

Integrability of close encounters in the spatial restricted three-body problem *

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Abstract

We extend to the spatial case a technique of integration of the close encounters formulated by Tullio Levi-Civita for the planar restricted three-body problem. We consider the Hamiltonian introduced in the Kustaanheimo-Stiefel regularization and construct a complete integral of the related Hamilton-Jacobi equation by means of a series convergent in a neighbourhood of the collisions with the primary or secondary body.

1 Introduction

A. Motivations. Two cornerstone models of physics are the two-body and the three-body problems, which share the same physics –Newton’s law of gravitation– but have a completely different mathematical development. While in the two-body problem all the orbits are classified according to the values of its constants of motion, in the three-body problem the constants carry out only a reduction to an integral manifold. Poincaré worked out the complexity of the dynamics on these manifolds; let us mention the deep fundamental results about the non-existence of global analytic first integrals and the non predictability due to homoclinic chaos [30]. An additional issue to the complexity of the three-body problem is due to the gravitational singularities, which not only involve collision solutions, but also close encounters. There is a rich recent literature about the complex dynamics of the three-body problem whenever close encounters are concerned, with strong links to the dynamics of comets, near-earth asteroids, and space mission design; see for example [32, 9, 13, 7, 19, 35, 20, 23, 12, 14, 4, 15, 16, 17, 29]. In particular, an individual close encounter is sufficient to produce a resonance

*Preprint version submitted for publication in Communications in Contemporary Mathematics, DOI: 10.1142/S0219199721500401, © World Scientific Publishing Company, <https://www.worldscientific.com/worldscinet/ccm>

transition and sequences of such transitions produce orbits which are unpredictable on time scales which are very short compared to the secular ones. These peculiar orbits of the three-body problem are typically observed for the comets of the Jupiter family, characterized by resonant transitions occurring at the close encounters with Jupiter (see paragraph E for more details). The numerical integration of these orbits is highly critical, due to the strong amplification of the separation of nearby solutions occurring at each close encounter. Therefore, it is essential to approximate analytically the arcs of solutions of the three-body problem passing close to a gravitational singularity as much precisely as possible. For the spatial case this problem was formulated by Tisserand (see the Remark, and paragraph E). Our paper is about this issue, which has been solved for the planar circular restricted three-body problem by Levi-Civita [25, 26].

More precisely, consider the circular restricted three-body problem defined by the motion of a body P of infinitesimally small mass in the gravitation field of two massive bodies P_1 and P_2 , the primary and secondary body respectively, which rotate uniformly around their common center of mass. In a rotating frame, the Hamiltonian of the problem is:

$$h(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2} + p_x y - p_y x - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}, \quad (1)$$

where $r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$ denote the distances of P from P_1, P_2 ; notice that as usual the units of mass, length and time have been chosen so that the masses of P_1 and P_2 are $1 - \mu$ and μ ($\mu \leq 1/2$) respectively, their coordinates are $(x_1, 0, 0) = (-\mu, 0, 0)$, $(x_2, 0, 0) = (1 - \mu, 0, 0)$ and their revolution period is 2π . Let us consider first the planar motions. Levi-Civita performed the integration of the close encounters in the planar circular restricted three-body problem through the introduction of a transformation which nowadays bears the name of Levi-Civita (LC hereafter) regularization. Explicitly:

$$x = x_j + u_1^2 - u_2^2 \quad (2)$$

$$y = 2u_1 u_2 \quad (3)$$

$$dt = r_j ds, \quad (4)$$

where (2), (3) are equivalent to the complex transformation:

$$x + iy = x_j + (u_1 + iu_2)^2,$$

while (4) is a parametrization of the physical time t into the proper time s . In the last part of the paper [26] Levi-Civita proved the existence of a local integral of the Hamilton-Jacobi equation of the Hamiltonian representing the planar circular restricted three-body problem regularized with (2), (3), (4), which we call the Levi-Civita Hamiltonian, in a neighbourhood of the collision singularity at P_j . The complete integral is constructed as a series analytic at $(u_1, u_2) = (0, 0)$, whose coefficients can be explicitly computed iteratively up to any arbitrary

large order. From this series, he proved the existence of a second first integral for the problem, independent of the Hamiltonian, defined in a neighbourhood of the collision singularity at P_j . Therefore, the local integration of planar close encounters has been solved by series¹. The extension of the Levi-Civita regularization to the spatial restricted three-body problem has been done by Kustaanheimo and Stiefel [21, 22] many decades after Levi-Civita, but the local integrability of the regularized Hamiltonian, which we call the Kustaanheimo-Stiefel Hamiltonian, at collisions has never been addressed. Here, our purpose is precisely to extend to the spatial case the point of view followed by Levi-Civita, thus offering a complete integrability of the spatial problem in a neighbourhood of the collision singularities.

Remark. Obviously, the local integrability of close encounters does not mean that the three-body problem is integrable. These local integrations are interesting because they are defined in a neighbourhood of a collision set, and allow to solve the (open) problem of close encounters, which we formulate as follows². Let σ be arbitrarily small; for any motion $(x(t), y(t), z(t))$ entering the ball $B_{(x_j, 0, 0)}(\sigma) \subset \mathbb{R}^3$ (centered at $(x_j, 0, 0)$ of radius σ) at time t_0 and leaving it at time t_1 , express $(x(t_1), y(t_1), z(t_1), p_x(t_1), p_y(t_1), p_z(t_1))$ as an explicit function of $(x(t_0), y(t_0), z(t_0), p_x(t_0), p_y(t_0), p_z(t_0))$. We recall that, while there is a rich literature about the collision manifolds of N -body problems, the problem of close encounters is of primal importance for astronomical applications such as the dynamics of comets, of near-Earth asteroids, and modern space mission design (see paragraph E of this Introduction for a detailed discussion).

B. Statement of the main result. The regularizations of the equations of motion in the spatial problems are more complicate than those of the planar problem, see for example, [27]. As for the Levi-Civita regularization, the Kustaanheimo-Stiefel regularization (KS hereafter) is defined by the introduction of a transformation on the space variables and by a time-reparametrization; but the KS space transformation is more complicate than the LC space transformation, since it is a map from a space of redundant variables u_1, u_2, u_3, u_4 to a space of Cartesian variables q_1, q_2, q_3 . In fact, for an algebraic reason that we better explain below,

¹As a matter of fact, Levi-Civita constructed the solution of the Hamilton-Jacobi equation only for the collision singularity at P_1 . Nevertheless, Levi-Civita's argument is valid also in a neighbourhood of the singularity at the secondary body P_2 with some relevant differences. For example, we notice that while the series at P_1 is analytic also in $\mu = 0$, the series at P_2 is not.

²This problem appeared in the literature, with a slightly different formulation, already in an earl paper by Tisserand about the dynamics of comets: “*Le problème de la détermination des grandes perturbations d’une comète par Jupiter revient donc au suivant, qui est très simple, au moins par son énoncé: On donne les éléments elliptiques ou paraboliques d’une comète, $a_0, e_0, \varpi_0, \dots$, au moment où elle pénètre dans la sphère d’activité de Jupiter. Il faut en déduire les éléments $a_1, e_1, \varpi_1, \dots$, au moment où elle en sort.*” [33], pag. 243. For more details about classic and modern astronomical motivations to the problem of close encounters we refer to paragraph E of this Introduction).

the generalization of the space transformation (2), (3) to the spatial case is related with the extension of complex numbers to a space of quaternions. Precisely, following [21, 22], we introduce the projection map:

$$\begin{aligned} \pi : \mathbb{R}^4 &\longrightarrow \mathbb{R}^3 \\ (u_1, u_2, u_3, u_4) &\longmapsto \pi(u_1, u_2, u_3, u_4) = (q_1, q_2, q_3), \end{aligned} \quad (5)$$

where $(q_1, q_2, q_3, 0) = A(u)u$, and:

$$A(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \quad (6)$$

is a matrix which plays a central role in the KS regularization, it is a linear homogeneous function of u_1, \dots, u_4 and satisfies $A(u)A^T(u) = |u|^2 \mathcal{I}$. Matrices with such properties exist only for $n = 1, 2, 4, 8$ (see [18]; the relation to quaternions is well exploited in [21, 22, 3, 36, 8]). For example, for $n = 2$ the matrix:

$$A_2(u) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$$

defines the transformation (2),(3) through $(x - x_j, y) = A_2(u)u$, and the lack of such a matrix for $n = 3$ is the reason for the definition of the KS regularization in a 4-dimensional space. Then, for any motion in the KS variables we introduce the parametrization of time (4); notice that, again, we have $r_j = |u|^2$. The space and time transformations (4), (5) have been used to represent the regularized equations of motions of the spatial circular restricted three-body problem in various forms (see [3] for a review of the subject, and Section 2 for a revisitiation).

To better accomplish the technique of integration introduced in [26] we first perform the phase-space translation

$$X = x - x_j \quad , \quad Y = y \quad , \quad Z = z \quad , \quad P_X = p_x \quad , \quad P_Y = p_y - x_j \quad , \quad P_Z = p_z, \quad (7)$$

conjugating h to the Hamiltonian (to fix ideas we present all these computations for $j = 2$, so that the reference system defined above will be called planetocentric):

$$\begin{aligned} H(X, Y, Z, P_X, P_Y, P_Z) &= \frac{P_X^2 + P_Y^2 + P_Z^2}{2} + P_X Y - P_Y X - \frac{\mu}{\sqrt{X^2 + Y^2 + Z^2}} \\ &- (1 - \mu) \left(\frac{1}{\sqrt{(X+1)^2 + Y^2 + Z^2}} - 1 + X \right) - (1 - \mu) - \frac{(1 - \mu)^2}{2}, \end{aligned} \quad (8)$$

the constant terms being kept for comparison with the values of the original Hamiltonian h . The KS regularization is obtained from the space transformation (5) with $(q_1, q_2, q_3) = (X, Y, Z)$ and, in Section 2, we show that it can be formulated in the following Hamiltonian form:

$$\mathcal{K}(u, U; E) = \frac{1}{8} |U - b_{(0,0,1)}(u)|^2 - \frac{1}{2} |u|^2 |(0, 0, 1) \times \pi(u)|^2 - |u|^2 E_\mu - \mu$$

$$- (1 - \mu) |u|^2 \left(\frac{1}{|\pi(u) + (1, 0, 0)|} - 1 + \pi(u) \cdot (1, 0, 0) \right), \quad (9)$$

where $U = (U_1, U_2, U_3, U_4)$ denote the conjugate momenta to $u = (u_1, u_2, u_3, u_4)$, the vector potential $b_\omega(u)$ (in (9) we have $\omega = (0, 0, 1)$), is defined by

$$b_\omega(u) = 2A^T(u)\Lambda_\omega A(u)u, \quad \Lambda_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (10)$$

and:

$$E_\mu = E + (1 - \mu) + \frac{(1 - \mu)^2}{2}.$$

The Hamiltonian $\mathcal{K}(u, U; E)$ is a regularization of the spatial three-body problem at P_2 . This means that the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}(U, u; E)$ with initial conditions satisfying:

- (i) $u(0) \neq 0$;
- (ii) $l(u(0), U(0)) = 0$, where

$$l(u, U) = u_4 U_1 - u_3 U_2 + u_2 U_3 - u_1 U_4 \quad (11)$$

is called the *bilinear form*;

- (iii) $\mathcal{K}(u(0), U(0); E) = 0$,

are conjugate, for s in a small neighbourhood of $s = 0$, via equations (4), (5) to solutions $(X(t), Y(t), Z(t), P_X(t), P_Y(t), P_Z(t))$ of the Hamilton equations of (8).

Our integration of the close encounters in the spatial circular restricted three-body problem is established on the construction of a complete integral $W(u, \nu; E, \mu)$ of the Hamilton-Jacobi equation of $\mathcal{K}(u, U; E)$, defined for all the values of the parameters $\nu = (\nu_1, \dots, \nu_4)$ in a neighbourhood of the sphere $|\nu| = 1$, and analytic in a neighbourhood of $u = 0$. Our main result is the following:³

Theorem 1. *For fixed values of E_* and of $\mu_* > 0$, there exists a complete integral $W(u, \nu; E, \mu)$ of the Hamilton-Jacobi equation:*

$$\mathcal{K} \left(u, \frac{\partial W}{\partial u}(u, \nu; E, \mu); E \right) = \mu(|\nu|^2 - 1) \quad (12)$$

depending on the four parameters ν and on E, μ , which is analytic for E, μ, ν in the set:

$$\{|\mu - \mu_*| < a, \quad |E - E_*| < b, \quad ||\nu| - 1| < c\}$$

³Theorem 1, and Theorem 2 below, have been announced in [5].

and u in the (complex) ball:

$$\{u \in \mathbb{C}^4 : |u| < d\}$$

with suitable constants $a, b, c, d > 0$ (depending only on E_*, μ_*). The coefficients of the Taylor expansions of W with respect to the variables u can be explicitly computed iteratively to any arbitrary order; in particular we have:

$$W = \sqrt{8\mu} \sum_{j=1}^4 \nu_j u_j + \mathcal{O}_3(u). \quad (13)$$

The complete integral W of the Hamilton-Jacobi equation will be used to define a canonical transformation:

$$(n, \nu) = (\hat{n}(u, U; E, \mu), \hat{\nu}(u, U; E, \mu))$$

through the system

$$U_\ell = \frac{\partial W}{\partial u_\ell}(u, \nu; E, \mu), \quad \ell = 1, \dots, 4 \quad (14)$$

$$n_\ell = \frac{\partial W}{\partial \nu_\ell}(u, \nu; E, \mu), \quad \ell = 1, \dots, 4. \quad (15)$$

conjugating $\mathcal{K}(u, U; E)$ to the Hamiltonian:

$$\hat{\mathcal{K}}(n, \nu) = \mu(|\nu|^2 - 1).$$

Therefore, the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}(u, U; E)$ are obtained from the equation:

$$(n(0) + 2\mu \nu(0)s, \nu(0)) = (\hat{n}(u(s), U(s); E, \mu), \hat{\nu}(u(s), U(s); E, \mu)). \quad (16)$$

Formula (16) provides all the solutions of the spatial circular restricted three-body problem in a neighbourhood of the collision set \mathcal{C}_2 .

C. On the proof of Theorem 1. The proof of Theorem 1 will be achieved through several steps: first, a geometric analysis of the KS Hamiltonian is needed to identify the parameters ν_1, \dots, ν_4 , providing the conserved momenta of Hamiltonian $\hat{\mathcal{K}}(n, \nu)$; second, an analytic part based on the Cauchy-Kowaleski theorem is used to provide analytic solutions to the Hamilton-Jacobi equation. The geometric analysis is the original heart of our proof and is completely new with respect to the work of Levi-Civita. In fact, while the geometric part required by the planar case is rather simpler, for the spatial case we need to represent in the space of the variables (u_1, \dots, u_4) the rotations of the euclidean space (q_1, q_2, q_3) with matrices which are in $\text{SO}(4)$ and leave invariant the bilinear form. Moreover, the subgroup of $\text{SO}(4)$ that we obtain this way must be parameterized by parameters ν_1, \dots, ν_4 , constrained to the unit sphere, such that the inversion of the

system of equations (14), (15) has no singularities (which arise if, for example, we parameterize the subgroup with three Euler angles). The analytic part is instead the argument that we extend from the planar problem, with an additional care for the global definition of the family of particular solutions found.

D. Complete integrability in the Cartesian phase-space. An additional interesting question concerns the existence of Cartesian local first integrals $F(x, y, z, p_x, p_y, p_z)$ independent of $h(x, y, z, p_x, p_y, p_z)$ defined in a set $\mathcal{B} \setminus \mathcal{C}_j$, where \mathcal{C}_j is a collision set:

$$\mathcal{C}_j = \{(x, y, z, p_x, p_y, p_z) : (x, y, z) = (x_j, 0, 0)\}, \quad j = 1, 2,$$

and \mathcal{B} is a neighbourhood of \mathcal{C}_j .⁴ First, we remark that the existence of Cartesian first integrals is not granted a priori from the existence of first integrals of the KS Hamiltonian; for example $|\nu|^2$ and $l(n, \nu)$ do not provide, with evidence, Cartesian first integrals. But neither the momenta ν_ℓ provide Cartesian first integrals. The deep reason is that the map π has not a global smooth inversion defined in a neighbourhood of $q = (x - x_j, y, z) = 0$ (see [17], where a similar problem is addressed for the global definition of chaos indicators for the spatial three body problem), so it can happen that functions $F(n, \nu)$ which are first integrals for $\hat{\mathcal{K}}$ do not define global Cartesian smooth functions in any neighbourhood of the collision set \mathcal{C}_j . Precisely, while we are not able to define Cartesian representatives of ν_ℓ , $\ell = 1, \dots, 4$, which are smooth in a neighbourhood of \mathcal{C}_j , we find that the functions:

$$\begin{aligned} N_X &= \nu_1 n_4 - \nu_4 n_1 \\ N_Y &= \frac{1}{2}(\nu_1 n_3 - n_1 \nu_3 + n_2 \nu_4 - n_4 \nu_2) \\ N_Z &= \frac{1}{2}(\nu_1 n_2 - n_1 \nu_2 + n_4 \nu_3 - n_3 \nu_4) \end{aligned} \quad (17)$$

are first integrals and have Cartesian representatives $\mathcal{N}_X, \mathcal{N}_Y, \mathcal{N}_Z$ globally defined and smooth in a neighbourhood of the collision sets. We consider the set of three first integrals:

$$\left(H, \mathcal{N}^2 := \mathcal{N}_X^2 + \mathcal{N}_Y^2 + \mathcal{N}_Z^2, \mathcal{N}_Z \right)$$

We notice that, since $\mathcal{N}^2, \mathcal{N}_Z$ are first integrals, we have:

$$\{H, \mathcal{N}^2\} = 0, \quad \{H, \mathcal{N}_Z\} = 0.$$

The Poisson bracket $\{H, \mathcal{N}_Z\} = 0$ is sufficient to grant the complete integrability of the planar circular restricted three-body problem in a neighbourhood of its collision singularities. It remains to understand if even the spatial problem is

⁴Through this paper, whenever we will refer to a subset of the Cartesian phase-space which is a neighbourhood of the collision set, we will precisely refer to a set $\mathcal{B} \setminus \mathcal{C}_j$, where \mathcal{B} is a neighbourhood of the collision set.

completely integrable. At this regard, we notice that in the space of the variables n, ν , we have:

$$\{N^2, N_Z\} = l(n, \nu)a(n, \nu) \quad , \quad N^2 = N_X^2 + N_Y^2 + N_Z^2, \quad (18)$$

so that the two integrals are in involution on the level set $l(n, \nu) = 0$. The atypical Poisson bracket in (18) seems a rule for the KS regularization. For example, the elementary Poisson brackets of $q = \hat{q}(u), p = \hat{p}(u, U)$ defined from $\hat{q}(u) = \pi(u)$, $(\hat{p}_1, \hat{p}_2, \hat{p}_3, 0) = \frac{1}{2|u|^2}A(u)U$, satisfy:

$$\{\hat{q}_i, \hat{p}_j\} = \delta_{ij}, \quad \{\hat{q}_i, \hat{q}_j\} = 0, \quad \{\hat{p}_i, \hat{p}_j\} = l(u, U)\phi_{ij}(u, U), \quad i, j = 1, 2, 3. \quad (19)$$

From (18) and (19) we will prove the following:

Theorem 2. *The set of first integrals $(H, \mathcal{N}^2, \mathcal{N}_Z)$ is complete.*

E. Astronomical motivations: close encounters. Astronomers were faced with the problem of close encounters few years after the publication of Newton's *Philosophiae Naturalis Principia Mathematica*, to understand the motion of comets. Comets are visible from Earth when they are close to the Sun, therefore apparitions at different epochs correspond to the same comet if they are linked by the same orbit. While Newton's theory allowed Halley and Clairaut to link the former apparitions of 1531, 1607, 1682 of Halley's comet and predict its return for 1759, the dramatic effect of close encounters became more evident with the discovery in 1770 of the Lexell's comet. Despite the orbit of Lexell's comet was elliptic with period of about 5.6 years, the comet was not seen in the next 10 years (not either afterwards). Lexell recognized that the comet had likely never had been seen before, because of a close encounter with Jupiter in 1767, and maybe it would be never be seen again because of another close encounter estimated for 1779. By studying the possible orbits of the comet after the latter close encounter, Le Verrier found that the future orbit of the comet was unpredictable [24]. The method used by Le Verrier was very modern, since he tried to reproduce the orbit of the comet by linking the orbits of the two different Sun-comet and Jupiter-comet two-body problems; for deep enough close encounters with Jupiter, the linkage expands the small experimental errors in the measure of the orbital parameters to complete indetermination. Tisserand, who was among the first ones to remark the need of a mathematical explanation of the problem (see footnote 2), found an approximate integral of motion constraining the possible large variations of the orbital parameters [33] as the effect of a close encounter, but still the problem remained highly undetermined. More recently, Öpik ([28], see [35] for a recent revisitation) developed Le Verrier's method and formulated a more refined predictive theory of close encounters which, despite the good agreement with numerical integrations, still needs a mathematical justification (see also [34]). The short-term indeterminism in the orbit of Lexell comet is not an exception, but is typical of comets having fast close encounters with

the planets. For example, a deep close encounter with Jupiter which occurred in 1959 is responsible of the indetermination of the past orbit of comet⁵ 67P which can be obtained from backward numerical integrations, for epochs exceeding few centuries [15, 16]. Modern topics of cometary dynamics where close encounters are relevant raised in the investigations about the formation of solar system. In the modern picture of the Solar system there is a population of icy bodies outside Neptune's orbit, of relatively small eccentricities and inclinations, which is potentially a reservoir of periodic comets (see, for example, [11] and references therein). This picture poses the mathematical problem of proving that the orbital resonances and chains of close encounters with the giant planets reduce the perihelion distance of these icy objects from values larger than Neptune's aphelion to distances shorter than 3 AU, where the body shows its cometary activity. Close encounters are important also for astrodynamics, since they are used in the technique of gravity assist to change the energy of a spacecraft: interplanetary missions to the giant planets have been possible only thanks to close encounters with the planets. The results that we prove in this paper could be exploited in these problems. In fact, by considering applications of close encounters, we notice that we have two relevant spheres: a sphere B where the gravitational interaction with the body P_2 is dominant (for example, this can be identified using the Hill's sphere), and a smaller sphere $B_2(\sigma) \subset B$ where the close encounters are integrated by series. So, we have a spherical neighbourhood of the Planets in the physical space, with radius depending on the energy of the incoming orbit, where one can compute the close encounter, or the incoming and outgoing orbits, with any needed precision. The crossing of the interspace $B \setminus B_2(\sigma)$ between the two spheres and of the region complementary to B where the gravitational interaction with P_1 is dominant needs to be studied with perturbation methods, such as those used in [9, 13].

The paper is organized as follows. In Section 2 we revisit the definition of the KS transformation with respect to any spatial frame centered at P_j and arbitrarily rotated; Section 3 is devoted to the identification of suitable parameters for the definition of a complete integral of the Hamilton-Jacobi equation of $\mathcal{K}(u, U : E)$; in Section 4 we prove the existence of particular solutions of the Hamilton-Jacobi equation; in Section 5 we prove the existence of a complete integral, thus proving Theorem 1, and we use it to define a canonical transformation; in Section 6 we discuss the existence of Cartesian first integrals and we prove Theorem 2; in Appendix 1 we revisit the integration of close encounters in the planar three-body problem done by Levi-Civita in [26]; in Appendix 2 we review a basic formulation of the Cauchy-Kowaleski theorem.

⁵Comet 67P/Churyumov-Gerasimenko has been the target of the recent mission Rosetta.

2 The KS Hamiltonian revisited

In order to solve the problem of close encounters in the spatial case we need to introduce the KS transformation with respect to any spatial frame centered at P_j and arbitrarily rotated, while in the usual KS transformation the Cartesian coordinates are referred to a rotating spatial frame with x axis containing the primaries P_1, P_2 and the z axis orthogonal to their orbit plane. In addition, we consider also an arbitrary scaling of the coordinates by a factor $\lambda > 0$; the scaling will be needed to define the parameters of the solutions of the Hamilton-Jacobi equation.

The Lagrangian formulation in the Cartesian variables. We start from the Lagrange function of the spatial circular restricted three-body problem:

$$\begin{aligned}
L_C(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{y}x - \dot{x}y + \frac{1}{2}(x^2 + y^2) \\
&\quad + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2 + z^2}} \\
&= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + (\dot{x}, \dot{y}, \dot{z}) \wedge (0, 0, 1) \cdot (x, y, z) + \frac{1}{2} |(0, 0, 1) \wedge (x, y, z)|^2 \\
&\quad + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2 + z^2}} \tag{20}
\end{aligned}$$

and, for any arbitrary matrix $\mathcal{R} \in SO(3)$ and any $\lambda > 0$, we define the coordinates transformation:

$$(x - x_j, y, z) = \lambda \mathcal{R} q, \tag{21}$$

where $q = (q_1, q_2, q_3)$ and (to fix ideas) $x_j = x_2 = 1 - \mu$, which extends to the transformation on the generalized velocities:

$$(\dot{x}, \dot{y}, \dot{z}) = \lambda \mathcal{R} \dot{q}. \tag{22}$$

By transforming the Lagrangian L_C with (21), (22), and by dropping the constants as well as the terms which are independent on the q_i and linear in the \dot{q}_i (which do not contribute to the Lagrange equations) we obtain the Lagrangian:

$$\begin{aligned}
L(q, \dot{q}) &= \frac{1}{2} \lambda^2 |\dot{q}|^2 + \lambda^2 (\dot{q} \wedge \omega) \cdot q + \frac{1}{2} \lambda^2 |\omega \wedge q|^2 + \frac{\mu}{\lambda |q|} \\
&\quad + (1 - \mu) \left(\frac{1}{|\lambda q + e|} + \lambda q \cdot e \right), \tag{23}
\end{aligned}$$

where $\omega = \mathcal{R}^T(0, 0, 1)$, $e = \mathcal{R}^T(1, 0, 0)$.

The redundant variables $\mathbf{u}_1, \dots, \mathbf{u}_4$. Redundant variables are easily introduced in the Lagrangian formalism (see, for example, [1]). As a first step, we compute the function:

$$\hat{L}(u, \dot{u}) = L\left(\pi(u), \frac{\partial \pi}{\partial u}(u)\dot{u}\right)$$

using the formulas:

$$\begin{aligned} (q_1, q_1, q_3, 0) &= A(u)u \\ (\dot{q}_1, \dot{q}_2, \dot{q}_3, 0) &= A(\dot{u})u + A(u)\dot{u} = 2A(u)\dot{u} - 2(0, 0, 0, l(u, \dot{u})), \end{aligned}$$

where $l(u, \dot{u})$ is the bilinear form defined in (11). We obtain:

$$\begin{aligned} \hat{L}(u, \dot{u}) &= 2\lambda^2 |u|^2 |\dot{u}|^2 - 2\lambda^2 l(u, \dot{u})^2 + \lambda^2 b_\omega(u) \cdot \dot{u} \\ &+ \frac{1}{2}\lambda^2 |\omega \wedge \pi(u)|^2 + \frac{\mu}{\lambda |u|^2} + (1 - \mu) \left(\frac{1}{|\lambda \pi(u) + e|} + \lambda \pi(u) \cdot e \right), \end{aligned} \quad (24)$$

where $b_\omega(u)$, is the vector potential already defined in (10).

Let us compare the solutions of the Lagrange equations of $\hat{L}(u, \dot{u})$, which we write in the form:

$$[\hat{L}]_i(u, \dot{u}, \ddot{u}) = 0 \quad , \quad \forall i = 1, \dots, 4$$

where:

$$[\hat{L}]_i(u, \dot{u}, \ddot{u}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{u}_i} - \frac{\partial L}{\partial u_i},$$

with the solutions of the Lagrange equations of $L(q, \dot{q})$, which we write in the form:

$$[L]_j(q, \dot{q}, \ddot{q}) = 0 \quad , \quad \forall j = 1, 2, 3$$

where:

$$[L]_j(q, \dot{q}, \ddot{q}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}.$$

Proposition 1. *If $u(t)$ is a solution of the Lagrange equations of $\hat{L}(u, \dot{u})$ with $u(0) \neq 0$, then $q(t) = \pi(u(t))$ is a solution of the Lagrange equations of L as soon as $u(t) \neq 0$.*

Proof of Proposition 1. For any smooth curve $u(t)$, we have:

$$[\hat{L}](u(t), \dot{u}(t), \ddot{u}(t)) = \left(\frac{\partial \pi}{\partial u} \right)^T [L](\pi(u(t)), \frac{d}{dt} \pi(u(t)), \frac{d^2}{dt^2} \pi(u(t)))$$

where $[\hat{L}] \in \mathbb{R}^4$, $[L] \in \mathbb{R}^3$ are the vectors of components $[\hat{L}]_i, [L]_j$ respectively.

Since for $u \neq 0$, the Kernel of the matrix $\frac{\partial \pi}{\partial u}^T$ contains only the vector $(0, 0, 0)$, any solution $u(t)$ of the Lagrange equations of \hat{L} (i.e. satisfying $[\hat{L}] = (0, 0, 0, 0)$)

projects to a solution $q(t) = \pi(u(t))$ of the Lagrange equations of L as soon as $u(t) \neq 0$. \square

The Legendre transform defined by \hat{L} is not invertible, since the quadratic form $2|u|^2|\dot{u}|^2 - 2l(u, \dot{u})^2$ in the generalized velocities \dot{u} is degenerate; therefore the definition of the Hamiltonian formalism is more tricky than usual. To remove the degeneracy we consider the modified Lagrangian:

$$\begin{aligned} \mathcal{L}(u, \dot{u}) &= \hat{L}(u, \dot{u}) + 2\lambda^2 l(u, \dot{u})^2 = 2\lambda^2 |u|^2 |\dot{u}|^2 + \lambda^2 b_\omega(u) \cdot \dot{u} \\ &+ \frac{1}{2}\lambda^2 |\omega \wedge \pi(u)|^2 + \frac{\mu}{\lambda |u|^2} + (1 - \mu) \left(\frac{1}{|\lambda\pi(u) + e|} + \lambda\pi(u) \cdot e \right), \end{aligned} \quad (25)$$

whose Legendre transform:

$$U = \frac{\partial \mathcal{L}}{\partial \dot{u}} = \lambda^2 (4|u|^2 \dot{u} + b_\omega(u)), \quad (26)$$

where $U = (U_1, U_2, U_3, U_4)$ denote the momenta conjugate to $u = (u_1, u_2, u_3, u_4)$, is non-degenerate for $u \neq 0$.

Proposition 2. *If $u(t)$ is a solution of the Lagrange equations of $\mathcal{L}(u, \dot{u})$ with initial conditions $u(0), \dot{u}(0)$ satisfying $u(0) \neq 0$ and $l(u(0), \dot{u}(0)) = 0$, then it is also a solution of the Lagrange equations of $\hat{L}(u, \dot{u})$ as soon as $u(t) \neq 0$.*

Before proving the Proposition, we remark that the Lagrangian $\mathcal{L}(u, \dot{u})$ is invariant with respect to the one-parameter family of transformations:

$$u \longmapsto \mathcal{S}_\alpha^0 u \quad (27)$$

where $\mathcal{S}_\alpha^0 \in SO(4)$ is defined by

$$\mathcal{S}_\alpha^0 = \begin{pmatrix} \cos \alpha & 0 & 0 & -\sin \alpha \\ 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 \\ \sin \alpha & 0 & 0 & \cos \alpha \end{pmatrix}, \quad (28)$$

whose orbits define the fibers of the projection π , i.e. $\pi(\mathcal{S}_\alpha^0 u) = \pi(u)$ for all α . Precisely, for all u, \dot{u}, α , we have:

$$\mathcal{L}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \dot{u}) = \mathcal{L}(u, \dot{u}).$$

As a consequence, by Noether's theorem, the function:

$$J(u, \dot{u}) = \left(\frac{d}{d\alpha} \mathcal{S}_\alpha^0 u \right) \Big|_{\alpha=0} \cdot \frac{\partial \mathcal{L}}{\partial \dot{u}} = (-u_4, u_3, -u_2, u_1) \cdot (4\lambda^2 |u|^2 \dot{u} + \lambda^2 b_\omega(u))$$

is a first integral for the Lagrange equations of \mathcal{L} . Moreover, since: $(-u_4, u_3, -u_2, u_1) \cdot b_\omega(u)$ vanishes identically, then:

$$J_o(u, \dot{u}) = |u|^2 l(u, \dot{u})$$

is a first integral for the Lagrange equations of \mathcal{L} .

Proof of Proposition 2. Let us consider a solution $u(t)$ of the Lagrange equations of \mathcal{L} with $u(0) \neq 0$ and $l(u(0), \dot{u}(0)) = 0$. Since $J_0(u, \dot{u})$ is constant along the solution, as soon as $u(t) \neq 0$ we have also $l(u(t), \dot{u}(t)) = 0$, as well as $l(u(t), \ddot{u}(t)) = 0$.

We claim that $u(t)$ solves also the Lagrange equations of $\hat{\mathcal{L}}$. In fact, we have:

$$\begin{aligned} [\hat{\mathcal{L}}]_i &= [\mathcal{L}]_i - 2\lambda^2 \left(\frac{d}{dt} \frac{\partial}{\partial \dot{u}_i} l^2(u, \dot{u}) - \frac{\partial}{\partial u_i} l^2(u, \dot{u}) \right) \\ &= [\mathcal{L}]_i - 4\lambda^2 \left(\frac{d}{dt} \left(l(u, \dot{u}) \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) \right) - l(u, \dot{u}) \frac{\partial}{\partial u_i} l(u, \dot{u}) \right) \\ &= [\mathcal{L}]_i - 4\lambda^2 \left(l(u, \ddot{u}) \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) + l(u, \dot{u}) \frac{d}{dt} \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) - l(u, \dot{u}) \frac{\partial}{\partial u_i} l(u, \dot{u}) \right) \end{aligned}$$

and when computed along the solution $u(t)$ (so that $[\mathcal{L}]_i = 0$, $l(u, \dot{u}) = 0$, $l(u, \ddot{u}) = 0$) we have also:

$$[\hat{\mathcal{L}}]_i(u(t), \dot{u}(t), \ddot{u}(t)) = 0.$$

□

Finally, we remark that for any initial condition $(q(0), \dot{q}(0))$ with $q(0) \neq 0$ we have the freedom of choosing the initial conditions $(u(0), \dot{u}(0))$ satisfying:

$$\pi(u(0)) = q(0) \quad , \quad \frac{\partial \pi}{\partial u}(u(0)) \dot{u}(0) = \dot{q}(0) \quad , \quad l(u(0), \dot{u}(0)) = 0.$$

In fact, if $l(u(0), \dot{u}(0)) \neq 0$, since the Kernel of $\frac{\partial \pi}{\partial u}(u)$ is generated by $\hat{u} = (u_4, -u_3, u_2, -u_1)$ we have the freedom of adding to $\dot{u}(0)$ a vector $\xi \hat{u}$ and to select $\xi \in \mathbb{R}$ so that:

$$l(u(0), \dot{u}(0) + \xi \hat{u}) = l(u(0), \dot{u}(0)) + \xi |u(0)|^2 = 0.$$

The KS Hamiltonian. The Legendre transform (26), which is invertible for all $|u| \neq 0$, conjugates the Lagrangian system defined by \mathcal{L} to the Hamiltonian system with Hamilton function:

$$\begin{aligned} K(u, U) &= \frac{1}{8\lambda^2 |u|^2} |U - \lambda^2 b_\omega(u)|^2 - \frac{1}{2} \lambda^2 |\omega \wedge \pi(u)|^2 \\ &\quad - \frac{\mu}{\lambda |u|^2} - (1 - \mu) \left(\frac{1}{|\lambda \pi(u) + e|} + \lambda \pi(u) \cdot e \right), \end{aligned} \quad (29)$$

where $U = (U_1, U_2, U_3, U_4)$ are the conjugate momenta to $u = (u_1, u_2, u_3, u_4)$. Let us compute the bilinear equality $l(u, \dot{u}) = 0$ in the Hamiltonian formulation; for all $u \neq 0$ we have:

$$l(u, \dot{u}) = \frac{1}{4\lambda^2 |u|^2} l(u, U - \lambda^2 b_\omega(u)) = \frac{1}{4\lambda^2 |u|^2} (l(u, U) - \lambda^2 l(u, b_\omega(u))).$$

Since $l(u, b_\omega(u)) = 0$ identically, the bilinear equality $l(u, \dot{u}) = 0$ is equivalent to the condition $l(u, U) = 0$.

The Hamiltonian $K(u, U)$ is still singular at $u = 0$; to remove the singularity we perform the iso-energetic reduction. For any value E we introduce the Hamiltonian:

$$\begin{aligned} \mathcal{K}_{\lambda\mathcal{R}}(u, U) &= |u|^2 \left(K(u, U) - E - \frac{(1-\mu)^2}{2} \right) \\ &= \frac{1}{8\lambda^2} |U - \lambda^2 b_\omega(u)|^2 - \frac{1}{2} \lambda^2 |u|^2 |\omega \wedge \pi(u)|^2 - \mu \lambda^{-1} |u|^2 \left(E + (1-\mu) + \frac{(1-\mu)^2}{2} \right) \\ &\quad - (1-\mu) |u|^2 \left(\frac{1}{|\lambda\pi(u) + e|} - 1 + \lambda\pi(u) \cdot e \right), \end{aligned} \quad (30)$$

which we call the KS Hamiltonian.

The solutions $u(s), U(s)$ of the Hamilton equations:

$$\begin{aligned} u'_j &= \frac{\partial}{\partial U_j} \mathcal{K}_{\lambda\mathcal{R}} \\ U'_j &= -\frac{\partial}{\partial u_j} \mathcal{K}_{\lambda\mathcal{R}} \quad , \quad j = 1, \dots, 4 \end{aligned} \quad (31)$$

with initial conditions $u(0) \neq 0$ and $\mathcal{K}_{\lambda\mathcal{R}}(u(0), U(0)) = 0$ are conjugate by the time transformation:

$$t(s) = \int_0^s |u(\sigma)|^2 d\sigma$$

to solutions of the Hamilton equations of $K(u, U)$ as soon as $u(s) \neq 0$. We also notice that $\mathcal{K}_{\lambda\mathcal{R}}(u, U)$ is invariant with respect to the one-parameter family of transformations

$$(u, U) \mapsto (\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 U),$$

i.e. we have:

$$\mathcal{K}_{\lambda\mathcal{R}}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 U) = \mathcal{K}_{\lambda\mathcal{R}}(u, U).$$

As a consequence, $l(u, U)$ is a first integral for this Hamiltonian system.

We remark that for $\lambda = 1, \mathcal{R} = \mathcal{I}$ the Hamiltonian:

$$\begin{aligned} \mathcal{K}_{\mathcal{I}}(u, U) &= \frac{1}{8} |U - b_{(0,0,1)}(u)|^2 - \frac{1}{2} |u|^2 |(0,0,1) \wedge \pi(u)|^2 - \mu |u|^2 \left(E + (1-\mu) + \frac{(1-\mu)^2}{2} \right) \\ &\quad - (1-\mu) |u|^2 \left(\frac{1}{|\pi(u) + (1,0,0)|} - 1 + \pi(u) \cdot (1,0,0) \right) \end{aligned} \quad (32)$$

provides an Hamiltonian formulation of the traditional KS regularization; see, for example, [10, 3] for alternative derivations.

3 The Hamilton-Jacobi equation for the KS Hamiltonian: the parameters space

Our aim is to define a complete integral of the Hamilton-Jacobi equation:

$$\mathcal{K}_{\mathcal{I}}\left(u, \frac{\partial W}{\partial u}\right) = \kappa, \quad (33)$$

which is analytic in a neighbourhood of $u = 0$, obtained from a family of solutions of (33) depending on suitable four parameters. Therefore, we proceed by defining families of particular solutions \tilde{W} of the Hamilton-Jacobi equations:

$$\mathcal{K}_{\lambda\mathcal{R}}\left(u, \frac{\partial \tilde{W}}{\partial u}\right) = \kappa$$

where $\mathcal{R} \in SO(3)$ is an arbitrary rotation matrix of the euclidean three-dimensional space and $\lambda > 0$, with \tilde{W} vanishing identically on an hyperplane defined by the choice of \mathcal{R} .

Remark. This procedure depends on four free parameters related to $\lambda > 0$ and to the matrix $\mathcal{R} \in SO(3)$, which in the end will provide the four parameters needed to define a complete solution of the Hamilton-Jacobi equation. The first idea to extend the argument of Levi-Civita would seem that of using the group $SO(4)$ to transform the KS Hamiltonian $\mathcal{K}_{\mathcal{I}}$, and then to define families of particular solutions \tilde{W} of the Hamilton-Jacobi equations:

$$\tilde{\mathcal{K}}\left(u, \frac{\partial \tilde{W}}{\partial u}\right) = \kappa$$

where $\tilde{\mathcal{K}}(u, U) = \tilde{\mathcal{K}}_{\mathcal{I}}(Su, SU)$ with $S \in SO(4)$, with \tilde{W} vanishing identically on an hyperplane defined by the choice of S . The problem is that, for arbitrary matrix $S \in SO(4)$, the bilinear form $l(u, U)$ is not invariant, i.e. $l(Su, SU) \neq l(u, U)$ on some u, U . We therefore follow a different strategy.

We have therefore to find a family of transformations on \mathbb{R}^4 such that:

- they project on the linear transformations of the three-dimensional euclidean space

$$(X, Y, Z) \mapsto \lambda\mathcal{R}(X, Y, Z)$$

with $\lambda > 0$ and $\mathcal{R} \in SO(3)$;

- their canonical extensions to the momenta leave invariant the diagram about the conjugation of Hamiltonians represented in figure 1;
- their canonical extensions to the momenta leave invariant the bilinear form $l(u, U)$ (up to the multiplication with a constant different from zero).

We find that the matrices:

$$\mathcal{S}_\nu = \begin{pmatrix} \nu_1 & -\nu_2 & -\nu_3 & -\nu_4 \\ \nu_2 & \nu_1 & -\nu_4 & \nu_3 \\ \nu_3 & \nu_4 & \nu_1 & -\nu_2 \\ \nu_4 & -\nu_3 & \nu_2 & \nu_1 \end{pmatrix},$$

with $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4 \setminus 0$, satisfy:

$$\mathcal{S}_\nu \mathcal{S}_\nu^T = |\nu|^2 \mathcal{I}, \quad (34)$$

and define linear transformations of \mathbb{R}^4 which project on linear transformation of the three-dimensional space so that, for any $u \in \mathbb{R}^4$, we have:

$$\pi(\mathcal{S}_\nu u) = \mathcal{R}_\nu \pi(u) \quad (35)$$

where:

$$\mathcal{R}_\nu = \begin{pmatrix} \nu_1^2 - \nu_2^2 - \nu_3^2 + \nu_4^2 & -2(\nu_1\nu_2 + \nu_3\nu_4) & -2(\nu_1\nu_3 - \nu_2\nu_4) \\ 2(\nu_1\nu_2 - \nu_3\nu_4) & \nu_1^2 - \nu_2^2 + \nu_3^2 - \nu_4^2 & -2(\nu_2\nu_3 + \nu_1\nu_4) \\ 2(\nu_1\nu_3 + \nu_2\nu_4) & -2(\nu_2\nu_3 - \nu_1\nu_4) & \nu_1^2 + \nu_2^2 - \nu_3^2 - \nu_4^2 \end{pmatrix} \quad (36)$$

is a matrix satisfying:

$$\mathcal{R}_\nu \mathcal{R}_\nu^T = |\nu|^4 \mathcal{I}, \quad (37)$$

which depends on the ν_j as in the Euler-Rodrigues formula.

Moreover, for all $(u, U) \in T^*\mathbb{R}^4$, we have:

$$l(\mathcal{S}_\nu u, \mathcal{S}_\nu U) = |\nu|^2 l(u, U) \quad .$$

We therefore consider the set of matrices:

$$\mathcal{S} = \cup_{\nu \in \mathbb{R}^4 \setminus 0} \mathcal{S}_\nu$$

and the map:

$$\begin{aligned} \Pi : \mathcal{S} &\longrightarrow SO(3) \\ \mathcal{S}_\nu &\longmapsto \Pi(\mathcal{S}_\nu) = \frac{1}{|\nu|^2} \mathcal{R}_\nu. \end{aligned}$$

The map Π is surjective. We have the following:

Proposition 3. *For any matrix $\mathcal{S}_\nu \in \mathcal{S}$ we have the identity:*

$$\mathcal{K}_{\mathcal{I}}(\mathcal{S}_\nu u, \mathcal{S}_\nu^{-T} U) = |\nu|^2 \mathcal{K}_{|\nu|^2 \Pi(\mathcal{S}_\nu)}(u, U). \quad (38)$$

Proof of Proposition 3. Let us denote $u = \mathcal{S}_\nu \tilde{u}, U = \mathcal{S}_\nu \tilde{U}$; we have the following identities:

- $|u| = |\nu| |\tilde{u}|$;
- $|\pi(u) + (1, 0, 0)| = |\pi(S_\nu \tilde{u}) + (1, 0, 0)| = \left| |\nu|^2 \Pi(S_\nu) \pi(\tilde{u}) + (1, 0, 0) \right|$
 $= \left| |\nu|^2 \left[\pi(\tilde{u}) + \Pi(S_\nu)^T (1, 0, 0) \right] \right|$;
- $\pi(u) \cdot (1, 0, 0) = \pi(S_\nu \tilde{u}) \cdot (1, 0, 0) = |\nu|^2 \pi(\tilde{u}) \cdot \Pi(S_\nu)^T (1, 0, 0)$;
- $|(0, 0, 1) \wedge \pi(u)| = |(0, 0, 1) \wedge \pi(S_\nu \tilde{u})| = |\nu|^2 |(0, 0, 1) \wedge \Pi(S_\nu) \pi(\tilde{u})|$
 $= |\nu|^2 |\Pi(S_\nu)^T (0, 0, 1) \wedge \pi(\tilde{u})|$,

which are proved from (34) and (35). Finally, we prove:

$$\left| S_\nu^{-T} \tilde{U} - b_{(0,0,1)}(S_\nu \tilde{u}) \right|^2 = \frac{1}{|\nu|^2} \left| \tilde{U} - |\nu|^4 b_{\Pi(S_\nu)^T(0,0,1)}(\tilde{u}) \right|^2. \quad (39)$$

From direct computation, for any $u \in \mathbb{R}^4$, we obtain:

$$A(S_\nu u) S_\nu = \hat{\mathcal{R}}_\nu A(u)$$

with:

$$\hat{\mathcal{R}}_\nu = \begin{pmatrix} \nu_1^2 - \nu_2^2 - \nu_3^2 + \nu_4^2 & -2(\nu_1 \nu_2 + \nu_3 \nu_4) & -2(\nu_1 \nu_3 - \nu_2 \nu_4) & 0 \\ 2(\nu_1 \nu_2 - \nu_3 \nu_4) & \nu_1^2 - \nu_2^2 + \nu_3^2 - \nu_4^2 & -2(\nu_2 \nu_3 + \nu_1 \nu_4) & 0 \\ 2(\nu_1 \nu_3 + \nu_2 \nu_4) & -2(\nu_2 \nu_3 - \nu_1 \nu_4) & \nu_1^2 + \nu_2^2 - \nu_3^2 - \nu_4^2 & 0 \\ 0 & 0 & 0 & |\nu|^2 \end{pmatrix}.$$

As a consequence, using (34) and by recalling the definition (10) of the vector potential b_ω , we have:

$$\begin{aligned} \left| S_\nu^{-T} \tilde{U} - b_{(0,0,1)}(S_\nu \tilde{u}) \right|^2 &= \left| \frac{1}{|\nu|^2} S_\nu \tilde{U} - 2A(S_\nu \tilde{u})^T \Lambda_{(0,0,1)} A(S_\nu \tilde{u}) S_\nu \tilde{u} \right|^2 \\ &= \frac{1}{|\nu|^2} \left| \tilde{U} - 2S_\nu^T A(S_\nu \tilde{u})^T \Lambda_{(0,0,1)} A(S_\nu \tilde{u}) S_\nu \tilde{u} \right|^2 \\ &= \frac{1}{|\nu|^2} \left| \tilde{U} - 2A(\tilde{u})^T \hat{\mathcal{R}}_\nu^T \Lambda_{(0,0,1)} \hat{\mathcal{R}}_\nu A(\tilde{u}) \tilde{u} \right|^2 = \frac{1}{|\nu|^2} \left| \tilde{U} - 2|\nu|^4 A(\tilde{u})^T \frac{\hat{\mathcal{R}}_\nu^T}{|\nu|^2} \Lambda_{(0,0,1)} \frac{\hat{\mathcal{R}}_\nu}{|\nu|^2} A(\tilde{u}) \tilde{u} \right|^2 \\ &= \frac{1}{|\nu|^2} \left| \tilde{U} - |\nu|^4 b_{\Pi(S_\nu)^T(0,0,1)}(\tilde{u}) \right|^2 \end{aligned}$$

where the last equality is a consequence of the fact that, for any $\omega \in \mathbb{R}^3$, the matrix:

$$\tilde{\Lambda}_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

$$\begin{array}{ccc}
L_{\mathcal{I}}(q, \dot{q}) & \xrightarrow{q=\mathcal{R}\tilde{q}} & L_{\mathcal{R}}(\tilde{q}, \dot{\tilde{q}}) \\
\downarrow KS & & \downarrow KS \\
\mathcal{L}_{\mathcal{I}}(u, \dot{u}), l(u, \dot{u}) = 0 & & \mathcal{L}_{\mathcal{R}}(\tilde{u}, \dot{\tilde{u}}), l(\tilde{u}, \dot{\tilde{u}}) = 0 \\
\downarrow Legendre & & \downarrow Legendre \\
\mathcal{K}_{\mathcal{I}}(u, U), l(u, U) = 0 & \dots\dots\dots & \mathcal{K}_{\mathcal{R}}(\tilde{u}, \tilde{U}), l(\tilde{u}, \tilde{U}) = 0
\end{array}$$

Figure 1: For any $S \in \mathcal{S}$ and $\mathcal{R} = \Pi(S)$ the diagram is commutative.

represents the linear transformations of \mathbb{R}^3 :

$$\tilde{\Lambda}_\omega \underline{x} = \omega \wedge \underline{x}$$

for all $\underline{x} \in \mathbb{R}^3$; then, for all $\underline{x} \in \mathbb{R}^3$ we have also:

$$\Pi(S_\nu)^T \tilde{\Lambda}_{(0,0,1)} \Pi(S_\nu) \underline{x} = \Pi(S_\nu)^T ((0,0,1) \wedge (\Pi(S) \underline{x})) = (\Pi(S_\nu)^T (0,0,1)) \wedge \underline{x},$$

and therefore

$$\Pi(S_\nu)^T \tilde{\Lambda}_\omega \hat{\Pi}(S_\nu) = \tilde{\Lambda}_{\Pi(S_\nu)^T(0,0,1)}.$$

From all the previous equalities we obtain (38). \square

4 The Hamilton-Jacobi equation for the KS Hamiltonian: particular solutions

In this Section we prove the existence of particular solutions \tilde{W} of the Hamilton-Jacobi equation:

$$\mathcal{K}_{|\nu|^2 \Pi(S_\nu)} \left(u, \frac{\partial \tilde{W}}{\partial u} \right) = \frac{\kappa}{|\nu|^2}, \quad (40)$$

with the following properties:

- the solutions $\tilde{W}(u; E, \mu, \kappa, \nu_1, \dots, \nu_4)$ are defined for any value of the parameters $(E, \mu, \kappa, \nu_1, \dots, \nu_4)$ in a set $\mathcal{D}_{\alpha, \dots, \delta}(E_*, \mu_*)$ defined by fixed values E_* and $\mu > 0$, and by suitably small $\alpha, \beta, \gamma, \delta > 0$:

$$|\mu - \mu_*| < \alpha$$

$$|E - E_*| < \beta$$

$$|\kappa| < \gamma$$

$$\{\nu \in \mathbb{R}^4 : 1 - \delta < |\nu| < 1 + \delta\},$$

and for any value of the parameters in this set it is analytic in the same common domain:

$$u \in \mathbb{C}^4 : |u| < \sigma,$$

with $\sigma > 0$ (depending only on $E_*, \mu_*, \alpha, \dots, \delta$).

- they satisfy:

$$\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \dots, \nu_4) = 0 \quad (41)$$

for all u_2, u_3, u_4 in a neighbourhood of 0.

- they are analytic also with respect to the parameters.

We remark that the domain above considered is local in the variables u and in the parameters E, μ, κ , but is not local in the parameters ν which are naturally defined in a neighbourhood of \mathbb{S}^3 . Therefore, since the proof will be obtained from the Cauchy-Kowaleski theorem, which grants the existence of local analytic solutions of PDE, we have to pay some care in proving the global character of the solutions obtained from the Cauchy-Kowaleski theorem with respect to the parameters ν_i .

In order to apply the Cauchy-Kowaleski theorem, we first rewrite the HJ equation (40) as follows:

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial u_1} &= |\nu|^4 b_{1,\omega}(u) \pm \sqrt{8} |\nu| \left(\mu + \kappa + \frac{1}{2} |\nu|^6 |u|^2 |\omega \wedge \pi(u)|^2 + |u|^2 |\nu|^2 E_\mu \right. \\ &\quad \left. + (1 - \mu) |u|^2 |\nu|^2 \left(\frac{1}{|\nu|^2 \pi(u) + e} - 1 + |\nu|^2 \pi(u) \cdot e \right) \right. \\ &\quad \left. - \frac{1}{8 |\nu|^2} \sum_{j=2}^4 \left(\frac{\partial \tilde{W}}{\partial u_j} - |\nu|^4 b_{j,\omega}(u) \right)^2 \right)^{\frac{1}{2}} \end{aligned} \quad (42)$$

where $E_\mu = E + (1 - \mu) + \frac{(1-\mu)^2}{2}$, $\omega = \Pi(S_\nu)^T(0, 0, 1)$, $e = \Pi(S_\nu)^T(1, 0, 0)$. We solve the previous equation by selecting the positive sign in front of the square root (the minus would provide a different solution), and therefore we consider the function:

$$\begin{aligned} F(u_1, \dots, u_4, p_1, p_2, p_3; E, \mu, \kappa, \nu_1, \dots, \nu_4) &= |\nu|^4 b_{1,\omega}(u) \\ &\quad + \sqrt{8} |\nu| \left(\mu + \kappa + \frac{1}{2} |\nu|^6 |u|^2 |\omega \wedge \pi(u)|^2 + |u|^2 |\nu|^2 E_\mu \right. \\ &\quad \left. + (1 - \mu) |u|^2 |\nu|^2 \left(\frac{1}{|\nu|^2 \pi(u) + e} - 1 + |\nu|^2 \pi(u) \cdot e \right) \right) \end{aligned}$$

$$-\frac{1}{8|\nu|^2} \sum_{j=2}^4 \left(p_{j-1} - |\nu|^4 b_{j,\omega}(u) \right)^2 \Big)^{\frac{1}{2}} \quad (43)$$

which depends parametrically on $E, \mu, \kappa, \nu_1, \dots, \nu_4$. For any fixed E_*, μ_* with $\mu_* > 0$ there exist $\alpha_0, \dots, \delta_0$ and σ_0 such that F is analytic for all $(E, \kappa, \nu_1, \dots, \nu_4) \in \mathcal{D}_{\alpha_0, \dots, \delta_0}$ in the set $|u| < \sigma_0$.

We first apply the Cauchy-Kovaleskaia theorem to the first-order PDE:

$$\frac{\partial \tilde{W}}{\partial u_1} = F \left(u_1, \dots, u_4, \frac{\partial \tilde{W}}{\partial u_2}, \frac{\partial \tilde{W}}{\partial u_3}, \frac{\partial \tilde{W}}{\partial u_4}; E, \mu, \kappa, \nu_1, \dots, \nu_4 \right) \quad (44)$$

where $E, \mu, \kappa, \nu_1, \dots, \nu_4$ are fixed in some set $\mathcal{D}_{\alpha_1, \dots, \delta_1}$, with the boundary condition (41):

$$\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \dots, \nu_4) = 0$$

for u_2, u_3, u_4 in a neighbourhood of $u = 0$. We obtain (see Section 8) the existence of a unique solution $\tilde{W}(u; E, \mu, \kappa, \nu_1, \dots, \nu_4)$ of such PDE problem which is analytic in a neighbourhood of $u = 0$, and the radius of convergence of the series:

$$\tilde{W} = \sum_{i_1, \dots, i_4 \geq 0} c_{i_1, \dots, i_4}(E, \mu, \kappa, \nu) u_1^{i_1} \dots u_4^{i_4} \quad (45)$$

is common for all the values of the parameters in the set $\mathcal{D}_{\alpha_1, \dots, \delta_1}$. The coefficients $c_{i_1, \dots, i_4}(E, \mu, \kappa, \nu)$ can be computed iteratively in the order $i_1 + \dots + i_4$, and since they are functions globally defined in $\mathcal{D}_{\alpha_1, \dots, \delta_1}$, the series (45) is globally defined in the $\mathcal{D}_{\alpha_1, \dots, \delta_1}$. In particular, we have:

$$\tilde{W} = \sqrt{8(\mu + \kappa)|\nu|^2} u_1 + \frac{E\mu|\nu|^3}{\sqrt{\mu + \kappa}} u_1 \sqrt{2} \left(\frac{u_1^2}{3} + u_2^2 + u_3^2 + u_4^2 \right) + u_1 \mathcal{O}_3(u). \quad (46)$$

It remains to establish the regularity of the function \tilde{W} defined the series (45) with respect to the parameters E, μ, κ, ν . Therefore we apply a second time the Cauchy-Kowaleski theorem to the first-order PDE (44) by considering the independent variables $(u_1, u_2, u_3, u_4, E, \mu, \kappa, \nu_1, \dots, \nu_4)$ in a neighbourhood of $(u_1, u_2, u_3, u_4, E, \mu, \kappa, \nu_1, \dots, \nu_4) = (0, 0, 0, 0, E_*, \mu_*, 0, \nu_1^*, \dots, \nu_4^*)$ with $\nu^* \in \mathbb{S}^3$, with the boundary condition:

$$\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \dots, \nu_4) = 0$$

for u_2, u_3, u_4 in a neighbourhood of $u = 0$ and for all E, μ, κ, ν in a neighbourhood of $E_*, \mu_*, 0, \nu^*$. We obtain (see Section 8) the existence of a unique solution $\tilde{W}_1(u; E, \mu, \kappa, \nu_1, \dots, \nu_4)$ of such PDE problem which is analytic in a neighbourhood of $(u, E, \mu, \kappa, \nu) = (0, E_*, \mu_*, 0, \nu_*)$, with series expansion:

$$\tilde{W}_1 = \sum_{i_1, \dots, i_{11} \geq 0} d_{i_1, \dots, i_{11}}(E_*, \mu_*, \nu_*) u_1^{i_1} \dots u_4^{i_4} (E - E_*)^{i_5} (\mu - \mu_*)^{i_6} \kappa^{i_7} (\nu_1 - \nu_1^*)^{i_8} \dots (\nu_4 - \nu_4^*)^{i_{11}}$$

converging within a radius $\rho(E_*, \mu_*, \nu_*)$ depending only on (E_*, μ_*, ν_*) . But since \tilde{W}_1 is also a solution of the PDE problem where the E, μ, κ, ν are given parameters, and \tilde{W}_1 satisfy the same boundary condition (41), from uniqueness we obtain

$$\tilde{W}_1(u; E, \mu, \kappa, \nu_1, \dots, \nu_4) = \tilde{W}(u; E, \mu, \kappa, \nu_1, \dots, \nu_4),$$

and this proves the analyticity of the global solution \tilde{W} for any value of the parameters in some $\mathcal{D}_{\alpha, \dots, \beta}$ and for some $|u| \leq \sigma$.

5 The Hamilton-Jacobi equation for the KS Hamiltonian: a complete integral

Theorem 1 follows from the following:

Proposition 4. *For fixed values of E_* and of $\mu_* > 0$, there exists a complete integral $W(u, \nu; E, \mu)$ of the Hamilton-Jacobi equation (33) depending on the four parameters ν and two additional parameters E, μ , with*

$$\kappa = \mu(|\nu|^2 - 1).$$

and analytic for E, μ, ν in the set:

$$\{|\mu - \mu_*| < a, \quad |E - E_*| < b, \quad \nu \in \mathbb{R}^4 : ||\nu| - 1| < c\}$$

and u in the (complex) ball:

$$B_\sigma = \{u \in \mathbb{C}^4 : |\nu| < d\}$$

with suitable $a, b, c, d > 0$. The coefficients of the Taylor expansions of W with respect to the variables u can be explicitly computed iteratively; in particular we have:

$$W = \sqrt{8\mu} \sum_{j=1}^4 \nu_j u_j + \mathcal{O}_3(u). \quad (47)$$

Proof of Proposition 4. The complete integral is defined by:

$$W(u; E, \mu, \nu) = \tilde{W}(|\nu|^{-2} S_\nu^T u; E, \mu, \kappa_\nu, \nu),$$

with $\kappa_\nu = \mu(|\nu|^2 - 1)$, where $\tilde{W}(\tilde{u}; E, \mu, \kappa, \nu)$ denotes the solution of the Hamilton-Jacobi equation (40):

$$\mathcal{K}_{|\nu|^2 \Pi(S_\nu)} \left(\tilde{u}, \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa, \nu) \right) = \frac{\kappa}{|\nu|^2}, \quad (48)$$

as it has been defined in the previous section. In fact, since we have:

$$\frac{\partial W}{\partial u}(u; E, \mu, \nu) = |\nu|^{-2} S_\nu \frac{\partial \tilde{W}}{\partial \tilde{u}}(|\nu|^{-2} S_\nu^T u, E, \mu, \kappa_\nu, \nu),$$

using Proposition 1, and setting $u = S_\nu \tilde{u}$, we obtain

$$\begin{aligned} \mathcal{K}_{\mathcal{I}} \left(u, \frac{\partial W}{\partial u}(u; E, \mu, \nu) \right) &= \mathcal{K}_{\mathcal{I}} \left(S_\nu \tilde{u}, S_\nu^{-T} \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa_\nu, \nu) \right) \\ &= |\nu|^2 \mathcal{K}_{|\nu|^2 \Pi(S_\nu)} \left(\tilde{u}, \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa_\nu, \nu) \right) = \kappa_\nu = \mu(|\nu|^2 - 1). \end{aligned}$$

By replacing in (46) κ with κ_ν and u with $|\nu|^{-2} S_\nu^T u$ we obtain (47). Therefore, the determinant:

$$j_4(u, \nu; E, \mu) = \det \left(\frac{\partial W}{\partial u_i \partial \nu_j} \right)$$

satisfies:

$$j_4(0, \nu; E, \mu) = 64\mu^2. \quad (49)$$

Therefore, W is a complete integral of the Hamilton-Jacobi equation in a neighbourhood of $u = 0$. \square

Let us analyze some consequences of Theorem 1.

For any $\nu \in \mathbb{S}^3$, which corresponds to $\kappa = 0$, the function W defines the foliation:

$$\Gamma_\nu = \left\{ (u, U) \in T^*\mathbb{R}^4 : |u| < \sigma, \quad U_j = \frac{\partial W}{\partial u_j}(u; E, \mu, \nu) \right\},$$

which is locally invariant (the solutions with initial conditions in a leaf Γ_ν can flow out of it in the future and/or in the past). Since we are interested in motions of the KS Hamiltonian $\mathcal{K}_{\mathcal{I}}$ which project on motions of the three-body problem, and since the leaves Γ_ν are foliated by the first integral $l(u, U)$, we consider:

$$\tilde{\Gamma}_\nu = \left\{ (u, U) \in T^*\mathbb{R}^4 : |u| < \sigma, \quad U_j = \frac{\partial W}{\partial u_j}(u; E, \mu, \nu), \quad l(u, U) = 0 \right\}.$$

Proposition 5. *For any $\nu \in \mathbb{S}^3$, $\tilde{\Gamma}_\nu$ is a manifold of dimension 3 in a neighbourhood of $(u, U) = (0, \sqrt{8\mu} \nu)$.*

Proof of Proposition 5. The set $\tilde{\Gamma}_\nu$ is obtained from the solutions (u, U) of the system:

$$F_1(u, U) = 0, \quad F_5(u, U) = 0,$$

where:

$$F_j(u, U) = U_j - \frac{\partial W}{\partial u_j}(u; E, \mu, \nu), \quad j = 1, \dots, 4$$

$$F_5(u, U) = l(u, U),$$

with $|u| < \sigma$. Since from (47) we have:

$$F_j = U_j - \sqrt{8\mu}\nu_j + \mathcal{O}_2(u) \quad , \quad j = 1, \dots, 4,$$

the restriction of the Jacobian matrix of the map $F = (F_1, \dots, F_5)$ to $\tilde{\Gamma}_\nu$ has the representation:

$$\begin{aligned} \mathcal{J}(u, U)|_{\tilde{\Gamma}_\nu} &= \begin{pmatrix} \nabla_u F_1 & \nabla_u F_2 & \nabla_u F_3 & \nabla_u F_4 & \nabla_u F_5 \\ \nabla_U F_1 & \nabla_U F_2 & \nabla_U F_3 & \nabla_U F_4 & \nabla_U F_5 \end{pmatrix} \Big|_{\tilde{\Gamma}_\nu} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & U_4 \\ 0 & 0 & 0 & 0 & -U_3 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & -U_1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Big|_{\tilde{\Gamma}_\nu} + \mathcal{O}_1(u) = \sqrt{8\mu} \begin{pmatrix} 0 & 0 & 0 & 0 & \nu_4 \\ 0 & 0 & 0 & 0 & -\nu_3 \\ 0 & 0 & 0 & 0 & \nu_2 \\ 0 & 0 & 0 & 0 & -\nu_1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \mathcal{O}_1(u). \end{aligned}$$

Since $|\nu| = 1$, the rank of the matrix $\mathcal{J}(u, U)|_{\tilde{\Gamma}_\nu}$ is equal to 5 in a neighbourhood of $(u, U) = (0, \sqrt{8\mu}\nu)$. \square

It remains therefore to represent the motions on the 3-dimensional locally invariant manifolds $\tilde{\Gamma}_\nu$, and this will be done by defining from the function W a suitable canonical transformation. Precisely, we consider the system:

$$U_i = \frac{\partial W}{\partial u_i}(u, \nu; E, \mu), \quad i = 1, \dots, 4 \quad (50)$$

$$n_i = \frac{\partial W}{\partial \nu_i}(u, \nu; E, \mu), \quad i = 1, \dots, 4. \quad (51)$$

which is well defined since the function W can be differentiated with respect to the variables ν_i .

Inversion of the sub-system (50). We first consider the sub-system formed by equations (50):

$$U_i = \frac{\partial W}{\partial u_i}(u, \nu; E, \mu), \quad i = 1, 2, 3, 4. \quad (52)$$

From (49), (47) and the analyticity of W with respect to u, ν , for any $\nu^* \in \mathbb{S}^3$ we have the local inversion of the sub-system (50) with respect to the variables ν , in a neighbourhood of $(u, \nu) = (0, \nu^*)$:

$$\nu = \hat{\nu}(u, U; E, \mu),$$

and the functions $\hat{\nu}$ are analytic. As a matter of fact, we have the stronger result:

Lemma 1. *The sub-system (50) has a global analytic inversion:*

$$\nu = \hat{\nu}(u, U; E, \mu),$$

defined for u, U so that u is in some complex ball $B(d_0)$ and for U in the image of the map:

$$\nu \mapsto \frac{\partial W}{\partial u}(u, \nu; E, \mu)$$

with $\nu \in \Omega_{c_0} = \{\nu : \|\nu\| - 1 < c_0\}$ with some suitable c_0, d_0 .

Proof of lemma 1. We first proof that for fixed E, μ , for all u suitably close to $u = 0$, and for suitably small c_1 , the map:

$$\begin{aligned} \Psi_u : \Omega_{c_1} &\longrightarrow \mathbb{R}^4 \\ \nu &\longmapsto \frac{\partial W}{\partial u}(u, \nu; E, \mu) \end{aligned} \quad (53)$$

is injective. From (47), we have the representation:

$$\Psi_u(\nu) = \sqrt{8\mu} \nu + \psi_u(u, \nu; E, \mu)$$

with $\psi_u(u, \nu; E, \mu) = \mathcal{O}_2(u)$.

For arbitrary $k \geq 3$, we extend the map Ψ_u to a map

$$\begin{aligned} \Psi_u^k : \mathcal{B}(1 + c_1) &\longrightarrow \mathbb{R}^4 \\ \nu &\longmapsto \Psi_u^k(\nu) = \sqrt{8\mu} \nu + \phi^k(|\nu|) \psi_u(u, \nu; E, \mu) \end{aligned} \quad (54)$$

where $\mathcal{B}(1 + c_1)$ is the real ball centered at $\nu = 0$ of radius $1 + c_1$ and

$$\phi^k : [0, 1 + c_1] \longrightarrow \mathbb{R}^4$$

is a \mathcal{C}^k -smooth function such that $\phi^k(x) = 1$ if $x \in [1 - c_1/2, 1 + c_1]$, $\phi^k(x) = 0$ if $x \in [0, 1 - c_1]$, and in the interval $(1 - c_1, 1 - c_1/2)$ increases smoothly and monotonically from 0 to 1. For any fixed k , by restricting eventually the domain of u , we have that the map Ψ_u^k is convex in the set $\mathcal{B}(1 + c_1)$. Then, from a result on the global inversion of convex maps (see Theorem 4.2, page 137, of [2]), the map Ψ_u^k is injective. But this implies that the also the map:

$$\begin{aligned} \Psi_u : \Omega_{\frac{c_1}{2}} &\longrightarrow \mathbb{R}^4 \\ \nu &\longmapsto \frac{\partial W}{\partial u}(u, \nu; E, \mu) \end{aligned} \quad (55)$$

is injective (in fact, if $\Psi_u(\nu') = \Psi_u(\nu'')$ with $\nu', \nu'' \in \Omega_{\frac{c_1}{2}}$, then we have also $\Psi_u^k(\nu') = \Psi_u^k(\nu'')$ and therefore $\nu' = \nu''$) and therefore has the inverse:

$$\Psi_u^{-1} : \Psi_u(\Omega_{\frac{c_1}{2}}) \longrightarrow \Omega_{\frac{c_1}{2}}.$$

From the local inversion theorem the inverse map is analytic. \square

The canonical transformation. The inversion of the system of equations (50) provides the functions:

$$\begin{aligned} \nu_i &= \hat{\nu}_i(u, U; E, \mu) \\ n_i &= \hat{n}_i(u, U; E, \mu) \quad i = 1, 2, 3, 4 \end{aligned} \quad (56)$$

which define a canonical transformation:

$$(n, \nu) = \chi_4(u, U)$$

conjugating $\mathcal{K}_{\mathcal{I}}$ to the Hamiltonian:

$$\hat{\mathcal{K}}(n, \nu) = \mu(|\nu|^2 - 1).$$

Therefore, the momenta ν_i are constants of motion and the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}_{\mathcal{I}}$ are obtained from the inversion of:

$$(n(0) + 2\mu \nu(0)s, \nu(0)) = \chi_4(u(s), U(s)).$$

The bilinear relation. From the identity:

$$W(u, \nu; E, \mu) = W(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu), \quad \forall \alpha \in \mathbb{R}$$

by differentiating both sides with respect to α and computing in $\alpha = 0$ we obtain:

$$l \left(\frac{\partial W}{\partial u}(u, \nu, E, \mu), u \right) + l \left(\frac{\partial W}{\partial \nu}(u, \nu, E, \mu), \nu \right) = 0$$

and therefore we have $l(u, U) = 0$ if and only if $l(\hat{n}, \hat{\nu}) = 0$. Consistently, $l(n, \nu)$ is a first integral of the Hamilton equations of $\hat{\mathcal{K}}(n, \nu)$.

6 The first integrals in the space of the Cartesian variables

In the previous section we have constructed four first integrals $\hat{\nu}_i(u, U; E, \mu)$ of the KS Hamiltonian which are analytic in a neighbourhood of the collision set, represented in the space of coordinates u, U by:

$$C = \{(u, U) \in T^*\mathbb{R}^4 : u = 0, \quad \|U\| = \sqrt{8\mu}\}.$$

It is therefore interesting to know if, from the $\hat{\nu}_i$, it is possible to construct first integrals $N_i(X, Y, Z, P_X, P_Y, P_Z)$ defined in the Cartesian phase-space of the variables (X, Y, Z, P_X, P_Y, P_Z) introduced in Section 2, eq. (7).

Following [17], we first show that from each $\hat{\nu}_i$ we construct a family of local first integrals defined only in a neighbourhood of any point (X, Y, Z, P_X, P_Y, P_Z) , with (X, Y, Z) in a neighbourhood of $(0, 0, 0)$; from this family, we construct 2 first integrals (independent on the energy E) which are globally defined in a complete neighbourhood of $(X, Y, Z) = (0, 0, 0)$.

A phase-spaces projection. We introduce a projection from the space:

$$T^*\mathbb{R}_0^4 = \{(u, U) \in T^*\mathbb{R}^4 : |u| \neq 0, \quad l(u, U) = 0\}$$

to the Cartesian phase space of the variables (X, Y, Z, P_X, P_Y, P_Z) introduced in Section 2, eq. (7). We denote:

$$(X, Y, Z, P_X, P_Y, P_Z) = \tilde{\pi}(u, U)$$

where $(X, Y, Z) = \pi(u)$ and:

$$(P_X, P_Y, P_Z, 0) = \frac{1}{2|u|^2} A(u)U \quad (57)$$

Local inversions of the phase-space projection. We consider a local inversion of $(X, Y, Z) = \pi(u)$:

$$\pi^{-1} : \mathcal{W} \longrightarrow \mathbb{R}^4$$

$$(X, Y, Z) \longmapsto u = \pi^{-1}(X, Y, Z)$$

with $\mathcal{W} \subseteq \mathbb{R}^3 \setminus \{0\}$ open set, and define:

$$(u, U) = \chi(X, Y, Z, P_X, P_Y, P_Z)$$

where $u = \pi^{-1}(X, Y, Z)$ and, from (57):

$$U = 2A(u)^T (P_X, P_Y, P_Z, 0).$$

We introduce the matrix:

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (58)$$

so that $l(u, U) = u \cdot \Omega U$. We notice that we have:

$$l(u, U) = 2u \cdot \Omega A(u)^T (P_X, P_Y, P_Z, 0) = 2(A(u)\Omega^T u) \cdot (P_X, P_Y, P_Z, 0) = 0,$$

since $A(u)\Omega^T u$ is a four dimensional vector with only the fourth component different from zero. Therefore, for any choice of π^{-1} , the phase-space local inversion χ is well defined in $T^*\mathbb{R}_0^4$.

An atlas of local inversions. Following [17] (where a similar result is proved between the Cartesian state-space with coordinates $x, y, z, \dot{x}, \dot{y}, \dot{z}$ and the state-space of the KS variables u, u') we define an atlas of two local inversions of the map π defined in $\mathbb{R}^3 \setminus (0, 0, 0)$:

Lemma 2. *Consider the maps*

$$\begin{aligned}\pi_-^{-1} : D_- = \mathbb{R}^3 \setminus \{(X, 0, 0) : X \geq 0\} &\longrightarrow \mathbb{R}^4 \\ \pi_+^{-1} : D_+ = \mathbb{R}^3 \setminus \{(X, 0, 0) : X \leq 0\} &\longrightarrow \mathbb{R}^4\end{aligned}$$

defined by

$$\begin{aligned}\pi_-^{-1}(X, Y, Z) &= \left(\frac{Y}{\sqrt{2(r-X)}}, \frac{\sqrt{r-X}}{\sqrt{2}}, 0, \frac{Z}{\sqrt{2(r-X)}} \right) \\ \pi_+^{-1}(X, Y, Z) &= \left(\frac{\sqrt{r+X}}{\sqrt{2}}, \frac{Y}{\sqrt{2(r+X)}}, \frac{Z}{\sqrt{2(r+X)}}, 0 \right),\end{aligned}$$

where $r = \sqrt{X^2 + Y^2 + Z^2}$, as well as their phase-space extensions:

$$\begin{aligned}\chi_{\pm} : (\mathbb{R}^3 \setminus 0) \times \mathbb{R}^3 &\longrightarrow T^*\mathbb{R}_0^4 \\ (X, Y, Z, P_X, P_Y, P_Z) &\longmapsto (u, U) = \chi_{\pm}(X, Y, Z, P_X, P_Y, P_Z)\end{aligned}$$

defined by:

$$\begin{aligned}\chi_-^{-1}(X, Y, Z, P_X, P_Y, P_Z) &= (\pi_-^{-1}(X, Y, Z), 2A(\pi_-^{-1}(X, Y, Z))^T(P_X, P_Y, P_Z, 0)) \\ \chi_+^{-1}(X, Y, Z, P_X, P_Y, P_Z) &= (\pi_+^{-1}(X, Y, Z), 2A(\pi_+^{-1}(X, Y, Z))^T(P_X, P_Y, P_Z, 0)).\end{aligned}$$

Then, for every (X, Y, Z, P_X, P_Y, P_Z) in the domain of χ_-^{-1} we have:

$$(X, Y, Z, P_X, P_Y, P_Z) = \chi \circ \chi_-^{-1}(X, Y, Z, P_X, P_Y, P_Z),$$

for every (X, Y, Z, P_X, P_Y, P_Z) in the domain of χ_+^{-1} we have:

$$(X, Y, Z, P_X, P_Y, P_Z) = \chi \circ \chi_+^{-1}(X, Y, Z, P_X, P_Y, P_Z),$$

and, for every (X, Y, Z, P_X, P_Y, P_Z) in the intersection of the domains of χ_-^{-1} and χ_+^{-1} exists $\alpha \in \mathbb{R}$ (depending only on (X, Y, Z)) such that, by denoting

$$(u_{\pm}, U_{\pm}) = \chi_{\pm}^{-1}(X, Y, Z, P_X, P_Y, P_Z),$$

we have:

$$u_+ = \mathcal{S}_{\alpha}^0 u_- \quad , \quad U_+ = \mathcal{S}_{\alpha}^0 U_- \quad . \quad (59)$$

Proof of Lemma 2. We prove that indeed we have $U_+ = \mathcal{S}_\alpha^0 U_-$. Since $u_+ = \mathcal{S}_\alpha^0 u_-$, we have:

$$\begin{aligned} U_+ &= 2A(u_+)^T(P_X, P_Y, P_Z) = 2A(\mathcal{S}_\alpha^0 u_-)^T(P_X, P_Y, P_Z, 0) \\ &= \mathcal{S}_\alpha^0 A(u_-)^T(P_X, P_Y, P_Z, 0) = \mathcal{S}_\alpha^0 U_-. \end{aligned}$$

□

Cartesian representatives of the $\hat{\nu}_i, \hat{n}_i$. Let us fix E, μ , and consider the set $\mathcal{D}_E \subseteq (D \setminus 0) \times \mathbb{R}^3$ where D is a suitable small neighbourhood of $(0, 0, 0)$ and for any $(X, Y, Z, P_X, P_Y, P_Z) \in \mathcal{D}_E$ we have $H(X, Y, Z, P_X, P_Y, P_Z) = E$. In the sets:

$$\mathcal{D}_E^\pm = \mathcal{D}_E \cap (D_\pm \times \mathbb{R}^3)$$

we define:

$$\begin{aligned} \tilde{\nu}_{i,\pm}(X, Y, Z, P_X, P_Y, P_Z) &= \hat{\nu}_i(\chi_\pm^{-1}(X, Y, Z, P_X, P_Y, P_Z); E, \mu) \\ \tilde{n}_{i,\pm}(X, Y, Z, P_X, P_Y, P_Z) &= \hat{n}_i(\chi_\pm^{-1}(X, Y, Z, P_X, P_Y, P_Z); E, \mu). \end{aligned}$$

Since these functions are constructed using the local inversions χ_\pm^{-1} , they satisfy the identity:

$$l(\tilde{n}_\pm(X, Y, Z, P_X, P_Y, P_Z), \tilde{\nu}_\pm(X, Y, Z, P_X, P_Y, P_Z)) = 0,$$

and since they are constructed from the solutions of the Hamilton-Jacobi equation on the zero energetic level of the KS Hamiltonian, they also satisfy:

$$|\tilde{\nu}_{i,\pm}(X, Y, Z, P_X, P_Y, P_Z)| = 1.$$

Let us denote by:

$$(u_\pm, U_\pm) = \chi_\pm^{-1}(X, Y, Z, P_X, P_Y, P_Z)$$

the pre-images, by α the angle such that:

$$u_+ = \mathcal{S}_\alpha^0 u_- \quad , \quad U_+ = \mathcal{S}_\alpha^0 U_-,$$

and:

$$\begin{aligned} \nu_+ &= \hat{\nu}(u_+, U_+; E, \mu) \quad , \quad \nu_- = \hat{\nu}(u_-, U_-; E, \mu) \\ n_+ &= \hat{n}(u_+, U_+; E, \mu) \quad , \quad n_- = \hat{n}(u_-, U_-; E, \mu). \end{aligned}$$

We prove:

$$\nu_+ = \mathcal{S}_\alpha^0 \nu_- \quad , \quad n_+ = \mathcal{S}_\alpha^0 n_- \tag{60}$$

Since:

$$U_+ = \frac{\partial W}{\partial u}(u_+, \nu_+; E, \mu)$$

$$U_- = \frac{\partial W}{\partial u}(u_-, \nu_-; E, \mu)$$

we have:

$$\frac{\partial W}{\partial u}(u_-, \nu_-; E, \mu) = (\mathcal{S}_\alpha^0)^T \frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u_-, \nu_+; E, \mu). \quad (61)$$

We use the previous equation to establish the relation between ν_+ and ν_- .

Let us consider the complete integral $W(u, \nu; E, \mu)$ of the Hamilton-Jacobi equation:

$$\mathcal{K}_{\mathcal{I}} \left(u, \frac{\partial W}{\partial u}(u, \nu; E, \mu) \right) = \mu(|\nu|^2 - 1)$$

defined in Section 5. In particular, for any ν in a suitable small neighbourhood of the sphere $|\nu| = 1$, the function $W(u, \nu; E, \mu)$ is analytic in a neighbourhood of $u = 0$ and, if also $u \cdot \nu = 0$, we have:

$$W(u, \nu; E, \mu) = 0 \quad .$$

For any $\alpha \in \mathbb{R}$, let us define the function:

$$\hat{W}_\alpha(u, \nu; E, \mu) = W(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu).$$

We prove:

$$W(u, \nu; E, \mu) = \hat{W}_\alpha(u, \nu; E, \mu).$$

In fact, since \mathcal{S}_α^0 acts as a symmetry for the Hamiltonian $\mathcal{K}_{\mathcal{I}}$,

$$\mathcal{K}_{\mathcal{I}}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 U) = \mathcal{K}_{\mathcal{I}}(u, U),$$

we have:

$$\begin{aligned} \mathcal{K}_{\mathcal{I}} \left(u, \frac{\partial \hat{W}_\alpha}{\partial u}(u, \nu; E, \mu) \right) &= \mathcal{K}_{\mathcal{I}} \left(u, (\mathcal{S}_\alpha^0)^T \frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu) \right) \\ &= \mathcal{K}_{\mathcal{I}} \left(\mathcal{S}_\alpha^0 u, \frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu) \right) = \mu(|\mathcal{S}_\alpha^0 \nu|^2 - 1) = \mu(|\nu|^2 - 1) \end{aligned}$$

and therefore $\hat{W}_\alpha(u, \nu; E, \mu)$ is a solution of the Hamilton-Jacobi equation. Also, $\hat{W}_\alpha(u, \nu; E, \mu) = 0$ on the hyperplane $u \cdot \nu = 0$. Therefore W, \hat{W}_α are both solutions of the same Hamilton-Jacobi equation; they are both analytic in a common neighbourhood of $u = 0$; they both vanish on the same hyperplane. Therefore, they coincide in their common domain:

$$W(u, \nu; E, \mu) = W(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu),$$

and in particular we have the identity:

$$\frac{\partial W}{\partial u}(u, \nu; E, \mu) = (\mathcal{S}_\alpha^0)^T \frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u, \mathcal{S}_\alpha^0 \nu; E, \mu).$$

Therefore, from eq. (61), we have:

$$\frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u_-, \mathcal{S}_\alpha^0 \nu_-; E, \mu) = \frac{\partial W}{\partial u}(\mathcal{S}_\alpha^0 u_-, \nu_+; E, \mu)$$

and from Lemma 1: $\nu_+ = \mathcal{S}_\alpha^0 \nu_-$. Finally, we have:

$$\begin{aligned} n_- &= \frac{\partial W}{\partial \nu}(u_-, \nu_-; E, \mu) = (\mathcal{S}_\alpha^0)^T \frac{\partial W}{\partial \nu}(\mathcal{S}_\alpha^0 u_-, \mathcal{S}_\alpha^0 \nu_-; E, \mu) \\ &= (\mathcal{S}_\alpha^0)^T \frac{\partial W}{\partial \nu}(u_+, \nu_+; E, \mu) = (\mathcal{S}_\alpha^0)^T n_+. \end{aligned}$$

From local to global first integrals. The functions $\tilde{\nu}_\pm, \tilde{n}_\pm$ constructed above indeed depend on the chart \mathcal{D}_E^\pm , and therefore are not globally defined in \mathcal{D}_E . We here aim to construct, from the functions $\nu(u, U, E), n(u, U, E)$, first integrals in the Cartesian coordinates which are globally defined in \mathcal{D}_E . First of all, we consider the dynamics in the ν, n variables:

$$\nu_i(s) = \nu_i(0) \quad , \quad n_i(s) = n_i(0) + 2\mu\nu_i(0)s$$

and we notice that the functions:

$$\begin{aligned} N_X &= \nu_1 n_4 - \nu_4 n_1 \\ N_Y &= \frac{1}{2}(\nu_1 n_3 - n_1 \nu_3 + n_2 \nu_4 - n_4 \nu_2) \\ N_Z &= \frac{1}{2}(\nu_1 n_2 - n_1 \nu_2 + n_4 \nu_3 - n_3 \nu_4) \end{aligned}$$

are first integrals. Since they are all invariant by composition with the map $(n, \nu) \mapsto (\mathcal{S}_\alpha^0 n, \mathcal{S}_\alpha^0 \nu)$ for any α , their local representatives:

$$N_X^\pm(X, Y, Z, P_X, P_Y, P_Z) = (\tilde{\nu}_1^\pm \tilde{n}_4^\pm - \tilde{\nu}_4^\pm \tilde{n}_1^\pm)(X, Y, Z, P_X, P_Y, P_Z)$$

$$N_Y^\pm(X, Y, Z, P_X, P_Y, P_Z) = \frac{1}{2}(\tilde{\nu}_1^\pm \tilde{n}_3^\pm - \tilde{\nu}_3^\pm \tilde{n}_1^\pm + \tilde{\nu}_4^\pm \tilde{n}_2^\pm - \tilde{\nu}_2^\pm \tilde{n}_4^\pm)(X, Y, Z, P_X, P_Y, P_Z)$$

$$N_Z^\pm(X, Y, Z, P_X, P_Y, P_Z) = \frac{1}{2}(\tilde{\nu}_1^\pm \tilde{n}_2^\pm - \tilde{\nu}_2^\pm \tilde{n}_1^\pm + \tilde{\nu}_4^\pm \tilde{n}_3^\pm - \tilde{\nu}_3^\pm \tilde{n}_4^\pm)(X, Y, Z, P_X, P_Y, P_Z)$$

satisfy, for all $(X, Y, Z, P_X, P_Y, P_Z) \in \mathcal{D}_E^+ \cap \mathcal{D}_E^-$:

$$\begin{aligned} N_X^+(X, Y, Z, P_X, P_Y, P_Z) &= N_X^-(X, Y, Z, P_X, P_Y, P_Z) \quad , \\ N_Y^+(X, Y, Z, P_X, P_Y, P_Z) &= N_Y^-(X, Y, Z, P_X, P_Y, P_Z) \quad , \\ N_Z^+(X, Y, Z, P_X, P_Y, P_Z) &= N_Z^-(X, Y, Z, P_X, P_Y, P_Z) \quad , \end{aligned}$$

and therefore are the local representatives of a functions $\mathcal{N}_X, \mathcal{N}_Y, \mathcal{N}_Z$ globally defined in \mathcal{D}_E . Now we allow E change in a small neighbourhood \mathcal{E} of a given E_* , and we consider the set of three first integrals:

$$\left(H, \mathcal{N}^2 := \mathcal{N}_X^2 + \mathcal{N}_Y^2 + \mathcal{N}_Z^2, \mathcal{N}_Z \right)$$

defined in $\cup_{E \in \mathcal{E}} \mathcal{D}_E$. We have the following:

Theorem. *The set of first integrals $(H, \mathcal{N}^2, \mathcal{N}_Z)$ is complete.*

Proof. Let us prove that $(H, \mathcal{N}^2, \mathcal{N}_Z)$ are independent in a set $\cup_{E \in \mathcal{E}} \mathcal{D}_E$. We first prove that $\mathcal{N}^2, \mathcal{N}_Z$ are independent on E , by showing that they are not constant on the energy levels $H(X, Y, Z, P_X, P_Y, P_Z) = E$.

For any arbitrary small ε , in the set

$$\{(X, Y, Z, P_X, P_Y, P_Z) \in \cup_{E \in \mathcal{E}} \mathcal{D}_E, \quad 0 < \|(X, Y, Z)\| < \varepsilon^2\} \quad (62)$$

we have:

$$\begin{aligned} \mathcal{N}_X &= P_Y Z - P_Z Y + D_X \\ \mathcal{N}_Y &= P_X Z - P_Z X + D_Y \\ \mathcal{N}_Z &= P_X Y - P_Y X + D_Z \\ \mathcal{N}^2 &= (P_Y Z - P_Z Y)^2 + (P_X Z - P_Z X)^2 + (P_X Y - P_Y X)^2 + D^2 \end{aligned} \quad (63)$$

where the functions D_X, D_Y, D_Z have sup-norm bounded by order ε^3 and D^2 bounded by order ε^4 in the set (62). Therefore, if we fix the value of E and one between $\mathcal{N}_Z, \mathcal{N}^2$, the third integral is not constant in the level set of the first two.

Let us now compute the Poisson brackets. Since $\mathcal{N}_Z, \mathcal{N}^2$ are first integrals for the Hamilton equations of H , we have:

$$\{H, \mathcal{N}_Z\} = 0, \quad \{H, \mathcal{N}^2\} = 0.$$

It remains to compute the Poisson bracket $\{\mathcal{N}^2, \mathcal{N}_Z\}$. Let us denote by $\hat{q}(u), \hat{p}(u, U)$ the functions defined by:

$$\hat{q}(u) = \pi(u), \quad (\hat{p}(u, U), 0) = \frac{1}{2|u|^2} A(u)U.$$

We notice the remarkable property of the Poisson brackets:

$$\{\hat{q}_i, \hat{p}_j\} = \delta_{ij}, \quad \{\hat{q}_i, \hat{q}_j\} = 0, \quad \{\hat{p}_i, \hat{p}_j\} = l(u, U) \phi_{ij}(u, U), \quad i, j = 1, 2, 3. \quad (64)$$

and from:

$$\{\mathcal{N}^2, \mathcal{N}_Z\} = l(n, \nu) a(n, \nu), \quad (65)$$

we prove $\{\mathcal{N}^2, \mathcal{N}_Z\} = 0$. In fact, since $\mathcal{N}^2, \mathcal{N}_Z$ they are invariant by composition with the map $(n, \nu) \mapsto (\mathcal{S}_\alpha^0 n, \mathcal{S}_\alpha^0 \nu)$ for any α , we have:

$$\hat{N}_Z(u, U; E) = N_z(\chi_4(u, U; E)) = \mathcal{N}_Z(\hat{q}(u, U), \hat{p}(u, U))$$

$$\hat{N}^2(u, U; E) = N^2(\chi_4(u, U; E)) = \mathcal{N}^2(\hat{q}(u, U), \hat{p}(u, U)).$$

By denoting with \mathbb{E}_k the standard symplectic matrix of \mathbb{R}^{2k} and $q = (X, Y, Z), p = (P_X, P_Y, P_Z)$, we have:

$$\begin{aligned} \{\hat{N}^2(u, U; E), \hat{N}_Z(u, U; E)\} &= \left(\frac{\partial N^2}{\partial u}, \frac{\partial N^2}{\partial U} \right) \cdot \left(\mathbb{E}_4 \left(\frac{\partial N_Z}{\partial u}, \frac{\partial N_Z}{\partial U} \right) \right) \\ &= \left(\frac{\partial \mathcal{N}^2}{\partial q}, \frac{\partial \mathcal{N}^2}{\partial p} \right) \cdot \left(J(u, U)^T \mathbb{E}_4 J(u, U) \left(\frac{\partial \mathcal{N}_Z}{\partial q}, \frac{\partial \mathcal{N}_Z}{\partial p} \right) \right) \end{aligned}$$

where J is the 8×6 Jacobian matrix of $(\hat{q}(u), \hat{p}(u, U))$. From the Poisson brackets (64) we notice that the 6×6 matrix $J(u, U)^T \mathbb{E}_4 J(u, U)$ is not identically equal to \mathbb{E}_3 , but when it is computed on (u, U) satisfying $l(u, U) = 0$ we have: $J(u, U)^T \mathbb{E}_4 J(u, U) = \mathbb{E}_3$. But, from (65), for $l(u, U) = 0$ we also have $\{\hat{N}^2(u, U; E), \hat{N}_Z(u, U; E)\} = 0$. Finally, since we have identified the preimages of \hat{q}, \hat{p} satisfying $l(u, U) = 0$, we have: $\{\mathcal{N}^2, \mathcal{N}_Z\} = 0$. \square

7 Appendix 1: a revisit of the integrability of the LC Hamiltonian in a neighbourhood of the collision singularities

Let us consider the Hamiltonian of the planar circular restricted three-body problem in the planetocentric reference frame (see (8) for comparison):

$$\begin{aligned} H_2(X, Y, P_X, P_Y) &= \frac{P_X^2 + P_Y^2}{2} + P_X Y - P_Y X - \frac{\mu}{\sqrt{X^2 + Y^2}} \\ &- (1 - \mu) \left(\frac{1}{\sqrt{(X+1)^2 + Y^2}} - 1 + X \right) - (1 - \mu) - \frac{(1 - \mu)^2}{2}. \end{aligned} \quad (66)$$

Following Levi-Civita we first define the canonical transformation:

$$(X, Y, P_X, P_Y) = \mathcal{Y}(u_1, u_2, U_1, U_2)$$

where:

$$X = u_1^2 - u_2^2, \quad Y = 2u_1 u_2$$

represents the equations (2), (3) in the planetocentric reference frame, and:

$$P_X = \frac{U_1 u_1 - U_2 u_2}{2|u|^2}, \quad P_Y = \frac{U_1 u_2 + U_2 u_1}{2|u|^2}$$

the canonical extension to the momenta U_1, U_2 . The transformation \mathcal{Y} conjugates H_2 to the Hamiltonian:

$$K_2(u_1, u_2, U_1, U_2) = \frac{1}{8|u|^2} \left(U_1 + 2|u|^2 u_2 \right)^2 + \frac{1}{8|u|^2} \left(U_2 - 2|u|^2 u_1 \right)^2 - \frac{1}{2} |u|^4 - \frac{\mu}{|u|^2}$$

$$-(1-\mu) \left(\frac{1}{\sqrt{1+2(u_1^2-u_2^2)+|u|^4}} - 1 + u_1^2 - u_2^2 \right) - (1-\mu) - \frac{(1-\mu)^2}{2}. \quad (67)$$

To remove the singularity at $u = 0$ we perform the iso-energetic reduction: for any value E of the Hamiltonian, we introduce the LC Hamiltonian:

$$\begin{aligned} \mathcal{K}_2(u, U; E) &= |u|^2 (K_2(u, U) - E) = \frac{1}{8} (U_1 + 2|u|^2 u_2)^2 + \frac{1}{8} (U_2 - 2|u|^2 u_1)^2 \\ &\quad - \frac{1}{2} |u|^6 - \mu - |u|^2 \left(E + (1-\mu) + \frac{(1-\mu)^2}{2} \right) \\ &\quad - (1-\mu) |u|^2 \left(\frac{1}{\sqrt{1+2(u_1^2-u_2^2)+|u|^4}} - 1 + u_1^2 - u_2^2 \right). \end{aligned} \quad (68)$$

The LC Hamiltonian is regular at $u = (0, 0)$, and the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}_2(u, U)$:

$$\frac{d}{ds} u_i = \frac{\partial \mathcal{K}_2}{\partial U_i}, \quad \frac{d}{ds} U_i = -\frac{\partial \mathcal{K}_2}{\partial u_i},$$

with initial conditions satisfying $u(0) \neq 0$ and $\mathcal{K}_2(u, U) = 0$, are conjugate, in a neighbourhood of $s = 0$, to solutions $(X(t), Y(t), P_x(t), P_y(t))$ of the Hamilton equations of (66) after the replacement of the proper time s with the time t through the formula:

$$t(s) = \int_0^s |u(\tau)|^2 d\tau. \quad (69)$$

A complete integral of the Hamilton-Jacobi equation:

$$\mathcal{K}_2 \left(u, \frac{\partial W}{\partial u}; E \right) = \kappa, \quad (70)$$

is a family of solutions of (70) depending on two parameters, satisfying the usual non-transversality property. We identify the two parameters in⁶ κ and, following T. Levi-Civita, an angle α related to the rotations of the plane u_1, u_2 . We have the following proposition: there exists a family of solutions $W(u_1, u_2, \alpha, \kappa)$ of the Hamilton-Jacobi equation (70), defined for $\alpha \in \mathbb{S}^1$, κ in a neighbourhood of $\kappa = 0$, and analytic for u_1, u_2 in neighbourhood of $(0, 0)$ of radius $\sigma > 0$, with σ depending from E and μ . Moreover:

(i) the Taylor series of W :

$$W = \sum_{n_1, n_2} W_{n_1, n_2}(\alpha, \mu, \kappa, E) u_1^{n_1} u_2^{n_2}$$

⁶In [26] only the case $\kappa = 0$ was considered, which actually is the only value which grants the conjugation between the solutions of the regularized and non-regularized equations.

has coefficients periodic in α , which can be computed iteratively to any order $n_1 + n_2$. In particular, we have:

$$W = \sqrt{8(\mu + \kappa)}(u_1 \cos \alpha + u_2 \sin \alpha) + \mathcal{O}_2(u_1, u_2). \quad (71)$$

(ii) By denoting:

$$j_2(u_1, u_2, \alpha, \kappa) = \det \begin{pmatrix} \frac{\partial W}{\partial u_1 \partial \alpha} & \frac{\partial W}{\partial u_1 \partial \kappa} \\ \frac{\partial W}{\partial u_2 \partial \alpha} & \frac{\partial W}{\partial u_2 \partial \kappa} \end{pmatrix},$$

from (71), we obtain $j_2(0, 0, \alpha, \kappa) = 4$. Therefore we have $j_2(0, 0, \alpha, \kappa) \neq 0$ in a neighbourhood of $(u_1, u_2) = (0, 0)$, uniformly in α and for κ in a small neighbourhood of $\kappa = 0$.

As a consequence, the system:

$$\begin{aligned} U_1 &= \frac{\partial W}{\partial u_1}(u_1, u_2, \alpha, \kappa) \\ U_2 &= \frac{\partial W}{\partial u_2}(u_1, u_2, \alpha, \kappa) \\ \beta &= -\frac{\partial W}{\partial \alpha}(u_1, u_2, \alpha, \kappa) \\ K &= s + \frac{\partial W}{\partial \kappa}(u_1, u_2, \alpha, \kappa) \end{aligned} \quad (72)$$

defines by inversion a s -dependent canonical transformation

$$(\alpha, \kappa, \beta, K) = \chi_2(s, \alpha, \kappa, \beta, K),$$

conjugating the Hamiltonian \mathcal{K}_2 to the zero-value Hamiltonian $\hat{\mathcal{K}}_2(s, \alpha, \kappa, \beta, K) = 0$. In particular, by selecting the value $\kappa = 0$, equations (72) provide the solution to the problem of planar close encounters.

The proof of the existence of the complete integral W has been done in [26] as follows. Consider the canonical transformation

$$u = \mathcal{R}_\alpha \tilde{u} \quad , \quad U = \mathcal{R}_\alpha \tilde{U}$$

where

$$\mathcal{R}_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad (73)$$

conjugating $\mathcal{K}_2(u, U; E)$ to the Hamiltonian:

$$\begin{aligned} \tilde{\mathcal{K}}_2(\tilde{u}, \tilde{U}; \alpha, E) &= \frac{1}{8} \left(\tilde{U}_1 + 2|\tilde{u}|^2 \tilde{u}_2 \right)^2 + \frac{1}{8} \left(\tilde{U}_2 - 2|\tilde{u}|^2 \tilde{u}_1 \right)^2 \\ &\quad - \frac{1}{2} |\tilde{u}|^6 - \mu - |u|^2 \left(E + (1 - \mu) + \frac{(1 - \mu)^2}{2} \right) \end{aligned}$$

$$-(1-\mu)|\tilde{u}|^2 \left(\frac{1}{\sqrt{1+2(\tilde{u}_1^2 - \tilde{u}_2^2)\cos 2\alpha + |\tilde{u}|^4}} - 1 + (\tilde{u}_1^2 - \tilde{u}_2^2)\cos 2\alpha \right), \quad (74)$$

and look for a particular solution $\tilde{W}(\tilde{u}, \alpha, \kappa)$ of the Hamilton–Jacobi equation:

$$\tilde{K}_2 \left(\tilde{u}, \frac{\partial \tilde{W}}{\partial u}; \alpha, E \right) = \kappa$$

satisfying:

$$\tilde{W}(0, \tilde{u}_2, \alpha, \kappa) = 0 \quad (75)$$

for all u_2 in a neighbourhood of 0. The existence of a solution to this problem which is analytic in a neighbourhood of $\tilde{u} = 0$, (with a common analyticity radius to all α) is quoted in [26] as a consequence on a general result about the regularity of the solutions of first order PDE, which we identify in the Cauchy–Kowaleski theorem (see [6], and the Appendix). The complete integral $W(u; \alpha, \kappa)$ is then defined by:

$$W = \tilde{W}(\mathcal{R}_\alpha^T u; \alpha, \kappa).$$

As it is usual in the Cauchy–Kowaleski theorem, the coefficients of the series expansion of W in u_1, u_2 can be computed iteratively up to any arbitrary order.

8 Appendix 2: the Cauchy-Kowaleski theorem

We consider the first order PDE:

$$\frac{\partial W}{\partial q_1} = F \left(q_1, q_2, \dots, q_n, \frac{\partial W}{\partial q_2}, \dots, \frac{\partial W}{\partial q_n} \right) \quad (76)$$

where $F(q_1, \dots, q_n, p_1, \dots, p_{n-1})$ is analytic in a neighbourhood of $q = (q_1, \dots, q_n) = 0$, $(p_1, \dots, p_{n-1}) = 0$. We call the plane $q_1 = 0$ the initial plane in the space of the variables q ; then, we consider the Cauchy’s problem of finding a solution $W(q)$ of the PDE (76) satisfying the given initial condition:

$$W(0, q_2, \dots, q_n) = \phi(q_2, \dots, q_n) \quad (77)$$

in a suitable neighbourhood of $(q_2, \dots, q_n) = (0, \dots, 0)$, where ϕ is a given function analytic in a neighbourhood of $(q_2, \dots, q_n) = (0, \dots, 0)$. The Cauchy–Kowaleski theorem states (see for example [6]) that the Cauchy problem has a unique solution analytic in a suitable small neighbourhood of $q = 0$. We will continue our discussion in the case which is useful for our purposes, defined by the special choice of the initial condition:

$$\phi(q_2, \dots, q_n) = 0.$$

The proof is obtained by constructing first a formal series expansion:

$$W = \sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} q_1^{i_1} \cdots q_n^{i_n} \quad (78)$$

for the solution as follows. From: $W(0, q_2, \dots, q_n) = \phi(q_2, \dots, q_n) = 0$ we immediately obtain:

$$\frac{\partial^{i_2}}{\partial q_{j_2}^{i_2}} \cdots \frac{\partial^{i_n}}{\partial q_{j_n}^{i_n}} W(0, q_2, \dots, q_n) = 0 \quad , j_2, \dots, j_n = 2, \dots, n$$

for all $i_2, \dots, i_n \geq 1$; correspondingly, we have $c_{0, i_2, \dots, i_n} = 0$. The coefficients with $i_1 \neq 0$ are computed iteratively on the order $i_1 + \dots + i_n$ by differentiating (76) and by computing the result at $q = 0$.

Then, the proof of the absolute convergence of the expansion (78) in a neighbourhood of $q = 0$ is obtained by using the method of majorants. To apply the method, one first observes that the terms c_{i_1, \dots, i_n} computed as indicated above can be represented as polynomials of the terms of the Taylor expansions of F and ϕ at $q = 0$, and the coefficients of these polynomials are non negative numbers. By exploiting this property one constructs a PDE problem whose solution can be given explicitly (and so its analyticity can be directly checked) and is a majorant of W . For the purposes of our paper, it is crucial to remark that all the differential equations whose solution have the same majorant converge in a common domain of $q = 0$.

Acknowledgements. This research has been supported by ERC project 677793 Stable and Chaotic Motions in the Planetary Problem. The author F.C. acknowledges also the project MIUR-PRIN 2017S35EHN titled ‘‘Regular and stochastic behaviour in dynamical systems’’. The author M.G. acknowledges also the project MIUR-PRIN 20178CJA2B titled ‘‘New frontiers of Celestial Mechanics: theory and applications’’.

References

- [1] Arnold V.I., Kozlov V.V. and Neishtadt A.I., Mathematical aspects of classical and celestial mechanics, *volume 3 of Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.
- [2] Berger Ma. and Berger Me., Perspectives in Nonlinearity, an introduction to nonlinear analysis, W.A. Benjamin Inc. publ., Amsterdam, 1968.
- [3] Celletti A., The Levi-Civita, KS and Radial-Inversion Regularizing Transformations, in Singularities in Gravitational Systems, Lecture Notes in Physics 590, D. Benest and Cl. Froeschlé eds., Springer, p. 25-48, 2002.

- [4] , Celletti, A. and Pucacco, G. and Stella, D., Lissajous and Halo orbits in the Restricted Three-Body Problem, *J. Nonlinear Science*, 25(2), 343-370, 2015.
- [5] Cardin F., Guzzo M., Integrability of the spatial restricted three-body problem near collisions (an announcement), *Rend. Lincei Mat. Appl.* 30, 195-204, 2019.
- [6] Courant R., Hilbert D., *Methods of Mathematical Physics, Vol. II*, Wiley-VCH Verlag GmbH & Co. KGaA, 1962.
- [7] Conley C., Low Energy Transit Orbits in the Restricted Three-Body Problems. *SIAM J. Appl. Math.*, 16(4), 732-746, 1967.
- [8] ElBialy M.S., The Kustaanheimo-Stiefel map, the Hopf fibration and the square root map on \mathbb{R}^3 and \mathbb{R}^4 , *J. Math. Anal. Appl.*, vol. 332, 2007.
- [9] Font J., Nunes A., Simó C., Consecutive quasi-collisions in the planar circular RTBP, *Nonlinearity*, 15, 115, 2002.
- [10] Froeschlé C., Numerical Studies of Dynamical Systems with Three Degrees of Freedom, *Astron. & Astrophys.*, vol. 4, 115-128, 1970.
- [11] Gladman B., The Kuiper belt and the solar system's comet disk, *Science*, vol. 307, 71-75, 2005.
- [12] Gomez, G. and Koon, W.S. and Lo, M.W. and Marsden, J.E. and Masdemont, J. and Ross, S.D., Connecting orbits and invariant manifolds in the spatial restricted three-body problem, *Nonlinearity*, 17, 1571-1606, 2004.
- [13] Guardia M., Kaloshin V., Zhang J.: Asymptotic Density of Collision Orbits in the Restricted Circular Planar 3 Body Problem, *Archive for Rational Mechanics and Analysis*, vol. 233, Issue 2, pp 799-836, 2019.
- [14] Guzzo M., Lega E., "On the identification of multiple close-encounters in the planar circular restricted three body problem." *Monthly Notices of the Royal Astronomical Society*, 428, 2688-2694, 2013.
- [15] Guzzo M. and Lega E., A study of the past dynamics of comet 67P/Churyumov-Gerasimenko with fast Lyapunov indicators, *Astronomy & Astrophysics*, 579, A79, 2015.
- [16] Guzzo M. and Lega E., Scenarios for the dynamics of comet 67P/Churyumov-Gerasimenko over the past 500 kyr, *MNRAS* 469, S321-S328, 2017.
- [17] Guzzo M. and Lega E., Geometric chaos indicators and computations of the spherical hypertube manifolds of the spatial circular restricted three-body problem, *Physica D*, vol. 373, 35-58, 2018.

- [18] Hurwitz A., Math. Werke II, 565, 1933.
- [19] Jorba A., Masdemont J., Dynamics in the center manifold of the restricted three-body problem, *Physica D* 132, 189-213, 1999.
- [20] Koon, W.S. and Lo, M.W. and Marsden, J.E. and Ross, S.D., *Dynamical Systems, The Three-Body Problem and Space Mission Design*, Springer Verlag, New York, 2007.
- [21] Kustaanheimo P., Spinor regularisation of the Kepler motion, *Annales Universitatis Turkuensis A* 73, 1-7. Also *Publications of the Astronomical Observatory Helsinki* 102, 1964.
- [22] Kustaanheimo P. and Stiefel E.L., Perturbation theory of Kepler motion based on spinor regularization, *Journal fur die Reine und Angewandte Mathematik* 218, 204-219, 1965.
- [23] , Lega, E. and Guzzo, M. and Froeschlé, C., Detection of close encounters and resonances in three-body problems through Levi-Civita regularization, *MNRAS*, 418, 107-113, 2011.
- [24] Le Verrier U.J., Théorie de la comete périodique de 1770, *Annales de l'Observatoire imperial de Paris, Memoires*, t. 3. Paris: Mallet-Bachelier, p. 203-270, 1-12, 1857.
- [25] Levi-Civita T., Sur la régularisation qualitative du problème restreint des trois corps, *Verhandl. des III Intern. Math.-Kongresses, Heidelberg*, 402-408, 1904.
- [26] Levi-Civita T., Sur la régularisation qualitative du problème restreint des trois corps, *Acta Math.*, vol. 30, 305-327, 1906.
- [27] Moser J., Regularization of Kepler's problem and the averaging method of a manifold. *Comm. on Pure and Appl. Math.*, vol. 23, 609-636, 1970.
- [28] Öpik E.J., *Interplanetary Encounters*, Elsevier, New York, 1976.
- [29] Paez R. and Guzzo M., A study of temporary captures and collisions in the Circular Restricted Three-Body Problem with normalizations of the Levi-Civita Hamiltonian, *I. J. Nonl. Mech.*, Vol. 120, 103417, 2020.
- [30] Poincaré H., *Les méthodes nouvelles de la mécanique céleste*, vol. 1-3. Gauthier-Villars, Paris 1892, 1893, 1899.
- [31] Siegel C.L., Moser J.K., *Lectures on Celestial Mechanics*, Reprint of the 1971 edition, Springer Verlag Berlin Heidelberg New York, 1971.

- [32] Simó C., Dynamical systems methods for space missions on a vicinity of collinear libration points, in Simó, C., editor, *Hamiltonian Systems with Three or More Degrees of Freedom* (S'Agaró, 1995), volume 533 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pp. 223-241, Dordrecht. Kluwer Acad. Publ., 1999.
- [33] Tisserand F.F., Sur la théorie de la capture de comètes périodiques, *Bulletin Astronomique*, vol. 6, p. 241-257, 1889.
- [34] Tommei G., Canonical elements for Opik theory, *Celestial Mechanics & Dynamical Astronomy*, Volume 94, Number 2, pp. 173-195, 2006.
- [35] Valsecchi G.B., Close Encounters in Öpik Theory, in *Lecture Notes in Physics, Singularities in Gravitational Systems*, D. Benest and Cl. Froeschlé editors, Springer, 2002.
- [36] Waldvogel J., Quaternions and the Perturbed Kepler Problem, *Celest. Mech. Dyn. Astron.*, vol. 95, 201-212, 2006.