# A categorical reading of the numerical existence property in constructive foundations

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**Abstract**

We propose here an analysis based on syntactic categories and internal categories of existence properties. These metamathematical properties are peculiar of constructive theories, since they bring the internal notion of existence back to the external one, in accordance with the informal paradigm for constructivism known under the acronym BHK. Category theory is a powerful tool to analyse this phenomenon, since a category is an environment which allows to describe effectively internal and external notions and their relationship.

**Keywords**: Constructivism; Internal categories; Existence properties.

## 1 Existence in constructive mathematics

In his "*A Constructive Manifesto*", chapter 1 section 3 p.11 in Bishop and Bridges (1985), Bishop was clear about his view on existence in mathematics:

> Constructive existence is much more restrictive than the ideal existence of classical mathematics. The only way to show that an object exists is to give a finite routine for finding it, whereas in classical mathematics other methods can be used.

This view is clearly in contrast with a majority formalist view, defended e.g. by Poincaré (Poincaré (1906), troisième article, III, and Troelstra and van Dalen (1988) p.19), according to which

> Existence can mean only one thing: freedom from contradiction.

or by Hilbert (in a letter to Frege, see e.g. p.69 of Shapiro (2005)):

> if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by them exist.

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Indeed, this mainstream attitude directly leads to the identification of $\exists x\, P(x)$ with $\neg\forall x\neg P(x)$ and $\neg\neg\exists x\, P(x)$, which is exactly one of the methods to which Bishop referred to in the quotation above; the position of constructivism towards these methods is very well expressed by Bridges' words in Bridges (2008), section 3.1:

> how could a proof of the impossibility of the non-existence of a certain object x describe a mental construction of x?

This strong philosophical (and methodological) constructive view on existence can be translated in formal mathematical and metamathematical terms, as we will see in the next sections.

It is sort of intuitive to understand that the constructive notion of existence is captured by mathematical foundational theories for which the distance between the mathematical level and the metamathematical one (in which the first is defined) is minimized. Here we will use the descriptive and expressive power of category theory to illustrate this fact in a more structured way: we will move from syntax and mathematical theories to categories and internal categories, respectively, adopting a methodology similar (to some respects) to that of algebraic set theory (see e.g. Simpson (1999) or Maschio (2015)).

## 2 A paradigm for constructive proofs: BHK

In most textbooks about constructive mathematics (e.g. in Troelstra and van Dalen (1988), chapter 1 section 3 p.9 and in Bridges and Vîţă (2006), chapter 1 section 1.1. p.3), the underlying logical system is explained by means of an informal interpretation of what is a constructive proof of a compound formula, known under the name of BHK[1]. According to this interpretation:

($\wedge_{\mathsf{BHK}}$) a proof of $P \wedge Q$ is a pair $\langle p, q\rangle$ with $p$ a proof of $P$ and $q$ a proof of $Q$;

($\vee_{\mathsf{BHK}}$) a proof of $P\vee Q$ is a pair $(i, r)$ consisting of a proof $r$ and a label $i$ declaring whether that proof is a proof of $P$ or a proof of $Q$;

($\rightarrow_{\mathsf{BHK}}$) a proof of $P \rightarrow Q$ is a procedure $f$ turning proofs $p$ of $P$ into proofs $f(p)$ of $Q$;

($\exists_{\mathsf{BHK}}$) a proof of $\exists x\, P(x)$ is a pair $\langle a, p\rangle$ consisting of an object $a$ and a proof $p$ of $P(a)$;

($\forall_{\mathsf{BHK}}$) a proof of $\forall x\, P(x)$ is a procedure $f$ which associates to each object $a$ a proof $f(a)$ of $P(a)$.

This interpretation does not say what is a proof of an atomic formula and it is clearly informal. However, as we will see in the next two sections, there are at least two ways to make it "formal": one is syntactical, the other semantical.

[1] The acronym BHK comes from the names of three mathematicians which contributed to the constructive approach to mathematics, namely Brouwer, Heyting and Kolmogorov.

## 3 A semantic counterpart of BHK

In order to concretely accomplish BHK, proofs should enjoy at least these two properties: a pair of proofs must be a proof and some (partial) functions sending proofs into proofs must be proofs. Mathematically, we can render these requirements as follows: if $\mathbb{P}$ is a collection of proofs in BHK sense, then there must be an injective pairing function $\mathsf{pair} : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, and $\mathbb{P}$ must be endowed with a (partial) function from $\mathbb{P}$ to the set of partial functions from $\mathbb{P}$ to $\mathbb{P}$.

There is a meaningful structure validating these requirements: natural numbers. In fact there is a recursive bijective encoding of pairs of natural numbers by means of natural numbers ($p : (n, m) \mapsto 2^n(2m + 1) - 1$) with projections $p_1$ and $p_2$, and every natural number $n$ represents a recursive (partial) function $\{n\}$ from $\mathbb{N}$ to $\mathbb{N}$, whenever a Gödelian encoding is fixed.

Using this structure on natural numbers, one can define the so-called Kleene realizability which is a rigorous semantical account of BHK for Heyting arithmetics, where proofs are interpreted as natural numbers. Kleene's realizability relation, which is represented by a formula $x \Vdash P$ ("$x$ realizes $P$"), is defined by induction on the complexity of the formula $P$:[2]

$x \Vdash P \equiv^{def} P$ for atomic formulas $P$;

($\wedge_{real}$) $x \Vdash P \wedge Q \equiv^{def} p_1(x) \Vdash P \wedge p_2(x) \Vdash Q$, that is, a realizer for $P \wedge Q$ is a natural number encoding a pair of natural numbers in which the first component realizes $P$ and the second realizes $Q$;

($\vee_{real}$) $x \Vdash P \vee Q \equiv^{def} (p_1(x) = 0 \wedge p_2(x) \Vdash P) \vee (p_1(x) = 1 \wedge p_2(x) \Vdash Q)$, that is, a realizer for $P \vee Q$ is a natural number enconding a pair in which the first component is a label which tells whether the second component is a realizer of $P$ or a realizer of $Q$;

($\to_{real}$) $x \Vdash P \to Q \equiv^{def} \forall y(y \Vdash P \to \{x\}(y) \downarrow \wedge \{x\}(y) \Vdash Q)$, that is, a realizer of $P \to Q$ is the code of a recursive function sending realizers of $P$ to realizers of $Q$;

($\exists_{real}$) $x \Vdash \exists z\, P \equiv^{def} p_2(x) \Vdash P[p_1(x)/z]$, that is, a realizer of $\exists z\, P$ is a natural number encoding a pair in which the second component is a realizer of the formula obtained from $P$ by substituting $z$ with the first component;

($\forall_{real}$) $x \Vdash \forall z\, P \equiv^{def} \forall z(\{x\}(z) \downarrow \wedge \{x\}(z) \Vdash P)$, that is, a realizer of $\forall z\, P$ is a code of a total recursive function sending each natural number $n$ to a realizer of $P[n/z]$.

Using Kleene realizability in Kleene (1945), in Troelstra (1971) it was proved that $\mathsf{HA} \vdash \exists x(x \Vdash \varphi) \Leftrightarrow \mathsf{HA} + \mathsf{ECT}_0 \vdash \varphi$, where HA is Heyting arithmetic and $\mathsf{ECT}_0$ is the so-called *Extended Church's Thesis* (see Troelstra and van

[2] We always assume $x$ and $y$ not to occur in the formulas of which the realizability relation is defined.

Dalen (1988), chapter 4 section 4, p.199). In particular this provides a relative consistency proof of

$$\mathsf{HA} + \text{“All definable functions between natural numbers are computable”}$$

with respect to HA.[3]

There are many other notions of realizability arising from similar algebraic structures, which are called *partial combinatory algebras* (see e.g. Van Oosten (2008), chapter 1).

It should also be noticed that there exist in literature Kleene realizability models for intuitionistic set theories like Intuitionistic Zermelo-Fraenkel set theory IZF (see e.g. Friedman (1973), Rosolini (1982) and McCarty (1986)); however in these cases one need to modify the interpretations of primitive formulas and quantifiers; in particular, since a realizer is a natural number, one cannot incorporate a witness for an existential statement (which would be a set) into it.

# 4 A syntactic counterpart of BHK

Per Martin-Löf introduced his intuitionistic type theory (see Martin-Löf (1984), Nordström et al. (1990)) in the early 70's. From the first lines of p.1 in Martin-Löf (1975) one can understand his goal and the particular attention dedicated to the meaning of existential statements:

> The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop. The language of the theory is richer that the languages of traditional intuitionistic systems in permitting proofs to appear as parts of propositions so that the propositions of the theory can express properties of proofs (and not only individuals, like in first order predicate logic). This makes it possible to strengthen the axioms for existence, disjunction absurdity and identity. In the case of existence, this possibility seems first to have been indicated by Howard, whose proposed axioms are special cases of the existential elimination rule of the present theory.

Concretely, in Martin-Löf type theory a dependent sum type constructor $\Sigma$ is defined by the following four rules:

1. a formation rule
$$\frac{A\,type \qquad B(x)\,type\,[x \in A]}{(\Sigma x \in A)B(x)\,type}$$

[3] The statement “All definable functions between natural numbers are computable” is not expressible as a formula in HA, but as a collection of formulas

$$\forall x \exists ! y\, \varphi(x,y) \to \exists e \forall x (\varphi(x, \{e\}(x)))$$

for $\varphi(x,y)$ formula of HA.

which states that one can form the dependent sum $(\Sigma x \in A)B(x)$ of a family $B(x)$ of types indexed over a type $A$;

2. an introduction rule

$$\frac{a \in A \qquad b \in B(a)}{\langle a, b\rangle \in (\Sigma x \in A)B(x)}$$

which states that all pairs $\langle a, b\rangle$ with $a \in A$ and $b \in B(a)$ are terms of type $(\Sigma x \in A)B(x)$;

3. an elimination rule

$$\frac{\begin{array}{l} d \in (\Sigma x \in A)B(x) \\ C(z)\,type\,[z \in (\Sigma x \in A)B(x)] \\ c(x, y) \in C(\langle x, y\rangle)[x \in A, y \in B(x)] \end{array}}{\mathsf{El}_\Sigma(d, c(x, y)) \in C(d)}$$

which essentially says that nothing else is in $(\Sigma x \in A)B(x)$ (in order to assign an element of $C(d)$ to each $d \in (\Sigma x \in A)B(x)$, it is sufficient to assign an element of $C(\langle x, y\rangle)$ to each $x \in A$ and $y \in B(x)$);

4. an equality rule

$$\frac{\begin{array}{l} a \in A \qquad b \in B(a) \\ C(z)\,type\,[z \in (\Sigma x \in A)B(x)] \\ c(x, y) \in C(\langle x, y\rangle)[x \in A, y \in B(x)] \end{array}}{\mathsf{El}_\Sigma(\langle a, b\rangle, c(x, y)) = c(a, b) \in C(\langle a, b\rangle)}.$$

One of the key features of Martin-Löf type theory is the so called "propositions-as-types" paradigm: logic and mathematics are identified. For example, the dependent sum type $\Sigma$ is used to represent the existential quantifier $\exists$, which, as a consequence, satisfies the following rules, which are obtained or derived from the rules above, by reading some types $P$ as propositions and by interpreting the relative judgements of the form $p \in P$ as "$p$ is a proof of $P$".

$$\frac{A\,type \qquad P(x)\,prop\,[x \in A]}{(\exists x \in A)P(x)\,prop} \qquad \frac{a \in A \qquad b \text{ is a proof of } P(a)}{\langle a, b\rangle \text{ is a proof of } (\exists x \in A)P(x)}$$

$$\frac{d \text{ is a proof of } (\exists x \in A)P(x)}{\pi_1(d) := \mathsf{El}_\Sigma(d, x(x, y)) \in A} \qquad \frac{d \text{ is a proof of } (\exists x \in A)P(x)}{\pi_2(d) := \mathsf{El}_\Sigma(d, y(x, y)) \text{ is a proof of } P(\pi_1(d))}$$

Hence, in Martin-Löf type theory the identification between $\Sigma$ and $\exists$ imposes the validity of the request about existential statements in BHK. Other constructors of Martin-Löf type theory are designed in order to accomplish BHK via the propositions-as-types paradigm.

In other type theories, like in the Minimalist Foundation **MF** (see Maietti and Sambin (2005), Maietti (2009)), which was introduced in order to provide a core foundation compatible with the most relevant classical and intuitionistic, predicative and impredicative, foundations, the paradigm propositions-as-types is not adopted. In the formulation of **MF** there is a distinction between two kinds of types: logical (propositions and small propositions) and mathematical (collections and sets) and the existential propositions satisfy rules which are similar to those of Σ-types above; however the elimination rule works only toward propositions. Hence one cannot in general produce the witness required by BHK. However one can show that **MF** admits a Kleene realizability interpretation (see Maietti and Maschio (2015), Ishihara et al. (2018)).

# 5 Existence properties

In first-order theories the requirement on existential quantifiers from BHK cannot be imposed in the formulation of the theory itself, as it is in fact done in Martin-Löf type theory. However, it can be controlled a posteriori, after being reformulated as a metamathematical property. The point is to find the right metamathematical formulation. In the literature there are many proposals; the difference between them consists in what they consider a *witness* for an existential statement should be.

In the first case witnesses are definable entities in the theory.

**Definition 5.1.** *A first-order theory* $\mathcal{T}$ *has the* existence property (**EP**) *if whenever* $\mathcal{T} \vdash \exists x\, P(x)$*, there exists a formula* $Q(x)$*, such that*

$$\mathcal{T} \vdash \exists! x\, Q(x) \wedge \forall x(Q(x) \to P(x)).$$

The existence property **EP** essentially means that if something satisfying a property is proven to exist in $\mathcal{T}$, then something definable in $\mathcal{T}$ can be proven to satisfy that property.

The intuitionistic set theory IZF (see Friedman and Ščedrov (1985)) and the constructive Zermelo-Fraenkel set theory CZF (see Swan (2014)) do not have **EP**, while, as we will see in a few lines, Heyting arithmetic HA has it. Classical theories like Peano arithmetic PA and ZF+V=L, that is Zermelo-Fraenkel set theory with the additional axiom that states that all sets are constructible, also have **EP**. If $\mathsf{PA} \vdash \exists x\, P(x)$, then one can take $Q(x)$ to be $P(x) \wedge \forall y(P(y) \to x \leq y)$ which works because of the minimum principle which is provable in PA. In ZF+V=L one can do essentially the same, because there one can define a well-ordering on the universe class V.

Although at first sight **EP** could be considered a good candidate to express the BHK requirement about existential quantifiers, one could object that the "unique existence" required in the definition could be proven in $\mathcal{T}$ by means of indirect methods, thus producing a *witness* being only apparently "concrete".

Another option could be to consider a *term existence property* in which witnesses are simply terms not containing variables. However this is not of great interest in this framework:[4] terms representing definable objects can indeed be added to a first-order theory leaving it essentially equivalent and turning, in the end, term existence property into existence property.

[4]The internal language of a doctrine in category theory is an example of framework in which terms have a clear "stable" meaning, that is "arrows of the base category", and where, hence, term existence property would be meaningful.

Looking for something sufficiently simple to be considered "stable" from the external point of view, one comes to numerals, that is natural (meta)numbers. In fact the notion of numeral requires only the ability of juxtaposing symbols, which is a minimal requirement for being able to formulate a first-order theory. In this sense we can think of numerals as a good notion of witnesses. However they can only be used as witnesses for those formulas in which the free variable represents a natural number in the sense of the theory $\mathcal{T}$:

**Definition 5.2.** *A first-order theory of natural numbers $\mathcal{T}$ has the* numerical existence property (**nEP**) *if, for every formula $P(x)$,*[5] *there exists a numeral* $\mathbf{n}$ *such that $\mathcal{T} \vdash P(\mathbf{n})$, whenever $\mathcal{T} \vdash \exists x\, P(x)$. $\mathcal{T}$ has the* unique numerical existence property ($\mathbf{nEP}_!$) *if, for every formula $P(x)$, there exists a numeral* $\mathbf{n}$ *such that $\mathcal{T} \vdash P(\mathbf{n})$, whenever $\mathcal{T} \vdash \exists! x\, P(x)$.*

*A first-order theory of sets $\mathcal{T}$, in which the existence of the set $\omega$ is provable, has the* numerical existence property (**nEP**) *if, for every formula $P(x)$, there exists a numeral* $\mathbf{n}$ *such that $\mathcal{T} \vdash P(\mathbf{n})$, whenever $\mathcal{T} \vdash \exists x \in \omega\, P(x)$. $\mathcal{T}$ has the* unique numerical existence property ($\mathbf{nEP}_!$) *if, for every formula $P(x)$, there exists a numeral* $\mathbf{n}$ *such that $\mathcal{T} \vdash P(\mathbf{n})$, whenever $\mathcal{T} \vdash \exists! x \in \omega\, P(x)$.*[6]

The numerical existence property **nEP** essentially means that if a natural number satisfying a property is proven to exist in $\mathcal{T}$, then a numeral can be proven to satisfy that property. $\mathbf{nEP}_!$ essentially means that definable natural numbers exactly coincide with numerals.

Peano arithmetic PA, Zermelo-Fraenkel set theory ZF and, in general, classical first-order theories $\mathcal{T}$ of numbers or sets (if consistent) do not have the numerical existence property, not even the unique one. Indeed one can consider an independent sentence $I$ (which exists by Gödel's first incompleteness theorem): clearly $\mathcal{T} \vdash \exists x((x = 0 \wedge \neg I) \vee (x = 1 \wedge I))$ as a consequence of the law of excluded middle; however there cannot be a numeral $\mathbf{n}$ such that $\mathcal{T} \vdash (\mathbf{n} = 0 \wedge \neg I) \vee (\mathbf{n} = 1 \wedge I)$, since in that case $\mathbf{n}$ would be 0 or 1 and we could hence prove $\neg I$ or $I$ in $\mathcal{T}$. Heyting arithmetic HA has the numerical existence property: this was proven by means of realizability by Kleene (see Kleene (1945)). CZF and IZF also have the numerical existence property, as was proven in Rathjen (2005) Theorem 1.2. and in Beeson (1985) chapter VIII section 9, respectively. Clearly, for first-order theories of natural numbers $\mathbf{nEP} \equiv \mathbf{EP} + \mathbf{nEP}_!$ (hence HA has **EP**), while for first-order theories of sets $\mathbf{EP} + \mathbf{nEP}_! \Rightarrow \mathbf{nEP}$, but the converse does not necessarily hold.

# 6 Categories of definable classes

The next step consists in organizing the content of a theory in a category of definable classes and to introduce some useful subcategories.

[5] When we write a formula $P(x_1, ..., x_n)$ we mean that $P$ contains at most $x_1, ..., x_n$ as free variables.

[6] In set theory, $P(\mathbf{n})$ is defined as follows: $P(0) \equiv^{def} \exists x(\forall y(y \notin x) \wedge P(x))$ and for every natural (meta)number $n$, $P(\mathbf{n+1}) \equiv^{def} \exists x(\forall y(y \in x \leftrightarrow y \in \mathbf{n} \vee y = \mathbf{n}) \wedge P(x))$.

We define the category $\mathsf{DC}[\mathcal{T}]$ of the definable classes of $\mathcal{T}$ as follows:

1. we first fix two variables $x, y$;
2. the objects of $\mathsf{DC}[\mathcal{T}]$ are formal expressions $\{x|\, P(x)\}$ where $P(x)$ is a formula; we identify objects $\{x|\, P(x)\}$ and $\{x|\, Q(x)\}$ with $P(x)$ and $Q(x)$ provable to be equivalent in $\mathcal{T}$;
3. an arrow from $\{x|\, P(x)\}$ to $\{x|\, Q(x)\}$ is a formula $F(x, y)$, such that
   (a) $F(x, y) \vdash_{\mathcal{T}} P(x) \wedge Q(y)$;
   (b) $F(x, y) \wedge F(x, z) \vdash_{\mathcal{T}} y = z$ (where $z$ is a fresh variable);
   (c) $P(x) \vdash_{\mathcal{T}} \exists y F(x, y)$;

   and we identify formulas provable to be equivalent in $\mathcal{T}$;
4. the composition $G(x, y) \circ F(x, y)$ is defined as $\exists z(F(x, z) \wedge G(z, y))$ where $z$ is a fresh variable;
5. the identity arrow of an object $\{x|\, P(x)\}$ is defined as the formula $P(x) \wedge x = y$.

For the theory $\mathcal{T}$ we can also define a category $\mathsf{DC}_{term}[\mathcal{T}]$ having the same objects as $\mathsf{DC}[\mathcal{T}]$, but for which an arrow from $\{x|\, P(x)\}$ to $\{x|\, Q(x)\}$ is an equivalence class $[t(x)]_{\simeq_{P(x)}}$ of terms $t(x)$,[7] such that $P(x) \vdash_{\mathcal{T}} Q(t(x))$ with respect to the relation $\simeq_{P(x)}$ for which $t(x) \simeq_{P(x)} s(x)$ when $P(x) \vdash_{\mathcal{T}} t(x) = s(x)$; the composition $[s(x)]_{\simeq_{Q(x)}} \circ [t(x)]_{\simeq_{P(x)}}$ of two arrows is defined by $[s(t(x))]_{\simeq_{P(x)}}$, while the identity $\mathsf{id}_{\{x|\, P(x)\}}$ is given by $[x]_{\simeq_{P(x)}}$.

The category $\mathsf{DC}_{term}[\mathcal{T}]$ is clearly a subcategory of $\mathsf{DC}[\mathcal{T}]$: just consider the functor sending each $\{x|\, P(x)\}$ to itself and each $[t(x)]_{\simeq_{P(x)}}$ to $P(x) \wedge y = t(x)$.

If $\mathcal{T}$ is a theory of natural numbers having at least 0 and the successor symbol $\mathsf{s}$ as primitive function symbols, we denote with $\mathsf{DC}_{nat}[\mathcal{T}]$ the subcategory of $\mathsf{DC}_{term}[\mathcal{T}]$ which have the same objects, the same definitions of composition and identity, but only those arrows which are representable by terms obtained using the variable $x$ and the function symbols 0 and $\mathsf{s}$.

If $\mathcal{T}$ is a theory of sets, we first translate terms $\tau$ of the language obtained using the variable $x$ and function symbols 0 and $\mathsf{s}$ into formulas $[[\tau]]$ of $\mathcal{T}$ as follows: $[[0]] \equiv^{def} \forall z\, (z \notin y)$, $[[x]] \equiv^{def} x = y$ and $[[\mathsf{s}(\tau(x))]] \equiv^{def} \exists z\, ([[\tau]][z/y] \wedge \forall u(u \in y \leftrightarrow u = z \vee u \in z))$.

The objects of $\mathsf{DC}_{nat}[\mathcal{T}]$ are defined as those objects $\{x|\, P(x)\}$ of $\mathsf{DC}[\mathcal{T}]$ such that $P(x) \vdash_{\mathcal{T}} x \in \omega$; an arrow in $\mathsf{DC}_{nat}[\mathcal{T}]$ from $\{x|\, P(x)\}$ to $\{x|\, Q(x)\}$ is an arrow of $\mathsf{DC}[\mathcal{T}]$ of the form $[[\tau]] \wedge P(x)$ for some term $\tau$ of the language obtained using the variable $x$ and function symbols 0 and $\mathsf{s}$. One can prove that compositions and identities inherited from $\mathsf{DC}[\mathcal{T}]$ work with this restriction.

[7] We will write $t(x_1, ...x_n)$ if the term $t$ contains at most $x_1$,...,$x_n$ as variables.

For every pair of objects $A = \{x|\, P(x)\}$ and $B = \{x|\, Q(x)\}$ in $\mathsf{DC}[\mathcal{T}]$ an injection $J^{A,B}_{nat}$ can be defined as follows. If $\mathcal{T}$ is a theory of natural numbers:

$$J^{A,B}_{nat} : \mathsf{DC}_{nat}[\mathcal{T}](A,B) \to \mathsf{DC}[\mathcal{T}](A,B)$$

$$[t(x)]_{\simeq_P} \mapsto P(x) \wedge y = t(x)$$

If $\mathcal{T}$ is a theory of sets and $A$ and $B$ are objects of $\mathsf{DC}_{nat}[\mathcal{T}](A,B)$, $J^{A,B}_{nat}$ is the obvious inclusion of $\mathsf{DC}_{nat}[\mathcal{T}](A,B)$ in $\mathsf{DC}[\mathcal{T}](A,B)$.

If the theory $\mathcal{T}$ has at least a definable element and a definable encoding of ordered pairs, that is, if we assume that there exist a formula $I(x)$ such that $\mathcal{T} \vdash \exists! x I(x)$ and a formula $Pr(x,y,z)$ with three free variables $x$, $y$, $z$ such that

1. $Pr(x,y,z) \wedge Pr(x,y,z') \vdash_{\mathcal{T}} z = z'$;
2. $Pr(x,y,z) \wedge Pr(x',y',z) \vdash_{\mathcal{T}} x = x' \wedge y = y'$;
3. $\vdash_{\mathcal{T}} \forall x \forall y \exists z\, Pr(x,y,z)$,

then $\mathsf{DC}[\mathcal{T}]$ is a cartesian category: a terminal object 1 given by $\{x|\, I(x)\}$ and a product of $\{x|\, P(x)\}$ and $\{x|\, Q(x)\}$ is given by the object $\{x|\, \exists y \exists z (P(y) \wedge Q(z) \wedge Pr(y,z,x))\}$ together with the obvious projections.

This is the case for all standard theories of natural numbers and of sets: in HA or PA the formula $I(x)$ can be taken to be $x = 0$ and $Pr(x,y,z)$ to be $z = 2^x(2y+1)$[8]; in set theories like CZF, IZF and ZFC, $I(x)$ can be taken to be $\forall y (y \notin x)$ and $Pr(x,y,z)$ to be $z = \{\{x\}, \{x,y\}\}$ [9].

In the rest of the chapter we will always implicitely assume $\mathcal{T}$ to be a first-order classical or intuitionistic theory of sets or of numbers having at least a definable element and an encoding of ordered pairs.

# 7 Existence properties, categorically

We now show what properties of the categories introduced in the previous section correspond to the existential properties introduced in section 5.

Before proving our characterization, let us recall some categorical notions: an arrow $e$ in a category $\mathbb{C}$ is a *regular epi* if there exist arrows $f$ and $g$ in $\mathbb{C}$ of which $e$ is the coequalizer, that is $e \circ f = e \circ g$ and for every arrow $e'$ such that $e' \circ f = e' \circ g$, there exists a unique arrow $r$ such that $r \circ e = e'$; an arrow $e : A \to B$ in $\mathbb{C}$ is a *split epi* if there exists an arrow $e'$ such that $e \circ e' = \mathsf{id}_B$.

**Theorem 7.1.** *Let $\mathcal{T}$ be a theory of natural numbers or a theory of sets as in the previous sections and let $P(x)$ be a formula of $\mathcal{T}$. Then*

1. *$\mathcal{T} \vdash \exists x\, P(x)$ if and only if the unique arrow from $\{x|\, P(x)\}$ to 1 in $\mathsf{DC}[\mathcal{T}]$ is a regular epi;*

[8] The exponential $2^{(-)}$ can be adequately represented by a definable relation.
[9] Here $z = \{x,y\}$ is an shorthand for $\forall u (u \in z \leftrightarrow (u = x \vee u = y)))$

2. *$\mathcal{T} \vdash \exists! x\, P(x)$ if and only if $I(x) \wedge P(y) : 1 \to \{x|\, Q(x)\}$ is a well-defined arrow in $\mathsf{DC}[\mathcal{T}]$ for every $Q(x)$ such that $P(x) \vdash_{\mathcal{T}} Q(x)$;*

3. *$\mathcal{T}$ has **EP** if and only if every regular epi in $\mathsf{DC}[\mathcal{T}]$ with codomain 1 is a split epi in $\mathsf{DC}[\mathcal{T}]$;*

4. *if $\mathcal{T}$ is a theory of natural numbers, $[t(x)]_{\simeq_{I(x)}} : 1 \to \{x|\, P(x)\}$ is an arrow in $\mathsf{DC}_{nat}[\mathcal{T}]$ if and only if there exists a numeral $\mathbf{n}$ such that $\mathbf{n} \simeq_{I(x)} t(x)$;*

5. *if $\mathcal{T}$ is a theory of sets and $[[\tau]] \wedge \forall u(u \notin x) : 1 \to \{x|\, P(x)\}$ is an arrow in $\mathsf{DC}_{nat}[\mathcal{T}]$, there exists a numeral $\mathbf{n}$ such that $[[\tau]] \wedge \forall u(u \notin x)$ and $[[\mathbf{n}]] \wedge \forall u(u \notin x)$ represent the same arrow from 1 to $\{x|\, P(x)\}$;*

6. *$\mathcal{T}$ has **nEP** if and only if every regular epi in $\mathsf{DC}[\mathcal{T}]$ from an object $A$ to 1 is a split epi with right inverse of the form $J^{1,A}_{nat}(f)$ with $f$ in $\mathsf{DC}_{nat}[\mathcal{T}]$;*

7. *$\mathcal{T}$ has **nEP**$_!$ if and only if $J^{1,A}_{nat}$ is a bijiection for every $A$ in $\mathsf{DC}_{nat}[\mathcal{T}]$.*

*Proof.* 1. Suppose that $\mathcal{T} \vdash \exists x\, P(x)$. Then one can prove that the unique arrow from $\{x|P(x)\}$ to 1 is the coequalizer of the two projections from $\{x|P(x)\} \times \{x|P(x)\}$ to $\{x|P(x)\}$. Conversely, suppose that the unique arrow ! from $\{x|P(x)\}$ to 1 is the coequalizer of two arrows $f$ and $g$; then if we consider the unique arrow $i$ from $\{x|P(x)\}$ to $\{x|I(x) \wedge \exists y\, P(y)\}$, then clearly $i \circ f = i \circ g$. In particular, it follows (since ! is the coequalizer of $f$ and $g$) that there exists an arrow from 1 to $\{x|I(x) \wedge \exists y\, P(y)\}$; this entails that $\mathcal{T} \vdash \exists x\, P(x)$.

2. If $\mathcal{T} \vdash \exists! x\, P(x)$ and $P(x) \vdash_{\mathcal{T}} Q(x)$, then $I(x) \wedge P(y) \vdash_{\mathcal{T}} I(x) \wedge Q(y)$, $(I(x) \wedge P(y)) \wedge (I(x) \wedge P(z)) \vdash_{\mathcal{T}} y = z$ and $I(x) \vdash_{\mathcal{T}} \exists y(I(x) \wedge P(y))$; conversely, if $I(x) \wedge P(y) : 1 \to \{x|\, Q\}$ is a well-defined arrow in $\mathsf{DC}[\mathcal{T}]$ for every $Q(x)$ such that $P(x) \vdash_{\mathcal{T}} Q(x)$, then in particular $I(x) \vdash_{\mathcal{T}} \exists y(I(x) \wedge P(y))$ and $I(x) \wedge P(y) \wedge P(z) \vdash_{\mathcal{T}} y = z$; since $\mathcal{T} \vdash \exists x\, I(x)$, then $\mathcal{T} \vdash \exists! y\, P(y)$.

3. Suppose that $\mathcal{T}$ has **EP** and suppose that $P(x) \wedge I(y)$, which is the unique arrow from $\{x|\, P(x)\}$ to 1 in $\mathsf{DC}[\mathcal{T}]$, is a regular epi in $\mathsf{DC}[\mathcal{T}]$; then by point 1. we have that $\mathcal{T} \vdash \exists x\, P(x)$; as a consequence of **EP**, we have that there exists $Q(x)$ such that $\mathcal{T} \vdash \exists! x\, Q(x)$ and $Q(x) \vdash_{\mathcal{T}} P(x)$. These conditions together with point 2. allow to conclude that $I(x) \wedge Q(y)$ is a well-defined arrow from 1 to $\{x|P(x)\}$. Clearly this arrow is a right inverse of the unique arrow from $\{x|\, P(x)\}$ to 1. Conversely, suppose that $P(x)$ is a formula for which $\mathcal{T} \vdash \exists x\, P(x)$. By point 1. the unique arrow from $\{x|\, P(x)\}$ to 1 in $\mathsf{DC}[\mathcal{T}]$ is a regular epi, hence it is a split epi. This means that there is an arrow $F(x, y)$ from 1 to $\{x|P(x)\}$. One can see immediately that, since $\mathcal{T} \vdash \exists! x\, I(x)$, the formula $F(x, y)$ is equivalent in $\mathcal{T}$ to $I(x) \wedge \exists z\, F(z, y)$. If we take $Q(x)$ to be $\exists z(F(z, x))$, then the requirement of **EP**, applied to $P(x)$, is satisfied by $Q(x)$.

4. and 5. follow from the very definition of $\mathsf{DC}_{nat}[\mathcal{T}]$ and of numerals. From these, points 6. and 7. follow immediately. □

# 8 Internalizing $\mathsf{DC}[\mathcal{T}]$ in itself

From now on, we will consider only theories $\mathcal{T}$ which enjoy a primitive recursive internal Gödelian encoding of their syntax by means of natural numbers. We also use, with abuse of notations, symbols for recursive function between natural numbers (including a primitive recursive bijective encoding of natural numbers $\mathsf{p}$ with primitive recursive projections $\mathsf{p}_1$ and $\mathsf{p}_2$), since they can be adequately represented in $\mathcal{T}$. In particular, every variable $\xi$ in the syntax of $\mathcal{T}$ is encoded by a numeral $\boldsymbol{\xi}$. One can hence define a formula $\mathsf{dc}(x) \equiv^{def} \mathsf{form}(x) \wedge \forall y(\mathsf{free}(y,x) \to y = \mathbf{x})$ which expresses the fact that $x$ is the code of a formula of $\mathcal{T}$ having at most $x$ as free variable, in such a way that the definable class $\Delta\Gamma_0 := \{x|\, \mathsf{dc}(x)\}$, which is an object of $\mathsf{DC}[\mathcal{T}]$, is an internalization of the collection of objects of $\mathsf{DC}[\mathcal{T}]$ itself. However, in the definition of $\mathsf{DC}[\mathcal{T}]$, we have identified definable classes which were given by provably equivalent formulas. We hence need to take this into account internally by means of the obvious internal equivalence relation $\equiv_0$:

$$\{x|\, \exists y \exists z(x = \mathsf{p}(y,z) \wedge \mathsf{dc}(y) \wedge \mathsf{dc}(z) \wedge \mathsf{der}(y,z) \wedge \mathsf{der}(z,y))\} \to \Gamma\Delta_0 \times \Gamma\Delta_0$$

where $\mathsf{der}(x,y)$ is a formula expressing the fact that the formula encoded by $y$ can be derived from that encoded by $x$ in $\mathcal{T}$.

Analogously, one can define a formula $\mathsf{fr}(x)$ expressing the fact that $x$ is the code of a definable functional relation.

However, in order to encode the collection of arrows of $\mathsf{DC}[\mathcal{T}]$, we need to keep track of their codomains (which can not be reconstructed otherwise). We hence consider the collection

$$\Delta\Gamma_1 := \{x|\, \exists y \exists z(x = \mathsf{p}(y,z) \wedge \mathsf{fr}(y) \wedge \mathsf{dc}(z) \wedge \mathsf{der}(y, \mathsf{sub}(z, \mathbf{y}, \mathbf{x})))\}$$

(where $\mathsf{sub}(z, \mathbf{y}, \mathbf{x})$ is a term representing a code for the formula encoded by $z$ in which the variable $x$ is substituted with $y$) which is indeed an internal account of the collection of arrows of $\mathsf{DC}[\mathcal{T}]$ once we consider the obvious internal equivalence relation $\equiv_1$ with domain

$$\{x|\, \exists y \exists z(x = \mathsf{p}(y,z) \wedge y \varepsilon \Delta\Gamma_1 \wedge z \varepsilon \Delta\Gamma_1 \wedge$$

$$\mathsf{der}(\mathsf{p}_1(y), \mathsf{p}_1(z)) \wedge \mathsf{der}(\mathsf{p}_1(z), \mathsf{p}_1(y)) \wedge \mathsf{der}(\mathsf{p}_2(y), \mathsf{p}_2(z)) \wedge \mathsf{der}(\mathsf{p}_2(z), \mathsf{p}_2(y)))\}.$$ [10]

In $\mathsf{DC}[\mathcal{T}]$ one can also define internally arrows corresponding to the domain, codomain, identity and composition operations (where we use the notation $\overline{\phantom{x}}$ to denote the Gödelian encoding in terms of primitive recursive functions of connectives, quantifiers and equality symbols):

1. the domain arrow is $\delta_0 := x\varepsilon\Delta\Gamma_1 \wedge y = \overline{\exists \mathbf{y}} \mathsf{p}_1(x) : \Delta\Gamma_1 \to \Delta\Gamma_0$;
2. the codomain arrow is $\delta_1 := x\varepsilon\Delta\Gamma_1 \wedge y = \mathsf{p}_2(x) : \Delta\Gamma_1 \to \Delta\Gamma_0$;
3. the identity arrow is $\mathsf{ID} := x\varepsilon\Delta\Gamma_0 \wedge y = \mathsf{p}(x \overline{\wedge} (\mathbf{x} \overline{\equiv} \mathbf{y}), x) : \Delta\Gamma_0 \to \Delta\Gamma_1$;

[10] If $C = \{x|\, P(x)\}$ is a definable class, then we write $t\varepsilon C$ as a shorthand for $P(t)$.

4. the composition arrow $\square : \mathsf{Pb}(\delta_1, \delta_0) \to \Delta\Gamma_1$, where $\mathsf{Pb}(\delta_1, \delta_0)$ denotes the obvious choice of a pullback for $\delta_1$ and $\delta_0$, is more complicated but can be easily formulated with some patience.

What really matters is that $\Delta\Gamma[\mathcal{T}] := ((\Delta\Gamma_0, \equiv_0), (\Delta\Gamma_1, \equiv_1), \delta_0, \delta_1, \mathsf{ID}, \square)$ is essentially an internal category (see e.g. MacLane and Moerdijk (1992) chapter V section 7) in the elementary quotient completion (see Maietti and Rosolini (2012)) of $\mathsf{DC}[\mathcal{T}]$ with respect to the doctrine of its subobjects (subobjects in $\mathsf{DC}[\mathcal{T}]$ can be represented by comprehension), since the arrows $\delta_0$, $\delta_1$, $\mathsf{ID}$ and $\square$ respect the internal equivalence relations $\equiv_0$ and $\equiv_1$.

One can notice that in the case in which a particular *parametric version* of **EP** holds for formulas restricted to natural numbers (e.g. in classical set theory and in Peano arithmetic), one can avoid internal equivalence relations and choose representatives via a formula, obtaining an internal account of $\mathsf{DC}[\mathcal{T}]$ as one of its internal categories. More precisely, if $\mathsf{Nat}(x)$ is a formula in $\mathcal{T}$ expressing that $x$ is a natural number, the particular parametric version of **EP** we consider is the following: whenever $P(x, y)$ is a formula with at most $x$ and $y$ as free variables such that $P(x, y) \vdash_{\mathcal{T}} \mathsf{Nat}(x) \wedge \mathsf{Nat}(y)$, there exists another formula $Q(x, y)$ with at most $x$ and $y$ as free variables such that

1. $Q(x, y) \vdash_{\mathcal{T}} P(x, y)$;
2. $\mathcal{T} \vdash \forall x(\exists y\, P(x, y) \to \exists! y\, Q(x, y))$;
3. $\forall y(P(x, y) \leftrightarrow P(x', y)) \wedge Q(x, z) \wedge Q(x', z') \vdash_{\mathcal{T}} z = z'$.

# 9 Numerical existence property and the relation between $\mathsf{DC}[\mathcal{T}]$ and $\Delta\Gamma[\mathcal{T}]$

In this section we recall a result in Maschio (2020) which connects internal categories and the numerical existence property.

First, we can observe that, whenever one has a internal equivalence relation $r : R \to I \times I$ in a finitely complete category $\mathbb{C}$, then the subset

$$\{(\pi_1 \circ r \circ f, \pi_2 \circ r \circ f) \,|\, f \in \mathsf{Hom}_{\mathbb{C}}(1, R)\} \subseteq \mathsf{Hom}_{\mathbb{C}}(1, I) \times \mathsf{Hom}_{\mathbb{C}}(1, I)$$

is an equivalence relation which we denote with $\mathsf{Ext}(r)$. Using this fact, one can define a category $\mathsf{Ext}(\Delta\Gamma[\mathcal{T}])$ "externalising" the internal category $\Delta\Gamma[\mathcal{T}]$ in $\mathsf{DC}[\mathcal{T}]$ as follows:

1. the collection of objects of $\mathsf{Ext}(\Delta\Gamma)$ is $\mathsf{Hom}_{\mathsf{DC}[\mathcal{T}]}(1, \Delta\Gamma_0)/\mathsf{Ext}(\equiv_0)$;
2. the collection of objects of $\mathsf{Ext}(\Delta\Gamma)$ is $\mathsf{Hom}_{\mathsf{DC}[\mathcal{T}]}(1, \Delta\Gamma_1)/\mathsf{Ext}(\equiv_1)$;
3. the domain function $\Delta_0$ is defined by $\Delta_0([f]) := [\delta_0 \circ f]$;
4. the codomain function $\Delta_1$ is defined by $\Delta_1([f]) := [\delta_1 \circ f]$;
5. the identity function $\mathsf{id}$ is defined by $\mathsf{id}([f]) := [\mathsf{ID} \circ f]$;

6. the composition function $\circ$ is defined by $[g] \circ [f] := [\square \circ \langle f, g \rangle]$[11].

In general one can define a functor $\mathbf{J} : \mathsf{DC}[\mathcal{T}] \to \mathsf{Ext}(\Delta\Gamma[\mathcal{T}])$ as follows:

1. If $\{x|\, P(x)\}$ is an object of $\mathsf{DC}[\mathcal{T}]$, then the formula $P(x)$ is encoded by a numeral $\mathbf{cod}(P(x))$ for which clearly $\mathcal{T} \vdash \mathsf{dc}(\mathbf{cod}(P(x)))$. We hence define $\mathbf{J}(\{x|\, P(x)\})$ as the equivalence class represented by the arrow

$$I(x) \wedge y = \mathbf{cod}(P(x)) : 1 \to \Delta\Gamma_0;$$

2. if $F(x, y) : \{x|\, P(x)\} \to \{x|\, Q(x)\}$ is an arrow in $\mathsf{DC}[\mathcal{T}]$, then $F(x, y)$ is encoded by a numeral $\mathbf{cod}(F(x, y))$ and $Q(x)$ is encoded by a numeral $\mathbf{cod}(Q(x))$ and, hence, $\mathcal{T} \vdash p(\mathbf{cod}(F(x, y)), \mathbf{cod}(Q(x)))\varepsilon\Delta\Gamma_1$. We hence send $F(x, y) : \{x|\, P(x)\} \to \{x|\, Q(x)\}$ to the equivalence class represented by the arrow

$$I(x) \wedge y = \mathsf{p}(\mathbf{cod}(F(x, y)), \mathbf{cod}(Q(x))) : 1 \to \Delta\Gamma_1.$$

We can hence enunciate the following result which is proven in Maschio (2020) in Theorem 7.4

**Theorem 9.1.** $\mathbf{J}$ *is an isomorphism if* $\mathcal{T}$ *has the* **nEP***.*

# 10 A categorical reading of numerical existence property in constructive foundations

A category $\mathbb{C}$ with enough structure (e.g. a Heyting category or a topos) can be thought of as a mathematical universe in which one can perform "internal mathematics". The internal mathematics performed in different categories satisfy different principles and each category has its own internal groups, rings, preorders... As we have seen, categories can also have their own internal categories. These internal categories have a completely different nature than the category in which they live (e.g. trivially objects of the external category form a set or a class, while in general the objects of objects of its internal categories need not be sets or classes). These two different levels of categories can be thought of as representations of the two levels corresponding to metamathematics and mathematics.

In our case the external category, $\mathsf{DC}[\mathcal{T}]$, is in fact defined in the metamathematical level: its objects are equivalence (meta)classes of formal expressions (forming a countable (meta)set) and the same holds for arrows. The internalization $\Delta\Gamma[\mathcal{T}]$ is an internal category of $\mathsf{DC}[\mathcal{T}]$ of which the objects do not form a (meta)set, although they form a class of elements from the point of view of the theory $\mathcal{T}$. We could roughly say that $\mathsf{DC}[\mathcal{T}]$ is an ordinary category from the point of view of the metamathematician, while $\Delta\Gamma[\mathcal{T}]$ is an ordinary category from the point of view of the mathematician working in the theory $\mathcal{T}$.

[11] We denote with $\langle f, g \rangle$ the unique arrow determined by the definition of pullback.

We can think of $\Delta\Gamma[\mathcal{T}]$ as the best representation of the category $\mathsf{DC}[\mathcal{T}]$ that a mathematician working in $\mathcal{T}$ (and using only the tools of $\mathcal{T}$) can obtain. The metamathematician knows both the category $\mathsf{DC}[\mathcal{T}]$ and the internal category $\Delta\Gamma[\mathcal{T}]$ and he could ask himself whether from its point of view $\Delta\Gamma[\mathcal{T}]$ is a good representation of $\mathsf{DC}[\mathcal{T}]$. This, as we have seen, is done by means of a simulation of the notion of elements for objects of the category, that is by means of global elements.

Theorem 9.1 essentially means that whenever the numerical existence property is satisfied (which happens essentially only for some constructive theories), then the metamathematician can consider $\Delta\Gamma[\mathcal{T}]$ a perfect representation of $\mathsf{DC}[\mathcal{T}]$. This in some sense breaks some portions of the floor separating the mathematical level and the metamathematical one.

As we have already said, categorical language is not necessary to understand this, however it provides a clearer picture, a concrete representation in which syntactical aspects are organized in such a way that they can form structures which are more familiar to mathematicians.

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