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Is every pseudo-orbit of some homeomorphism near an exact orbit of a nearby homeomorphism?

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Abstract
We propose a relaxed notion of shadowing. In particular, for a homeomorphism f on a metric space, we ask whether every approximate orbit is near some actual orbit of some nearby system. We distinguish the cases where the nearby homeomorphism is close or arbitrarily close to f. We prove the relations between these notions and ordinary shadowing and present various examples. We finally discuss an application, a C^0-closing lemma for chain recurrent points of a homeomorphism on a topological manifold, not necessarily compact. This result leads to a characterization of explosion phenomena for various recurrent sets on such manifolds.

Mathematics Subject Classification Primary 37B65; Secondary 37B20

1 Introduction

Shadowing is a fundamental property of dynamical systems, both for its theoretical significance and for numerical applications. Roughly speaking, a dynamical system has the shadowing property if every approximate orbit (a so-called pseudo-orbit) is near some actual one. We refer to [19, 20, 24] for a detailed review on shadowing.

The first results involving pseudo-orbits were obtained by Anosov [2], Bowen [7] and Conley [10] in the qualitative study of dynamical systems. Moreover, the shadowing property is closely related to toplogical stability, since it is well-known that topological stability implies

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shadowing (see [24, 28]). From a theoretical point of view, it is worth recalling the relations between shadowing and genericity; even if shadowing is not C^1-generic [6], it is generic in the C^0-topology of homeomorphisms on a smooth manifold [26]. This result allows us to study shadowing from a purely topological dynamics point of view (see e.g. [1, 3, 13]), as in the present paper.

Clearly, the notion of pseudo-orbit plays a fundamental role in numerical applications. In fact, a numerical approximation of a map introduces a small error at each iterate. Since the errors can accumulate, we need to know whether the observed behavior can actually occur. If a system has the shadowing property then sufficiently precise pseudo-orbits can be shadowed. We refer for example to [11, 18, 21] for a discussion of this topic.

The (ordinary) notion of shadowing has been relaxed in different ways, by modifying the definition of pseudo-orbit, modifying the definition of an actual orbit shadowing a pseudo-orbit, or both. For example, we could require that the actual orbit be close to the pseudo-orbit only on average, or for "most" iterates; or that the sum of the errors in the pseudo-orbit be small. Specific examples can be found in [8, 14, 24]. The aim of the present paper is to present a relaxed notion of shadowing, essentially motivated from modeling of physical systems. In particular, for a homeomorphism f on a metric space, we ask whether every approximate orbit is near some actual orbit of some nearby system. We distinguish the cases of "nearby shadowing", when the homeomorphism with the ε-shadowing point is ε-close to f (see Definition 2), and "very nearby shadowing", when such a homeomorphism is arbitrarily close to f (see Definition 3). We prove the relations between these notions and ordinary shadowing and give various examples. As an application of nearby shadowing, we prove a C^0-closing lemma for chain recurrent points of homeomorphisms on topological manifolds, not necessarily compact. Finally, as a corollary, we obtain a characterization of explosion phenomena on such spaces.

2 Preliminaries

Let (X, d) be a metric space. We recall the notion of shadowing in the set $\mathcal{H}(X)$ of homeomorphisms, endowed with the metric

$$d_{\mathcal{H}}(f, g) = \sup_{x\in X} d(f(x), g(x)) + \sup_{x\in X} d(f^{-1}(x), g^{-1}(x))$$

$\forall f, g \in \mathcal{H}(X)$.

Definition 1 Let $f \in \mathcal{H}(X)$.

(a) Let $\delta > 0$. A bisequence $\xi = (x_i)_{-\infty}^{+\infty}$ of points in X is a (f, δ)-pseudo-orbit if

$$d(f(x_i), x_{i+1}) \le \delta \quad \forall i \in \mathbb{Z}\,.$$

(b) Let $\varepsilon > 0$. A point $y \in X$ (f, ε)-shadows a bisequence $\xi = (x_i)_{-\infty}^{+\infty}$ if

$$d(f^i(y), x_i) \le \varepsilon \quad \forall i \in \mathbb{Z}\,.$$

(c) The homeomorphism f has the (ordinary) shadowing property if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that every (f, δ)-pseudo-orbit is (f, ε)-shadowed by some $y \in X$.

The aim of the present paper is to generalize ordinary shadowing by asking whether every approximate orbit is near some actual orbit of some nearby system. The motivation is from modeling of physical systems. The assumptions for a model, no matter how precise, will not exactly match physical reality, and so the resulting map f will be only a close approximation to the actual underlying map. Thus it is reasonable to ask whether an f-pseudo-orbit can be shadowed by an orbit of a nearby map g.

In order to introduce precise definitions, let

$$V_\varepsilon(f) := \{g \in \mathcal{H}(X) : d_{\mathcal{H}}(f, g) \leq \varepsilon\}.$$

Definition 2 We say that f has the *nearby shadowing property* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any (f, δ)-pseudo-orbit ξ, there exists a $g \in V_\varepsilon(f)$ such that ξ is (g, ε)-shadowed by some $y \in X$.

Definition 3 We say that f has the *very nearby shadowing property* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any (f, δ)-pseudo-orbit ξ and any $\alpha > 0$, there exists a $g \in V_\alpha(f)$ such that ξ is (g, ε)-shadowed by some $y \in X$.

The difference is that for nearby shadowing, the homeomorphism with the ε-shadowing point is within ε of f, while for very nearby shadowing there must be an ε-shadowing homeomorphism arbitrarily close to f. It is clear that ordinary shadowing implies very nearby shadowing (just take $g = f$ in every case), and that very nearby shadowing implies nearby shadowing. We also notice that in [12, 17], the authors define a property similar to but stronger than nearby shadowing. They require that a single g be able to shadow all (f, δ)-pseudo-orbits, while for nearby shadowing, g can depend on the choice of pseudo-orbit. They also restrict the possible g's to nearby elements of a parametrized family (tent or quadratic maps, specifically).

The paper is organized as follows. We will see that nearby shadowing is strictly weaker than very nearby shadowing, and so strictly weaker than ordinary shadowing as well (Examples 4 and 5). Very nearby shadowing is equivalent to ordinary shadowing on compact spaces (Theorem 6), but strictly weaker on noncompact ones (Example 7). Moreover, Example 5 gives the opportunity to notice that having homeomorphisms arbitrarily close with the shadowing property is not sufficient to have nearby shadowing. Finally, in Theorem 9, we prove that nearby shadowing is a sufficient condition to obtain a C^0-closing lemma for chain recurrent points in a topological manifold, not necessarily compact. This theorem turns out to give a characterization of the so-called explosion phenomena (see Corollary 13).

3 Ordinary shadowing vs. (very) nearby shadowing

We start by showing that—in general—nearby shadowing is strictly weaker than very nearby shadowing.

Example 4 We construct the space X, a subspace of the interval $[-1, 1]$ with the usual topology, as follows. Start from

$$\begin{cases} x_n = 1 - \frac{1}{n} & \text{for } n \geq 2 \\ y_m = \frac{1}{m} & \text{for } m \geq 3 \end{cases}$$

then shift each point one unit to the left, setting

$$\begin{cases} x_{-n} = -1 + x_n = -\frac{1}{n} & \text{for } n \geq 2 \\ y_{-m} = -1 + y_m = -1 + \frac{1}{m} & \text{for } m \geq 3 \end{cases}$$

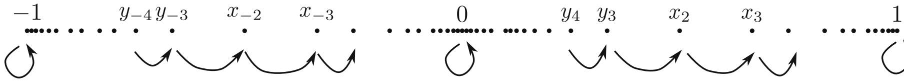


Fig. 1 Dynamics of f in Example 4

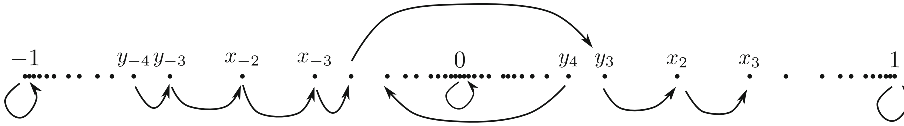


Fig. 2 Dynamics of g_k for $k = 4$ in Example 4

Let

$$X = \{-1\} \cup \{y_{-m}\}_{m=3}^{\infty} \cup \{x_{-n}\}_{n=2}^{\infty} \cup \{0\} \cup \{y_m\}_{m=3}^{\infty} \cup \{x_n\}_{n=2}^{\infty} \cup \{1\}$$

and define the homeomorphism $f \in \mathcal{H}(X)$ by fixing -1, 0, and 1 and moving every other point to its immediate right. We refer to Fig. 1.
Since X is discrete except near the fixed points, pseudo-orbits can differ from actual orbits only near the fixed points. Aside from the fixed points, all actual orbits either move from -1 to 0 or from 0 to 1. The only pseudo-orbits that cannot be f-shadowed are those starting near -1, pausing near 0 for an arbitrary number of iterates, and moving right to 1. We can, however, create a nearby homeomorphism such that a single point near to 0 on the left jumps over to the right, and then continues on an f-orbit. By choosing that point at the right distance from 0, we can make the g-orbit linger near 0 for the correct amount of time. More precisely, for $k \geq 4$ define $g_k \in \mathcal{H}(X)$ as follows:

$$\begin{cases} g_k(x_{-n}) = y_{n-1} & \text{for } n = k \\ g_k(y_n) = x_{-(n+1)} & \text{for } n = k \\ g_k = f & \text{otherwise.} \end{cases}$$

We refer to Fig. 2.
We see that

$$d_{\mathcal{H}}(f, g_k) = 2d(x_{-k}, y_{k-1}) = 2\left|\frac{1}{k} + \frac{1}{k-1}\right|,$$

and—by choosing $k \geq 4$ appropriately—we can make the g_k-orbit linger near the point 0 for the required amount of time to shadow the pseudo-orbit.
Finally, note that f does not have very nearby shadowing, since only one g_k will shadow a given pseudo-orbit.

We remark that it is possible to construct a similar example on the (entire) interval $[-1, 1]$ by taking f to be the time-one map of the flow generated by $\frac{dx}{dt} = \sin^2 \pi x$.

In the next example the nearby shadowing property does not hold at all.

Example 5 The previous homeomorphism can be turned into a homeomorphism on the circle by identifying the points -1 and 1. This example does not have nearby shadowing: a pseudo-orbit can visit each fixed point infinitely often, and spend an arbitrarily long time there, while that behavior is impossible for an orbit of a homeomorphism. Similarly, if we take f to be the identity (say on the circle), then we can construct pseudo-orbits that repeatedly reverse direction, which again is impossible for orbits of homeomorphisms.

According to [29], shadowing is generic for circle homeomorphisms. Thus, Example 5 also shows that having homeomorphisms arbitrarily close with the shadowing property is not sufficient to have nearby shadowing. In fact, if it were, then on any of the many spaces for which shadowing is dense in the set of homeomorphisms, every homeomorphism would have nearby shadowing.

We continue by proving that very nearby shadowing is equivalent to ordinary shadowing on compact spaces.

Theorem 6 *Let (X, d) be a compact metric space and let $f \in \mathcal{H}(X)$. Then f has the very nearby shadowing property if and only if it has the ordinary shadowing property.*

Proof We need to show that very nearby shadowing implies ordinary shadowing. Given $\varepsilon > 0$, let $\delta > 0$ be such that for any (f, δ)-pseudo-orbit $\xi = (x_i)_{-\infty}^{+\infty}$ and any $\alpha > 0$, there exists a $g_\alpha \in V_\alpha(f)$ such that ξ is (g_α, ε)-shadowed by some $y_\alpha \in X$. This means that

$$d(g_\alpha^i(y_\alpha), x_i) \leq \varepsilon \quad \forall i \in \mathbb{Z}. \tag{1}$$

Take a sequence such that

$$\lim_{j \to +\infty} \alpha_j = 0.$$

By compactness, we can assume (by passing to a subsequence if necessary) that there exists a point $y \in X$ such that

$$\lim_{j \to +\infty} y_{\alpha_j} = y.$$

Then for any fixed $i \in \mathbb{Z}$,

$$\lim_{j \to +\infty} g_{\alpha_j}^i(y_{\alpha_j}) = f^i(y).$$

Thus we have

$$d(f^i(y), x_i) = \lim_{j \to +\infty} d(g_{\alpha_j}^i(y_{\alpha_j}), x_i) \leq \lim_{j \to +\infty} \varepsilon = \varepsilon$$

where in the inequality we have used hypothesis (1). Therefore we have proved that the (f, δ)-pseudo-orbit $\xi = (x_i)_{-\infty}^{+\infty}$ is (f, ε)-shadowed by the point y, that is, f has the ordinary shadowing property. □

With the next example, we show that the equivalence stated in the previous theorem does not hold in noncompact spaces. The example is given by the doubling map $f(x) = 2x$ on $(0, +\infty)$.

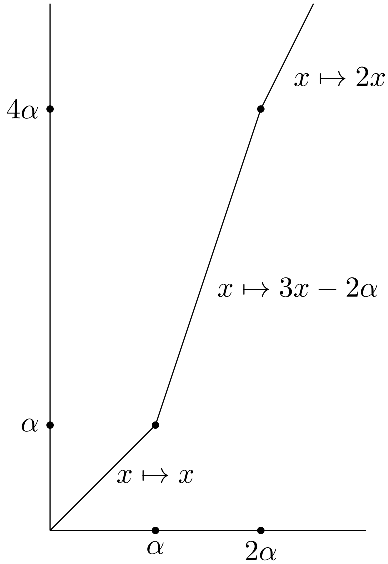
Example 7 On $X = (0, \infty)$, let consider $f(x) = 2x$. This map does not have the shadowing property since, for any $\delta > 0$, we can take the (f, δ)-pseudo-orbit $(\ldots, \delta, \delta, \ldots)$ which is not shadowed by any f-orbit. In such a case, pseudo-orbits that stay near 0 forever are the only ones that are not f-shadowable.[1] However, as proved here below, f does have very nearby shadowing. For any $\alpha > 0$, let define the homeomorphism g_α by

$$g_\alpha(x) = \begin{cases} x & \text{if } x \leq \alpha \\ 3x - 2\alpha & \text{if } \alpha \leq x \leq 2\alpha \\ 2x & \text{if } x \geq 2\alpha \end{cases}$$

(see Fig. 3).

[1] We observe that, on the extended domain $[0, \infty)$, the doubling map fixes the origin and therefore has the shadowing property.

Fig. 3 The homeomorphism g_α



So we have that

$$d_{\mathcal{H}}(f, g_\alpha) = |f(\alpha) - g_\alpha(\alpha)| + |f^{-1}(\alpha) - g_\alpha^{-1}(\alpha)| = 2\alpha.$$

We observe that an (f, δ)-pseudo-orbit $(x_i)_{-\infty}^{+\infty}$ that is not f-shadowable must stay near 0 and so must satisfy $x_i \leq \delta$ for all $i \in \mathbb{Z}$. Thus it is δ-shadowed by the g_δ-orbit of the point $y = \delta$, namely $(\ldots, \delta, \delta, \delta, \ldots)$.

One application of nearby shadowing is the following result, which is a kind of C^0-closing lemma (see [27] for an exhaustive discussion of this result). We need the notion of chain recurrent point for a homeomorphism (the definition applies more generally to continuous maps on a metric space).

Definition 8 Let $f \in \mathcal{H}(X)$. A point $x \in X$ is chain recurrent for f if for any $\varepsilon > 0$, there is a (finite) (f, ε)-pseudo-orbit starting and ending at x.

Theorem 9 *Let M be a topological manifold and let $f \in \mathcal{H}(M)$ have the nearby shadowing property. If x is chain recurrent for f, then for any $\alpha > 0$, there is a homeomorphism $g \in V_\alpha(f)$ such that x is a periodic point for g.*

Remark 10 We underline that there are versions of the C^0-closing lemma for chain recurrent points on compact topological manifolds, both for continuous maps [5, Lemma 5], and, in dimension two and above, for homeomorphisms and diffeomorphisms [5, Lemma 5 and proof of Lemma 4]. Moreover, in the non-compact case, we recall the following C^0-closing lemmas: for chain recurrent points of a C^r planar flow, whose fixed points are hyperbolic [23, Page 175] (see also [9]); for generalized recurrent points of a specific C^r planar flow [22, Theorem 1.9]; for non-wandering points of differential [16] and discrete dynamical systems [15, Theorem 3].

The main lines of the proof are inspired by the arguments for nonwandering points in [25, Theorem 10.4].

Proof The idea is simple: we find a nearby homeomorphism with a finite orbit that starts and ends near x, then perturb it so that it starts and ends at x. More precisely, assume that x is not already periodic. Fix $\alpha > 0$ and consider the closed ball $B := \mathrm{cl}(B(x, 3\alpha))$ of center x and radius 3α. Since B is compact and f, f^{-1} are continuous, $f|_B$ and $f^{-1}|_B$ are uniformly continuous. Consequently, we can take $0 < \alpha' < \frac{\alpha}{5}$ such that for every $a, b \in B$:

$$d(a, b) \leq \alpha' \Rightarrow \max\left(d(f(a), f(b)), d(f^{-1}(a), f^{-1}(b))\right) \leq \frac{\alpha}{5}. \tag{2}$$

Since f has nearby shadowing, there exists $0 < \delta < \alpha'$ such that for any (f, δ)-pseudo-orbit, there is a homeomorphism in $V_{\alpha'}(f)$ that α'-shadows it. Since x is chain recurrent for f, there is a finite (f, δ)-pseudo-orbit $(x_0 = x, x_1, \dots, x_n = x)$ based in x, and so we can find $g_1 \in V_{\alpha'}(f)$ and $y \in M$ such that the orbit segment $(y, g_1(y), \dots, g_1^n(y))$ α'-shadows it. This means—in particular—that

$$d(x, y) \leq \alpha' \quad \text{and} \quad d(x, g_1^n(y)) \leq \alpha'. \tag{3}$$

Let l and l' be the segments in the Euclidean space $\mathbb{R}^n$ ($n \geq 1$ is the dimension of M) with endpoints x, y and $x, g_1^n(y)$ respectively. By estimates (3), we can take an open neighborhood U of $l \cup l'$ contained in B. Moreover (truncating the orbit segment $(y, g_1(y). \dots g_1^j(y), \dots g_1^n(y))$ if necessary), we can choose U not containing $g_1^j(y)$, for every $j = 1, \dots, n-1$.

By Lemma 10.1 from [25], there exist a homeomorphism h_1 that moves x to y and a homeomorphism h_2 that moves $g_1^n(y)$ to x, both equal to the identity outside U and such that

$$d_{\mathcal{H}}(h_i, Id) \leq \alpha' \qquad i = 1, 2. \tag{4}$$

Finally, define

$$g := h_2 \circ g_1 \circ h_1.$$

Then $g^n(x) = x$. Moreover,

$$d_{\mathcal{H}}(f, g) \leq d_{\mathcal{H}}(f, g_1) + d_{\mathcal{H}}(g_1, g_1 \circ h_1) + d_{\mathcal{H}}(g_1 \circ h_1, \underbrace{h_2 \circ g_1 \circ h_1}_{=g}).$$

Clearly, $d_{\mathcal{H}}(f, g_1) \leq \alpha'$ since $g_1 \in V_{\alpha'}(f)$. Moreover, by (2) and (4):

$$d_{\mathcal{H}}(g_1, g_1 \circ h_1) \leq d_{\mathcal{H}}(g_1, f) + d_{\mathcal{H}}(f, f \circ h_1) + d_{\mathcal{H}}(f \circ h_1, g_1 \circ h_1) \leq \alpha' + \frac{\alpha}{5} + \alpha'.$$

In fact $f \circ h_1 = f$ outside B since $h_1 = id$ outside B and therefore:

$$d_{\mathcal{H}}(f, f \circ h_1) = d_{\mathcal{H}}(f|_B, f \circ h_1|_B) \leq \frac{\alpha}{5}$$

because $d_{\mathcal{H}}(h_i, Id) \leq \alpha'$ and (2) holds. Finally, by (4) again:

$$d_{\mathcal{H}}(g_1 \circ h_1, \underbrace{h_2 \circ g_1 \circ h_1}_{=g}) \leq \alpha'.$$

Thus

$$d_{\mathcal{H}}(f, g) \leq 4\alpha' + \frac{\alpha}{5} \leq \alpha$$

as desired. □

The C^0-closing lemma obtained in Theorem 9 provides an interesting characterization of explosion phenomena on a topological manifold (not necessarily compact). Let us start by introducing the notions of closed recurrent set and explosions. We refer to [5, Page 323] and [4, Definition 2.1].

Definition 11 We call $\mathcal{K}$ a closed recurrent set map if $\mathcal{K}$ is a map from the set of homeomorphisms $\mathcal{H}(X)$ to the power set of X, $\mathcal{P}(X)$:

$$\mathcal{H}(X) \ni f \mapsto \mathcal{K}(f) \in \mathcal{P}(X)\,,$$

so that for all f, $\mathcal{K}(f)$ is some closed set and

$$\mathrm{Per}(f) \subseteq \mathcal{K}(f) \subseteq \mathcal{CR}(f)$$

where $\mathrm{Per}(f)$ and $\mathcal{CR}(f)$ are respectively the set of periodic points and the set of chain recurrent points for f.

Examples of closed recurrent set maps are $f \mapsto \overline{\mathrm{Per}(f)}$, or $f \mapsto \mathcal{CR}(f)$, or the nonwandering set, or the strong chain recurrent set, or the generalized recurrent set. See, for example, [4, Section 2].

Definition 12 Let $f \in \mathcal{H}(X)$ and $\mathcal{K}$ be a closed recurrent set map such that $\mathcal{K}(f) \neq X$. We say that f does not permit $\mathcal{K}$-explosions if for any open neighborhood U of $\mathcal{K}(f)$ in X there exists a neighborhood V of f in $\mathcal{H}(X)$ such that if $g \in V$ then $\mathcal{K}(g) \subset U$.

The absence of explosions of the chain recurrent set, see e.g. [5, Theorem F] or [4, Theorem 2.2], and the C^0-closing lemma of Theorem 9 imply the following.

Corollary 13 *Let M be a topological manifold and $f \in \mathcal{H}(M)$ with the nearby shadowing property. Let $\mathcal{K}$ be a closed recurrent set map such that $\mathcal{K}(f) \neq M$. If f does not permit $\mathcal{K}$-explosions, then $\mathcal{K}(f) = \mathcal{CR}(f)$. Moreover, if M is compact, then f does not permit $\mathcal{K}$-explosions if and only if $\mathcal{K}(f) = \mathcal{CR}(f)$.*

Proof We follow the proof of [4, Proposition 3.1]. We start by assuming that f does not permit $\mathcal{K}$-explosions. That is, for any open neighborhood U of $\mathcal{K}(f)$ there exists a neighborhood V of f in $\mathcal{H}(M)$ such that for any $g \in V$ we have $\mathcal{K}(g) \subset U$. Thus

$$\mathcal{K}(f) \subseteq \mathcal{CR}(f) \subseteq \bigcup_{g\in V} \mathrm{Per}(g) \subseteq \bigcup_{g\in V} \mathcal{K}(g) \subseteq U\,,$$

where the second inclusion comes from Theorem 9, which can be applied since we suppose that f has the nearby shadowing property. Since the neighborhood U of $\mathcal{K}(f)$ is arbitrary, we conclude that $\mathcal{K}(f) = \mathcal{CR}(f)$.
Assume now that M is compact. Because of the compactness of M, if $\mathcal{K}(f) = \mathcal{CR}(f)$, then f does not permit $\mathcal{K}$-explosions since—by [5, Theorem F] or [4, Theorem 2.2]—f does not permit $\mathcal{CR}$-explosions. This concludes the proof. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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