ORIGINAL ARTICLE

MATHEMATISCHE
NACHRICHTEN

Embedded trace operator for infinite metric trees

Valentina Franceschi[1] | **Kiyan Naderi**[2] | **Konstantin Pankrashkin**[2]

[1]Dipartimento di Matematica Tullio Levi-Civita, Università di Padova, Padova, Italy

[2]Fakultät V, Institut für Mathematik, Carl von Ossietzky Universität Oldenburg, Oldenburg, Germany

Correspondence
Konstantin Pankrashkin, Fakultät V, Institut für Mathematik, Carl von Ossietzky Universität Oldenburg, Ammerländer Heerstr. 114–118, 26129 Oldenburg, Germany.
Email: konstantin.pankrashkin@uol.de

Funding information
GNAMPA

Abstract

We consider a class of infinite weighted metric trees obtained as perturbations of self-similar regular trees. Possible definitions of the boundary traces of functions in the Sobolev space on such a structure are discussed by using identifications of the tree boundary with a surface. Our approach unifies some constructions proposed by Maury, Salort, and Vannier for discrete weighted dyadic trees (expansion in orthogonal bases of harmonic functions on the graph and using Haar-type bases on the domain representing the boundary), and by Nicaise and Semin and Joly, Kachanovska, and Semin for fractal metric trees (approximation by finite sections and identification of the boundary with a interval): We show that both machineries give the same trace map, and for a range of parameters we establish the precise Sobolev regularity of the traces. In addition, we introduce new geometric ingredients by proposing an identification with arbitrary Riemannian manifolds. It is shown that any compact manifold admits a suitable multiscale decomposition and, therefore, can be identified with a metric tree boundary in the context of trace theorems.

KEYWORDS
fractional Sobolev space, metric graph, multiscale analysis, quantum graph, trace theorem, triangulation of a manifold

1 | INTRODUCTION

In this paper, we discuss some aspects of the trace theory for Sobolev spaces on infinite metric graphs. By a metric graph we mean a geometric configuration that arises if one replaces the edges of a discrete graph by intervals, and one introduces a differential operator on such a structure by defining a differential expression on each interval and by imposing a gluing condition at each node. A metric graph together with a differential operator on it is often called a quantum graph. While quantum graphs represent by now a well-established theory [3, 4, 18, 42], the most attention was concentrated on the study of regular configurations with suitable lower bounds on the edge lengths and other parameters: In that case, it is known that gluing conditions at the nodes are sufficient to define a self-adjoint operator or a non–self-adjoint one with good properties [25]. More recent papers [11, 12, 16, 19, 23, 40, 45, 60] initiated the discussion of the most general quantum graphs, which shows that in many cases additional "boundary conditions at the external boundary" must be imposed. It should be noted that the notion of boundary for general graphs is not obvious, which is a well-known issue for both

metric and discrete infinite graphs [12, 33, 41, 46, 66]; we recall that metric and discrete graphs show a number of common features [13, 41, 44, 53], and in case of equilateral metric graphs even a kind of unitary equivalence between respective Laplacians can be established, see the first paper by von Below [63] and later developments, for example, in [54, 55, 64].

If the boundary is naturally defined (e.g., for a tree, the set of infinite paths starting at a fixed vertex can be naturally viewed as the boundary), one arrives at the question of the description of possible boundary conditions, which requires a construction of suitable function spaces at the boundary as well as a rigorous definition of boundary values for functions defined on the graph. For some classes of infinite trees, the abstract boundary can be endowed with a metric structure, which gives rise to Sobolev-type spaces and associated trace theorems. The paper [5] deals with dyadic discrete trees: One identifies the tree boundary with the ring $\mathbb{Z}_2$ of 2-adic numbers and establishes a trace theorem for a class of weights. A similar construction was applied in [7, 32, 35–39] for more general trees and more general function spaces as well: In all these works, the boundary of a discrete tree is viewed as a totally disconnected metric space with, respectively, defined function spaces on it. The paper [12] makes first steps in defining the boundary of a metric graph as a totally disconnected metric space, but no precise trace theorem was established. Further trace results for a class of metric graphs with unit edge lengths were established in [38, 39] with the unit length edges, still for an abstractly defined boundary. At the same time, if a metric graph is considered as a model of a structure embedded into a space, then in many cases, its boundary can be naturally identified with a surface: As an example, one may think of a dandelion clock, whose surface can be viewed as a sphere in the three-dimensional Euclidean space. The aim of this work is to elaborate a theory of boundary traces on metric trees using *concrete geometric realizations* of the abstract tree boundary. Our analysis includes a description of the *range* of the trace operator (and not just its boundedness), which seems to be new for metric graphs.

The idea of such an identification is not completely new, and our work is mainly motivated by the papers [30, 47, 52, 61] dealing with the analysis of Dirichlet-to-Neumann operators and wave equations on trees viewed as a model of human lung. In particular, our main object (geometric tree) is directly borrowed from [30]. One may also mention the paper [6], in which abstract metric spaces are obtained as the limits of embedded discrete structures (hyperbolic tilings), which leads to some precise mapping theorems as well. Nevertheless, while the tilings used show some ideological similarities with the multiscale decompositions employed in our work (see below), our precise situation (trace map for a metric tree) is not covered directly by the constructions of [6]. We also mention the papers [14, 26] in which abstract versions of multiscale decompositions (usually referred to as dyadic or p-adic cubes) were used for other purposes such as boundedness criteria for singular integral operators: We refer to Kigami's monograph [34] for a variety of further applications. While some elements of our analysis look very similar to the ideas used in the analysis in metric spaces with the help of p-adic cubes, the objectives are not same: We have two given objects (a metric tree and a Riemannian manifold) with smooth structures that are already prescribed, and the goal is to interpret one of them as the boundary of the second one. This setting is quite different to what was done for the abstract metric spaces, where the key step is the definition of a smooth structure using suitable embedded trees/graphs.

The paper [47], which was our main starting point, deals with weigthed discrete Laplacians on an infinite dyadic tree, and it established a trace theorem for discrete Sobolev spaces by identifying the boundary with a Euclidean domain. Note that the later paper [5] has shown the equivalence of the resulting mapping theorems with the approach based on abstract tree boundaries viewed as ultrametric spaces. The works [30, 52, 61] proposed a modified model with the help of the continuous weighted Laplacians, and an identification of the boundary trace with an interval was addressed. Moreover in [61] the Sobolev regularity was partially studied. The notion of the boundary trace was then used as a theoretical tool in [52, 61] in order to establish the equivalence between various definitions of the Sobolev spaces on the fractal trees, which further served for numerical approximation of infinite trees by their finite truncations when solving the wave equation [27–29, 31]. It should be said that the approaches of [30] and [30, 52, 61] to the definition of the boundary trace were quite different: The paper [47] uses an orthonormal basis of harmonic functions (so that the definition of the boundary trace of an arbitrary function is recovered from its expansion in this basis), while [30, 52, 61] used more explicit approximations by finite truncations. As a by-result of our analysis, one obtains that both the approaches are equivalent and allow for inclusion of Riemannian manifolds and not just Euclidean domains.

We now describe our configuration and the main results in greater detail. Let $p \in \mathbb{N}$ with $p \geq 2$ be given and a root o be given. We glue to o an edge $e_{0,0}$ represented by an interval of length $\ell_{0,0}$, the second vertex of $e_{0,0}$ will be called $X_{0,0}$. If all $e_{n,k}$ and $X_{n,k}$ with $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, p^n - 1\}$ are already constructed, then to each $X_{n,k}$ we attach p new edges $e_{n+1,pk+j}$, with $j \in \{0, \ldots, p-1\}$, having lengths $\ell_{n+1,pk+j}$, and the pendant vertices of $e_{n+1,pk+j}$, to be denoted by $X_{n+1,pk+j}$, will be viewed as children of $X_{n,k}$. This process continues infinitely, which creates a infinite rooted metric tree $\mathcal{T}$. The subtree of $\mathcal{T}$ starting at $X_{n,k}$, that is, the subtree spanned by the offsping of $X_{n,k}$ (the children, the children of the children, etc.), will be denoted by $\mathcal{T}_{n,k}$. See Figure 1.
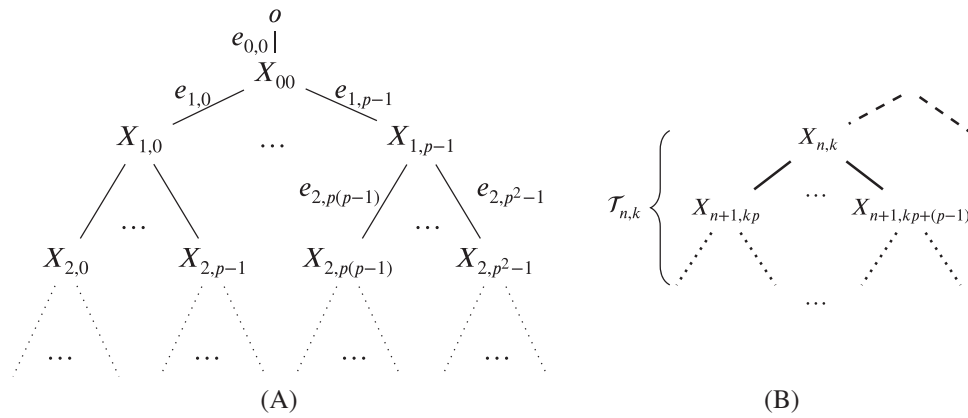
FIGURE 1 (A) The tree $\mathcal{T}$. (B) A subtree $\mathcal{T}_{n,k}$.

For subsequent constructions, it will be useful to introduce coordinates on $\mathcal{T}$. Denote by $L_{n,k}$ the distance between the root o and $X_{n,k}$, that is, the length of the unique path between o and $X_{n,k}$ obtained by summing the lengths of all edges in the path. Then by (n,k,t) with $t \in [L_{n,k} - \ell_{n,k}, L_{n,k}]$, we denote the point of $e_{n,k}$, which is at the distance $L_{n,k} - t$ from $X_{n,k}$. In this notation,

$$X_{n,k} = (n,k,L_{n,k}) = (n+1, pk+j, L_{n,k}) \text{ for any } j \in \{0, \dots, p-1\}.$$

Let $w : \mathcal{T} \to (0, \infty)$ be a locally bounded measurable function, which will be used as an integration weight: For $f : \mathcal{T} \to \mathbb{C}$, one defines

$$\int_{\mathcal{T}} f \, d\mu := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{L_{n,k}-\ell_{n,k}}^{L_{n,k}} f(n,k,t) \, w(n,k,t) \, dt,$$

then

$$L^2(\mathcal{T}) := \left\{ f : \mathcal{T} \to \mathbb{C} : \|f\|_{L^2(\mathcal{T})}^2 := \int_{\mathcal{T}} |f|^2 \, d\mu < \infty \right\}.$$

Due to the above definition, the set of vertices has zero measure. Therefore, each measurable function $f : \mathcal{T} \to \mathbb{C}$ can be identified with a family of functions $(f_{n,k})$,

$$f_{n,k} := f(n,k,\cdot) : (L_{n,k} - \ell_{n,k}, L_{n,k}) \to \mathbb{C}, \quad n \in \mathbb{N}_0, \quad k \in \{0, \dots, p^n-1\}.$$

Then $f = (f_{n,k})$ belongs to $L^2(\mathcal{T})$ if and only if

$$\|f\|_{L^2(\mathcal{T})}^2 := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{L_{n,k}-\ell_{n,k}}^{L_{n,k}} |f_{n,k}(t)|^2 w_{n,k}(t) \, dt < \infty.$$

If $f = (f_{n,k})$ is such that all $f_{n,k}$ have locally integrable distributional derivatives $f'_{n,k}$, we denote $f' := (f'_{n,k})$. The first Sobolev space $H^1(\mathcal{T})$ on $\mathcal{T}$ is then introduced as

$$H^1(\mathcal{T}) := \{ f \in L^2(\mathcal{T}) : f \text{ is continuous with } f' \in L^2(\mathcal{T}) \}, \qquad \|f\|_{H^1(\mathcal{T})}^2 := \|f\|_{L^2(\mathcal{T})}^2 + \|f'\|_{L^2(\mathcal{T})}^2.$$

Moreover, we denote

$$H_c^1(\mathcal{T}) := \{ f \in H^1(\mathcal{T}) : \text{ there exists } N \in \mathbb{N} \text{ such that } f_{n,k} \equiv 0 \text{ for all } (n,k) \text{ with } n > N \},$$

$$H_0^1(\mathcal{T}) := \text{ the closure of } H_c^1(\mathcal{T}) \text{ in } H^1(\mathcal{T}).$$
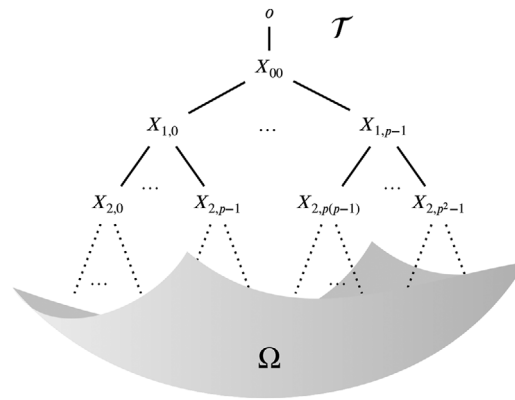
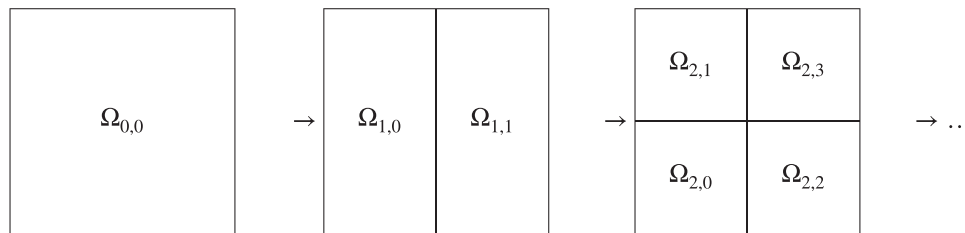**FIGURE 2** The boundary of $\mathcal{T}$ viewed as a surface Ω.

FIGURE 3 An example of a multiscale decomposition (for $p = 2$).

One arrives at the following quite natural questions:

(a) Do we have $H_0^1(\mathcal{T}) = H^1(\mathcal{T})$?
(b) If not, can we characterize the functions in $H_0^1(\mathcal{T})$ by their "behavior of infinity," that is, by the behavior of $f = (f_{n,k}) \in H^1(\mathcal{T})$ for $n \to \infty$?
(c) Can this "behavior at infinity" be characterized by a function defined on some set Ω viewed as the "boundary" of $\mathcal{T}$?

Remark that the H^1-norm on $\mathcal{T}$ represents the sesquilinear form of the Neumann Laplace operator Δ, which is important for the study of various diffusion processes on $\mathcal{T}$. One can also consider first a thickened version $\mathcal{T}_\varepsilon$ of $\mathcal{T}$ (i.e., one embeds $\mathcal{T}$ in $\mathbb{R}^n$ and takes the ε-neighborhood) and consider the associated Neumann Laplacian Δ_ε, then one has a suitably defined convergence of Δ_ε to Δ as $\varepsilon \to 0$, see [57, 58]. The problem (a) is related to the question whether the boundary of $\mathcal{T}$ is penetrable, that is, whether one can impose alternative boundary conditions at the tree boundary. In fact, problem (a) was already addressed for several classes of metric graphs, see [40, Section 3] and [41, Theorem 7.24], but none of them covers our situation in full generality. The problems (b) and (c) are closely related to a concrete representation of such conditions and to the existence and uniqueness of solutions of the associated boundary value problems. We provide answers to the above questions by identifying the abstract boundary $\partial\mathcal{T}$ with a geometric object, more precisely, an open set Ω with compact closure in a d-dimensional Riemannian manifold (in particular, Ω is allowed to be an arbitrary compact Riemannian manifold), see Figure 2. The main assumption on Ω is that it admits a special decomposition: There exists $\Omega_{n,k} \subset \Omega$, $n \in \mathbb{N}_0$, $k \in \{0, \dots, p^n - 1\}$, constructed as follows. One sets $\Omega_{0,0} := \Omega$. If some $\Omega_{n,k}$ is constructed, one chooses p nonempty disjoint subsets $\Omega_{n+1,pk+j} \subset \Omega_{n,k}$, $j \in \{0, \dots, p-1\}$, such that

$$\left| \Omega_{n,k} \setminus \bigcup_{j=0}^{p-1} \Omega_{n+1,pk+j} \right| = 0,$$

and this process continues infinitely (Figure 3). In addition, one needs to impose some geometric conditions on $\Omega_{n,k}$ for large n: Informally, all $\Omega_{n,k}$ must have approximately the same volume, and their shape is not allowed to become "too complicated." A decomposition satisfying all necessary assumptions will be called *a regular strongly balanced p-multiscale decomposition* of Ω (we refer to Subsection 3.2 for rigorous definitions concerning Euclidean open sets and to Subsection 3.4 for an extension to the case of manifolds).
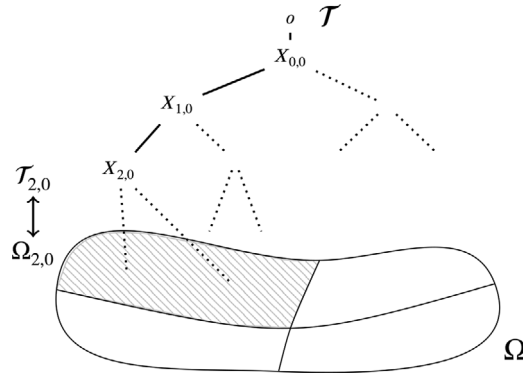
FIGURE 4   Identifying $\mathcal{T}_{n,k}$ with $\Omega_{n,k}$.

Remark that the combinatorial structure of the family $(\Omega_{n,k})$ repeats the combinatorial structure of the family of subtrees $(\mathcal{T}_{n,k})$: For arbitrary (n, k) and (n', k'), one has

(1) $\Omega_{n,k} \subset \Omega_{n',k'}$ if and only if $\mathcal{T}_{n,k} \subset \mathcal{T}_{n',k'}$,
(2) $\Omega_{n,k} \cap \Omega_{n',k'} \neq \emptyset$ if and only if $\mathcal{T}_{n,k} \cap \mathcal{T}_{n',k'} \neq \emptyset$,

and this observation is used to create a link between the functions defined on $\mathcal{T}$ and those defined on Ω. More precisely, one imagines that the boundary of $\mathcal{T}$ is glued to Ω in such a way that the boundary of each $\mathcal{T}_{n,k}$ is glued to $\Omega_{n,k}$. In this case, if a function f on $\mathcal{T}$ has a constant value $\alpha_{n,k}$ along some $\mathcal{T}_{n,k}$ and is zero on all other subtrees $\mathcal{T}_{n,j}$ with $j \neq k$, then it natural to identify the boundary trace of f with the function $\alpha_{n,k} \mathbb{1}_{\Omega_{n,k}}$ (Figure 4). It appears that this somewhat naive definition can be given a rigorous form, and a part of our main results can be summarized as follows:

Theorem 1.1. *Assume that there exist constants $\alpha > 0, 0 < \ell < 1$, and $c \geq 1$ such that for any $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, p^n - 1\}$, there holds*

$$c^{-1} \ell^n \leq \ell_{n,k} \leq c \ell^n, \quad c^{-1} \alpha^n \leq w_{n,k} \leq c \alpha^n. \tag{1}$$

Then $H_0^1(\mathcal{T}) \neq H^1(\mathcal{T})$ if and only if

$$\ell < \alpha p < \frac{1}{\ell}. \tag{2}$$

Assume that (2) is satisfied and let Ω be a nonempty open set with compact closure in a d-dimensional Riemannian manifold of bounded geometry. We denote by $H^s(\Omega)$ the associated fractional Sobolev spaces of order $s \geq 0$ and require that Ω admits a regular strongly balanced p-multiscale decomposition $(\Omega_{n,k})$ as defined in Subsection 3.4. Denote

$$\sigma := \frac{1}{2}\left(1 - \frac{\log \ell - \log \alpha}{\log p}\right) > 0$$

and let $0 \leq s < \frac{1}{2}$ such that $s \leq \sigma d$, then for any $f \in H^1(\mathcal{T})$, there exists the limit

$$\gamma_\Omega^\mathcal{T} f := \lim_{N \to \infty} \sum_{K=0}^{p^N - 1} f(X_{N,K}) \mathbb{1}_{\Omega_{N,K}} \in H^s(\Omega).$$

The embedded trace operator defined by $\gamma_\Omega^\mathcal{T} : H^1(\mathcal{T}) \to H^s(\Omega)$ is a bounded linear operator with $\ker \gamma_\Omega^\mathcal{T} = H_0^1(\Omega)$, and

$$\gamma_\Omega^\mathcal{T}\left(H^1(\mathcal{T})\right) = H^{\sigma d}(\Omega) \text{ if } \sigma d < \frac{1}{2}.$$

Remark 1.2. The right-hand inequality in (2) means precisely that the tree $\mathcal{T}$ has finite total weight.

Remark 1.3. If Ω is a Euclidean open set, then one can show that the linear map $\gamma_\Omega^{\mathcal{T}}$ given by the same expression is bounded and surjective as a map from $H^1(\mathcal{T})$ to $A^{\sigma d}(\Omega)$ for *any* value of σ, where $A^{\sigma d}(\Omega)$ is a so-called approximation space (which happens to coincide with $H^{\sigma d}$ if $\sigma d < \frac{1}{2}$): We refer to Subsection 4.3 for more detailed formulations. This settles the open question [52, section 5, Question 2] about the range of the embedded trace operator for our class of metric graphs, even for a more general geometric trace realization.

For $d = 1$ our result is very close to the construction of the bounded trace operator in [61, Theorem 5.4.13] and [30, sections 3.1–3.2], but even in this case our result is stronger (for the class of trees we consider) as we show its surjectivity for a range of parameters.

A large part of the paper is devoted to the proof of the assertions of Theorem 1.1 for the case $c = 1$ in (1). For this special case, the tree $\mathcal{T}$ will be denoted by $\mathbb{T}$ and called *geometric tree* following the convention proposed in [30]. The advantage of the geometric tree is that it allows for a decomposition into a direct sum of one-dimensional problems, and trace theorems in one dimension are much simpler to study. Such a decomposition is well known [51, 57], but we need a number of explicit formulas for various intermediate transformation operators, which are missing in the existing literature, so we opted for a self-contained presentation in Section 2. This part of analysis is concluded by constructing an abstract trace operator in Subsection 2.5, which maps $H^1(\mathbb{T})$ into a discrete ℓ^2-type space inherited from the direct sum decomposition.

In Section 3, we introduce approximation spaces $A^r(\Omega)$, which consist of the functions defined on Ω that can be "well approximated" by linear combinations of indicator functions of some subsets of Ω. In Subsection 3.1, we recall the most important constructions for fractional Sobolev spaces, which are used in the analysis. In Subsection 3.2, we introduce special decompositions of Euclidean domains and define the associated approximation spaces. In Subsection 3.3, we show that in some important cases the approximation spaces coincide with the usual fractional Sobolev spaces. The constructions of Subsections 3.2 and 3.3 are an adaptation of the respective 2-adic spaces in [47], which were in turn motivated by more general considerations coming from the wavelet analysis [15, 48]. In Subsection 3.4, we transfer these constructions to the case of open sets on manifolds using the traditional approach with local charts. We note that Sections 2 and 3 are independent from each other. They also contain a lot of introductory material and we hope that they can be of independent interest beyond the immediate scope of this work.

In Section 4, we make last steps in the construction of the embedded trace operator. First, in Subsection 4.1, we identify the ℓ^2-space from the construction of the abstract trace operator with the approximation spaces $A^r(\Omega)$ using an identification of suitable bases. The embedded trace operator is then obtained as the superposition of this identification with the abstract trace operator, and the resulting properties are summarized in Subsection 4.2. At this point, all assertions of Theorem 1.1 are proved for the geometric tree $\mathbb{T}$, and in Subsection 4.3, we transfer them to the general $\mathcal{T}$ using a coordinate change.

All preceding results require the existence of decompositions of open sets or manifolds into pieces with special properties; it seems that these questions were not addressed in sufficient generality in earlier works. In the last Section 5, we show that such decompositions exist for large classes of Ω, in particular, for all convex polyhedrons, all convex smooth domains and all compact manifolds. This is done by adapting the existing results from very diverse areas of analysis to the context of multiscale decompositions.

We consider this work as an initial component for the systematic analysis of boundary value and transmission problems on infinite metric graphs, which will be continued in several directions. A key role in our analysis is played by the decomposition of trees into a direct sum of one-dimensional problems. It was noted in [8] that such decomposition actually exists for a much larger class of metric graphs, so we hope that at least some elements of our analysis will be useful beyond the context of trees. The possibility of the identification of the tree boundary with a prescribed surface gives a possible approach to describe the interaction between fractal trees touching each other along some interface and to include fractal building blocks in the so-called hybrid spaces [2, 9, 10, 17, 56, 59]. Such applications will be covered in ongoing works.

2 | ANALYSIS ON GEOMETRIC TREES

2.1 | Tree structure and function spaces

In this section, we analyze in greater detail the "ideal" case $\ell_{n,k} := \ell^n$ and $w_{n,k} = \alpha^n$ for all (n, k), that is, with $c = 1$ in (1). The corresponding tree will be denoted $\mathbb{T}$ (as opposite to $\mathcal{T}$ for the general case) and called a *geometric tree*. The geometric
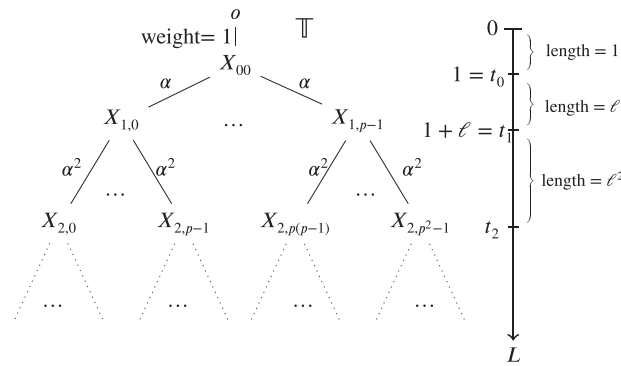
FIGURE 5   The structure of a geometric tree.

trees have a lot of symmetries, which will be exploited for the analysis, and some expressions can be written in a slightly different form.

Remark that the underlying combinatorial graph is $G := (V, E)$, with the set of vertices V and the set of edges E given by

$$V := \{o\} \cup \{X_{n,k} : n \in \mathbb{N}_0, \ k \in \{0, 1, \dots, p^n - 1\}\},$$

$$E := \{e_{n,k} : n \in \mathbb{N}_0, \ k \in \{0, 1, \dots, p^n - 1\}\},$$

$$e_{n,k} := \begin{cases} (0, X_{0,0}), & n = 0, \\ (X_{n-1,[\log_p k]}, X_{n,k}), & n \geq 0, \end{cases}$$

where $[t]$ stands for the integer part of $t \in [0, \infty)$, that is, the largest integer not exceeding t. Remark that each edge $e_{n,k}$ connects each $X_{n,k}$ with its uniquely defined parent, which is $X_{n-1,[\log_p k]}$ for $n \geq 1$ and o for $n = 0$. All vertices except the root have the degree $p + 1$ (i.e., have $p + 1$ neighbors: p children and 1 parent), and the degree of the root is 1. For $p = 1$, the graph G is simply a half-infinite chain, so we assume from now on that $p \geq 2$.

Consider the numbers

$$t_{-1} := 0, \quad t_n = \sum_{k=0}^{n} \ell^k, \quad n \in \mathbb{N}_0, \quad L := \lim_{n \to \infty} t_n \in (0, \infty]. \tag{3}$$

By construction, the numbers $t_{-1} < t_0 < t_1 < \dots$ subdivide $(0, L)$ into the infinitely many intervals (t_{n-1}, t_n) of length ℓ^n, $n \in \mathbb{N}_0$ (Figure 5). The combinatorial tree G is related to the metric tree $\mathbb{T}$ as follows: We identify each $e_{n,k}$ with a copy of $[t_{n-1}, t_n]$ using the convention that the endpoint of $e_{n,k}$ is identified with the initial point of each of its children. In other words,

$$\mathbb{T} := \{((n, k), t) : n \in \mathbb{N}_0, \ k \in \{0, \dots, p^n - 1\}, \ t \in [t_{n-1}, t_n]\} / \sim$$

for the identification $\sim$ defined by

$$((n, k), t_n) \sim ((n + 1, pk + j), t_n), \quad n \in \mathbb{N}_0, k \in \{0, \dots, p^n - 1\}, j \in \{0, \dots, p - 1\}.$$

For what follows for $x, y \in \mathbb{T}$, we write

(1) $x \leq y$ if the path from o to y passes through x (equivalently one can say that y belongs to the offspring of x),
(2) $x < y$ if $x \leq y$ and $x \neq y$.

We will also need to consider some special subgraphs of $\mathbb{T}$. For $n \in \mathbb{N}$, denote

$$\mathbb{T}^n := \text{ the tree truncated after the } n\text{th generation,}$$
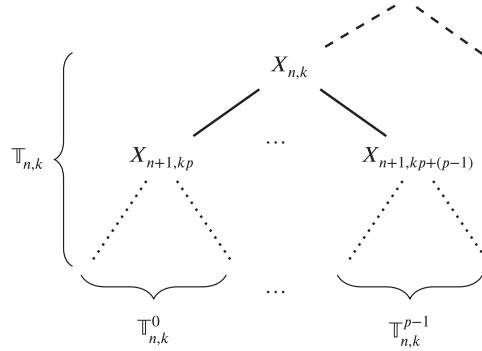
FIGURE 6 The subtrees $\mathbb{T}_{n,k}$ and $\mathbb{T}_{n,k}^j$.

that is, $\mathbb{T}^n$ is composed of all edges $e_{m,k}$ with $m \le n$. For $n \in \mathbb{N}$, $k \in \{0, \dots, p^n - 1\}$, consider

$$\mathbb{T}_{n,k} := \{x \in \mathbb{T} : X_{n,k} \le x\}.$$

Remark that

$$\mathbb{T}_{n,k} = \bigcup_{j=0}^{p-1} \mathbb{T}_{n,k}^j, \qquad \mathbb{T}_{n,k}^j := e_{n+1,pk+j} \cup \mathbb{T}_{n+1,pk+j},$$

see Figure 6. By construction, each $\mathbb{T}_{n,k}^j$ is a rooted metric tree (having the same combinatorial structure as $\mathbb{T}$ itself) with $X_{n,k}$ being the root. The vertices of $\mathbb{T}_{n,k}^j$ are $X_{n,k}$ and $X_{n+m,kp^m+jp^{m-1}+r}$ with $m \in \mathbb{N}$ and $r \in \{0, \dots, p^{m-1} - 1\}$.

The set $\mathbb{T}$ becomes a metric space if one considers the natural distance ρ,

$$\rho(x, y) := \text{the length of the unique path between } x, y \in \mathbb{T}, \qquad |x| := \rho(x, o) \text{ for } x \in \mathbb{T},$$

which gives rise to the notion of a continuous function on $\mathbb{T}$. The number L in (3) is usually referred to as the height of $\mathbb{T}$, and

> for all subsequent constructions we assume
>
> $$L < \infty \qquad \text{or, equivalently,} \qquad \ell < 1.$$

We consider the measure μ on $\mathbb{T}$, which coincides with $\alpha^n \, dt$ along $e_{n,k}$, where dt is the one-dimensional Lebesgue measure and $\alpha > 0$ is a fixed constant. A function $f : \mathbb{T} \to \mathbb{C}$ is measurable if each of its components

$$f_{n,k} : [t_{n-1}, t_n] \ni t \mapsto f((n,k), t) \in \mathbb{C}, \qquad f_{n,k} := f|_{e_{n,k}} \text{ for short,}$$

is measurable; in most cases, we will identify f with the set of its components $(f_{n,k})$. The integral of such f over $\mathbb{T}$ with respect to μ is then given by

$$\int_{\mathbb{T}} f \, d\mu := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \alpha^n \int_{t_{n-1}}^{t_n} f_{n,k}(t) \, dt.$$

The above integration gives rise to the naturally defined space $L^2(\mathbb{T})$:

$$L^2(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \text{ measurable} : \|f\|_{L^2(\mathbb{T})}^2 := \int_{\mathbb{T}} |f|^2 \, d\mu < \infty \right\},$$

$$\int_{\mathbb{T}} |f|^2 \, d\mu := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \alpha^n \|f_{n,k}\|_{L^2(e_{n,k})}^2, \qquad \|f_{n,k}\|_{L^2(e_{n,k})}^2 := \int_{t_{n-1}}^{t_n} |f_{n,k}(t)|^2 \, dt.$$

In addition, we consider the Sobolev-type space $H^1(\mathbb{T})$ defined by

$$H^1(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : f_{n,k} \in H^1(t_{n-1}, t_n) \text{ for any } (n,k), f' := (f'_{n,k}) \in L^2(\mathbb{T}) \text{ and } f \text{ is continuous on } \mathbb{T} \right\}.$$

Recall that $H^1(t_{n-1}, t_n) \subset C^0([t_{n-1}, t_n])$ due to Sobolev embedding theorem, so the continuity of f on $\mathbb{T}$ in the above definition of $H^1(\mathbb{T})$ actually means the continuity at the vertices,

$$f_{n,k}(t_n) = f_{n+1,pk+j}(t_n) \text{ for all } n \in \mathbb{N}_0, k \in \{0, \dots, p^n - 1\}, j \in \{0, \dots, p - 1\}.$$

We equip $H^1(\mathbb{T})$ with the scalar product $\langle \cdot, \cdot \rangle_{H^1(\mathbb{T})}$ defined by

$$\langle f, g \rangle_{H^1(\mathbb{T})} := \langle f, g \rangle_{L^2(\mathbb{T})} + \langle f', g' \rangle_{L^2(\mathbb{T})}$$

and the induced norm $\| \cdot \|_{H^1(\mathbb{T})}$, then one easily checks that $H^1(\mathbb{T})$ becomes a Hilbert space. The following result from [30, section 3.5] will be important below:

Lemma 2.1. *For any $p \in \mathbb{N}$, $\ell \in (0,1)$, $\alpha > 0$, the embedding $H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is compact.*

Now we denote

$$H^1_c(\mathbb{T}) := \left\{ f = (f_{n,k}) \in H^1(\mathbb{T}) : \exists N \in \mathbb{N} \text{ such that } f_{n,k} \equiv 0 \text{ for } n > N \right\},$$

$$H^1_0(\mathbb{T}) := \text{the closure of } H^1_c(\mathbb{T}) \text{ in } H^1(\mathbb{T}).$$

2.2 | Orthogonal decompositions

All constructions in this section are essentially from the paper [51], but we need several intermediate objects, which did not appear there explicitly, so we prefer to give a complete argument.

We now introduce several subspaces of $L^2(\mathbb{T})$ determined through additional invariance properties. We start with the space of radial functions,

$$L^2_{\text{rad}}(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \text{ for any } x, y \in \mathbb{T} \text{ with } |x| = |y| \text{ one has } f(x) = f(y) \right\},$$

which will be considered with the induced scalar product. Remark that $f \in L^2_{\text{rad}}(\mathbb{T})$ means the existence of a function $F : (0, L) \to \mathbb{C}$ with $f(x) = F(|x|)$ for all $x \in \mathbb{T}$, which means that the components $(f_{n,k})$ satisfy $f_{n,k}(t) = F(t)$ for all $e_{n,k} \in E$ and $t \in (t_{n-1}, t_n)$. This yields

$$\|f\|^2_{L^2(\mathbb{T})} = \sum_{n=0}^{\infty} \alpha^n \sum_{k=0}^{p^n-1} \int_{t_{n-1}}^{t_n} |f_{n,k}(t)|^2 \, dt = \sum_{n=0}^{\infty} \alpha^n \sum_{k=0}^{p^n-1} \int_{t_{n-1}}^{t_n} |F(t)|^2 \, dt = \sum_{n=0}^{\infty} \alpha^n p^n \int_{t_{n-1}}^{t_n} |F(t)|^2 \, dt = \int_0^L |F(t)|^2 q(t) \, dt$$

for the weight function

$$q : (0, L) \mapsto (0, \infty), \quad q(t) = (\alpha p)^n \text{ for } t \in (t_{n-1}, t_n).$$

The above computation shows that the map

$$U_{\text{rad}} : L^2((0, L), q(t) \, dt) \to L^2_{\text{rad}}(\mathbb{T}), \qquad U_{\text{rad}} F : \mathbb{T} \ni x \mapsto F(|x|),$$

is a unitary operator. Furthermore, consider the roots of unity:

$$\theta_s := e^{\frac{2\pi i}{p} s}, \quad s \in \{0, \dots, p - 1\},$$

and define, for $e_{n,k} \in E$ and $s \in \{1, \dots, p-1\}$,

$$L^2_{n,k,s}(\mathbb{T}) := \big\{ f \in L^2(\mathbb{T}) : \ f|_{\mathbb{T} \setminus \mathbb{T}_{n,k}} = 0 \text{ and for any } j, j' \in \{0, \dots, p-1\} \text{ and}$$

$$x \in \mathbb{T}^j_{n,k} \text{ and } y \in \mathbb{T}^{j'}_{n,k} \text{ with } |x| = |y| \text{ one has } \theta_s^{-j} f(x) = \theta_s^{-j'} f(y) \big\}.$$

Observe that each function $f \in L^2_{n,k,s}(\mathbb{T})$ is radial on each subtree $\mathbb{T}^j_{n,k}$, which means that $f(x)$ only depends on the distance $\rho(x, X_{n,k})$ for all $x \in \mathbb{T}^j_{n,k}$. More precisely, for some functions $F_j : (t_n, L) \to \mathbb{C}$, $j \in \{0, \dots, p-1\}$, there holds $f(x) = F_j(|x|)$ for all $x \in \mathbb{T}^j_{n,k}$, and, in addition, $F_j = \theta_s^j F_0$ for each $j \in \{0, \dots, p-1\}$, so f is uniquely determined by $F := F_0$. By construction we have

$$\int_{\mathbb{T}^j_{n,k}} |f|^2 \, \mathrm{d}\mu = \sum_{m=1}^{\infty} \alpha^{n+m} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} \left| f_{n+m, k p^m + j p^{m-1} + r}(t) \right|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \alpha^{n+m} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} \left| F_j(t) \right|^2 \mathrm{d}t$$

$$= \sum_{m=1}^{\infty} \alpha^{n+m} p^{m-1} \int_{t_{n+m-1}}^{t_{n+m}} \left| F_j(t) \right|^2 \mathrm{d}t$$

$$= p^{-n-1} \sum_{m=n+1}^{\infty} (\alpha p)^m \int_{t_{m-1}}^{t_m} \left| F_j(t) \right|^2 \mathrm{d}t = p^{-n-1} \int_{t_n}^{L} \left| F_j(t) \right|^2 q(t) \, \mathrm{d}t,$$

and

$$\int_{\mathbb{T}} |f|^2 \, \mathrm{d}\mu = \int_{\mathbb{T}_{n,k}} |f|^2 \, \mathrm{d}\mu = \sum_{j=0}^{p-1} \int_{\mathbb{T}^j_{n,k}} |f|^2 \, \mathrm{d}\mu = p^{-n-1} \sum_{j=0}^{p-1} \int_{t_n}^{L} \left| F_j(t) \right|^2 q(t) \, \mathrm{d}t = p^{-n-1} \sum_{j=0}^{p-1} \int_{t_n}^{L} \left| \theta_s^j F(t) \right|^2 q(t) \, \mathrm{d}t$$

$$= p^{-n-1} \sum_{j=0}^{p-1} \int_{t_n}^{L} |F(t)|^2 q(t) \, \mathrm{d}t = p^{-n} \int_{t_n}^{L} |F(t)|^2 q(t) \, \mathrm{d}t.$$

This computation shows that the map

$$U_{n,k,s} : L^2((t_n, L), p^{-n} q(t) \, \mathrm{d}t) \to L^2_{n,k,s}(\mathbb{T}),$$

$$U_{n,k,s} F : \ x \mapsto \begin{cases} \theta_s^j F(|x|), & \text{if } x \in \mathbb{T}^j_{n,k} \text{ for some } j \in \{0, \dots, p-1\}, \\ 0, & \text{otherwise}, \end{cases}$$

is a unitary operator. We further denote

$$H^1_{\mathrm{rad}}(\mathbb{T}) := H^1(\mathbb{T}) \cap L^2_{\mathrm{rad}}(\mathbb{T}), \quad H^1_{n,k,s}(\mathbb{T}) := H^1(\mathbb{T}) \cap L^2_{n,k,s}(\mathbb{T}),$$

and note the obvious implications

$$f \in H^1_{\mathrm{rad}}(\mathbb{T}) \Rightarrow f' \in L^2_{\mathrm{rad}}(\mathbb{T}), \quad f \in H^1_{n,k,s}(\mathbb{T}) \Rightarrow f' \in L^2_{n,k,s}(\mathbb{T}). \tag{4}$$

Theorem 2.2 [51]. *One has the orthogonal direct sum decomposition*

$$L^2(\mathbb{T}) = L^2_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} L^2_{n,k,s}(\mathbb{T}). \tag{5}$$

Proof. (A) We begin with the orthogonality. Let $f \in L^2_{n,k,s}(\mathbb{T})$ for some (n, k, s), then $f = U_{n,k,s} F$ for some $F \in L^2((t_n, L), p^{-n} q(t) \, \mathrm{d}t)$.

(A.1) Let $g \in L^2_{\mathrm{rad}}(\mathbb{T})$, then $g = U_{\mathrm{rad}}G$ for some $G \in L^2\left((0, L), q(t)\,\mathrm{d}t\right)$ and

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \sum_{j=0}^{p-1} \int_{\mathbb{T}_{n,k}^j} f\bar{g}\,\mathrm{d}\mu = \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{n+m} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} f_{n+m,kp^m+jp^{m-1}+r}(t)\, \overline{g_{n+m,kp^m+jp^{m-1}+r}(t)}\,\mathrm{d}t$$

$$= \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{n+m} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} \theta_s^j F(t)\overline{G(t)}\,\mathrm{d}t = \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{n+m} p^{m-1} \int_{t_{n+m-1}}^{t_{n+m}} \theta_s^j F(t)\overline{G(t)}\,\mathrm{d}t$$

$$= \sum_{j=0}^{p-1} p^{-n-1}\theta_s^j \sum_{m=n+1} (\alpha p)^m \int_{t_{m-1}}^{t_m} F(t)\overline{G(t)}\,\mathrm{d}t = p^{-n-1}\left(\sum_{j=0}^{p-1} \theta_s^j\right) \int_{t_n}^{L} F(t)\overline{G(t)}\,q(t)\,\mathrm{d}t$$

while

$$\sum_{j=0}^{p-1} \theta_s^j = \frac{1-\theta_s^p}{1-\theta_s} \equiv \frac{1-1}{1-\theta_s} \equiv 0, \tag{6}$$

so $\langle f, g\rangle_{L^2(\mathbb{T})} = 0$.

(A.2) Now let $g \in L^2_{n',k',s'}(\mathbb{T})$ with $(n, k, s) \neq (n', k', s')$, then $g = U_{n',k',s'}G$ for some $G \in L^2\left((t_n, L), p^{-n}q(t)\,\mathrm{d}t\right)$. We have three possiblities:

(A.2.1) None of $X_{n,k} \leq X_{n',k'}$ and $X_{n',k'} \leq X_{n,k}$ holds. In this case, one has $\mathbb{T}_{n,k} \cap \mathbb{T}_{n',k'} = \emptyset$, so f and g have disjoint supports and $\langle f, g\rangle_{L^2(\mathbb{T})} = 0$.

(A.2.2) One has $X_{n,k} < X_{n',k'}$ or $X_{n,k} < X_{n',k'}$. To be definite, assume that $X_{n',k'} < X_{n,k}$, then $\mathbb{T}_{n,k} \subset \mathbb{T}_{n',k'}^{j'}$ for some $j' \in \{0, \dots, p-1\}$.

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f\bar{g}\,\mathrm{d}\mu = \int_{\mathbb{T}_{n,k}} f\bar{g}\,\mathrm{d}\mu = \sum_{j=0}^{p-1} \int_{\mathbb{T}_{n,k}^j} f\bar{g}\,\mathrm{d}\mu$$

$$= \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} f_{n+m,kp^m+jp^{m-1}+r}(t)\, \overline{g_{n+m,kp^m+jp^{m-1}+r}(t)}\,\mathrm{d}t.$$

By assumptions in each summand, we have $f_{n+m,kp^m+jp^{m-1}+r}(t) = \theta_s^j F(t)$ and $g_{n+m,kp^m+jp^{m-1}+r}(t) = \theta_{s'}^{j'} G(t)$, so similarly to the preceding computation, we obtain

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} \sum_{r=0}^{p^{m-1}-1} \theta_s^j \theta_{s'}^{-j'} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} F(t)\overline{G(t)}\,\mathrm{d}t = \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} p^{m-1} \theta_s^j \theta_{s'}^{-j'} \int_{t_{n+m-1}}^{t_{n+m}} F(t)\overline{G(t)}\,\mathrm{d}t$$

$$= \sum_{j=0}^{p-1} \sum_{m=n+1}^{\infty} (\alpha p)^m p^{-n-1}\theta_s^j \theta_{s'}^{-j'} \int_{t_{m-1}}^{t_m} F(t)\overline{G(t)}\,\mathrm{d}t = p^{-n-1}\theta_{s'}^{-j'} \underbrace{\sum_{j=0}^{p-1} \theta_s^j}_{=0 \text{ by } (6)} \int_{t_n}^{L} F(t)\overline{G(t)}\,q(t)\,\mathrm{d}t = 0.$$

(A.2.3) One has $X_{n,k} = X_{n',k'}$. In this case, $s \neq s'$,

$$f_{n+m,kp^m+jp^{m-1}+r}(t) = \theta_s^j F(t), \quad g_{n+m,kp^m+jp^{m-1}+r}(t) = \theta_{s'}^j G(t),$$

for all $m \in \mathbb{N}$, $j \in \{0, \dots, p-1\}$ and $r \in \{0, \dots, p^{m-1}-1\}$, and

$$\langle f, g \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f \bar{g} \, d\mu = \int_{\mathbb{T}_{n,k}} f \bar{g} \, d\mu = \sum_{j=0}^{p-1} \int_{\mathbb{T}_{n,k}^j} f \bar{g} \, d\mu$$

$$= \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} \sum_{r=0}^{p^{m-1}-1} \int_{t_{n+m-1}}^{t_{n+m}} f_{n+m, kp^m + jp^{m-1} + r}(t) \overline{g_{n+m, kp^m + jp^{m-1} + r}(t)} \, dt$$

$$= \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} \sum_{r=0}^{p^{m-1}-1} \theta_s^j \theta_{s'}^{-j} \int_{t_{n+m-1}}^{t_{n+m}} F(t) \overline{G(t)} \, dt = \sum_{j=0}^{p-1} \sum_{m=1}^{\infty} \alpha^{m+n} p^{m-1} \theta_s^j \theta_{s'}^{-j} \int_{t_{n+m-1}}^{t_{n+m}} F(t) \overline{G(t)} \, dt$$

$$= \sum_{j=0}^{p-1} \left(\frac{\theta_s}{\theta_{s'}} \right)^j \sum_{m=n+1}^{} (\alpha p)^m p^{-n-1} \int_{t_{m-1}}^{t_m} F(t) \overline{G(t)} \, dt = p^{-n-1} \sum_{j=0}^{p-1} \left(\frac{\theta_s}{\theta_{s'}} \right)^j \int_{t_n}^{L} F(t) \overline{G(t)} q(t) \, dt,$$

while

$$\sum_{j=0}^{p-1} \left(\frac{\theta_s}{\theta_{s'}} \right)^j = \frac{1 - \left(\frac{\theta_s}{\theta_{s'}} \right)^p}{1 - \frac{\theta_s}{\theta_{s'}}} \equiv \frac{1 - 1}{1 - \frac{\theta_s}{\theta_{s'}}} = 0,$$

so $\langle f, g \rangle_{L^2(\mathbb{T})} = 0$.

The orthogonality of the decomposition is completely proved.

(B) In order to prove the totality of the decomposition, it is sufficient to show that any function supported on a single edge belongs to the right-hand side of (5). So let $e_{m,r} \in E$ and $h \in L^2(t_{m-1}, t_m)$. Define $v \in L^2(\mathbb{T})$ by

$$v_{n,k} := \begin{cases} h, & (n,k) = (m,r), \\ 0, & \text{otherwise}, \end{cases}$$

then we need to show that v belongs to the right-hand side of (5).

Consider the p^m-dimensional subspace

$$S := \left\{ u \in L^2(\mathbb{T}) : u_{n,k} \in \mathbb{C}h \text{ for } n = m \text{ and } k = 0, \dots, p^m - 1, u_{n,k} = 0 \text{ for } n \neq m \right\},$$

then by construction, we have $v \in S$. In addition,

$$S \cap L^2_{n,k,s}(\mathbb{T}) = \{0\} \text{ for } n \geq m \text{ and any } (k,s),$$

$$\dim \left(S \cap L^2_{\text{rad}}(\mathbb{T}) \right) = 1,$$

$$\dim \left(S \cap L^2_{n,k,s}(\mathbb{T}) \right) = 1 \text{ for } n < m \text{ and any } (k,s).$$

This yields

$$\dim \left[S \cap \left(L^2_{\text{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} L^2_{n,k,s}(\mathbb{T}) \right) \right] = \dim \left(S \cap L^2_{\text{rad}}(\mathbb{T}) \right) + \sum_{n=0}^{m-1} \sum_{k=0}^{p^n-1} \sum_{s=1}^{p-1} \dim \left(S \cap L^2_{n,k,s}(\mathbb{T}) \right)$$

$$= 1 + \sum_{n=0}^{m-1} \sum_{k=0}^{p^n-1} \sum_{s=1}^{p-1} 1 = 1 + \sum_{n=0}^{m-1} p^n(p-1) = p^m = \dim S,$$

which shows that $S \subset L^2_{\text{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} L^2_{n,k,s}(\mathbb{T})$, and due to $v \in S$, we arrive at the conclusion. $\square$

By using the inclusions (4) we immediately obtain the following:

Corollary 2.3. *One has the orthogonal decomposition*

$$H^1(\mathbb{T}) = H^1_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H^1_{n,k,s}(\mathbb{T}).$$

Corollary 2.4. *The map*

$$U := U_{\mathrm{rad}} \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} U_{n,k,s} : L^2((0,L), q(t)\,\mathrm{d}t) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} L^2((t_n, L), p^{-n}q(t)\,\mathrm{d}t) \to L^2_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} L^2_{n,k,s}(\mathbb{T}).$$

is a unitary operator.

In order to have a similar transformation for H^1-spaces, we recall that for an interval $(a,b) \subset \mathbb{R}$ and a piecewise continuous weight function $v : (a,b) \to (0,\infty)$, one usually denotes

$$H^1((a,b), v(t)\,\mathrm{d}t) := \big\{ f \in L^2((a,b), v(t)\,\mathrm{d}t) : f' \in L^2((a,b), v(t)\,\mathrm{d}t) \big\},$$

which becomes a Hilbert space if considered with the scalar product

$$\langle f, g \rangle_{H^1((a,b), v(t)\,\mathrm{d}t)} = \langle f, g \rangle_{L^2((a,b), v(t)\,\mathrm{d}t)} + \langle f', g' \rangle_{L^2((a,b), v(t)\,\mathrm{d}t)} \equiv \int_a^b \Big(f(t)\overline{g(t)} + f'(t)\overline{g'(t)} \Big) v(t)\,\mathrm{d}t.$$

In addition we make the following observations:

(1) If $F \in L^2((0,L), q(t)\,\mathrm{d}t)$, then:
  - $U_{\mathrm{rad}}F$ is continuous on $\mathbb{T}$ if and only if F is continuous on $(0,L)$,
  - $U_{\mathrm{rad}}F \in H^1(\mathbb{T})$ if and only if $F \in H^1((0,L), q(t)\,\mathrm{d}t)$,
  - and in this case we have

$$\|(U_{\mathrm{rad}}F)'\|^2_{L^2(\mathbb{T})} = \|U_{\mathrm{rad}}(F')\|^2_{L^2(\mathbb{T})} = \|F'\|^2_{L^2((0,L), q(t)\,\mathrm{d}t)};$$

(2) if $F \in L^2((t_n, L), p^{-n}q(t)\,\mathrm{d}t)$, then:
  - $U_{n,k,s}F$ is continuous on $\mathbb{T}$ if and only if F is continuous on $[t_n, L]$ with $F(t_n) = 0$,
  - in this case $U_{n,k,s}F \in H^1(\mathbb{T})$ if and only if $F \in H^1((t_n, L), p^{-n}q(t)\,\mathrm{d}t)$,
  - and in this case $\|(U_{n,k,s}F)'\|^2_{L^2(\mathbb{T})} = \|U_{n,k,s}(F')\|^2_{L^2(\mathbb{T})} = \|F'\|^2_{L^2((t_n, L), p^{-n}q(t)\,\mathrm{d}t)}.$

Therefore, it will be convenient to denote

$$\tilde{H}^1((t_n, L), p^{-n}q(t)\,\mathrm{d}t) := \big\{ f \in H^1((t_n, L), p^{-n}q(t)\,\mathrm{d}t) : f(t_n) = 0 \big\},$$

then it follows that

$$U_{\mathrm{rad}} : H^1((0,L), q(t)\,\mathrm{d}t) \to H^1_{\mathrm{rad}}(\mathbb{T}), \quad U_{n,k,s} : \tilde{H}^1((t_n, L), p^{-n}q(t)\,\mathrm{d}t) \to H^1_{n,k,s}(\mathbb{T})$$

are unitary operators, which gives the following:

Corollary 2.5. *The map*

$$U := U_{\mathrm{rad}} \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} U_{n,k,s}$$

defines a unitary operator

$$H^1((0,L), q(t)\,\mathrm{d}t) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} \tilde{H}^1((t_n,L), p^{-n}q(t)\,\mathrm{d}t) \to H^1_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H^1_{n,k,s}(\mathbb{T}) \equiv H^1(\mathbb{T}).$$

2.3 | Embeddings and equivalent norms

We denote

$$\tilde{H}^1(\mathbb{T}) := \left\{ f \in H^1(\mathbb{T}) : f(o) = 0 \right\},$$

which is a closed subspace of $H^1(\mathbb{T})$.

Lemma 2.6 (Poincaré inequality). *There is a constant $C > 0$ such that*

$$\|f\|_{L^2(\mathbb{T})} \leq C\|f'\|_{L^2(\mathbb{T})} \text{ for all } f \in \tilde{H}^1(\mathbb{T}). \tag{7}$$

Proof. For (7), it is sufficient to show that the number

$$a := \inf_{f \in \tilde{H}^1(\mathbb{T}),\, f \not\equiv 0} \frac{\|f'\|^2_{L^2(\mathbb{T})}}{\|f\|^2_{L^2(\mathbb{T})}}$$

is strictly positive (then one can choose $C := 1/\sqrt{a}$). Remark that

$$\tilde{H}^1(\mathbb{T}) \times \tilde{H}^1(\mathbb{T}) \ni (f,g) \mapsto \langle f', g' \rangle_{L^2(\mathbb{T})}$$

is a closed, symmetric, nonnegative, densely defined sesquilinear form in $L^2(\mathbb{T})$, so it generates a nonnegative self-adjoint operator A in $L^2(\mathbb{T})$, and the above number a is the bottom of the spectrum of A. By Lemma 2.1, the embedding $\tilde{H}^1(T) \hookrightarrow L^2(\mathbb{T})$ is compact, then A has compact resolvent, and a is its smallest eigenvalue due to the min–max principle. In particular, the infimum in the definition of a is attained on some eigenfunction $F \in \tilde{H}^1(\mathbb{T})$. We have obviously $a \geq 0$. Assume that $a = 0$, then $\|F'\|^2_{L^2(\mathbb{T})} = 0$. This means that $F'_{n,k} = 0$ for all (n,k), that is, all $F_{n,k}$ are constant. The continuity of F shows that F is constant on $\mathbb{T}$, and $F \in \tilde{H}^1(\mathbb{T})$ means that $F(o) = 0$ and then $F \equiv 0$, which is impossible. Hence, $a > 0$. $\square$

In view of Lemma 2.6, the norm induced by the $\tilde{H}^1$-scalar product

$$\langle f, g \rangle_{\tilde{H}^1(\mathbb{T})} := \langle f', g' \rangle_{L^2(\mathbb{T})}$$

is equivalent to the initial $H^1(\mathbb{T})$-norm. Moreover, we have the direct sum decomposition

$$\tilde{H}^1(\mathbb{T}) = \tilde{H}^1_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} \tilde{H}^1_{n,k,s}(\mathbb{T}),$$

with

$$\tilde{H}^1_{\mathrm{rad}}(\mathbb{T}) := \tilde{H}^1(\mathbb{T}) \cap H^1_{\mathrm{rad}}(\mathbb{T}) \equiv \{f \in H^1_{\mathrm{rad}}(\mathbb{T}) : f(o) = 0\},$$

$$\tilde{H}^1_{n,k,s}(\mathbb{T}) := \tilde{H}^1(\mathbb{T}) \cap H^1_{n,k,s}(\mathbb{T}) \equiv \{f \in H^1_{n,k,s}(\mathbb{T}) : f(o) = 0\} \equiv H^1_{n,k,s}(\mathbb{T}),$$

which is orthogonal with respect to both $\langle \cdot, \cdot \rangle_{H^1(\mathbb{T})}$ and $\langle \cdot, \cdot \rangle_{\tilde{H}^1(\mathbb{T})}$.

We further remark that for $F \in H^1((0, L), q(t)\, dt)$, the inclusion $U_{\mathrm{rad}}F \in \tilde{H}^1_{\mathrm{rad}}(\mathbb{T})$ is equivalent to $F(0) = 0$. So if we additionally denote

$$\tilde{H}^1((0, L), q(t)\, dt) := \{F \in H^1((0, L), q(t)\, dt) : F(0) = 0\},$$

then we conclude that the map

$$U := U_{\mathrm{rad}} \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} U_{n,k,s}$$

is an isomorphism between $\tilde{H}^1((0, L), q(t)\, dt) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} \tilde{H}^1((t_n, L), p^{-n}q(t)\, dt)$ equipped with the usual H^1 scalar products and

$$\tilde{H}^1(\mathbb{T}) \equiv \tilde{H}^1_{\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H^1_{n,k,s}(\mathbb{T}) \tag{8}$$

viewed with the $\tilde{H}^1$ scalar product.

We will additionally introduce the space

$$\tilde{H}^1_c(\mathbb{T}) = \{f \in H^1_c(\mathbb{T}) : f(o) = 0\}, \quad \tilde{H}^1_0(\mathbb{T}) = \{f \in H^1_0(\mathbb{T}) : f(o) = 0\},$$

then it is standard to see that $\tilde{H}^1_0(\mathbb{T})$ is the closure of $\tilde{H}^1_c(\mathbb{T})$ in $\tilde{H}^1(\mathbb{T})$. In particular, $\tilde{H}^1_0(\mathbb{T})$ is a closed subspace of $H^1(\mathbb{T})$, so using the orthogonal decomposition (8), we arrive at the decomposition

$$\tilde{H}^1_0(\mathbb{T}) = \tilde{H}^1_{0|\mathrm{rad}}(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H^1_{0|n,k,s}(\mathbb{T}) \tag{9}$$

with

$$\tilde{H}^1_{0|\mathrm{rad}}(\mathbb{T}) := \tilde{H}^1_{\mathrm{rad}}(\mathbb{T}) \cap H^1_0(\mathbb{T}), \quad H^1_{0|n,k,s}(\mathbb{T}) := H^1_{n,k,s}(\mathbb{T}) \cap H^1_0(\mathbb{T}),$$

which is orthogonal with respect to the $\tilde{H}^1$-scalar product. Further remark that for $F \in \tilde{H}^1((0, L), q(t)\, dt)$, the inclusion $U_{\mathrm{rad}}F \in H^1_c(\mathbb{T})$ is equivalent to

$$F \in \tilde{H}^1_c((0, L), q(t)\, dt) \equiv \{F \in \tilde{H}^1((0, L), q(t)\, dt) : \exists a \in (0, L) \text{ such that } F(t) = 0 \text{ for all } t \geq a\}.$$

Recall that U_{rad} is an isomorphism between $\tilde{H}^1((0, L), q(t)\, dt)$ and $\tilde{H}^1_{\mathrm{rad}}(\mathbb{T})$, and $\tilde{H}^1_{\mathrm{rad}}(\mathbb{T}) \cap H^1_c(\mathbb{T})$ is dense in $\tilde{H}^1_{0|\mathrm{rad}}(\mathbb{T})$, so it follows that

$$U^{-1}_{\mathrm{rad}} \tilde{H}^1_{0|\mathrm{rad}}(\mathbb{T}) = \text{the closure of } \tilde{H}^1_c((0, L), q(t)\, dt) \text{ in } \tilde{H}^1((0, L), q(t)\, dt) =: \tilde{H}^1_0((0, L), q(t)\, dt).$$

Remark that the usual mollifying procedure shows that

$$\tilde{H}^1_0((0, L), q(t)\, dt) = \text{the closure of } C^\infty_c(0, L) \text{ in } \tilde{H}^1((0, L), q(t)\, dt),$$

which will be useful below.

Similarly one shows that for any (n, k, s), there holds

$$U_{n,k,s}^{-1} H_{0|n,k,s}^1(\mathbb{T}) = \tilde{H}_0^1((t_n, L), p^{-n}q(t)\,dt) := \text{the closure of } C_c^\infty(t_n, L) \text{ in } \tilde{H}^1((t_n, L), p^{-n}q(t)\,dt),$$

and we conclude that the map

$$U := U_{\mathrm{rad}} \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} U_{n,k,s}$$

is an isomorphism between

$$\tilde{H}_0^1((0, L), q(t)\,dt) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} \tilde{H}_0^1((t_n, L), p^{-n}q(t)\,dt)$$

equipped with the usual H^1 scalar products and

$$\tilde{H}_0^1(\mathbb{T}) \equiv \tilde{H}_{0|\mathrm{rad}}^1(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H_{0|n,k,s}^1(\mathbb{T})$$

viewed with the $\tilde{H}^1$ scalar product.

2.4 | Harmonic functions

By construction, $\tilde{H}_0^1(\mathbb{T})$ is a closed subspace of $\tilde{H}^1(\mathbb{T})$, so let us introduce the subspace

$$\tilde{H}_\Delta^1(\mathbb{T}) := \text{the orthogonal complement of } \tilde{H}_0^1(\mathbb{T}) \text{ in } \tilde{H}^1(\mathbb{T}).$$

We would like to understand the structure of this subspace, in particular, to construct an orthonormal basis (recall that the orthogonality is understood with respect to the $\tilde{H}^1$-scalar product).

Remark first that in view of the orthogonal decompositions (8) and (9), we have the orthogonal decomposition

$$\tilde{H}_\Delta^1(\mathbb{T}) \equiv \tilde{H}_{\Delta|\mathrm{rad}}^1(\mathbb{T}) \oplus \bigoplus_{e_{n,k} \in E} \bigoplus_{s=1}^{p-1} H_{\Delta|n,k,s}^1(\mathbb{T}), \tag{10}$$

$$\tilde{H}_{\Delta|\mathrm{rad}}^1(\mathbb{T}) := \text{the orthogonal complement of } \tilde{H}_{0|\mathrm{rad}}^1(\mathbb{T}) \text{ in } \tilde{H}_{\mathrm{rad}}^1(\mathbb{T}),$$

$$H_{\Delta|n,k,s}^1(\mathbb{T}) := \text{the orthogonal complement of } H_{0|n,k,s}^1(\mathbb{T}) \text{ in } H_{n,k,s}^1(\mathbb{T}).$$

Let $f_{\mathrm{rad}} \in \tilde{H}_{\mathrm{rad}}^1(\mathbb{T})$, then $f_{\mathrm{rad}} \in \tilde{H}_{\Delta|\mathrm{rad}}^1(\mathbb{T})$ if and only if

$$\langle f_{\mathrm{rad}}', g' \rangle_{L^2(\mathbb{T})} \equiv \langle f_{\mathrm{rad}}, g \rangle_{\tilde{H}^1(\mathbb{T})} = 0 \text{ for all } g \in \tilde{H}_{0|\mathrm{rad}}^1(\mathbb{T}). \tag{11}$$

We write $f_{\mathrm{rad}} = U_{\mathrm{rad}} F_{\mathrm{rad}}$ with $F_{\mathrm{rad}} \in \tilde{H}^1((0, L), q(t)\,dt)$ and $g = U_{\mathrm{rad}} G$ with $G \in \tilde{H}_0^1((0, L), q(t)\,dt)$ and recall that $C_c^\infty(0, L)$ is dense in $\tilde{H}_0^1((0, L), q(t)\,dt)$, then it follows that (11) holds if and only if

$$\left\langle (U_{\mathrm{rad}} F_{\mathrm{rad}})', (U_{\mathrm{rad}} G)' \right\rangle_{L^2(\mathbb{T})} = 0 \text{ for all } G \in C_c^\infty(0, L).$$

As $(U_{rad}F_{rad})' = U_{rad}(F'_{rad})$ and $(U_{rad}G)' = U_{rad}(G')$ and U_{rad} is unitary as an operator $L^2((0,L), q(t)\,dt) \to L^2_{rad}(\mathbb{T})$, this is equivalent to

$$\int_0^L F'_{rad}(t)\overline{G'(t)}q(t)\,dt \equiv \langle f'_{rad}, g'\rangle_{L^2((0,L),q(t)\,dt)} = 0 \text{ for all } G \in C_c^\infty(0,L), \tag{12}$$

which means that $(qF'_{rad})' = 0$ in $(0,L)$ in the sense of distributions, which shows that $F'_{rad} = c/q$ for some $c \in \mathbb{C}$. We also recall that F_{rad} must be continuous with $F_{rad}(0) = 0$, so F_{rad} is completely determined by the value c of its derivative in $(0,1)$: For any $n \in \mathbb{N}_0$ and $t \in (t_{n-1}, t_n)$, one has

$$F'_{rad}(t) = \frac{c}{(\alpha p)^n}, \quad F_{rad}(t) = c\left(\frac{t - t_{n-1}}{(\alpha p)^n} + \sum_{k=0}^{n-1}\left(\frac{\ell}{\alpha p}\right)^k\right) \equiv c\left(\frac{t - t_{n-1}}{(\alpha p)^n} + \frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}}\right).$$

It remains to check if $F_{rad} \in \tilde{H}^1((0,L), q(t)\,dt)$, for $c \neq 0$. We have

$$\|F'_{rad}\|^2_{L^2((0,L),q(t)\,dt)} = \int_0^L |F'_{rad}(t)|^2 q(t)\,dt = \sum_{n=0}^\infty (\alpha p)^n \int_{t_{n-1}}^{t_n} |F'_{rad}(t)|^2\,dt$$

$$= \sum_{n=0}^\infty (\alpha p)^n \frac{|c|^2}{(\alpha p)^{2n}} \underbrace{(t_n - t_{n-1})}_{=\ell^n} = |c|^2 \sum_{n=0}^\infty \left(\frac{\ell}{\alpha p}\right)^n,$$

which is finite if and only if $\ell < \alpha p$, which we assume from now on: In this case,

$$\|F'_{rad}\|^2_{L^2((0,L),q(t)\,dt)} = |c|^2\left(1 - \frac{\ell}{\alpha p}\right)^{-1}. \tag{13}$$

To compute $\|F_{rad}\|^2_{L^2((0,L),q(t)\,dt)}$, we first represent, for any $n \in \mathbb{N}_0$ and $t \in (t_{n-1}, t_n)$,

$$|F_{rad}(t)|^2 = |c|^2\left(\frac{(t - t_{n-1})^2}{(\alpha p)^{2n}} + 2\frac{t - t_{n-1}}{(\alpha p)^n}\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}} + \left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}}\right)^2\right),$$

therefore,

$$\int_{t_{n-1}}^{t_n} |F_{rad}(t)|^2\,dt = |c|^2\left(\frac{\ell^{3n}}{3(\alpha p)^{2n}} + \frac{\ell^{2n}}{(\alpha p)^n}\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}} + \ell^n\left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}}\right)^2\right)$$

and

$$\|F_{rad}\|^2_{L^2((0,L),q(t)\,dt)} = \int_0^L |F_{rad}(t)|^2 q(t)\,dt = \sum_{n=0}^\infty (\alpha p)^n \int_{t_{n-1}}^{t_n} |F_{rad}(t)|^2\,dt$$

$$= |c|^2\left(\frac{1}{3}\sum_{n=0}^\infty \left(\frac{\ell^3}{\alpha p}\right)^n + \sum_{n=0}^\infty \ell^{2n}\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}} + \sum_{n=0}^\infty (\alpha p\ell)^n\left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^n}{1 - \frac{\ell}{\alpha p}}\right)^2\right),$$

which is finite if and only if $\ell^3 < \alpha p$ and $\alpha p \ell < 1$ (we recall that $\ell < 1$ by the initial assumption). For subsequent computations, it will be useful to normalize F_{rad} in $\tilde{H}^1((0, L), q(t)\,dt)$, in view of (13) this amounts to the choice

$$c := \sqrt{1 - \frac{\ell}{\alpha p}},$$

and gives

$$F_{\mathrm{rad}}(t) = \sqrt{1 - \frac{\ell}{\alpha p}}\left(\frac{t - t_{n-1}}{(\alpha p)^n} + \sum_{k=0}^{n-1}\left(\frac{\ell}{\alpha p}\right)^k\right), \quad t \in (t_{n-1}, t_n), \quad n \in \mathbb{N}_0. \tag{14}$$

We summarize these computations as

$$\tilde{H}^1_{\mathrm{rad}}(T) \neq \tilde{H}^1_{0|\mathrm{rad}}(T) \text{ if and only if } \ell < \alpha p < \frac{1}{\ell}, \qquad \tilde{H}^1_{\Delta|\mathrm{rad}}(\mathbb{T}) = \begin{cases} \mathbb{C} U_{\mathrm{rad}} F_{\mathrm{rad}}, & \ell < \alpha p < \frac{1}{\ell}, \\ 0, & \text{otherwise.} \end{cases}$$

The spaces $\tilde{H}^1_{\Delta|n,k,s}$ are studied in the same way. Let us fix an admissible triple (n, k, s) and recall that $U_{n,k,s} C_c^\infty(t_n, L)$ is dense in $\tilde{H}^1_{0|n,k,s}(\mathbb{T})$, then a function $f_{n,k,s} \in \tilde{H}^1_{n,k,s}(\mathbb{T})$ belongs to $\tilde{H}^1_{\Delta|n,k,s}(\mathbb{T})$ if and only if $f_{n,k,s} = U_{n,k,s} F_{n,k,s}$ with $F \in \tilde{H}^1((t_n, L), p^{-n} q(t)\,dt)$ satisfying

$$\langle (U_{n,k,s} F_{n,k,s})', (U_{n,k,s} G)' \rangle_{L^2(\mathbb{T})} = 0 \text{ for all } G \in C_c^\infty(t_n, L),$$

which can be equivalently rewritten as

$$\langle U_{n,k,s}(F'_{n,k,s}), U_{n,k,s}(G') \rangle_{L^2(\mathbb{T})} \equiv \langle F'_{n,k,s}, G' \rangle_{L^2((t_n, L), p^{-n} q(t)\,dt)} \equiv p^{-n} \int_{t_n}^L F'_{n,k,s}(t)\overline{G'(t)} q(t)\,dt = 0$$

for all $G \in C_c^\infty(t_n, L)$, which means $(q F'_{n,k,s})' = 0$ in (t_n, L). As $F_{n,k,s}$ can be chosen independent of (k, s), we write simply

$$F_n := F_{n,k,s}.$$

So we obtain $F'_n = c/q$ in (t_n, L) with some $c \in \mathbb{C}$, and for $t \in (t_{n+m-1}, t_{n+m})$ with $m \in \mathbb{N}$, one has

$$F'_n(t) = \frac{c}{(\alpha p)^{n+m}}, \quad F_n(t) = c\left(\frac{t - t_{n+m-1}}{(\alpha p)^{n+m}} + \sum_{k=1}^{m-1}\left(\frac{\ell}{\alpha p}\right)^{n+k}\right) \equiv c\left(\frac{t - t_{n+m-1}}{(\alpha p)^{n+m}} + \left(\frac{\ell}{\alpha p}\right)^{n+1} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^{m-1}}{1 - \frac{\ell}{\alpha p}}\right).$$

In order to check if $F_n \in \tilde{H}^1((t_n, L), p^{-n} q(t)\,dt)$, we again compute

$$\|F'_n\|^2_{L^2((t_n, L), p^{-n} q(t)\,dt)} = p^{-n} \sum_{m=1}^\infty \int_{t_{n+m-1}}^{t_{n+m}} |F'_n(t)|^2 (\alpha p)^{n+m}\,dt$$

$$= |c|^2 p^{-n} \sum_{m=1}^\infty \ell^{n+m} (\alpha p)^{-2(n+m)} (\alpha p)^{n+m} = |c|^2 p^{-n} \sum_{m=1}^\infty \left(\frac{\ell}{\alpha p}\right)^{n+m},$$

which is finite if and only if $\ell < \alpha p$, and in that case

$$\|F'_n\|^2_{L^2((t_n, L), p^{-n} q(t)\,dt)} = \frac{|c|^2}{p^n}\left(\frac{\ell}{\alpha p}\right)^{n+1} \frac{1}{1 - \frac{\ell}{\alpha p}}. \tag{15}$$

Furthermore, in this case for $t \in (t_{n+m-1}, t_{n+m})$ with $m \in \mathbb{N}$, one has

$$|F_n(t)|^2 = |c|^2 \left(\frac{(t - t_{n+m-1})^2}{(\alpha p)^{2n+2m}} + 2(t - t_{n+m-1}) \frac{\ell^{n+1}}{(\alpha p)^{2n+m+1}} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^m}{1 - \frac{\ell}{\alpha p}} + \left(\frac{\ell}{\alpha p}\right)^{2n+2} \left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^m}{1 - \frac{\ell}{\alpha p}} \right)^2 \right),$$

therefore,

$$\int_{t_{n+m-1}}^{t_{n+m}} \left| F_n(t) \right|^2 \mathrm{d}t = |c|^2 \left[\frac{1}{3} \frac{\ell^{3m+3n}}{(\alpha p)^{2n+2m}} + \ell^{2n+2m} \frac{\ell^{n+1}}{(\alpha p)^{2n+m+1}} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^{m-1}}{1 - \frac{\ell}{\alpha p}} + \ell^{m+n} \left(\frac{\ell}{\alpha p}\right)^{2n+2} \left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^{m-1}}{1 - \frac{\ell}{\alpha p}} \right)^2 \right],$$

and

$$\|F_n\|^2_{L^2((t_n,L), p^{-n}q(t)\,\mathrm{d}t)} = p^{-n} \sum_{m=1}^{\infty} (\alpha p)^{n+m} \int_{t_{n+m-1}}^{t_{n+m}} \left| F_n(t) \right|^2 \mathrm{d}t$$

$$= |c|^2 \left[\frac{1}{3} \left(\frac{\ell^3}{\alpha p}\right)^{n+m} + \frac{\ell^{3n+2m+1}}{(\alpha p)^{n+1}} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^{m-1}}{1 - \frac{\ell}{\alpha p}} + (\alpha p \ell)^{n+m} \left(\frac{\ell}{\alpha p}\right)^{2n+2} \left(\frac{1 - \left(\frac{\ell}{\alpha p}\right)^{m-1}}{1 - \frac{\ell}{\alpha p}} \right)^2 \right]$$

is finite if and only if $\ell^3 < \alpha p$ and $\alpha p \ell < 1$ (as one always has $\ell < 1$). For what follows, we normalize $F_{n,k,s}$ to have unit norm in $\tilde{H}^1((t_n, L), p^{-n}q(t)\,\mathrm{d}t)$: as follows from (15), this means the choice

$$c := p^n \left(\frac{\alpha}{\ell}\right)^{\frac{n}{2}} \sqrt{\frac{\alpha p - \ell}{\ell}}$$

and then

$$F_n(t) = p^n \left(\frac{\alpha}{\ell}\right)^{\frac{n}{2}} \sqrt{\frac{\alpha p - \ell}{\ell}} \left(\frac{t - t_{n+m-1}}{(\alpha p)^{n+m}} + \sum_{k=1}^{m-1} \left(\frac{\ell}{\alpha p}\right)^{n+k} \right), \quad t \in (t_{n+m-1}, t_{n+m}) \text{ with } m \in \mathbb{N}. \tag{16}$$

Therefore,

$$\tilde{H}^1_{n,k,s}(T) \neq \tilde{H}^1_{0|n,k,s}(T) \text{ if and only if } \ell < \alpha p < \frac{1}{\ell}, \qquad \tilde{H}^1_{\Delta|n,k,s}(\mathbb{T}) = \begin{cases} \mathbb{C} U_{n,k,s} F_n, & \ell < \alpha p < \frac{1}{\ell}, \\ 0, & \text{otherwise.} \end{cases}$$

We summarize the preceding computations:

Lemma 2.7. *One has $\tilde{H}^1_\Delta(\mathbb{T}) \neq \{0\}$ or, equivalently, $\tilde{H}^1_0(T) \neq \tilde{H}^1(\mathbb{T})$, if and only if*

$$\ell < \alpha p < \frac{1}{\ell}. \tag{17}$$

If this condition is satisfied, then the functions

$$\phi_{\mathrm{rad}} := U_{\mathrm{rad}} F_{\mathrm{rad}} \text{ and } \phi_{n,k,s} := U_{n,k,s} F_n \text{ with } e_{n,k} \in E \text{ and } s \in \{1, \ldots, p-1\}$$

with F_{rad} from (14) and F_n from (16) form an orthonormal basis in $\tilde{H}^1_\Delta(\mathbb{T})$.

We remark that the criterion (17) for $\tilde{H}_0^1(T) \neq \tilde{H}^1(\mathbb{T})$ and, equivalently, for $H_0^1(T) \neq H^1(\mathbb{T})$, was already obtained in [30] (even for a more general configuration) by different methods, but in our case, it appears naturally as a by-result of the construction of the orthonormal basis.

For subsequent constructions, we will need the limits

$$F_{\text{rad}}^\infty := \lim_{t \to L^-} F_{\text{rad}}(t), \quad F_n^\infty := \lim_{t \to L^-} F_n(t). \tag{18}$$

As F_{rad} and F_n are increasing functions, we have

$$F_{\text{rad}}^\infty = \lim_{n \to \infty} F_{\text{rad}}(t_n) = \lim_{n \to \infty} \sqrt{1 - \frac{\ell}{\alpha p}} \sum_{k=0}^n \left(\frac{\ell}{\alpha p}\right)^k$$

$$= \sqrt{1 - \frac{\ell}{\alpha p}} \lim_{n \to \infty} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^{n+1}}{1 - \frac{\ell}{\alpha p}} = \sqrt{1 - \frac{\ell}{\alpha p}} \frac{1}{1 - \frac{\ell}{\alpha p}} = \sqrt{\frac{\alpha p}{\alpha p - \ell}} \tag{19}$$

and, similarly,

$$F_n^\infty = \lim_{m \to \infty} F_n(t_{n+m}) = \lim_{m \to \infty} p^n \left(\frac{\alpha}{\ell}\right)^{\frac{n}{2}} \sqrt{\frac{\alpha p - \ell}{\ell}} \sum_{k=1}^m \left(\frac{\ell}{\alpha p}\right)^{n+k}.$$

Due to

$$\sum_{k=1}^m \left(\frac{\ell}{\alpha p}\right)^{n+k} = \left(\frac{\ell}{\alpha p}\right)^{n+1} \sum_{k=0}^{m-1} \left(\frac{\ell}{\alpha p}\right)^k = \left(\frac{\ell}{\alpha p}\right)^{n+1} \frac{1 - \left(\frac{\ell}{\alpha p}\right)^m}{1 - \frac{\ell}{\alpha p}} \xrightarrow{m \to \infty} \left(\frac{\ell}{\alpha p}\right)^{n+1} \frac{\alpha p}{\alpha p - \ell},$$

we obtain

$$F_n^\infty = p^n \left(\frac{\alpha}{\ell}\right)^{\frac{n}{2}} \sqrt{\frac{\alpha p - \ell}{\ell}} \left(\frac{\ell}{\alpha p}\right)^{n+1} \frac{\alpha p}{\alpha p - \ell} \equiv \sqrt{\frac{\ell}{\alpha p - \ell}} \left(\frac{\ell}{\alpha}\right)^n. \tag{20}$$

2.5 | Abstract trace operator

Everywhere in this subsection we assume that

the inequalities (17) are satisfied.

Now we construct the first version of the trace operator. We define

$$\mathcal{Z} := \{\text{rad}\} \cup \left\{(n, k, s) : e_{n,k} \in E, s \in \{1, \dots, p - 1\}\right\}$$

and define

$$\nu : \mathcal{Z} \to \{\text{rad}\} \cup \mathbb{N}_0, \quad \nu(\text{rad}) := \text{rad}, \quad \nu(n, k, s) := n.$$

For numerical operations, it will be useful to identify

$$\text{rad} := -1.$$

Recall that we have the orthogonal decomposition $\tilde{H}^1(\mathbb{T}) = \tilde{H}_0^1(\mathbb{T}) \oplus \tilde{H}_\Delta^1(\mathbb{T})$ and let $\tilde{P}_\Delta : \tilde{H}^1(\mathbb{T}) \to \tilde{H}_\Delta^1(\mathbb{T})$ be the orthogonal projector. By Lemma 2.7, the map

$$\Theta : \ell^2(\mathcal{Z}) \ni (a_z) \mapsto \sum_{z \in \mathcal{Z}} a_z U_z F_z \in H_\Delta^1(\mathbb{T})$$

is unitary. Remark that the behavior of $U_z F_z$ near the boundary of $\mathbb{T}$ is uniquely quantified by z and the limiting values $F_{\nu(z)}^\infty$ defined in (18). Therefore, it is reasonable to consider the multiplication operator

$$M : \ell^2(\mathcal{Z}) \to \ell^2(\mathcal{Z}), \quad (a_z) \mapsto \left(p^{-\frac{\nu(z)}{2}} F_{\nu(z)}^\infty a_z \right).$$

The explicit expressions (19) and (20) show that the coefficients $p^{-\frac{\nu(z)}{2}} F_{\nu(z)}^\infty$ are strictly positive and uniformly bounded, therefore, the operator M is injective and bounded. However, it is not surjective. Namely, for $r \geq 0$, introduce

$$\ell_r^2(\mathcal{Z}) := \left\{ (a_z) \in \ell^2(\mathcal{Z}) : (p^{r\nu(z)} a_z) \in \ell^2(\mathcal{Z}) \right\}, \quad \|(a_z)\|_{\ell_r^2(\mathcal{Z})} := \|(p^{r\nu(z)} a_z)\|_{\ell^2(\mathcal{Z})}. \tag{21}$$

Denote

$$\sigma := \frac{1}{2 \log p} \log \frac{\alpha p}{\ell} \equiv \frac{1}{2} \left(1 - \frac{\log \ell - \log \alpha}{\log p} \right) > 0,$$

then

$$F_{\nu(z)}^\infty = \sqrt{\frac{\ell}{\alpha p - \ell}} \, p^{-\sigma \nu(z)} \text{ for all } z \in \mathcal{Z},$$

and it follows that $M : \ell^2(\mathcal{Z}) \to \ell_\sigma^2(\mathcal{Z})$ is a bounded bijective operator. This gives the abstract trace operator

$$\tau := M\Theta^{-1}\tilde{P}_\Delta : \tilde{H}^1(\mathbb{T}) \to \ell_\sigma^2(\mathcal{Z}), \tag{22}$$

which is a linear operator that is bounded and surjective by construction. Remark that $M\Theta^{-1}$ is injective, which shows that $\ker \tau = \ker \tilde{P}_\Delta = \tilde{H}_0^1(\mathbb{T})$. In addition we extend τ to $H^1(\mathbb{T})$: For any $f \in H^1(\mathbb{T})$, set

$$\tau f := \tau \tilde{f} \tag{23}$$

with any $\tilde{f} \in \tilde{H}^1(\mathbb{T})$ such that $f = \tilde{f}$ in $\mathbb{T} \setminus \mathbb{T}^n$ for some $n \in \mathbb{N}_0$.

Theorem 2.8. *The abstract trace operator $\tau : H^1(\mathbb{T}) \to \ell_\sigma^2(\mathcal{Z})$ is well defined, linear, bounded, surjective, with $\ker \tau = H_0^1(\mathbb{T})$.*

Proof. Let $f \in H^1(\mathbb{T})$ and $\tilde{f}, \tilde{g} \in \tilde{H}^1(\mathbb{T})$ such that $f = \tilde{f}$ in $\mathbb{T} \setminus \mathbb{T}^n$ for some $n \in \mathbb{N}_0$ and $f = \tilde{g}$ in $\mathbb{T} \setminus \mathbb{T}^m$ for some $m \in \mathbb{N}_0$. Without loss of generality assume $n \leq m$, then $\tilde{f} = \tilde{g}$ in $\mathbb{T} \setminus \mathbb{T}^m$, that is, $\tilde{f} - \tilde{g} = 0$ in $\mathbb{T} \setminus \mathbb{T}^m$. This means that $\tau(\tilde{f} - \tilde{g}) = 0$, i.e. $\tau \tilde{f} = \tau \tilde{g}$. This shows that τ is a well-defined map.

Let $\varphi : (0, L) \to \mathbb{R}$ be a C^∞ function such that $\varphi = 0$ in $(0, \frac{1}{2})$ and $\varphi = 1$ in $(\frac{3}{4}, L)$. For $f \in H^1(\mathbb{T})$, the function $\tilde{f} : \mathbb{T} \ni x \mapsto \varphi(|x|)f(x)$ belongs to $\tilde{H}^1(\mathbb{T})$ and coincides with f in $\mathbb{T} \setminus \mathbb{T}^0$, so one has $\tau f = \tau \tilde{f}$. As $f \mapsto \tilde{f}$ is linear, it follows that the extended τ is also linear.

To show the boundedness, it is sufficient to show the boundedness of the map $H^1(\mathbb{T}) \ni f \mapsto \tilde{f} \in \tilde{H}^1(\mathbb{T})$. For any $f \in H^1(\mathbb{T})$, one has the identities

$$\tilde{f}_{0,0} = \varphi f, \quad \tilde{f}_{n,k} = f_{n,k} \text{ for } (n, k) \neq (0, 0).$$

This gives

$$\|\tilde{f}\|^2_{\tilde{H}^1(\mathbb{T})} = \sum_{e_{n,k} \in E} (\alpha p)^n \|\tilde{f}'_{n,k}\|^2_{L^2(t_{n-1}, t_n)}$$

$$= \|\tilde{f}'_{0,0}\|^2_{L^2(0,1)} + \sum_{\substack{e_{n,k} \in E \\ (n,k) \neq (0,0)}} (\alpha p)^n \|f'_{n,k}\|^2_{L^2(t_{n-1}, t_n)} \leq \|\tilde{f}'_{0,0}\|^2_{L^2(0,1)} + \|f\|^2_{H^1(\mathbb{T})},$$

and

$$\|\tilde{f}'_{0,0}\|^2_{L^2(0,1)} = \|\varphi f'_{0,0} + \varphi' f_{0,0}\|^2_{L^2(0,1)} \leq 2\|\varphi f'_{0,0}\|^2_{L^2(0,1)} + 2\|\varphi' f_{0,0}\|^2_{L^2(0,1)}$$

$$\leq 2\|\varphi\|^2_\infty \|f'_{0,0}\|^2_{L^2(0,1)} + 2\|\varphi'\|^2_\infty \|f_{0,0}\|^2_{L^2(0,1)} \leq 2b\|f_{0,0}\|^2_{H^1(0,1)} \text{ with } b := \max\{\|\varphi\|^2_\infty, \|\varphi'\|^2_\infty\},$$

so one obtains $\|\tilde{f}\|^2_{\tilde{H}^1(\mathbb{T})} \leq 2b\|f_{0,0}\|^2_{H^1(0,1)} + \|f\|^2_{H^1(\mathbb{T})} \leq (2b+1)\|f\|^2_{H^1(\mathbb{T})}$, which gives the result.

The map $H^1(\mathbb{T}) \ni f \mapsto \tilde{f} \in \tilde{H}^1(\mathbb{T})$ is surjective, which shows that the range of τ is the same as before.

If $f \in H^1_0(\mathbb{T})$, then there exist $f_n \in H^1_c(\mathbb{T})$ with $f_n \xrightarrow{H^1(\mathbb{T})} f$. Then $\tilde{f}_n \in \tilde{H}^1_c(\mathbb{T})$ with $\tilde{f}_n \xrightarrow{H^1(\mathbb{T})} \tilde{f}$, so $\tilde{f} \in \tilde{H}^1_0(\mathbb{T})$ and $\tau f = \tau \tilde{f} = 0$.

On the other hand, if $\tau f = 0$, then $\tau \tilde{f} = 0$ and $\tilde{f} \in \tilde{H}^1_0(\mathbb{T})$. Then there exist $g_n \in \tilde{H}^1_c(\mathbb{T})$ with $g_n \xrightarrow{H^1(\mathbb{T})} \tilde{f}$. The function $g : \mathbb{T} \ni x \mapsto (1 - \varphi(|x|)) f(x)$ is supported in $\mathbb{T}^0$ and, hence, it belongs to $H^1_c(\mathbb{T})$. Therefore, $g + g_n \in H^1_c(\mathbb{T})$ with $g + g_n \xrightarrow{H^1(\mathbb{T})} g + \tilde{f} = f$, so $f \in H^1_c(\mathbb{T})$. $\square$

The above definition of τ is involved due to the application of the orthogonal projector and the expansion into an orthonormal basis. Let us show that it can be recovered using more elementary operations.

Lemma 2.9. *One has continuous embeddings*

$$\tilde{H}^1((0, L), q(t)\,dt) \hookrightarrow C^0([0, L]), \qquad \tilde{H}^1((t_n, L), p^{-n} q(t)\,dt) \hookrightarrow C^0([t_n, L]) \text{ for any } n \in \mathbb{N}_0,$$

where the right-hand sides are endowed with $\|\cdot\|_\infty$.

Proof. The continuity inside the respective intervals is clear due to the one-dimensional Sobolev theorem, and it remains to establish norm estimates. Let $f \in \tilde{H}^1((0, L), q(t)\,dt)$ and $t \in (0, L)$, then

$$|f(t)|^2 = \left| \int_0^t f'(s)\,ds \right|^2 \leq \left(\int_0^L |f'(s)|\,ds \right)^2 = \left(\int_0^L \frac{1}{\sqrt{q(s)}} \cdot |f'(s)| \sqrt{q(s)}\,ds \right)^2 \leq \int_0^L \frac{ds}{q(s)} \cdot \int_0^L |f'(s)|^2 q(s)\,ds.$$

The second factor on the right-hand side is $\|f\|^2_{\tilde{H}^1((0,L), q(t)\,dt)}$, while

$$\int_0^L \frac{ds}{q(s)} = \sum_{n=0}^\infty \int_{t_{n-1}}^{t_n} \frac{ds}{(\alpha p)^n} = \sum_{n=0}^\infty \left(\frac{\ell}{\alpha p} \right)^n =: a < \infty.$$

As $t \in (0, L)$ was arbitrary, this yields $\|f\|_\infty \leq \sqrt{a}\|f\|_{\tilde{H}^1((0,L), q(t)\,dt)}$.

If $f \in \tilde{H}^1((t_n, L), p^{-n} q(t)\,dt)$ with some $n \in \mathbb{N}_0$, then its extension to $(0, L)$ by zero $\tilde{f}$ belongs to $f \in \tilde{H}^1((0, L), q(t)\,dt)$, and one uses the first part of the proof. $\square$

Lemma 2.10. *Let* $\tau' : H^1(\mathbb{T}) \to \ell^2(\mathcal{Z})$ *be a bounded linear map such that*

(a) $\tau' f = 0$ *for any* $f \in H^1_c(\mathbb{T})$,

(b) for any $F \in H^1\left((0, L), q(t)\, dt\right)$, one has

$$(\tau' U_{\mathrm{rad}} F)_z = \begin{cases} \lim_{t \to L^-} F(t), & z = \mathrm{rad}, \\ 0, & \text{otherwise,} \end{cases}$$

(c) for any $\lambda \in \mathcal{Z} \setminus \{\mathrm{rad}\}$ and any $F \in \tilde{H}^1\left((t_{\nu(\lambda)}, L), p^{-\nu(\lambda)} q(t)\, dt\right)$, one has

$$(\tau' U_\lambda F)_z = p^{-\frac{\nu(\lambda)}{2}} \begin{cases} \lim_{t \to L^-} F(t), & z = \lambda, \\ 0, & \text{otherwise,} \end{cases}$$

then $\tau' = \tau$. Moreover, these properties are satisfied by τ.

Proof. We first remark that the limits on the right-hand sides of (b) and (c) are well defined by Lemma 2.9; for (b), one uses the fact that F coincides with some function $\tilde{F} \in \tilde{H}^1\left((0, L), q(t)\, dt\right)$ in $(\frac{1}{2}, L)$.

(i) The boundedness of τ' and the condition (a) give $H_0^1(\mathbb{T}) \subset \ker \tau'$. For any $\lambda, z \in \mathcal{Z}$, we have

$$(\tau' \phi_\lambda)_z = (\tau' U_\lambda F_{\nu(\lambda)})_z = \begin{cases} \lim_{t \to L^-} F_{\nu(\lambda)}(t) p^{-\frac{\nu(z)}{2}}, & z = \lambda, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} F_{\nu(\lambda)}^\infty p^{-\frac{\nu(z)}{2}}, & z = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

On the other side, by definition we have $(\Theta^{-1} \phi_\lambda)_z = \delta_{\lambda, z}$ (where $\delta_{\lambda, z}$ are the usual Kronecker symbols) and, therefore,

$$(\tau \phi_\lambda)_z = \begin{cases} p^{-\frac{\nu(\lambda)}{2}} F_{\nu(\lambda)}^\infty, & z = \lambda, \\ 0, & \text{otherwise,} \end{cases}$$

which coincides with $\tau' \phi_\lambda$. As the linear span of $\tilde{H}_0^1(\mathbb{T})$ and $(\phi_\lambda)_{\lambda \in \mathcal{Z}}$ is dense in $\tilde{H}^1(\mathbb{T})$ and τ' is bounded, it follows that $\tau' = \tau$ on $\tilde{H}^1(\mathbb{T})$.

(ii) Let $\varphi : (0, L) \to \mathbb{R}$ be a C^∞ function such that $\varphi = 0$ in $(0, \frac{1}{2})$ and $\varphi = 1$ in $(\frac{3}{4}, L)$. For $f \in H^1(\mathbb{T})$, consider $\tilde{f} : \mathbb{T} \ni x \mapsto \varphi(|x|) f(x) \in \tilde{H}^1(\mathbb{T})$. As $\tilde{f} - f = 0$ in $\mathbb{T} \setminus \mathbb{T}^0$, we have $\tilde{f} - f \in H_c^1(\mathbb{T})$, and

$$\tau' f \stackrel{\text{(a)}}{=} \tau' \tilde{f} \stackrel{\text{(i)}}{=} \tau \tilde{f} \stackrel{(23)}{=} \tau f.$$

(iii) It remains to check that τ satisfies the three properties. Remark that (a) holds by construction. Let F be as in (b), then one can represent $F := c F_{\mathrm{rad}} + G$ with $c \in \mathbb{C}$ and $\lim_{t \to L^-} G(t) = 0$. By linearity we have

$$(\tau U_{\mathrm{rad}} F)_z = \tau(c U_{\mathrm{rad}} F_{\mathrm{rad}} + U_{\mathrm{rad}} G) = c F_{\mathrm{rad}}^\infty \delta_{\mathrm{rad}, z} + \tau U_{\mathrm{rad}} G \equiv \left(\lim_{t \to L^-} F(t) \right) \delta_{\mathrm{rad}, z} + \tau U_{\mathrm{rad}} G.$$

Therefore, to show (b), it is sufficient to show that $\tau U_{\mathrm{rad}} G = 0$. For that we take $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, with $\varphi(t) = 1$ for $t \leq 0$ and $\varphi(t) = 0$ for $t \geq 1$, and for $n \in \mathbb{N}$, consider the functions

$$\varphi_n : t \mapsto \varphi\left(\frac{t - t_{n-1}}{\ell^n} \right), \quad G_n := \varphi_n G \in H_c^1((0, L), q(t)\, dt).$$

As for any n there holds $\tau U_{\mathrm{rad}} G_n = 0$, it is sufficient to show that $G_n \to G$ in $H^1\left((0, L), q(t)\, dt\right)$ as $n \to \infty$.

The dominated convergence implies $G_n \to G$ in $L^2\left((0, L), q(t)\, dt\right)$ as $n \to \infty$. We have $G_n' = \varphi_n' G + \varphi_n G'$ and the second summand converges to G' in $L^2\left((0, L), q(t)\, dt\right)$ as $n \to \infty$. It remains to check $\varphi_n' G \to 0$ in $L^2\left((0, L), q(t)\, dt\right)$ for $n \to \infty$.

We have

$$(\varphi_n' G)(t) = \frac{1}{\ell^n} \varphi'\left(\frac{t - t_{n-1}}{\ell^n}\right) G(t),$$

and the function vanishes outside $(t_{n-1}, t_n) \subset (t_{n-1}, L)$. If follows that

$$\|\varphi_n' G\|^2_{L^2((0,L), q(t)\, dt)} \leq \frac{\|\varphi'\|_\infty}{\ell^{2n}} \int_{t_{n-1}}^L |G(t)|^2 q(t)\, dt.$$

As G vanishes at L, for all $t \in (t_{n-1}, L)$, we have

$$|G(t)|^2 = \left|\int_t^L G'(s)\, ds\right| = \left|\int_t^L G'(s)\sqrt{q(s)}\frac{1}{\sqrt{q(s)}}\, ds\right| \leq \int_t^L |G'(s)|^2 q(s)\, ds \int_t^L \frac{ds}{q(s)} \leq \int_{t_{n-1}}^L |G'(s)|^2 q(s)\, ds \int_{t_{n-1}}^L \frac{ds}{q(s)},$$

and we obtain

$$\|\varphi_n' G\|^2_{L^2((0,L), q(t)\, dt)} \leq C_n \int_{t_{n-1}}^L |G'(t)|^2 q(s)\, ds, \qquad C_n := \frac{\|\varphi'\|_\infty}{\ell^{2n}} \int_{t_{n-1}}^L \frac{ds}{q(s)} \int_{t_{n-1}}^L q(s)\, ds. \tag{24}$$

In order to show the sought convergence, it suffices to show that C_n remain bounded for large n. For that, using the explicit expression of q and the relations (17), we compute

$$\int_{t_{n-1}}^L \frac{ds}{q(s)} \int_{t_{n-1}}^L q(s)\, ds = \sum_{k=n-1}^\infty (\ell\alpha p)^k \cdot \sum_{k=n-1}^\infty \left(\frac{\ell}{\alpha p}\right)^k$$

$$= (\ell\alpha p)^{n-1}(1 - \ell\alpha p)^{-1}\left(\frac{\ell}{\alpha p}\right)^{n-1}\left(1 - \frac{\ell}{\alpha p}\right)^{-1} \equiv \ell^{2n-2}(1 - \ell\alpha p)^{-1}\left(1 - \frac{\ell}{\alpha p}\right)^{-1},$$

which gives

$$C_n = \frac{\|\varphi'\|_\infty}{\ell^2}(1 - \ell\alpha p)^{-1}\left(1 - \frac{\ell}{\alpha p}\right)^{-1},$$

that is, C_n are independent of n. This concludes the proof of (b) for τ, and the property (c) is proved in the same way. $\qquad\square$

We complement the preceding observations by the following approximation result, which will be useful for the geometric interpretation of the embedded trace:

Lemma 2.11. *For $f \in H^1(\mathbb{T})$ and $N \in \mathbb{N}$, let f_N be the extension of $f|_{\mathbb{T}^N}$ by constants, that is,*

$$f_N(x) := \begin{cases} f(x), & x \in \mathbb{T}^N, \\ f(X_{N,K}), & x \in \mathbb{T}_{N,K}, \ K \in \{0, \ldots, p^N - 1\}, \end{cases}$$

then $f_N \xrightarrow{N \to \infty} f$ in $H^1(\mathbb{T})$, in particular, $\tau f = \lim_{N\to\infty} \tau f_N$ in $\ell^2_\sigma(\mathcal{Z})$.

Proof. Due to the one-dimensional Sobolev inequality for any $N \in \mathbb{N}$, one can find some $B_N > 0$ such that $|f(X_{N,K})| \leq B_N\|f\|_{H^1(\mathbb{T})}$ for any $f \in H^1(\mathbb{T})$ and any $K \in \{0, \ldots, p^N - 1\}$. We have

$$\|f_N\|^2_{L^2(\mathbb{T})} = \|f\|^2_{L^2(\mathbb{T}^N)} + \sum_{K=0}^{p^N-1} |f(X_{N,K})|^2 \int_{\mathbb{T}_{N,K}} 1\, d\mu,$$

while

$$\int_{\mathbb{T}_{N,K}} 1 \, d\mu = \sum_{j=0}^{p-1} \sum_{n=N+1}^{\infty} \sum_{k=0}^{p^n-1} (\alpha\ell)^n \equiv p \sum_{n=N+1}^{\infty} (p\alpha\ell)^n =: s_N \overset{(17)}{<} \infty. \tag{17}$$

It follows that $\|f_N\|_{L^2(\mathbb{T})}^2 \le \|f\|_{L^2(\mathbb{T}^N)}^2 + p^{N+1} B_N^2 s_N \|f\|_{H^1(\mathbb{T})}^2 < \infty$, that is, $f_N \in L^2(\mathbb{T})$ for any N. At the same time,

$$(f_N')_{n,k} = \begin{cases} f_{n,k}', & n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $|f_N'| \le |f'|$, which yields $\|f_N'\|_{L^2(\mathbb{T})} \le \|f'\|_{L^2(\mathbb{T})}$ and $f_N \in H^1(\mathbb{T})$. In addition,

$$\|f_N' - f'\|_{L^2(\mathbb{T})} = \|f'\|_{L^2(\mathbb{T}\setminus\mathbb{T}^N)} \xrightarrow{N\to\infty} 0. \tag{25}$$

Now let $C > 0$ be the constant from the Poincaré inequality (7). By construction, one has $f_N - f \in \tilde{H}^1(\mathbb{T})$, therefore, due to (25),

$$\|f_N - f\|_{L^2(\mathbb{T})} \le C \|f_N' - f'\|_{L^2(\mathbb{T})} \xrightarrow{N\to\infty} 0,$$

which concludes the proof. □

3 | APPROXIMATION SPACES

3.1 | Excursus about Sobolev spaces

Let us briefly recall various definitions and basic facts related to fractional Sobolev spaces H^s with $s \in (0,1)$ on open sets and manifolds.

Let $\Omega \subset \mathbb{R}^d$ be a nonempty open subset and $k \in \mathbb{N}_0$, then the kth Sobolev space $H^k(\Omega)$ is defined as

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) : \ \partial^\alpha f \in L^2(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k \right\},$$

which is a Hilbert space if equipped with the scalar product

$$\langle f, g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \le k} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2(\Omega)}.$$

For $\Omega = \mathbb{R}^d$, we obtain an equivalent definition via the Fourier transform. Namely for $s \in [0, \infty)$, define the sth Sobolev space on Ω by

$$H^s(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : \ \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^d) \right\} \text{ with } \langle \xi \rangle := \sqrt{1 + |\xi|^2},$$

where $\hat{f}$ is the Fourier transform of f, which becomes a Hilbert space if equipped with the norm

$$\langle f, g \rangle_{H^s(\mathbb{R}^d),\wedge} = \left\langle \langle \xi \rangle^s \hat{f}, \langle \xi \rangle^s \hat{g} \right\rangle_{L^2(\mathbb{R}^d)}.$$

For $s \in \mathbb{N}_0$, the two above definitions of $H^s(\mathbb{R}^d)$ coincide and the two norms are equivalent.

Let $\Omega \subset \mathbb{R}^d$ be a bounded nonempty open subset and $s \in (0,1)$, then the sth Sobolev space on $H^s(\Omega)$ is defined as the space of the restrictions on Ω of the functions from $H^s(\mathbb{R}^d)$ with the quotient norm

$$\|f\|_{H^s(\Omega),*} = \inf_{g \in H^s(\mathbb{R}^d), \ g|_\Omega = f} \|g\|_{H^s(\mathbb{R}^d)}.$$

We will need various equivalent characterizations of these spaces as well as several equivalent norms [1]. Recall that Ω is said to be with Lipschitz boundary if for any $p \in \partial\Omega$, there exist $\varepsilon > 0$, $a > 0$, a Lipschitz function h defined on the open ball $B_\varepsilon(0) \subset \mathbb{R}^{d-1}$ with $h(0) = 0$ and $|h(y_1, \dots, y_{d-1})| < a$ for all $(y_1, \dots, y_{d-1}) \in B_\varepsilon(0)$, and Cartesian coordinates $(y_1, \dots, y_m)$ centered at p such that

$$\Omega \cap \{(y_1, \dots, y_d) : (y_1, \dots, y_{d-1}) \in B_\varepsilon(0), |y_d| < 2a\} = \{(y_1, \dots, y_d) : (y_1, \dots, y_{d-1}) \in B_\varepsilon(0), h(y_1, \dots, y_{d-1}) < y_d < 2a\}.$$

The first reformulation comes from the interpolation theory, see [62, Chapter 34]. Let X and Y be normalized spaces with $X \subset Y$ and $s \in (0, 1)$. Choose any $b > 1$ and for $f \in Y$ and $t > 0$, define

$$K(f, t) = \inf_{g \in X} (\|f - g\|_Y + t\|g\|_X), \qquad F^s := (F^s)_{j \in \mathbb{N}} \text{ with } F_j^s := b^{js} K(f, b^{-j}), \tag{26}$$

then the interpolated space $[Y, X]_s$ is defined by

$$[Y, X]_s := \left\{ f \in Y : \|f\|_{[Y,X]_s}^2 := \|f\|_Y^2 + \|F^s\|_{\ell^2}^2 < \infty \right\}, \tag{27}$$

and for any $0 < s < s' \leq 1$, one has

$$H^s(\Omega) = \left[L^2(\Omega), H^{s'}(\Omega) \right]_{\frac{s}{s'}} \tag{28}$$

with an equivalence of the associated norms, see [15, Theorem 3.5.1]

If Ω has Lipschitz boundary, then

$$H^s(\Omega) = \left\{ f \in L^2(\Omega) : [f]_{H^s(\Omega)}^2 := \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} \, dx \, dy < \infty \right\}, \tag{29}$$

while the seminorms $f \mapsto \|F^s\|_{\ell^2}$ and $[\cdot]_{H^s(\Omega)}$ are equivalent, see [62, Chapter 36].

Another group of equivalent characterizations comes from the theory of Besov spaces, which we discuss following [15, Chapter 3]. For $f \in L^2(\Omega)$, define its modulus of smoothness by

$$w(f, t) := \sup_{|h| \leq t} \|f(\cdot + h) - f\|_{L^2(\Omega_h)}, \quad t > 0, \quad \Omega_h := \Omega \cap (\Omega + h). \tag{30}$$

For $s \in (0, 1)$ one defines the Besov seminorm $[f]_{B,s}$ of f by

$$[f]_{B,s}^2 := \int_0^1 t^{-2s} w(f, t)^2 \frac{dt}{t},$$

then the Besov space $B_{2,2}^s(\Omega)$ is defined by

$$B_{2,2}^s(\Omega) := \left\{ f \in L^2(\Omega) : \|f\|_{B_{2,2}^s}^2 := \|f\|_{L^2}^2 + [f]_{B,s}^2 < \infty \right\}.$$

Let $b > 1$, then the substitution $t := b^{-x}$ gives

$$\int_0^1 t^{-2s} w(f, t)^2 \frac{dt}{t} = (\log b) \int_0^\infty b^{2sx} w(f, b^{-x})^2 \, dx,$$

and using the monotonicity of $w(f, t)$ in t, we obtain the inequalities

$$\frac{1}{b^2} \sum_{j=0}^{\infty} b^{2sj} w(f, p^{-j})^2 \leq \sum_{j=0}^{\infty} b^{2sj} w(f, b^{-j-1})^2 \leq \sum_{j=0}^{\infty} \int_{j}^{j+1} b^{2sx} w(f, b^{-x})^2 \, \mathrm{d}x$$

$$\leq \sum_{j=0}^{\infty} b^{2s(j+1)} w(f, b^{-j})^2 \leq b^{2s} \sum_{j=0}^{\infty} b^{2sj} w(f, b^{-j})^2,$$

which shows that the seminorm

$$f \mapsto \left\| b^{sj} w(f, b^{-j}) \right\|_{\ell^2} \tag{31}$$

is equivalent to the above Besov seminorm. If, in addition, the set Ω has Lipschitz boundary, then

$$B_{2,2}^s(\Omega) = \left[L^2(\Omega), H^1(\Omega) \right]_s \equiv H^s(\Omega),$$

and the seminorms $f \mapsto \|b^{sj} w(f, b^{-j})\|_{\ell^2}$ and $[f]_{H^s(\Omega)}$ are equivalent.

We summarize the above considerations as follows:

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open subset with Lipschitz boundary, $0 < s < s' < 1$. For $f \in L^2(\Omega)$, define*

$$W^s := (W_j^s)_{j \in \mathbb{N}}, \qquad\qquad F^{s,s'} := (F_j^{s,s'})_{j \in \mathbb{N}},$$

$$W_j^s := p^{\frac{sj}{d}} w(f, p^{-\frac{j}{d}}), \qquad\qquad F_j^{s,s'} := p^{js/d} K_{s'}(f, p^{-\frac{js'}{d}}),$$

$$K_{s'}(f, t) = \inf_{g \in H^{s'}(\Omega)} \left(\|f - g\|_{L^2(\Omega)} + t \|g\|_{H^{s'}(\Omega)} \right),$$

then $\| \cdot \|_{L^2} + [\cdot]_{H,s}$ given by (29) and $f \mapsto \|f\|_{L^2} + \|W^s\|_{\ell^2}$ and $f \mapsto \|f\|_{L^2} + \|F^{s,s'}\|_{\ell^2}$ are equivalent norms on $H^s(\Omega)$.

Remark that the definitions of W_j^s and $F_j^{s,s'}$ correspond to the choice $b := p^{\frac{1}{d}}$ in (31) and $b := b := p^{\frac{s'}{d}}$ in (26), respectively.

Finally, if Ω is bounded with Lipschitz boundary and $0 \leq s < \frac{1}{2}$, then $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$, see, for example, eq. (2.220) in [49].

3.2 | Multiscale decompositions

Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open subset. Our next aim is to decompose Ω in a very special (but still quite natural) way. We adapt the construction proposed in [47] for $p = 2$, which is in turn a geometric realization of the approximation spaces used in the wavelet analysis, see, for example, [15, Chapter 2] or [48, Chapter 2].

By a *p-multiscale decomposition* of Ω, we mean a collection $(\Omega_{n,k})_{n \in \mathbb{N}_0, k \in \{0, \dots, p^n - 1\}}$ of nonempty subsets of Ω such that

(A1) $\Omega_{0,0} = \Omega$;
(A2) for any $n \in \mathbb{N}_0$, the sets $\Omega_{n,0}, \dots, \Omega_{n,p^n-1}$ are mutually disjoint;
(A3) for any $n \in \mathbb{N}_0$ and $k = 0, \dots, p^n - 1$, one has

$$\Omega_{n+1,pk+j} \subset \Omega_{n,k} \text{ for any } j \in \{0, \dots, p - 1\}, \quad \left| \Omega_{n,k} \setminus \bigcup_{j=0}^{p-1} \Omega_{n+1,pk+j} \right| = 0.$$

The above conditions can be viewed as a hierarchical decomposition procedure: One sets $\Omega_{0,0} := \Omega$, and if for some n all $\Omega_{n,k}$ are already constructed, then one decomposes each $\Omega_{n,k}$ (up to zero measure sets) into p disjoint pieces $\Omega_{n+1,pk+j}$, $j \in \{0, \dots, p-1\}$. In order to have a control of the size of $\Omega_{n,k}$, we introduce further classes of conditions.

A decomposition $(\Omega_{n,k})$ is *weakly balanced* if

(A4) there is $C_0 \geq 1$ such that

$$\frac{1}{C_0} \frac{|\Omega|}{p^n} \leq |\Omega_{n,k}| \leq C_0 \frac{|\Omega|}{p^n}$$

for all $n \in \mathbb{N}_0$ and $k \in \{0, \dots, p^n - 1\}$,

and is *strongly balanced* if it satisfies the stronger condition

(A4*) $|\Omega_{n,k}| = \dfrac{|\Omega|}{p^n}$ for all $n \in \mathbb{N}_0$ and $k \in \{0, \dots, p^n - 1\}$.

Finally, a decomposition $(\Omega_{n,k})$ is called *regular* if it satisfies the following two conditions:

(A5) There exists $c_1 > 0$ such that for all $n \in \mathbb{N}_0$ and $k \in \{0, \dots, p^n - 1\}$, one has

$$\operatorname{diam} \Omega_{n,k} \leq c_1 p^{-\frac{n}{d}};$$

(A6) there exists $c_2 > 0$ such that for all $h \in \mathbb{R}^d$, $n \in \mathbb{N}_0$ and $k \in \{0, \dots, p^n - 1\}$, one has

$$\left| \Omega_{n,k} \setminus (\Omega_{n,k} + h) \right| \leq c_2 |h| p^{-\frac{n(d-1)}{d}}.$$

Very roughly, the last two conditions say that the shape of $\Omega_{n,k}$ cannot become "too complicated" for large n. For the rest of the subsection, we assume that:

$$\boxed{\begin{array}{l} \Omega \subset \mathbb{R}^d \text{ is a bounded open set with Lipschitz boundary which admits} \\ \text{a regular weakly balanced } p\text{-multiscale decomposition } \mathcal{O} := (\Omega_{n,k}). \end{array}} \quad (32)$$

This covers a large class of Ω: We refer to Section 5 for a more detailed discussion.

The above conditions (A1)–(A3) are very standard and say that the family $(\Omega_{n,k})$ is a so-called dyadic cube, which are a very popular tool in the geometric analysis on metric spaces, see, for example, Kigami's book [34]. Usually these standard requirements are supplied with additional conjectures on the size and the shape of the pieces to guarantee additional properties of induced function spaces, see, for example, [14, 26] for further alternatives. We remark that our set of conditions (A4)–(A6) is chosen to guarantee a kind of compatibility of the partition with the Euclidian smooth structure and fractional Sobolev spaces, as will be seen from the subsequent constructions.

Let us establish further properties of $\mathcal{O}$.

Lemma 3.2. *Under the assumption (32), there holds*

$$K := \sup_{(n,k)} \#\left\{ j : \operatorname{dist}(\Omega_{n,j}, \Omega_{n,k}) \leq p^{-\frac{n}{d}} \right\} < \infty.$$

Proof. Let us pick some (n,k). Recall that by assumption (A5) we have the inequality $\operatorname{diam} \Omega_{n,j} \leq c_1 p^{-\frac{n}{d}}$ for all j. Now let j be such that $\operatorname{dist}(\Omega_{n,j}, \Omega_{n,k}) \leq p^{-\frac{n}{d}}$, then there exist $x_{n,j} \in \Omega_{n,j}$ and $x_{n,k} \in \Omega_{n,k}$ with $|x_{n,j} - x_{n,k}| < 2p^{-\frac{n}{d}}$. It follows

that for any $x \in \Omega_{n,j}$, one has the inequalities

$$|x - x_{n,k}| \leq |x - x_{n,j}| + |x_{n,j} - x_{n,k}| < \operatorname{diam} \Omega_{n,j} + 2p^{-\frac{n}{d}} \leq (c_1 + 2)p^{-\frac{n}{d}},$$

which shows the inclusion $\Omega_{n,j} \subset B_{(c_1+2)p^{-\frac{n}{d}}}(x_{n,k})$. Using $|\Omega_{n,j}| \geq C_0^{-1}|\Omega|p^{-n}$, the number of possible js is bounded from above by the number

$$\frac{\left|B_{(c_1+2)p^{-\frac{n}{d}}}(x_{n,k})\right|}{C_0^{-1}|\Omega|p^{-n}} \equiv \frac{C_0\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \frac{\left((c_1+2)p^{-\frac{n}{d}}\right)^d}{|\Omega|p^{-n}} \equiv \frac{C_0\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \frac{(c_1+2)^d}{|\Omega|}.$$

$\square$

For $n \in \mathbb{N}_0$, define

$$V_n := \operatorname{span}\left\{\mathbb{1}_{\Omega_{n,k}} : k = 0, \ldots, p^n - 1\right\} \subset L^2(\Omega), \qquad P_n := \text{the orthogonal projector on } V_n \text{ in } L^2(\Omega); \qquad (33)$$

in other words,

$$P_n f = \sum_{k=0}^{p^n-1} \frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} f \, \mathrm{d}x \, \mathbb{1}_{\Omega_{n,k}}.$$

Due to the assumption (A3) for any n, we have

$$\mathbb{1}_{\Omega_{n,k}} = \sum_{j=0}^{p-1} \mathbb{1}_{\Omega_{n+1,pk+j}} \text{ a.e.,}$$

which shows that $(V_n)_{n \in \mathbb{N}_0}$ is a strictly increasing sequence of closed subspaces. We will be interested in approximating arbitrary f by $P_n f$ with large n.

Lemma 3.3. *For any $f \in L^2(\Omega)$, one has $f = \lim_{n \to \infty} P_n f$.*

Proof. This is an adaptation of [47, Lemma 4.5].

(i) Let $g \in C^0(\overline{\Omega})$ and $\varepsilon > 0$. As g is uniformly continuous on $\overline{\Omega}$, one can find some $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for all $x, y \in \Omega$ with $|x - y| < \delta$. By (A5) one can find some $N \in \mathbb{N}$ such that $\operatorname{diam} \Omega_{n,k} < \delta$ for all $n \geq N$ and all k. Now let $n \geq N$ and pick $x_{n,k} \in \Omega_{n,k}$, then for any $x \in \Omega_{n,k}$, there holds $|g(x) - g(x_{n,k})| < \varepsilon$. Therefore,

$$\|g - g_n\|_\infty < \varepsilon \text{ for } g_n := \sum_{k=0}^{p^n-1} g(x_{n,k})\mathbb{1}_{\Omega_{n,k}} \in V_n,$$

which yields $\|g - P_n g\|_{L^2(\Omega)} \leq \|g - g_n\|_{L^2(\Omega)} \leq \|g - g_n\|_\infty^2 \sqrt{|\Omega|} \leq \varepsilon\sqrt{|\Omega|}$ for all $n \geq N$.

This shows that $\|g - P_n g\|_{L^2(\Omega)} \xrightarrow{n \to \infty} 0$ for any $g \in C^0(\overline{\Omega})$.

(ii) Let $f \in L^2(\Omega)$. Let $\varepsilon > 0$, then there exists $g \in C^0(\overline{\Omega})$ with $\|f - g\|_{L^2(\Omega)} < \varepsilon$. By (i) there is $N \in \mathbb{N}$ such that $\|g - P_n g\|_{L^2(\Omega)} < \varepsilon$ for all $n \geq N$. Then for all $n \geq N$, one has

$$\|f - P_n f\|_{L^2(\Omega)} \leq \|f - g\|_{L^2(\Omega)} + \|g - P_n g\|_{L^2(\Omega)} + \|P_n(g - f)\|_{L^2(\Omega)} \leq 2\|f - g\|_{L^2(\Omega)} + \|g - P_n g\|_{L^2(\Omega)} < 3\varepsilon,$$

which shows the claim. $\square$

We now introduce the approximation spaces $A^r(\Omega)$ consisting of the functions $f \in L^2(\Omega)$ such that the speed of convergence in Lemma 3.3 can be controlled in some special way. The construction is standard, see, for example, [15, section 3.5], but we need to recall the precise role of various parameters.

Definition 3.4. Let $r > 0$. For $f \in L^2(\Omega)$ set

$$\xi := (\xi_n)_{n \in \mathbb{N}_0}, \quad \xi_n := p^{\frac{nr}{d}} \operatorname{dist}_{L^2(\Omega)}(f, V_n) \equiv p^{\frac{nr}{d}} \|f - P_n f\|_{L^2(\Omega)}.$$

Then the *approximation space* $A^r(\Omega)$ and its norm are defined by

$$A^r(\Omega) = \{f \in L^2(\Omega) : \xi \in \ell^2\}, \quad \|f\|^2_{A^r(\Omega)} = \|P_0 f\|^2_{L^2(\Omega)} + \|\xi\|^2_{\ell^2}. \tag{34}$$

Remark that

$$\|f\|^2_{A^r(\Omega)} \geq \|P_0 f\|^2_{L^2(\Omega)} + |\xi_0|^2 \equiv \|P_0 f\|^2_{L^2(\Omega)} + \|f - P_0 f\|^2_{L^2(\Omega)} \equiv \|f\|^2_{L^2(\Omega)}.$$

Recall that the space $A^r(\Omega)$ depends on the decomposition $(\Omega_{n,k})$, but it is not reflected in the notation.

For what follows, it will be useful to work with another equivalent norm on $A^r(\Omega)$. Using the spaces V_n from (33), for any $n \in \mathbb{N}_0$, we introduce

$$U_n := \begin{cases} V_0, & n = 0, \\ V_n \cap V_{n-1}^\perp, & n \geq 1. \end{cases} \tag{35}$$

By construction, this gives the orthogonal decomposition $L^2(\Omega) = \bigoplus_{n=0}^{\infty} U_n$, the orthogonal projector Q_n on U_n is given by

$$Q_n := \begin{cases} P_0, & n = 0, \\ P_n - P_{n-1}, & n \geq 1, \end{cases}$$

and for each $n \in \mathbb{N}_0$ and $f \in L^2(\Omega)$, we have

$$f = \sum_{k=0}^{\infty} Q_k f = P_n f + \sum_{k=n+1}^{\infty} Q_k f, \tag{36}$$

while the summands on the right-hand side are mutually orthogonal. The following result is a particular case of [15, Theorem 3.5.3], but we include it for the sake of completeness.

Lemma 3.5. *Let $r > 0$. For $f \in L^2(\Omega)$, set*

$$\zeta := (\zeta_n)_{n \in \mathbb{N}_0}, \quad \zeta_n := p^{\frac{nr}{d}} \|Q_n f\|_{L^2(\Omega)}.$$

Then $\| \cdot \|_{A^r(\Omega)}$ and $f \mapsto \|\zeta\|_{\ell^2}$ are equivalent norms on $A^r(\Omega)$.

Proof. Recall that by definition there holds $\|f\|^2_{A^r(\Omega)} = \|Q_0 f\|^2_{L^2} + \sum_{n=0}^{\infty} \|f - P_n f\|^2_{L^2(\Omega)} p^{2\frac{nr}{d}}$. We have

$$\|\zeta\|^2_{\ell^2} \equiv \|Q_0 f\|^2_{L^2(\Omega)} + \sum_{n=1}^{\infty} \|Q_n f\|^2_{L^2(\Omega)} p^{2\frac{nr}{d}} \overset{(36)}{\leq} \|Q_0 f\|^2_{L^2(\Omega)} + \sum_{n=1}^{\infty} \|f - P_{n-1} f\|^2_{L^2(\Omega)} p^{2\frac{nr}{d}}$$

$$\equiv \|Q_0 f\|^2_{L^2(\Omega)} + p^{2\frac{nr}{d}} \sum_{n=0}^{\infty} \|f - P_n f\|^2_{L^2(\Omega)} p^{2\frac{nr}{d}} \leq p^{2\frac{r}{d}} \left(\|Q_0 f\|^2_{L^2(\Omega)} + \sum_{n=0}^{\infty} \|f - P_n f\|^2_{L^2(\Omega)} p^{2\frac{nr}{d}} \right) \equiv p^{2\frac{r}{d}} \|f\|^2_{A^r(\Omega)}.$$

At the same time,

$$\|f\|_{A^r(\Omega)}^2 = \|Q_0 f\|_{L^2}^2 + \sum_{n=0}^{\infty} \|f - P_n f\|_{L^2(\Omega)}^2 p^{2\frac{nr}{d}} \overset{(36)}{=} \|Q_0 f\|_{L^2(\Omega)}^2 + \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \|Q_k f\|_{L^2(\Omega)}^2 p^{2\frac{nr}{d}}$$

$$= \|Q_0 f\|_{L^2(\Omega)}^2 + \sum_{k=1}^{\infty} \|Q_k f\|_{L^2(\Omega)}^2 \sum_{n=0}^{k-1} p^{2\frac{nr}{d}} = \|Q_0 f\|_{L^2(\Omega)}^2 + \sum_{k=1}^{\infty} \|Q_k f\|_{L^2}^2 \frac{p^{2\frac{kr}{d}} - 1}{p^{2\frac{r}{d}} - 1}$$

$$\leq \|Q_0 f\|_{L^2(\Omega)}^2 + \frac{1}{p^{2\frac{r}{d}} - 1} \sum_{k=1}^{\infty} p^{2\frac{kr}{d}} \|Q_k f\|_{L^2}^2$$

$$\leq \max\left(1, \frac{1}{p^{2\frac{r}{d}} - 1}\right) \sum_{k=0}^{\infty} p^{2\frac{kr}{d}} \|Q_k f\|_{L^2(\Omega)}^2 \equiv \max\left(1, \frac{1}{p^{2\frac{r}{d}} - 1}\right) \|\zeta\|_{\ell^2}^2.$$

$\square$

3.3 | Relating approximation spaces to Sobolev spaces

Now we are going to compare the above equivalent norms on $A^r(\Omega)$ with suitable Sobolev norms. All results of this section are suitable adaptations of respective results from [47, section 4] for $p = 2$ and for slightly different definitions of multiscale decompositions and the spaces A^r. Recall that the seminorm $[\cdot]_{H^s}$ was defined in (29).

Lemma 3.6. *For any $0 < s < 1$, there exists $B > 0$ such that for any $n \in \mathbb{N}_0$ and $f \in H^s(\Omega)$ there holds $\|f - P_n f\|_{L^2(\Omega)} \leq Bp^{-\frac{ns}{d}}[f]_{H^s(\Omega)}$.*

Proof. Let $n \in \mathbb{N}_0$ and $k \in \{0, \dots, p^n - 1\}$. Recall that $P_n f$ is piecewise constant,

$$P_n f(x) = \frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} f(y)\,\mathrm{d}y \text{ as } x \in \Omega_{n,k}.$$

Using Cauchy–Schwarz inequality, we estimate

$$\int_{\Omega_{n,k}} |f(x) - P_n f(x)|^2 \,\mathrm{d}x = \int_{\Omega_{n,k}} \left|f(x) - \frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} f(y)\,\mathrm{d}y\right|^2 \mathrm{d}x = \int_{\Omega_{n,k}} \left|\frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} (f(x) - f(y))\,\mathrm{d}y\right|^2 \mathrm{d}x$$

$$\leq \int_{\Omega_{n,k}} \frac{1}{|\Omega_{n,k}|^2} |\Omega_{n,k}| \int_{\Omega_{n,k}} |f(x) - f(y)|^2 \,\mathrm{d}y\,\mathrm{d}x = \frac{1}{|\Omega_{n,k}|} \iint_{\Omega_{n,k} \times \Omega_{n,k}} |f(x) - f(y)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

For all $x, y \in \Omega_{n,k}$ with $x \neq y$, one has $|x - y| \leq \operatorname{diam} \Omega_{n,k}$, which gives

$$\int_{\Omega_{n,k}} |f(x) - P_n f(x)|^2 \,\mathrm{d}x \leq \frac{(\operatorname{diam} \Omega_{n,k})^{d+2s}}{|\Omega_{n,k}|} \iint_{\Omega_{n,k} \times \Omega_{n,k}} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \equiv \frac{(\operatorname{diam} \Omega_{n,k})^{d+2s}}{|\Omega_{n,k}|} [f]_{H^s(\Omega)}^2.$$

Recall due to the choice of $\Omega_{n,k}$ (see Subsection 3.2) and (A4), we have

$$\frac{(\operatorname{diam} \Omega_{n,k})^{d+2s}}{|\Omega_{n,k}|} \leq c_1 p^{-\frac{n}{d}}, \quad |\Omega_{n,k}| \geq \frac{|\Omega|}{C_0 p^n},$$

therefore,

$$\frac{(\operatorname{diam} \Omega_{n,k})^{d+2s}}{|\Omega_{n,k}|} \leq \frac{C_0 (c_1 p^{-\frac{n}{d}})^{d+2s} p^n}{|\Omega|} = Bp^{-\frac{2sn}{d}}, \quad B := \frac{C_0 c_1^{d+2s}}{|\Omega|}$$

resulting in $\|f - P_n\|^2_{L^2(\Omega_{n,k})} \leq B p^{-\frac{2sn}{d}} [f]^2_{H^s(\Omega_{n,k})}$. Then

$$\|f - P_n f\|^2_{L^2(\Omega)} = \sum_{k=0}^{p^n-1} \|f - P_n f\|^2_{L^2(\Omega_{n,k})} \leq B p^{-\frac{2ns}{d}} \sum_{k=0}^{b^n-1} [f]^2_{H^s(\Omega_{n,k})} \leq B p^{-\frac{ns}{d}} \|f\|_{H^s(\Omega)}.$$

□

This allows one to show the first embedding result:

Theorem 3.7. *Assume (32), then $H^s(\Omega) \hookrightarrow A^r(\Omega)$ for any $0 \leq r \leq s < 1$.*

Proof. We equip $H^s(\Omega)$ with the interpolation norm (Proposition 3.1)

$$\|f\|^2_{H^s} := \|f\|^2_{L^2(\Omega)} + \left\| p^{\frac{js}{d}} K_{s'}(f, ^{-\frac{js'}{d}}) \right\|^2_{\ell^2}, \quad 0 < s < s' < 1.$$

Let $f \in L^2(\Omega)$. As P_j are orthogonal projections, for any $j \in \mathbb{N}_0$ and any $g \in H^{s'}(\Omega) \subset L^2(\Omega)$, one has

$$\|f - P_j f\|_{L^2(\Omega)} \leq \|f - P_j g\|_{L^2(\Omega)} \leq \|f - g\|_{L^2(\Omega)} + \|g - P_j g\|_{L^2(\Omega)}.$$

Using Lemma 3.6 to estimate the last summand, we conclude that one can choose some $C \geq 1$ such that

$$\|f - P_j f\|_{L^2(\Omega)} \leq \|f - g\|_{L^2(\Omega)} + C b^{-\frac{js'}{d}} \|g\|_{H^{s'}(\Omega)} \text{ for all } j \in \mathbb{N}_0,$$

which immediately gives

$$\|f - P_j f\|_{L^2(\Omega)} \leq C K_{s'}(f, b^{-\frac{js'}{d}}). \tag{37}$$

Now let $f \in H^s(\Omega)$, then

$$\|f\|^2_{A^r(\Omega)} = \|P_0 f\|^2_{L^2(\Omega)} + \sum_{j=0}^{\infty} p^{2\frac{jr}{d}} \|f - P_j f\|^2_{L^2(\Omega)}$$

$$\text{use (37): } \leq C\left(\|f\|^2_{L^2(\Omega)} + \sum_{j=0}^{\infty} p^{2\frac{jr}{d}} K_{s'}(f, p^{-\frac{js'}{d}})^2 \right) \equiv C\left(\|f\|^2_{L^2(\Omega)} + \sum_{j=0}^{\infty} p^{2(r-s)\frac{j}{d}} p^{\frac{2sj}{d}} K_{s'}(f, p^{-\frac{js'}{d}})^2 \right)$$

$$\text{use } r \leq s: \leq C\left(\|f\|^2_{L^2(\Omega)} + \sum_{j=0}^{\infty} p^{\frac{2sj}{d}} K_{s'}(f, p^{-\frac{js'}{d}})^2 \right) \equiv C\left(\|f\|^2_{L^2(\Omega)} + \left\| p^{\frac{js}{d}} K_{s'}(f, ^{-\frac{js'}{d}}) \right\|^2_{\ell^2} \right) \equiv C \|f\|^2_{H^s(\Omega)}.$$

□

In order to obtain embedding results in the other direction, more work is needed. Recall that the modulus of smoothness $w(f, t)$ was defined in (30).

Lemma 3.8. *There exists $C \geq 2$ such that for all $n \in \mathbb{N}_0$, $t > 0$, and $f \in V_n$, there holds $w(f, t) \leq C p^{\frac{n}{2d}} t^{\frac{1}{2}} \|f\|_{L^2(\Omega)}$.*

Proof. Consider first the case $t \in (0, p^{-\frac{n}{d}})$ for some $n \in \mathbb{N}_0$. Let $f \in V_n$, then

$$f = \sum_{j=0}^{p^n-1} f_{n,j} \mathbb{1}_{\Omega_{n,j}}, \quad f_{n,j} \in \mathbb{C}, \quad \|f\|^2_{L^2(\Omega)} = \sum_{j=0}^{p^n-1} |f_{n,j}|^2 |\Omega_{n,j}| \geq \frac{|\Omega|}{C_0 p^n} \sum_{j=0}^{p^n-1} |f_{n,j}|^2, \tag{38}$$

where we used (A4) in the last step. Recall that by (A6) for any (n, k), there holds

$$|\Omega_{n,k} \setminus (\Omega_{n,k} - h)| \equiv \|\mathbb{1}_{\Omega_{n,k}-h} - \mathbb{1}_{\Omega_{n,k}}\|_{L^1(\Omega_{n,k})} \leq c_2 |h| p^{-\frac{n(d-1)}{d}}, \quad h \in \mathbb{R}^d, \tag{39}$$

and by Lemma 3.2 for some $K > 0$, we have

$$\#\left\{j : \operatorname{dist}(\Omega_{n,j}, \Omega_{n,k}) \le p^{-\frac{n}{d}}\right\} \le K \text{ for any } (n,k). \tag{40}$$

Remark that $f(y) = f_{n,k}$ for any $y \in \Omega_{n,k}$. In particular, if $x \in \Omega_{n,k}$ and $x + h \in \Omega_{n,k}$, then $f(x+h) = f(x)$. It follows that for any $x \in \Omega_{n,k}$, one has

$$|f(x+h) - f(x)| = \left|\mathbb{1}_{\Omega_{n,k}}(x+h) - \mathbb{1}_{\Omega_{n,k}}(x)\right| \cdot |f(x+h) - f(x)| \equiv \left|\mathbb{1}_{\Omega_{n,k}-h}(x) - \mathbb{1}_{\Omega_{n,k}}(x)\right| \cdot |f(x+h) - f(x)|. \tag{41}$$

Now let $h \in \mathbb{R}^d$ with $|h| \le t$. Then for any $x \in \Omega_{n,k}$, we estimate

$$|f(x+h) - f(x)|^2 = \left|\sum_{j=0}^{p^n-1} f_{n,j}\mathbb{1}_{\Omega_{n,j}}(x+h) - f_{n,k}\right|^2 \le \left(|f_{n,k}| + \sum_{j=0}^{p^n-1}|f_{n,j}|\mathbb{1}_{\Omega_{n,j}}(x+h)\right)^2$$

$$\le \left(|f_{n,k}| + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le|h|}|f_{n,j}|\right)^2 \le \left(|f_{n,k}| + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le p^{-\frac{n}{d}}}|f_{n,j}|\right)^2$$

use (40) and Cauchy–Schwarz: $$\le (K+1)\left(|f_{n,k}|^2 + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le p^{-\frac{n}{d}}}|f_{n,j}|^2\right).$$

Using this inequality on the right-hand side of (41), we obtain

$$\int_{\Omega_{n,k}}|f(x+h) - f(x)|^2\,\mathrm{d}x \le (K+1)\|\mathbb{1}_{\Omega_{n,k}-h} - \mathbb{1}_{\Omega_{n,k}}\|_{L^1(\Omega_{n,k})}\left(|f_{n,k}|^2 + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le p^{-\frac{n}{d}}}|f_{n,j}|^2\right)$$

use (39): $$\le (K+1)c_2|h|p^{-\frac{n(d-1)}{d}}\left(|f_{n,k}|^2 + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le p^{-\frac{n}{d}}}|f_{n,j}|^2\right).$$

By summing over all k, we arrive at

$$\int_{\Omega}|f(x+h) - f(x)|^2\,\mathrm{d}x = \sum_{k=0}^{p^n-1}\int_{\Omega_{n,k}}|f(x+h) - f(x)|^2\,\mathrm{d}x \le (K+1)c_2|h|p^{-\frac{n(d-1)}{d}}$$

$$\times \sum_{k=0}^{p^n-1}\left(|f_{n,k}|^2 + \sum_{j:\,\operatorname{dist}(\Omega_{n,j},\Omega_{n,k})\le p^{-\frac{n}{d}}}|f_{n,j}|^2\right)$$

use (40): $$\le (K+1)^2 c_2|h|p^{-\frac{n(d-1)}{d}}\sum_{k=0}^{p^n-1}|f_{n,k}|^2$$

use (38): $$\le (K+1)^2 c_2|h|p^{-\frac{n(d-1)}{d}}\frac{C_0 p^n}{|\Omega|}\|f\|_{L^2(\Omega)}^2 \le c^2 p^{\frac{n}{d}}|h|\,\|f\|_{L^2(\Omega)}^2, \quad c := (K+1)\sqrt{\frac{C_0 c_2}{|\Omega|}}.$$

The last estimate holds for all h with $|h| \le t$, which yields $w(f,t) \le c p^{\frac{n}{2d}}t^{\frac{1}{2}}\|f\|_{L^2(\Omega)}^2$ for all $t \in (0, p^{-\frac{n}{d}})$.

For $t \geq p^{-\frac{n}{d}}$ we simply estimate $w(t, f) \leq 2\|f\|_{L^2(\Omega)} \leq 2p^{\frac{n}{2d}} t^{\frac{1}{2}} \|f\|_{L^2(\Omega)}$ and obtain the claim by taking $C := \max(2, c)$. $\qquad\square$

Theorem 3.9. *Let (32) be satisfied, then for any $0 < s < \frac{1}{2}$ and $s \leq r$, one has $A^r(\Omega) \hookrightarrow H^s(\Omega)$.*

Proof. We equip $H^s(\Omega)$ with the Besov norm

$$\|f\|_{B^s}^2 := \|f\|_{L^2(\Omega)}^2 + \left\| p^{\frac{sj}{d}} w(f, p^{-\frac{j}{d}}) \right\|_{\ell^2}^2$$

and recall that for any $t > 0$, the modulus of smoothness $w(f, t)$ satisfies the triangle inequality with respect to f. For each j, we have

$$f = \sum_{k=0}^{j} Q_k f + (f - P_j f), \qquad w(f, p^{-\frac{j}{d}}) \leq \sum_{k=0}^{j} w(Q_k, p^{-\frac{j}{d}}) + w(f - P_j f, p^{-\frac{j}{d}}).$$

Remark that $Q_k f \in V_k$, so by Lemma 3.8 we estimate with some $C \geq 2$,

$$w(Q_k, p^{-\frac{j}{d}}) \leq C p^{\frac{k}{2d} - \frac{j}{2d}} \|Q_k f\|_{L^2(\Omega)}, \quad w(f - P_j f, p^{-\frac{j}{d}}) \leq C \|f - P_j f\|_{L^2(\Omega)}.$$

In addition, for $k \geq 1$, we have $P_k P_{k-1} = P_{k-1}$, therefore,

$$\|Q_k f\|_{L^2(\Omega)} = \|P_k f - P_{k-1} f\|_{L^2(\Omega)} = \|P_k(f - P_{k-1} f)\|_{L^2(\Omega)} \leq \|f - P_{k-1} f\|_{L^2(\Omega)},$$

and then (recall that $Q_0 = P_0$)

$$w(f, p^{-\frac{j}{d}}) \leq C p^{-\frac{j}{2d}} \|P_0 f\|_{L^2(\Omega)} + C \sum_{k=1}^{j} p^{\frac{k}{2d} - \frac{j}{2d}} \|f - P_{k-1} f\|_{L^2(\Omega)} + C \|f - P_j f\|_{L^2(\Omega)}$$

$$= C p^{-\frac{j}{2d}} \|P_0 f\|_{L^2(\Omega)} + C p^{\frac{1}{2d}} \sum_{k=0}^{j-1} p^{\frac{k}{2d} - \frac{j}{2d}} \|f - P_k f\|_{L^2(\Omega)} + C \|f - P_j f\|_{L^2(\Omega)}$$

$$\leq B p^{-\frac{j}{2d}} \|P_0 f\|_{L^2(\Omega)} + B \sum_{k=0}^{j} p^{\frac{k}{2d} - \frac{j}{2d}} \|f - P_k f\|_{L^2(\Omega)}$$

with $B := C p^{\frac{1}{2d}} > C$. It follows that for any j, one has

$$p^{\frac{sj}{d}} w(f, p^{-\frac{j}{d}}) \leq B(F_j' + F_j) \tag{42}$$

with sequences $F' := (F_j')$ and $F := (F_j)$ given by

$$F_j' := p^{\frac{sj}{d} - \frac{j}{2d}} \|P_0 f\|_{L^2(\Omega)} \equiv p^{(s - \frac{1}{2})\frac{j}{d}} \|P_0 f\|_{L^2(\Omega)}, \qquad F_j := p^{\frac{sj}{d}} \sum_{k=0}^{j} p^{\frac{k}{2d} - \frac{j}{2d}} \|f - P_k f\|_{L^2(\Omega)}.$$

Due to $s < \frac{1}{2}$, we have

$$\|F'\|_{\ell^2}^2 = \sum_{j=0}^{\infty} |F_j'|^2 = a \|P_0 f\|_{L^2(\Omega)}^2 \text{ with } a := \sum_{j=0}^{\infty} p^{2(s - \frac{1}{2})\frac{j}{d}} < \infty.$$

In order to control F_j, we represent it as

$$F_j = \sum_{k=0}^{j} p^{\frac{sj}{d}+\frac{k}{2d}-\frac{j}{2d}-\frac{rk}{d}} p^{\frac{rk}{d}} \|f - P_k f\|_{L^2(\Omega)}.$$

We estimate the exponents using $s \leq r$:

$$\frac{sj}{d} + \frac{k}{2d} - \frac{j}{2d} - \frac{rk}{d} = \left(s - \frac{1}{2}\right)\frac{j}{d} + \left(\frac{1}{2} - r\right)\frac{k}{d} \leq \left(s - \frac{1}{2}\right)\frac{j}{d} + \left(\frac{1}{2} - s\right)\frac{k}{d} \equiv \left(s - \frac{1}{2}\right)\frac{j-k}{d},$$

and arrive at

$$F_j \leq \sum_{k=0}^{j} p^{(s-\frac{1}{2})\frac{j-k}{d}} p^{\frac{rk}{d}} \|f - P_k f\|_{L^2(\Omega)}. \tag{43}$$

Define $\alpha := (\alpha_j)_{j\in\mathbb{Z}}$ and $\beta := (\beta_j)_{j\in\mathbb{Z}}$ by

$$\alpha_j := \begin{cases} 0, & j < 0, \\ p^{(s-\frac{1}{2})\frac{j}{d}}, & j \geq 0, \end{cases} \qquad \beta_j := \begin{cases} 0, & j < 0, \\ p^{\frac{rj}{d}} \|f - P_j f\|_{L^2(\Omega)}, & j \geq 0, \end{cases}$$

then (43) takes the form $F_j \leq (\alpha * \beta)_j$ for any $j \in \mathbb{N}_0$, with $*$ being the convolution product. Using Young's convolution inequality, we obtain $\|F\|_{\ell^2} \leq \|\alpha * \beta\|_{\ell^2(\mathbb{Z})} \leq \|\alpha\|_{\ell^1(\mathbb{Z})}\|\beta\|_{\ell^2(\mathbb{Z})}$. Then it follows by (42) that

$$\|f\|_{B^s}^2 = \left\| p^{\frac{sj}{d}} w(f, p^{-\frac{j}{d}}) \right\|_{\ell^2}^2 \leq 2B\|F'\|_{\ell^2} + 2B\|F\|_{\ell^2}^2 \leq 2Ba\|P_0 f\|_{L^2(\Omega)}^2 + 2B\|\alpha\|_{\ell^1(\mathbb{Z})}^2 \|\beta\|_{\ell^2(\mathbb{Z})}^2.$$

Recall that $\|P_0 f\|_{L^2(\Omega)}^2 + \|\beta\|_{\ell^2(\mathbb{Z})}^2 = \|f\|_{A^r(\Omega)}^2$, therefore, $\|f\|_{B^s}^2 \leq 2B \max(a, \|\alpha\|_{\ell^1(\mathbb{Z})}^2)\|f\|_{A^r(\Omega)}$ for all $f \in A^r(\Omega)$. □

By combining Theorems 3.7 and 3.9, we obtain the following result.

Corollary 3.10. *Under the assumption (32), there holds $A^r(\Omega) = H^r(\Omega)$ for $0 \leq r < \frac{1}{2}$.*

3.4 | Extension to open sets in manifolds

Let (S, g) be an d-dimensional Riemannian manifold of bounded geometry. For the construction of Sobolev spaces $H^s(S)$ with $s \geq 0$, we refer to [22]. Let $\Omega \subset S$ be a nonempty open set such that $\overline{\Omega} \subset S$ is compact (the case $\Omega = S$ is possible, if S itself is compact). The Sobolev space $H^s(\Omega)$ on Ω is then defined as the space of the restrictions on Ω of the functions from $H^s(S)$ with the quotient norm

$$\|f\|_{H^s(\Omega),*} = \inf_{F\in H^s(S),\, F|_\Omega = f} \|F\|_{H^s(S)}.$$

It follows from the general construction of Sobolev spaces that:

(1) for any open $\Omega_0 \subset \Omega$, the linear map $H^s(\Omega) \ni f \mapsto f|_{\Omega_0} \in H^s(\Omega_0)$ is bounded;
(2) if for some local chart $\Phi : \mathbb{R}^d \supset \tilde{O} \mapsto O \subset S$ one has $\overline{\Omega} \subset O$, then the map $f \mapsto f \circ \Phi$ is an isomorphism between $H^s(\Omega)$ and $H^s(\tilde{\Omega})$, with $\tilde{\Omega} := \Phi^{-1}(\Omega) \subset \mathbb{R}^d$.

We say that nonempty subsets $(\Omega_{n,k})_{n\in\mathbb{N}_0, k=0,\ldots,p^n-1} \subset \Omega$ form a *p-multiscale decomposition of Ω*, if the following conditions hold:

(B1) $\Omega_{0,0} = \Omega$;

(B2) for any $n \in \mathbb{N}_0$, the sets $\Omega_{n,0}, \ldots, \Omega_{n,p^n-1}$ are mutually disjoint;

(B3) for any $n \in \mathbb{N}_0$ and $k = 0, \ldots, p^n - 1$, one has

$$\Omega_{n+1,pk+j} \subset \Omega_{n,k} \text{ for any } j \in \{0, \ldots, p-1\}, \quad \left| \Omega_{n,k} \setminus \bigcup_{j=0}^{p-1} \Omega_{n+1,pk+j} \right| = 0.$$

This decomposition is called *regular and weakly balanced* if it satisfies additionally the following:

(B4) For some $N \in \mathbb{N}_0$, each $\overline{\Omega_{N,K}}$ with $K \in \{0, \ldots, p^N - 1\}$ is covered by a local chart $\Phi_{N,K}$ on S such that the sets

$$\tilde{\Omega}_{N,K} := \Phi_{N,K}^{-1}(\Omega_{N,K})$$

are bounded open sets with Lipschitz boundaries in $\mathbb{R}^d$.

(B5) For each $K \in \{0, \ldots, p^N - 1\}$, the sets

$$\tilde{\Omega}_{n,k}^{N,K} := \tilde{\Omega}_{N+n,p^n K+k}, \quad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, p^n - 1\},$$

form a *regular weakly balanced p-multiscale decomposition of $\tilde{\Omega}_{N,K}$*,

and it is called *regular and strongly balanced* if one has in addition, for N from (B4),

(B6) $|\Omega_{N,K}| = p^{-N}|\Omega|$ for all $K \in \{0, \ldots, p^N - 1\}$,

(B7) for each $K \in \{0, \ldots, p^N - 1\}$ the sets

$$\tilde{\Omega}_{n,k}^{N,K} := \tilde{\Omega}_{N+n,p^n K+k}, \quad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, p^n - 1\},$$

form a *regular strongly balanced p-multiscale decomposition of $\tilde{\Omega}_{N,K}$*.

For the rest of the subsection, we assume that:

$$\boxed{\begin{array}{l} \Omega \subset S \text{ is an open set with compact closure which admits a regular} \\ \text{weakly balanced } p\text{-multiscale decomposition } (\Omega_{n,k}) \end{array}} \tag{44}$$

and let N and $\tilde{\Omega}_{n,k}$ be as in (B4)–(B5). Then the sets

$$\Omega_{n,k}^{N,K} := \Omega_{N+n,p^n K+k}, \quad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, p^n - 1\}, \tag{45}$$

form a regular balanced multiscale decomposition of $\Omega_{N,K}$ for $K \in \{0, \ldots, p^N - 1\}$. In addition, the decomposition $(\Omega_{n,k})$ gives rise to the projectors P_n and the spaces V_n and $A^r(\Omega)$ defined in the same way as in the Euclidean case: For $n \in \mathbb{N}_0$, define

$$V_n := \text{span}\left\{ \mathbb{1}_{\Omega_{n,k}} : k = 0, \ldots, p^n - 1 \right\} \subset L^2(\Omega),$$

$$U_n := \begin{cases} V_0, & n = 0, \\ V_n \cap V_{n-1}^\perp, & n \geq 1, \end{cases} \tag{46}$$

$$P_n := \text{the orthogonal projector on } V_n \text{ in } L^2(\Omega),$$

$$Q_n := \text{the orthogonal projector on } U_n \text{ in } L^2(\Omega).$$

Remark that Lemmas 3.3 and 3.5 are transferred directly to this new setting. We establish several additional properties of the spaces $A^r(\Omega)$.

Lemma 3.11. *Let $A^r(\Omega_{N,K})$ be the approximation spaces associated with the decompositions $(\Omega_{n,k}^{N,K})$ from 45, then for any $r > 0$ and $N \in \mathbb{N}$, the map*

$$
J \ : \ A^r(\Omega) \ni f \mapsto (f_{N,K})_{K \in \{0,\dots,p^n-1\}} \in \bigoplus_{K=0}^{p^n-1} A^r(\Omega_{N,K}), \quad f_{N,K} := f|_{\Omega_{N,K}},
$$

is an isomorphism.

Proof. For $f \in L^2(\Omega)$, we have

$$
\frac{1}{|\Omega_{n,k}^{N,K}|} \int_{\Omega_{n,k}^{N,K}} f \, \mathrm{dvol}_g = \alpha_{N+n,p^n K+k} \qquad \text{for } \alpha_{n,k} := \frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} f \, \mathrm{dvol}_g.
$$

For $n \in \mathbb{N}_0$ let $P_n^{N,K}$ be the orthogonal projector on

$$
V_n^{N,K} := \mathrm{span} \left\{ \mathbb{1}_{\Omega_{n,k}^{N,K}} \ : \ k \in \{0,\dots,p^n-1\} \right\} \subset L^2(\Omega_{N,K})
$$

in $L^2(\Omega_{N,K})$, then

$$
P_n^{N,K} f_{N,K} = \sum_{k=0}^{p^n-1} \frac{1}{|\Omega_{n,k}^{N,K}|} \int_{\Omega_{n,k}^{N,K}} f \, \mathrm{dvol}_g \, \mathbb{1}_{\Omega_{n,k}^{N,K}} \equiv \sum_{k=0}^{p^n-1} \alpha_{N+n,p^n K+k} \mathbb{1}_{\Omega_{N+n,p^n K+k}}.
$$

At the same time,

$$
P_{N+n}f = \sum_{j=0}^{p^{N+n}-1} \alpha_{n,j} \mathbb{1}_{\Omega_{n,j}} = \sum_{K=0}^{p^N-1} \sum_{k=0}^{p^n-1} \alpha_{N+n,p^n K+k} \, \mathbb{1}_{\Omega_{N+n,p^n K+k}} = \sum_{K=0}^{p^N-1} (P_n^{N,K} f_{N,K}) \mathbb{1}_{\Omega_{N,K}},
$$

$$
f = \sum_{K=0}^{p^N-1} f_{N,K} \, \mathbb{1}_{\Omega_{N,K}}, \qquad f - P_{N+n}f = \sum_{K=0}^{p^N-1} (f_{N,K} - P_n^{N,K} f_{N,K}) \, \mathbb{1}_{\Omega_{N,K}}.
$$

As the summands in the last sum have disjoint supports, they are orthogonal in $L^2(\Omega)$, and

$$
\|f - P_{N+n}f\|_{L^2(\Omega)}^2 = \sum_{K=0}^{p^N-1} \|f_{N,K} - P_n^{N,K} f_{N,K}\|^2. \tag{47}
$$

For any $r \geq 0$, we have

$$
\sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - P_{N+n}f\|_{L^2(\Omega)}^2 \overset{(47)}{=} \sum_{K=0}^{p^N-1} \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f_{N,K} - P_n^{N,K} f_{N,K}\|^2 \equiv \sum_{K=0}^{p^N-1} \|f_{N,K}\|_{A^r(\Omega_{N,K})}^2. \tag{48}
$$

Therefore, if $f \in A^r(\Omega)$, then

$$
\|Jf\|^2 = \sum_{K=0}^{p^N-1} \|f_{N,K}\|_{A^r(\Omega_{N,K})}^2 \overset{(48)}{=} \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - P_{N+n}f\|_{L^2(\Omega)}^2
$$

$$
= p^{-2\frac{Nr}{d}} \sum_{n=N}^{\infty} p^{2\frac{nr}{d}} \|f - P_n f\|_{L^2(\Omega)}^2 \leq p^{-2\frac{Nr}{d}} \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - P_n f\|_{L^2(\Omega)}^2 = p^{-2\frac{Nr}{d}} \|f\|_{A^r(\Omega)}^2,
$$

which shows that J is bounded, and it is clearly injective. For any $f_{N,K} \in A^r(\Omega_{N,K})$, the function

$$f := \sum_{K=0}^{p^N-1} f_{N,K} \, \mathbb{1}_{\Omega_{N,K}}$$

belongs to $L^2(\Omega)$, and

$$\|f\|_{A^r(\Omega)}^2 = \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - P_n\|_{L^2(\Omega)}^2 = \sum_{n=0}^{N-1} p^{2\frac{nr}{d}} \|f - P_n\|_{L^2(\Omega)}^2 + p^{2\frac{Nr}{d}} \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - P_{N+n}\|_{L^2(\Omega)}^2,$$

$$\text{use (48):} = \sum_{n=0}^{N-1} p^{2\frac{nr}{d}} \|f - P_n\|_{L^2(\Omega)}^2 + p^{2\frac{Nr}{d}} \sum_{K=0}^{p^N-1} \|f_{N,K}\|_{A^r(\Omega_{N,K})}^2 < \infty,$$

which shows that $f \in A^r(\Omega)$. Therefore, J is surjective as well, and it follows by the closed graph theorem that J is an isomorphism. $\square$

Lemma 3.12. *Assume that (B4) holds with $N = 0$, that is, there exists a local chart $\Phi : \mathbb{R}^d \supset \tilde{O} \mapsto O \subset S$ such that $\overline{\Omega} \subset O$ and the sets $\widetilde{\Omega}_{n,k} := \Phi^{-1}(\Omega_{n,k})$ form a regular weakly balanced p-multiscale decomposition of $\widetilde{\Omega} := \Phi^{-1}(\Omega)$. Consider the associated spaces $A^r(\tilde{\Omega})$, then for any $0 \le r < 1$, the map $A^r(\Omega) \ni f \mapsto \tilde{f} := f \circ \Phi \in A^r(\widetilde{\Omega})$ is an isomorphism.*

Proof. Recall that there exists $\tilde{c}_1 > 0$ such that

$$\text{diam } \tilde{\Omega}_{n,k} \le \tilde{c}_1 p^{-\frac{n}{d}} \text{ for all } n \in \mathbb{N}_0 \text{ and } k \in \{0, \dots, p^n - 1\}. \tag{49}$$

For a function f defined on Ω and the function $\tilde{f} := f \circ \Phi$ defined on $\tilde{\Omega}$, one has

$$\int_\Omega f(y) \, \mathrm{dvol}_g(y) = \int_{\tilde{\Omega}} \tilde{f}(u) J_\Phi(u) \, du, \qquad J_\Phi(u) := \sqrt{\det\left(g_\Phi\left(\partial_j \Phi(u), \partial_k \Phi(u)\right)\right)_{j,k \in \{1,\dots,d\}}}, \tag{50}$$

and there exist $b_1, b_2 > 0$ such that $b_1 \le J_\Phi(u) \le b_2$ for all $u \in \tilde{\Omega}$. It follows that the map $\Theta : f \mapsto \tilde{f}$ defines an isomorphism between $L^2(\Omega)$ and $L^2(\tilde{\Omega})$,

$$b_1 \|\Theta f\|_{L^2(\tilde{\Omega})}^2 \le \|\Theta f\|_{L^2(\Omega)}^2 \le b_2 \|\Theta f\|_{L^2(\tilde{\Omega})}^2 \text{ for all } f \in L^2(\Omega),$$

in particular,

$$b_1 |\tilde{\Omega}_{n,k}| \le |\Omega_{n,k}| \le b_2 |\tilde{\Omega}_{n,k}| \text{ for all } (n,k). \tag{51}$$

For $n \in \mathbb{N}_0$, let $\tilde{P}_n$ denote the orthogonal projector in $L^2(\tilde{\Omega})$ on the subspace

$$\tilde{V}_n := \mathrm{span}\left\{ \mathbb{1}_{\tilde{\Omega}_{n,k}} : k \in \{0, \dots, p^n - 1\} \right\},$$

then

$$P_n f = \sum_{k=0}^{p^n-1} \frac{1}{|\Omega_{n,k}|} \int_{\Omega_{n,k}} f(y) \, \mathrm{dvol}_g(y),$$

$$\text{use (50):} \quad \Theta P_n f(y) \equiv (P_n f)(\Phi(y)) = \sum_{k=0}^{p^n-1} \frac{1}{|\Omega_{n,k}|} \int_{\tilde{\Omega}_{n,k}} \tilde{f}(u) J_\Phi(u) \, du \, \mathbb{1}_{\tilde{\Omega}_{n,k}}(y),$$

$$\tilde{P}_n \Theta f(y) \equiv \tilde{P}_n \tilde{f}(y) = \sum_{k=0}^{p^n-1} \frac{1}{|\tilde{\Omega}_{n,k}|} \int_{\tilde{\Omega}_{n,k}} \tilde{f}(u) \, du \, \mathbb{1}_{\tilde{\Omega}_{n,k}}(y).$$

Therefore,

$$(\Theta P_n f - \tilde{P}_n \Theta f)(y) = \sum_{k=0}^{p^n-1} \int_{\tilde{\Omega}_{n,k}} \tilde{f}(u)\left(\frac{J_\Phi(u)}{|\Omega_{n,k}|} - \frac{1}{|\tilde{\Omega}_{n,k}|}\right) \mathrm{d}u\, \mathbb{1}_{\tilde{\Omega}_{n,k}}(y). \tag{52}$$

As J_Φ is a smooth function on a neighborhood of the closure of $\tilde{\Omega}$, it is a Lipschitz function, and one finds some $a > 0$ with

$$|J_\Phi(u) - J_\Phi(u')| \le a|u - u'| \text{ for all } u, u' \in \tilde{\Omega}.$$

Pick any $u_{n,k} \in \tilde{\Omega}_{n,k}$ and denote $J_{n,k} := J_\Phi(u_{n,k})$, then for any $u \in \tilde{\Omega}_{n,k}$ there holds

$$|J_\Phi(u) - J_{n,k}| \equiv |J_\Phi(u) - J_\Phi(u_{n,k})| \le a|u - u_{n,k}| \le a \operatorname{diam} \tilde{\Omega}_{n,k} \overset{(49)}{\le} \tilde{a} p^{-\frac{n}{d}}, \quad \tilde{a} := a\tilde{c}_1.$$

We have

$$\left||\Omega_{n,k}| - J_{n,k}|\tilde{\Omega}_{n,k}|\right| = \left|\int_{\tilde{\Omega}_{n,k}} (J_\Phi(y) - J_{n,k})\, \mathrm{d}y\right| \le \int_{\tilde{\Omega}_{n,k}} |J_\Phi(y) - J_{n,k}|\, \mathrm{d}y \le \tilde{a} p^{-\frac{n}{d}} |\tilde{\Omega}_{n,k}|,$$

and we can find $a_{n,k} \in [-\tilde{a}, \tilde{a}]$ such that $|\Omega_{n,k}| = \left(J_{n,k} + a_{n,k} p^{-\frac{n}{d}}\right) |\tilde{\Omega}_{n,k}|$. Then for any $u \in \tilde{\Omega}_{n,k}$, we have

$$\left|\frac{J_\Phi(u)}{|\Omega_{n,k}|} - \frac{1}{|\tilde{\Omega}_{n,k}|}\right| = \left|\frac{J_\Phi(u)|\tilde{\Omega}_{n,k}| - |\Omega_{n,k}|}{|\Omega_{n,k}||\tilde{\Omega}_{n,k}|}\right|$$

$$\equiv \frac{\left|\left(J_\Phi(u) - J_{n,k} - a_{n,k} p^{-\frac{n}{d}}\right)\right| |\tilde{\Omega}_{n,k}|}{|\Omega_{n,k}||\tilde{\Omega}_{n,k}|} \le \frac{(\tilde{a} + |a_{n,k}|)p^{-\frac{n}{d}}}{|\Omega_{n,k}|} \overset{(51)}{\le} \frac{b p^{-\frac{n}{d}}}{|\tilde{\Omega}_{n,k}|} \text{ with } b := \frac{2\tilde{a}}{b_1}.$$

It follows from (52) that

$$\left|(\Theta P_n f - \tilde{P}_n \Theta f)(y)\right| \le \sum_{k=0}^{p^n-1} \frac{b p^{-\frac{n}{d}}}{|\tilde{\Omega}_{n,k}|} \int_{\tilde{\Omega}_{n,k}} |\tilde{f}(u)|\, \mathrm{d}u\, \mathbb{1}_{\tilde{\Omega}_{n,k}}(y) \equiv b p^{-\frac{n}{d}} (\tilde{P}_n \Theta|f|)(y).$$

It follows that

$$\|\Theta P_n f - \tilde{P}_n \Theta f\|_{L^2(\tilde{\Omega})}^2 \le b^2 p^{-\frac{2n}{d}} \|\tilde{P}_n \Theta|f|\|_{L^2(\tilde{\Omega})}^2 \le b^2 p^{-\frac{2n}{d}} \|\Theta|f|\|_{L^2(\tilde{\Omega})}^2 \overset{(51)}{\le} b^2 b_1^{-1} p^{-\frac{2n}{d}} \|f\|_{L^2(\Omega)}^2. \tag{53}$$

Now let $f \in A^r(\Omega)$, then

$$\|\tilde{f} - \tilde{P}_n \tilde{f}\|_{L^2(\tilde{\Omega})}^2 \equiv \|\Theta f - \tilde{P}_n \Theta f\|_{L^2(\tilde{\Omega})}^2 = \left\|(\Theta f - \Theta P_n f) + (\Theta P_n f - \tilde{P}_n \Theta f)\right\|_{L^2(\tilde{\Omega})}^2$$

$$\le 2\|\Theta(f - P_n f)\|_{L^2(\tilde{\Omega})}^2 + 2\|\Theta P_n f - \tilde{P}_n \Theta f\|_{L^2(\tilde{\Omega})}^2$$

use (51) and (53): $\le 2b_1^{-1}\|f - P_n f\|_{L^2(\Omega)} + 2b^2 b_1^{-1} p^{-\frac{2n}{d}} \|f\|_{L^2(\Omega)},$

$$\|\tilde{P}_0 \tilde{f}\|_{L^2(\tilde{\Omega})}^2 \equiv \|\tilde{P}_0 \Theta f\|_{L^2(\tilde{\Omega})}^2 = \|\Theta P_0 f - (\Theta P_0 f - \tilde{P}_0 \Theta f)\|_{L^2(\tilde{\Omega})}^2 \le 2\|\Theta P_0 f\|_{L^2(\tilde{\Omega})}^2 + 2\|\Theta P_0 f - \tilde{P}_0 \Theta f\|_{L^2(\tilde{\Omega})}^2;$$

use (51) and (53): $\le 2b_1^{-1}\|P_0 f\|_{L^2(\Omega)} + 2b^2 b_1^{-1}\|f\|_{L^2(\Omega)}.$

Therefore,

$$\|\tilde{f}\|^2_{A^r(\tilde{\Omega})} = \|\tilde{P}_0\tilde{f}\|^2_{L^2(\tilde{\Omega})} + \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|\tilde{f} - \tilde{P}_n\tilde{f}\|^2_{L^2(\tilde{\Omega})}$$

$$\leq 2b_1^{-1} \left(\underbrace{\|P_0\tilde{f}\|^2_{L^2(\Omega)} + \sum_{n=0}^{\infty} p^{2\frac{nr}{d}} \|f - \tilde{P}_nf\|^2_{L^2(\Omega)}}_{\equiv \|f\|^2_{A^r(\Omega)}} \right) + 2b^2b_1^{-1} \left(1 + \underbrace{\sum_{n=0}^{\infty} p^{\frac{2n}{d}(r-1)}}_{=:S<\infty} \right) \underbrace{\|f\|^2_{L^2(\Omega)}}_{\leq \|f\|^2_{A^r(\Omega)}}$$

$$\leq \left(2b_1^{-1} + 2b^2b_1^{-1}(1+S) \right) \|f\|^2_{A^r(\Omega)},$$

which shows that $\tilde{f} \in A^r(\tilde{\Omega})$ and that Θ defines a bounded operator $A^r(\Omega) \to A^r(\tilde{\Omega})$, and remark that it is injective by construction.

On the other hand, let $\tilde{f} \in A^r(\tilde{\Omega})$ and $f \in L^2(\Omega)$ with $\tilde{f} = \Theta f$. Then

$$\|f - P_nf\|^2_{L^2(\Omega)} \overset{(51)}{\leq} b_2\|\Theta f - \Theta P_nf\|^2_{L^2(\tilde{\Omega})} = b_2\|(\Theta f - \tilde{P}_n\Theta f) + (\tilde{P}_n\Theta f - \Theta P_nf)\|^2_{L^2(\tilde{\Omega})}$$

$$\text{use } \Theta f = \tilde{f}: \leq 2b_2\|\tilde{f} - \tilde{P}_n\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2\|\tilde{P}_n\Theta f - \Theta P_nf\|^2_{L^2(\tilde{\Omega})}$$

$$\text{use (53)}: \leq 2b_2\|\tilde{f} - \tilde{P}_n\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2b^2b_1^{-1}p^{-\frac{2n}{d}}\|f\|^2_{L^2(\Omega)}$$

$$\text{use (51)}: \leq 2b_2\|\tilde{f} - \tilde{P}_n\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2^2b^2b_1^{-1}p^{-\frac{2n}{d}}\|\tilde{f}\|^2_{L^2(\tilde{\Omega})},$$

$$\|P_0f\|^2_{L^2(\Omega)} \overset{(51)}{\leq} b_2\|\Theta P_0f\|^2_{L^2(\tilde{\Omega})} = b_2\|\tilde{P}_0\Theta f - (\tilde{P}_n\Theta f - \Theta P_0f)\|^2_{L^2(\tilde{\Omega})}$$

$$\text{use } \Theta f = \tilde{f}: \leq 2b_2\|\tilde{P}_0\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2\|\tilde{P}_n\Theta f - \Theta P_nf\|^2_{L^2(\tilde{\Omega})}$$

$$\text{use (53)}: \leq 2b_2\|\tilde{P}_0\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2b^2b_1^{-1}\|f\|^2_{L^2(\Omega)}$$

$$\text{use (51)}: \leq 2b_2\|\tilde{P}_0\tilde{f}\|^2_{L^2(\tilde{\Omega})} + 2b_2^2b^2b_1^{-1}\|\tilde{f}\|^2_{L^2(\Omega)}.$$

Therefore,

$$\|f\|^2_{A^r(\Omega)} = \|P_0f\|^2_{L^2(\tilde{\Omega})} + \sum_{n=0}^{\infty} p^{2\frac{nr}{d}}\|f - P_nf\|^2_{L^2(\tilde{\Omega})}$$

$$\leq 2b_2 \left(\underbrace{\|P_0\tilde{f}\|^2_{L^2(\Omega)} + \sum_{n=0}^{\infty} p^{2\frac{nr}{d}}\|\tilde{f} - \tilde{P}_n\tilde{f}\|^2_{L^2(\Omega)}}_{\equiv \|\tilde{f}\|^2_{A^r(\tilde{\Omega})}} \right) + 2b_2^2b^2b_1^{-1} \left(1 + \underbrace{\sum_{n=0}^{\infty} p^{\frac{2n}{d}(r-1)}}_{=:S<\infty} \right) \underbrace{\|\tilde{f}\|^2_{L^2(\tilde{\Omega})}}_{\leq \|\tilde{f}\|^2_{A^r(\tilde{\Omega})}}$$

$$\leq \left(2b_2 + b_2^2b^2b_1^{-1}(1+S) \right) \|\tilde{f}\|^2_{A^r(\tilde{\Omega})},$$

which shows that $f \in A^r(\Omega)$, and, consequently, that $\Theta^{-1} : A^r(\tilde{\Omega}) \to A^r(\Omega)$ is everywhere defined and bounded. □

Now we can transfer the relations between A^r and H^s known for the Euclidian case to the case of manifolds.

Theorem 3.13. *Assume (44). Then for all $r \geq 0$ and $0 < s < 1$, there holds*

$$A^r(\Omega) \hookrightarrow H^s(\Omega) \text{ if } s < \frac{1}{2} \text{ and } s \leq r, \tag{54}$$

$$H^s(\Omega) \hookrightarrow A^r(\Omega) \text{ if } 0 \leq r \leq s < 1. \tag{55}$$

In particular,

$$A^r(\Omega) = H^r(\Omega) \text{ for } 0 \leq r < \frac{1}{2}.$$

Proof. As $A^r(\Omega) \hookrightarrow A^{r'}(\Omega)$ for $r \geq r'$, it is sufficient to prove (54) under the additional assumption $r < 1$. We first use the map

$$A^r(\Omega) \ni f \mapsto (f_{N,K})_{K \in \{0,\dots,p^N-1\}} \in \bigoplus_{K=0}^{p^N-1} A^r(\Omega_{N,K}), \quad f_{N,K} := f|_{\Omega_{N,K}},$$

which is an isomorphism by Lemma 3.11. For each K, the map

$$A^r(\Omega_{N,K}) \ni f_{N,K} \mapsto \tilde{f}_{N,K} := f_{N,K} \circ \Phi_{N,K} \in A^r(\tilde{\Omega}_{N,K})$$

is also an isomorphism by Lemma 3.12, and $A^r(\tilde{\Omega}_{N,K}) \ni \tilde{f}_{N,K} \to \tilde{f}_{N,K} \in H^s(\tilde{\Omega}_{N,K})$ is an embedding by Theorem 3.9. In addition,

$$H^s(\tilde{\Omega}_{N,K}) \ni \tilde{f}_{N,K} \mapsto \tilde{f}_{N,K} \circ \Phi_{N,K}^{-1} \equiv f_{N,K} \in H^s(\Omega_{N,K})$$

is an isomorphism due to the construction of Sobolev spaces. Therefore, we have shown that

$$A^r(\Omega) \ni f \mapsto (f_{N,K})_{K \in \{0,\dots,p^N-1\}} \in \bigoplus_{K=0}^{p^N-1} H^s(\Omega_{N,K})$$

is an embedding. We now recall that due to $s < \frac{1}{2}$, the subspaces $C_c^\infty(\tilde{\Omega}_{N,K})$ are dense in $H^s(\tilde{\Omega}_{N,K})$, which in turn means that $C_c^\infty(\Omega_{N,K})$ are dense in $H^s(\Omega_{N,K})$, therefore, the operator $J_{N,K}$ of extension by zero from $\Omega_{N,K}$ to Ω extends by density from $C_c^\infty(\Omega_{N,K})$ to an embedding $J_{N,K} : H^s(\Omega_{N,K}) \to H^s(\Omega)$. Then

$$J : \bigoplus_{K=0}^{p^N-1} H^s(\Omega_{N,K}) \ni (\varphi_{N,K})_{K \in \{0,\dots,p^N-1\}} \mapsto \sum_{k=0}^{p^N-1} J_{N,K} \varphi_{N,K} \in H^s(\Omega)$$

is an embedding, which finishes the proof of (54).

For (55), we consider the following maps:

$$H^s(\Omega) \ni f \mapsto (f_{N,K})_{K \in \{0,\dots,p^N-1\}} \in \bigoplus_{K=0}^{p^N-1} H^s(\Omega_{N,K}) \text{ with } f_{N,K} := f|_{\Omega_{N,K}}, \tag{a}$$

$$H^s(\Omega_{N,K}) \ni f_{N,K} \mapsto \tilde{f}_{N,K} := f_{N,K} \circ \Phi_{N,K} \in H^s(\tilde{\Omega}_{N,K}), \tag{b}$$

$$H^s(\tilde{\Omega}_{N,K}) \ni \tilde{f}_{N,K} \mapsto \tilde{f}_{N,K} \in A^r(\tilde{\Omega}_{N,K}), \tag{c}$$

$$A^r(\tilde{\Omega}_{N,K}) \ni \tilde{f}_{N,K} \mapsto \tilde{f}_{N,K} \circ \Phi_{N,K}^{-1} \equiv f_{N,K} \in A^r(\Omega_{N,K}), \tag{d}$$

$$\bigoplus_{K=0}^{p^N-1} A^r(\Omega_{N,K}) \ni (f_{N,K})_{K \in \{0,\dots,p^N-1\}} \mapsto f \in A^r(\Omega). \tag{e}$$

The map (a) is an embedding due to the definition of Sobolev spaces (in fact, even as an isomorphism due to the first half of the proof), (b) is an isomorphism due to the definition of Sobolev spaces, (c) is an embedding by Theorem 3.7, (d) is an isomorphism by Lemma 3.12, and (e) is an ismorphism by Lemma 3.11. Taking the composition one arrives at the conclusion. $\qquad\square$

We discuss the existence of multiscale decompositions and the additional condition (B7) for some classes Ω in Section 5.

4 | EMBEDDED TRACES

4.1 | Abstract trace space as an approximation space

Let Ω be an open set in $\mathbb{R}^d$ (as in Subsection 3.2) or in a d-dimensional manifold (as in Subsection 3.4) admitting a p-multiscale decomposition $(\Omega_{n,k})$. We introduce an operator of identification I_Ω between the functions defined on $\mathcal{Z}$ (see Subsection 2.5) and the functions defined on Ω as follows. First, for each $z \in \mathcal{Z}$, consider the basis sequences

$$e_z := (\delta_{z,\zeta})_{\zeta \in \mathcal{Z}}.$$

Then we consider the linear map

$$I_\Omega : \operatorname{span}\{e_z : z \in \mathcal{Z}\} \to \operatorname{span}\{\mathbb{1}_{\Omega_{n,k}} : n \in \mathbb{N}_0, \ k = 0, \dots, p^n - 1\},$$

$$I_\Omega : e_z \mapsto \begin{cases} \mathbb{1}_\Omega, & z = \operatorname{rad}, \\ p^{\frac{n}{2}} \sum_{j=0}^{p-1} \theta_s^j \mathbb{1}_{\Omega_{n+1,pk+j}}, & z = (n,k,s). \end{cases} \tag{56}$$

Proposition 4.1 (Euclidean case). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and the decomposition $(\Omega_{n,k})$ be regular and <u>strongly balanced</u>. Then for any $r \geq 0$, the map I_Ω extends by continuity to an isomorphism between $\ell_r^2(\Omega)$ and $A^{rd}(\Omega)$.*

Proof.

(i) The linear span of e_ζ is dense in $\ell_r^2(\mathcal{Z})$, and $\langle e_z, e_\zeta \rangle_{\ell_r^2(\mathcal{Z})} = p^{2r\nu(z)}\delta_{z,\zeta}$ for all $z, \zeta \in \mathcal{Z}$.

(ii) Now remark that $I_\Omega e_{\operatorname{rad}} \in V_0$ and $I_\Omega e_{n,k,s} \in V_{n+1}$. At the same time (using the fact that the decomposition is strongly balanced),

$$\int_{\Omega_{n,k}} I_\Omega e_{n,k,s}(x)\,dx = p^{\frac{n}{2}} \sum_{j=0}^{p-1} \theta_s^j |\Omega_{n+1,pk+j}| = p^{\frac{n}{2}} \frac{|\Omega_{n,k}|}{p} \sum_{j=0}^{p-1} \theta_s^j = 0,$$

and for any $k_0 \neq k$, one has

$$\int_{\Omega_{n,k_0}} I_\Omega e_{n,k,s}(x)\,dx = 0$$

as $e_{n,k,s}$ vanishes identically in Ω_{n,k_0}. This shows that $I_\Omega e_{n,k,s}$ is orthogonal in $L^2(\Omega)$ to all $\mathbb{1}_{\Omega_{n,k_0}}$, $k_0 \in \{0, \dots, p-1\}$, in other words, $I_\Omega e_{n,k,s} \perp V_n$. Therefore, we have shown that $I_\Omega e_{\operatorname{rad}} \in U_0$ and $I_\Omega e_{n,k,s} \in U_{n+1}$ or, in other words,

$$I_\Omega e_z \in U_{\nu(z)+1} \text{ for all } z \in \mathcal{Z}. \tag{57}$$

As the subspaces U_j are mutually orthogonal in $L^2(\Omega)$, one has $\langle I_\Omega e_z, I_\Omega e_\zeta \rangle_{L^2(\Omega)} = 0$ for $\nu(z) \neq \nu(\zeta)$. In addition, if $z = (n, k_1, s_1)$ and $\zeta = (n, k_2, s_2)$ with $k_1 \neq k_2$, then $I_\Omega e_z$ and $I_\Omega e_\zeta$ have disjoint supports (contained in the disjoint

sets Ω_{n,k_1} and Ω_{n,k_2}), so one has again $\langle I_\Omega e_z, I_\Omega e_\zeta \rangle_{L^2(\Omega)} = 0$. Finally,

$$\langle I_\Omega e_{n,k,s}, I_\Omega e_{n,k,s'} \rangle_{L^2(\Omega)} = p^n \sum_{j=0}^{p-1} \theta_s^j \overline{\theta_{s'}^j} |\Omega_{n+1,pk+j}| = p^n \frac{|\Omega_{n,k}|}{p} \sum_{j=0}^{p-1} \left(\frac{\theta_s}{\theta_{s'}} \right)^j = p^n |\Omega_{n,k}| \delta_{s,s'}.$$

Altogether we obtain

$$\langle I_\Omega e_z, I_\Omega e_\zeta \rangle_{L^2(\Omega)} = |\Omega| \, \delta_{z,\zeta}, \quad z, \zeta \in \mathcal{Z}. \tag{58}$$

(iii) We will equip $A^{rd}(\Omega)$ with the norm

$$\|f\|_{A,rd}^2 := \sum_{n=0}^{\infty} p^{2rn} \|Q_n f\|_{L^2(\Omega)}^2,$$

see Lemma 3.5. For $N \in \mathbb{N}$, we denote $W_N := \text{span}\{e_z : \nu(z) \le N-1\} \subset \ell_r^2(\mathcal{Z})$. Let $f \in W_N$, then

$$f = \sum_{\nu(z) \le N-1} f_z e_z, \quad f_z \in \mathbb{C}, \qquad I_\Omega f = \sum_{n=0}^{N} F_n, \quad F_n := \sum_{\nu(z)=n-1} f_z I_\Omega e_z.$$

Due to (57) one has $Q_n I_\Omega f = F_n$ for all $n \le N$ and $Q_n I_\Omega f = 0$ for $n \ge N+1$, therefore,

$$\|f\|_{A,rd}^2 = \sum_{n=0}^{N} p^{2rn} \|F_n\|_{L^2(\Omega)}^2 \overset{(58)}{=} \sum_{n=0}^{N} p^{2rn} |\Omega| \sum_{\nu(z)=n-1} |f_z|^2 = p^{2r} |\Omega| \sum_{z \in \mathcal{Z}} p^{2r\nu(z)} |f_z|^2 \equiv p^{2r} |\Omega| \|(f_z)\|_{\ell_r^2(\mathcal{Z})}^2,$$

which shows that I_Ω is an isometry (up to a constant factor), in particular, it is bounded and extends an isometry between of $\ell_r^2(\mathcal{Z})$ and some closed subspace $\text{ran}\, I_\Omega \subset A^{rd}(\Omega)$.

(iv) It remains to show that $\text{ran}\, I_\Omega = A^{rd}(\Omega)$. Remark that by construction, we have $\dim W_N = p^n$. At the same time, $I_\Omega W_N \subset V_N$, so we obtain $p^N = \dim W_N = \dim I_\Omega(W_N) \le \dim V_N = p^N$, which shows that $I_\Omega(W_N) = V_N$ for any N. As N can be arbitrarily large, $\text{ran}\, I_\Omega$ contains any finite linear combination of $\mathbb{1}_{\Omega_{n,k}}$. As these linear combinations span a dense subset of $A^{rd}(\Omega)$ and $\text{ran}\, I_\Omega$ is closed, we have $\text{ran}\, I_\Omega = A^{rd}(\Omega)$. $\qquad\square$

Proposition 4.2 (Manifold case). *Let Ω be an open set with compact closure in a manifold of bounded geometry and the decomposition $(\Omega_{n,k})$ be regular and* *strongly balanced*. *Let $0 \le s < 1$ with $s \le rd$, then the map I_Ω extends by continuity to an embedding $\ell_r^2(\Omega) \hookrightarrow A^s(\Omega)$. For $s = rd < 1$, this embedding is an isomorphism.*

Proof. Let N, $\Phi_{N,K}$ and $\tilde{\Omega}_{n,k}$ be as in (B4)–(B5).

(i) If $N = 0$, then the map $\tilde{I}_\Omega : \xi \mapsto (I_\Omega \xi) \circ \Phi_{0,0}$ is covered by Proposition 4.1 and defines an isomorphism between $\ell_r^2(\mathcal{Z})$ and $A^{rd}(\tilde{\Omega})$, for $\tilde{\Omega} := \tilde{\Omega}_{0,0} \equiv \Phi_{0,0}^{-1}(\Omega)$.

If $rd < 1$, then it follows by Lemma 3.12 that I_Ω is an isomorphism between $\ell_r^2(\mathcal{Z})$ and $A^{rd}(\Omega)$, and $A^{rd}(\Omega) \hookrightarrow A^s(\Omega)$ for $0 \le s \le rd$.

If $rd \ge 1$, using $A^{rd}(\tilde{\Omega}) \hookrightarrow A^s(\tilde{\Omega})$, we obtain $\tilde{I}_\Omega : \ell_r^2(\mathcal{Z}) \hookrightarrow A^s(\tilde{\Omega})$, and Lemma 3.12 gives $I_\Omega : \ell_r^2(\mathcal{Z}) \hookrightarrow A^s(\Omega)$.

(ii) Now assume that $N \ge 1$ and consider

$$W_N := \text{span}\{e_z : \nu(z) \le N-1\}, \qquad O_N := \text{span}\{\mathbb{1}_{\Omega_{N,K}} : K = 0, \dots, p^N - 1\},$$

then by construction, one has $I_\Omega(W_N) \subset O_N$. We will equip $A^{rd}(\Omega)$ with the norm

$$\|f\|_{A,rd}^2 := \sum_{n=0}^{\infty} p^{2rn} \|Q_n f\|_{L^2(\Omega)}^2,$$

see Proposition 36. The computations (i)–(iii) in the proof of Proposition 4.1 show that

$$\|I_\Omega f\|^2_{A,rd} = p^{2r} |\Omega| \|(f_z)\|^2_{\ell^2_r(\mathcal{Z})} \text{ for any } f \in W_N.$$

In particular, $I_\Omega : W_N \to O_N$ is injective. As both W_N and O_N have the same dimension p^N, the map $I_\Omega : W_N \to O_N$ is a linear isomorphism, and one can find a basis $b^0, \dots, b^{p^N-1}$ in W_N such that

$$I_\Omega b^K = p^{\frac{N}{2}} \mathbb{1}_{\Omega_{N,K}} \text{ for each } K \in \{0, \dots, p^N - 1\}.$$

(iii) For $\xi \in \ell^2_r(\mathcal{Z})$, define $\xi^N \in W_N$ by

$$\xi^N_z := \begin{cases} \xi_z, & \nu(z) \leq N - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\gamma_K(\xi)$, $K \in \{0, \dots, p^N - 1\}$, be the coordinates of ξ^N in the basis (b^K). Now we consider the map

$$\Psi : \ell^2_r(\mathcal{Z}) \ni (\xi_z) \mapsto \eta \equiv (\eta^0, \dots, \eta^{p^N-1}) \in \bigoplus_{K=0}^{p^N-1} \ell^2_r(\mathcal{Z}),$$

$$\eta^K_z = \begin{cases} \gamma_K(\xi), & z = \text{rad}, \\ \xi_{N+n,p^n K+k,s}, & z = (n,k,s), \end{cases} \quad K \in \{0, \dots, p^N - 1\}.$$

By construction Ψ is an isomorphism, with Ψ^{-1} given by

$$\left(\Psi^{-1}(\eta^0, \dots, \eta^{p^N-1}) \right)_z = \begin{cases} \sum_{K=0}^{p^N-1} \eta^K_{\text{rad}} b^K_z, & \nu(z) \leq N - 1, \\ \eta^K_{n-N,k-p^{n-N}K,s}, & z = (n,k,s), \ n \geq N, \\ & p^{n-N}K \leq k < p^{n-N}(K+1). \end{cases}$$

One computes

$$\left(\Psi^{-1}(\underbrace{0, \dots, 0}_{K-1 \text{ times}}, e_\lambda, 0, \dots, 0) \right)_z = \begin{cases} b^K_z, & \lambda = \text{rad}, \ \nu(z) \leq N - 1, \\ 1, & \lambda = (n,k,s), \ z = (n+N, p^n K+k, s), \\ 0, & \text{otherwise,} \end{cases}$$

or, equivalently,

$$\Psi^{-1}(\underbrace{0, \dots, 0}_{K-1 \text{ times}}, e_\lambda, 0, \dots, 0) = \begin{cases} b^K, & \lambda = \text{rad}, \\ e_{(n+N, p^n K+k, s)}, & \lambda = (n,k,s). \end{cases}$$

Due to the definition of I_Ω, one has then

$$I_\Omega \Psi^{-1}(\underbrace{0, \dots, 0}_{K-1 \text{ times}}, e_\lambda, 0, \dots, 0) = \begin{cases} p^{\frac{N}{2}} \mathbb{1}_{\Omega_{N,K}}, & \lambda = \text{rad}, \\ p^{\frac{N+n}{2}} \sum_{j=0}^{p-1} \theta^j_s \mathbb{1}_{\Omega_{N+n+1, p^{n+1}K+pk+j}} \end{cases}$$

$$= p^{\frac{N}{2}} J_{N,K} I_{\Omega_{N,K}} e_\lambda,$$

where

(1) $J_{N,K}$ is the operator of extension by zero from $\Omega_{N,K}$ to Ω,
(2) $I_{\Omega_{N,K}} : \ell_r^2(\mathcal{Z}) \to A^{rd}(\Omega_{N,K})$ is the identification operator for the decomposition $(\Omega_{N+k,p^nK+k})_{n,k}$, which is already covered by (i).

(iv) The above computations show that $I_\Omega \Psi^{-1}$ acts as

$$I_\Omega \Psi^{-1}(\eta^0, \ldots, \eta^{p^N-1}) = p^{\frac{N}{2}} \sum_{K=0}^{p^K-1} J_{N,K} I_{\Omega_{N,K}} \eta^K.$$

By (i), each $I_{\Omega_{N,K}}$ is an isomorphism (for $s = rd < 1$) or an embedding (for all other cases), and it follows by Lemma 3.11 that

$$I_\Omega \Psi^{-1} : \bigoplus_{K=0}^{p^N-1} \ell_r^2(\mathcal{Z}) \to A^{rd}(\Omega)$$

is an isomorphism (for $s = rd < 1$) or an embedding (for all other cases), and then $I_\Omega \equiv (I_\Omega \Psi^{-1})\Psi$ preserves the same properties. $\qquad\square$

4.2 | Embedded trace operator

For all assertions in this subsection, let Ω be an open set with compact closure in a d-dimensional manifold of bounded geometry S admitting a regular strongly balanced p-multiscale decomposition $(\Omega_{n,k})$ as described in Subsections 3.2 and 3.4.

Recall (Theorem 2.8) that we have constructed an abstract trace operator

$$\tau : H^1(\mathbb{T}) \to \ell_\sigma^2(\mathcal{Z}), \qquad \sigma := \frac{1}{2\log p} \log \frac{\alpha p}{\ell} \equiv \frac{1}{2}\left(1 - \frac{\log \ell - \log \alpha}{\log p}\right) > 0, \tag{59}$$

which is bounded and surjective with $\ker \tau = H_0^1(\mathbb{T})$. We recall that $p \in \mathbb{N}$ with $p \geq 2$ and that the parameters α and ℓ satisfy

$$0 < \ell < 1, \quad \ell < \alpha p < \frac{1}{\ell}, \tag{60}$$

see Lemma 2.7. We define the identification/embedding operator

$$I_\Omega : \ell_\sigma^2(\mathcal{Z}) \to A^s(\Omega)$$

as in Propositions 4.1 and 4.2. This gives rise to the *embedded trace operator*

$$\gamma_\Omega := I_\Omega \tau : H^1(\mathbb{T}) \to A^s(\Omega),$$

with the following options for s:

(1) If Ω is a d-dimensional Euclidean open set (as in Subsection 3.2), then γ_Ω is a bounded linear operator for any $0 \leq s \leq \sigma d$, surjective for $s = \sigma d$.
(2) If Ω is an open set in d-dimensional manifold (as in Subsection 3.4), then γ_Ω is a bounded linear operator for any $0 \leq s < 1$ such that $s \leq \sigma d$, surjective for $s = \sigma d < 1$.

In all these cases, one has by construction

$$\ker \gamma_\Omega = \ker \tau = H_0^1(\mathbb{T}).$$

In addition, using the identification between the approximation and Sobolev spaces (Theorem 3.13) we obtain the following:

(3) If Ω is an open set in d-dimensional manifold, then

$$\gamma_\Omega : H^1(\mathbb{T}) \hookrightarrow H^s(\Omega) \text{ for and } 0 \le s < \tfrac{1}{2} \text{ with } s \le \sigma d.$$

In particular,

$$\gamma_\Omega\big(H^1(\mathbb{T})\big) = H^{\sigma d}(\Omega) \text{ if } \sigma d < \tfrac{1}{2}. \tag{61}$$

Remark 4.3. It is useful to check that the condition $\sigma d < \tfrac{1}{2}$ in (61) can really be satisfied under the restrictions (59) and (60). In view of (59), the condition can be rewritten as

$$\frac{d}{\log p} \log \frac{\alpha p}{\ell} < 1 \quad \text{i.e.} \quad \frac{\alpha p}{\ell} < p^{\frac{1}{d}},$$

so together with (60), we arrive at

$$0 < \ell < 1, \quad 1 < \frac{\alpha p}{\ell} < \min\Big\{\frac{1}{\ell^2}, p^{\frac{1}{d}}\Big\}.$$

Therefore, if one fixes arbitrary $\ell \in (0,1)$ and $p \in \mathbb{N}$ with $p \ge 2$, the required condition is satisfied for

$$\frac{\ell}{p} < \alpha < \frac{\ell}{p} \min\Big\{\frac{1}{\ell^2}, p^{\frac{1}{d}}\Big\},$$

that is, for a nontrivial range of α.

Finally, we give a more illustrative description of the embedded trace operator, which uses more classical terms:

Theorem 4.4 (Embedded trace using limit values). *Let $0 \le s < \tfrac{1}{2}$ with $s \le \sigma d$, then for any $f \in H^1(\mathbb{T})$, there holds*

$$\gamma_\Omega f = \lim_{N \to \infty} \sum_{K=0}^{p^N-1} f(X_{N,K}) \mathbb{1}_{\Omega_{N,K}},$$

where the limit is taken in $H^s(\Omega)$.

Proof.

(i) Let $f \in H^1(\mathbb{T})$. For $N \in \mathbb{N}$, consider $f_N : \mathbb{T} \to \mathbb{C}$ defined by

$$f_N(x) := \begin{cases} f(x), & x \in \mathbb{T}^N, \\ f(X_{N,k}), & x \in \mathbb{T}_{N,K}, \ K \in \{0, \dots, p^N - 1\}. \end{cases}$$

In Lemma 2.11, we have shown that $f_N \in H^1(\mathbb{T})$ with $f_N \xrightarrow{N \to \infty} f$ in $H^1(\mathbb{T})$. Due to the boundedness of γ_Ω we have $\gamma_\Omega f_N \xrightarrow{N \to \infty} \gamma_\Omega f$ in $H^s(\Omega)$. Therefore, it is sufficient to show that for any $N \in \mathbb{N}$, one has

$$\gamma_\Omega f_N = \sum_{K=0}^{p^N-1} f(X_{N,K}) \mathbb{1}_{\Omega_{N,K}}. \tag{62}$$

(ii) Let $N \in \mathbb{N}$ be fixed. Pick a function $F \in H^1((0, L), q(t)\,dt)$ such that $F(t) = 0$ for $t \leq t_N$ and $F(t) = 1$ for $t \geq t_{N+1}$. Define $\varphi := U_{\mathrm{rad}} F \in H^1(\mathbb{T})$, that is, $\varphi : \mathbb{T} \ni x \mapsto F(|x|)$, then $\varphi \mathbb{1}_{\mathbb{T}_{N,K}} \in H^1(\mathbb{T})$ for any $K \in \{0, \dots, p^N - 1\}$. We will show that

$$\gamma_\Omega(\varphi \mathbb{1}_{\mathbb{T}_{N,K}}) = \mathbb{1}_{\Omega_{N,K}} \text{ for any } K \in \{0, \dots, p^N - 1\}. \tag{63}$$

In fact, if (63) is proved, then (62) follows directly: One has

$$f_N = \sum_{K=0}^{p^N-1} f(X_{N,K}) \varphi \, \mathbb{1}_{\mathbb{T}_{N,K}} \text{ in } \mathbb{T} \setminus \mathbb{T}^{N+1},$$

which implies

$$\gamma_\Omega f_N = \sum_{K=0}^{p^N-1} f(X_{N,K}) \gamma_\Omega \left(\varphi \mathbb{1}_{\mathbb{T}_{N,K}} \right) \overset{(63)}{=} \sum_{K=0}^{p^N-1} f(X_{N,K}) \mathbb{1}_{\Omega_{N,K}}.$$

(iii) It remains to prove (63). Consider

$$S_N := \mathrm{span}\{\varphi \mathbb{1}_{\mathbb{T}_{N,K}} : K = 0, \dots, p^N - 1\} \subset H^1(\mathbb{T}).$$

The functions $\varphi \mathbb{1}_{\mathbb{T}_{N,K}}$ form a basis of S_N, so $\dim S_N = p^N$. Now we remark that for any $z \in \mathcal{Z}$, one has the inclusion $\dim S \cap H_z^1(\mathbb{T}) \subset \mathbb{C} U_z F$, and

$$\dim \left(S \cap H_{\mathrm{rad}}^1(\mathbb{T}) \right) = 1, \quad \dim \left(S \cap H_{n,k,s}^1(\mathbb{T}) \right) = \begin{cases} 1, & n \leq N - 1, \\ 0, & n \geq N. \end{cases}$$

Due to the orthogonal decomposition $H^1(\mathbb{T}) = \bigoplus_{z \in \mathcal{Z}} H_z^1(\mathbb{T})$, we conclude that

$$S_N = \mathrm{span}\{U_z F : z \in \mathcal{Z}, \nu(z) \leq N - 1\},$$

and the functions $U_z F$ with $\nu(z) \leq N - 1$ form a basis in S_N. Recall that by Lemma 2.10 we have $\tau U_z F = p^{-\frac{\nu(z)}{2}} e_z$ for $\nu(z) \leq N - 1$, and then, using (56),

$$\gamma_\Omega U_z F = p^{-\frac{\nu(z)}{2}} I_\Omega e_z = \begin{cases} \mathbb{1}_\Omega, & z = \mathrm{rad}, \\ \sum_{j=0}^{p-1} \theta_s^j \mathbb{1}_{\Omega_{n+1,pk+j}}, & z = (n, k, s), \end{cases} \quad \nu(z) \neq N - 1.$$

Recall that $U_{\mathrm{rad}} F = \varphi$ and that for $(n, k, s) \in \mathcal{Z}$ with $n \leq N - 1$, one has

$$U_{n,k,s} F = \sum_{j=0}^{p-1} \theta_s^j \varphi \mathbb{1}_{\mathbb{T}_{n+1,pk+j}}.$$

Now let us define a linear map $R : S_N \to L^2(\Omega)$ by

$$R(\varphi \mathbb{1}_{\mathbb{T}_{N,K}}) := \mathbb{1}_{\Omega_{N,K}} \text{ for any } K \in \{0, \dots, p^N - 1\}.$$

Using the linearity, one obtains

$$R(U_{\text{rad}}F) = R(\varphi) = R\left(\sum_{K=0}^{p^N-1} \varphi \mathbb{1}_{\mathbb{T}_{N,K}}\right) = \sum_{K=0}^{p^N-1} R(\varphi \mathbb{1}_{\mathbb{T}_{N,K}}) = \sum_{K=0}^{p^N-1} \mathbb{1}_{\Omega_{N,K}} = \mathbb{1}_{\Omega},$$

and for any $(n, k, s) \in \mathcal{Z}$ with $n \leq N - 1$,

$$R(\varphi \mathbb{1}_{\mathbb{T}_{n+1,pk+j}}) = R\left(\sum_{K:\, \mathbb{T}_{N,K} \subset \mathbb{T}_{n+1,pk+j}} \varphi \mathbb{T}_{N,K}\right) = \sum_{K:\, \mathbb{T}_{N,K} \subset \mathbb{T}_{n+1,pk+j}} R(\varphi \mathbb{T}_{N,K})$$

$$= \sum_{K:\, \mathbb{T}_{N,K} \subset \mathbb{T}_{n+1,pk+j}} \mathbb{1}_{\Omega_{N,K}} = \sum_{K:\, \Omega_{N,K} \subset \Omega_{n+1,pk+j}} \mathbb{1}_{\Omega_{N,K}} = \mathbb{1}_{\Omega_{n+1,pk+j}},$$

$$R(U_{n,k,s}F) = \sum_{j=0}^{p-1} \theta_s^j R(\varphi \mathbb{1}_{\mathbb{T}_{n+1,pk+j}}) = \sum_{j=0}^{p-1} \theta_s^j \mathbb{1}_{\Omega_{n+1,pk+j}},$$

which shows that $R(U_z F) = \gamma_\Omega U_z F$ for any $z \in \mathcal{Z}$ with $\nu(z) \leq N - 1$. As $U_z F$ form a basis of S_N, it follows that $R = \gamma_\Omega|_{S_N}$. In particular, $\gamma_\Omega(\varphi \mathbb{1}_{\mathbb{T}_{N,K}}) = R(\varphi \mathbb{1}_{\mathbb{T}_{N,K}})$ for all $K \in \{0, \dots, p^N - 1\}$, which shows (63) and concludes the proof. $\square$

Remark 4.5. In Theorem 4.4, one can also take the limit in $A^s(\Omega)$ with any s such that $\gamma_\Omega : H^1(\Omega) \to A^s(\Omega)$ is bounded: The proof remains unchanged.

4.3 | Proof of Theorem 1.1

By now we have proved all assertions of Theorem 1.1 for the special case $\mathcal{T} = \mathbb{T}$. Recall that in Theorem 1.1 we require the condition (1), that is,

$$c^{-1} \leq \frac{\ell_{n,k}}{\ell^n} \leq c, \quad c^{-1} \leq \frac{w_{n,k}}{\alpha_n} \leq c, \tag{64}$$

and that $\mathbb{T}$ corresponds to $c = 1$. In order to cover the case of general $\mathcal{T}$, we employ a suitable bijection between $\mathcal{T}$ and $\mathbb{T}$. Namely, define $\varphi : \mathbb{T} \to \mathcal{T}$ by

$$\varphi(n, k, t) := \left(n, k, L_{n,k} - \ell_{n,k} + \frac{t - t_{n-1}}{\ell^n} \ell_{n,k}\right),$$

then φ maps the vertices $X_{n,k}$ on $\mathbb{T}$ to the same vertices on $\mathcal{T}$, the restrictions $\varphi|_{e_{n,k}}$ are dilations by constant factors, and both φ and φ^{-1} are continuous.

If $f : \mathcal{T} \to \mathbb{C}$, consider $g := f \circ \varphi : \mathbb{T} \to \mathbb{C}$. Remark that f is continuous if and only if g is continuous. Furthermore, if $f = (f_{n,k})$ and $g = (g_{n,k})$, then

$$\|f\|_{L^2(\mathcal{T})}^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{L_{n,k}-\ell_{n,k}}^{L_{n,k}} |f_{n,k}(s)|^2 w_{n,k}(s)\, ds,$$

$$\|g\|_{L^2(\mathbb{T})}^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \alpha^n \int_{t_{n-1}}^{t_n} |g_{n,k}(t)|^2\, dt$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \alpha^n \int_{t_{n-1}}^{t_n} \left|g_{n,k}\left(n, k, L_{n,k} - \ell_{n,k} + \frac{t - t_{n-1}}{\ell^n} \ell_{n,k}\right)\right|^2\, dt = \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \alpha^n \frac{\ell^n}{\ell_{n,k}} \int_{L_{n,k}-\ell_{n,k}}^{L_{n,k}} |g_{n,k}(s)|^2\, ds.$$

In view of (64), we have

$$c^{-2}\frac{\ell^n}{\ell_{n,k}} \leq c^{-1}\alpha^n \leq w_{n,k} \leq c\alpha^n \leq c^2\alpha^n \frac{\ell^n}{\ell_{n,k}},$$

and we infer

$$c^{-2}\|g\|^2_{L^2(\mathbb{T})} \leq \|f\|^2_{L^2(\mathcal{T})} \leq c^2\|g\|^2_{L^2(\mathbb{T})}. \tag{65}$$

In addition, $f_{n,k}$ is weakly differentiable if and only if $g_{n,k}$ is weakly differentiable, and then

$$g'_{n,k} = \frac{\ell_{n,k}}{\ell^n}(f'\circ\varphi)_{n,k}.$$

By (64), it follows that $c^{-1}|f'\circ\varphi| \leq |g'| \leq c|f'\circ\varphi|$, and then

$$c^{-4}\|f'\|^2_{L^2(\mathcal{T})} \overset{(65)}{\leq} c^{-2}\|f'\circ\varphi\|^2_{L^2(\mathbb{T})}\|g'\|^2_{L^2(\mathbb{T})} \leq c^2\|f'\circ\varphi\|^2_{L^2(\mathbb{T})} \overset{(65)}{\leq} c^4\|f'\|^2_{L^2(\mathcal{T})}.$$

It follows that the linear operator $\Theta : f \mapsto f\circ\varphi$ is an isomorphism between $L^2(\mathcal{T})$ and $L^2(\mathbb{T})$ as well as between $H^1(\mathcal{T})$ and $H^1(\mathbb{T})$. In addition, it is bijective from $H^1_c(\mathcal{T})$ and $H^1_c(\mathbb{T})$ by construction, so it is also an isomorphism between $H^1_0(\mathcal{T})$ and $H^1_0(\mathbb{T})$. This shows that $H^1(\mathcal{T}) \neq H^1_0(\mathcal{T})$ if and only if $H^1(\mathbb{T}) \neq H^1_0(\mathbb{T})$, which is equivalent to the inequalities (17).

Due to Theorem 4.4, we actually have $\gamma^{\mathcal{T}}_\Omega = \gamma_\Omega\circ\Theta$, so the properties of γ_Ω from the preceding subsection are directly transferred to $\gamma^{\mathcal{T}}_\Omega$. In particular:

(1) $\ker\gamma^{\mathcal{T}}_\Omega = \Theta^{-1}(\ker\gamma_\Omega) = \Theta^{-1}\left(H^1_0(\mathbb{T})\right) = H^1_0(\mathcal{T})$,
(2) if Ω is a d-dimensional Euclidean open set admitting a regular strongly balanced p-multiscale decomposition (Subsection 3.2), then $\gamma^{\mathcal{T}}_\Omega : H^1(\mathcal{T}) \to A^s(\Omega)$ is a bounded linear operator for any $0 \leq s \leq \sigma d$, and it is surjective for $s = \sigma d$,
(3) if Ω is an open set in d-dimensional manifold and admitting a regular strongly balanced p-multiscale decomposition (Subsection 3.4), then:
   - $\gamma_\Omega : H^1(\mathcal{T}) \to A^s(\Omega)$ is a bounded linear operator for any $0 \leq s < 1$ such that $s \leq \sigma d$, surjective if $s = \sigma d < 1$,
   - $\gamma_\Omega : H^1(\mathcal{T}) \to H^s(\Omega)$ is a bounded linear operator for any $0 \leq s < \frac{1}{2}$ such that $s \leq \sigma d$, surjective if $s = \sigma d < \frac{1}{2}$.

All assertions are proved.

5 | EXISTENCE OF REGULAR BALANCED DECOMPOSITIONS

The construction of the embedded trace in the preceding subsection requires the existence of a regular strongly balanced p-multiscale decomposition. Let us show that such decompositions really exist for a wide class of Ω. Our approach also indicates how to construct such a decomposition for a given domain or manifold. Our construction is strictly adapted to the required properties stated in Sections 3.2 and 3.4. We note that the existence of dyadic cubes with various additional properties in abstract metric spaces is discussed, for example, in [14, 26].

Example 5.1 (Hypercubes). Let us show first how to construct a regular strongly balanced p-multiscale decomposition of the d-dimensional hypercube

$$Q^{(d)} := (0,1)^d \subset \mathbb{R}^d.$$

For $d = 1$, we decompose iteratively each interval into p equal subintervals to obtain the decomposition

$$Q^{(1)}_{n,k} = (kp^{-n}, (k+1)p^{-n}), \quad n \in \mathbb{N}_0, \quad k \in \{0, \dots, p^n - 1\}. \tag{66}$$

For $d \geq 2$, we obtain a decomposition by dividing alternately each side into p equal parts. First, set $\Omega_{0,0} := \Omega$. Now assume that $Q_{n,k}^{(d)}$ are already constructed for some $n \in \mathbb{N}_0$ and all $k \in \{0, \dots, p^n - 1\}$ and that for each (n, k), one has

$$Q_{n,k}^{(d)} = Q_{n_1,k_1}^{(1)} \times \cdots \times Q_{n_d,k_d}^{(1)}$$

with suitable $n_s \in \mathbb{N}_0$ and $k_s \in \{0, \dots, p^{n_s} - 1\}$. Let $i \in \{1, \dots, d\}$ be such that $(n + 1) \equiv i \mod d$, then we obtain $Q_{n+1,pk+j}^{(d)}$ with $j \in \{0, \dots, p - 1\}$ by subdividing the ith side $Q_{n_i,k_i}^{(1)}$ of $Q_{n,k}^{(d)}$ into p equal subintervals

$$\tilde{I}_j := Q_{n_i+1,pk_i+j}^{(1)}, \quad j \in \{0, \dots, p - 1\},$$

and then by setting, for each $j \in \{0, \dots, p - 1\}$,

$$Q_{n+1,pk+j}^{(d)} = Q_{n_1,k_1}^{(1)} \times Q_{n_{i-1},k_{i-1}}^{(1)} \times \tilde{I}_j \times Q_{n_{i+1},k_{i+1}}^{(1)} \times \cdots \times Q_{n_d,k_d}^{(1)}.$$

Let us show that this decomposition is regular and strongly balanced. The assumptions (A1)–(A3) are obviously satisfied, as well as (A4*), as on each passage from $Q_{n,k}^{(d)}$ to $Q_{n+1,k}^{(d)}$ one divides the volumes exactly by p. In order to check (A5)–(A6), we remark that for $n \geq d + 1$, each $Q_{n,k}^{(d)}$ has the form $I_1 \times \cdots \times I_d$, where I_j are intervals with

$$p^{-K} \leq |I_j| \leq p^{1-K} \text{ for } (K - 1)d + 1 \leq n \leq Kd, \quad K \in \mathbb{N}_0.$$

We infer

$$p^{-\frac{n+d}{d}} \leq |I_j| \leq p^{-\frac{n-d}{d}} \tag{67}$$

and it follows that

$$\operatorname{diam} Q_{n,k}^{(d)} \equiv \sqrt{|I_1|^2 + \cdots + |I_d|^2} \leq \sqrt{d}\, p^{-\frac{n-d}{d}} \equiv p\sqrt{d}\, p^{-\frac{n}{d}},$$

that is, (A5) is satisfied. Now let $h = (h_1, \dots, h_d) \in \mathbb{R}^d$, then

$$Q_{n,k}^{(d)} \setminus (Q_{n,k}^{(d)} + h) = (I_1 \times I_2 \times \cdots \times I_d) \setminus ((I_1 + h_1) \times (I_2 + h_2) \times \cdots \times (I_d + h_d))$$

$$\subset (I_1 \setminus (I_1 + h_1)) \times I_2 \times \cdots \times I_d$$

$$\cup\, I_1 \times (I_2 \setminus (I_2 + h_2)) \times \cdots \times I_d$$

$$\dots \cup\, I_1 \times I_2 \times \cdots \times (I_d \setminus (I_d + h_d)).$$

We have $|I_k \setminus (I_k + h_k)| \leq |h_k| \leq |h|$, which gives the volume estimate

$$|Q_{n,k}^{(d)} \setminus (Q_{n,k}^{(d)} + h)| \leq \sum_{k=1}^{d} |h_k| \prod_{j \neq k} |I_j| \overset{(67)}{\leq} \sum_{k=1}^{d} |h_k| \left(p^{-\frac{n-d}{d}} \right)^{d-1} \leq dp^{d-1}|h|p^{-n\frac{d-1}{d}}$$

and shows (A6).

Example 5.2 (Piecewise smooth star-shaped open sets). One says that a bounded open set $\Omega \subset \mathbb{R}^d$ belongs to the class (H) if:

(1) Ω is star-shaped with respect to a point $x_0 \in \Omega$.
(2) There exist $\varepsilon > 0$ with $B_\varepsilon(x_0) \subset \Omega$ and a finite partition of $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ such that for each $j = 1, \dots, n$:

- each Ω_j is a cone with vertex at x_0,
- $\partial \Omega_j \cap \partial \Omega$ is a C^1 surface,
- the set $B_\varepsilon(x_0) \cap \Omega_j$ is convex.

(3) There exists $\delta > 0$ such that $\nu(x) \cdot (x - x_0) \geq \delta$ for all $x \in \partial \Omega$, where $\nu(x)$ denotes the outward unit normal to $\partial \Omega$ at x (defined almost everywhere on $\partial \Omega$).

Remark that the class (H) contains all convex polyhedrons and all convex open sets with smooth boundaries. It is shown in [20, Theorem 5.4] that for arbitrary Ω, Ω' in (H) with $|\Omega| = |\Omega'|$, there exists a bi-Lipschitz bijection $\Phi : \Omega' \to \Omega$ with $|\det D\Phi| = 1$ (i.e., Φ preserves the volumes). Note that for many special classes of Ω, Ω' like cubes, balls, cylinders, simplices such a map Φ can be given by explicit formulas, see [21, 24] and references therein.

If Ω' admits a regular strongly balanced p-multiscale decomposition $(\Omega'_{n,k})$, then the sets $\Omega_{n,k} := \Phi(\Omega'_{n,k})$ form a regular strongly balanced p-multiscale decomposition of Ω, as the conditions (A5)–(A6) remain true under bi-Lipschitz transformations. In particular, for each Ω in (H), one can take a hypercube Q with $|Q| = |\Omega|$ and translate a decomposition of Q (Example 5.1) into a decomposition of Ω.

This discussion shows that any open set Ω of the class (H) admits a regular strongly balanced p-multiscale decomposition.

Example 5.3 (Composed open sets). Let Ω be an open set with compact closure in a d-dimensional manifold of bounded geometry S. Assume that Ω can be decomposed (up to zero measure sets) into disjoint open pieces W_j, $j = 1, \ldots, p^N$ such that

(1) all W_j have the same volume,
(2) there exist local charts $\Phi_j : \mathbb{R}^d \supseteq \tilde{O}_j \to O_j \subset S$ with $\overline{W_j} \subset O_j$,
(3) the sets $\tilde{W}_j := \Phi_j^{-1}(W_j) \subset \mathbb{R}^d$ are with Lipschitz boundaries and admit regular strongly balanced p-multiscale decompositions,

then the decompositions of $\tilde{W}_j$ are first transferred to W_j with the help of Φ_j and then suitably renumerated to produce a regular strongly balanced p-multiscale decomposition of the whole Ω.

Example 5.4 (Compact manifolds). By combining the preceding observations, one can show that each compact manifold admits a regular strongly balanced p-multiscale decomposition. The idea comes from Benoît Kloeckner's comments in the MathOverflow discussion [43].

Let (Ω, g) be a compact d-dimensional Riemannian manifold. It is known [65] that Ω admits a triangulation: There exist disjoint open $W_1, \ldots, W_N \subset \Omega$ with

$$\left| \Omega \setminus (W_1 \cup \cdots \cup W_N) \right| = 0$$

and local charts $\Phi_j : \mathbb{R}^d \supseteq \tilde{O}_j \mapsto O_j \subset \Omega$ with $\overline{W_j} \subset O_j$ such that the sets $\tilde{\Omega}_j := \Phi_j^{-1}(W_j)$ are d-dimensional simplices. Without loss of generality we assume that $N = p^n$ (otherwise one cuts some of the simplices $\tilde{\Omega}_j$ into smaller subsimplices to obtain a required number). Then one can find a smooth function $f : \Omega \to (0, \infty)$ such that

$$\int_{W_j} f \, \mathrm{dvol}_g = \frac{|\Omega|}{N}.$$

By [50] there exists a diffeomorphism $\phi : \Omega \to \Omega$ with $\phi_*(f \, \mathrm{dvol}_g) = \mathrm{dvol}_g$. The open sets $\Omega_j := \phi(W_j)$ satisfy

$$|\Omega_j| = \int_{\Omega_j} 1 \, \mathrm{dvol}_g = \int_{\phi(W_j)} 1 \, \mathrm{dvol}_g = \int_{W_j} f \, \mathrm{dvol}_g = \frac{|\Omega|}{N},$$

that is, they have the same volume and exhaust Ω up to a zero-measure subset. In addition, each Ω_j is covered by the local chart $\Psi_j := \phi \circ \Phi_j$ with $\Psi_j^{-1}(\Omega_j) = \tilde{\Omega}_j$. As discussed in Example 5.2, each simplex $\tilde{\Omega}_j$ admits a regular strongly balanced p-multiscale decomposition. This decomposition is transferred to Ω_j with the help of Ψ_j, and the resulting decompositions of Ω_j are then combined into a regular strongly balanced p-multiscale decomposition of Ω.

ACKNOWLEDGMENTS

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Pure Appl. Math. (Amsterdam), vol. 140, 2nd ed., Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305.

[2] P. A. Ruiz et al. *Analysis on hybrid fractals*, Commun. Pure Appl. Anal. **19** (2020), no. 1, 47–84.

[3] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Math. Surveys Monogr., vol. 186, Amer. Math. Soc., Providence, RI, 2013, pp. xiv+270, DOI 10.1090/surv/186.

[4] G. Berkolaiko, R. Carlson, S. A. Fulling, and P. Kuchment (eds.), *Quantum graphs and their applications*, Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, DOI 10.1090/conm/415.

[5] F. Bernicot, B. Maury, and D. Salort, *A 2-adic approach of the human respiratory tree*, Netw. Heterog. Media **5** (2010), no. 3, 405–422.

[6] A. Björn, J. Björn, and N. Shanmugalingam, *Extension and trace results for doubling metric measure spaces and their hyperbolic fillings*, J. Math. Pures Appl. (9) **159** (2022), 196–249.

[7] A. Björn, J. Björn, J. T. Gill, and N. Shanmugalingam, *Geometric analysis on Cantor sets and trees*, J. Reine Angew. Math. **725** (2017), 63–114.

[8] J. Breuer and N. Levi, *On the decomposition of the Laplacian on metric graphs*, Ann. Henri Poincaré **21** (2020), no. 2, 499–537.

[9] J. Brüning, V. Geyler, and K. Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, Rev. Math. Phys. **20** (2008), no. 1, 1–70.

[10] J. Brüning and V. A. Geyler, *Scattering on compact manifolds with infinitely thin horns*, J. Math. Phys. **44** (2003), no. 2, 371–405.

[11] R. Carlson, *Boundary value problems for infinite metric graphs*, Analysis on graphs and its applications, Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 355–368, DOI 10.1090/pspum/077/2459880.

[12] R. Carlson, *Robin boundary conditions for the Laplacian on metric graph completions*, Methods Funct. Anal. Topology **28** (2022), no. 1, 12–24.

[13] C. Cattaneo, *The spectrum of the continuous Laplacian on a graph*, Monatsh. Math. **124** (1997), no. 3, 215–235.

[14] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601–628.

[15] A. Cohen, *Numerical analysis of wavelet methods*, Studies in Mathematics and Its Applications, vol. 32, North-Holland Publishing Co., Amsterdam, 2003, pp. xviii+336.

[16] M. Düfel, J. B. Kennedy, D. Mugnolo, M. Plümer, and M. Täufer, *Boundary conditions matter: on the spectrum of infinite quantum graphs*, 2022, preprint arXiv:2207.04024.

[17] P. Exner, *The von Neumann way to treat systems of mixed dimensionality*, Rep. Math. Phys. **55** (2005), no. 1, 79–92.

[18] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplyaev (eds.), *Analysis on graphs and its applications*, Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. xiv+705, DOI 10.1090/pspum/077.

[19] P. Exner et al. *Spectral theory of infinite quantum graphs*, Ann. Henri Poincaré **19** (2018), no. 11, 3457–3510.

[20] I. Fonseca and G. Parry, *Equilibrium configurations of defective crystals*, Arch. Ration. Mech. Anal. **120** (1992), no. 3, 245–283.

[21] J. A. Griepentrog, W. Höppner, H.-C. Kaiser, and J. Rehberg, *A bi-Lipschitz continuous, volume preserving map from the unit ball onto a cube*, Note Mat. **28** (2008), no. 1, 177–193.

[22] N. Große and C. Schneider, *Sobolev spaces on Riemannian manifolds with bounded geometry: general coordinates and traces*, Math. Nachr. **286** (2013), no. 16, 1586–1613.

[23] S. Haeseler, M. Keller, D. Lenz, and R. Wojciechowski, *Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions*, J. Spectr. Theory **2** (2012), no. 4, 397–432.

[24] A. Holhoş and D. Roşca, *Uniform refinable 3D grids of regular convex polyhedrons and balls*, Acta Math. Hungar. **156** (2018), no. 1, 182–193.

[25] A. Hussein, D. Krejčiřík, and P. Siegl, *Non-self-adjoint graphs*, Trans. Amer. Math. Soc. **367** (2015), no. 4, 2921–2957.

[26] T. Hytönen and A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. **126** (2012), no. 1, 1–33.

[27] P. Joly and M. Kachanovska, *Transparent boundary conditions for wave propagation in fractal trees: convolution quadrature approach*, Numer. Math. **146** (2020), no. 2, 281–334.

[28] P. Joly and M. Kachanovska, *Local transparent boundary conditions for wave propagation in fractal trees (I). Method and numerical implementation*, SIAM J. Sci. Comput. **43** (2021), no. 6, A3760–A3788.

[29] P. Joly and M. Kachanovska, *Local transparent boundary conditions for wave propagation in fractal trees (II). Error and complexity analysis*, SIAM J. Numer. Anal. **60** (2022), no. 2, 529–557.

[30] P. Joly, M. Kachanovska, and A. Semin, *Wave propagation in fractal trees. Mathematical and numerical issues*, Netw. Heterog. Media **14** (2019), no. 2, 205–264.

[31] P. Joly and A. Semin, *Mathematical and numerical modeling of wave propagation in fractal trees*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 19-20, 1047–1051.

[32] V. A. Kaimanovich, *Random walks on Sierpiński graphs: hyperbolicity and stochastic homogenization*, Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, pp. 145–183.

[33] M. Keller, D. Lenz, and R. a. K. Wojciechowski, *Graphs and discrete Dirichlet spaces*, Grundlehren der mathematischen Wissenschaften, vol. 358, Springer, Cham, 2021, pp. xv+668, DOI 10.1007/978-3-030-81459-5.

[34] J. Kigami, *Geometry and analysis of metric spaces via weighted partitions*, Lecture notes in mathematics, vol. 2265, Springer, Cham.

[35] J. Kigami, *Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees*, Adv. Math. **225** (2010), no. 5, 2674–2730.

[36] S.-L. Kong, K.-S. Lau, and T.-K. L. Wong, *Random walks and induced Dirichlet forms on self-similar sets*, Adv. Math. **320** (2017), 1099–1134.

[37] S.-L. Kong, K.-S. Lau, and T.-K. L. Wong, *Random walks and induced Dirichlet forms on compact spaces of homogeneous type*, Analysis, probability and mathematical physics on fractals, Fractals Dyn. Math. Sci. Arts Theory Appl., vol. 5, World Sci. Publ., Hackensack, NJ, 2020, pp. 273–296.

[38] P. Koskela, K. N. Nguyen, and Z. Wang, *Trace operators on regular trees*, Anal. Geom. Metr. Spaces **8** (2020), no. 1, 396–409.

[39] P. Koskela, K. N. Nguyen, and Z. Wang, *Trace and density results on regular trees*, Potential Anal. **57** (2022), no. 1, 101–128.

[40] A. Kostenko, D. Mugnolo, and N. Nicolussi, *Self-adjoint and Markovian extensions of infinite quantum graphs*, J. Lond. Math. Soc. (2) **105** (2022), no. 2, 1262–1313.

[41] A. Kostenko and N. Nicolussi, *Laplacians on infinite graphs*, Memoirs of the European Mathematical Society, vol. 3, EMS Press, Berlin, 2023, pp. viii+232, DOI 10.4171/mems/3.

[42] P. Kurasov, *Spectral geometry of graphs*, Operator Theory: Advances and Applications, vol. 293, Springer, Basel, 2024, DOI 10.1007/978-3-662-67872-5.

[43] T. Lefeuvre, *Triangulation with simplices of same volume*, MathOverflow, https://mathoverflow.net/q/237500.

[44] D. Lenz and K. Pankrashkin, *New relations between discrete and continuous transition operators on (metric) graphs*, Integral Equations Operator Theory **84** (2016), no. 2, 151–181.

[45] D. Lenz, C. Schubert, and I. Veselić, *Unbounded quantum graphs with unbounded boundary conditions*, Math. Nachr. **287** (2014), no. 8-9, 962–979.

[46] D. Lenz, F. Sobieczky, and W. Woess, (eds.) *Random walks, boundaries and spectra*, Progress in Probability, vol. 64, Birkhäuser/Springer Basel AG, Basel, 2011, pp. xxvi+324, DOI 10.1007/978-3-0346-0244-0.

[47] B. Maury, D. Salort, and C. Vannier, *Trace theorems for trees and application to the human lungs*, Netw. Heterog. Media **4** (2009), no. 3, 469–500.

[48] Y. Meyer, *Wavelets and operators*, Cambridge Stud. Adv. Math., vol. 37, Cambridge University Press, Cambridge, 1992, pp. xvi+224.

[49] I. Mitrea and M. Mitrea, *Multi-layer potentials and boundary problems for higher-order elliptic systems in Lipschitz domains*, Lecture notes in mathematics, vol. 2063, Springer, Heidelberg, 2013, pp. x+424, DOI 10.1007/978-3-642-32666-0.

[50] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.

[51] K. Naimark and M. Solomyak, *Geometry of Sobolev spaces on regular trees and the Hardy inequalities*, Russ. J. Math. Phys. **8** (2001), no. 3, 322–335.

[52] S. Nicaise and A. Semin, *Density and trace results in generalized fractal networks*, ESAIM Math. Model. Numer. Anal. **52** (2018), no. 3, 1023–1049.

[53] K. Pankrashkin, *Spectra of Schrödinger operators on equilateral quantum graphs*, Lett. Math. Phys. **77** (2006), no. 2, 139–154.

[54] K. Pankrashkin, *Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures*, J. Math. Anal. Appl. **396** (2012), no. 2, 640–655.

[55] K. Pankrashkin, *An example of unitary equivalence between self-adjoint extensions and their parameters*, J. Funct. Anal. **265** (2013), no. 11, 2910–2936.

[56] K. Pankrashkin, S. Roganova, and N. Yeganefar, *Resolvent expansions on hybrid manifolds*, Integral Equations Operator Theory **71** (2011), no. 2, 199–223.

[57] Y. Pinchover, G. Wolansky, and D. Zelig, *Spectral properties of Schrödinger operators on radial N-dimensional infinite trees*, Israel J. Math. **165** (2008), 281–328.

[58] O. Post, *Spectral analysis on graph-like spaces*, Lecture notes in mathematics, vol. 2039, Springer, Heidelberg, 2012, pp. xvi+431, DOI 10.1007/978-3-642-23840-6.

[59] O. Post, *Boundary pairs associated with quadratic forms*, Math. Nachr. **289** (2016), no. 8-9, 1052–1099.

[60] C. Schubert, C. Seifert, J. Voigt, and M. Waurick, *Boundary systems and (skew-)self-adjoint operators on infinite metric graphs*, Math. Nachr. **288** (2015), no. 14-15, 1776–1785.

[61] A. Semin, *Propagation d'ondes dans des jonctions de fentes minces*, Ph.D. thesis, University Paris-Sud, 2010.

[62] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture notes of the Unione Matematica Italiana, vol. 3, Springer, Berlin; UMI, Bologna, 2007, pp. xxvi+218.

[63] J. von Below, *A characteristic equation associated to an eigenvalue problem on c^2-networks*, Linear Algebra Appl. **71** (1985), 309–325.

[64] J. von Below and J. A. Lubary, *Harmonic functions on locally finite networks*, Results Math. **45** (2004), no. 1-2, 1–20.

[65] J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. (2) **41** (1940), 809–824.

[66] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Math., vol. 138, Cambridge University Press, Cambridge, 2000, pp. xii+334, DOI 10.1017/CBO9780511470967.