

Singular behavior for a multi-parameter periodic Dirichlet problem

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Abstract: We consider a Dirichlet problem for the Poisson equation in a periodically perforated domain. The geometry of the domain is controlled by two parameters: a real number $\epsilon > 0$, proportional to the radius of the holes, and a map ϕ , which models the shape of the holes. So, if g denotes the Dirichlet boundary datum and f the Poisson datum, we have a solution for each quadruple (ϵ, ϕ, g, f) . Our aim is to study how the solution depends on (ϵ, ϕ, g, f) , especially when ϵ is very small and the holes narrow to points. In contrast with previous works, we do not introduce the assumption that f has zero integral on the fundamental periodicity cell. This brings in a certain singular behavior for ϵ close to 0. We show that, when the dimension n of the ambient space is greater than or equal to 3, a suitable restriction of the solution can be represented with an analytic map of the quadruple (ϵ, ϕ, g, f) multiplied by the factor $1/\epsilon^{n-2}$. In case of dimension $n = 2$, we have to add $\log \epsilon$ times the integral of $f/2\pi$.

Keywords: Dirichlet problem, integral equation method, Poisson equation, periodically perforated domain, singularly perturbed domain, real analytic continuation in Banach spaces.

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1 Introduction

The object of this paper is to study a singular perturbation of a Dirichlet-Poisson problem in a periodically perforated domain. The aim is to show that the solution can be written as a combination of real analytic maps and—possibly singular but completely known—elementary functions of the perturbation parameters. The geometry of the problem is controlled by two parameters: a positive real number ϵ that determines the size of the holes and a shape function ϕ that deforms the boundary of a certain reference domain Ω into the shape of the holes. To wit, the holes are shifted copies of $\epsilon\phi(\partial\Omega)$ and thus, for ϵ that tends to zero, they shrink down to points. Next, we take a Dirichlet datum g and a Poisson datum f in such a way that the problem has a unique solution for each choice of the four variables ϵ , ϕ , g , and f . So it makes sense to ask ourselves what we can say of the map that takes a quadruple (ϵ, ϕ, g, f) to the corresponding solution. In particular, we want to see what happens when ϵ is close to zero and the holes are shrinking to points.

The interest for periodic problems for the Laplace equations is (in part) motivated by their relevance in the applications. For instance, problems of this kind arise in the study of effective properties of composite materials. The reader may find some examples in the works of Ammari, Kang, and Lim [2], Ammari, Kang, and Touibi [3], Drygaś, Gluzman, Mityushev, and Nawalaniec [10], Gluzman, Mityushev, and Nawalaniec [14], and Kapanadze, Mishuris, and Pesetskaya [15, 16].

Indeed, we have ourselves already written on this topic. In particular, the problem of this paper is very similar to those of [34] and [35]. There is a critical difference though: in [34] we take the Poisson datum f equal to 0 and in [35] the function f is required to have zero integral on the periodicity cell, whereas here we abandon this assumption. As a consequence, we have to deal with a specific singular behavior that appears for ϵ close to zero: If the ambient space has dimension $n \geq 3$, the solution shows a singularity of the order of $1/\epsilon^{n-2}$, and, for $n = 2$, a $(\log \epsilon)$ -singularity. Also, in the previous works [34, 35] the number ϵ was solely responsible for the geometric deformation of the problem, and thus only homothetic transformations of the holes were allowed. Here, instead, the holes can change shape according to the function ϕ and we analyze the joint dependence on the set of variables (ϵ, ϕ, g, f) .

We now describe our problem in detail. We fix once for all

$$n \in \mathbb{N} \setminus \{0, 1\} \quad \text{and} \quad q_{11}, \dots, q_{nn} \in]0, +\infty[,$$

where \mathbb{N} denotes the set of natural numbers including 0. We set

$$q := \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix} \quad \text{and} \quad Q := \prod_{j=1}^n]0, q_{jj}[\subseteq \mathbb{R}^n.$$

The set Q is the fundamental periodicity cell and the diagonal matrix q is the periodicity matrix associated with the cell Q .

We consider a set $\Omega \subseteq \mathbb{R}^n$ satisfying the following assumption:

$$\begin{aligned} \Omega &\text{ is a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{1,\alpha} \\ &\text{ such that } \mathbb{R}^n \setminus \bar{\Omega} \text{ is connected.} \end{aligned} \tag{1}$$

In (1), as well as in the rest of the paper, α is a fixed number in the open interval $]0, 1[$ and the symbol $\bar{\cdot}$ denotes the closure of a set. For the definition of sets and functions of the Schauder class $C^{j,\alpha}$ ($j \in \mathbb{N}$) we refer, e.g., to Gilbarg and Trudinger [13].

The boundary $\partial\Omega$ of Ω plays the role of a reference set and the boundary of the holes is obtained rescaling and shifting the image of $\partial\Omega$ under a suitable map ϕ . The set of the functions ϕ that we allow is denoted by $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ and consists of the functions of $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) := (C^{1,\alpha}(\partial\Omega))^n$ that are injective and have injective differential at all points of $\partial\Omega$. We can verify that $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ is open in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ (see, e.g., Lanza de Cristoforis and Rossi [26, Lem. 2.2, p. 197] and [25, Lem. 2.5, p. 143]). Moreover, if $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, then the Jordan-Leray separation theorem (see, e.g., Deimling [9, Thm. 5.2, p. 26]) ensures that $\phi(\partial\Omega)$ splits \mathbb{R}^n into exactly two open connected components. We denote by $\mathbb{I}[\phi]$ the bounded one and we can verify that $\mathbb{I}[\phi]$ is a set of class $C^{1,\alpha}$ (cf. [25, Lem. 2.6, p. 144]). To put it simple, the modified set $\mathbb{I}[\phi]$ keeps the same regularity of the reference set Ω . Incidentally, we observe that this might not be the case with sets of a lower regularity. For example, a bi-Lipshitz image of the boundary of a Lipschitz set might not be the boundary of a Lipschitz set (cf., e.g., McLean [33, Fig. 2 (iii)]).

Next we fix a point

$$p \in Q,$$

which is the point where the hole in the reference periodicity cell shrinks to. It will be convenient to consider perturbations around a fixed

$$\phi_0 \in \mathcal{A}_{\partial\Omega}^{1,\alpha}.$$

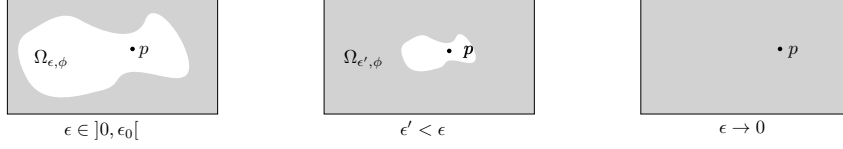


Figure 1: A 2-dimensional example of perforated reference periodicity cell. The hole $\Omega_{\epsilon, \phi}$ shrinks toward the point p when ϵ tends to 0.

There is no loss of generality in this choice, because ϕ_0 can be any function of $\mathcal{A}_{\partial\Omega}^{1, \alpha}$. The advantage is that there exist a real number $\epsilon_0 > 0$ and an open neighborhood \mathcal{O}_{ϕ_0} of ϕ_0 in $\mathcal{A}_{\partial\Omega}^{1, \alpha}$ such that

$$p + \epsilon \mathbb{I}[\phi] \subseteq Q \quad \forall (\epsilon, \phi) \in]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0}. \quad (2)$$

Then, for these ϵ 's and ϕ 's we can define the hole

$$\Omega_{\epsilon, \phi} := p + \epsilon \mathbb{I}[\phi] \quad \forall (\epsilon, \phi) \in]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0},$$

which is contained in Q , has size proportional to ϵ , and shape determined by ϕ . When ϵ tends to 0, the hole shrinks toward the point p while its shape changes according to ϕ (see Figure 1).

The periodic set of holes is given by

$$\mathbb{S}[\Omega_{\epsilon, \phi}] := \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega_{\epsilon, \phi}),$$

and the periodic domain where we define the Poisson equation is

$$\mathbb{S}[\Omega_{\epsilon, \phi}]^- := \mathbb{R}^n \setminus \overline{\mathbb{S}[\Omega_{\epsilon, \phi}]} \quad \forall (\epsilon, \phi) \in]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0}.$$

In other words, the domain $\mathbb{S}[\Omega_{\epsilon, \phi}]^-$ is obtained removing from \mathbb{R}^n the periodic set of holes $\overline{\mathbb{S}[\Omega_{\epsilon, \phi}]}$ (see Figure 2). When ϵ approaches zero, the hole in the cell $qz + Q$ shrinks toward $qz + p$.

We now introduce suitable spaces for the functional data of the problem. For the right-hand side of the Dirichlet boundary condition we take a function

$$g \in C^{1, \alpha}(\partial\Omega),$$

which we properly transplant to be defined on $\partial\Omega_{\epsilon, \phi} = p + \epsilon\phi(\partial\Omega)$. As for the Poisson datum, regularity has to be chosen more carefully. Lanza de Cristoforis in [19] and Preciso in [38, 39] shown that Roumieu analytic functions produce analytic composition

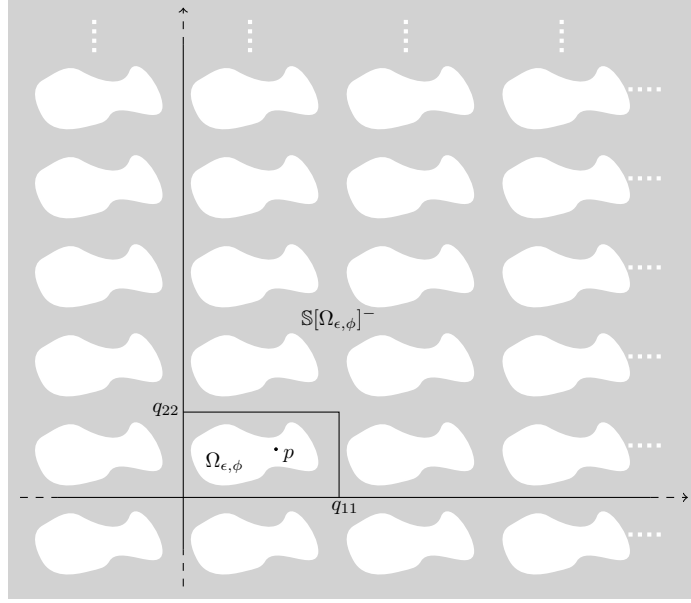


Figure 2: A 2-dimensional example of periodically perforated set $\mathbb{S}[\Omega_{\epsilon, \phi}]^-$.

operators and analytic Newtonian potentials, a feature that will come handy later on in our analysis. Moreover, since we are dealing with a periodic problem, we have to take periodicity into account. So, the right-hand side of the Poisson equation will be a function

$$f \in C_{q, \omega, \rho}^0(\mathbb{R}^n),$$

where $\rho > 0$ is fixed and $C_{q, \omega, \rho}^0(\mathbb{R}^n)$ denotes the Roumieu class of q -periodic analytic functions (see (5) for the exact definition of $C_{q, \omega, \rho}^0(\mathbb{R}^n)$, see also [35]).

All the ingredients are now introduced and we can state our Dirichlet-Poisson problem. For a quadruple $(\epsilon, \phi, g, f) \in]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1, \alpha}(\partial\Omega) \times C_{q, \omega, \rho}^0(\mathbb{R}^n)$, we look at

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{S}[\Omega_{\epsilon, \phi}]^-, \\ u(x + qz) = u(x) & \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon, \phi}]^-}, \forall z \in \mathbb{Z}^n, \\ u(x) = g \circ \phi^{(-1)}(\epsilon^{-1}(x - p)) & \forall x \in \partial\Omega_{\epsilon, \phi}. \end{cases} \quad (3)$$

It is well known that problem (3) has a unique solution in the space $C^{1, \alpha}(\overline{\mathbb{S}[\Omega_{\epsilon, \phi}]^-})$ (see Theorem 3.1, see also [35, Prop. 2.2]). To emphasize the dependence on (ϵ, ϕ, g, f) , we denote it by

$$u[\epsilon, \phi, g, f].$$

Then our goal is to describe the map

$$]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1, \alpha}(\partial\Omega) \times C_{q, \omega, \rho}^0(\mathbb{R}^n) \ni (\epsilon, \phi, g, f) \mapsto u[\epsilon, \phi, g, f] \in C^{1, \alpha}(\overline{\mathbb{S}[\Omega_{\epsilon, \phi}]^-}).$$

We observe, however, that the space in the right-hand side depends on ϵ , and so it is not suited to be the codomain of a function that depends on ϵ itself. To fix this inconvenience, we take an open bounded set V compactly contained in $\mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$ and for ϵ sufficiently small we consider the restriction of $u[\epsilon, \phi, g, f]$ to \bar{V} . In this way $u[\epsilon, \phi, g, f]_{|\bar{V}}$ belongs to $C^2(\bar{V})$, a space that does not depend on ϵ .

In our main Theorem 4.1 we show that, up to taking a smaller $\epsilon_0 > 0$ and a smaller open neighborhood \mathcal{O}_{ϕ_0} of ϕ_0 , the map that takes (ϵ, ϕ, g, f) to $u[\epsilon, \phi, g, f]_{|\bar{V}}$ is described by the formula

$$u[\epsilon, \phi, g, f]_{|\bar{V}} = \frac{1}{\epsilon^{n-2}} \mathfrak{U}[\epsilon, \phi, g, f] + \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(x) dx,$$

where $\delta_{2,n}$ is the Kronecker delta symbol and where

$$\mathfrak{U} :]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n) \rightarrow C^2(\bar{V})$$

is real analytic (here $C^2(\bar{V})$ could be replaced with $C^k(\bar{V})$ for any $k > 0$). The formula above makes evident that for $n = 2$ the singular behavior of the solution only appears as soon as f has non-zero integral over Q . The same is true for $n \geq 3$, as we can deduce comparing this result with those in [35].

With the regularity chosen for f and g we can solve the problem in the framework of Schauder spaces. It would be possible to use Sobolev spaces instead of Schauder spaces (as in [4], for example, where we studied a perturbation near a vertex point). Doing so we might relax the regularity assumption on g and—with a straightforward modification of the analysis that we will present—obtain results similar to those in described above. It seems more difficult, instead, to relax the assumptions on f .

Usually, boundary value problems in singularly perturbed domains are studied with the methods of Asymptotic Analysis, as we can find in the works of Kozlov, Maz'ya and Movchan [17], Maz'ya, Movchan, and Nieves [30], Mazya, Nazarov and Plamenewskii [31, 32], Novotny and Sokołowski [37], and so on. In this paper we adopt a different approach, named *Functional Analytic Approach*, which is more suited to obtain representation formulas in terms of analytic functions (see [6] for a detailed introduction). For this specific problem we will employ some periodic potential theory and we will exploit the idea of [36] (later developed in [24] and [29]) of using the periodic Newtonian potential corrected with a suitable multiple of the periodic fundamental solution of the Laplace equation. The singular behavior that arises when f has non-zero integral over Q is

related to the fact that, in that case, the problem

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{R}^n, \\ u(x + qz) = u(x) & \forall x \in \mathbb{R}^n, \forall z \in \mathbb{Z}^n, \end{cases} \quad (4)$$

has no solution (as one can check applying the Divergence Theorem to u in Q). Problem (4) may be seen as the limit of (3) as ϵ goes to 0. Indeed, in a recent paper Feppon and Ammari [11] have related the appearance of singular behaviors to the failure of similar compatibility conditions. The problems studied in [11] are not too far from the one that we analyze in this paper, but involve some specific generalized periodicity conditions. We also mention that the result that we present here is, in a sense, a periodic counterpart of a result obtained by Lanza de Cristoforis [20, 21] for a bounded domain with a single small hole.

Incidentally, we point out that the application of the *Functional Analytic Approach* to perturbations of parabolic equations is still an open problem (to us). The potential theoretic tools to deal with periodic and non-periodic problems for the heat equations have been developed in [22, 27, 28]. Nevertheless, the anisotropy between space and time prevents from a direct extension of the method. It is a future goal of the authors and collaborators to extend the methods of the present paper to parabolic problems, starting from regular perturbation and then passing to singular ones.

The paper is organized as follows: In Section 2 we list some preliminary results of periodic potential theory that are used in Section 3 to transform problem (3) into an equivalent system of integral equations. In Section 4 we prove our main Theorem 4.1.

2 Preliminaries of potential theory

As mentioned in the introduction, we use periodic potential theory to transform problem (3) into an equivalent system of integral equations. More precisely, we use the periodic double layer potential, whose definition differs from that of the classical double layer potential because we replace the fundamental solution of the Laplace operator $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ with a periodic analog. This will be a q -periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z \in \mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n},$$

where δ_{qz} denotes the Dirac distribution with mass in qz and where $|\cdot|_n$ denotes the n -dimensional measure of a set. To define $S_{q,n}$ we can take

$$S_{q,n}(x) := - \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|Q|_n 4\pi^2 |q^{-1}z|^2} e^{2\pi i(q^{-1}z) \cdot x},$$

where the series converges in the sense of distributions on \mathbb{R}^n (cf., e.g., Ammari and Kang [1, p. 53], [6, §2.1]). It can be shown that $S_{q,n}$ is real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$ and is locally integrable in \mathbb{R}^n (cf., e.g., [6, Thms. 12.3, 12.4]).

We will also find useful to write $S_{q,n}$ as the sum of the classical fundamental solution of the Laplacian

$$S_n(x) := \begin{cases} \frac{1}{s_2} \log |x| & \forall x \in \mathbb{R}^2 \setminus \{0\}, \text{ if } n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n \geq 3, \end{cases}$$

and a remainder $R_{q,n} := S_{q,n} - S_n$ which is regular around the origin. Here above s_n is the $(n-1)$ -dimensional measure of the boundary $\partial\mathbb{B}_n(0,1)$ of the unit ball $\mathbb{B}_n(0,1)$ in \mathbb{R}^n . We note that $R_{q,n}$ has an analytic extension to $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$, which we still denote by $R_{q,n}$, and that

$$\Delta R_{q,n} = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \delta_{qz} - \frac{1}{|Q|_n}$$

in the sense of distributions (see, e.g., [6, Thm. 12.4]).

We now recall the definition of the classical (non-periodic) double layer potential. We introduce another set $\tilde{\Omega}$, which we use as a dummy for our definitions: $\tilde{\Omega}$ is a bounded open connected subset of \mathbb{R}^n of class $C^{1,\alpha}$ such that $\mathbb{R}^n \setminus \overline{\tilde{\Omega}}$ is connected. The classical double layer potential supported on $\tilde{\Omega}$ and with density $\theta \in C^{1,\alpha}(\partial\tilde{\Omega})$ is defined by

$$w_{\tilde{\Omega}}[\theta](t) := - \int_{\partial\tilde{\Omega}} \nu_{\tilde{\Omega}}(s) \cdot DS_n(t-s)\theta(s) d\sigma_s \quad \forall t \in \mathbb{R}^n,$$

where $\nu_{\tilde{\Omega}}$ denotes the outward unit normal to $\partial\tilde{\Omega}$ and the symbol “ \cdot ” denotes the scalar product in \mathbb{R}^n . As is well known, the restriction $w_{\tilde{\Omega}}[\theta]|_{\tilde{\Omega}}$ extends to a function $w_{\tilde{\Omega}}^+[\theta]$ in $C^{1,\alpha}(\overline{\tilde{\Omega}})$ and the restriction $w_{\tilde{\Omega}}[\theta]|_{\mathbb{R}^n \setminus \overline{\tilde{\Omega}}}$ extends to a function $w_{\tilde{\Omega}}^-[\theta]$ in $C_{loc}^{1,\alpha}(\mathbb{R}^n \setminus \tilde{\Omega})$. Moreover, on the boundary we have the jump formula:

$$w_{\tilde{\Omega}}^\pm[\theta]|_{\partial\tilde{\Omega}} = \pm \frac{1}{2}\theta + w_{\tilde{\Omega}}[\theta]|_{\partial\tilde{\Omega}} \quad \forall \theta \in C^{1,\alpha}(\partial\tilde{\Omega})$$

(cf. Folland [12, Ch. 3], [6, § 4.5]).

To define the periodic double layer potential we need some more notation about periodic domains. If $\tilde{\Omega}_Q$ is an arbitrary subset of \mathbb{R}^n such that $\overline{\tilde{\Omega}_Q} \subseteq Q$ (another dummy set), we define

$$\mathbb{S}[\tilde{\Omega}_Q] := \bigcup_{z \in \mathbb{Z}^n} (qz + \tilde{\Omega}_Q) = q\mathbb{Z}^n + \tilde{\Omega}_Q, \quad \mathbb{S}[\tilde{\Omega}_Q]^- := \mathbb{R}^n \setminus \overline{\mathbb{S}[\tilde{\Omega}_Q]}.$$

Then a function u from $\overline{\mathbb{S}[\tilde{\Omega}_Q]}$ or from $\overline{\mathbb{S}[\tilde{\Omega}_Q]^-}$ to \mathbb{R} is q -periodic if $u(x + qz) = u(x)$ for all x in the domain of definition of u and for all $z \in \mathbb{Z}^n$. For $j \in \{0, 1\}$ and $\alpha \in]0, 1[$, we denote by $C_q^{j,\alpha}(\overline{\mathbb{S}[\tilde{\Omega}_Q]})$ and $C_q^{j,\alpha}(\overline{\mathbb{S}[\tilde{\Omega}_Q]^-})$ the spaces of q -periodic functions of class $C^{j,\alpha}$ in $\overline{\mathbb{S}[\tilde{\Omega}_Q]}$ and in $\overline{\mathbb{S}[\tilde{\Omega}_Q]^-}$, respectively (cf. [6, p. 491]).

If $\tilde{\Omega}_Q$ is of class $C^{1,\alpha}$, then the periodic double layer potential with density $\mu \in C^{1,\alpha}(\partial\tilde{\Omega}_Q)$ is defined by

$$w_{q,\tilde{\Omega}_Q}[\mu](x) := - \int_{\partial\tilde{\Omega}_Q} \nu_{\tilde{\Omega}_Q}(y) \cdot DS_{q,n}(x-y)\mu(y) d\sigma_y \quad \forall x \in \mathbb{R}^n,$$

and we see that the expression in the right-hand side differs from that in the definition of $w_{\tilde{\Omega}}[\mu]$ because we replace S_n with $S_{q,n}$.

It is well known that the restriction $w_{q,\tilde{\Omega}_Q}[\mu]|_{\mathbb{S}_q[\tilde{\Omega}_Q]}$ extends to a function $w_{q,\tilde{\Omega}_Q}^+[\mu]$ of $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\tilde{\Omega}_Q]})$ and the restriction $w_{q,\tilde{\Omega}_Q}[\mu]|_{\mathbb{S}_q[\tilde{\Omega}_Q]^-}$ extends to a function $w_{q,\tilde{\Omega}_Q}^-[\mu]$ of $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\tilde{\Omega}_Q]^-})$. Moreover, on the boundary of $\tilde{\Omega}_Q$ we have the jump formula

$$w_{q,\tilde{\Omega}_Q}^\pm[\mu]|_{\partial\tilde{\Omega}_Q} = \pm \frac{1}{2}\mu + w_{q,\tilde{\Omega}_Q}[\mu]|_{\partial\tilde{\Omega}_Q} \quad \forall \mu \in C^{1,\alpha}(\partial\tilde{\Omega}_Q)$$

(cf., e.g., [6, Thm. 12.10]).

As mentioned in the introduction, we will use Roumieu analytic functions. The advantage is that the composition operator

$$(u, v) \mapsto u \circ v$$

is real analytic in the pair of (u, v) if u is taken in a Roumieu class and v in a Schauder space (see Proposition 2.2 below). Also, Roumieu analytic functions produce Roumieu analytic Newtonian potentials (see Theorem 2.1). So, for all bounded open subsets $\tilde{\Omega}$ of \mathbb{R}^n and $\rho > 0$, we set

$$C_{\omega,\rho}^0(\tilde{\Omega}) := \left\{ u \in C^\infty(\tilde{\Omega}) : \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\tilde{\Omega})} < +\infty \right\},$$

and

$$\|u\|_{C_{\omega,\rho}^0(\bar{\Omega})} := \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\bar{\Omega})} \quad \forall u \in C_{\omega,\rho}^0(\bar{\Omega}),$$

where $|\beta| := \beta_1 + \dots + \beta_n$ is the length of the multi-index $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. As is well known, the Roumieu class $(C_{\omega,\rho}^0(\bar{\Omega}), \|\cdot\|_{C_{\omega,\rho}^0(\bar{\Omega})})$ is a Banach space.

If $k \in \mathbb{N}$, then we set

$$C_q^k(\mathbb{R}^n) := \{u \in C^k(\mathbb{R}^n) : u(x + qz) = u(x) \quad \forall x \in \mathbb{R}^n, \forall z \in \mathbb{Z}^n\},$$

and

$$C_q^\infty(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) : u(x + qz) = u(x) \quad \forall x \in \mathbb{R}^n, \forall z \in \mathbb{Z}^n\}.$$

Similarly, if $\rho > 0$, we set

$$C_{q,\omega,\rho}^0(\mathbb{R}^n) := \left\{ u \in C_q^\infty(\mathbb{R}^n) : \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\bar{Q})} < +\infty \right\}, \quad (5)$$

and

$$\|u\|_{C_{q,\omega,\rho}^0(\mathbb{R}^n)} := \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\bar{Q})} \quad \forall u \in C_{q,\omega,\rho}^0(\mathbb{R}^n).$$

We can see that the periodic Roumieu class $(C_{q,\omega,\rho}^0(\mathbb{R}^n), \|\cdot\|_{C_{q,\omega,\rho}^0(\mathbb{R}^n)})$ is a Banach space.

It is common to use Newtonian potentials to convert boundary value problems for the Poisson equation into boundary value problems for the Laplace equation. To keep this tradition alive we need to introduce a periodic analog of the Newtonian potential: if $h \in C_q^0(\mathbb{R}^n)$, then we set

$$P_q[h](x) := \int_Q S_{q,n}(x-y)h(y) dy \quad \forall x \in \mathbb{R}^n.$$

Some of the properties of the periodic Newtonian potential are listed in the following theorem (we refer to [5] for an exhaustive overview).

Theorem 2.1. *The following statements hold.*

(i) *Let $f \in C_q^1(\mathbb{R}^n)$. Then $P_q[f] \in C_q^2(\mathbb{R}^n)$ and*

$$\Delta P_q[f](x) = f(x) - \frac{1}{|Q|_n} \int_Q f(y) dy \quad \forall x \in \mathbb{R}^n.$$

(ii) *Let $\rho > 0$. Then there exists $\rho' \in]0, \rho]$ such that $P_q[f] \in C_{q,\omega,\rho'}^0(\mathbb{R}^n)$ for all $f \in C_{q,\omega,\rho}^0(\mathbb{R}^n)$ and such that $P_q[\cdot]$ is linear and continuous from $C_{q,\omega,\rho}^0(\mathbb{R}^n)$ to $C_{q,\omega,\rho'}^0(\mathbb{R}^n)$.*

Then we introduce a slight variant of Preciso [38, Prop. 4.2.16, p. 51] and [39, Prop. 1.1, p. 101] on the real analyticity of a composition operator (see also Lanza de Cristoforis [18, Prop. 2.17, Rem. 2.19] and [20, Prop. 9, p. 214]).

Proposition 2.2. *Let $m, h, k \in \mathbb{N}$, $h, k \geq 1$. Let $\alpha \in]0, 1]$, $\rho > 0$. Let Ω', Ω'' be bounded open connected subsets of $\mathbb{R}^h, \mathbb{R}^k$, respectively. Let Ω'' be of class C^1 . Then the operator T defined by*

$$T[u, v] := u \circ v$$

for all $(u, v) \in C_{\Omega', \rho}^0(\overline{\Omega'}) \times C^{m, \alpha}(\overline{\Omega''}, \Omega')$ is real analytic from the open subset $C_{\Omega', \rho}^0(\overline{\Omega'}) \times C^{m, \alpha}(\overline{\Omega''}, \Omega')$ of $C_{\Omega', \rho}^0(\overline{\Omega'}) \times C^{m, \alpha}(\overline{\Omega''}, \mathbb{R}^h)$ to $C^{m, \alpha}(\overline{\Omega''})$.

Finally, we need a last technical lemma about the real analytic dependence of certain maps related to the change of variables in integrals and to the pullback of the outer normal field. For a proof we refer to Lanza de Cristoforis and Rossi [25, p. 166] and to Lanza de Cristoforis [20, Prop. 1].

Lemma 2.3. *Let α, Ω be as in (1). Then the following statements hold.*

(i) *For each $\psi \in \mathcal{A}_{\partial\Omega}^{1, \alpha}$, there exists a unique $\tilde{\sigma}[\psi] \in C^{0, \alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\psi] > 0$ and*

$$\int_{\psi(\partial\Omega)} w(s) d\sigma_s = \int_{\partial\Omega} w \circ \psi(y) \tilde{\sigma}[\psi](y) d\sigma_y, \quad \forall w \in L^1(\psi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ from $\mathcal{A}_{\partial\Omega}^{1, \alpha}$ to $C^{0, \alpha}(\partial\Omega)$ is real analytic.

(ii) *The map from $\mathcal{A}_{\partial\Omega}^{1, \alpha}$ to $C^{0, \alpha}(\partial\Omega, \mathbb{R}^n)$ that takes ψ to $\nu_{1[\psi]} \circ \psi$ is real analytic.*

3 Formulation of problem (3) in terms of integral equations

First we convert problem (3) into a Dirichlet problem for the Laplace equation. In order to do so, we would like to use a function whose Laplacian equals the right-hand side $f \in C_{q, \omega, \rho}^0(\mathbb{R}^n)$ of the first equation in problem (3). The natural candidate would be the periodic Newtonian potential $P_q[f]$. However, we have

$$\Delta P_q[f](x) = f(x) - \frac{1}{|Q|_n} \int_Q f(y) dy \quad \forall x \in \mathbb{S}[\Omega_{\epsilon, \phi}]^-. \quad (6)$$

So we need to get rid of the term $\frac{1}{|Q|_n} \int_Q f(y) dy$ in equation (6): We note that the function from $\mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$ to \mathbb{R} that takes x to $-S_{q, n}(x - p) \int_Q f(y) dy$ is q -periodic

and analytic, and that

$$\Delta \left[-S_{q,n}(x-p) \int_Q f(y) dy \right] = \frac{1}{|Q|_n} \int_Q f(y) dy \quad \forall x \in \mathbb{R}^n \setminus (p + q\mathbb{Z}^n).$$

As a consequence,

$$\Delta \left[P_q[f](x) - S_{q,n}(x-p) \int_Q f(y) dy \right] = f(x) \quad \forall x \in \mathbb{S}[\Omega_{\epsilon,\phi}]^- \quad (7)$$

and we can use the corrected Newtonian potential

$$P_q[f](x) - S_{q,n}(x-p) \int_Q f(y) dy$$

to transform problem (3) into a Dirichlet problem for the Laplace equation, that in turn we can analyze using the periodic double layer potential.

This is what we do to prove the following Theorem 3.1: first we transform the Dirichlet-Poisson problem into a Dirichlet-Laplace problem, and then we represent the solution as the sum of a constant and a double layer potential with a density that satisfies a certain boundary integral equation pulled-back to $\partial\Omega$.

Theorem 3.1. *Let $\alpha \in]0, 1[$. Let $\rho > 0$. Let $p \in Q$. Let Ω be as in (1). Let $(\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. Let $\epsilon_0, \mathcal{O}_{\phi_0}$ be as in (2). Let $(\epsilon, \phi, g, f) \in]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. Then problem (3) has a unique solution $u[\epsilon, \phi, g, f]$ in $C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-})$, which is delivered by the formula*

$$\begin{aligned} u[\epsilon, \phi, g, f](x) &:= \omega(\epsilon, \phi, g, f, x) + P_q[f](x) - S_{q,n}(x-p) \int_Q f(y) dy \\ &\quad + \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-}, \end{aligned} \quad (8)$$

where

$$\omega(\epsilon, \phi, g, f, x) := w_{q,\Omega_{\epsilon,\phi}}^-[\theta \circ \phi^{(-1)}(\epsilon^{-1}(\cdot - p))](x) + c \quad \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-}$$

and where (θ, c) is the unique solution in $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ of the system of integral equations

$$\begin{aligned} -\frac{1}{2}\theta(t) - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS_n(\phi(t) - \phi(s))\theta(s)\tilde{\sigma}[\phi](s) d\sigma_s \\ - \epsilon^{n-1} \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DR_{q,n}(\epsilon(\phi(t) - \phi(s)))\theta(s)\tilde{\sigma}[\phi](s) d\sigma_s + c \\ = g(t) - \int_Q S_{q,n}(p + \epsilon\phi(t) - y)f(y) dy \\ + \frac{1}{\epsilon^{n-2}} S_n(\phi(t)) \int_Q f(y) dy + R_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy \quad \forall t \in \partial\Omega, \end{aligned} \quad (9)$$

$$\int_{\partial\Omega} \theta \tilde{\sigma}[\phi] d\sigma = 0. \quad (10)$$

Proof. By (7) we can see that a function $u \in C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-})$ solves problem (3) if and only if the function

$$u(x) - P_q[f](x) + S_{q,n}(x-p) \int_Q f(y) dy - \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-}$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta\omega(x) = 0 & \forall x \in \mathbb{S}[\Omega_{\epsilon,\phi}]^-, \\ \omega(x+qz) = \omega(x) & \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-}, \forall z \in \mathbb{Z}^n, \\ \omega(x) = g \circ \phi^{(-1)}(\epsilon^{-1}(x-p)) - P_q[f](x) \\ \quad + S_{q,n}(x-p) \int_Q f(y) dy - \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy & \forall x \in \partial\Omega_{\epsilon,\phi}. \end{cases} \quad (11)$$

By [6, Prop. 12.24] the solution of problem (11) exists, is unique, belongs to $C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-})$, and can be written as

$$\omega(x) = w_{q,\Omega_{\epsilon,\phi}}^-[\theta \circ \phi^{(-1)}(\epsilon^{-1}(\cdot-p))](x) + c \quad \forall x \in \overline{\mathbb{S}[\Omega_{\epsilon,\phi}]^-},$$

where (θ, c) is the unique pair in $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ such that $\int_{\partial\Omega} \theta \tilde{\sigma}[\phi] d\sigma = 0$ and

$$\begin{aligned} & -\frac{1}{2}\theta \circ \phi^{(-1)}(\epsilon^{-1}(x-p)) + w_{q,\Omega_{\epsilon,\phi}}^-[\theta \circ \phi^{(-1)}(\epsilon^{-1}(\cdot-p))](x) + c \\ & = g \circ \phi^{(-1)}(\epsilon^{-1}(x-p)) - P_q[f](x) \\ & \quad + S_{q,n}(x-p) \int_Q f(y) dy - \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad \forall x \in \partial\Omega_{\epsilon,\phi}. \end{aligned}$$

By a change of variable, the last equation can be rewritten as

$$\begin{aligned} & -\frac{1}{2}\theta(t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_{\Omega_{\epsilon,\phi}}(p + \epsilon\phi(s)) DS_{q,n}(p + \epsilon\phi(t) - (p + \epsilon\phi(s))) \theta(s) \tilde{\sigma}[\phi](s) d\sigma_s + c \\ & = g(t) - \int_Q S_{q,n}(p + \epsilon\phi(t) - y) f(y) dy \\ & \quad + S_{q,n}(p + \epsilon\phi(t) - p) \int_Q f(y) dy - \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad \forall t \in \partial\Omega. \end{aligned} \quad (12)$$

Then we note that

$$\begin{aligned} S_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy & = \frac{1}{\epsilon^{n-2}} S_n(\phi(t)) \int_Q f(y) dy + \delta_{2,n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \\ & \quad + R_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy \quad \forall t \in \partial\Omega \end{aligned}$$

and, since

$$\nu_{\Omega_\epsilon, \phi}(p + \epsilon\phi(s)) = \nu_{\mathbb{I}[\phi]} \circ \phi(s) \quad \forall s \in \partial\Omega$$

(cf., e.g., Lanza de Cristoforis [21, Lem. 3.1]), equation (12) can be rewritten as

$$\begin{aligned} & -\frac{1}{2}\theta(t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) DS_{q,n}(\epsilon(\phi(t) - \phi(s)))\theta(s)\tilde{\sigma}[\phi](s) d\sigma_s + c \\ & = g(t) - \int_Q S_{q,n}(p + \epsilon\phi(t) - y)f(y) dy \\ & + \frac{1}{\epsilon^{n-2}}S_n(\phi(t)) \int_Q f(y) dy + R_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy \quad \forall t \in \partial\Omega, \end{aligned}$$

which is easily seen to be equivalent to (9). \square

We denote by $(\theta_{\epsilon, \phi, g, f}, c_{\epsilon, \phi, g, f})$ the unique solution of (9)–(10). We would like, however, to have integral equations that are defined also for $\epsilon = 0$, and system (9)–(10) is not. So we rescale and take

$$(\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#) := \epsilon^{n-2}(\theta_{\epsilon, \phi, g, f}, c_{\epsilon, \phi, g, f})$$

for all $(\epsilon, \phi, g, f) \in]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. We see that $(\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#)$ coincides with the unique pair $(\theta^\#, c^\#) \in C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ such that

$$\begin{aligned} & -\frac{1}{2}\theta^\#(t) - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS_n(\phi(t) - \phi(s))\theta^\#(s)\tilde{\sigma}[\phi](s) d\sigma_s \tag{13} \\ & - \epsilon^{n-1} \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DR_{q,n}(\epsilon(\phi(t) - \phi(s)))\theta^\#(s)\tilde{\sigma}[\phi](s) d\sigma_s + c^\# \\ & = \epsilon^{n-2}g(t) - \epsilon^{n-2} \int_Q S_{q,n}(p + \epsilon\phi(t) - y)f(y) dy \\ & + S_n(\phi(t)) \int_Q f(y) dy + \epsilon^{n-2}R_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy \quad \forall t \in \partial\Omega, \\ & \int_{\partial\Omega} \theta^\# \tilde{\sigma}[\phi] d\sigma = 0, \tag{14} \end{aligned}$$

and (13)–(14) makes sense also for $\epsilon = 0$.

Using the pair $(\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#)$ instead of $(\theta_{\epsilon, \phi, g, f}, c_{\epsilon, \phi, g, f})$ we obtain from Theorem 3.1 the following alternative representation formula for $u[\epsilon, \phi, g, f]$.

Corollary 3.2. *Let $\alpha \in]0, 1[$. Let $\rho > 0$. Let $p \in Q$. Let Ω be as in (1). Let $(\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. Let $\epsilon_0, \mathcal{O}_{\phi_0}$ be as in (2). Let $(\epsilon, \phi, g, f) \in]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. Then the unique solution $u[\epsilon, \phi, g, f]$ in $C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_\epsilon, \phi]^-})$*

of problem (3) (which is also given by (8)) can be written as

$$\begin{aligned} u[\epsilon, \phi, g, f](x) &= \frac{1}{\epsilon^{n-2}} w_{q, \Omega_\epsilon, \phi}^- [\theta_{\epsilon, \phi, g, f}^\# \circ \phi^{(-1)}(\epsilon^{-1}(\cdot - p))](x) + \frac{1}{\epsilon^{n-2}} c_{\epsilon, \phi, g, f}^\# \\ &\quad + P_q[f](x) - S_{q, n}(x - p) \int_Q f(y) dy \\ &\quad + \delta_{2, n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad \forall x \in \overline{\mathbb{S}[\Omega_\epsilon, \phi]^-}, \end{aligned}$$

where $(\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#)$ is the unique solution in $C^{1, \alpha}(\partial\Omega) \times \mathbb{R}$ of system (13)-(14).

For (ϵ, ϕ, g, f) that tends to $(0, \phi_0, g_0, f_0)$ system (13)-(14) turns into the following “limiting system of integral equations,”

$$\begin{aligned} -\frac{1}{2} \theta^\#(t) - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi_0]} \circ \phi_0(s) \cdot DS_n(\phi_0(t) - \phi_0(s)) \theta^\#(s) \tilde{\sigma}[\phi_0](s) d\sigma_s + c^\# & \quad (15) \\ &= \delta_{2, n} g_0(t) - \delta_{2, n} \int_Q S_{q, n}(p - y) f_0(y) dy \\ &\quad + S_n(\phi_0(t)) \int_Q f_0(y) dy + \delta_{2, n} R_{q, n}(0) \int_Q f_0(y) dy \quad \forall t \in \partial\Omega, \\ \int_{\partial\Omega} \theta^\# \tilde{\sigma}[\phi_0] d\sigma &= 0. \quad (16) \end{aligned}$$

In the following Theorem 3.3 we prove that (15)-(16) has a solution and that such solution is unique. As we shall see, system (15)-(16) is related to a specific boundary value problem, which we call the “limiting boundary value problem.” The proof of Theorem 3.3 follows the guidelines of the proof of [34, Lem. 3.4].

Theorem 3.3. *Let $\alpha \in]0, 1[$. Let $\rho > 0$. Let $p \in Q$. Let Ω be as in (1). Let $(\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1, \alpha} \times C^{1, \alpha}(\partial\Omega) \times C_{q, \omega, \rho}^0(\mathbb{R}^n)$. Let $\tilde{\tau}$ be the unique solution in $C^{0, \alpha}(\partial\Omega)$ of*

$$\begin{cases} -\frac{1}{2} \tau(t) + \int_{\partial\Omega} \nu_{\mathbb{I}[\phi_0]} \circ \phi_0(t) \cdot DS_n(\phi_0(t) - \phi_0(s)) \tau(s) \tilde{\sigma}[\phi_0](s) d\sigma_s = 0 & \forall t \in \partial\Omega, \\ \int_{\partial\Omega} \tau \tilde{\sigma}[\phi_0] d\sigma = 1. \end{cases} \quad (17)$$

Then the following statements hold.

(i) *The limiting system (15)-(16) has one and only one solution $(\tilde{\theta}^\#, \tilde{c}^\#)$ in $C^{1, \alpha}(\partial\Omega) \times \mathbb{R}$. Moreover,*

$$\tilde{c}^\# = \int_{\partial\Omega} g_n^\# \tilde{\tau} \tilde{\sigma}[\phi_0] d\sigma,$$

where, for all $t \in \partial\Omega$ we have

$$g_n^\#(t) := \begin{cases} g_0(t) - \int_Q S_{q, n}(p - y) f_0(y) dy + S_n(\phi_0(t)) \int_Q f_0(y) dy \\ \quad + R_{q, n}(0) \int_Q f_0(y) dy & \text{if } n = 2, \\ S_n(\phi_0(t)) \int_Q f_0(y) dy & \text{if } n \geq 3. \end{cases}$$

(ii) *The limiting boundary value problem*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \overline{\mathbb{I}[\phi_0]}, \\ u(x) = g_n^\# \circ \phi_0^{(-1)}(x) & \forall x \in \phi_0(\partial\Omega), \\ \lim_{x \rightarrow \infty} u(x) = \tilde{c}^\#, \end{cases} \quad (18)$$

has one and only one solution $\tilde{u}^\#$ in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n \setminus \mathbb{I}[\phi_0])$. Moreover,

$$\tilde{u}^\#(x) = w_{\mathbb{I}[\phi_0]}^-[\tilde{\theta}^\# \circ \phi_0^{(-1)}](x) + \tilde{c}^\# \quad \forall x \in \mathbb{R}^n \setminus \mathbb{I}[\phi_0]. \quad (19)$$

Proof. By classical potential theory (cf., e.g., Folland [12, Ch. 3], [6, Lem. 6.34]) and by the theorem of change of variable in integrals, we can see that problem (17) has a unique solution $\tilde{\tau} \in C^{0,\alpha}(\partial\Omega)$. Then the validity of statement (i) follows by [6, Thm. 6.37 (i)] and, once more, by the theorem of change of variable in integrals.

To prove statement (ii) we first observe that problem (18) has at most one continuous solution (this follows by a classical argument based on the Maximum Principle). Then, by the properties of the classical double layer potentials and exploiting the fact that $(\tilde{\theta}^\#, \tilde{c}^\#)$ is a solution of (15)-(16), we can see that the function $\tilde{u}^\#$ in (19) is harmonic and satisfies the second and the third conditions in (18) (cf. [6, § 4.5]). Thus it coincides with the unique solution of (18). \square

We might be curious to know what is the constant $\tilde{c}^\#$ that appears in Theorem 3.3. In the following remark we attempt an explanation.

Remark 3.4. *By the computations of [8, Lem. 7.2], we can verify that if $n = 2$, then $\tilde{u}^\#$ coincides with the unique solution of*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \overline{\mathbb{I}[\phi_0]}, \\ u(x) = g_n^\# \circ \phi_0^{(-1)}(x) & \forall x \in \phi_0(\partial\Omega), \\ \sup_{x \in \mathbb{R}^2 \setminus \mathbb{I}[\phi_0]} |u(x)| < +\infty, \end{cases}$$

and that

$$\tilde{c}^\# = \lim_{x \rightarrow \infty} \tilde{u}^\#(x).$$

If, instead, $n \geq 3$, then an extension of the 2-dimensional argument of [8, Lem. 7.2] to the $(n \geq 3)$ -dimensional case shows that

$$\tilde{c}^\# = \frac{1}{(2-n)s_n} \int_Q f_0(y) dy \left(\lim_{x \rightarrow \infty} |x|^{n-2} H_0(x) \right)^{-1},$$

where H_0 is the unique function of $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n \setminus \mathbb{I}[\phi_0])$ such that

$$\begin{cases} \Delta H_0(x) = 0 & \forall x \in \mathbb{R}^n \setminus \overline{\mathbb{I}[\phi_0]}, \\ H_0(x) = 1 & \forall x \in \phi_0(\partial\Omega), \\ \lim_{x \rightarrow \infty} H_0(x) = 0 \end{cases}$$

and where we can see that the limit

$$\lim_{x \rightarrow \infty} |x|^{n-2} H_0(x)$$

exists and belongs to $]0, +\infty[$ (see, e.g., Folland [12, Chap. 2, Prop. 2.74]). In particular, we have

$$\tilde{c}^\# \neq 0 \text{ as soon as } \int_Q f_0(y) dy \neq 0.$$

In the following Theorem 3.5 we consider the map that takes (ϵ, ϕ, g, f) to the pair $(\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#)$ and prove that it has a real analytic continuation in a neighborhood of $(0, \phi_0, g_0, f_0)$. The proof of Theorem 3.5 exploits the Implicit Function Theorem for real analytic maps in Banach spaces (cf., e.g., Deimling [9, Thm. 15.3]).

Theorem 3.5. *Let $\alpha \in]0, 1[$. Let $\rho > 0$. Let $p \in Q$. Let Ω be as in (1). Let $(\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$. Let $\epsilon_0, \mathcal{O}_{\phi_0}$ be as in (2). Then there exist $\epsilon_{\#,1} \in]0, \epsilon_0[$, an open neighborhood \mathcal{O}'_{ϕ_0} of ϕ_0 in $\mathcal{A}_{\partial\Omega}^{1,\alpha}$, an open neighborhood \mathcal{U}_0 of (g_0, f_0) in $C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$, and a real analytic map*

$$(\Theta_\#, C_\#) :]-\epsilon_{\#,1}, \epsilon_{\#,1}[\times \mathcal{O}'_{\phi_0} \times \mathcal{U}_0 \rightarrow C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$$

such that

$$\begin{aligned} (\Theta_\#[\epsilon, \phi, g, f], C_\#[\epsilon, \phi, g, f]) &= (\theta_{\epsilon, \phi, g, f}^\#, c_{\epsilon, \phi, g, f}^\#) \quad \forall (\epsilon, \phi, g, f) \in]0, \epsilon_{\#,1}[\times \mathcal{O}'_{\phi_0} \times \mathcal{U}_0, \\ (\Theta_\#[0, \phi_0, g_0, f_0], C_\#[0, \phi_0, g_0, f_0]) &= (\tilde{\theta}^\#, \tilde{c}^\#). \end{aligned}$$

Proof. Let $\Lambda_\# := (\Lambda_{\#,1}, \Lambda_{\#,2})$ be the map from $]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ defined by

$$\begin{aligned} \Lambda_{\#,1}[\epsilon, \phi, g, f, \theta^\#, c^\#](t) &:= -\frac{1}{2}\theta^\#(t) - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS_n(\phi(t) - \phi(s))\theta^\#(s)\tilde{\sigma}[\phi](s) d\sigma_s \\ &\quad - \epsilon^{n-1} \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DR_{q,n}(\epsilon(\phi(t) - \phi(s)))\theta^\#(s)\tilde{\sigma}[\phi](s) d\sigma_s + c^\# \\ &\quad - \epsilon^{n-2}g(t) + \epsilon^{n-2} \int_Q S_{q,n}(p + \epsilon\phi(t) - y)f(y) dy \end{aligned}$$

$$-S_n(\phi(t)) \int_Q f(y) dy - \epsilon^{n-2} R_{q,n}(\epsilon\phi(t)) \int_Q f(y) dy \quad \forall t \in \partial\Omega,$$

$$\Lambda_{\#,2}[\epsilon, \phi, g, f, \theta^\#, c^\#] := \int_{\partial\Omega} \theta^\# \tilde{\sigma}[\phi] d\sigma,$$

for all $(\epsilon, \phi, g, f, \theta^\#, c^\#) \in]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$.

We first note that equation $\Lambda_{\#}[0, \phi_0, g_0, f_0, \theta^\#, c^\#] = 0$ in the unknown $(\theta^\#, c^\#) \in C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ is equivalent to the limiting system (15)-(16), which has one and only one solution $(\tilde{\theta}^\#, \tilde{c}^\#)$ in $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$. Similarly, if $(\epsilon, \phi, g, f) \in]0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$, then equation $\Lambda_{\#}[\epsilon, \phi, g, f, \theta^\#, c^\#] = 0$ in the unknown $(\theta^\#, c^\#) \in C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ is equivalent to the system (13)-(14) and has one and only one solution $(\theta_{\epsilon,\phi,g,f}^\#, c_{\epsilon,\phi,g,f}^\#)$ belonging to $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$.

Then we observe that $\Lambda_{\#}$ is real analytic in a neighborhood of $(0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#)$. Namely, it is real analytic from $]-\epsilon_0, \epsilon_0[\times \mathcal{O}_{\phi_0} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$. This follows by the real analyticity results for the double layer potential of Lanza de Cristoforis and Rossi [26, Thm. 4.11 (iii)], by real analyticity results for integral operators with real analytic kernel (cf. [23]), by the regularity result for volume potentials of Theorem 2.1, by the analyticity results for the composition operator of Valent [40, Thm. 5.2, p. 44], by Proposition 2.2, by Lemma 2.3, and by standard calculus in Banach spaces.

Since we plan to use the Implicit Function Theorem, we now consider the partial differential $\partial_{(\theta^\#, c^\#)} \Lambda_{\#}[0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#]$ of $\Lambda_{\#}$ at $(0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#)$ with respect to the variable $(\theta^\#, c^\#)$. By standard calculus in Banach spaces we have

$$\begin{aligned} & \partial_{(\theta^\#, c^\#)} \Lambda_{\#,1}[0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#](\bar{\theta}, \bar{c})(t) \\ &= -\frac{1}{2} \bar{\theta}(t) - \int_{\partial\Omega} \nu_{1[\phi_0]} \circ \phi_0(s) \cdot DS_n(\phi_0(t) - \phi_0(s)) \bar{\theta}(s) \tilde{\sigma}[\phi_0](s) d\sigma_s + \bar{c} \quad \forall t \in \partial\Omega, \\ & \partial_{(\theta^\#, c^\#)} \Lambda_{\#,2}[0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#](\bar{\theta}, \bar{c}) = \int_{\partial\Omega} \bar{\theta} \tilde{\sigma}[\phi_0] d\sigma, \end{aligned}$$

for all $(\bar{\theta}, \bar{c}) \in C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$. By arguing as in the proofs of Theorem 3.3 (i) and of [6, Prop. 13.10], we can see that $\partial_{(\theta^\#, c^\#)} \Lambda_{\#}[0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#]$ is a bijection. Then by the Open Mapping Theorem, the operator $\partial_{(\theta^\#, c^\#)} \Lambda_{\#}[0, \phi_0, g_0, f_0, \tilde{\theta}^\#, \tilde{c}^\#]$ is also a homeomorphism from $C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$ to itself.

We can invoke the Implicit Function Theorem for real analytic maps in Banach spaces (cf., e.g., Deimling [9, Thm. 15.3]) and deduce the existence of $\epsilon_{\#,1}$, \mathcal{O}'_{ϕ_0} , \mathcal{U}_0 , and $(\Theta_{\#}, C_{\#})$ as in the statement. \square

4 A functional analytic representation theorem for the solution

We are ready to prove our main Theorem 4.1. As mentioned in the introduction, we will write the map $(\epsilon, \phi, g, f) \mapsto u[\epsilon, \phi, g, f]_{\bar{V}}$ as a combination of real analytic maps of (ϵ, ϕ, g, f) and—possibly singular but completely known—elementary functions of ϵ . In particular, we will focus on the case where (ϵ, ϕ, g, f) is close to a quadruple $(0, \phi_0, g_0, f_0)$ with the size parameter ϵ equal to 0, which is interesting also because of the singular behavior that appears when $\int_Q f_0 dx \neq 0$. Theorem 4.1 is a consequence of Theorem 3.5 on the analytic continuation of $(\epsilon, \phi, g, f) \mapsto (\theta_{\epsilon, \phi, g, f}^{\#}, c_{\epsilon, \phi, g, f}^{\#})$ and of the representation formula for $u[\epsilon, \phi, g, f]$ of Corollary 3.2.

Theorem 4.1. *Let $\alpha \in]0, 1[$. Let $\rho > 0$. Let $p \in Q$. Let Ω be as in (1). Let $(\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1, \alpha} \times C^{1, \alpha}(\partial\Omega) \times C_{q, \omega, \rho}^0(\mathbb{R}^n)$. Let $\epsilon_{\#, 1}$, \mathcal{O}'_{ϕ_0} , \mathcal{U}_0 be as in Theorem 3.5. Let V be a bounded open subset of $\mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$. Then there exist $\epsilon_{\#, 2} \in]0, \epsilon_{\#, 1}[$, an open neighborhood \mathcal{O}''_{ϕ_0} of ϕ_0 in $\mathcal{A}_{\partial\Omega}^{1, \alpha}$ contained in \mathcal{O}'_{ϕ_0} , and a real analytic map \mathfrak{U} from $]-\epsilon_{\#, 2}, \epsilon_{\#, 2}[\times \mathcal{O}''_{\phi_0} \times \mathcal{U}_0$ to $C^2(\bar{V})$ such that*

$$\bar{V} \subseteq \mathbb{S}[\Omega_{\epsilon, \phi}]^- \quad \forall (\epsilon, \phi) \in]-\epsilon_{\#, 2}, \epsilon_{\#, 2}[\times \mathcal{O}''_{\phi_0} \quad (20)$$

and

$$u[\epsilon, \phi, g, f]_{|\bar{V}} = \frac{1}{\epsilon^{n-2}} \mathfrak{U}[\epsilon, \phi, g, f] + \delta_{2, n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \quad (21)$$

$$\forall (\epsilon, \phi, g, f) \in]0, \epsilon_{\#, 2}[\times \mathcal{O}''_{\phi_0} \times \mathcal{U}_0.$$

Moreover,

$$\mathfrak{U}[0, \phi_0, g_0, f_0](x) = \tilde{c}^{\#} + \delta_{2, n} P_q[f_0](x) - \delta_{2, n} S_{q, n}(x - p) \int_Q f_0(y) dy \quad \forall x \in \bar{V}. \quad (22)$$

Proof. Clearly, (20) holds true for $\epsilon_{\#, 2}$ and \mathcal{O}''_{ϕ_0} small enough. Then we note that

$$\begin{aligned} u[\epsilon, \phi, g, f](x) &= -\epsilon \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS_{q, n}(x - p - \epsilon\phi(s)) \Theta_{\#}[\epsilon, \phi, g, f](s) \tilde{\sigma}[\phi](s) d\sigma_s \\ &\quad + \frac{1}{\epsilon^{n-2}} C_{\#}[\epsilon, \phi, g, f] + P_q[f](x) - S_{q, n}(x - p) \int_Q f(y) dy \\ &\quad + \delta_{2, n} \frac{\log \epsilon}{2\pi} \int_Q f(y) dy \end{aligned}$$

for all $x \in \bar{V}$ and all $(\epsilon, \phi, g, f) \in]0, \epsilon_{\#,2}[\times \mathcal{O}''_{\phi_0} \times \mathcal{U}_0$ and we set

$$\begin{aligned} \mathfrak{U}[\epsilon, \phi, g, f](x) &:= -\epsilon^{n-1} \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS_{q,n}(x - p - \epsilon\phi(s)) \Theta_{\#}[\epsilon, \phi, g, f](s) \tilde{\sigma}[\phi](s) d\sigma_s \\ &+ C_{\#}[\epsilon, \phi, g, f] + \epsilon^{n-2} P_q[f](x) - \epsilon^{n-2} S_{q,n}(x - p) \int_Q f(y) dy \end{aligned}$$

for all $x \in \bar{V}$ and all $(\epsilon, \phi, g, f) \in]-\epsilon_{\#,2}, \epsilon_{\#,2}[\times \mathcal{O}''_{\phi_0} \times \mathcal{U}_0$. By Theorem 2.1, by Lemma 2.3, by real analyticity results for integral operators with real analytic kernel (cf. [23]), and by standard calculus in Banach spaces, we deduce that \mathfrak{U} is a real analytic map from $]-\epsilon_{\#,2}, \epsilon_{\#,2}[\times \mathcal{O}''_{\phi_0} \times \mathcal{U}_0$ to $C^2(\bar{V})$ such that equality (21) holds. Since $C_{\#}[0, \phi_0, g_0, f_0] = \tilde{c}_{\#}$, we also deduce the validity of (22). \square

If we fix $(\phi, g, f) = (\phi_0, g_0, f_0) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times C^{1,\alpha}(\partial\Omega) \times C_{q,\omega,\rho}^0(\mathbb{R}^n)$ with

$$\int_Q f_0 dx \neq 0,$$

then formula (21) shows that for $n = 2$ the function $u[\epsilon, \phi_0, g_0, f_0]$ displays a singular behavior of order $\log \epsilon$ as ϵ tends to 0. Under the same assumptions on the triple (ϕ, g, f) , we can see that, for $n \geq 3$, $u[\epsilon, \phi_0, g_0, f_0]$ has a singularity of order ϵ^{2-n} as ϵ tends to 0. This can be deduced from (21) and (22) and remembering that $\tilde{c}_{\#} \neq 0$ (cf. Remark 3.4).

Also, the fact that \mathfrak{U} is real analytic means that we can expand $\mathfrak{U}[\epsilon, \phi, g, f]$ into a power series that converges (in norm) for (ϵ, ϕ, g, f) in a neighborhood of $(0, \phi_0, g_0, f_0)$. The approach presented in this paper can be used to compute the corresponding coefficients (see, e.g., [6, 7, 8]).

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