\sin[2\overline{\alpha}(\tau_x)]\sin[2\overline{\alpha}(\tau_z)]\Big\} \\
&+ \frac{2}{2\ell+1}\sum_{L_1L_2} I^{2,-2,0}_{\ell L_1 L_2}\Big[C^{\alpha\alpha|(1)}_{L_2}\big|_{xz} I^{2,-2,0}_{\ell L_1 L_2}\Big(C^{EE|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\Big\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] \\
&\qquad - (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\Big\} \\
&+ C^{BB|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\left\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] + (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big) \\
&+ C^{\alpha E|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz} C^{\alpha E|(1)}_{L_2,\,\mathrm{EM}}\big|_{zx} I^{2,0,-2}_{\ell L_1 L_2}\left\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] - (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big],
\end{aligned}
\tag{2.150}
$$

---

$$
\begin{aligned}
C^{BB|(1)}_{\ell,\mathrm{obs}}\big|_{xz} = &\left[1 - 2V^{(1)}_{\alpha}\big|_{xx} - 2V^{(1)}_{\alpha}\big|_{zz}\right]\Big\{C^{BB|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\cos[2\overline{\alpha}(\tau_x)]\cos[2\overline{\alpha}(\tau_z)] \\
&\qquad + C^{EE|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\sin[2\overline{\alpha}(\tau_x)]\sin[2\overline{\alpha}(\tau_z)]\Big\} \\
&+ \frac{2}{2\ell+1}\sum_{L_1L_2} I^{2,-2,0}_{\ell L_1 L_2}\Big[C^{\alpha\alpha|(1)}_{L_2}\big|_{xz} I^{2,-2,0}_{\ell L_1 L_2}\Big(C^{EE|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\Big\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] \\
&\qquad + (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\Big\} \\
&+ C^{BB|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\left\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] - (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big) \\
&+ C^{\alpha E|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz} C^{\alpha E|(1)}_{L_2,\,\mathrm{EM}}\big|_{zx} I^{2,0,-2}_{\ell L_1 L_2}\left\{\cos[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z) + (-1)^{\ell+L_1+L_2}\cos[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big],
\end{aligned}
\tag{2.151}
$$

---

$$
\begin{aligned}
C^{EB|(1)}_{\ell,\mathrm{obs}}\big|_{xz} = &\left[1 - 2V^{(1)}_{\alpha}\big|_{xx} - 2V^{(1)}_{\alpha}\big|_{zz}\right]\Big\{C^{EE|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\cos[2\overline{\alpha}(\tau_x)]\sin[2\overline{\alpha}(\tau_z)] \\
&\qquad - C^{BB|(1)}_{\ell,\,\mathrm{EM}}\big|_{xz}\sin[2\overline{\alpha}(\tau_x)]\cos[2\overline{\alpha}(\tau_z)]\Big\} \\
&+ \frac{2}{2\ell+1}\sum_{L_1L_2} I^{2,-2,0}_{\ell L_1 L_2}\Big[C^{\alpha\alpha|(1)}_{L_2}\big|_{xz} I^{2,-2,0}_{\ell L_1 L_2}\Big(C^{BB|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\Big\{\sin[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] \\
&\qquad - (-1)^{\ell+L_1+L_2}\sin[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\Big\} \\
&- C^{EE|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz}\left\{\sin[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] + (-1)^{\ell+L_1+L_2}\sin[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big) \\
&- C^{\alpha E|(1)}_{L_1,\,\mathrm{EM}}\big|_{xz} C^{\alpha E|(1)}_{L_2,\,\mathrm{EM}}\big|_{zx} I^{2,0,-2}_{\ell L_1 L_2}\left\{\sin[2\overline{\alpha}(\tau_x) - 2\overline{\alpha}(\tau_z)] - (-1)^{\ell+L_1+L_2}\sin[2\overline{\alpha}(\tau_x) + 2\overline{\alpha}(\tau_z)]\right\}\Big].
\end{aligned}
\tag{2.152}
$$

For completeness, it is then possible to find similar formulas also for the CMB angular power spectra involving a single polarization field, i.e. $C_{\ell,\text{obs}}^{TE|(1)}$ and $C_{\ell,\text{obs}}^{TB|(1)}$. The procedure is the same previously described: one has to start from the following general cross-correlator:

$$\begin{aligned}&\langle T^{\text{obs}}_{\ell_1 m_1}(\tau_x)a^{\text{obs}}_{j,\ell_2 m_2}(\tau_z)\rangle = \\ &= \sum_{s=\pm 2}\frac{e^{is\bar{\alpha}(\tau_z)}}{2}\sum_{LM}\int \mathrm{d}^2\hat{n}\,{}_sY^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\,{}_sY_{LM}(\hat{\mathbf{n}})\mathscr{R}^{(s)}_{jk}\langle T^{\text{EM}}_{\ell_1 m_1}(\tau_x)a^{\text{EM}}_{k,LM}(\tau_z)e^{is\delta\alpha[\tau_z,(\tau_0-\tau_z)\hat{\mathbf{n}}]}\rangle,\end{aligned} \tag{2.153}$$

and then expand the exponential containing the anisotropic cosmic birefringence angle at the quadratic order with Eq. (2.127). By using again the Isserlis theorem, we can unpack the resulting four-point correlation function in terms of angular power spectra, and the final result can be simplified by exploiting Eqs. (2.133) and (2.138). At the very end, we find

$$C^{Tj|(1)}_{\ell,\text{obs}}\big|_{xz} = \left[1-2V^{(1)}_{\alpha}\big|_{zz}\right]\sum_{s=\pm 2}\frac{e^{is\bar{\alpha}(\tau_z)}}{2}\mathscr{R}^{(s)}_{jk}C^{Tk|(1)}_{\ell,\,\text{EM}}\big|_{xz}, \tag{2.154}$$

which yields

$$C^{TE|(1)}_{\ell,\text{obs}}\big|_{xz} = \left[1-2V^{(1)}_{\alpha}\big|_{zz}\right]\cos[2\bar{\alpha}(\tau_z)]C^{TE|(1)}_{\ell,\,\text{EM}}\big|_{xz}, \tag{2.155}$$

$$C^{TB|(1)}_{\ell,\text{obs}}\big|_{xz} = \left[1-2V^{(1)}_{\alpha}\big|_{zz}\right]\sin[2\bar{\alpha}(\tau_z)]C^{TB|(1)}_{\ell,\,\text{EM}}\big|_{xz}. \tag{2.156}$$

Let us notice that by setting $x = y =$ reco, one should find some formulas valid when only the contribution coming from recombination is considered, and by also disregarding the cross-correlation $C^{\alpha E|(1)}_{\ell\,\text{EM}}$, then Eqs. (2.150)-(2.152) and Eqs. (2.155)-(2.156) just reduce to the standard formulas that can be found e.g. in Ref. [79]. We are now in the position to plot the CMB angular power spectra affected by cosmic birefringence. In order to do this we have numerically evaluated Eqs. (2.119)-(2.123), by computing each component via Eqs. (2.150)-(2.152) and Eqs. (2.155)-(2.156). For this purpose, we have again exploited our modified version of CLASS to calculate the spectra of anisotropic birefringence and the isotropic angle from the two epochs, i.e. recombination and reionization. Let us just mention that we have neglected all the unlensed unrotated terms coming from different sources (i.e. "rec-rei" and "rei-rec"). Since we expect the CMB radiation transfer functions for the recombination and the reionization contributions to peak at very different redshifts, it is reasonable to neglect such cross-correlations. The final results are plotted in Figs. 2.4-2.5 up to $\ell_{\max} = 200$ just for sake of simplicity, since the evaluation of Wigner $3j$-symbols is numerically time-consuming. If we look for instance at Fig. 2.4, we observe that the $C^{EE|(1)}_{\ell,\text{obs}}$ spectrum predicted by the $\Lambda$CDM model (i.e. in absence of cosmic birefringence) is larger than the rotated ones: this is due to the fact that the birefringence mechanism induces a mixing of the $E$ and $B$ polarization modes, and so it partially removes power from the former to transfer it to the latter. By direct inspection of Figs. 2.4-2.5, we see that the impact of cosmic birefringence on CMB power spectra is consistent with our previous considerations made in Sec. 2.3.1. For instance, as shown in Fig. 2.4, the angular power spectrum of the $E$ modes of CMB polarization deviates from the ΛCDM one (i.e. without birefringence) for $m_\chi = 10^{-31}$ eV at low multipoles. Indeed the dominant modification comes from isotropic cosmic birefringence, and according to Fig. 2.2 an axion field of such mass is clearly present also after reionization, causing such an impact on the low multipoles of $C_{\ell,\mathrm{obs}}^{EE|(1)}$. Analogously, according again to Fig. 2.2, an axion with mass $m_\chi = 10^{-28}$ eV does not experience a relevant time evolution after reionization: indeed, this is clearly noticed in Fig. 2.4, where the reionization contribution to $C_{\ell,\mathrm{obs}}^{EE|(1)}$ at low multipoles is not significantly converted to $C_{\ell,\mathrm{obs}}^{BB|(1)}$.

### 2.4.2 ACB from Recombination and Reionization

By looking at Eqs. (2.119)-(2.123), whose components are given by the formulas collected in Eqs. (2.150)-(2.152) and Eqs. (2.155)-(2.156), we can see that cosmic birefringence enters in the expressions for the CMB rotated spectra with both the recombination and reionization contributions. Thus, it is important to understand how much the two signals differ. To see this, we have used again our modified `CLASS` code to plot the angular power spectra of anisotropic birefringence from the two epochs in Fig. 2.6. Let us notice that different masses of the axion field imply a different contribution to the reionization signal: indeed, it turns out that for a sufficiently light axion scalar field, the contribution from reionization can be larger than that one from recombination at least at low multipoles for the spectra we considered, i.e. $C_\ell^{\alpha\alpha|(1)}$, $C_\ell^{\alpha T|(1)}$, and $C_\ell^{\alpha E|(1)}$. On the other hand, as the axion mass increases, the two contributions become comparable to each other.

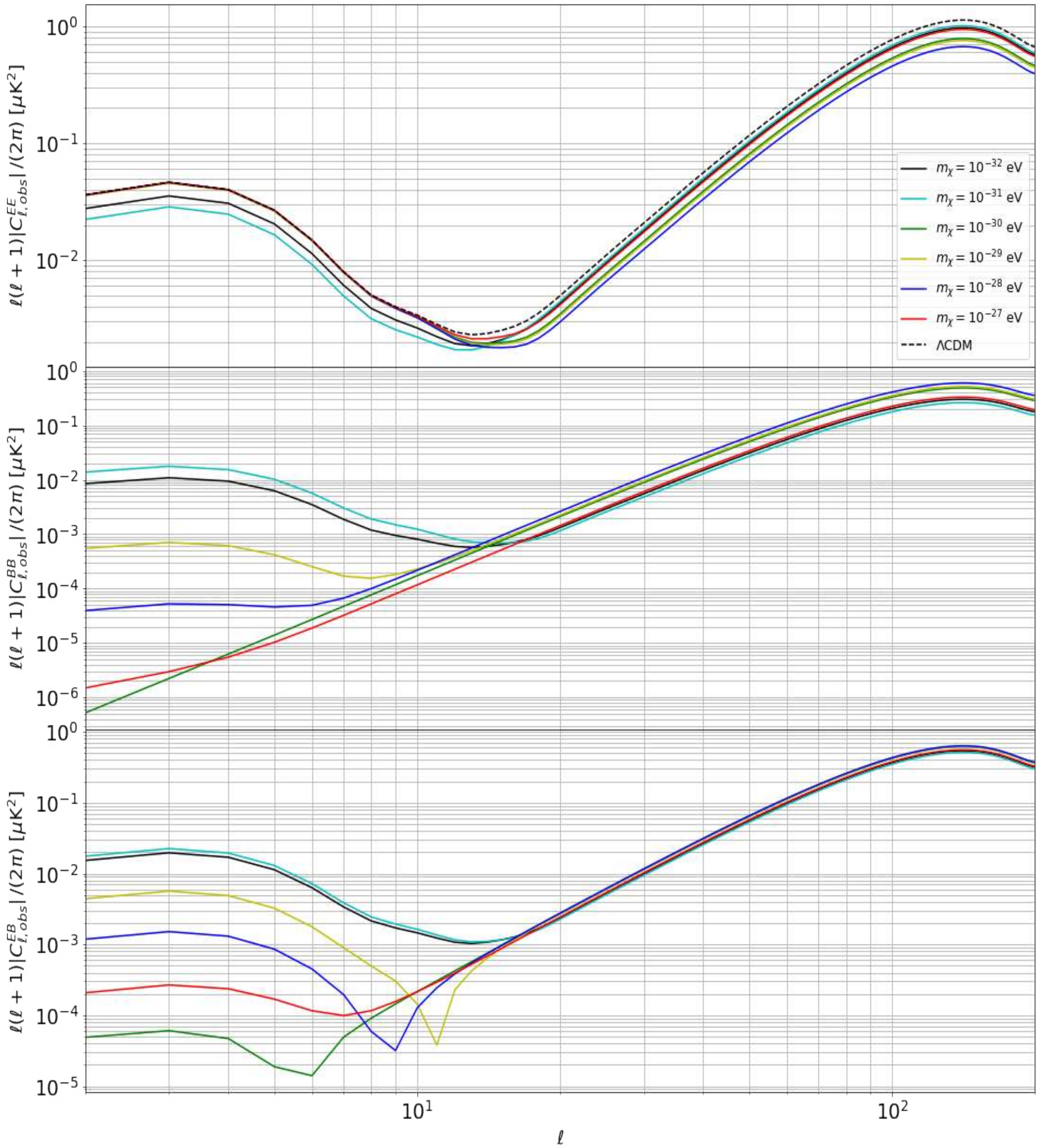


**Figure 2.4:** Absolute value of the unlensed power spectra of CMB polarization affected by isotropic and anisotropic cosmic birefringence, from recombination and reionization, for the model defined by Eq. (2.89) with the same set of parameters used in Fig. 2.3. The tensor-to-scalar ratio is set equal to zero, so that there are no primordial *B* modes. In absence of parity-violating mechanisms, the unlensed *EB* spectrum with no birefringence is predicted to be zero by the standard ΛCDM model.

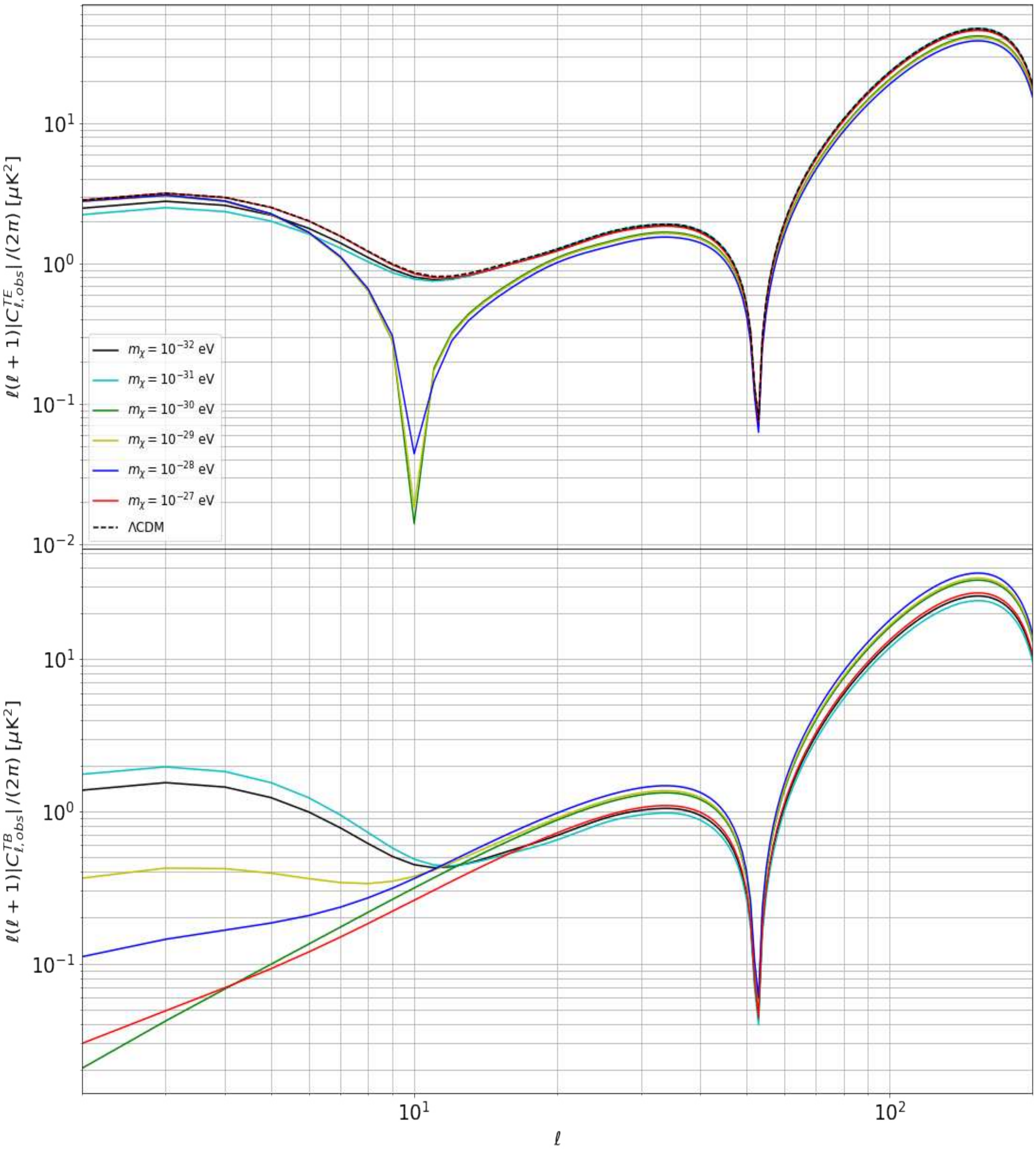


**Figure 2.5:** Absolute value of the unlensed cross-correlations of CMB temperature with polarization affected by isotropic and anisotropic cosmic birefringence, from recombination and reionization, for the model defined by Eq. (2.89) with the same set of parameters used in Fig. 2.3. The tensor-to-scalar ratio is set equal to zero, so that there are no primordial *B* modes. In absence of parity-violating mechanisms, the unlensed *TB* spectrum with no birefringence is predicted to be zero by the standard ΛCDM model.

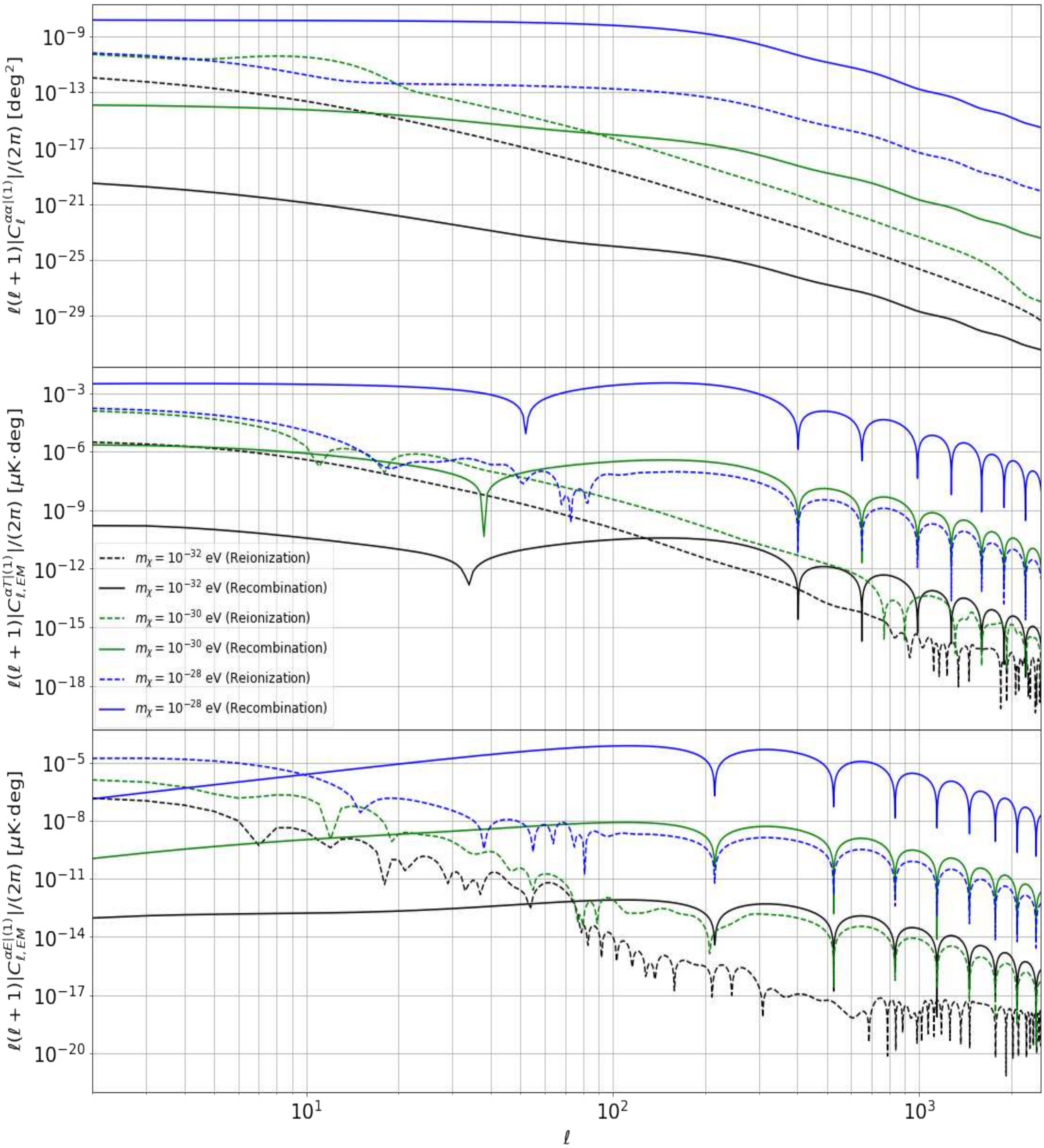


**Figure 2.6:** Absolute value of the angular power spectra involving anisotropic cosmic birefringence for the model defined by Eq. (2.89), coming from recombination (solid line) and from reionization (dashed line) for the same set of parameters used in Fig. 2.3. The mixed terms (e.g. recombination-reionization) are not considered here.

# 3

# Cross-Bispectra of Anisotropic Birefringence

In the previous chapter we have introduced the phenomenon of cosmic birefringence, and computed the angular cross-correlation between its anisotropic component and the standard CMB observables. In this chapter instead, we are going farther, proposing a new kind of promising observable involving ACB. In fact, in Ref. [82] the authors have calculated the CMB three-point angular correlation functions (in short the **angular bispectra**), for the temperature and birefringent polarization fields, i.e. by taking into account the cosmic birefringence effects. As we did in Sec. 2.4.1, the authors have performed such calculations under the assumption that the temperature, the unrotated polarization fields of CMB and the anisotropic rotation angle are all Gaussian random fields. Nevertheless, they have found that observed CMB bispectra would arise if two-point cross-correlations of $T$ with $\delta\alpha$, and $E$ with $\delta\alpha$ are non-vanishing. Instead, we are going here to follow Ref. [85], and to calculate the three-point angular correlations between $\delta\alpha$ and the CMB observables, showing that still keeping the Gaussian assumption for these fields, there exist non-zero observed bispectra even in absence of an unrotated cross-correlation between the birefringence angle and $T$, $E$, $B$. Furthermore, we will show in this chapter that the $\langle \delta\alpha\, TB \rangle$ and $\langle \delta\alpha\, EB \rangle$ bispectra are the three-point angular correlation functions with the largest signal-to-noise ratio. However, before to start, let us perform some brief warm-up computations in order to show our motivations in going beyond the simple two-point correlation functions. In Sec. 2.3.2 we have computed the two-point angular cross-correlations between $\delta\alpha$ and the unrotated CMB fields: at the contrary, here, we want co compute those with the birefringent ones. Of course, since the temperature field is not affected by cosmic birefringence, the only relevant quantities are the Stokes parameters associated with the linear polarization. Therefore, the most general cross-correlation of the anisotropic component of birefringence angle with CMB polarization modes is obtained by slightly modifying Eq. (2.126),

$$\langle \alpha^{(1)*}_{\ell_1 m_1}(\tau_x) a^{\rm obs}_{j,\ell_2 m_2}(\tau_z)\rangle = \sum_{s_2} \frac{e^{is_2\bar{\alpha}(\tau_2)}}{2} \sum_{L_2 M_2} \int \mathrm{d}^2\hat{n}\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2 M_2}(\hat{\mathbf{n}}_2) \\ \mathscr{R}^{(s_2)}_{jk} \left\langle \alpha^{(1)*}_{\ell_1 m_1}(\tau_x) a^{\rm EM}_{k,L_2 M_2}(\tau_z) e^{is_2\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_2]} \right\rangle . \tag{3.1}$$

The ensemble average on the right-hand side of the above equation can be easily evaluated by approximating the exponential to unity, since we want to work at the leading order in $\delta\alpha$. Thus, the term within angular brackets on the right-hand side of Eq. (3.1) simply reduces to

$$\langle \alpha^{(1)*}_{\ell_1 m_1}(\tau_x) a^{\rm EM}_{k,L_2 M_2}(\tau_z) e^{is_2\delta\alpha^{(1)}[(\tau_0-\tau_2)\hat{\mathbf{n}}_2]}\rangle \simeq C^{\alpha k|(1)}_{L_2,\,\rm EM}|_{xz}\delta_{\ell_1 L_2}\delta_{m_1 M_2} . \tag{3.2}$$

We then substitute the result above in Eq. (3.1) to obtain

$$\langle \alpha^{(1)*}_{\ell_1 m_1}(\tau_x) a^{\rm obs}_{j,\ell_2 m_2}(\tau_z)\rangle = \frac{1}{2}\sum_{s_2} e^{is_2\bar{\alpha}(\tau_z)} \int \mathrm{d}^2\hat{n}\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{\ell_1 m_1}(\hat{\mathbf{n}}_2) \mathscr{R}^{(s_2)}_{jk} C^{\alpha k|(1)}_{\ell_1,\,\rm EM}|_{xz} . \tag{3.3}$$

Therefore, by exploiting the orthogonality relation of spin-weighted spherical harmonics and performing the summation over $s_2 = \pm 2$, we get the following set of relations:

$$C^{\alpha E}_{\ell,\rm obs}|_{xz} = C^{\alpha E|(1)}_{\ell,\,\rm EM}|_{xz} \cos\left[2\bar{\alpha}(\tau_z)\right] - C^{\alpha B|(1)}_{\ell,\,\rm EM}|_{xz} \sin\left[2\bar{\alpha}(\tau_z)\right] , \tag{3.4}$$

$$C^{\alpha B}_{\ell,\rm obs}|_{xz} = C^{\alpha E|(1)}_{\ell,\,\rm EM}|_{xz} \sin\left[2\bar{\alpha}(\tau_z)\right] + C^{\alpha B|(1)}_{\ell,\,\rm EM}|_{xz} \cos\left[2\bar{\alpha}(\tau_z)\right] . \tag{3.5}$$

We can easily see that the observed angular power spectra are simply obtained from the EM ones by performing a spatial rotation. This result is telling us that the rotated angular power spectra would vanish if the unrotated correlations are absent. This conclusion albeit trivial, is interesting also because it provides an additional motivation to investigate higher-order correlation functions. Indeed, as we will see very soon, we will reach a very different conclusion for the angular bispectra. This chapter mainly follows Ref. [85], and its structure is organized as follows. In Sec. 3.1 we explicitly compute the birefringent three-point angular cross-correlation functions involving $\delta\alpha$ and CMB observables. In Sec. 3.2 we provide as an example some plots showing the behavior of the angular bispectra obtained in Sec. 3.1 for a scale-invariant model of cosmic birefringence. In Sec. 3.3 we estimate the signal-to-noise ratio for the angular bispectra we have computed in Sec. 3.1.

**Table 3.1:** Bispectra involving the anisotropic birefringence angle $\delta\alpha$ and the CMB anisotropy maps, $T$, $E$, $B$, ordered according to the number of polarization fields $N_P$.

| $N_P$ | | | | |
|---|---|---|---|---|
| 0 | $\delta\alpha\,\delta\alpha\,T$ | $\delta\alpha\,TT$ | | |
| 1 | $\delta\alpha\,\delta\alpha\,E$ | $\delta\alpha\,\delta\alpha\,B$ | $\delta\alpha\,TE$ | $\delta\alpha\,TB$ |
| 2 | $\delta\alpha\,EE$ | $\delta\alpha\,EB$ | $\delta\alpha\,BB$ | |

## 3.1 Birefringent Angular Bispectra

We now move on by evaluating all the three-point functions involving correlations between the anisotropic birefringence angle $\delta\alpha$ and the CMB maps. For this reason, we are going to calculate the ensemble averages for the combinations listed in Tab. 3.1. Since cosmic birefringence only affects the Stokes parameters $Q$ and $U$, it is clear that the observed correlators $\langle\delta\alpha\,\delta\alpha\,T\rangle$ and $\langle\delta\alpha\,TT\rangle$ correspond to the unrotated ones, and so they are non-zero only in the presence of some intrinsic (primordial) underlying non-Gaussianity, a case that we are not considering in this thesis, according to our previous assumptions. Anyway, as we are going to show, differently from what occurs for the two-point correlation functions, vanishing primordial (three-point) correlation functions do not prevent the possibility to have non-vanishing observed three-point correlation functions for the other combinations. Therefore, let us focus on the three-point functions listed in the second and the third line of Tab. 3.1.

### 3.1.1 One Polarization Field

In analogy with what we have done before, the bispectra involving a single polarization field in Tab. 3.1 can be evaluated by firstly calculating the following general quantity:

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) b^{(1)}_{\ell_2 m_2}(\tau_y) a^{\mathrm{obs}(\tau_z)}_{j,\ell_3 m_3}\rangle = \sum_{s_3} \frac{e^{is_3\bar{\alpha}(\tau_z)}}{2} \sum_{L_3 M_3} \int \mathrm{d}\hat{n}_3 \, {}_{s_3}Y^*_{\ell_3 m_3}(\hat{\mathbf{n}}_3) \, {}_{s_3}Y_{L_3 M_3}(\hat{\mathbf{n}}_3) \mathscr{R}^{(s_3)}_{jk} \\ \left\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) b^{(1)}_{\ell_2 m_2}(\tau_y) a^{\mathrm{EM}}_{k,L_3 M_3} e^{is_3\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_3]} \right\rangle ,\end{aligned} \tag{3.6}$$

and then specializing for $b_{\ell m} = \alpha^{(1)}_{\ell m}$ and $T^{(1)}_{\ell m}$. From Eq. (3.6) is it easy to understand why we can have non-vanishing three-point correlation functions even under the Gaussian assumption: the ensemble average within the angular integral is effectively a **trispectrum**, i.e. a four-point correlation function in the harmonic space, which is in general non-zero for Gaussian random fields. In order to see this more clearly, we now expand the exponential as

$$e^{is_3\delta\alpha^{(1)}(\tau_0-\tau_z)\hat{\mathbf{n}}_3]} \simeq 1 + is_3 \sum_{L_3'M_3'} \alpha^{(1)}_{L_3'M_3'}(\tau_z)_0 Y_{L_3'M_3'}(\hat{\mathbf{n}}_3). \tag{3.7}$$

We now want to investigate a specific case, i.e. that for which we have no two-point cross-correlation between $\delta\alpha$ and the CMB maps. Such a situation is possible for example when the potential is suppressed by a small axion mass, as discussed in Sec. 2.3.2, but here we want more in general to study if it is possible to source some non-Gaussian correlation functions, even without any unrotated cross-correlation. According to this, it is then not difficult to show that the only non-vanishing contribution to the ensemble average on the right-hand side of Eq. (3.6) is given by

$$\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) b^{(1)}_{\ell_2 m_2}(\tau_y) a^{\rm EM}_{k,L_3M_3}(\tau_z) e^{is_3\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_3]}\rangle = is_3\delta_{\ell_2 L_3}\delta_{m_2,-M_3} C^{\alpha\alpha}_{\ell_1}|_{xz} C^{bk|(1)}_{\ell_2,\,\rm EM}|_{yz\,0} Y^*_{\ell_1 m_1}(\hat{\mathbf{n}}_3), \tag{3.8}$$

where we have exploited Eq. (2.129), and neglected all the terms proportional to the unrotated cross-correlations between $\alpha$ and the CMB fields. For the same reason, since our aim is to study what happens when the two-point cross-correlations between the anisotropic birefringence angle and the CMB anisotropies are absent, we can already infer that the only non-vanishing bispectra involving a single polarization field are those with $b^{(1)}_{\ell m} = T^{(1)}_{\ell m}$, and so from now on we replace the generic field $b$ with the CMB temperature anisotropies. We now substitute Eq. (3.8) in Eq. (3.6), and we perform the integration over the solid angle by means of Eq. (2.141), so that Eq. (3.6) simply reduces to

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) T^{(1)}_{\ell_2 m_2}(\tau_y) a^{\rm obs}_{j,\ell_3 m_3}(\tau_z)\rangle &= \\ &= \frac{i}{2}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} \sum_{s_3=\pm 2} s_3\, e^{is_3\overline{\alpha}(\tau_z)} \mathscr{R}^{(s_3)}_{jk} C^{\alpha\alpha|(1)}_{\ell_1}|_{xz} C^{Tk|(1)}_{\ell_2,\,\rm EM}|_{yz} I^{0,s_3,-s_3}_{\ell_1\ell_2\ell_3}\,.\end{aligned} \tag{3.9}$$

After performing the summation over $s_3 = \pm 2$, we find the following expressions for the observed $\langle \delta\alpha\, TE\rangle$ and $\langle \delta\alpha\, TB\rangle$ angular bispectra:

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) T^{(1)}_{\ell_2 m_2}(\tau_y) E^{\rm obs}_{\ell_3 m_3}(\tau_z)\rangle = \Big\{ i\left[1-(-1)^{\ell_T}\right]\cos\left[2\alpha(\tau_z)\right] - \left[1+(-1)^{\ell_T}\right]\sin\left[2\overline{\alpha}(\tau_z)\right]\Big\}\\ C^{\alpha\alpha|(1)}_{\ell_1}|_{xz} C^{TE|(1)}_{\ell_2,\,\rm EM}|_{yz} I^{0,2,-2}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix},\end{aligned} \tag{3.10}$$

with, by recalling Eq. (2.143), $\ell_T = \ell_1 + \ell_2 + \ell_3$ being the total multipole number, and

$$\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) T^{(1)}_{\ell_2 m_2}(\tau_y) B^{\rm obs}_{\ell_3 m_3}(\tau_z)\rangle = \left\{ \left[1+(-1)^{\ell_T}\right]\cos\left[2\overline{\alpha}(\tau_z)\right] + i\left[1-(-1)^{\ell_T}\right]\sin\left[2\overline{\alpha}(\tau_z)\right] \right\} C^{\alpha\alpha|(1)}_{\ell_1}\big|_{xz} C^{TE|(1)}_{\ell_2,\,\rm EM}\big|_{yz} I^{0,2,-2}_{\ell_1\ell_2\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{3.11}$$

where we have disregarded the terms proportional to $C^{TB}_\ell$ because of our assumptions. It is interesting to note that we obtain some non-zero bispectra between $\delta\alpha$ and CMB maps, even assuming no correlation between them at the two-(three) point level. Moreover, Eqs. (3.10)-(3.11) contain terms of mixed parity for $\ell_T$, where the imaginary part of the bispectra is non-vanishing only if $\ell_T$ is an odd number, encoding parity-breaking signatures, as we will show in Sec. 3.2. Moreover, let us notice that the two bispectra are non-vanishing even for $\overline{\alpha} = 0$, which corresponds to a regime of purely anisotropic cosmic birefringence. From Eqs. (3.10)-(3.11) we see that setting $\overline{\alpha} = 0$ turns off one of the two terms associated with the parity properties of $\ell_T$, but the other term survives: this feature is suggesting that having a zero isotropic birefringence angle fixes the parity properties of the bispectrum itself, but it still produces a non-vanishing signal.

### 3.1.2 Two Polarization Fields

We now move on to consider the more complicated bispectra listed in the last line of Tab. 3.1. Similarly to the previous case, we have

$$\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) a^{\rm obs}_{i,\ell_2 m_2}(\tau_y) a^{\rm obs}_{j,\ell_3 m_3}(\tau_z)\rangle = \frac{1}{4}\sum_{s_2 s_3} e^{i[s_2\overline{\alpha}(\tau_y)+s_3\overline{\alpha}(\tau_z)]} \sum_{L_2M_2}\sum_{L_3M_3}\int d\hat{n}_2 \int d\hat{n}_3\, {}_{s_2}Y^*_{\ell_2 m_2}(\hat{\mathbf{n}}_2)\, {}_{s_2}Y_{L_2M_2}(\hat{\mathbf{n}}_2)\, {}_{s_3}Y^*_{\ell_3 m_3}(\hat{\mathbf{n}}_3)\, {}_{s_3}Y_{L_3M_3}(\hat{\mathbf{n}}_3) \mathscr{R}^{(s_2)}_{ik}\mathscr{R}^{(s_3)}_{jl}\left\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) a^{\rm EM}_{k,L_2M_2}(\tau_y) a^{\rm EM}_{l,L_3M_3}(\tau_z) e^{is_2\delta\alpha^{(1)}[(\tau_0-\tau_y)\hat{\mathbf{n}}_2]} e^{is_3\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_3]}\right\rangle, \tag{3.12}$$

where, expanding the Taylor-expanding the complex exponentials,

$$e^{is_2\delta\alpha^{(1)}[(\tau_0-\tau_y)\hat{\mathbf{n}}_2]} e^{is_3\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_3]} \simeq \simeq 1 + is_2\sum_{L_2'M_2'}\alpha^{(1)}_{L_2'M_2'}(\tau_y)_0 Y_{L_2'M_2'}(\hat{\mathbf{n}}_2) + is_3\sum_{L_3'M_3'}\alpha^{(1)}_{L_3'M_3'}(\tau_z)_0 Y_{LM}(\hat{\mathbf{n}}_3), \tag{3.13}$$

and by exploiting again Eq. (2.129), we work out the ensemble average within the integral as

$$\begin{aligned}&\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) a^{\rm EM}_{k,L_2M_2}(\tau_y) a^{\rm EM}_{l,L_3M_3}(\tau_z) e^{is_2\delta\alpha^{(1)}[(\tau_0-\tau_y)\hat{\mathbf{n}}_2]} e^{is_3\delta\alpha^{(1)}[(\tau_0-\tau_z)\hat{\mathbf{n}}_3]}\rangle \simeq \\ &\simeq \frac{i}{2}\left[C^{\alpha\alpha|(1)}_{\ell_1}|_{xy} + C^{\alpha\alpha|(1)}_{\ell_1}|_{xz}\right] C^{kl|(1)}_{L_2,\,{\rm EM}}|_{yz}\delta_{L_2L_3}\delta_{M_2,-M_3}\left[s_2\,{}_0Y^*_{\ell_1m_1}(\hat{\mathbf{n}}_2) + s_3\,{}_0Y^*_{\ell_1m_1}(\hat{\mathbf{n}}_3)\right].\end{aligned} \tag{3.14}$$

We now replace Eq. (3.14) in Eq. (3.12) and, by performing the angular integration with the help of Eq. (2.73) and Eq. (2.141), we finally get

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) a^{\rm obs}_{i,\ell_2m_2}(\tau_y) a^{\rm obs}_{j,\ell_3m_3}(\tau_z)\rangle = \frac{i}{8}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\sum_{s_2s_3} e^{i[s_2\bar{\alpha}(\tau_y)+s_3\bar{\alpha}(\tau_z)]}\mathscr{R}^{(s_2)}_{ik}\mathscr{R}^{(s_3)}_{jl} \\ \left[C^{\alpha\alpha|(1)}_{\ell_1}|_{xy} + C^{\alpha\alpha|(1)}_{\ell_1}|_{xz}\right]\left[s_2 C^{kl|(1)}_{\ell_3,\,{\rm EM}}|_{yz} I^{0,s_2,-s_2}_{\ell_1\ell_2\ell_3} + s_3 C^{kl|(1)}_{\ell_2,\,{\rm EM}}|_{yz} I^{0,-s_3,s_3}_{\ell_1\ell_2\ell_3}\right].\end{aligned} \tag{3.15}$$

Thus, we can use the expression in Eq. (3.15) to compute the bispectra listed in the last row of Tab. 3.1. Let us notice that we get a relatively simple expression because, according to our assumptions, we have set equal to zero all the two-point cross-correlations of the anisotropic cosmic birefringent angle with CMB temperature and polarization modes, and because we are working at the leading order in $\delta\alpha$: differently from the case involving a single polarization field, this time no one of the configurations in the last line of Tab. 3.1 is vanishing. After some algebra we obtain

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) E^{\rm obs}_{\ell_2m_2}(\tau_y) E^{\rm obs}_{\ell_3m_3}(\tau_z)\rangle = \frac{i}{4}\sum_{s_2s_3} e^{is_2\bar{\alpha}(\tau_y)+is_3\bar{\alpha}(\tau_z)}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} \\ \Big[C^{\alpha\alpha}_{\ell_1}|_{xy} I^{0,s_2,-s_2}_{\ell_1\ell_2\ell_3}\left(s_2 C^{EE|(1)}_{\ell_3,\,{\rm EM}}|_{yz} - s_3 C^{BB|(1)}_{\ell_3,\,{\rm EM}}|_{yz}\right) \\ + C^{\alpha\alpha|(1)}_{\ell_1}|_{xz} I^{0,-s_3,s_3}_{\ell_1\ell_2\ell_3}\left(s_3 C^{EE|(1)}_{\ell_2\,{\rm EM}}|_{yz} - s_2 C^{BB|(1)}_{\ell_2,\,{\rm EM}}|_{yz}\right)\Big],\end{aligned} \tag{3.16}$$

$$\begin{aligned}\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) B^{\rm obs}_{\ell_2m_2}(\tau_y) B^{\rm obs}_{\ell_3m_3}(\tau_z)\rangle = \frac{i}{4}\sum_{s_2s_3} e^{is_2\bar{\alpha}(\tau_y)+is_3\bar{\alpha}(\tau_z)}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} \\ \Big[C^{\alpha\alpha|(1)}_{\ell_1}|_{xy} I^{0,s_2,-s_2}_{\ell_1\ell_2\ell_3}\left(s_2 C^{BB|(1)}_{\ell_3\,{\rm EM}}|_{yz} - s_3 C^{EE|(1)}_{\ell_3,\,{\rm EM}}|_{yz}\right) + \\ + C^{\alpha\alpha|(1)}_{\ell_1}|_{xz} I^{0,-s_3,s_3}_{\ell_1\ell_2\ell_3}\left(s_3 C^{BB|(1)}_{\ell_2,\,{\rm EM}}|_{yz} - s_2 C^{EE|(1)}_{\ell_2,\,{\rm EM}}|_{yz}\right)\Big],\end{aligned} \tag{3.17}$$

$$\langle \alpha^{(1)}_{\ell_1 m_1}(\tau_x) E^{\rm obs}_{\ell_2 m_2}(\tau_y) B^{\rm obs}_{\ell_3 m_3}(\tau_z)\rangle = \frac{1}{4}\sum_{s_2 s_3} e^{is_2\bar{\alpha}(\tau_y)+is_3\bar{\alpha}(\tau_z)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$
$$\left[ C^{\alpha\alpha|(1)}_{\ell_1}\big|_{xy}\, I^{0,s_2,-s_2}_{\ell_1\ell_2\ell_3} \left( \frac{s_2 s_3}{2} C^{EE|(1)}_{\ell_3,\,\rm EM}\big|_{yz} - 2 C^{BB|(1)}_{\ell_3,\,\rm EM}\big|_{yz} \right) + \right. \tag{3.18}$$
$$\left. + C^{\alpha\alpha|(1)}_{\ell_1}\big|_{xz}\, I^{0,-s_3,s_3}_{\ell_1\ell_2\ell_3} \left( \frac{s_2 s_3}{2} C^{BB|(1)}_{\ell_2,\,\rm EM}\big|_{yz} - s_2 C^{EE|(1)}_{\ell_2,\,\rm EM}\big|_{yz} \right) \right].$$

Interestingly enough, all the bispectra collected in Eqs. (3.16)-(3.18) contain terms of different parity, since they are proportional to $[1 \mp (-1)^{\ell_T}]$ and thus vanish for $\ell_T =$ even (odd). This is a manifest signature that these objects encode parity-violating features, that are due to the parity-breaking nature of cosmic birefringence. Moreover, as we could expect, all the bispectra that we have computed are proportional to the self-correlator $C^{\alpha\alpha|(1)}_{\ell}$, which obviously depends on the specific model which induces the birefringent mechanism. Furthermore, it is interesting to see that, thanks to the symmetry properties of the Wigner $3j$-symbols, the $\langle \delta\alpha\, EE\rangle$ and $\langle \delta\alpha\, BB\rangle$ angular bispectra are invariant under the index permutation $\ell_2 \leftrightarrow \ell_3$. We point out this feature, because in general angular correlation functions involving different fields are not symmetric under the simultaneous interchange of their three multipole numbers $\ell_1\ \ell_2\ \ell_3$ (for example the *TTE*, *TET*, and *ETT* combinations of a bispectrum of CMB temperature and polarization would correspond to three distinct bispectra , as discussed e.g. in Refs. [198–201]). We will use this property in Sec. 3.3, when we will estimate the signal-to-noise ratio of the cross-bispectra we have computed here.

## 3.2 Reduced Bispectra

In order to explicitly evaluate some of the angular bispectra, we now adopt a phenomenological approach. Differently from Chap. 2, we consider here a scale-invariant model of cosmic birefringence, for which the self-correlator of $\delta\alpha$ reads:

$$C^{\alpha\alpha|(1)}_{\ell} \equiv \frac{C^{(1)}_{\alpha\alpha}}{\ell(\ell+1)}, \tag{3.19}$$

where $C^{(1)}_{\alpha\alpha}$ is a model-dependent parameter which encodes the physics of the axion field $\chi$ and that quantifies the amplitude of the anisotropic component of the birefringence angle. Since we are working under the assumption of statistical isotropy, our bispectra should be invariant under spatial rotations, and this requires the angular bispectra should be proportional to the Wigner $3j$-symbol (see e.g. Ref. [202]):

$$\langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3}\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B^{XYZ}_{\ell_1\ell_2\ell_3}, \tag{3.20}$$

where $B^{XYZ}_{\ell_1\ell_2\ell_3}$ is the **angular averaged bispectrum** and $X, Y, Z$ denote $\delta\alpha$, $T, E, B$. However, in order to extract the physical information we display the so-called **reduced bispectra** $b^{XYZ}_{\ell_1\ell_2\ell_3}$, that are related to the angular averaged one via

$$B^{XYZ}_{\ell_1\ell_2\ell_3} = G_{\ell_1\ell_2\ell_3} b^{XYZ}_{\ell_1\ell_2\ell_3}, \tag{3.21}$$

where the function $G_{\ell_1\ell_2\ell_3}$ is defined as:

$$\begin{aligned} G_{\ell_1\ell_2\ell_3} \equiv -2\sqrt{\frac{(\ell_2+2)!(\ell_3+2)!}{(\ell_2-2)!(\ell_3-2)!}} \Big\{ & \ell_1(\ell_1+1)\left[\ell_3(\ell_3+1)+\ell_2(\ell_2+1)-\ell_1(\ell_1+1)\right]+ \\ & + \ell_2(\ell_2+1)\left[\ell_3(\ell_3+1)-\ell_2(\ell_2+1)+\ell_1(\ell_1+1)\right]+ \\ & + (\ell_3+2)(\ell_3-1)\left[\ell_1(\ell_1+1)-\ell_2(\ell_2+1)-\ell_3(\ell_3+1)\right] \Big\}^{-1} I^{0,2,-2}_{\ell_1\ell_2\ell_3}. \end{aligned} \tag{3.22}$$

Indeed such an expression matches the more common one for $\ell_T =$ even, as it can be shown via standard techniques in quantum theory of angular momentum,

$$G_{\ell_1\ell_2\ell_3} \xrightarrow{\ell_T = \text{even}} I^{0,0,0}_{\ell_1\ell_2\ell_3}, \tag{3.23}$$

and can be found by exploiting the recursive formulas for the Wigner $3j$-symbols (see e.g Refs. [183, 203]). The definition of Eq. (3.22) is more general, since it remains non-zero for $\ell_T =$ odd. Differently, $I^{0,0,0}_{\ell_1\ell_2\ell_3}$ is vanishing for $\ell_T =$ odd (see e.g. Ref. [204]). However, it is now time to understand how the parity of $\ell_T$ is related to the the parity of the bispectrum itself (see e.g. Refs. [19, 23]): this can be understood by taking the complex conjugate of Eq. (3.20),

$$\langle a^*_{X,\ell_1 m_1} a^*_{Y,\ell_2 m_2} a^*_{Z,\ell_3 m_3} \rangle = \langle a_{X,\ell_1,-m_1} a_{Y,\ell_2,-m_2} a_{Z,\ell_3,-m_3} \rangle, \tag{3.24}$$

where we have exploited that the harmonic coefficients of the CMB observables obey a reality condition:

$$a^*_{X,\ell_1 m_1} = (-1)^{m_1} a_{X,\ell_1,-m_1}. \tag{3.25}$$

As shown in Eq. (2.67), the Wigner-$3j$ symbol automatically ensures the following selection rule on the $m$'s:

$$m_1 + m_2 + m_3 = 0, \tag{3.26}$$

so that we can express the complex conjugate of the angular bispectrum by means of Eq. (3.20) as before, but with all the signs in front of the $m$'s flipped:

$$\left[\langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3}\rangle\right]^* = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} B^{XYZ}_{\ell_1\ell_2\ell_3} = (-1)^{\ell_T} \langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3}\rangle \,, \tag{3.27}$$

where we have used one of the properties collected in Eq. (2.143). This means that we can write the transformation properties of the angular bispectrum under complex-conjugation (c.c.) as

$$\text{ang. bispectrum} \xrightarrow{\text{c.c.}} \begin{cases} +\text{ itself} & \text{if } \ell_T = \text{even}, \\ -\text{ itself} & \text{if } \ell_T = \text{odd}. \end{cases} \tag{3.28}$$

This is equivalent to say that: if $\ell_T$ is equal to an even number, then the angular bispectrum has to be a purely real quantity, whereas if $\ell_T$ is equal to an odd number, then the angular bispectrum has to be a purely imaginary quantity. Therefore, the most general bispectrum, without any assumption about the parity of $\ell_T$ can be written as

$$B^{XYZ}_{\ell_1\ell_2\ell_3} = B^{XYZ,\text{even}}_{\ell_1\ell_2\ell_3} + iB^{XYZ,\text{odd}}_{\ell_1\ell_2\ell_3}, \tag{3.29}$$

where the subscripts "even" and "odd" refers to the parity of $\ell_T$. Moreover, this decomposition also plays a role in defining the overall parity of the angular bispectrum, since

$$\langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3}\rangle \xrightarrow{\text{parity}} (-1)^{\ell_T + N_B} \langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3}\rangle \,, \tag{3.30}$$

being $N_B$ the number of $B$-mode polarization CMB fields involved in the ensemble average. Hence, it is clear that parity is violated if $\ell_T + N_\text{B}$ is equal to an odd number. Since $N_\text{B}$ is fixed from the type of bispectrum one wants to evaluate, it follows that parity is automatically broken if the correlation function can be written as the sum of terms of mixed parity of $\ell_T$. Therefore, from Eq. (3.29), we can now infer that the most general parity-violating CMB bispectrum is a complex quantity, since it involves components of different parity for $\ell_T$. Coming back to our specific case, we have shown in Sec. 3.1 that our angular bispectra contain both parity-even and parity-odd components, so with the help of Eqs. (3.20)-(3.21), we can find the expression of the reduced bispectra associated with Eqs. (3.10)-(3.11) and Eqs. (3.16)-(3.18):

$$b^{XYZ}_{\ell_1\ell_2\ell_3} = b^{XYZ,\text{even}}_{\ell_1\ell_2\ell_3} + ib^{XYZ,\text{odd}}_{\ell_1\ell_2\ell_3}. \tag{3.31}$$

Therefore, we can adopt such a decomposition and plot the reduced bispectra starting from the angular three-point correlation functions we have computed in Sec. 3.1. As done e.g. in Ref. [82], in order to display our bispectra we fix two of the three different $\ell$'s by using the following configurations:

$$\ell_T = \text{even} : \qquad \{\ell_1, \ell_2, \ell_3\} = \{4, \ell, \ell + 4\} \tag{3.32}$$

$$\ell_T = \text{odd} : \qquad \{\ell_1, \ell_2, \ell_3\} = \{4, \ell, \ell + 3\} \tag{3.33}$$

that automatically determine the overall parity properties, and ensure the triangular selection rule guaranteed by the Wigner 3*j*-symbol. We have numerically computed the reduced angular bispectra by using Eq. (2.109), and the Boltzmann code CLASS to evaluate the CMB angular power spectra: the results are shown in Figs. 3.1-3.2, where we have assumed only scalar perturbations, but we have taken into account the weak gravitational lensing. Moreover, we have not considered the contribution coming from reionization, but just the recombination one. Let us notice that the oscillating behavior exhibited by the bispectra involving a single polarization field in Fig. 3.1 is due to the the fact that these objects are proportional to the CMB cross-correlator $C_{\ell,\,\mathrm{EM}}^{TE|(1)}$ (see e.g. Ref. [144]). As mentioned before, an interesting result worth to be noticed is that even by assuming no isotropic cosmic birefringence, anyhow a non-vanishing reduced bispectra with a certain parity is generated.

## 3.3 Estimation of the Signal-to-Noise Ratio

In this section, we are going to compute the **signal-to-noise ratio** (SNR) for the birefringent bispectra of Eqs. (3.10)-(3.11) and (3.16)-(3.18). According to its definition, the SNR is the ratio of the signal power to the noise power, and so, in order to estimate the uncertainty in the measurement of the bispectra, we invoke the **Cramer-Rao inequality**, which states that the variance of an unbiased estimator for a given theoretical parameter cannot be less than the diagonal element of the inverse **Fisher matrix** (see e.g. Ref. [205]). By working in analogy with what is usually done in the context of primordial non-Gaussianity (see e.g. Refs. [150, 200, 202, 206–208]), we consider here the simplest scenario where the form of a given bispectrum is considered known and the only parameter of interest is its overall amplitude. Since we are considering a single parameter, the Fisher matrix reduces to being just a number (see e.g. Ref. [209]):

$$\mathsf{F}_{(XYZ)} = \sum_{\ell_1 \le \ell_2 \le \ell_3} \sum_{\ell_1' \le \ell_2' \le \ell_3'} \sum_{ii'} B^{i}_{\ell_1\ell_2\ell_3} \left[\mathsf{Cov}(\hat{B}^{i}_{\ell_1\ell_2\ell_3}, \hat{B}^{i'}_{\ell_1'\ell_2'\ell_3'})\right]^{-1} B^{i'}_{\ell_1'\ell_2'\ell_3'} . \tag{3.34}$$

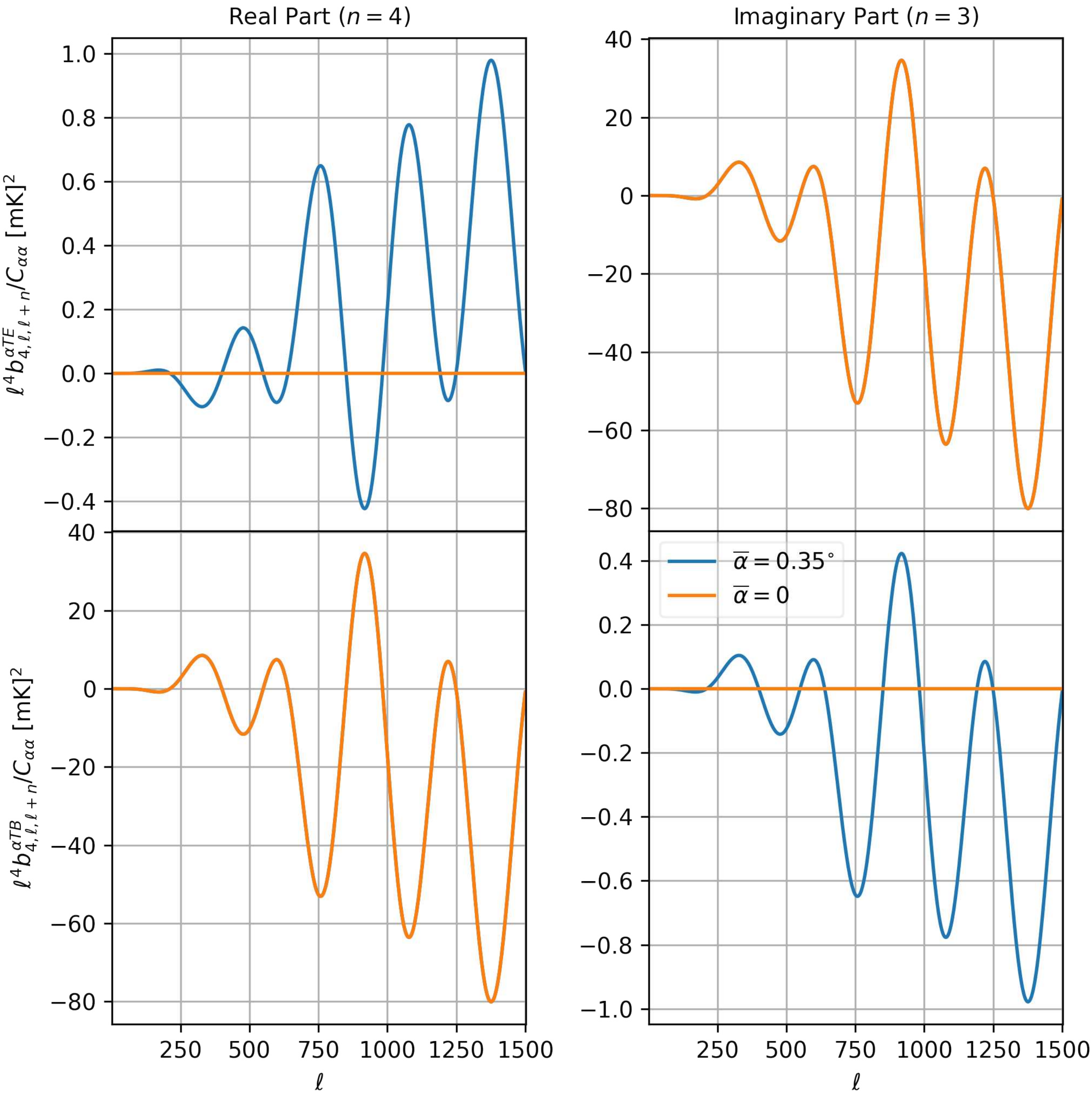


**Figure 3.1:** Real and imaginary components of the reduced bispectra involving one polarization fields, in the purely anisotropic regime ($\overline{\alpha} = 0$) and for a fixed value of the isotropic birefringence angle.

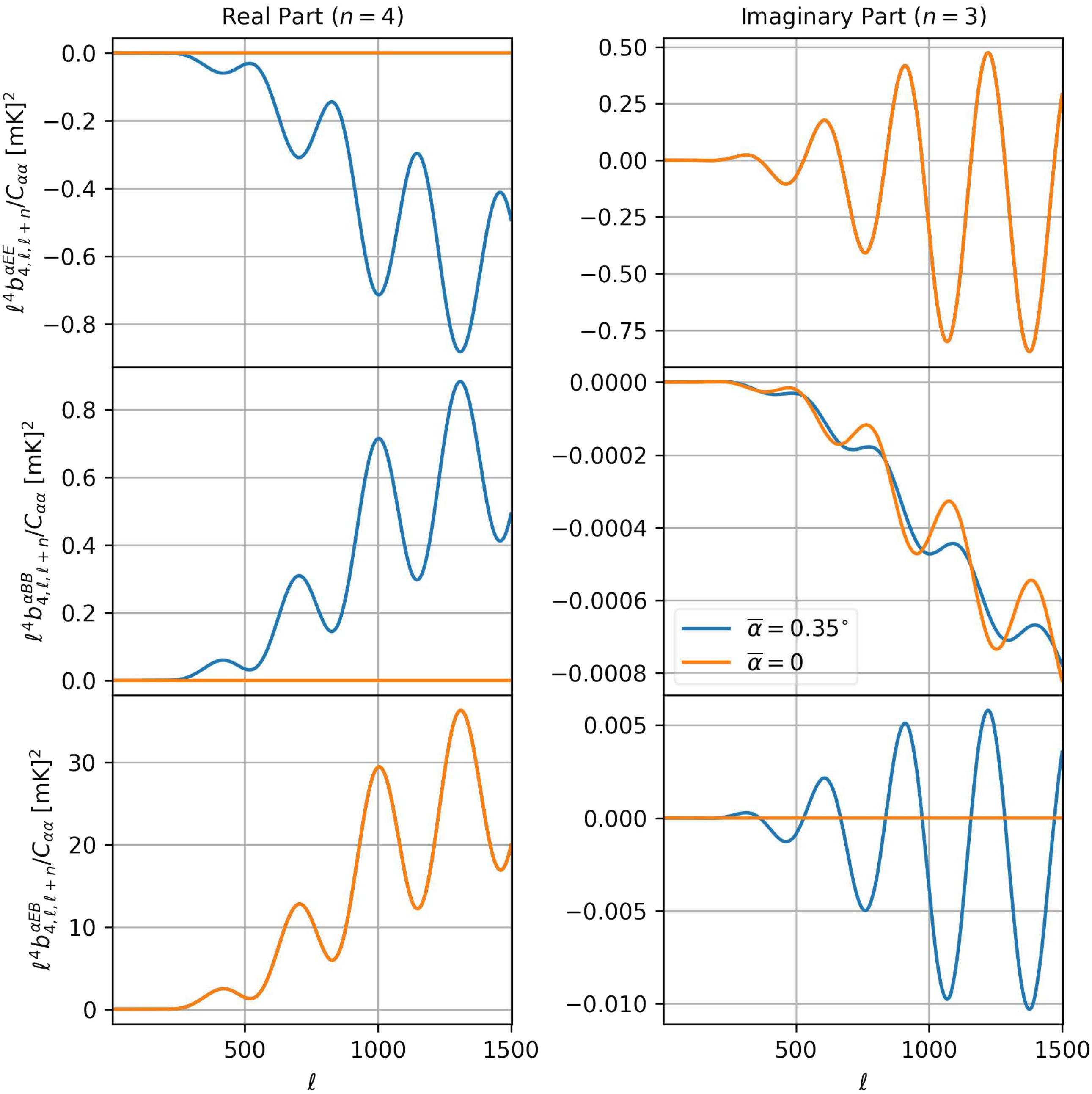


**Figure 3.2:** Real and imaginary components of the reduced bispectra involving two polarization fields, in the purely anisotropic regime ($\overline{\alpha} = 0$) and for a fixed value of the isotropic birefringence angle.

where $X, Y, Z = \delta\alpha, T, E, B$ and the index

$$i = (X, Y, Z), (Y, Z, X), (Z, X, Y), \dots \tag{3.35}$$

labels all the possible non-redundant permutations of a fixed triplet of fields (i.e. they are 6 when $X \neq Y \neq Z$, 1 when $X = Y = Z$ and 3 otherwise). $\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3}$ is an unbiased estimator for the observed angular averaged bispectrum (see e.g. Ref. [210]),

$$\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3} \equiv \sum_{m_1m_2m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a^{X,\mathrm{obs}}_{\ell_1m_1} a^{Y,\mathrm{obs}}_{\ell_2m_2} a^{Z,\mathrm{obs}}_{\ell_3m_3}, \tag{3.36}$$

and $\mathrm{Cov}(\hat{B}^{i}_{\ell_1\ell_2\ell_3}, \hat{B}^{i'}_{\ell'_1\ell'_2\ell'_3})$ is the covariance matrix element. Differently from what is done e.g. in Ref. [199], we are not summing over all the possible combinations of different fields, but we are treating separately each contribution from the five non-vanishing bispectra that we have found in Sec. 3.1 (a similar approach has been adopted for the CMB bispectra induced by weak gravitational lensing in Ref. [211]). The covariance matrix element is defined as

$$\mathrm{Cov}(\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3}, \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3}) \equiv \langle \hat{B}^{XYZ}_{\ell_1\ell_2\ell_3} \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3} \rangle - \langle \hat{B}^{XYZ}_{\ell_1\ell_2\ell_3} \rangle \langle \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3} \rangle. \tag{3.37}$$

As we are going to show, the first term encodes several contributions, and some of them are at least quadratic in $\delta\alpha$, whereas the second term is at least quartic in $\delta\alpha$ because, as can be seen from Eqs. (3.10)-(3.11) and (3.16)-(3.18), all our bispectra are proportional to $C^{\alpha\alpha|(1)}_{\ell_1}$. For this reason we can disregard the second term on the right-hand side of Eq. (3.37), and approximate the covariance matrix element as

$$\mathrm{Cov}(\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3}, \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3}) \simeq \sum_{m_1m_2m_3} \sum_{m'_1m'_2m'_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell'_1 & \ell'_2 & \ell'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \left\langle a^X_{\ell_1m_1} a^Y_{\ell_2m_2} a^Z_{\ell_3m_3} a^{X'}_{\ell'_1m'_1} a^{Y'}_{\ell'_2m'_2} a^{Z'}_{\ell'_3m'_3} \right\rangle_{\mathrm{obs}}, \tag{3.38}$$

In our case, one of the three fields $X, Y, Z$ is always set to be $\delta\alpha$, and since we are working at the leading perturbative order, here we can neglect the factor $\exp(\pm 2i\delta\alpha)$ appearing in Eq. (2.124). As a consequence, according to our assumptions, there are no connected terms arising from non-Gaussian contributions in the six-point correlation function on the right-hand side of Eq. (3.38), and so we can again exploit the Isserlis' theorem. Furthermore, as shown in Eqs. (3.4)-(3.5), the observed two-point cross-correlations between $\delta\alpha$ and the CMB maps are simply a rotation of the unrotated ones, and so, according to our phenomenological assumptions, they vanish too. Thanks to all these approximations, the covariance matrix reduces to

$$
\begin{aligned}
&\mathsf{Cov}(\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3}, \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3}) \simeq \sum_{m_1 m_2 m_3} \sum_{m'_1 m'_2 m'_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell'_1 & \ell'_2 & \ell'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \\
&\Big[ \langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} \rangle \langle a_{Z,\ell_3 m_3} a_{X',\ell'_1 m'_1} \rangle \langle a_{Y',\ell'_2 m'_2} a_{Z',\ell'_3 m'_3} \rangle + \langle a_{X,\ell_1 m_1} a_{Z,\ell_3 m_3} \rangle \langle a_{Y,\ell_2 m_2} a_{X',\ell'_1 m'_1} \rangle \langle a_{Y',\ell'_2 m'_2} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{X',\ell'_1 m'_1} \rangle \langle a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3} \rangle \langle a_{Y',\ell'_2 m'_2} a_{Z',\ell'_3 m'_3} \rangle + \langle a_{X,\ell_1 m_1} a_{Y',\ell'_2 m'_2} \rangle \langle a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3} \rangle \langle a_{X',\ell'_1 m'_1} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{Z',\ell'_3 m'_3} \rangle \langle a_{Y,\ell_2 m_2} a_{Z,\ell_3 m_3} \rangle \langle a_{X',\ell'_1 m'_1} a_{Y',\ell'_2 m'_2} \rangle + \langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} \rangle \langle a_{Z,\ell_3 m_3} a_{Y',\ell'_2 m'_2} \rangle \langle a_{X',\ell'_1 m'_1} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{Y,\ell_2 m_2} \rangle \langle a_{Z,\ell_3 m_3} a_{Z',\ell'_3 m'_3} \rangle \langle a_{X',\ell'_1 m'_1} a_{Y',\ell'_2 m'_2} \rangle + \langle a_{X,\ell_1 m_1} a_{Z,\ell_3 m_3} \rangle \langle a_{Y,\ell_2 m_2} a_{Y',\ell'_2 m'_2} \rangle \langle a_{X',\ell'_1 m'_1} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{Z,\ell_3 m_3} \rangle \langle a_{Y,\ell_2 m_2} a_{Z',\ell'_3 m'_3} \rangle \langle a_{X',\ell'_1 m'_1} a_{Y',\ell'_2 m'_2} \rangle + \langle a_{X,\ell_1 m_1} a_{X',\ell'_1 m'_1} \rangle \langle a_{Y,\ell_2 m_2} a_{Y',\ell'_2 m'_2} \rangle \langle a_{Z,\ell_3 m_3} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{X',\ell'_1 m'_1} \rangle \langle a_{Y,\ell_2 m_2} a_{Z',\ell'_3 m'_3} \rangle \langle a_{Z,\ell_3 m_3} a_{Y',\ell'_2 m'_2} \rangle + \langle a_{X,\ell_1 m_1} a_{Y',\ell'_2 m'_2} \rangle \langle a_{Y,\ell_2 m_2} a_{X',\ell'_1 m'_1} \rangle \langle a_{Z,\ell_3 m_3} a_{Z',\ell'_3 m'_3} \rangle \\
&+ \langle a_{X,\ell_1 m_1} a_{Y',\ell'_2 m'_2} \rangle \langle a_{Y,\ell_2 m_2} a_{Z',\ell'_3 m'_3} \rangle \langle a_{Z,\ell_3 m_3} a_{X',\ell'_1 m'_1} \rangle + \langle a_{X,\ell_1 m_1} a_{Z',\ell'_3 m'_3} \rangle \langle a_{Y,\ell_2 m_2} a_{X',\ell'_1 m'_1} \rangle \langle a_{Z,\ell_3 m_3} a_{Y',\ell'_2 m'_2} \rangle \\
&\qquad + \langle a_{X,\ell_1 m_1} a_{Z',\ell'_3 m'_3} \rangle \langle a_{Y,\ell_2 m_2} a_{Y',\ell'_2 m'_2} \rangle \langle a_{Z,\ell_3 m_3} a_{X',\ell'_1 m'_1} \rangle \Big]_{\text{obs}} .
\end{aligned}
\tag{3.39}
$$

Because of statistical isotropy, we can rewrite the two-point correlation functions in terms of the angular power spectra: by doing this, the first nine terms in the square brackets on the right-hand side of Eq. (3.39) become proportional to (see e.g. Ref [212])

$$
\sum_{mm'} \begin{pmatrix} L & \ell & \ell \\ 0 & m & -m \end{pmatrix} \begin{pmatrix} L & \ell' & \ell' \\ 0 & m' & -m' \end{pmatrix} = \sum_{mm'} \frac{(-1)^{\ell+\ell'-m-m'}}{\sqrt{(2\ell+1)(2\ell'+1)}} \delta_{L0} . \tag{3.40}
$$

However, the observable multipoles start from $L \geq 2$, and so this means that the term above gives no contribution in the estimation of the signal-to-noise ratio. Therefore we are only left with

$$
\begin{aligned}
\mathsf{Cov}(\hat{B}^{XYZ}_{\ell_1\ell_2\ell_3}, \hat{B}^{X'Y'Z'}_{\ell'_1\ell'_2\ell'_3}) = {} & (-1)^{\ell_T} C^{XX'}_{\ell_1,\text{obs}} C^{YY'}_{\ell_2,\text{obs}} C^{ZZ'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_1}\delta_{\ell_2\ell'_2}\delta_{\ell_3\ell'_3} + C^{XX'}_{\ell_1,\text{obs}} C^{YZ'}_{\ell_2,\text{obs}} C^{ZY'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_1}\delta_{\ell_2\ell'_3}\delta_{\ell_3\ell'_2} \\
& + C^{XY'}_{\ell_1,\text{obs}} C^{YX'}_{\ell_2,\text{obs}} C^{ZZ'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_2}\delta_{\ell_2\ell'_1}\delta_{\ell_3\ell'_3} + (-1)^{\ell_T} C^{XY'}_{\ell_1,\text{obs}} C^{YZ'}_{\ell_2,\text{obs}} C^{ZX'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_2}\delta_{\ell_2\ell'_3}\delta_{\ell_3\ell'_1} \\
& + (-1)^{\ell_T} C^{XZ'}_{\ell_1,\text{obs}} C^{YX'}_{\ell_2,\text{obs}} C^{ZY'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_3}\delta_{\ell_2\ell'_1}\delta_{\ell_3\ell'_2} + C^{XZ'}_{\ell_1,\text{obs}} C^{YY'}_{\ell_2,\text{obs}} C^{ZX'}_{\ell_3,\text{obs}} \delta_{\ell_1\ell'_3}\delta_{\ell_2\ell'_2}\delta_{\ell_3\ell'_1} ,
\end{aligned}
\tag{3.41}
$$

where we have used Eq. (2.145). It can seem not formally trivial to obtain the inverse covariance matrix starting from Eq. (3.41). Anyway, we know that the covariance matrix element is non-vanishing only when connecting the same triplets, i.e. when $(\ell_1, \ell_2, \ell_3)$ is equal to $(\ell'_1, \ell'_2, \ell'_3)$ or to a permutation of it. Thus, since we restrict the summation in the $(\ell_1 \leq \ell_2 \leq \ell_3)$ and $(\ell'_1 \leq \ell'_2 \leq \ell'_3)$ domains, we can observe that the covariance matrix is already diagonal in the triplets space, so that we can rewrite Eq. (3.34) as

$$
\mathsf{F}_{(XYZ)} = \sum_{\ell_1 \leq \ell_2 \leq \ell_3} \sum_{ii'} B^{i}_{\ell_1\ell_2\ell_3} \left[ \mathsf{Cov}(\hat{B}^{i}_{\ell_1\ell_2\ell_3}, \hat{B}^{i'}_{\ell_1\ell_2\ell_3}) \right]^{-1} B^{i'}_{\ell_1\ell_2\ell_3} . \tag{3.42}
$$

Let us just mention that the procedure of the domain restriction in the triplet space we have adopted from the beginning of this section is physically correct: this is due to the fact that any angular averaged bispectrum is symmetric under the simultaneous interchange of its three multipole numbers $\ell_1, \ell_2, \ell_3$ and its three field indices $X, Y, Z$ (see e.g. Ref. [198]). Thus, in order to extract the information content, it is enough to study just the subspace $\ell_1 \leq \ell_2 \leq \ell_3$, since we are already summing over all the possible field permutations. We have now to specify the general formula of Eq. (3.42) for the five bispectra collected in Eqs. (3.10)-(3.11) and (3.16)-(3.18). Before doing this, we make a further approximation: since the total angular averaged bispectrum is the sum of different terms, it is reasonable to expect that the dominant contribution in the signal would come from those terms that are non-vanishing even for $\bar{\alpha} = 0$. Hence, in this section we consider a regime of purely anisotropic cosmic birefringence, which allows us to replace $C_{\ell,\mathrm{obs}} \simeq C^{(1)}_{\ell\,\mathrm{EM}}$ within the covariance in Eq. (3.42). Moreover, it is convenient to express the signal-to-noise ratio by means of a matrix formalism by defining a proper quadratic form, which should involve data vectors containing all the permutations of the given bispectrum and a suitable expression for the covariance matrix element:

$$\begin{aligned}\mathsf{Cov}^{XYZ}_{\ell_1\ell_2\ell_3} = \Big\{ & C^{XX'}_{\ell_1} C^{YZ'}_{\ell_2} C^{ZY'}_{\ell_3} \delta_{\ell_2\ell_3} + C^{XY'}_{\ell_1} C^{YX'}_{\ell_2} C^{ZZ'}_{\ell_3} \delta_{\ell_1\ell_2} + C^{XZ'}_{\ell_1} C^{YY'}_{\ell_2} C^{ZX'}_{\ell_3} \delta_{\ell_1\ell_3}\delta_{\ell_2\ell_3} \\ & + (-1)^{\ell_T} \Big[ C^{XX'}_{\ell_1} C^{YY'}_{\ell_2} C^{ZZ'}_{\ell_3} + C^{XY'}_{\ell_1} C^{YZ'}_{\ell_2} C^{ZX'}_{\ell_3} \delta_{\ell_1\ell_2}\delta_{\ell_2\ell_3} + C^{XZ'}_{\ell_1} C^{YX'}_{\ell_2} C^{ZY'}_{\ell_3} \delta_{\ell_1\ell_3}\delta_{\ell_2\ell_3} \Big] \Big\}_{\mathrm{EM}} .\end{aligned} \tag{3.43}$$

Therefore, we can substitute such a general expression in the definition of the Fisher matrix, so that we obtain five formulas for the squared signal-to-noise ratios (one for each birefringent bispectrum). We report the results of the numerical evaluation of in Tab. 3.2: they have been obtained by summing up to $\ell_{\max} = 200$ and by considering an ideal regime with zero instrumental noise. Our choice for $\ell_{\max} = 200$ is dictated by two reasons: in some realistic models for birefringence with a Chern-Simons term these are the typical multipole values up to which the power-spectrum of the anisotropic birefringence angle $C^{\alpha\alpha}_\ell$ is approximately scale-invariant, which is the kind of spectrum we are using here as a toy-model; secondly we are going to specialize our Fisher forecast to a typical **LiteBIRD**-like satellite mission. In Tab. 3.2 we have reported both the signal-to-noise ratio in units of $\sqrt{C_{\alpha\alpha}}$ and according to the current tightest upper observational constraints on the amplitude of a scale-invariant angular power spectrum of anisotropic cosmic birefringence from ACTPol and SPTpol (see Refs. [95, 96], respectively):

$$C_{\alpha\alpha} < 6.3 \times 10^{-5}\ \mathrm{rad}^2, \qquad \text{(95\% C.L., ACTPol, SPTpol).} \tag{3.44}$$

Present constraints on anisotropic birefringence, provided as amplitude $C_{\alpha\alpha}$ of the scale-invariant spectrum of $\delta\alpha(\hat{\mathbf{n}})$, are also given by *Planck* and Bicep-Keck data (see Refs. [97, 100, 102, 103]). Other compatible,

**Table 3.2:** Numerical estimation of the SNR for the birefringent bispectra in the ideal case (zero instrumental noise) in the purely anisotropic regime.

| **Bispectrum** | **SNR** (in units of $\sqrt{C_{\alpha\alpha}}$) | **SNR** (if $C_{\alpha\alpha} \sim 6 \times 10^{-5}$) |
|---|---|---|
| $\delta\alpha\, TE$ | $\approx 80$ | $\approx 0.62$ |
| $\delta\alpha\, TB$ | $\approx 1926$ | $\approx 14.92$ |
| $\delta\alpha\, EB$ | $\approx 3680$ | $\approx 28.51$ |
| $\delta\alpha\, EE$ | $\approx 83$ | $\approx 0.64$ |
| $\delta\alpha\, BB$ | $\approx 4$ | $\approx 0.03$ |

even though weaker, constraints on this parameter are provided by Polarbear and WMAP observations (see Refs. [91, 93], respectively). Future CMB observations are expected to improve the current bounds on cosmic birefringence by orders of magnitude, ad discussed e.g. in Ref. [213]. Similarly, we can now examine the detection possibility of our bispectra for a future CMB experiment, like the LiteBIRD satellite (see e.g. Ref. [214]), as we have mentioned before. Here, we analyze an idealized experimental configuration where foregrounds are neglected. Thus, the signal-to-noise ratio is evaluated adding this time the instrumental noise to the power spectra appearing in Eq. (3.43). By assuming a Gaussian form for the experimental **window function** of beam $\theta$, and by considering a white instrumental noise, we can use the **Knox's formula** for the CMB correlators (see Ref. [215]) :

$$C_\ell^{XY} \mapsto C_\ell^{XY} + \mu\, w^{-1} \exp\left(\frac{\ell^2\theta^2}{8\ln 2}\right) \tag{3.45}$$

where $\mu$ is a numerical factor which reads one for the $TT$ spectrum, 2 for the $EE$ and $BB$ spectra, and zero otherwise, and $\sqrt{1/w}$ is the power noise. In fact, CMB cross-correlations have no noise contribution, since the noises from different maps are not correlated (see e.g. Ref. [216]). A more complicated expression has to be considered instead for the auto-spectrum of anisotropic cosmic birefringence (see Ref. [80]):

$$C_\ell^{\alpha\alpha} \mapsto C_\ell^{\alpha\alpha} + \left\{ \sum_{L_1 L_2} \frac{\pi(2L_1+1)(2L_2+1)(C_{L_1}^{EE})^2 e^{-(L_1^2+L_2^2)\theta^2/(8\ln 2)}}{\left[C_{L_1}^{BB} e^{-L_1^2\theta^2/(8\ln 2)} + 2\,w^{-1}\right]\left[C_{L_2}^{EE} e^{-L_2^2\theta^2/(8\ln 2)} + 2\,w^{-1}\right]} \begin{pmatrix} L_1 & \ell & L_2 \\ 2 & 0 & -2 \end{pmatrix}^2 \right\}^{-1} . \tag{3.46}$$

By substituting Eqs. (3.45)-(3.46) in Eq. (3.42) and by multiplying the overall result by the **fraction of the sky** $f_{\rm sky}$ to which the experiment is sensitive, we can estimate the SNR according for a LiteBIRD-like experiment to the following instrumental parameters (see Refs. [83, 217]):

$$\theta = 30', \qquad w^{-1/2} = 4.5\,\mu\text{K-arcmin}, \qquad f_{\rm sky} = 0.7. \tag{3.47}$$

From Tab. 3.3 we can see that the bispectra involving a single $B$-mode in the polarization pattern, i.e. $\langle \delta\alpha\, TB \rangle$

**Table 3.3:** Numerical estimation of the SNR for the birefringent bispectra (including the LiteBIRD satellite instrumental noise) in the purely anisotropic regime.

| **Bispectrum** | **SNR** (if $C_{\alpha\alpha} \sim 6 \times 10^{-5}$) |
|---|---|
| $\delta\alpha\, TE$ | $\approx 0.0661$ |
| $\delta\alpha\, TB$ | $\approx 4.0635$ |
| $\delta\alpha\, EB$ | $\approx 7.5658$ |
| $\delta\alpha\, EE$ | $\approx 0.0543$ |
| $\delta\alpha\, BB$ | $\approx 0.0004$ |

and $\langle \delta\alpha\, EB \rangle$, are the more promising for what concerns a possible future detection. This is due to the form assumed by the covariance matrix in Eq. (3.43) for these two specific cases, which, strictly speaking, once inverted results in a matrix of fractions with denominators that are smaller than in the case of the other bispectra. The reason for that is the dependence of the covariance matrix elements on quantities like $C_\ell^{TB}$ or $C_\ell^{EB}$ that are null by hypothesis (and also the fact that the covariance matrix will contain terms proportional to the power spectrum of the $B$ modes). Moreover, it is not surprising that SNR for the $\langle \delta\alpha\, EB \rangle$ bispectrum is larger than that for the $\langle \delta\alpha\, TB \rangle$, since in the former case the covariance matrix elements depend on the CMB temperature power spectrum, whose amplitude is estimated to be larger than that of $C_\ell^{EE}$. These results and considerations further motivate our choice of performing a Fisher forecast for a LiteBIRD-like experiment, that is a $B$-mode devoted satellite mission. They also justify our choice of analyzing the SNR for specific combinations of the various fields involved in the observations, since, according to our results, we do expect that the bispectra involving a single $B$-mode would provide the dominant contribution to the total SNR. The results shown in Tab. 3.3 are indeed quite promising, showing in principle that the constraints they could provide are comparable to the present limits we have on anisotropic birefringence. A few further comments are in order here. We have checked that the SNR remains very small either if we start from $l_{\min} = 10$ up to $l_{\max} = 200$, or in the case where we stop at $l_{\max} = 10$, which is indeed telling us that the main contribution to the SNR comes from squeezed configurations where, e.g. $\ell_1 \ll \ell_2 \sim \ell_3$. Also, as mentioned in Sec. 3.2, we accounted only for the recombination epoch as the time of polarization generation. We do expect that adding the reionization epoch as well would not dramatically modify the SNR. Indeed we have verified that this is the case, in the simplifying assumption that the power spectrum of anisotropic birefringence from the reionization epoch is scale invariant and with the same amplitude as that adopted in (3.19). For example, in Ref. [85], we have found that for the $\langle \alpha EB \rangle$ bispectrum the SNR slightly increases to $\mathrm{SNR}_{\alpha EB}^{\bar{\alpha}=0} = 8.0334$.

# 4
# Conclusions

In this thesis we have discussed how it is possible to probe fundamental physics through cosmological observables. In particular, we have shown that the CMB polarization anisotropies are sensitive to parity-violating extensions of Maxwell's electromagnetism, able to induce a rotation of the linear polarization plane of photons during propagation, causing cosmic birefringence. After having reviewed the basic notions of modern electrodynamics, general relativity and of the CMB physics in Chap. 1, we have focused on showing the impact of the isotropic and anisotropic birefringence on the Stokes parameters of the cosmic microwave background radiation. Indeed, in Chap. 2, we have reviewed Refs. [77, 171], and performed a tomographic analysis of cosmic birefringence, by studying the dynamics of an axion-like field $\chi$: We have solved the equation of motion of the background term $\bar{\chi}(\tau)$ and of its inhomogeneous fluctuation $\delta\chi^{(1)}(\tau, \mathbf{x})$ for several values of the axion mass $m_\chi$, that enter in the axion potential as described by Eq. (2.89). Our approach has allowed us to clarify how cosmic birefringence affects the CMB observables. For instance, we have found that different values for the axion mass imprint very different signatures in the birefringence signal, making the tomographic approach to ACB a powerful probe of the axion field underlying physics, in analogy with what has been found in previous studies in literature, e.g. in Refs. [74–76], but where the analysis was restricted just to the isotropic case of cosmic birefringence. A relevant message we want to convey is that our tomographic treatment of ACB is able to make manifest unique features of the birefringence anisotropies with respect to the purely isotropic case: indeed, we have shown that, although a large axion mass prevents the possibility to have isotropic cosmic birefringence, this behavior is not mimicked by the anisotropic counterpart. As can be seen by comparing Fig. 2.2 with Fig. 2.3, this is due to the fact the larger the axion mass is, larger the amplitudes of the $C_\ell^{\alpha\alpha|(1)}$, $C_\ell^{\alpha T|(1)}$ and $C_\ell^{\alpha E|(1)}$ are. This fact has a very intriguing consequence: since

CMB observations have constrained the ACB amplitude below a certain threshold, it follows that the axion mass can be constrained too up to an upper value. This is a clear example that ACB can encode additional and complementary information, relevant also for the isotropic counterpart. Another important result we have found in Chap. 2 is that for low multipoles and for sufficiently small values of the axion mass, the reionization contribution to anisotropic cosmic birefringence is higher with respect to the recombination one, as can be seen by looking at Fig. 2.6. For this reason, a future development of our research could be trying to use the signal coming from reionization encoded in ACB as a probe of the axion parameters, as it has been already done in the purely isotropic regime. All the aforementioned results were possible thanks to the generalization of the standard formalism which describes how anisotropic cosmic birefringence affects the CMB angular power spectra, by including the reionization contribution. Indeed, the general formulas collected in Eqs. (2.150)-(2.152) and Eqs. (2.155)-(2.156) can be seen as a generalized version of the well known equations for ACB that it is possible to find e.g. in Ref. [79]. Thanks to our modified version of the Boltzmann code `CLASS`, we have been also able to numerically compute the rotated CMB spectra, which is one of the observables where we expect to test our theoretical predictions. However, this was just a part of the story, since relevant information can be found also in higher-order correlation functions with respect to the merely two-point ones. In fact, we have then moved forward, and in Chap. 3 we have reviewed Ref. [85]: in particular we have shown what is the relation between the observed angular correlation functions involving the anisotropic birefringence angle and the CMB maps, and their unrotated counterparts. The observed angular power spectra are simply obtained by a rotation of the primordial ones, but this simple relation cannot be extended to higher-order correlators. Indeed we have computed the angular three-point functions and the corresponding reduced bispectra: we have found that even by assuming that $\delta\alpha$, $T$ and the unrotated $E$ and $B$ fields are all Gaussian random fields, and although any two-point cross-correlation $C_\ell^{\alpha X}$ (with $X = T, E, B$) is taken to be zero, there exist non-vanishing parity-breaking bispectra. Moreover, from the results shown in Figs. 3.1-3.2, it is possible to see that there are non-vanishing contributions also in a purely anisotropic regime. We have also estimated the signal-to-noise ratio for the birefringent bispectra, showing that a future LiteBIRD-like experiment could be eventually able to detect the signals encoded in the $\langle \delta\alpha TB \rangle$ and $\langle \delta\alpha EB \rangle$ bispectra. To conclude, cosmic birefringence is certainly a topic which has become more and more relevant in cosmology, especially thanks to the hints of detection coming from the latest analysis of the *Planck* data: this was one of the motivations for the theoretical works on which this thesis is based. The main goal of such a dissertation was to show the robust potentialities of this topic as a probe of parity-violating signatures in the Universe, even if the current cosmological paradigm is robust too. However, paraphrasing the famous epistemologist Thomas Kuhn, stressing a paradigm is the best way to strengthen it, because, if it survived, it will be because it has been subjected to critical thinking and not to the acceptance of dogmas. We really think that, despite its limitation, the content of this thesis was devoted to this holy purpose.

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