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Amending "A note on solving the fleet quickest routing problem on a grid graph"

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Abstract
A recent paper (Di Giacomo et al. Cent Eur J Oper Res 28:1069–1090, 2020) on the Fleet Quickest Routing Problem on Grid graphs (FQRP-G) claims that eight levels guarantee that a fleet of vehicles, simultaneously starting from the bottom level of the grid, can reach the top level by moving on Manhattan paths without ever stopping and without collisions, independently of the number of vehicles and columns of the grid and the configuration of vehicles' origins and destinations. In this amending note, we will analyse the results in Di Giacomo et al. (Cent Eur J Oper Res 28:1069–1090, 2020) and show that the routing rule leading to this claim cannot be applied to all instances of FQRP-G, and that it can cause collisions. The analysis points out sufficient conditions under which the proposed rule draws proven collision-free routes for FQRP-G using no more than eight levels. Computational experiments demonstrate the relevance of the eight-levels bound and the functionality of the routing rule in practice, showing that it correctly solves the majority of instances up to one thousand vehicles. From a theoretical perspective, the claim that eight (or any constant number of) levels are sufficient to solve any instance of FQRP-G remains an open issue.

Keywords Collision-free routing · Grid graphs · Manhattan paths · Routing rule

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1 Introduction

The fleet quickest routing problem on grid graphs (FQRP-G) considers n vehicles moving on a grid graph with m rows (or levels) and n columns. Each node is identified by a pair (p, q), where p and q are, respectively, the row and column indexes. Any vehicle i has to move from its origin, located in position $(1, i)$, to its destination in position $(m, \sigma(i))$. All vehicles start at the same time and all edges have the same crossing time. In order to avoid collisions, any two vehicles cannot visit the same node, or use the same edge, at the same time. The problem is to find a collision-free routing of the vehicles that minimizes the time needed by all the vehicles to reach their destinations. Under these conditions, it can be observed (Andreatta et al. 2010; Di Giacomo et al. 2020) that a solution is optimal if all vehicles move on origin-to-destination Manhattan paths (i.e., shortest paths on a grid) without any intermediate stop. Therefore, following (Andreatta et al. 2010; Cenci et al. 2017; Di Giacomo et al. 2020), we only consider as feasible (hence optimal) solutions the ones made of collision-free nonstop Manhattan paths. However, it is also observed that such solutions may not exist if m is not large enough. In Cenci et al. (2017), Di Giacomo et al. (2020) the number m of levels is thus considered as a parameter, that should be as small as possible in order to obtain low values of the optimal solution, but high enough to guarantee the existence of a feasible solution.

An interesting upper bound for m is proposed in Cenci et al. (2017), where it is shown that a solution to any instance of FQRP-G can be found using m^* levels, where m^* is an increasing function of n and it is equal to $3 + \lfloor \frac{n-1}{4} \rfloor$ if $n \geq 3$, 2 otherwise. The same article also proves that such m^* is indeed the minimum number of levels required by any feasible solution to FQRP-G, under the additional restrictive assumption that allowed Manhattan paths are *simple*, i.e., all the horizontal moves of a vehicle, if any, take place consecutively on the same level.

The bound for parameter m is further improved in Di Giacomo et al. (2020), where the following result is stated: *Eight levels are sufficient to solve any instance of FQRP-G* (Proposition 8 in Di Giacomo et al. 2020).

The proof proposed in Di Giacomo et al. (2020) also includes a routing rule (RR) that exploits the opportunity of using non-simple Manhattan paths to build n routes, one for each vehicle of any FQRP-G instance, on a grid with a constant number (i.e., 8, independent from n) of levels.

In this note, we identify some issues related to RR that may yield collisions in the output routing. By analysing these issues, we propose an amendment to Proposition 8 in Di Giacomo et al. (2020), stating two assumptions on considered FQRP-G instances that guarantee that eight levels are sufficient. In fact, we identify a class of FQRP-G instances for which the proofs provided in Di Giacomo et al. (2020) show that RR outputs collision-free routing. Out of this class, we also build counterexamples where RR indeed fails. In order to state the practical implications of the proposed amendment, we also perform an extensive empirical analysis to estimate the percentage of FQRP-G instances that satisfy the amending assumptions or, otherwise, the percentage of instances where RR anyway outputs collision-free routing: empirical evidence shows that, up to $n = 1000$, only in a minority of cases

RR generates collisions and thus that, although not general, $m = 8$ is a relevant bound from a practical perspective.

2 Summary of the relevant notation and the routing rule

We briefly revise some notions from Di Giacomo et al. (2020) that will be useful in the following (see the cited paper for details).

Given an instance of FQRP-G, its vehicles set can be partitioned into three subsets, according to the direction of their horizontal moves: $R_1 = \{i \in V : \sigma(i) > i\}$, $L_1 = \{i \in V : \sigma(i) < i\}, H = \{i \in V : \sigma(i) = i\}$.

Two vehicles i and j are in *node conflict* (resp. *arc conflict*) if there exist a Manhattan path for i and a Manhattan path for j that visit the same node (resp. cross the same edge in opposite directions) at the same time; if both vehicles use such paths, then they *collide* on a node (resp. on an arc), causing a node (resp. arc) *collision*. Notice that we here prefer to distinguish conflicts from collisions, even tough the term "collision" does not appear in Di Giacomo et al. (2020) and is sometimes replaced by "occurring conflict".

Following Di Giacomo et al. (2020), node conflicts are further classified into three types. Two vehicles are in *node conflict of type A* (or *A-conflict*) if the conflict node is unique and coincides with one of their destinations, meaning that a collision of type A only occurs on the top level when a vehicle has already arrived at its destination node, and the other vehicle crosses such node. Vehicles i and j are in *node conflict of type B* (or *B-conflict*) if they conflict on every node of the median column $\frac{i+j}{2}$, and the median column is neither column $\sigma(i)$ nor $\sigma(j)$. Vehicle i is *subject to a conflict of type C (or C-conflict) with* vehicle j if i and j conflict on every node of the median column $\frac{i+j}{2}$, and the median column $\frac{i+j}{2}$ coincides with $\sigma(i)$ (notation $i(j)$ is used). In such case, the collision can only be avoided if vehicle i reaches its destination column after (hence, at a higher level) vehicle j has left it.

A *C-conflict path* is a sequence of vehicles $(z_1, z_2, \dots, z_k)$ such that, for each vehicle z_i with $1 \le i \le k-1$, $z_i(z_{i+1})$, i.e., each vehicle is subject to a C-conflict with its successor, and also there exists another vehicle v such that $z_k(v)$. A vehicle z_i in such C-conflict path, with $2 \le i \le k$, is *sigma adjacent* if $|\sigma(z_i) - \sigma(z_{i-1})| = 1$.

The routing rule (RR) proposed in Di Giacomo et al. (2020) is an algorithm that, given an instance of FQRP-G, assigns a route to each vehicle using 8 levels, independently of the size n of the instance. According to RR, reported in Fig. 1 for ease of reference, vehicles in H move only vertically, whereas vehicles in R_1 and, respectively, L_1, perform their horizontal moves at odd and, respectively, even levels. The routes for vehicles in R_1 and L_1 depend on their positions in a C-conflict path: vehicles not included in C-conflict paths (see cases 2. and 3. of RR) and the first element of each C-conflict path (see cases 4.1. and 5.1.) use simple Manhattan paths; the routes of the remaining vehicles contain horizontal moves at two or three distinct levels and their shapes depend on the fact that the vehicles are sigma adjacent (cases 4.2. and 5.2.) or not (cases 4.3. and 5.3.). The columns where the vertical

Given an instance of FQRP-G, route any vehicle i in the grid as follows:

1. If $i \in H$, then it performs only vertical steps until its destination.
2. If $i \in R_1$ and it does not belong to a C-conflict path, then it performs all its horizontal moves on level 1 and, once arrived at its destination column, all its vertical moves.
3. If $i \in L_1$ and it does not belong to a C-conflict path, then it performs one vertical step, then all its horizontal moves on level 2 and, once arrived at its destination column, all the remaining vertical moves.
4. If $i \in R_1$ and it belongs to a C-conflict path, consider the following cases:
   - 4.1. i is the first element of the C-conflict path, i.e., there exists j such that $i(j)$ and there does not exist any k such that $k(i)$
     Vehicle i performs six vertical moves until level 7, then all its horizontal steps; once arrived at its destination column, the last vertical move.
   - 4.2. i is sigma adjacent
     Vehicle i moves horizontally on the first level until column $\sigma(i) - 1$; it then performs 4 vertical steps until level 5; then one more horizontal step reaching column $\sigma(i)$ and, in the end, all remaining vertical moves.
   - 4.3. i is not sigma adjacent and there is a vehicle v such that $v(i)$
     Vehicle i moves horizontally on the first level, until it reaches column $\sigma(v)$, then it performs two vertical steps until level 3, and then one more horizontal move until node $(3, \sigma(v) + 1)$. Then two more vertical moves until level 5, all the remaining horizontal moves and, in the end, the remaining vertical steps.
5. If $i \in L_1$ and it belongs to a C-conflict path, consider the following cases:
   - 5.1. i is the first element of the C-conflict path, i.e., there exists j such that $i(j)$ and there does not exist any k such that $k(i)$
     Vehicle i performs five vertical moves until level 6, then all its horizontal steps; once arrived at its destination column, the last vertical moves.
   - 5.2. i is sigma adjacent
     Vehicle i first performs a vertical step, then it moves horizontally on the second level until column $\sigma(i) + 1$. Then it performs two vertical moves reaching node $(4, \sigma(i) + 1)$, one horizontal step reaching column $\sigma(i)$, and then all its remaining vertical moves.
   - 5.3. i is not sigma adjacent and there is a vehicle v such that $v(i)$
     Vehicle i first performs a vertical step, and then moves horizontally on the second level, until column $\sigma(v)$. It then performs two vertical steps until level 4, then $\sigma(v) - \sigma(i) - 1$ horizontal moves until column $\sigma(i) + 1$, two vertical moves until level 6, the last horizontal move and, in the end, all the remaining vertical steps.

Fig. 1 The routing rule (RR) for FQRP-G proposed in Di Giacomo et al. (2020)

steps of a non sigma adjacent vehicle i take place also depend on a vehicle v such that $v(i)$. In particular, vertical steps occur between:

- levels 1 and 3 of column $\sigma(v)$, levels 3 and 5 of column $\sigma(v)+1$ and levels 5 and 8 of column $\sigma(i)$ (vehicles in R_1, case 4.3.) or
- levels 1 and 2 of column i, levels 2 and 4 of column $\sigma(v)$, levels 4 and 6 of column $\sigma(i)+1$ and levels 6 and 8 of column $\sigma(i)$ (vehicles in L_1, case 5.3.).

3 Critical review of the routing rule

In Di Giacomo et al. (2020), RR is used to show that there exists a routing rule that allows all vehicles to travel on a eight-levels grid using collision-free nonstop Manhattan paths. In this section, we will point out some issues related to RR whose main consequence is that, on the contrary, there are instances where RR leads to collisions.

3.1 Issue 1: RR does not uniquely identify the route of vehicles that belong to more than one C-conflict path

Let us consider vehicles i, j and k such that $j(i)$, $k(i)$, and i is subject to a C-conflict with another vehicle, i.e., i belongs to (at least) two C-conflict paths. If i falls into cases 4.3. or 5.3., its routing is different depending on the choice of the vehicle v mentioned in these cases, which is not uniquely identified by RR (it may be, for example, j or k).

This ambiguity is relevant because the arguments presented in Di Giacomo et al. (2020) only guarantee that the collision of i is avoided with the chosen v, with no discussion involving further C-conflicting vehicles, if any. Indeed, the following Counterexample 1 shows an instance with a vehicle in case 4.3. belonging to two C-conflict paths, such that one of the C-conflicts leads to a collision. A similar counterexample could be given for case 5.3.

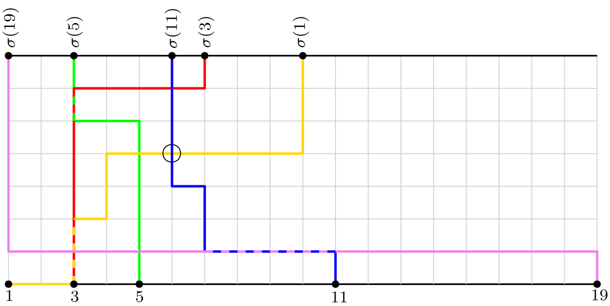

Fig. 2 Counterexample 1: RR generates a collision between vehicles 1 and 11

Counterexample 1 Consider an instance of FQRP-G on a grid with 19 columns and including vehicles origins and destinations as in Fig. 2. The following C-conflicts appear: 3(11), 11(1), 1(19) and 5(1). Therefore, vehicle 1 belongs to two C-conflict paths and case 4.3. does not specify whether its route is defined by either C-conflict 11(1) or 5(1), showing the ambiguity of RR. Here 5(1) is chosen, Fig. 2 shows the RR vehicle's routes and a C-collision between vehicles 1 and 11 at the circled node (5, 6).

We remark that, if i is sigma adjacent and therefore falls into cases 4.2. or 5.2., then its route is uniquely identified by RR. However, discussion in Di Giacomo et al. (2020) only guarantees that the C-collision of i related to sigma adjacency is avoided, while further possible C-collisions between i and any other vehicle z such that $z(i)$ are not analysed.

Observation 1 RR may generate C-collisions in case one vehicle belongs to two or more C-conflict paths.

3.2 Issue 2: The statement of Corollary 1 in Di Giacomo et al. (2020) is incomplete

Corollary 1 in Di Giacomo et al. (2020) considers two vehicles i and j in B-conflict and states that, assuming that each grid level allows horizontal movements in one direction only, *a sufficient condition to avoid their collision is that vehicle i (or j) does not perform any vertical step on the median column $\frac{i+j}{2}$*. The corollary is given, without proof, after Proposition 2 (Di Giacomo et al. 2020), which characterizes routes with no B-type collisions. In particular, Proposition 2 works under the condition that the minimum level used by i (or, resp., j) on column $\frac{i+j}{2}$ is strictly greater than the maximum level used by j (or, resp., i) on the same column. The corollary is indeed a direct consequence of such proposition, provided that vehicles i and j satisfy the same conditions, which however are not made explicit in its statement. In fact, if the paths of i and j satisfy the original statement of Corollary 1 in Di Giacomo et al. (2020), but they do not satisfy the aforementioned conditions, then vehicles in B-conflict may collide, as shown by Counterexample 2.

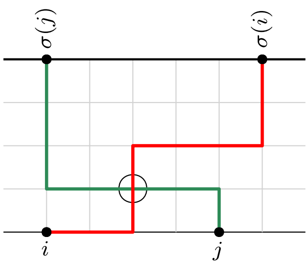

Fig. 3 Counterexample 2: a collision under the hypothesis of Corollary 1 in Di Giacomo et al. (2020)

Counterexample 2 Consider Fig. 3: the routes of the B-conflicting vehicles i and j have a collision and satisfy the hypothesis of Corollary 1 in Di Giacomo et al. (2020).

Issue 2 has impact, at least from a formal point of view, on the proof of Proposition 8 in Di Giacomo et al. (2020), where Corollary 1 is mentioned. In particular, to exclude that RR generates any B-collision between two vehicles $i \in R_1$ and $j \in L_1$, the proof starts by distinguishing two situations: either i and j both belong to C-conflict paths and are not sigma adjacent (that is, i is routed by case 4.3. and j by case 5.3.), or not. The first situation will be analysed in the following Issue 3. Corollary 1 is mentioned in the second situation, in particular, to exclude B-collisions when i falls into case 4.3. or j falls into case 5.3.: this part of the proof can be straightforwardly amended since B-collisions are excluded as a consequence of Proposition 2 in Di Giacomo et al. (2020), instead of Corollary 1.

3.3 Issue 3: RR may generate B-collisions

Let $i \in R_1$ and $j \in L_1$ be two vehicles in conflict of type B. The last part of the proof of Proposition 8 in Di Giacomo et al. (2020) deals with B-collisions between i and j when they both belong to a C-conflict path (that may be different for i and j) and are not sigma adjacent, i.e., RR routes i and j according to cases 4.3. and 5.3., respectively. In particular, the proof focuses on the case one vehicle moves vertically on the median column $b = \frac{i+j}{2}$, and the other crosses the same column horizontally. Here, the proof presents some flaws, that we illustrate by distinguishing two possible situations, depending on i or, respectively, j is the vehicle that moves vertically on column b. The potential collision where $i \in R_1$ moves vertically in b is explicitly taken into account by the proof, that properly identifies the collision node between levels 3 and 5, where the vertical moves of i on column b take place. Then, the proof claims that vehicle j crosses b at level either 2 or 6, which would exclude collisions. However, we notice that the path of j includes relevant horizontal moves even at level 4 (see case 5.3. of RR). Indeed, Counterexample 3 shows that these moves may cross column b and cause a B-collision between i and j. The case where j moves vertically in b is not explicitly treated in the proof. Even in this case, RR may generate collisions, as shown by Counterexample 4.

Counterexample 3 In the instance of FQRP-G represented in Fig. 4, the following C-conflicts appear: 13(17), 17(1), 11(15), 15(5) and 5(19). There are two disjoint C-conflict paths, (13,17) and (11,15,5), and Fig. 4 discloses the routes of involved vehicles, as from RR: the paths of vehicles 5 and 17 are drawn by cases 4.3. and 5.3., respectively, and have a B-collision at circled node (4,11).

Counterexample 4 Consider the instance of FQRP-G on a grid with 19 columns represented in Fig. 5. The following C-conflicts appear: 9(5), 5(15), 15(1), 7(3) and 3(19). There are two disjoint C-conflict paths, (9,5,15) and (7,3), and Fig. 5 discloses

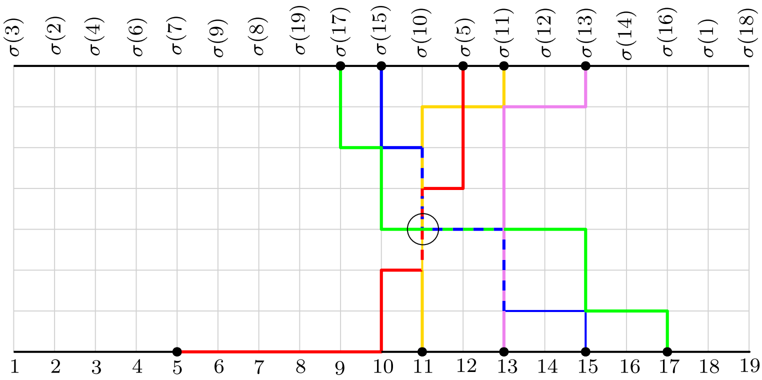

Fig. 4 Counterexample 3: RR generates a collision between vehicles 5 and 17

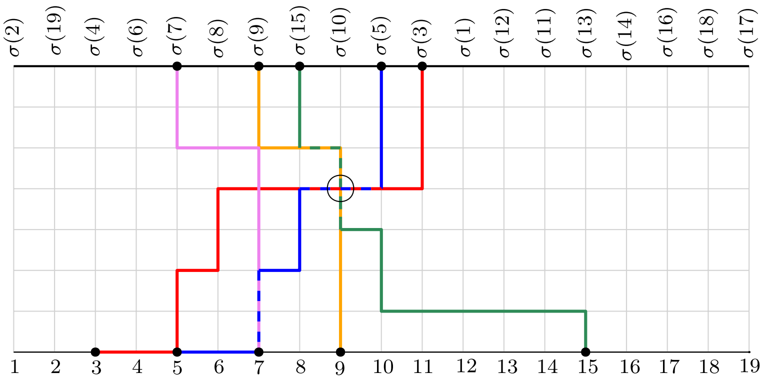

Fig. 5 Counterexample 4: RR generates a collision between vehicles 3 and 15

the routes of involved vehicles, as from RR: the paths of vehicles 15 and 3 are drawn by cases 5.3. and 4.3., respectively, and have a B-collision at circled node (5,9).

We remark that Issue 3 affects RR cases 4.3. and 5.3., which are also involved in Issue 1. However, in both the proposed counterexamples, each vehicle belongs to at most one C-conflict path and Issue 1 does not apply, so that B-collisions occur independently of Issue 1.

Observation 2 RR may generate B-collisions between two B-conflicting vehicles $i \in R_1$ and $j \in L_1$ if they both belong to C-conflict paths, are not the first vehicle of the respective C-conflict path and are not sigma adjacent, i.e., i is in case 4.3. and j is in case 5.3..

4 An amendment to Proposition 8 in [1]

We recall that Proposition 8 in Di Giacomo et al. (2020) affirms that eight levels are sufficient to solve any instance of FQRP-G, as a consequence of the correctness of RR. In view of the issues reported above, RR may generate collisions and cannot be used as argument in the proof of Proposition 8 in Di Giacomo et al. (2020). Of course, the same issues do not exclude that different arguments may exist to prove Proposition 8, like, e.g., a new routing rule that guarantees collision-free nonstop Manhattan paths using 8 (or less) levels. However, RR is still relevant, because available results from Di Giacomo et al. (2020) together with the analysis of the issues reported above allow us to amend the statement of Proposition 8 in Di Giacomo et al. (2020) and prove a weakened version of it.

To this end, we restrict our attention to instances of FQRP-G that satisfy the following assumptions.

Assumption 1 Each vehicle belongs to at most one C-conflict path.

Assumption 2 There are no B-conflicts between vehicles that (i) belong to C-conflict paths, (ii) are not the first vehicle of the respective C-conflict path and (iii) are not sigma adjacent.

For any instance of FQRP-G satisfying Assumption 1, the ambiguity of RR reported in Issue 1 is not relevant, as RR uniquely identifies the route of the vehicles and the C-collisions mentioned in Observation 1 are avoided. If such instance also satisfies Assumption 2, then, by the arguments in Issue 3, the B-collisions mentioned in Observation 2 are excluded as well. It follows that, under Assumptions 1 and 2, the proof of Proposition 8 in Di Giacomo et al. (2020) holds. By Issue 2, the only required adjustment is to refer to Proposition 2 in Di Giacomo et al. (2020) rather than to Corollary 1 in Di Giacomo et al. (2020).

The above discussion proves the following proposition, stating that Assumptions 1 and 2 together constitute a sufficient condition for RR to generate collision-free routing.

Proposition 1 *If an instance of FQRP-G satisfies Assumptions 1 and 2, then RR does not generate collisions.*

The following proposition summarizes preceding results.

Proposition 2 *Eight levels are sufficient to solve any instance of FQRP-G satisfying Assumptions 1 and 2.*

Table 1 Estimated impact of Assumptions 1 and 2 and functionality of RR in practice

n	10	25	50	75	100	150	200	300	400	500	1000
A1	99.1	93.3	84.1	75.4	67.9	54.7	44.1	28.7	18.7	12.1	1.4
A2	100.0	99.5	94.5	84.8	73.1	49.0	29.7	8.9	2.2	0.5	0.0
A1 & A2	99.1	92.8	80.2	65.9	52.6	30.7	16.4	3.9	0.8	0.1	0.0
Coll. free	100.0	99.6	98.8	98.0	97.2	95.5	93.9	90.6	87.6	84.6	71.2
Avg coll	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.2
Max coll	1	2	3	3	3	5	4	7	4	5	6

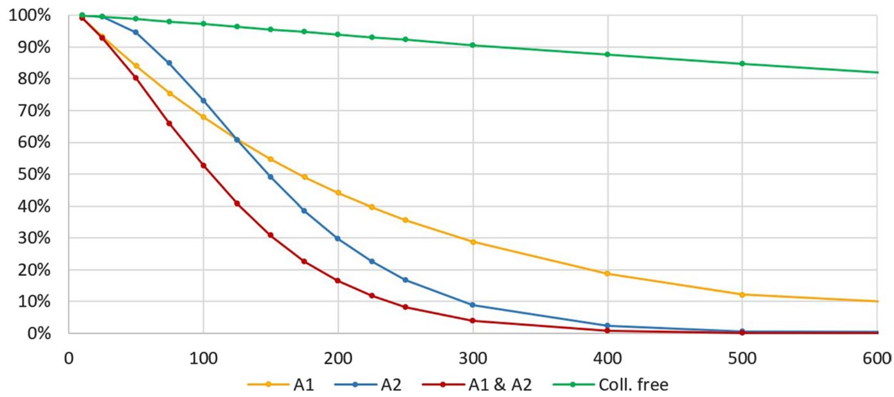

Fig. 6 Percentages of instances satisfying Assumption 1 (**A1**), Assumption 2 (**A2**), both assumptions (**A1 & A2**), and having a collision-free solution output by RR (**Coll. free**)

5 Computational results and conclusions

In order to analyze the impact of Propositions 1 and 2 on the results in Di Giacomo et al. (2020), we performed some computational experiments. First, we aim at estimating the percentage of instances that satisfy Assumptions 1 and/or 2. Moreover, we estimate the probability that RR can correctly solve an instance of FQRP-G, even in case Assumptions 1 or 2 are not satisfied. To this end, we implemented RR, solving the ambiguity related to Issue 1 by randomly choosing the vehicle v that shapes the routes of vehicles falling into cases 4.3. or 5.3. (code is available at Andreatta et al. 2023).

Experiments consider instances with number n of columns and vehicles between 10 and 1000. Since the number of all possible instances is $n!$, in order to keep the overall processing time within computational feasibility, we considered one million instances per size, each generated by assigning vehicles' destinations according to a random permutation of the column indexes. It follows that our samples are less representative as n increases, and the results are empirical estimates derived from the generated sample.

Results are summarized by Fig. 6 and details are reported in Table 1. The top row of the table indicates the instance size, the following three rows give the percentages of the one million sampled instances satisfying, respectively: Assumption 1, Assumption 2, and both assumptions. Row "Coll. free" reports the percentages of the one million sampled instances where RR generated collision-free routing, the last two rows refer to the remaining sampled instances and show the average and the maximum number of collisions observed in one instance in which at least one collision occurred. Fig. 6 plots the values of the first four rows of Table 1 for all the tested sizes but $n = 1000$.

In order to check if the obtained results depend on the specific set of instances considered, ten further independent repetitions of the experiments with different samples of one million instances have been performed: they took about one day computation overall, and each experiment consistently led to the same empirical estimates.

Up to size 50, more than 80% of the sample satisfies Assumptions 1 and 2, and therefore could be optimally solved by RR. Moreover, for the generated small-sized instances, RR always found collision-free routes, but in a few cases (less than 1.2%). The percentage of sampled instances satisfying the given assumptions drastically drops with larger instances, and it becomes negligible (less than 4%) after 300 vehicles. In fact, no generated instance with size equal to 1000 satisfied Assumption 2. The trend is justified by the fact that as the instance size increases, the likelihood of a vehicle conflicting with others also increases. Consequently, conditions that would violate the assumptions (vehicles belonging to more than one C-conflict path, or specific combinations of B- and C-conflicts) are more likely to arise, which explains the increased estimated frequency of violations. Nevertheless, our tests show that, at least in our sample, collisions are very rare, and RR is still able to find feasible routes for more than 90% of instances up to size 300. This fraction stays above 80% up to size 500, and above 70% with size 1000. We remark that RR optimally solves not only all instances satisfying Assumptions 1 and 2, but also many others that fail to satisfy these assumptions, thus showing that they are not necessary conditions to generate collision-free routing. Even more interestingly, the last rows of Table 1 state that the number of collisions observed in a single instance is always very small, even for the largest sizes we tested. For the instances where RR does not provide feasible routes, the average number of collisions is about one, and the maximum never exceeds seven. This means that we just need to remove a very small number of vehicles (one on average, no more than seven in the worst case) as to make the solution output by RR feasible.

Computational results demonstrate the relevance of RR as a fast algorithm that solves very large instances of FQRP-G, even if it has the drawback that it is possible that collisions might arise. The literature has consistently proposed alternative methods to solve FQRP-G, that differently trade off collision-free guarantee, running time and number of required levels. The algorithms *CaR* (Cenci et al. 2017) and *HeurA* (Andreatta et al. 2021) output collision-free solutions in very small running times even for large instances, but the number of required levels may be high and, in some cases, it is equal to the bound m^*. On the contrary, references (Andreatta et al. 2021; De Francesco and De Giovanni 2023) run an Integer Linear Programming model for each instance to find a collision-free routing that minimizes the number of required levels: experiments on 10 random instances per size (up to 500 vehicles), as well as on ad-hoc instances (up to about 200 vehicles) that contain longest possible C-conflict paths, show that solutions could be found using no more than four levels for horizontal moves, hence using a grid with $m = 5$. However, the approach incurs higher computational costs, making it prohibitive for large instances.

From a theoretical standpoint, the discussion in this paper shows that the bound of eight levels, stated in Di Giacomo et al. (2020) and valid under Assumptions 1 and 2, is not valid in general, even if it remains a relevant result in practice. The best known upper bound for FQRP-G is m^*, provided in Cenci et al. (2017) and depending on the number of columns in the grid. However, the experiments reported in this paper, as well as preliminary computational results in Andreatta et al. (2021), De Francesco and De Giovanni (2023), show that m^* may be disproportionately high, and stimulate further research towards a general bound that is tighter and, possibly, independent from the instance size.

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Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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