THE BRAUER-SIEGEL RATIO FOR PRIME CYCLOTOMIC FIELDS

NEELAM KANDHIL, ALESSANDRO LANGUASCO AND PIETER MOREE

ABSTRACT. The Brauer-Siegel theorem concerns the size of the product of the class number and the regulator of a number field K. We derive bounds for this product in case K is a prime cyclotomic field, distinguishing between whether there is a Siegel zero or not. In particular, we make a result of Tatuzawa (1953) more explicit. Our theoretical advancements are complemented by numerical illustrations that are consistent with our findings.

1. INTRODUCTION

Let K be a number field, $\mathcal{O}$ its ring of integers and s a complex variable. For $\Re(s) > 1$ the *Dedekind zeta function* is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where $\mathfrak{a}$ ranges over the non-zero ideals in $\mathcal{O}$, $\mathfrak{p}$ over the prime ideals in $\mathcal{O}$, and $N\mathfrak{a}$ denotes the *absolute norm* of $\mathfrak{a}$, that is the index of $\mathfrak{a}$ in $\mathcal{O}$. Note that $\zeta_{\mathbb{Q}}(s)$ is merely the Riemann zeta-function $\zeta(s)$. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and that it has a simple pole at $s = 1$. It has residue

$$\mathcal{R}(K) = \frac{2^{r_1}(2\pi)^{r_2} h(K) \operatorname{Reg}(K)}{\omega_K \sqrt{d_K}}, \tag{1}$$

where r_1 and r_2 denote the number of real, respectively complex embeddings of K, d_K the absolute value of the discriminant, ω_K the roots of unity in K, $\operatorname{Reg}(K)$ its regulator and $h(K)$ its class number. Formula (1) is called the *analytic class number formula*. In it the only mysterious quantity is $h(K)\operatorname{Reg}(K)$ and one could hope to get bounds on it via estimates of $\mathcal{R}(K)$. One has for example, under the Generalized Riemann Hypothesis and the strong Artin conjecture for $\zeta_K(s)/\zeta(s)$,

$$\Big(\frac{1}{2} + o(1)\Big)\frac{\zeta(n)}{e^{\gamma}\log\log d_K} \le \mathcal{R}(K) \le (2 + o(1))^{n-1}\big(e^{\gamma}\log\log d_K\big)^{n-1}, \tag{2}$$

where n denotes the degree of K, see [2, Section 3], and γ is Euler's constant. In 2015, Louboutin [21] (see also [20]) gave a weaker, but unconditional, bound for $\mathcal{R}(K)$. More precisely, he demonstrated that for any given $\varepsilon > 0$ and $n_0 \ge 5$, there exists a number ρ_0 such that for all number fields K of degrees $n \ge n_0$ and $d_K \ge \rho_0^n$, we have

$$\mathcal{R}(K) \le \frac{3}{\sqrt{n}}\Big(\frac{c(1+\varepsilon)e^{\gamma+\sqrt{6/n}}\log d_K}{n}\Big)^{n-1}, \tag{3}$$

where $c = \frac{1}{2}\big(1 - \frac{1}{\sqrt{5}}\big)$.

The following classical result, see, e.g., Lang [11, Ch. 16], provides some information on the size of $h(K)\operatorname{Reg}(K)$.

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Classical Theorem 1 (Brauer-Siegel). *Suppose K runs through a sequence of number fields that are normal over $\mathbb{Q}$ and satisfy $[K:\mathbb{Q}]/\log d_K \to 0$. Then*

$$\frac{\log(h(K)\operatorname{Reg}(K))}{\log\sqrt{d_K}} \to 1.$$

This result is very general and so not surprisingly far from optimal in certain specific situations. The goal of this paper is to investigate $h(K)\operatorname{Reg}(K)$ for prime cyclotomic fields $\mathbb{Q}(\zeta_q)$. From now on our focus is exclusively on these fields. In this case the Brauer-Siegel theorem yields

$$\log(h(q)\operatorname{Reg}(q)) \sim \log\sqrt{d_q} \sim \frac{q}{2}\log q, \quad (q\to\infty), \tag{4}$$

since $d_q = q^{q-2}$, where we have shortened $h(\mathbb{Q}(\zeta_q))$ to $h(q)$, $d_{\mathbb{Q}(\zeta_q)}$ to d_q and $\operatorname{Reg}(\mathbb{Q}(\zeta_q))$ to $\operatorname{Reg}(q)$. We will continue to use this notation in the remaining part of this work.

1.1. **Residue of the prime cyclotomic field.** Let $q \ge 3$ be a prime number and $K = \mathbb{Q}(\zeta_q)$ a *prime cyclotomic field*. We have the factorization

$$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s)\prod_{\chi\neq\chi_0} L(s,\chi), \tag{5}$$

where χ runs over the non-principal characters modulo q. Combination of (5) and (1) yields

$$\mathcal{R}(q) = \prod_{\chi\neq\chi_0} L(1,\chi) = \frac{2^{r_1}(2\pi)^{r_2}h(q)\operatorname{Reg}(q)}{\omega_q\sqrt{d_q}}. \tag{6}$$

Many mathematicians tried to exploit the identity in (6) in order to obtain information on the size of the class number $h(q)$.

1.2. **The Brauer-Siegel theorem and the Brauer-Siegel ratio.** In the case of prime cyclotomic fields, Tatuzawa [29, Theorem 3] in 1953 identified a suitable normalization function[1], analogue of the one for the Kummer ratio for the relative class number of the cyclotomic field, see, e.g. [10], as

$$H(q) := 2\sqrt{q}\Big(\frac{q}{2\pi}\Big)^{\frac{q-1}{2}} \tag{7}$$

so that

$$\mathcal{R}(q) = \frac{h(q)\operatorname{Reg}(q)}{H(q)}.$$

In analogy with the terminology used for the relative class number, from now on we will call $\mathcal{R}(q)$ the *Brauer-Siegel ratio* for a prime cyclotomic field, a term that seems to have been introduced by Ulmer [30] in the context of abelian varieties over function fields. In his Theorem 3, Tatuzawa proved that for every positive ε there exists $c(\varepsilon) > 0$ such that

$$\frac{c(\varepsilon)}{q^{\varepsilon}} < \mathcal{R}(q) < (\log q)^c, \tag{8}$$

where $c > 0$ is an absolute constant.

Here, adapting the technique used in Kandhil et al. [10] to study the order of magnitude of the Kummer ratio[2] for the relative class number of prime cyclotomic field, we bound the value of c in (8). We show that c is essentially at most 2 in the most general case; otherwise

[1] There's a typo in the statement of Theorem 3 of [29]; performing the computations and checking with formula (27.13) of Ribenboim [27] one gets $H(q)$ as defined in (7).

[2] This analogy also speaks in favor of the terminology Brauer-Siegel ratio.

it is less than 1. Moreover, we explicitly show the role of the *Siegel zero* (defined in Section 2.2) in both the bounds appearing in (8). In particular, the Siegel zero contribution will be expressed using the *exponential integral function* (defined as $E_1(x) := \int_x^\infty e^{-t}\,dt/t$ for $x > 0$).

Theorem 1. *Let $\ell(q)$ be a function that tends arbitrarily slow and monotonically to infinity as q tends to infinity. There is an effectively computable prime q_0 (possibly depending on ℓ) such that the following statements are true:*
1) If for some $q \ge q_0$ the family of Dirichlet L-series $L(s,\chi)$, with χ any non-principal character modulo q, has no Siegel zero, then

$$\frac{e^{-1.91}}{(\log q)^{1-\xi}} < \prod_{\chi\neq\chi_0} L(1,\chi) < e^{0.56}\,(\log q)^{1-\xi}, \tag{9}$$

for some absolute constant ξ.
2) If for some $q \ge q_0$ the family of Dirichlet L-series $L(s,\chi)$, with χ any non-principal character modulo q, has a Siegel zero β_0 then

$$\frac{e^{-1.91}e^{-E_1(1-\beta_0)}}{(\log q)^2\,\ell(q)} < \prod_{\chi\neq\chi_0} L(1,\chi) < e^{0.56}e^{-E_1(1-\beta_0)}\,(\log q)^2\,\ell(q).$$

The computations carried out in Section 5 for the odd primes up to 10^7 show a remarkable fit between $\mathcal{R}(q) = \prod_{\chi\neq\chi_0} L(1,\chi)$ and $c/(\log q)^{3/4}$, with $c \in (1/5, 2/3)$. In this respect, the scatter plot of the normalized values $\mathcal{R}(q)(\log q)^{3/4}$ presented in Figure 2 is particularly relevant. We think it is possible that the "true" order of magnitude for $\mathcal{R}(q)$ in Theorem 1 might be the one on the left hand side of (9) with $\xi = 1/4$.

Remark 1. All constants in Theorem 1 can be further sharpened by arguing as in Remark 5 below.

Remark 2. It is natural to expect a further improvement on assuming the validity of the Riemann Hypothesis for the family of Dirichlet L-series $L(s,\chi)$, with χ any non-principal character. However, this only leads to a minor improvement of the constants involved in the statement of Part 1 of Theorem 1 (more precisely, 0.56 and -1.91 can be replaced by -0.14, respectively -2.60). For this reason we omit the proof.

Remark 3. It is a consequence of Theorem 1 that asymptotically the upper bounds (2) and (3) are quite weak for prime cyclotomic fields. However, the *lower* bound in (2) seems reasonable sharp in this case.

Remark 4. We have $1 \ll E_1(1-\beta_0) < \varepsilon\log q + c(\varepsilon)$, where $c(\varepsilon)$ is ineffective. To prove this, we start by recalling that

$$E_1(x) = -\boldsymbol{\gamma} - \log x + \int_0^x (1-e^{-t})\frac{dt}{t} = -\boldsymbol{\gamma} - \log x - \sum_{k=1}^\infty \frac{(-x)^k}{(k!)k} \quad (x>0).$$

Since $0 < 1 - e^{-x} < x$, we infer from the first equality that

$$-\boldsymbol{\gamma} - \log x < E_1(x) < -\boldsymbol{\gamma} - \log x + x \quad (x>0).$$

On using that for every $\varepsilon > 0$ there exists a constant $c_1(\varepsilon)$ such that $\beta_0 < 1 - c_1(\varepsilon)q^{-\varepsilon}$, the bounds for $E_1(x)$ lead to $1 \ll E_1(1-\beta_0) < \varepsilon\log q + c(\varepsilon)$, where $c(\varepsilon)$ is ineffective. Using the weaker, but with an effective constant, estimate $\beta_0 < 1 - cq^{-1/2}(\log q)^{-2}$ we obtain that $1 \ll E_1(1-\beta_0) < \frac{1}{2}\log q + 2\log\log q + c_1$, where $c_1 > 0$ is an effective constant.

Clearly Theorem 1 has implications for the asymptotic estimates of $h(q)\operatorname{Reg}(q)$; for example, Part 2 and the estimates of Remark 4 yield

$$\log(h(q)\operatorname{Reg}(q)) = \frac{q}{2}\log q - \frac{q}{2}\log(2\pi) + O(\log\log q) \quad (q\to\infty),$$

improving on the Brauer-Siegel implication (4).

The paper is organized as follows: In Section 2 we recall results we need (mainly from prime number theory and in Section 3 we prove a useful lemma about a sum over prime powers in an arithmetic progression modulo a prime $q \geq 3$. Section 4 is devoted to the proof of Theorem 1. Section 5 describes an efficient algorithm to compute $\mathcal{R}(q)$ and some graphical representations regarding its distribution. Section 6 establishes some analogies between the prime sums over characters connected with $\log\mathcal{R}(q)$ and the ones for the Mertens' constants in arithmetic progressions.

2. Preliminaries

2.1. **Notations.** Throughout this article, we will use the following standard notations:

$$\pi(t) = \sum_{p\leq t} 1, \qquad \pi(t;d,b) = \sum_{\substack{p\leq t\\ p\equiv b \pmod d}} 1,$$

$$\theta(t;d,b) = \sum_{\substack{p\leq t\\ p\equiv b \pmod d}} \log p \quad \text{and} \quad \psi(t;d,b) = \sum_{\substack{n\leq t\\ n\equiv b \pmod d}} \Lambda(n),$$

where Λ denotes the von Mangoldt function and b and d are coprime.

2.2. **Siegel zeros.** Let $K \neq \mathbb{Q}$ be an algebraic number field having d_K as its absolute discriminant over the rational numbers. Then, see Stark [28, Lemma 3], $\zeta_K(s)$ has at most one zero in the region in the complex plane determined by

$$\Re(s) \geq 1 - \frac{1}{4\log d_K}, \qquad |\Im(s)| \leq \frac{1}{4\log d_K}.$$

If such a zero exists, it is real, simple and often called *Siegel zero*. When $K=\mathbb{Q}(\zeta_q)$, using (5) it is easy to see that the Siegel zero is attached to the family of Dirichlet L-series (mod q). In this case, the Dirichlet character χ such that $L(s,\chi)$ has the Siegel zero is called the *exceptional character* and it is a well known fact that it is quadratic.

The presence of a Siegel zero strongly influences the distribution of the primes in the progressions modulo q. We present two classical results in this direction we will make use of.

Classical Theorem 2 (Brun-Titchmarsh[3]). *Let $x, y > 0$ and a, q be positive integers such that $(a,q)=1$. Then*

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\varphi(q)\log(y/q)}, \tag{10}$$

for all $y > q$.

In particular, a key role is played by the constant 2 present in (10); from the works of Motohashi [25], Friedlander-Iwaniec [6], Ramaré [26, Theorems 6.5-6.6] and Maynard [23], it is well known that replacing such a constant with any value less than 2 is equivalent with assuming that there does not exist a Siegel zero for $\prod_{\chi\neq\chi_0} L(s,\chi)$.

[3]For a proof, see, e.g., Montgomery-Vaughan [24, Theorem 2].

In Part 1 of Theorem 1 we will in fact assume that $\prod_{\chi\neq\chi_0} L(s,\chi)$ has no Siegel zero and we will make use of the following result by Maynard [23, Proposition 3.5, second part].

Theorem (Maynard). *There is a fixed constant $\varepsilon > 0$ such that there exists an effectively computable constant q_1, such that if the set of the non-principal Dirichlet L-functions* (mod q)*, for $q > q_1$, does not have a Siegel zero then for $x \geq q^{7.999}$ and for any b co-prime with q we have that*

$$\Big|\psi(x;q,b) - \frac{x}{\varphi(q)}\Big| < \frac{(1-\varepsilon)x}{\varphi(q)}. \tag{11}$$

From now on $\log_2 x$ denotes $\log\log x$. We also record the following theorem of Dusart [5, Theorem 5.5] which plays a crucial role in the proof of our main result.

Theorem (Dusart). *Let $\mathcal{M}$ be the Meissel-Mertens constant given by the infinite sum*

$$\mathcal{M} := \boldsymbol{\gamma} + \sum_p \Big(\log\Big(1-\frac{1}{p}\Big) + \frac{1}{p}\Big). \tag{12}$$

Then we have for $x \geq 2278383$,

$$\Big|\sum_{p\leq x}\frac{1}{p} - \log_2 x - \mathcal{M}\Big| \leq \frac{0.2}{(\log x)^3}. \tag{13}$$

One has $\mathcal{M} \approx 0.261497212847643$, for more decimals see `https://oeis.org/A077761`.

3. A useful lemma

For q a prime and $b \in \{1,\ldots,q-1\}$, let

$$S_q(b) := \sum_{\substack{m\geq 2\\ p^m\equiv b \ (\mathrm{mod}\ q)}} \frac{1}{mp^m}. \tag{14}$$

This quantity will play a role in the proof of Theorem 1.

We will need the following lemma, the proof of which is similar to a well known result by Ankeny and Chowla, see the estimate of C_4 in [1].

Lemma 1. *There exists a constant $c > 0$ such that for every odd prime q and for every $b \in \{1,\ldots,q-1\}$ we have*

$$S_q(b) < \frac{1}{q}\Big(\frac{43}{13} - \frac{18}{13}\zeta(3) + \frac{c}{\log q}\Big), \tag{15}$$

where $S_q(b)$ is defined in (14)*.*

Proof. We split the sum in (14) into three subsums according to whether $p^m \leq q(\log q)^2$, $q(\log q)^2 < p^m < q^2$ or $p^m > q^2$. The contribution of the sums over the first range to the final estimate will be less than c_1/q, with a constant $c_1 > 0$ that will be explicitly determined, while the others contribute $\ll 1/(q\log q)$.

We first consider the case $p^m > q^2$. For any prime $p > q \geq 3$ we have

$$\sum_{m\geq 2}\frac{1}{mp^m} \leq \frac{1}{2}\frac{1}{p(p-1)} < \frac{1}{p^2}.$$

Moreover, for $p \le q$, the condition $p^m > q^2$ implies $m \ge 3$ and hence

$$\sum_{\substack{m\ge 2;\, p\le q\\ p^m>q^2}} \frac{1}{mp^m} \le \frac{1}{3}\sum_{\substack{m\ge 3;\, p\le q\\ p^m>q^2}} \frac{1}{p^m} = \frac{1}{3p^3}\sum_{j\ge 0}\frac{1}{p^j} \le \frac{1}{3q^2}\frac{p}{p-1} < \frac{1}{q^2}.$$

Combining these estimates, we have

$$\sum_{\substack{m\ge 2;\ p^m>q^2\\ p^m\equiv a \ (\mathrm{mod}\ q)}} \frac{1}{mp^m} \le \sum_{p>q}\sum_{m\ge 2}\frac{1}{mp^m} + \sum_{p\le q}\sum_{\substack{m\ge 2\\ p^m>q^2}}\frac{1}{mp^m} < \sum_{p>q}\frac{1}{p^2} + \sum_{p\le q}\frac{1}{q^2}$$

$$= 2\int_q^\infty \frac{\pi(t)}{t^3}dt \ll \int_q^\infty \frac{dt}{t^2\log t} \ll \frac{1}{q\log q}, \tag{16}$$

where we used the partial summation formula and the Chebyshev bound in the weaker form $\pi(t) \ll t/\log t$. From the proof of [8, Lemma 1], we have

$$\sum_{\substack{m\ge 2;\ q(\log q)^2<p^m<q^2\\ p^m\equiv a \ (\mathrm{mod}\ q)}} \frac{1}{mp^m} \le \sum_{m=2}^{\lfloor 4\log q\rfloor}\frac{1}{m}\frac{m}{q(\log q)^2} \ll \frac{1}{q\log q}.$$

We proceed now analogously to the proof of [8, Lemma 2]. Setting

$$\alpha(m) := \frac{1}{2}(m^2-m), \quad \beta(m) := \frac{1}{2}(m^2+m)-1$$

and

$$\kappa := \frac{43}{13} - \frac{18}{13}\zeta(3) = 1.64330\ldots, \tag{17}$$

we obtain

$$\sum_{\substack{m\ge 2;\, p^m\le q(\log q)^2\\ p^m\equiv a \ (\mathrm{mod}\ q)}} \frac{1}{mp^m} \le \sum_{m\ge 2}\frac{1}{m}\sum_{r=\alpha(m)}^{\beta(m)}\frac{1}{rq+a} \le \sum_{m\ge 2}\frac{1}{m}\sum_{r=\alpha(m)}^{\beta(m)}\frac{1}{rq}$$

$$\le \frac{1}{q}\Big(\frac{3}{4} + \sum_{m\ge 3}\frac{1}{m}\sum_{r=\alpha(m)}^{\beta(m)}\frac{1}{r}\Big) \le \frac{\kappa}{q}, \tag{18}$$

where we have isolated the contribution of the terms corresponding to $m = 2$ in the penultimate step and the final estimate follows from the proof of [10, Lemma 1]. Now (15) follows on combining (16)-(18). □

Remark 5. By isolating more initial terms of the sum over m in equation (18) the value of the leading constant κ in Lemma 1 can be reduced to 1.601. Directly computing the involved sum for $m \le 5000$, would lead to a value > 1.599688, and so the upper bound 1.601 is almost optimal (cf. Remark 1 in [10] for the treatment of a similar case).

4. Proof of Theorem 1

Using the Euler product for $L(1,\chi)$ with $\chi \ne \chi_0$, and Taylor's formula for $\log(1-u)$, we obtain

$$\log \mathcal{R}(q) = -\sum_{\chi\ne\chi_0}\sum_{p}\log\Big(1-\frac{\chi(p)}{p}\Big) = \sum_{\chi\ne\chi_0}\sum_{p}\sum_{m\ge 1}\frac{\chi(p^m)}{mp^m} = \Sigma_1 + \Sigma_2, \tag{19}$$

say, where Σ_1 is the contribution of the primes ($m = 1$) and Σ_2 that of the prime powers ($m \geq 2$).

We first estimate Σ_2. Using the orthogonality property of the Dirichlet characters, i.e.,

$$\frac{1}{q-1} \sum_{\chi \bmod q} \chi(a)\overline{\chi}(b) = \begin{cases} 1, & b \equiv a \pmod q, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\Sigma_2 = (q-1) \sum_{\substack{m \geq 2 \\ p^m \equiv 1 \pmod q}} \frac{1}{mp^m} - \sum_{\substack{m \geq 2 \\ p \neq q}} \frac{1}{mp^m}.$$

Recalling (14), it is easy to see that

$$\sum_{\substack{m \geq 2 \\ p \neq q}} \frac{1}{mp^m} = \sum_{b=1}^{q-1} S_q(b),$$

and hence

$$-\sum_{b=1}^{q-1} S_q(b) < \Sigma_2 < (q-1)S_q(1).$$

On invoking Lemma 1 we then obtain

$$|\Sigma_2| < \kappa + \frac{c}{\log q}, \tag{20}$$

where c is a positive constant and κ is defined in (17).

We now proceed to define some quantities that will be useful later to estimate Σ_1. For any $b \in \{1, \ldots, q-1\}$ and $x > 0$ let

$$S_q(b, x) := \sum_{\substack{p \leq x \\ p \equiv b \pmod q}} \frac{1}{p} \quad \text{and} \quad S(x) := \sum_{\substack{p \leq x \\ p \neq q}} \frac{1}{p}. \tag{21}$$

Using again the orthogonality of the Dirichlet characters, for any $x > 0$ we have

$$\sum_{\chi \neq \chi_0} \sum_{p \leq x} \frac{\chi(p)}{p} = (q-1)S_q(1, x) - S(x). \tag{22}$$

As a consequence we obtain

$$\Sigma_1 = \lim_{x \to \infty} \big((q-1)S_q(1, x) - S(x)\big).$$

We begin by estimating $S(x)$, followed by estimating $S_q(1, x)$ (which will bring the possible Siegel zero into play). From now on we will assume that q is a sufficiently large prime. Substituting $x = x_1 = q^{\ell(q)}$ in (13), we obtain

$$S(x_1) \geq \log_2 q + \log \ell(q) + 0.261497 + \frac{1}{\ell(q) \log q} - \frac{1}{q} \tag{23}$$

and

$$S(x_1) \leq \log_2 q + \log \ell(q) + 0.261498 + \frac{1}{\log q}. \tag{24}$$

We will use (20) and (23)-(24) in the proofs of both parts of Theorem 1.

4.1. **Proof of Theorem 1, Part 2.** We first prove Part 2. The starting point is (19). We split the prime sum Σ_1 in three subsums S_1, S_2, S_3 defined according to whether $p \le x_1$, $x_1 < p \le x_2$ or $p \ge x_2$, with $x_2 = e^q$ and $x_1 = q^{\ell(q)}$.

We start by estimating S_1: recalling [10, eq. (26)-(27)] and using (21), we obtain

$$(q-1)S_q(1,x) < 2\Big(\log_2\Big(\frac{x}{q}\Big) + C_1 + \frac{1}{\log q}\Big), \tag{25}$$

where $C_1 = -0.4152617906$ and $x \ge q^2$. Combining (22), (23)-(24) and (25) we have

$$\begin{aligned} S_1 := \sum_{\chi\ne\chi_0}\sum_{p\le x_1}\frac{\chi(p)}{p} &\le \log_2 q + \log\ell(q) + 2C_1 - 0.261497 + \frac{2}{\log q} \\ &< \log_2 q + \log\ell(q) - 1.09202 + \frac{2}{\log q} \end{aligned} \tag{26}$$

and

$$S_1 > -\log_2 q - \log\ell(q) - 0.261498 - \frac{1}{\log q}. \tag{27}$$

We will now proceed to estimate S_3. By orthogonality and the partial summation formula, we have

$$\begin{aligned} S_3 = \sum_{\chi\ne\chi_0}\sum_{p\ge x_2}\frac{\chi(p)}{p} &= (q-1)\sum_{\substack{p\ge x_2\\ p\equiv 1 \pmod q}}\frac{1}{p} - \sum_{\substack{p\ge x_2\\ p\ne q}}\frac{1}{p} \\ &= \frac{1}{q} + \lim_{y\to\infty}\frac{((q-1)\pi(y;q,1)-\pi(y))}{y} - \frac{(q-1)\pi(x_2;q,1)-\pi(x_2)}{x_2} \\ &\quad + \int_{x_2}^{\infty}\frac{(q-1)\pi(u;q,1)-\pi(u)}{u^2}du \ll q^2 e^{-c_1\sqrt{q}}, \end{aligned} \tag{28}$$

where $c_1 > 0$ is an absolute constant. In the final estimate, we have used both the prime number theorem and the Siegel-Walfisz theorem.

It remains to estimate S_2. Recall now (see, e.g., [4, Ch. 19]) that if χ is a non-principal character modulo q and $2 \le T \le x$, then

$$\theta(x,\chi) := \sum_{p\le x}\chi(p)\log p = -\delta_{\beta_0}\frac{x^{\beta_0}}{\beta_0} - \sideset{}{'}\sum_{|\gamma|\le T}\frac{x^{\rho}}{\rho} + O\Big(\frac{x(\log qx)^2}{T} + \sqrt{x}\Big), \tag{29}$$

where $\delta_{\beta_0} = 1$ if the Siegel zero β_0 exists and is zero otherwise, and $\sum'$ is the sum over all non-trivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$, with the exception of β_0 and its symmetric zero $1-\beta_0$.

By the partial summation formula and (29) with $T = q^4$, we have

$$\begin{aligned} S_2 := \sum_{\chi\ne\chi_0}\sum_{x_1<p\le x_2}\frac{\chi(p)}{p} &= \sum_{\chi\ne\chi_0}\Big(\frac{\theta(x_2,\chi)}{x_2\log x_2} - \frac{\theta(x_1,\chi)}{x_1\log x_1} + \int_{x_1}^{x_2}\theta(u,\chi)\frac{1+\log u}{(u\log u)^2}du\Big) \\ &= -\delta_{\beta_0}\int_{x_1}^{x_2}\frac{u^{\beta_0-2}}{\log u}du - \int_{x_1}^{x_2}\Big(\sum_{\chi\ne\chi_0}\sideset{}{'}\sum_{|\gamma|\le q^4}u^{\rho-2}\Big)\frac{du}{\log u} + (q-1)E_q, \end{aligned} \tag{30}$$

and

$$E_q \ll \int_{x_1}^{x_2}\Big(\frac{(\log qu)^2}{q^4 u} + \frac{1}{u^{3/2}}\Big)\frac{du}{\log u} \ll \frac{1}{q^2}.$$

By using [22, Lemmas 7 and 8][4], we obtain

$$\int_{x_1}^{x_2} \Big(\sum_{\chi\neq\chi_0}\sum_{|\gamma|\leq q^4}{}' u^{\rho-2}\Big)\frac{du}{\log u} \ll \frac{1}{\ell(q)}. \tag{31}$$

In this case we have that $\delta_{\beta_0} = 1$ in (30); we now proceed to evaluate the term depending on β_0. A direct computation using that $\log x_2 = q$ gives

$$\int_{x_1}^{x_2} \frac{u^{\beta_0-2}}{\log u}du = \int_{\log x_1}^{\log x_2} \frac{dt}{te^{(1-\beta_0)t}} = E_1(1-\beta_0) - \int_{1-\beta_0}^{(1-\beta_0)\log x_1} \frac{dt}{te^t} - E_1(q(1-\beta_0)),$$

where $E_1(u)$ denotes the exponential integral function. Recalling that $x_1 = q^{\ell(q)}$, where $\ell(q)$ tends to infinity arbitrarily slowly and monotonically as q tends to infinity, we have

$$\int_{1-\beta_0}^{(1-\beta_0)\log x_1} \frac{dt}{te^t} \leq \log_2 x_1 = \log_2 q + \log \ell(q) \quad \text{and} \quad E_1(q(1-\beta_0)) \ll \frac{1}{q}. \tag{32}$$

Inserting (31)-(32) into (30), we finally get

$$|S_2 + E_1(1-\beta_0)| \leq \log_2 q + \log \ell(q) + o(1). \tag{33}$$

Combining (26)-(28) and (33), in this case we obtain

$$\Sigma_1 + E_1(1-\beta_0) < 2\log_2 q + 2\log \ell(q) - 1.0920 \tag{34}$$

and

$$\Sigma_1 + E_1(1-\beta_0) > -2\log_2 q - 2\log \ell(q) - 0.2615. \tag{35}$$

Part 2 of Theorem 1 now follows on combining (19), (20) and (34)-(35) (recall that Σ_2 is bounded in (20)).

4.2. **Proof of Theorem 1, Part 1.** We proceed now to prove Part 1 of Theorem 1.

The quantities S_1, S_2, S_3 are the same ones defined into Section 4.1. We first remark that for S_3 we can re-use eq. (28). We now estimate S_2. In this case $\delta_{\beta_0} = 0$ and, arguing as in (30)-(31), we have

$$S_2 \ll \frac{1}{\ell(q)}. \tag{36}$$

We now estimate S_1: since $\delta_{\beta_0} = 0$, we can use a sharper version of the Brun-Titchmarsh theorem. In particular, we can use (11) with $\varepsilon = 2\xi$. Since $\theta(x; q, b) = \psi(x; q, b) + O(\sqrt{x})$, we conclude that, for $x \geq q^{7.999}$ and b coprime with q,

$$\theta(x; q, b) < 2(1-\xi)\frac{x}{\varphi(q)} + C\sqrt{x}, \tag{37}$$

where $C > 0$ is a suitable constant. Using (37) we can replace (25) with

$$(q-1)S_q(1, x) < 2(1-\xi)\log_2 x + 2C_1 + \frac{c}{\log q}, \tag{38}$$

for $x > q^8$, where $c > 0$ is an effective constant.

[4]Note that [22, Lemma 7] holds for every T and x_1 such that $\lim_{q\to\infty} \log(qT)/\log x_1 = 0$. This allows us to choose $T = q^4$ and $x_1 = q^{\ell(q)}$, where $\ell(q)$ tends to infinity arbitrarily slowly and monotonically as q tends to infinity. The final error term in [22, Lemma 8] is then $\ll 1/\ell(q) = o(1)$, as q tends to infinity.

A way to prove (38) for $x > q^8$ is the following. By the partial summation formula and using (37) we find

$$
\begin{aligned}
(q-1)\sum_{\substack{kq<p\le x\\ p\equiv 1 \ (\mathrm{mod}\ q)}}\frac{1}{p} &= (q-1)\Big(\frac{\theta(x;q,1)}{x\log x}-\frac{\theta(kq;q,1)}{kq\log(kq)}+\int_{kq}^{x}\theta(u;q,1)\frac{1+\log u}{(u\log u)^2}du\Big)\\
&< 2(1-\xi)\Big(\frac{1}{\log x}+\int_{kq}^{x}\Big(1+\frac{1}{\log u}\Big)\frac{du}{u\log u}\Big)+\frac{C}{q}\\
&\le 2(1-\xi)\big(\log_2 x-\log_2(kq)\big)+\frac{c}{\log q}\\
&\le 2(1-\xi)\log_2 x-2\log_2 k+\frac{c}{\log q}, \qquad (39)
\end{aligned}
$$

where $c>0$ is an effective constant. In deriving (39) we also used that $x>q^8$ is equivalent to $\sqrt{x}<x^{3/4}/q^2$. From eq. (25) of [10] we also have

$$
(q-1)\sum_{\substack{p\le kq\\ p\equiv 1 \ (\mathrm{mod}\ q)}}\frac{1}{p}\le\sum_{j=1}^{(k-1)/2}\frac{q-1}{2jq-1}<\frac{1}{2}\sum_{j=1}^{(k-1)/2}\frac{1}{j}=\frac{1}{2}H_{\frac{k-1}{2}}, \qquad (40)
$$

where $H_n := \sum_{j=1}^{n}\frac{1}{j}$ is the n-th harmonic number. Inequality (38) then follows on combining (39)-(40).

Using (36), (38), (23)-(24) and arguing as in (26)-(27), we can replace (34)-(35) with

$$
\Sigma_1 < (1-2\xi)\log_2 q+\log\ell(q)-1.0920 < (1-\xi)\log_2 q-1.0920, \qquad (41)
$$

respectively

$$
\Sigma_1 > -(1-2\xi)\log_2 q-\log\ell(q)-0.2615 > -(1-\xi)\log_2 q-0.2615. \qquad (42)
$$

Now Part 1 of Theorem 1 follows on combining (19), (20) and (41)-(42). □

5. Numerical results

The numerical results were obtained using the Fast Fourier Transform method already presented in [12], see also [15].

In particular, the fundamental formula for the odd Dirichlet characters case was already fully described in [10] and reads

$$
\sum_{\chi(-1)=-1}\log L(1,\chi)=\frac{q-1}{2}\Big(\log\pi-\frac{\log q}{2}\Big)+\sum_{\chi(-1)=-1}\log\Big|\sum_{a=1}^{q-1}\frac{a}{q}\chi(a)\Big|. \qquad (43)
$$

For even χ we used the formula:

$$
L(1,\chi)=2\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\overline{\chi}(a)\log\Big(\Gamma\Big(\frac{a}{q}\Big)\Big),
$$

where Γ denotes Euler's Gamma function and the *Gauß sum* $\tau(\chi) := \sum_{a=1}^{q}\chi(a)\,e(a/q)$, $e(x) := \exp(2\pi i x)$, verifies $|\tau(\chi)| = q^{1/2}$ (see e.g., Cohen [3, proof of Proposition 10.3.5]). Combination of these two formulas gives

$$
\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\log L(1,\chi)=\frac{q-3}{2}\Big(\log 2-\frac{\log q}{2}\Big)+\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\log\Big|\sum_{a=1}^{q-1}\overline{\chi}(a)\log\Big(\Gamma\Big(\frac{a}{q}\Big)\Big)\Big|. \qquad (44)
$$

Hence, $\log \mathcal{R}(q)$ is obtained by summing the quantities in (43)-(44) and $\mathcal{R}(q)$ is computed as $\exp(\log \mathcal{R}(q))$.

An alternative approach can be based on the formula $L(1,\chi) = -\frac{1}{q}\sum_{a=1}^{q-1}\chi(a)\digamma\left(\frac{a}{q}\right)$, where $\digamma(x) = (\Gamma'/\Gamma)(x)$ is the *digamma* function. Very similar computations then lead to

$$\log \mathcal{R}(q) = -(q-2)\log q + \sum_{\chi\neq\chi_0} \log \Bigl|\sum_{a=1}^{q-1} \chi(a)\digamma\Bigl(\frac{a}{q}\Bigr)\Bigr|. \tag{45}$$

In practice, though, it is better to use (43)-(44) because in half of the cases no evaluations of special functions are needed there. Moreover, the $\log\Gamma$ function is directly available in the C programming language. Nevertheless, formula (45) can be useful to double-check our results.

The summation over a in (43)-(44) can be handled using the FFT procedure and we can also embed here a *decimation in frequency strategy*, see, e.g., [10, 12, 15]. The FFT procedure requires $O(q)$ memory positions and the computation of $\log \mathcal{R}(q)$ via (43)-(44) has a computational cost of $O(q\log q)$ arithmetic operations plus the cost of computing $q-1$ values of the $\log\Gamma$ and logarithm functions and products.

For the evaluation of the computational error we refer to Section 6.3 of [10]. More statistics and details on computations regarding $L(1,\chi)$ can be found in [13, 14].

5.1. **Comments on the plots and on the histograms.** The actual values of $\mathcal{R}(q)$ presented in the herewith included plots and histograms were obtained for every odd prime q up to 10^7 using the FFTW [7] software library set to work with the *long double precision* (80 bits). Such results were then collected in some *comma-separated values* (csv) files and then all the plots and the histograms were obtained running on such stored data some suitable designed scripts written using Python (v. 3.11.7) and making use of the packages Pandas (v. 2.2.0) and Matplotlib (v. 3.8.2).

Figure 1 shows the values for $\mathcal{R}(q)$ for q up to 10^7; it is clear that $\mathcal{R}(q)$ essentially behaves as $c/(\log q)^{3/4}$, where $1/5 < c < 2/3$. This is compatible with the estimates in Theorem 1.

In Figure 2 we present their normalized values $\mathcal{R}(q)(\log q)^{3/4}$. Figures 3 shows the histograms of the same quantities.

5.2. **Computing the prime sums over Dirichlet characters.** Here we briefly show how to compute $\log \mathcal{R}(q)$ using prime sums over Dirichlet characters. This procedure is much less efficient than the one that uses the Fast Fourier Transform. However, for small values of q it can be used to double-check the results. Moreover, it also shows a way to independently compute Σ_1 and Σ_2.

Recalling (19), we first split the sum over primes (this is important to improve the convergence speed, see Lemma 2 below) so that

$$\begin{aligned}\sum_p \log\Bigl(1-\frac{\chi(p)}{p}\Bigr) &= \sum_{p\le P}\log\Bigl(1-\frac{\chi(p)}{p}\Bigr) + \sum_{p>P}\sum_{m\ge1}\frac{\chi(p^m)}{mp^m},\\ &= \sum_{p\le P}\log\Bigl(1-\frac{\chi(p)}{p}\Bigr) + \sum_{m\ge1}\frac{1}{m}\sum_{p>P}\frac{\chi^m(p)}{p^m},\end{aligned}$$

where $P = Aq$, A is a fixed positive integer[5] and χ is a non-principal character. Since the principal character is not involved, there are no convergence problems. We also used the multiplicativity of the Dirichlet characters.

[5]In practical computations A, and hence P too, cannot be not too large since to perform this step the whole set of prime numbers up to P must be generated.

Hence

$$\log \mathcal{R}(q) = \sum_{\chi\neq\chi_0}\sum_{p\le P}\log\Big(1-\frac{\chi(p)}{p}\Big) + \sum_{\chi\neq\chi_0}\sum_{m\ge 1}\frac{1}{m}\sum_{p>P}\frac{\chi^m(p)}{p^m} =: r(q,P)+S(q,P), \qquad (46)$$

say. Let $M \geq 2$ be an integer. Splitting the sum over m, we obtain

$$\begin{aligned} S(q,P) &= \sum_{\chi\neq\chi_0}\sum_{m=1}^{M}\frac{1}{m}\sum_{p>P}\frac{\chi^m(p)}{p^m} + \sum_{\chi\neq\chi_0}\sum_{m>M}\frac{1}{m}\sum_{p>P}\frac{\chi^m(p)}{p^m} \\ &=: S(q,P,M)+E_1(q,P,M), \end{aligned} \qquad (47)$$

say. By trivial bounds on χ, it is easy to prove that

$$|E_1(q,P,M)| \le \frac{P(q-1)}{M(M-1)(P-1)P^M}. \qquad (48)$$

Using the Möbius inversion formula, see, e.g., Cohen [3, Proposition 10.1.5], we can write

$$\sum_{p>P}\frac{\chi^m(p)}{p^m} = \sum_{k\ge 1}\frac{\mu(k)}{k}\log\big(L_P(km,\chi^{km})\big),$$

where, for $\Re(s) \geq 1$ and $\chi \neq \chi_0$, we have defined the truncated L-function as

$$L_P(s,\chi) := \prod_{p>P}\Big(1-\frac{\chi(p)}{p^s}\Big)^{-1} = L(s,\chi)\prod_{p\le P}\Big(1-\frac{\chi(p)}{p^s}\Big).$$

Hence

$$S(q,P,M) = \sum_{\chi\neq\chi_0}\sum_{m=1}^{M}\frac{1}{m}\sum_{k\ge 1}\frac{\mu(k)}{k}\log\big(L_P(km,\chi^{km})\big). \qquad (49)$$

Remark that $km = 1$ only if $k = m = 1$. Recalling $\chi \neq \chi_0$, in the previous formula we will never encounter the pole at 1 of the Riemann zeta function.

Let now $K \geq 1$ be an integer. Splitting the sum over k in (49), we have

$$\begin{aligned} S(q,P,M) &= \sum_{\chi\neq\chi_0}\sum_{m=1}^{M}\frac{1}{m}\sum_{k=1}^{K}\frac{\mu(k)}{k}\log\big(L_P(km,\chi^{km})\big) \\ &\quad + \sum_{\chi\neq\chi_0}\sum_{m=1}^{M}\frac{1}{m}\sum_{k>K}\frac{\mu(k)}{k}\log\big(L_P(km,\chi^{km})\big) \\ &=: S(q,P,M,K)+E_2(q,P,M,K), \end{aligned} \qquad (50)$$

say. Combining (46)-(47) and (50) we obtain

$$\log \mathcal{R}(q) = r(q,P)+S(q,P,M,K)+E_1(q,P,M)+E_2(q,P,M,K). \qquad (51)$$

Both $r(q,P)$ and $S(q,P,M,K)$ are finite sums and can be directly computed, while $E_1(q,P,M)$ and $E_2(q,P,M,K)$ must be estimated. For $E_1(q,P,M)$ we will use (48). To estimate $E_2(q,P,M,K)$, we will need the following lemma [17, Lemma 1].

Lemma 2. *Let $\chi \pmod q$ be a Dirichlet character and $n \geq 2$ be an integer. If $P \geq 1$ is an integer then*

$$\big|\log\big(L_P(n,\chi)\big)\big| \le \frac{P^{1-n}}{n-1}.$$

Using this lemma it is easy to see that

$$|E_2(q,P,M,K)| \le \frac{2P(q-1)}{K^2(P-1)(P^K-1)}, \tag{52}$$

where the estimate does not depend on M.

Let now $\Delta > 0$ be an integer. Taking $P = Aq$, by exploiting (48) and (52) one can choose M, K such that $|E_1(q,P,M)| + |E_2(q,P,M,K)| < 10^{-\Delta}$. Taking $P = Aq$, by exploiting (48) and (52) one can choose M, K such that $|E_1(q,P,M)| + |E_2(q,P,M,K)| < 10^{-\Delta}$, where $\Delta > 0$ is any prescribed integer. Hence, by computing $r(q,P) + S(q,P,M,K)$ in (51), one obtains $\log \mathcal{R}(q)$ with an accuracy of (at least) Δ decimals.

We remark that the same idea can be used for Σ_2, since it is enough to let m start from 2 in the analogue of $S(q,P)$. For Σ_1, the algorithm is simpler since there is no sum over m in (47), in $S(q,B,1)$ and in $S(q,B,1,K)$. Hence only $E_2(q,B,1,K)$ is present. Moreover, if one has already computed $\log \mathcal{R}(q)$ with sufficient accuracy using the FFT, then it is enough to obtain Σ_1 in order to have $\Sigma_2 = \log \mathcal{R}(q) - \Sigma_1$ too.

6. Connections and analogies with the Mertens' constants in arithmetic progressions

In this final section we establish some connections and analogies for the prime sum that defines $\log \mathcal{R}(q)$ with the ones involved in the definition of the Mertens and Meissel-Mertens constants in arithmetic progressions: $\mathcal{M}(q,b), \mathcal{B}(q,b), \mathcal{C}(q,b)$, $1 \le b < q$, $(b,q) = 1$, since they can be written with sums over Dirichlet characters that are similar to the one into (19).

Recalling that q is prime and using eq. (1-1) in [18], we have

$$\sum_{\substack{p\le x\\ p\equiv b \pmod q}} \frac{1}{p} = \frac{\log\log x}{q-1} + \mathcal{M}(q,b) + O_q\Big(\frac{1}{\log x}\Big),$$

where $x \to +\infty$. Moreover, a direct computation and Theorem 428 of Hardy-Wright [9] show that

$$(q-1)\mathcal{M}(q,b) = \mathcal{M} - \sum_{p\mid q}\frac{1}{p} + \sum_{\chi\ne\chi_0}\overline{\chi}(b)\sum_{p}\frac{\chi(p)}{p},$$

where $\mathcal{M}$ is the Meissel-Mertens constant defined in (12). From the previous equation, and using (19), it is clear that

$$\Sigma_1 = (q-1)\mathcal{M}(q,1) - \mathcal{M} + \sum_{p\mid q}\frac{1}{p}.$$

Moreover, $\mathcal{R}(q)$ is related with the constant for the Mertens' product in arithmetic progression. In [16, p. 38] it is proved that

$$\prod_{\substack{p\le x\\ p\equiv b \pmod q}} \Big(1-\frac{1}{p}\Big) \sim \frac{\mathcal{C}(q,b)}{(\log x)^{\frac{1}{q-1}}},$$

as q tends to infinity, where $\mathcal{C}(q,b)$ verifies

$$\mathcal{C}(q,b)^{q-1} = e^{-\gamma}\prod_{p}\Big(1-\frac{1}{p}\Big)^{\alpha(p;q,b)},$$

and $\alpha(p;q,b) = q-2$ if $p \equiv b \bmod q$ and $\alpha(p;q,b) = -1$ otherwise. In [17, p. 316] (or [18, eq. (2-4)]) it is also proved that

$$(q-1)\log \mathcal{C}(q,b) = -\boldsymbol{\gamma} + \log\frac{q}{q-1} - \sum_{\chi\neq\chi_0} \overline{\chi}(b) \sum_{m\geq 1} \frac{1}{m} \sum_{p} \frac{\chi(p)}{p^m}.$$

Besides some correction terms, the only difference between the formula for $\log \mathcal{C}(q,1)$ and the one in (19) for $\log \mathcal{R}(q)$ is that $\chi(p^m)$ in (19) must be replaced by $\chi(p)$.

Finally, for $\mathcal{B}(q,b)$ defined as

$$\mathcal{B}(q,b) := \sum_{p\equiv b \bmod q} \Bigl(\log\bigl(1-\frac{1}{p}\bigr) + \frac{1}{p}\Bigr),$$

in [18, eq. (2-3)] it is proved that

$$(q-1)\mathcal{B}(q,b) := \mathcal{B}(q) - \sum_{\chi\neq\chi_0} \overline{\chi}(b) \sum_{m\geq 2} \frac{1}{m} \sum_{p} \frac{\chi(p)}{p^m},$$

where $\mathcal{B}(q) := -\sum_{m\geq 2} \frac{1}{m} \sum_{(p,q)=1} \frac{1}{p^m}$, represents the contribution of the principal character $\chi_0 \pmod q$ and equals

$$\mathcal{B}(q) = \sum_{(p,q)=1} \Bigl(\log\bigl(1-\frac{1}{p}\bigr) + \frac{1}{p}\Bigr) = \mathcal{B} - \sum_{p\mid q} \Bigl(\log\bigl(1-\frac{1}{p}\bigr) + \frac{1}{p}\Bigr),$$

where $\mathcal{B} := \sum_p \bigl(\log(1-\frac{1}{p}) + \frac{1}{p}\bigr) = \mathcal{M} - \boldsymbol{\gamma}$. In this case too, besides some correction terms, the only difference between the formula for $\mathcal{B}(q,1)$ and the one in (19) for Σ_2 is that $\chi(p^m)$ in (19) must be replaced by $\chi(p)$.

The behavior of the prime sum over characters that involves $\chi^m(p) - \chi(p)$ is studied in [19], starting from eq. (6)-(7) there.

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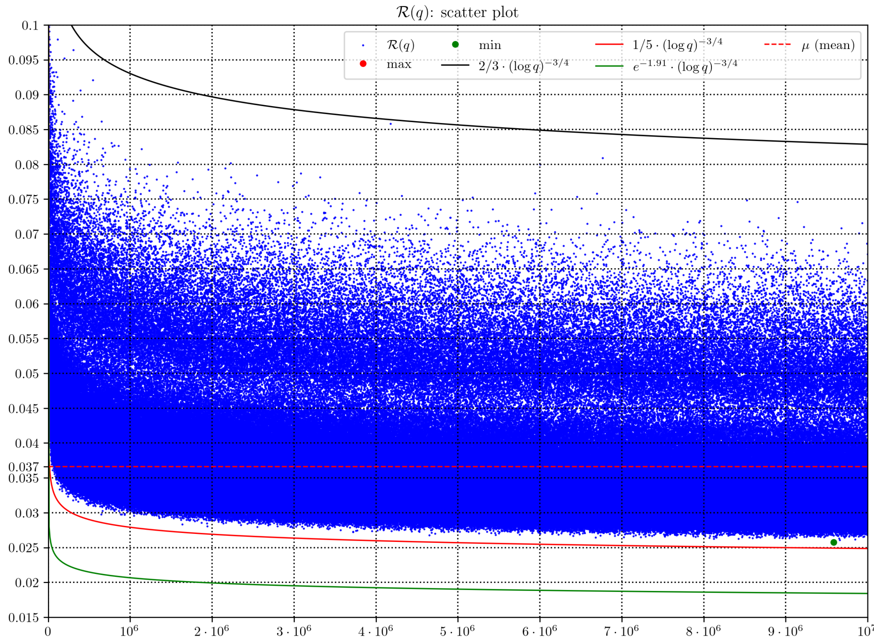

FIGURE 1. The values of $\mathcal{R}(q)$, q prime, $3 \le q \le 10^7$. The maximal value (red dot) is attained at $q = 3$ and its value is $0.604599\ldots$; much larger than the other plotted values. The red dashed line represents the mean value.

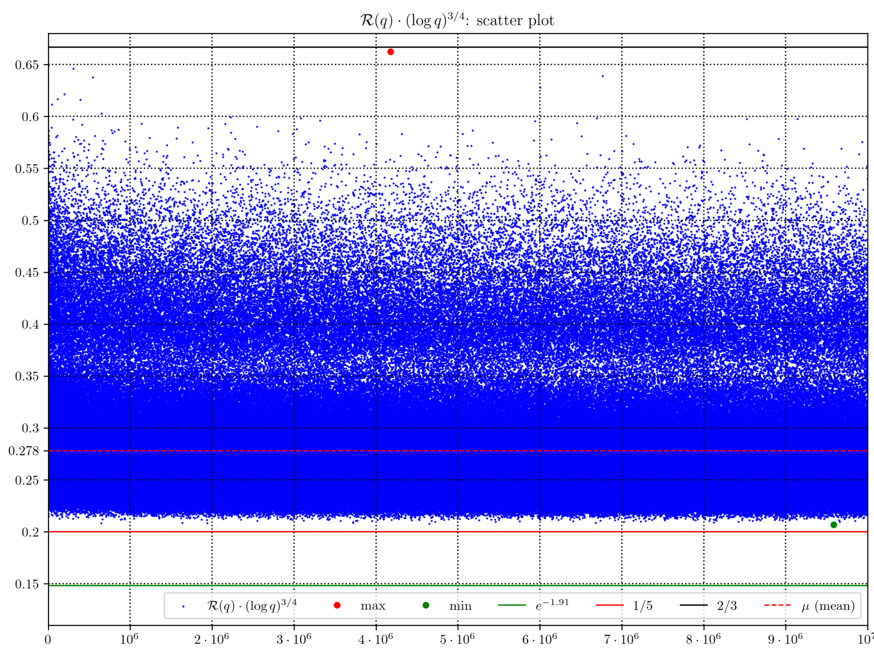

FIGURE 2. The values of $\mathcal{R}(q)(\log q)^{3/4}$, q prime, $3 \le q \le 10^7$. The red dashed line represents the mean value.

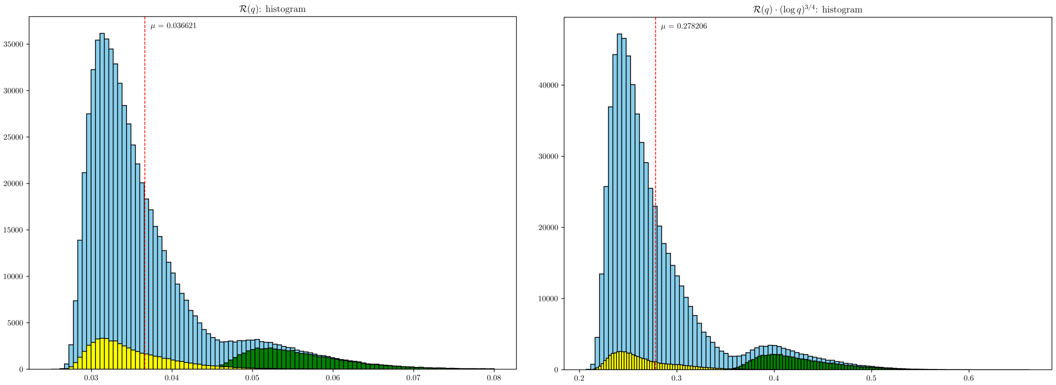

FIGURE 3. On the left: the values of $\mathcal{R}(q)$ (cerulean bars), q prime, $3 \le q \le 10^7$, but the contributions of the primes $q \ge 5$ such that $2q+1$ is prime (green bars) or $2q-1$ is prime (yellow bars) are superimposed. On the right: idem, but for the normalized values $\mathcal{R}(q)(\log q)^{3/4}$. The red dashed lines represent the mean values.

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Neelam Kandhil
University of Hong Kong, Department of Mathematics, Pokfulam, Hong Kong.
e-mail: `kandhil@hku.hk`

Alessandro Languasco
Università di Padova, Dipartimento di Matematica "Tullio Levi-Civita", via Trieste 63, 35121 Padova, Italy.
e-mail: `alessandro.languasco@unipd.it`; `alessandro.languasco@gmail.com`

Pieter Moree
Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.
e-mail: `moree@mpim-bonn.mpg.de`