Communications on Pure and Applied Analysis
Vol. 23, No. 2, February 2024, pp. 144–164
doi:10.3934/cpaa.2024004

MULTI-PARAMETER PERTURBATIONS FOR THE SPACE-PERIODIC HEAT EQUATION

Matteo Dalla Riva[1], Paolo Luzzini[2]
Riccardo Molinarolo[3] and Paolo Musolino[*2]

[1]Dipartimento di Ingegneria, Università degli Studi di Palermo
Viale delle Scienze, Ed. 8, 90128 Palermo, Italy

[2]Dipartimento di Matematica 'Tullio Levi-Civita', Università degli Studi di Padova
Via Trieste 63, 35121 Padova, Italy

[3]Dipartimento di Scienze Molecolari e Nanosistemi, Università Ca' Foscari Venezia
Via Torino 155, 30172 Venezia Mestre, Italy

(Communicated by Junping Shi)

Abstract. This paper is divided into three parts. The first part focuses on periodic layer heat potentials, demonstrating their smooth dependence on regular perturbations of the support of integration. In the second part, we present an application of the results from the first part. Specifically, we consider a transmission problem for the heat equation in a periodic domain and we show that the solution depends smoothly on the shape of the transmission interface, boundary data, and transmission parameters. Finally, in the last part of the paper, we fix all parameters except for the transmission parameters and outline a strategy to deduce an explicit expansion of the solution using Neumann-type series.

1. **Introduction.** Understanding how the properties of an object depend on its shape is a crucial aspect of many real-world problems, especially when seeking to achieve the optimal configuration for maximizing some sort of efficiency.

In mathematical jargon, the quest for optimal shapes is commonly known as "shape optimization," and it has garnered considerable attention in the mathematical literature. The interested reader can find ample references and results in the monographs by Henrot and Pierre [9] and Sokołowski and Zolésio [25].

2020 *Mathematics Subject Classification.* Primary: 35K20; 31B10. Secondary: 35B10; 47H30; 45A05.

Key words and phrases. heat equation, domain perturbation, layer potentials, transmission problem, shape sensitivity analysis, periodic domain, special nonlinear operators, Neumann series.

M. D. R., P. L. and P. M. are supported by the "INdAM GNAMPA Project" codice CUP_E53C22001930001 "Operatori differenziali e integrali in geometria spettrale" and of the project funded by the EuropeanUnion – NextGenerationEU under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.1 - Call PRIN 2022 No. 104 of February 2, 2022 of Italian Ministry of University and Research; Project 2022SENJZ3 (subject area: PE - Physical Sciences and Engineering) "Perturbation problems and asymptotics for elliptic differential equations: variational and potential theoretic methods". P. M. and R. M. are also supported by the SPIN Project "DOMain perturbation problems and INteractions Of scales - DOMINO" of the Ca' Foscari University of Venice and the support from EU through the H2020-MSCA-RISE-2020 project EffectFact, Grant agreement ID: 101008140.

*Corresponding author: Paolo Musolino.

From a mathematical standpoint, addressing such questions often involves studying how solutions to specific boundary value problems, as well as related quantities, are affected by perturbations of the domain of definition and other problem parameters. This leads us to analyze the mappings that connect a set of perturbation parameters to the solution of a boundary value problem. To undertake this project, having access to the toolbox of differential calculus is advantageous. Consequently, understanding the regularity properties of these maps becomes crucial. In other words, it is important to determine whether these maps are continuous, differentiable, or enjoy higher regularity properties, such as smoothness and analyticity.

These properties reveal different aspects of the perturbation and can be used in different ways: Continuity implies that small variations of the perturbation parameters correspond to small changes in the solution. Differentiability allows for characterizing the stationary points as critical points. These critical points are important in optimization problems as they represent potential optimal configurations. Smoothness and analyticity are stronger properties. With smoothness we can approximate the solution with its Taylor expansion in the perturbation parameter with any degree of accuracy, while with analyticity we can represent the solution as a convergent power series.

Now, a common approach for studying boundary value problems is the layer potential theoretic method, which employs integral operators to transform the original problem into a system of boundary integral equations. Eventually, this method allows us to obtain the solution as a sum of layer potentials.

As a result, an approach to understanding the perturbation sensitivity of a solution to a boundary value problem is by studying how the layer potentials and the integral operators depend upon such perturbations.

Many authors have explored this approach for elliptic equations. For example, Potthast [23] proved that layer potentials for the Helmholtz equation are Fréchet differentiable functions of the support of integration. Applications to scattering theory can be found, e.g., in Haddar and Kress [8] and Kirsch [14].

However, we observe that very few results prove regularities beyond differentiability. An exception is the works of Lanza de Cristoforis and his collaborators, dedicated to proving that layer potentials and integral operators depend analytically on domain perturbations. Here we mention Lanza de Cristoforis and Rossi [18] for the layer potentials for the Laplace equation, Lanza de Cristoforis and Rossi [19] for the Helmholtz equation, [3] for general second order equations, and [17] for the periodic case. Moreover, in [5] we have obtained a smoothness result for the heat layer potentials which, in the first part of the present paper, we will extend to the space-periodic heat layer potentials.

The method developed by Lanza de Cristoforis and collaborators was called the "functional analytic approach" (cf. [4]). It was used for both regular and singular perturbations, where a perturbation is classified as regular if it does not cause any loss of regularity in the domain, and as singular if it does.

Another approach to dealing with regular domain perturbations has recently appeared in the literature, relying on complex analysis techniques and aiming to prove the "shape holomorphy" of layer potential operators and integral operators. For applications of this approach, we refer the reader to Jerez-Hanckes, Schwab, and Zech [13], which deals with the electromagnetic wave scattering problem.

Apart from [5], all the above cited literature concerns elliptic equations. Notably, corresponding results for parabolic problems are more scarse. To the best of our knowledge the only exceptions are some works of Chapko, Kress and Yoon (see, e.g., [2]) and Hettlich and Rundell [10] for the Fréchet differentiability upon the domain of the solution of the heat equation with application to some inverse problems in heat conduction, and the already cited [5] for the infinite order smoothness of the layer heat potentials upon the support of integration.

In this paper, we adopt Lanza de Cristoforis' functional analytic approach to obtain higher order regularity results for the space-periodic version of layer heat potentials upon the support of integration. In particular, in the first part of the paper we investigate the space-periodic layer potentials for the heat equation and demonstrate that they depend smoothly on a pair (ϕ, μ), where ϕ is a function that characterizes the shape of the domain and μ is the (pull-back of the) density function. To achieve this, we build upon similar findings for the nonperiodic heat layer potentials established in [5]. To the best of our knowledge, this is the first paper to show such a result for space-periodic heat layer potentials, previous papers dealing with periodic layer potentials being dedicated to the case of elliptic operators.

In the subsequent sections, we showcase how the results obtained in the first part can be utilized to examine the shape sensitivity of solutions to boundary value problems. As an illustrative application, we consider a transmission problem for the heat equation in a space-periodic domain. We show that the solution depends smoothly on the shape of the transmission interface, as well as on the boundary data and the transmission parameters.

Lastly, in the final part of the paper, we revisit the space-periodic transmission problem studied in the previous section. However, this time, we fix all parameters except for the transmission parameters. Then we outline a strategy to deduce an explicit expansion of the solution using a Neumann-type series.

The paper is organized as follows: Section 2 introduces some notation and preliminaries. In Section 3, we review certain results from [5] concerning nonperiodic layer potentials. In Section 4, we derive analogous results for the space-periodic layer potentials. Section 5 investigates the perturbation sensitivity of solutions to a transmission problem in a space-periodic domain. Finally, in Section 6, we consider the scenario where all parameters are fixed, except for the transmission parameters.

2. **Preliminaries.**

From this point onward, we fix a value for n from the set $\mathbb{N} \setminus \{0, 1\}$, where $\mathbb{N}$ denotes the set of natural numbers, including zero. Additionally, we define a periodicity cell as follows:

$$Q := \prod_{j=1}^{n}]0, q_{jj}[,$$

where $q_{jj} > 0$ for all $j \in \{1, \dots, n\}$. We denote by q the diagonal matrix

$$q := \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix},$$

and by $|Q|_n = \prod_{i=1}^{n} q_{jj}$ the measure of the peridicity cell Q. Clearly

$$q\mathbb{Z}^n = \{qz : z \in \mathbb{Z}^n\}$$

is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q. A set $A \subseteq \mathbb{R}^n$ is said to be q-periodic if $A + qz = A$ for all $z \in \mathbb{Z}^n$. If A is a

q-periodic set, a function $f : A \to \mathbb{R}$ is said to be q-periodic if $f(\cdot + qz) = f(\cdot)$ for all $z \in \mathbb{Z}^n$.

If Ω is a subset of $\mathbb{R}^n$ then $\overline{\Omega}$, $\partial\Omega$, and ν_Ω denote the closure, boundary, and, where defined, the outward normal to Ω, respectively. If $\overline{\Omega} \subseteq Q$, then we set

$$\mathbb{S}[\Omega] := \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega, \qquad \mathbb{S}[\Omega]^- := \mathbb{R}^n \setminus \overline{\mathbb{S}[\Omega]}.$$

We observe that both $\mathbb{S}[\Omega]$ and $\mathbb{S}[\Omega]^-$ are q-periodic domains.

We will consider the heat equation

$$\partial_t u - \Delta u = 0$$

in domains that are space-periodic and our approach will rely on the space-periodic potential theory for the heat equation. Specifically, we will exploit space-periodic layer potentials obtained by replacing the classical fundamental solution of the heat equation with a periodic counterpart. As it is well known, a fundamental solution of the heat equation is defined as follows:

$$S_n(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } (t,x) \in (0,+\infty) \times \mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in ((-\infty,0] \times \mathbb{R}^n) \setminus \{(0,0)\}. \end{cases}$$

Then a q-periodic fundamental solution $S_{q,n} : (\mathbb{R} \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n) \to \mathbb{R}$ for the heat equation is defined by taking

$$S_{q,n}(t,x) := \begin{cases} \sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} & \text{if } (t,x) \in (0,+\infty) \times \mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in ((-\infty,0] \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n) \end{cases} \tag{2.1}$$

(see Pinsky [22, Ch. 4.2] for the case $n = 1$ and Bernstein, Ebert and Sören Kraußhar [1] for $n \geq 2$, see also [20]).

We will use the functional framework of Schauder classes. For the classical definitions of sets and functions belonging to class $C^{j,\alpha}$, with $\alpha \in (0,1)$ and $j \in \{0,1\}$, we refer to Gilbarg and Trudinger [7]. For the definition of time-dependent functions in the parabolic Schauder class $C^{\frac{j+\alpha}{2};j+\alpha}$ on $[0,T] \times \overline{\Omega}$ or $[0,T] \times \partial\Omega$ we refer to Ladyženskaja, Solonnikov, and Ural'ceva [15]. In essence, a function of class $C^{\frac{j+\alpha}{2};j+\alpha}$ is $\left(\frac{j+\alpha}{2}\right)$-Hölder continuous in the time variable, and (j,α)-Schauder regular in the space variable. We also denote by $C_0^{\frac{j+\alpha}{2};j+\alpha}$ the parabolic Schauder class of functions that vanish at time $t = 0$, and by $C_{0,q}^{\frac{j+\alpha}{2};j+\alpha}$ the subspace of $C_0^{\frac{j+\alpha}{2};j+\alpha}$ consisting of functions that are also q-periodic. The definition of parabolic Schauder classes can be extended to products of intervals and manifolds by using local charts. In the present paper all the functional spaces we consider consist of real valued functions.

We will adopt the following notation: If D is a subset of $\mathbb{R}^n$, $T > 0$ and h is a map from D to $\mathbb{R}^n$, we denote by h^T the map from $[0,T] \times D$ to $[0,T] \times \mathbb{R}^n$ defined by

$$h^T(t,x) := (t, h(x)) \qquad \forall (t,x) \in [0,T] \times D.$$

Let $\alpha \in (0,1)$ and assume that

$$\begin{aligned} &\Omega \text{ is a bounded connected open subset of } \mathbb{R}^n \text{ of class } C^{1,\alpha} \\ &\text{and has connected exterior } \Omega^- := \mathbb{R}^n \setminus \overline{\Omega}\,. \end{aligned} \tag{2.2}$$

We take Ω to be the reference shape, and to formalize domain perturbations, we consider specific classes of diffeomorphisms defined on the boundary $\partial\Omega$.

Precisely, we denote by $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ the set of functions of class $C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)$ that are injective together with their differential at all points of $\partial\Omega$. According to Lanza de Cristoforis and Rossi [19, Lem. 2.2, p. 197] and [18, Lem. 2.5, p. 143], $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ is an open subset of $C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)$.

For $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, e.g., [4, §A.4]). We denote the bounded connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega)$ by $\mathbb{I}[\phi]$ and the unbounded one by $\mathbb{E}[\phi]$. Moreover, we will use ν_ϕ to denote the outer unit normal to $\mathbb{I}[\phi]$.

Then we set

$$\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} := \left\{\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega} : \phi(\partial\Omega) \subseteq Q\right\},$$

and for brevity, we use the notation

$$\mathbb{S}[\phi] := \mathbb{S}[\mathbb{I}[\phi]], \qquad \mathbb{S}[\phi]^- := \mathbb{S}[\mathbb{I}[\phi]]^-$$

for all $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. Both $\mathbb{S}[\phi]$ and $\mathbb{S}[\phi]^-$ are q-periodic domains depending on the diffeomorphism ϕ (see Figure 1). Therefore, we can perturb the shape of $\mathbb{S}[\phi]$ and $\mathbb{S}[\phi]^-$ by changing the function ϕ.

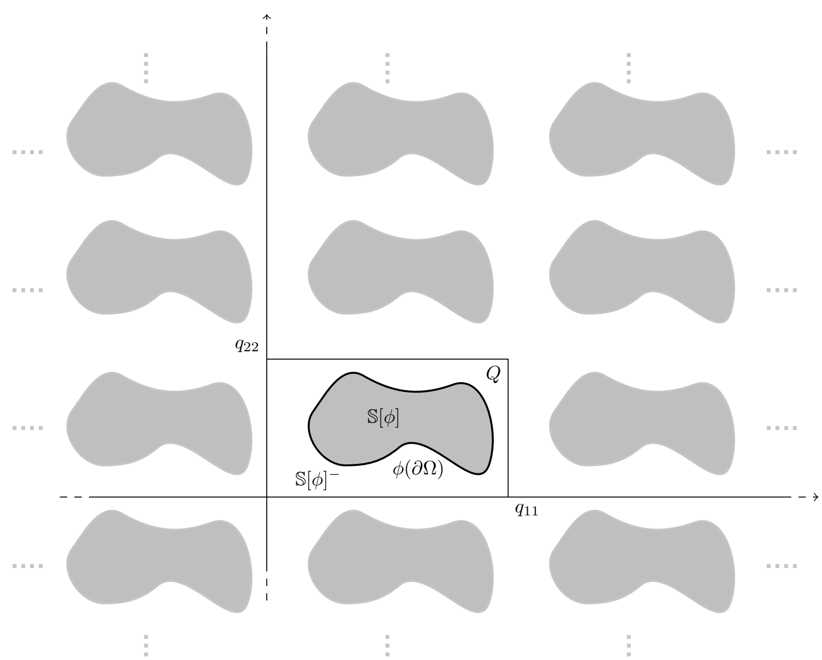

FIGURE 1. *The sets $\mathbb{S}[\phi]^-$, $\mathbb{S}[\phi]$, and $\phi(\partial\Omega)$ in case $n = 2$.*

We will consider integral operators supported on $\phi(\partial\Omega)$. To analyze their dependence on ϕ, we will perform a change of variables. For this purpose, we rely on the following technical lemma, which shows that the map related to the change of variables in the area element and the pullback $\nu_\phi \circ \phi$ of the outer normal field depend analytically on ϕ. A proof of this lemma can be found in Lanza de Cristoforis and Rossi [18, p. 166] and Lanza de Cristoforis [16, Prop. 1].

Lemma 2.1. *Let $\alpha \in (0,1)$ and Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ with connected exterior. Then the following statements hold.*

(i) For each $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$, there exists a unique $\tilde{\sigma}_n[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}_n[\phi] > 0$ and

$$\int_{\phi(\partial\Omega)} w(s)\, d\sigma_s = \int_{\partial\Omega} w \circ \phi(y) \tilde{\sigma}_n[\phi](y)\, d\sigma_y, \qquad \forall w \in L^1(\phi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}_n[\cdot]$ is real analytic from $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega)$.

(ii) The map from $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ which takes ϕ to $\nu_\phi \circ \phi$ is real analytic.

3. **Domain perturbations of classical layer potentials.** Our first goal is to prove that space-periodic layer potentials for the heat equation depend smoothly on perturbations of the support of integration. As previously mentioned in the introduction, related results have already been established in [5] for the non-periodic layer potentials. We intend to leverage those existing results and extend them to the periodic case.

Therefore, we begin by reviewing the findings of [5], which concern layer heat potentials supported on $[0,T] \times \phi(\partial\Omega)$ for some $T > 0$ and $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$, as well as integral operators acting between Schauder spaces on $[0,T] \times \phi(\partial\Omega)$. However, to treat ϕ as a variable and state smoothness results for ϕ-dependent functions, we need to work in a ϕ-independent functional setting. We will then pullback the layer potentials to the fixed domain $[0,T] \times \partial\Omega$ and, simultaneously, push forward the density functions from $[0,T] \times \partial\Omega$ to $[0,T] \times \phi(\partial\Omega)$.

To be precise, we consider the operators that take $\mu \in C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ to

$$V[\phi,\mu](t,\xi) := \int_0^t \int_{\phi(\partial\Omega)} S_n(t-\tau, \phi(\xi) - y) \mu \circ (\phi^T)^{(-1)}(\tau, y)\, d\sigma_y d\tau,$$

$$V_l[\phi,\mu](t,\xi) := \int_0^t \int_{\phi(\partial\Omega)} \partial_{x_l} S_n(t-\tau, \phi(\xi) - y) \mu \circ (\phi^T)^{(-1)}(\tau, y)\, d\sigma_y d\tau \qquad \forall l \in \{1,\dots,n\},$$

$$W^*[\phi,\mu](t,\xi) := \int_0^t \int_{\phi(\partial\Omega)} D_x S_n(t-\tau, \phi(\xi) - y) \cdot \nu_\phi(\xi) \mu \circ (\phi^T)^{(-1)}(\tau, y)\, d\sigma_y d\tau,$$

for all $(t,\xi) \in [0,T] \times \partial\Omega$. Additionally, for $\psi \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$ we define

$$W[\phi,\psi](t,\xi) := -\int_0^t \int_{\phi(\partial\Omega)} D_x S_n(t-\tau, \phi(\xi) - y) \cdot \nu_\phi(y) \psi \circ (\phi^T)^{(-1)}(\tau, y)\, d\sigma_y d\tau,$$

for all $(t,\xi) \in [0,T] \times \partial\Omega$. In the expressions above, $\partial_{x_l} S_n$ and $D_x S_n$ denote the x_l-derivative and the gradient of S_n with respect to the spatial variables, respectively.

The functions $V[\phi,\mu]$, $V_l[\phi,\mu]$, and $W^*[\phi,\mu]$ are the ϕ-pullbacks of the single-layer potential and of integral operators associated to its x_l-derivative and to its normal derivative. Instead $W[\phi,\psi]$ is the ϕ-pullback of the double-layer potential. They are defined on $[0,T] \times \partial\Omega$ and have densities given by $\mu \circ (\phi^T)^{(-1)}$ and $\psi \circ (\phi^T)^{(-1)}$.

In [5, Thm. 6.3], it has been proven that the operators $V[\phi,\cdot]$, $V_l[\phi,\cdot]$, $W^*[\phi,\cdot]$, and $W[\phi,\cdot]$ depend smoothly on the shape parameter ϕ. Specifically, we have the following result:

Theorem 3.1. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be as in (2.2). Then, the maps that take $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$ to the following operators are all of class C^∞:*

(i) $V[\phi,\cdot] \in \mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right)$,

(ii) $V_l[\phi,\cdot] \in \mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$ *for all* $l \in \{1,\dots,n\}$,

(iii) $W^*[\phi,\cdot] \in \mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$,

(iv) $W[\phi,\cdot] \in \mathcal{L}\left(C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right)$.

Theorem 3.1 presents an extension of similar results that were already known for layer potentials associated with elliptic equations to the parabolic setting. For example, Lanza de Cristoforis and Rossi [18, 19] established these results for the Laplace and Helmholtz equations, and [3] for general second-order equations. However, extending these results to the parabolic setting is not a trivial task. The main difficulty lies in the interaction between the time and space variables. Applying the strategy used in [18] to the parabolic case only yields a regularity result for C^2 perturbations of the domain, falling short of the desired $C^{1,\alpha}$ setting.

Another difference between the elliptic and parabolic cases is that in the elliptic scenario, the layer potentials exhibit analytic dependence on the shape parameter ϕ, while Theorem 3.1 only guarantees that they are infinitely differentiable maps. The reason for this lack of analyticity lies in the regularity of the fundamental solution S_n, which is C^∞ but not real analytic over the entire space $\mathbb{R}^{1+n}\setminus\{(0,0)\}$ due to its non-analytic behavior at $t=0$. In contrast, the fundamental solution of the Laplace equation, as well as other constant coefficient elliptic operators, is analytic in $\mathbb{R}^n\setminus\{0\}$.

As we shall see, such a difference implies a distinct behavior of the solutions to boundary value problems: analytic dependence on ϕ for the elliptic case *vs* C^∞-dependence for the parabolic case.

4. **Space-periodic layer heat potentials.** We now shift our focus to space-periodic layer heat potentials, where we replace the classical fundamental solution S_n of the heat equation with its periodization $S_{q,n}$ (see (2.1)). We will start by introducing the definition of periodic layer potentials. Next, we will review some properties established in [20]. Finally, we will utilize Theorem 3.1 to derive the corresponding regularity results for the ϕ-pullback of periodic layer potentials.

Let $\alpha \in (0,1)$ and $T>0$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\overline{\Omega}\subseteq Q$. For a density $\mu \in L^\infty\big([0,T]\times\partial\Omega\big)$, the q-periodic in space layer heat potentials are defined as

$$v_q[\mu](t,x) := \int_0^t \int_{\partial\Omega} S_{q,n}(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall\,(t,x)\in[0,T]\times\mathbb{R}^n,$$

and

$$w_q[\mu](t,x) := -\int_0^t \int_{\partial\Omega} D_x S_{q,n}(t-\tau,x-y)\cdot\nu_\Omega(y)\mu(\tau,y)\,d\sigma_y d\tau \quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^n.$$

The functions $v_q[\mu]$ and $w_q[\mu]$ are called respectively the q-periodic single- and double-layer heat potential. Moreover, we set

$$w^*_q[\mu](t,x) := \int_0^t \int_{\partial\Omega} D_x S_{q,n}(t-\tau,x-y)\cdot\nu_\Omega(x)\mu(\tau,y)\,d\sigma_y d\tau$$

$$\forall\,(t,x) \in [0,T] \times \partial\Omega.$$

The map $w_q^*[\mu]$ is related to the normal derivative of the q-periodic in space single-layer potential (see Theorem 4.1).

Periodic layer heat potentials enjoy properties similar to those of their standard counterpart. We collect them in the following two theorems. The proofs can be found in [20, Thms. 2, 3].

Theorem 4.1. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\overline{\Omega} \subseteq Q$. Then the following statements hold.*

(i) Let $\mu \in L^\infty([0,T] \times \partial\Omega)$. Then $v_q[\mu]$ is continuous, q-periodic in space and $v_q[\mu] \in C^\infty\big((0,T] \times (\mathbb{R}^n \setminus \partial\mathbb{S}[\Omega])\big)$. Moreover $v_q[\mu]$ solves the heat equation in $(0,T] \times (\mathbb{R}^n \setminus \partial\mathbb{S}[\Omega])$.

(ii) Let $v_q^+[\mu]$ and $v_q^-[\mu]$ denote the restrictions of $v_q[\mu]$ to $[0,T] \times \overline{\mathbb{S}[\Omega]}$ and to $[0,T] \times \overline{\mathbb{S}[\Omega]^-}$, respectively. The map from $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ to the space $C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T] \times \overline{\mathbb{S}[\Omega]}\big)$ that takes μ to $v_q^+[\mu]$ is linear and continuous. Likewise, the map from $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ to the space $C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T] \times \overline{\mathbb{S}[\Omega]^-}\big)$ that takes μ with $v_q^-[\mu]$ is also linear and continuous.

(iii) Let $\mu \in C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ and $l \in \{1,\dots,n\}$. Then the following jump relations hold:

$$\frac{\partial}{\partial\nu_\Omega} v_q^\pm[\mu](t,x) = \pm\frac{1}{2}\mu(t,x) + w_q^*[\mu](t,x),$$

$$\partial_{x_l} v_q^\pm[\mu](t,x) = \pm\frac{1}{2}\mu(t,x)\,(\nu_\Omega(x))_l + \int_0^t \int_{\partial\Omega} \partial_{x_l} S_{q,n}(t-\tau, x-y)\mu(\tau,y)\,d\sigma_y d\tau,$$

for all $(t,x) \in [0,T] \times \partial\Omega$.

Theorem 4.2. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\overline{\Omega} \subseteq Q$. Then the following statements hold.*

(i) Let $\mu \in L^\infty([0,T]\times\Omega)$. Then $w_q[\mu]$ is q-periodic in space, $w_q[\mu] \in C^\infty\big((0,T]\times (\mathbb{R}^n \setminus \partial\mathbb{S}[\Omega])\big)$, and $w_q[\mu]$ solves the heat equation in $(0,T] \times (\mathbb{R}^n \setminus \partial\mathbb{S}[\Omega])$.

(ii) Let $\mu \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$. Then the restriction $w_q[\mu]_{|[0,T]\times\mathbb{S}[\Omega]}$ can be extended uniquely to an element $w_q^+[\mu] \in C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T] \times \overline{\mathbb{S}[\Omega]}\big)$ and the restriction $w_q[\mu]_{|[0,T]\times\mathbb{S}[\Omega]^-}$ can be extended uniquely to an element $w_q^-[\mu] \in C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T] \times \overline{\mathbb{S}[\Omega]^-}\big)$. Moreover the following jump formulas hold:

$$w_q^\pm[\mu](t,x) = \mp\frac{1}{2}\mu(t,x) + w_q[\mu](t,x)\,,$$

$$\frac{\partial}{\partial\nu_\Omega} w_q^+[\mu](t,x) - \frac{\partial}{\partial\nu_\Omega} w_q^-[\mu](t,x) = 0,$$

for all $(t,x) \in [0,T] \times \partial\Omega$.

(iii) The map from $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ to the space $C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T]\times\overline{\mathbb{S}[\Omega]}\big)$ that takes μ to the function $w_q^+[\mu]$ is linear and continuous. Likewise, the map from $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$ to the space $C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T] \times \overline{\mathbb{S}[\Omega]^-}\big)$ that takes μ to the function $w_q^-[\mu]$ is also linear and continuous.

The main idea in the proof of Theorems 4.1 and 4.2 revolves around representing periodic layer potentials as the sum of their non-periodic counterparts and a remainder, which is an integral operator with a nonsingular kernel. This is feasible because the map

$$R_{q,n}(t,x) := S_{q,n}(t,x) - S_n(t,x), \qquad \forall\,(t,x) \in (\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q\mathbb{Z}^n) \tag{4.1}$$

can be extended by continuity to $(\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))$. Keeping the notation $R_{q,n}$ for this extension, we have that

$$R_{q,n} \in C^\infty((\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))).$$

In other words, $R_{q,n}$ is smooth in a neighborhood of the origin $(0,0)$. A proof of this assertion can be found in [20, Thm. 1].

The same idea can be used to recover the periodic counterpart of Theorem 3.1. We first need to introduce the pull-back of the boundary integral operators associated with q-periodic layer heat potentials. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that both Ω and Ω^- are connected. Let $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. For $\mu \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$, we consider the operators

$$V_q[\phi,\mu](t,\xi) := \int_0^t\int_{\phi(\partial\Omega)} S_{q,n}(t-\tau,\phi(\xi)-y)\mu\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau$$

$$V_{q,l}[\phi,\mu](t,\xi) := \int_0^t\int_{\phi(\partial\Omega)} \partial_{x_l}S_{q,n}(t-\tau,\phi(\xi)-y)\mu\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau \qquad \forall l\in\{1,\dots,n\}$$

$$W_q^*[\phi,\mu](t,\xi) := \int_0^t\int_{\phi(\partial\Omega)} D_xS_{q,n}(t-\tau,\phi(\xi)-y)\cdot\nu_\phi(\xi)\mu\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau,$$

for all $(t,\xi)\in[0,T]\times\partial\Omega$. Also, for $\psi \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ we set

$$W_q[\phi,\psi](t,\xi) := -\int_0^t\int_{\phi(\partial\Omega)} D_xS_{q,n}(t-\tau,\phi(\xi)-y)\cdot\nu_\phi(y)\psi\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau,$$

for all $(t,\xi)\in[0,T]\times\partial\Omega$. Similarly to the non-periodic scenario, the function $V_q[\phi,\mu]$ is the ϕ-pullback of the q-periodic single-layer potential restricted on the boundary $[0,T]\times\phi(\partial\Omega)$, while $V_{q,l}[\phi,\mu]$ and $W_q^*[\phi,\mu]$ are respectively related to its x_l and normal derivatives. The function $W_q[\phi,\psi]$ is instead related to the boundary behavior of the q-periodic double-layer potential.

We are now ready to present the main result of this section, concerning the smoothness of the mappings that associate ϕ with $V_q[\phi,\cdot]$, $V_{q,l}[\phi,\cdot]$, $W_q^*[\phi,\cdot]$, and $W_q[\phi,\cdot]$.

Theorem 4.3. *Let $\alpha\in(0,1)$ and $T>0$. Let Ω be as in (2.2). Then the maps that take $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to the following operators are all of class C^∞:*

(i) $V_q[\phi,\cdot]\in\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right)$,

(ii) $V_{q,l}[\phi,\cdot]\in\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$ *for all* $l\in\{1,\dots,n\}$,

(iii) $W_q^*[\phi,\cdot]\in\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$,

(iv) $W_q[\phi,\cdot]\in\mathcal{L}\left(C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right)$.

Proof. We confine ourselves to demonstrate the theorem for the map $\phi\mapsto V_q[\phi,\cdot]$ in point (i). The proof for the operators in (ii), (iii), and (iv) can be carried out by a straightforward adaptation of the argument presented below. In these cases, we will use statements (ii), (iii), and (iv) of Theorem 3.1, analogously to how we will use statement (i) of the same Theorem 3.1 in the forthcoming argument.

As shown in [20, Thm. 1], the map $R_{q,n}$ defined in (4.1) is of class C^∞ in the set $(\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))$. In particular, $R_{q,n}$ is smooth in a neighborhood of $(0,0)\in\mathbb{R}\times\mathbb{R}^n$.

Let $(\phi,\mu)\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. Clearly, definition (4.1) implies that

$$\begin{aligned}V_q[\phi,\mu](t,\xi) &= V[\phi,\mu](t,\xi)\\ &\quad+\int_0^t\int_{\phi(\partial\Omega)}R_{q,n}(t-\tau,\phi(\xi)-y)\mu\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau \qquad (4.2)\end{aligned}$$

for all $(t,\xi)\in[0,T]\times\partial\Omega$. By Theorem 3.1 (i), the map that takes $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to

$$V[\phi,\cdot]\in\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega),C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right)$$

is of class C^∞. We now consider the second term on the right-hand side of (4.2). By Lemma 2.1 we have

$$\begin{aligned}&\int_0^t\int_{\phi(\partial\Omega)}R_{q,n}(t-\tau,\phi(\xi)-y)\mu\circ(\phi^T)^{(-1)}(\tau,y)\,d\sigma_y d\tau\\ &=\int_0^t\int_{\partial\Omega}R_{q,n}(t-\tau,\phi(\xi)-\phi(\eta))\mu(\tau,y)\tilde{\sigma}_n[\phi](\eta)\,d\sigma_\eta d\tau.\end{aligned}$$

We note that

$$\phi(\xi)-\phi(\eta)\notin q\mathbb{Z}^n\setminus\{0\}\qquad\forall\,(\xi,\eta)\in\partial\Omega\times\partial\Omega.$$

Indeed, if it was that $(\xi,\eta)\in\partial\Omega\times\partial\Omega$ and $\phi(\xi)-\phi(\eta)\in q\mathbb{Z}^n\setminus\{0\}$, then we would have that $\phi(\xi)\in\phi(\partial\Omega)+q\mathbb{Z}^n\setminus\{0\}$, which clearly cannot be. Then, by Lemma 2.1 and by the results of [5, Lemma A.2, Lemma A.3] on non-autonomous composition operators and on time-dependent integral operators with non-singular kernels, we deduce that the map from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ to $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ that takes (ϕ,μ) to the function

$$K[\phi,\mu](t,\xi):=\int_0^t\int_{\partial\Omega}R_{q,n}(t-\tau,\phi(\xi)-\phi(\eta))\mu(\tau,y)\tilde{\sigma}_n[\phi](\eta)\,d\sigma_\eta d\tau\qquad\forall(t,\xi)\in[0,T]\times\partial\Omega,$$

is of class C^∞.

It remains to show that $\phi\mapsto K[\phi,\cdot]$ is C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to the operator space

$$\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega),C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\right).$$

Given that $K[\phi,\mu]$ is linear and continuous with respect to the variable μ, we have

$$K[\phi,\cdot]=d_\mu K[\phi,\mu]\qquad\forall(\phi,\mu)\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega),\qquad(4.3)$$

where the term on the right-hand side is the partial Frechet differential of $(\phi,\mu)\mapsto K[\phi,\mu]$ with respect to μ, evaluated at the point (ϕ,μ). Because $(\phi,\mu)\mapsto K[\phi,\mu]$ is a map of class C^∞, the map that takes (ϕ,μ) to $d_\mu K[\phi,\mu]$ is also of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ to $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega),C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega))$. Hence, the map $(\phi,\mu)\mapsto K[\phi,\cdot]$ is of class C^∞ by (4.3), and, since it does not depend on

μ, we conclude that $\phi \mapsto K[\phi,\cdot]$ is C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to the space $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega))$.

Hence, the validity of the theorem for the map $\phi \mapsto V[\phi,\cdot]$ in point (i) has now been proven. □

It is worth recalling that a result similar to Theorem 4.3 has been previously proven in [17] for periodic layer potentials corresponding to a general class of second-order elliptic equations. Later, these findings were used to study the effect of perturbations on physical quantities relevant to materials science and fluid mechanics. For instance, we refer to [6] which deals with the effective properties of periodic structures.

5. **A transmission problem.** The theorem presented in the preceding section, Theorem 4.3, serves as a toolkit to analyze the solution to boundary value problems for the heat equation in spatially periodic domains. The primary goal of using this theorem is to demonstrate the smooth dependence of such solutions on shape perturbations. As emphasized in the introduction, the feasibility of employing Theorem 4.3 for this purpose relies on the applicability of boundary integral operators and layer potentials to derive solutions for boundary value problems.

As an illustrative application, we consider a periodic transmission problem. We will demonstrate that its solution depends smoothly on the shape of the transmission interface, the boundary data, and the transmission parameters.

Now, let's introduce this specific problem. Consider $\alpha\in(0,1)$, $T>0$, and a bounded open subset Ω of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that both Ω and its exterior Ω^- are connected. Let $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. We fix the transmission parameters $\lambda^+,\lambda^->0$ and choose $f\in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ and $g\in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. With this setup, we proceed to consider the following transmission problem:

$$\begin{cases} \partial_t u^+ - \Delta u^+ = 0 & \text{in } (0,T]\times\mathbb{S}[\phi], \\ \partial_t u^- - \Delta u^- = 0 & \text{in } (0,T]\times\mathbb{S}[\phi]^-, \\ u^+(t,x+qz) = u^+(t,x) & \forall\,(t,x)\in[0,T]\times\overline{\mathbb{S}[\phi]},\ \forall\, z\in\mathbb{Z}^n, \\ u^-(t,x+qz) = u^-(t,x) & \forall\,(t,x)\in[0,T]\times\overline{\mathbb{S}[\phi]^-},\ \forall\, z\in\mathbb{Z}^n, \\ u^+ - u^- = f\circ(\phi^T)^{(-1)} & \text{on } [0,T]\times\partial\Omega, \\ \lambda^-\frac{\partial}{\partial\nu_\Omega}u^- - \lambda^+\frac{\partial}{\partial\nu_\Omega}u^+ = g\circ(\phi^T)^{(-1)} & \text{on } [0,T]\times\partial\Omega, \\ u^+(0,\cdot)=0 & \text{in } \overline{\mathbb{S}[\phi]}, \\ u^-(0,\cdot)=0 & \text{in } \overline{\mathbb{S}[\phi]^-}. \end{cases} \tag{5.1}$$

Problem (5.1) can be seen as the periodic version in $(0,T]\times\mathbb{S}[\phi]$ and $(0,T]\times\mathbb{S}[\phi]^-$ of the transmission problem for the heat equation considered in Hofmann, Lewis, and Mitrea [12]. We emphasize that there are other transmission problems for the heat equation that are relevant in applications, and in particular we refer to the one considered in Qiu, Rieder, Sayas, and Zhang [24].

In [21, Thm. 4] it has been proved that the solution (u^+,u^-) of (5.1) exists, is unique, and belongs to a suitable product of Schauder spaces. Moreover, this solution can be expressed as a pair of periodic single-layer heat potentials, and the densities of these potentials are solutions to a particular system of boundary integral equations. To be precise, the following result holds:

Theorem 5.1. *Let $\alpha\in(0,1)$ and $T>0$. Let Ω be as in (2.2). Let $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. Let $\lambda^+,\lambda^->0$ and $f\in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$, $g\in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. Then problem*

(5.1) *has a unique solution*

$$(u^+,u^-) \in C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\overline{\mathbb{S}[\phi]}) \times C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\overline{\mathbb{S}[\phi]^-}).$$

Moreover,

$$u^+ = v_q^+[\mu^+], \qquad u^- = v_q^-[\mu^-],$$

where (μ^+,μ^-) *is the unique solution in* $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\phi(\partial\Omega))\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\phi(\partial\Omega))$ *of the system of integral equations*

$$\begin{cases} v_q^+[\mu^+]_{|[0,T]\times\phi(\partial\Omega)} - v_q^-[\mu^-]_{|[0,T]\times\phi(\partial\Omega)} = f\circ(\phi^T)^{(-1)}, \\ \lambda^-\left(-\frac{1}{2}\mu^- + w_q^*[\mu^-]\right) - \lambda^+\left(\frac{1}{2}\mu^+ + w_q^*[\mu^+]\right) = g\circ(\phi^T)^{(-1)}. \end{cases} \tag{5.2}$$

Keeping in mind Theorem 5.1, we will use the notation

$$(u^+[\phi,\lambda^+,\lambda^-,f,g], u^-[\phi,\lambda^+,\lambda^-,f,g])$$

to denote the unique solution of problem (5.1).

Moreover, thanks to Theorem 5.1, we have a representation of the unique solution of the transmission problem as a pair of single-layer potentials with densities that solve the system of boundary integral equations in (5.2). Then, to understand how the solution depends upon variations of ϕ, λ^+, λ^-, f, and g, we plan to first understand how the densities depend on such parameters. To maintain consistency within the functional spaces, we have to perform a ϕ-pullback of the integral equations in (5.2). This transformation results in a system of ϕ-dependent integral equations defined on the fixed domain $[0,T]\times\partial\Omega$. This is achieved through a change of variables applied to (5.2), leading to the following proposition:

Proposition 5.2. *Let* $\alpha\in(0,1)$ *and* $T>0$. *Let* Ω *be as in* (2.2). *Let* $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. *Let* $\lambda^+,\lambda^->0$ *and* $f\in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$, $g\in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. *Then the unique solution*

$$\begin{aligned}(u^+[\phi,\lambda^+,\lambda^-,f,g],&u^-[\phi,\lambda^+,\lambda^-,f,g]) \\ &\in C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\overline{\mathbb{S}[\phi]}) \times C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\overline{\mathbb{S}[\phi]^-})\end{aligned}$$

of problem (5.1) *can be written as*

$$u^+[\phi,\lambda^+,\lambda^-,f,g] = v_q^+[\rho^+\circ(\phi^T)^{(-1)}] \qquad u^-[\phi,\lambda^+,\lambda^-,f,g] = v_q^-[\rho^-\circ(\phi^T)^{(-1)}],$$

where (ρ^+,ρ^-) *is the unique solution in* $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ *of the system of integral equations*

$$\begin{cases} V_q[\phi,\rho^+] - V_q[\phi,\rho^-] = f, \\ \lambda^-\left(-\frac{1}{2}\rho^- + W_q^*[\phi,\rho^-]\right) - \lambda^+\left(\frac{1}{2}\rho^+ + W_q^*[\phi,\rho^+]\right) = g. \end{cases} \tag{5.3}$$

Our next step is to understand the dependence of the solution (ρ^+,ρ^-) of (5.3) upon $(\phi,\lambda^+,\lambda^-,f,g)$. To achieve this, we first observe that system (5.3) can be equivalently reformulated as a single integral equation. In fact, by the linearity of the single-layer potential $V_q[\phi,\cdot]$, we can rewrite the first equation in (5.3) as

$$V_q[\phi,\rho^+-\rho^-] = f. \tag{5.4}$$

Then, by leveraging the invertibility of the single-layer potential (cf. [21, Thm. 2]) and using equality (5.4), we can express either ρ^+ or ρ^- in terms of the other. Substituting this expression into the second equation of (5.3), we arrive at the following proposition:

Proposition 5.3. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be as in (2.2). Take $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$. Assume $\lambda^+, \lambda^- > 0$ and take $f \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ and $g \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. Define the contrast transmission parameter $\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]$ by*

$$\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] := \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+}\,. \tag{5.5}$$

If $(\rho^+,\rho^-) \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega) \times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ is the unique solution of the system of integral equations (5.3), then ρ^- is the unique solution in $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ of the integral equation

$$\begin{aligned}\rho^- &- 2\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]W_q^*[\phi,\rho^-]\\ &= -\frac{2}{\lambda^-+\lambda^+}\left(\lambda^+\left(\frac{1}{2}I + W_q^*[\phi,\cdot]\right)\Big(V_q[\phi,\cdot]^{(-1)}(f)\Big) + g\right)\end{aligned} \tag{5.6}$$

and ρ^+ is given by

$$\rho^+ = \rho^- + V_q[\phi,\cdot]^{(-1)}(f). \tag{5.7}$$

Proof. As already noted, equation (5.7) follows by the first equation of (5.3) and by the linearity and invertibility of the operator $V_q[\phi,\cdot]$ from $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ to $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$ (cf. [21, Thm. 2]). Then, substituting (5.7) into the second equation in (5.3) and using the linearity of $W_q^*[\phi,\cdot]$, we obtain

$$\begin{aligned}\lambda^-\left(-\frac{1}{2}\rho^- + W_q^*[\phi,\rho^-]\right) - \lambda^+\left(\frac{1}{2}\rho^- + \frac{1}{2}V_q[\phi,\cdot]^{(-1)}(f)\right)&\\ - \lambda^+\left(W_q^*[\phi,\rho^-] + W_q^*\left[\phi, V_q[\phi,\cdot]^{(-1)}(f)\right]\right) &= g,\end{aligned}$$

which, after a rearrangement, yields

$$\begin{aligned}(\lambda^-+\lambda^+)&\left(-\frac{1}{2}\rho^-\right) + (\lambda^- - \lambda^+)W_q^*[\phi,\rho^-]\\ &= \lambda^+\left(\frac{1}{2}I + W_q^*[\phi,\cdot]\right)\Big(V_q[\phi,\cdot]^{(-1)}(f)\Big) + g.\end{aligned}$$

Multiplying both sides of the above equation by $-\frac{2}{\lambda^-+\lambda^+}$, we obtain (5.6), which, in view of [21, Lem. 2], is well known to have a unique solution (cf. the definition of $\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]$ in (5.5)). □

In the proof of Proposition 5.3, we utilized the invertibility of the operator $I - 2\gamma W_q^*[\phi,\cdot]$ for $\gamma \in (-1,1)$, a fact established in [21, Lem. 2]. Even for $\gamma = 1$, this operator remains invertible, as follows from [20, Lem. 6]. In the subsequent lemma, we demonstrate the invertibility of this operator for $\gamma = -1$ as well, thereby establishing its invertibility for all $\gamma \in [-1,1]$.

Lemma 5.4. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be as in (2.2). Let $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ and $\gamma \in [-1,1]$. Then the operator from $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ into itself that maps ρ to the function $\rho - 2\gamma W_q^*[\phi,\rho]$ is a linear homeomorphism.*

Proof. As previously noted, the assertion for $\gamma \in (-1,1)$ and $\gamma = 1$ follows by [21, Lem. 2] and [20, Lem. 6], respectively (note that for $\gamma \in (-1,1)$, there exist $\gamma^+,\gamma^- > 0$ such that $\gamma = (\gamma^- - \gamma^+)/(\gamma^- + \gamma^+)$). Thus, the task at hand is to demonstrate the statement for $\gamma = -1$.

Due to the compactness of $W_q^*[\phi,\cdot]$ (cf. [21, Thm. 1]), the operator $I-2\gamma W_q^*[\phi,\cdot]$ is a Fredholm operator of index zero. Consequently, to demonstrate that it is a linear homeomorphism, it suffices to prove its injectivity. So, let $\rho \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ be such that

$$\rho + 2W_q^*[\phi,\rho] = 0 \quad \text{on } [0,T]\times\partial\Omega.$$

By Theorem 4.1, the single-layer potential $v_q^+[\rho\circ(\phi^T)^{(-1)}]$ belongs to the space $C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}\big([0,T]\times\overline{\mathbb{S}[\phi]}\big)$ and is a solution of the following q-periodic homogeneous interior Neumann problem:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T]\times\mathbb{S}[\phi], \\ u(t,x+qz) = u(t,x) & \forall\,(t,x)\in[0,T]\times\overline{\mathbb{S}[\phi]},\ \forall\, z\in\mathbb{Z}^n, \\ \frac{\partial}{\partial\nu_\Omega}u = 0 & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot) = 0 & \text{in } \overline{\mathbb{S}[\phi]}\,. \end{cases} \tag{5.8}$$

We proceed to prove that $u=0$ is the sole solution of problem (5.8) by a standard energy argument. It will follow that $v_q^+[\rho\circ(\phi^T)^{(-1)}]=0$ and, by the invertibility of the restriction to $[0,T]\times\phi(\partial\Omega)$ of the single-layer potential (cf. [21, Thm. 2]), we will conclude that $\rho\circ(\phi^T)^{(-1)}=0$, and thus that $\rho=0$.

So, let $u\in C_{0,q}^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\overline{\mathbb{S}[\phi]})$ be a solution of (5.8). Let

$$e(t) := \int_\Omega (u(t,y))^2\,dy \qquad \forall t\in[0,T].$$

Given that u is uniformly continuous on $[0,T]\times\overline{\mathbb{S}[\phi]}$, we can see that $t\mapsto e(t)$ is continuous on $[0,T]$. Furthermore, we can demonstrate that e belongs to $C^1([0,T])$. A detailed proof is provided in [20, Lem. 5 and Prop. 2], and it is based on classical differentiation theorems for integrals depending on a parameter, along with a specific approximation of the support of integration (see Verchota [26, Thm. 1.12, p. 581]). Following the argument in the same reference ([20, Lem. 5 and Prop. 2]), we can also verify that

$$\begin{aligned} \frac{d}{dt}e(t) &= -2\int_\Omega |Du(t,y)|^2\,dy + 2\int_{\partial\Omega} u(t,y)\frac{\partial}{\partial\nu_\Omega}u(t,y)\,d\sigma_y \\ &= -2\int_\Omega |Du(t,y)|^2\,dy \quad \forall t\in(0,T), \end{aligned}$$

where the integral on $\partial\Omega$ vanishes thanks to the boundary condition in (5.8). Hence $\frac{d}{dt}e\le 0$ in $(0,T)$. Since $e\ge 0$ and $e(0)=0$, we conclude that $e(t)=0$ for all $t\in[0,T]$. Accordingly, $u=0$ on $[0,T]\times\overline{\Omega}$, and the q-periodicity of u implies $u=0$ on $[0,T]\times\overline{\mathbb{S}[\phi]}$. Hence

$$v_q^+[\rho\circ(\phi^T)^{(-1)}] = 0 \qquad \text{in } [0,T]\times\overline{\mathbb{S}[\phi]}\,,$$

a fact that, as explained above, concludes the proof of the statement. □

Taking inspiration from Proposition 5.3 and Lemma 5.4, we define the map

$$\Lambda : \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}\times(0,+\infty)^2\times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega) \to C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$$

given by

$$\Lambda[\phi,\lambda^+,\lambda^-,f,g] := \big(I - 2\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]W_q^*[\phi,\cdot]\big)^{(-1)}$$

$$\left(-\frac{2}{\lambda^- + \lambda^+}\left(\lambda^+\left(\frac{1}{2}I + W_q^*[\phi,\cdot]\right)\left(V_q[\phi,\cdot]^{(-1)}(f)\right) + g\right)\right),$$

with $\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]$ as in (5.5). Then the solution $\rho^-[\phi,\lambda^+,\lambda^-,f,g]$ to the integral equation in (5.6) is given by

$$\rho^-[\phi,\lambda^+,\lambda^-,f,g] = \Lambda[\phi,\lambda^+,\lambda^-,f,g], \tag{5.9}$$

and if we take

$$\rho^+[\phi,\lambda^+,\lambda^-,f,g] = \rho^-[\phi,\lambda^+,\lambda^-,f,g] + V_q[\phi,\cdot]^{(-1)}(f), \tag{5.10}$$

we see, by Proposition 5.3, that the pair

$$\left(\rho^+[\phi,\lambda^+,\lambda^-,f,g], \rho^-[\phi,\lambda^+,\lambda^-,f,g]\right)$$

is the unique solution of (5.3).

Our next objective is to establish a regularity result for the map that takes $(\phi, \lambda^+,\lambda^-,f,g)$ to $(\rho^+[\phi,\lambda^+,\lambda^-,f,g],\rho^-[\phi,\lambda^+,\lambda^-,f,g])$, which stems from the smooth dependence of layer potentials on perturbations in the integration's support of Theorem 4.3, coupled with the analyticity of the inversion map in Banach algebras. Subsequently, the regularity of the mapping

$$(\phi,\lambda^+,\lambda^-,f,g) \mapsto \left(\rho^+[\phi,\lambda^+,\lambda^-,f,g],\rho^-[\phi,\lambda^+,\lambda^-,f,g]\right)$$

will resolve into a regularity result for the mapping that relates $(\phi,\lambda^+,\lambda^-,f,g)$ with the solution of (5.1).

Proposition 5.5. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be as in (2.2). Then the map*

$$(\phi,\lambda^+,\lambda^-,f,g) \mapsto \left(\rho^+[\phi,\lambda^+,\lambda^-,f,g],\rho^-[\phi,\lambda^+,\lambda^-,f,g]\right)$$

is of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0,+\infty)^2 \times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega) \times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ to $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega) \times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$.

Proof. By Theorem 4.3, the map that takes ϕ to $V_q[\phi,\cdot]$ is of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to the space $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega))$, and the map that takes (ϕ,γ) to $I - 2\gamma W_q^*[\phi,\cdot]$ is of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (-1,1)$ to the space $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega))$. Since the map from $(0,+\infty)^2$ to $(-1,1)$ that takes (λ^+,λ^-) to $\frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+}$ is also of class C^∞, we deduce that the map from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0,+\infty)^2$ to the space $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega))$ that takes a triple $(\phi,\lambda^+,\lambda^-)$ to

$$I - 2\frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} W_q^*[\phi,\cdot]$$

is of class C^∞.

Now, the map that takes a linear invertible operator to its inverse is real analytic (cf. Hille and Phillips [11, Thms. 4.3.2 and 4.3.4]), and therefore of class C^∞. So, by the invertibility of the periodic single layer of [21, Thm. 2] and by Lemma 5.4 we deduce that the map from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ to

$$\mathcal{L}\left(C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$$

that takes ϕ to $V_q[\phi,\cdot]^{(-1)}$ and the map from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0,+\infty)^2$ to

$$\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)$$

that takes $(\phi, \lambda^+, \lambda^-)$ to $(I - 2\frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} W_q^*[\phi, \cdot])^{(-1)}$, are both of class C^∞.

Given the bilinearity and continuity of the evaluation map $(L, v) \mapsto L[v]$, which acts from

$$\mathcal{L}\left(C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)\right) \times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$$

to $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$, as well as from

$$\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)\right) \times C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$$

to $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$, we can deduce that the mapping $(\phi, f) \mapsto V_q[\phi, \cdot]^{(-1)}(f)$ is of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$ to $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ and, similarly, the map

$$(\phi, \lambda^+, f) \mapsto \lambda^+ \left(\frac{1}{2} I + W_q^*[\phi, \cdot]\right) \left(V_q[\phi, \cdot]^{(-1)}(f)\right)$$

is of class C^∞ from $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0, +\infty) \times C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$ to $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$.

By once again relying on the bilinearity and continuity of the evaluation map, we ultimately deduce that the map taking $(\phi, \lambda^+, \lambda^-, f, g)$ to

$$\left(I - 2\frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} W_q^*[\phi, \cdot]\right)^{(-1)} \left(-\frac{2}{\lambda^- + \lambda^+} \left(\lambda^+ \left(\frac{1}{2} I + W_q^*[\phi, \cdot]\right) \left(V_q[\phi, \cdot]^{(-1)}(f)\right) + g\right)\right)$$

is of class C^∞, where the domain is $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0, +\infty)^2 \times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega) \times C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$, and the codomain is $C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)$.

Hence, the smoothness of the map $(\phi, \lambda^+, \lambda^-, f, g) \mapsto \rho^-[\phi, \lambda^+, \lambda^-, f, g]$ follows directly from (5.9) and the definition of Λ. The smoothness of $(\phi, \lambda^+, \lambda^-, f, g) \mapsto \rho^+[\phi, \lambda^+, \lambda^-, f, g]$ is a consequence of (5.10). □

Theorem 5.2 provides a representation formula for the solution of problem (5.1) in terms of periodic single-layer potentials, while Proposition 5.5 demonstrates that the corresponding densities exhibit smooth dependence on the shape, boundary data, and transmission parameters. Specifically, we have the expressions

$$\begin{aligned} &u^+[\phi, \lambda^+, \lambda^-, f, g](t, x) \\ &= \int_0^t \int_{\partial\Omega} S_{q,n}(t - \tau, x - \phi(y)) \rho^+[\phi, \lambda^+, \lambda^-, f, g](\tau, y) \tilde{\sigma}_n[\phi](y) \, d\sigma_y d\tau, \end{aligned} \tag{5.11}$$

for all $(t, x) \in [0, T] \times \overline{\mathbb{S}[\phi]}$, and

$$\begin{aligned} &u^-[\phi, \lambda^+, \lambda^-, f, g](t, x) \\ &= \int_0^t \int_{\partial\Omega} S_{q,n}(t - \tau, x - \phi(y)) \rho^-[\phi, \lambda^+, \lambda^-, f, g](\tau, y) \tilde{\sigma}_n[\phi](y) \, d\sigma_y d\tau, \end{aligned} \tag{5.12}$$

for all $(t, x) \in [0, T] \times \overline{\mathbb{S}[\phi]^-}$, where $\rho^+[\phi, \lambda^+, \lambda^-, f, g]$ and $\rho^-[\phi, \lambda^+, \lambda^-, f, g]$ are maps of class C^∞ with respect to the variables $(\phi, \lambda^+, \lambda^-, f, g)$. We are ready to show the main result of this section, about the smooth dependence of the solution of (5.1) on $(\phi, \lambda^+, \lambda^-, f, g)$.

Theorem 5.6. *Let $\alpha \in (0, 1)$ and $T > 0$. Let Ω be as in (2.2). Let Ω^i and Ω^e be two bounded open subsets of $\mathbb{R}^n$. Let $\mathcal{B}^{1,\alpha}_{\partial\Omega,Q}$ be the open subset of $\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$ consisting of those diffeomorphisms ϕ such that*

$$\overline{\Omega^i} \subseteq \mathbb{S}[\phi], \quad \overline{\Omega^e} \subseteq \mathbb{S}[\phi]^-.$$

Then, the map

$$(\phi,\lambda^+,\lambda^-,f,g)\mapsto\Big(u^+[\phi,\lambda^+,\lambda^-,f,g]_{|[0,T]\times\overline{\Omega^i}},u^-[\phi,\lambda^+,\lambda^-,f,g]_{|[0,T]\times\overline{\Omega^e}}\Big)$$

is of class C^∞ *from* $\mathcal{B}^{1,\alpha}_{\partial\Omega,Q}\times(0,+\infty)^2\times C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)\times C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ *to* $C_0^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\overline{\Omega^i})\times C_0^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\overline{\Omega^e})$.

Proof. Without loss of generality we can assume that Ω^i and Ω^e are of class $C^{1,\alpha}$. The maps that associate a diffeomorphism ϕ with the functions

$$\overline{\Omega^i}\times\partial\Omega\ni(x,y)\mapsto x-\phi(y)\in\mathbb{R}^n$$

and

$$\overline{\Omega^e}\times\partial\Omega\ni(x,y)\mapsto x-\phi(y)\in\mathbb{R}^n$$

are both affine and continuous (and thus, smooth), from $\mathcal{B}^{1,\alpha}_{\partial\Omega,Q}$ to $C^{1,\alpha}(\overline{\Omega^i}\times\partial\Omega,\mathbb{R}^n\setminus q\mathbb{Z}^n)$ and $C^{1,\alpha}(\overline{\Omega^e}\times\partial\Omega,\mathbb{R}^n\setminus q\mathbb{Z}^n)$, respectively. By arguing as in the proof of [5, Lem. A.1 and Lem. A.3] regarding the regularity of superposition operators, we deduce that the maps that take ϕ to the functions

$$S_{q,n}(t,x-\phi(y))\qquad\forall[0,T]\times\overline{\Omega^i}\times\partial\Omega$$

and

$$S_{q,n}(t,x-\phi(y))\qquad\forall[0,T]\times\overline{\Omega^e}\times\partial\Omega$$

are of class C^∞ from $\mathcal{B}^{1,\alpha}_{\partial\Omega,Q}$ to $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times(\overline{\Omega^i}\times\partial\Omega))$ and to $C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times(\overline{\Omega^e}\times\partial\Omega))$, respectively. Indeed, we note that the results of [5, Lem. A.1 and Lem. A.3] remain valid also in the case of a manifold with a boundary.

Then, the statement follows by the representation formulas (5.11), (5.12) for the functions $u^\pm[\phi,\lambda^+,\lambda^-,f,g]$, by Proposition 5.5 on the smoothness of $\rho^\pm[\phi,\lambda^+,\lambda^-,f,g]$, by Lemma 2.1 on the analyticity of $\tilde{\sigma}_n[\phi]$, and by the regularity result on integral operators with non-singular kernels of [5, Lem. A.2], which continues to apply even in the case of a manifold with a boundary. □

6. **An expansion result by Neumann-type series.**

If we consider fixed values of $\phi\in\mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$, $f\in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$, and $g\in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\Omega)$, a combination of Proposition 5.3 and a modified version of Proposition 5.5 allows us to establish that the solution to problem (5.1) exhibits analytic dependence on the term $\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]$. Consequently, we can express the densities as convergent power series. Alternatively, this result can be achieved more directly by employing the Neumann series Theorem.

To be more precise, we can demonstrate that locally, around a fixed pair of parameters $(\lambda_0^+,\lambda_0^-)\in(0,+\infty)^2$, the densities can be expressed by means of a Neumann-type series. The terms of this series involve the difference of the terms $\lambda_{\mathbf{c}}[\lambda^+,\lambda^-]$ and $\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]$, as well as iterated compositions of the operator

$$\big(I-2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]W_q^*[\phi,\cdot]\big)^{(-1)}\circ W_q^*[\phi,\cdot].$$

Naturally, once we establish this result for the densities, by utilizing the representation formula of the solution in terms of space-periodic layer potentials, we can deduce a similar result for the solution. The detailed calculation is left to the zealous reader.

We will use the following notation: Given two Banach spaces X and Y and a bounded linear map $T : X \to Y$, we define

$$T^j := \underbrace{T \circ \cdots \circ T}_{j-\text{times}} \quad \text{for every } j \in \mathbb{N},$$

with the convention that $T^0 = I$.

In the theorem below, we fix $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$, $\lambda_0^+, \lambda_0^- > 0$, $f \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$, and $g \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\Omega)$ and we show a representation formula for $\rho^-[\phi, \lambda^+, \lambda^-, f, g]$ as a convergent power series depending on the difference of the terms $\lambda_{\mathbf{c}}[\lambda^+, \lambda^-]$ and $\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-]$. For the sake of exposition, for every $j \in \mathbb{N}$, we define the map

$$\mathcal{K}_j : \mathcal{A}^{1,\alpha}_{\partial\Omega,Q} \times (0, +\infty)^2 \to \mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \partial\Omega)),$$

given by

$$\mathcal{K}_j[\phi, \lambda_0^+, \lambda_0^-] := 2^j \left(\left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-] W_q^*[\phi, \cdot]\right)^{(-1)} \circ W_q^*[\phi, \cdot] \right)^j . \tag{6.1}$$

Then the following holds.

Theorem 6.1. *Let $\alpha \in (0,1)$ and $T > 0$. Let Ω be as in (2.2). Let $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$, $\lambda_0^+, \lambda_0^- > 0$, $f \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T] \times \partial\Omega)$, and $g \in C_0^{\frac{\alpha}{2};\alpha}([0,T] \times \Omega)$ be fixed.*

Then, there exists a positive constant $\varepsilon \in (0, +\infty)$ such that the following holds: For every $(\lambda^+, \lambda^-) \in (0, +\infty)^2$ such that

$$|\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-]| < \varepsilon, \tag{6.2}$$

with $\lambda_{\mathbf{c}}[\cdot, \cdot]$ defined by (5.5), we have

$$\begin{aligned}\rho^-[\phi, \lambda^+, \lambda^-, f, g] = &\left(\sum_{j=0}^{+\infty} (\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-])^j \mathcal{K}_j[\phi, \lambda_0^+, \lambda_0^-] \right) \\ &\circ \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-] W_q^*[\phi, \cdot]\right)^{(-1)} (\rho_0^-[\phi, \lambda^+, \lambda^-, f, g]),\end{aligned} \tag{6.3}$$

where the series

$$\sum_{j=0}^{+\infty} \zeta^j \mathcal{K}_j[\phi, \lambda_0^+, \lambda_0^-]$$

converges normally in $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega))$ for $|\zeta| < \varepsilon$ and where

$$\rho_0^-[\phi, \lambda^+, \lambda^-, f, g] := -\frac{2}{\lambda^- + \lambda^+} \left(\lambda^+ \left(\frac{1}{2} I + W_q^*[\phi, \cdot] \right) \left(V_q[\phi, \cdot]^{(-1)}(f) \right) + g \right). \tag{6.4}$$

Proof. Let $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega,Q}$, $\lambda_0^+, \lambda_0^- > 0$, $f \in C_0^{\frac{1+\alpha}{2};1+\alpha}([0,T]\times\partial\Omega)$, and $g \in C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. We first notice that, by the definition of ρ_0^- in (6.4), we can rewrite (5.6) as

$$\left(I - 2\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] W_q^*[\phi, \cdot]\right) \rho^-[\phi, \lambda^+, \lambda^-, f, g] = \rho_0^-[\phi, \lambda^+, \lambda^-, f, g], \tag{6.5}$$

for every $(\lambda^+, \lambda^-) \in (0, +\infty)^2$. We now consider the operator on the left-hand side of (6.5), which is $I - 2\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] W_q^*[\phi, \cdot] : C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega) \to C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$. By adding and subtracting the term $2\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-] W_q^*[\phi, \cdot]$ and factoring out the operator $I - 2\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-] W_q^*[\phi, \cdot]$, we can rewrite this operator as follows:

$$\begin{aligned}&I - 2\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] W_q^*[\phi, \cdot] \\ &= I - 2\lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-] W_q^*[\phi, \cdot] - 2(\lambda_{\mathbf{c}}[\lambda^+, \lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+, \lambda_0^-]) W_q^*[\phi, \cdot]\end{aligned}$$

$$
\begin{aligned}
&= \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right) \\
&\circ \Big(I - 2(\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]) \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \circ W_q^*[\phi,\cdot]\Big) . \qquad (6.6)
\end{aligned}
$$

In particular, by (6.6), we deduce that

$$
\begin{aligned}
&\left(I - 2\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] W_q^*[\phi,\cdot]\right)^{(-1)} \\
&= \Big(I - 2(\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]) \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \circ W_q^*[\phi,\cdot]\Big)^{(-1)} \\
&\quad \circ \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} . \qquad (6.7)
\end{aligned}
$$

Then, if we choose $\varepsilon > 0$ small enough, for example

$$
\varepsilon := \frac{1}{2\left\| \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \circ W_q^*[\phi,\cdot] \right\|_{\mathcal{L}\left(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)\right)}}, \qquad (6.8)
$$

we have that, for every $(\lambda^+,\lambda^-) \in (0,+\infty)^2$ such that (6.2) holds, the inverse of the operator

$$
I - 2(\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]) \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \circ W_q^*[\phi,\cdot]
$$

from $C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega)$ into itself can be written as a normally convergent Neumann series in $\mathcal{L}(C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega), C_0^{\frac{\alpha}{2};\alpha}([0,T]\times\partial\Omega))$. In fact, by (6.2) and (6.8), and by the Neumann series Theorem, we have that

$$
\begin{aligned}
&\Big(I - 2(\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-]) \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \circ W_q^*[\phi,\cdot]\Big)^{(-1)} \\
&= \sum_{j=0}^{+\infty} (\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-])^j \mathcal{K}_j[\phi,\lambda_0^+,\lambda_0^-] , \qquad (6.9)
\end{aligned}
$$

where for each $j \in \mathbb{N}$ the operator $\mathcal{K}_j[\cdot,\cdot,\cdot]$ is defined by (6.1). Finally, (6.5), (6.7) and (6.9) yield to the validity of (6.3). □

Remark 6.2. Let the assumptions of Theorem 6.1 hold. By equations (6.3) and (6.4), we have

$$
\begin{aligned}
&\rho^-[\phi,\lambda^+,\lambda^-,f,g] \\
&= -\frac{2\lambda^+}{\lambda^- + \lambda^+} \left(\sum_{j=0}^{+\infty} (\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-])^j \mathcal{K}_j[\phi,\lambda_0^+,\lambda_0^-] \right) \\
&\quad \circ \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} \left(\frac{1}{2} I + W_q^*[\phi,\cdot]\right) \left(V_q[\phi,\cdot]^{(-1)}(f)\right) \\
&\quad - \frac{2}{\lambda^- + \lambda^+} \left(\sum_{j=0}^{+\infty} (\lambda_{\mathbf{c}}[\lambda^+,\lambda^-] - \lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-])^j \mathcal{K}_j[\phi,\lambda_0^+,\lambda_0^-] \right) \\
&\quad \circ \left(I - 2\lambda_{\mathbf{c}}[\lambda_0^+,\lambda_0^-] W_q^*[\phi,\cdot]\right)^{(-1)} (g) ,
\end{aligned}
$$

for every $(\lambda^+,\lambda^-) \in (0,+\infty)^2$ such that condition (6.2) holds.

Acknowledgments. The authors are members of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA) of the "Istituto Nazionale di Alta Matematica" (INdAM). Part of this work was done while R.M. was visiting C3M - Centre for Computational Continuum Mechanics (Slovenia). R.M. wishes to thank C3M for the kind hospitality.

REFERENCES

[1] S. Bernstein, S. Ebert and R. Sören Kraußhar, On the diffusion equation and diffusion wavelets on flat cylinders and the n-torus, *Math. Methods Appl. Sci.*, **34** (2011), 428-441.

[2] R. Chapko, R. Kress and J. R. Yoon, On the numerical solution of an inverse boundary value problem for the heat equation, *Inverse Probl.*, **14** (1998), 853-867.

[3] M. Dalla Riva and M. Lanza de Cristoforis, A perturbation result for the layer potentials of general second order differential operators with constant coefficients, *J. Appl. Funct. Anal.*, **5** (2010), 10-30.

[4] M. Dalla Riva, M. Lanza de Cristoforis and P. Musolino, *Singularly Perturbed Boundary Value Problems: A Functional Analytic Approach*, Springer Nature, Cham, 2021.

[5] M. Dalla Riva and P. Luzzini, Regularity of layer heat potentials upon perturbation of the space support in the optimal Hölder setting, *Differ. Integral Equ.*, **36** (2023), 971-1003.

[6] M. Dalla Riva, P. Luzzini and P. Musolino, et al., Dependence of effective properties upon regular perturbations, in *Mechanics and Physics of Structured Media. Asymptotic and Integral Equations Methods of Leonid Filshtinsky*, Academic Press, Elsevier, London, 2022, pp. 271-301.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[8] H. Haddar and R. Kress, On the Fréchet derivative for obstacle scattering with an impedance boundary condition, *SIAM J. Appl. Math.*, **65** (2004), 194-208.

[9] A. Henrot and M. Pierre, *Variation et Optimisation de Formes*, Vol. **48** of Mathématiques & Applications (Berlin), Springer, Berlin, 2005.

[10] F. Hettlich and W. Rundell, Identification of a discontinuous source in the heat equation, *Inverse Probl.*, **17** (2001), 1465-1482.

[11] E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, vol. 31, American Mathematical Society, Providence, R. I., 1957.

[12] S. Hofmann, J. Lewis and M. Mitrea, Spectral properties of parabolic layer potentials and transmission boundary problems in nonsmooth domains, *Illinois J. Math.*, **47** (2003), 1345-1361.

[13] C. Jerez-Hanckes, C. Schwab and J. Zech, Electromagnetic wave scattering by random surfaces: shape holomorphy, *Math. Models Methods Appl. Sci.*, **27** (2017), 2229-2259.

[14] A. Kirsch, The domain derivative and two applications in inverse scattering theory, *Inverse Probl.*, **9** (1993), 81-96.

[15] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, **23** American Mathematical Society, Providence, R.I. 1968.

[16] M. Lanza de Cristoforis, Perturbation problems in potential theory, a functional analytic approach, *J. Appl. Funct. Anal.*, **2** (2007), 197-222.

[17] M. Lanza de Cristoforis and P. Musolino, A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients, *Far East J. Math. Sci.*, **52** (2011), 75-120.

[18] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double-layer potentials upon perturbation of the support and of the density, *J. Integral Equ. Appl.* **16** (2004), 137-174.

[19] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double-layer potentials for the Helmholtz equation upon perturbation of the support and of the density, In *Analytic methods of analysis and differential equations: AMADE 2006*, Camb. Sci. Publ., Cambridge, 2008, pp. 193-220.

[20] P. Luzzini, Regularizing properties of space-periodic layer heat potentials and applications to boundary value problems in periodic domains, *Math. Methods Appl. Sci.* **43** (2020), 5273-5294.

[21] P. Luzzini and P. Musolino, Periodic transmission problems for the heat equation, in *Integral Methods in Science and Engineering*, 211–223, Birkhäuser/Springer, Cham, 2019.

[22] M. A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, Reprint of the 2002 original, Graduate Studies in Mathematics, **102**, American Mathematical Society, Providence, RI, 2009.

[23] R. Potthast, Fréchet differentiability of boundary integral operators in inverse acoustic scattering, *Inverse Probl.*, **10** (1994), 431-447.

[24] T. Qiu, A. Rieder and F. J. Sayas, et al., Time-domain boundary integral equation modeling of heat transmission problems, *Numer. Math.*, **143** (2019), 223-259.

[25] J. Sokołowski and J.-P. Zolésio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Vol. **16** of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1992.

[26] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains, *J. Funct. Anal.*, **59** (1984), 572-611.

Received September 2023; revised November 2023; early access January 2024.