

Integrability of the n -dimensional Axially Symmetric Chaplygin Sphere

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Abstract—We consider the n -dimensional Chaplygin sphere under the assumption that the mass distribution of the sphere is axisymmetric. We prove that, for initial conditions whose angular momentum about the contact point is vertical, the dynamics is quasi-periodic. For $n = 4$ we perform the reduction by the associated $\text{SO}(3)$ symmetry and show that the reduced system is integrable by the Euler–Jacobi theorem.

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Dedicated to S.A. Chaplygin on the occasion of his 150th birthday

1. INTRODUCTION

The Chaplygin sphere is perhaps the most interesting example of an integrable system in non-holonomic mechanics. It concerns the motion of a sphere, whose centre of mass coincides with its geometric centre, that rolls without slipping on the plane. The integrability of the problem was proved by S. A. Chaplygin in his celebrated paper [4].

The n -dimensional generalisation of the problem was introduced by Fedorov and Kozlov in [8] where the authors conjecture that this generalisation is also integrable. To the author's best knowledge such a conjecture is only known to be true in the following cases:

- 1) the inertia tensor $\mathbb{I} : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ is spherical, i. e., a constant factor of the identity operator. In this simple case the dynamics is trivially integrable since the angular velocity remains constant along the motion;
- 2) the case treated by Jovanović in [12]. Here the initial condition is restricted to have *horizontal momentum*. Moreover, the inertia operator $\mathbb{I} : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ is assumed to be of a very specific type and, in particular, to map the space of rank-two matrices in $\mathfrak{so}(n)$ into itself (see [12, Eq. (49)]).

For $n \geq 4$ the inertia operator considered by Jovanović in case 2 above does not generally correspond to a physical inertia operator of a multi-dimensional rigid body unless such a body is axisymmetric (see [12, Remark 2] and [7, Appendix B]).

In this paper we further analyse the dynamics of the system under the assumption that the distribution of mass of the sphere is axisymmetric. In our approach we take advantage of the associated additional $\text{SO}(n-1)$ symmetry of the problem. This kind of symmetry analysis has already proved to be useful in determining new cases of integrability of the n -dimensional Veselova problem [7].

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Our main result is to prove that the dynamics of the problem for arbitrary n is quasi-periodic if the angular momentum about the contact point is *vertical*. We also consider general initial conditions in the case $n = 4$ and show that the reduction of the system by the additional $SO(3)$ symmetry is integrable by the Euler–Jacobi theorem.

The paper is organised as follows. We first recall the equations of motion and their main properties in Section 2. Next, in Section 3, we define the spaces of vertical and horizontal momentum both in 3 and nD . Section 4 studies the axisymmetric sphere for general n and Section 5 focuses on the case $n = 4$.

2. PRELIMINARIES

2.1. The Classical 3D Chaplygin Sphere

The homogeneity of the plane where the rolling takes place leads to a symmetry of the problem with respect to the action of the Euclidean group $SE(2)$. The reduced equations of motion are well known and given by

$$\dot{M} = M \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega, \tag{2.1}$$

where $M \in \mathbb{R}^3$ is the angular momentum of the sphere about the contact point, $\Omega \in \mathbb{R}^3$ is the angular velocity, $\gamma \in \mathbb{R}^3$ is the normal vector to the plane and “ \times ” denotes the cross product. All vectors M, Ω, γ are written in a body frame that is attached to the centre of the sphere and satisfy

$$M = \mathbb{I}(\Omega) + b\gamma \times (\Omega \times \gamma), \tag{2.2}$$

where the 3×3 matrix \mathbb{I} is the tensor of inertia, and $b = mr^2$ where $m, r > 0$ denote the mass and the radius of the sphere. We assume that the body frame is aligned with the principal axes of inertia so $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$, with $I_j > 0$ denoting the principal moments of inertia.

The system (2.1) possesses the trivial integral $\|\gamma\|^2$ and from now on we restrict our attention to $\|\gamma\|^2 = 1$, and interpret γ as an element of the unit sphere S^2 . For future reference we note that the first equation in (2.1) may be equivalently written as

$$\mathbb{I}(\dot{\Omega}) = \mathbb{I}(\Omega) \times \Omega - b\gamma \times (\dot{\Omega} \times \gamma). \tag{2.3}$$

For a fixed $\gamma \in S^2$, Eq. (2.2) defines a linear relation between M and Ω that we denote by $M = L_\gamma(\Omega)$. As may be checked directly, the determinant of L_γ satisfies

$$\det(L_\gamma) = c \left(1 - b \langle \tilde{\mathbb{I}}^{-1} \gamma, \gamma \rangle \right) > 0,$$

where $\tilde{\mathbb{I}} := \mathbb{I} + b\text{Id}_3$, the constant $c = \prod_{j=1}^3 (I_j + b) = \det(\tilde{\mathbb{I}})$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in \mathbb{R}^3 . The equations of motion may be written down in explicit form in terms of $(M, \gamma) \in \mathbb{R}^3 \times S^2$ by noting that the inversion of L_γ leads to

$$\Omega = L_\gamma^{-1}(M) = \tilde{\mathbb{I}}^{-1}M + \frac{bc \langle \tilde{\mathbb{I}}^{-1}M, \gamma \rangle}{\det(L_\gamma)} \tilde{\mathbb{I}}^{-1}\gamma. \tag{2.4}$$

First Integrals and Measure Preservation

The equations of motion (2.1) state that the expression of the vectors M and γ in the space frame is constant. This observation is trivial in the case of γ , but is quite remarkable in the case of M , and leads to the existence of the following first integrals:

$$f_1 = \langle M, \gamma \rangle, \quad f_2 = \|M\|^2.$$

In addition, the system preserves the energy $H = \frac{1}{2} \langle M, \Omega \rangle$ and possesses the invariant measure

$$\mu = \frac{1}{\sqrt{\det(L_\gamma)}} dM d\gamma = \sqrt{\det(L_\gamma)} d\Omega d\gamma.$$

Therefore, Eqs. (2.1), which define a vector field on the 5-dimensional phase space $P = \mathbb{R}^3 \times S^2 \ni (M, \gamma)$, possess 3 independent integrals f_1, f_2, H together with the invariant measure μ , and are thus integrable by the Euler–Jacobi theorem (see, e. g., [1]). The explicit integration of Eqs. (2.1) was obtained by Chaplygin in [4].

2.2. The n -dimensional Chaplygin Sphere

The multi-dimensional generalisation of the Chaplygin sphere was first considered by Fedorov and Kozlov [8]. In this case the n -dimensional sphere rolls without slipping on an $(n-1)$ -dimensional hyperplane whose homogeneity leads to an $\text{SE}(n-1)$ symmetry, and the reduced equations of the motion are given by

$$\dot{M} = [M, \Omega], \quad \dot{\gamma} = -\Omega\gamma, \quad (2.5)$$

where the angular momentum about the contact point M and the angular velocity Ω are now elements of the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric matrices and $[M, \Omega]$ denotes their commutator. As before, $\gamma \in \mathbb{R}^n$ is the Poisson vector that gives body coordinates of the unit normal to the hyperplane where the rolling takes place; in particular, $\|\gamma\| = 1$ throughout the motion, so we think of $\gamma \in \mathbb{S}^{n-1}$. All quantities are expressed in the body frame that is attached to the centre of the sphere. The relationship between M and Ω that generalises (2.2) is¹⁾

$$M = L_\gamma(\Omega) = \mathbb{I}(\Omega) + b(\Omega\gamma) \wedge \gamma, \quad (2.6)$$

where, as before, $b = mr^2$. The inertia tensor is now a map $\mathbb{I} : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ of the form

$$\mathbb{I}(\Omega) = \mathbb{J}\Omega + \Omega\mathbb{J}, \quad (2.7)$$

where \mathbb{J} is the *mass tensor*, which is a constant, symmetric, $n \times n$ matrix that depends on the mass distribution of the body. By an appropriate choice of a body frame, \mathbb{J} may be assumed to be diagonal with positive entries (see, e. g., Ratiu [15]).

For further reference we note that, in analogy with (2.3), the first equation in (2.5) may be rewritten as

$$\mathbb{I}(\dot{\Omega}) = [\mathbb{I}(\Omega), \Omega] - b(\dot{\Omega}\gamma) \wedge \gamma. \quad (2.8)$$

To the author's best knowledge, an explicit expression for L_γ^{-1} that generalises (2.4) for a general inertia tensor is unknown. In Proposition 3 below we give such a formula under the assumption that the inertia tensor is axisymmetric.

First Integrals and Measure Preservation

As in the 3D case, the equations of motion (2.5) state that the expressions of M and γ in the space frame are constant and this leads to the existence of several integrals of motion. To see this, note that for any $\sigma \in \mathbb{R}$, the matrix $M + \sigma\gamma\gamma^t$ undergoes an iso-spectral evolution:

$$\frac{d}{dt}(M + \sigma\gamma\gamma^t) = [M + \sigma\gamma\gamma^t, \Omega]. \quad (2.9)$$

As a consequence, the coefficients of the two-variable polynomial

$$p(\lambda, \sigma) = \det(M + \sigma\gamma\gamma^t - \lambda\text{Id}_n)$$

are first integrals. In addition to these integrals, the energy of the system is also preserved. In this n -dimensional case it is given by

$$H = \frac{1}{2}\langle M, \Omega \rangle_\kappa,$$

where $\langle \cdot, \cdot \rangle_\kappa$ is the Killing metric in $\mathfrak{so}(n)$ defined by $\langle \xi_1, \xi_2 \rangle_\kappa = -\frac{1}{2}\text{tr}(\xi_1\xi_2)$ for $\xi_1, \xi_2 \in \mathfrak{so}(n)$.

The work of Fedorov and Kozlov [8] shows that the n -dimensional system also possesses a smooth invariant measure that again may be written as

$$\mu = \frac{1}{\sqrt{\det(L_\gamma)}} dM d\gamma = \sqrt{\det(L_\gamma)} d\Omega d\gamma. \quad (2.10)$$

Despite the large number of first integrals and the existence of an invariant measure, the integrability of the system for $n > 3$ has only been established in a very particular case, described below, by Jovanović [12], and for spherical inertia tensors.

¹⁾We recall that the wedge product of column vectors $u, v \in \mathbb{R}^n$ is defined as $u \wedge v = uv^t - vu^t \in \mathfrak{so}(n)$.

3. VERTICAL AND HORIZONTAL MOMENTUM

3.1. The 3D Case

The integration of the equations carried out by Chaplygin [4] proceeds by first distinguishing two special classes of initial conditions that correspond to **vertical** and **horizontal momentum**. In the first case the vectors M and γ are parallel and in the other perpendicular. With this in mind we define the subsets \mathcal{V}, \mathcal{H} of the phase space $P = \mathbb{R}^3 \times S^2 \ni (M, \gamma)$ by

$$\mathcal{V} = \{(M, \gamma) \in P : M \times \gamma = 0\}, \quad \mathcal{H} = \{(M, \gamma) \in P : f_2 = \langle M, \gamma \rangle = 0\}. \tag{3.1}$$

These are submanifolds of P , of dimension 3 and 4, respectively, which are invariant by the flow of (2.1). Their intersection $\mathcal{V} \cap \mathcal{H}$ consists of initial conditions having $M = 0$, which corresponds to the sphere being at rest.

In his celebrated work [4], Chaplygin integrated the equations of motion for initial conditions in \mathcal{H} in terms of hyperelliptic functions on a genus 2 Riemann surface. He then showed that the integration for generic initial conditions that do not belong to \mathcal{H} nor \mathcal{V} , may be reduced to the case of horizontal momentum by means of an insightful change of variables.

The integration of the equations for vertical momentum is much simpler since the equations of motion in this case reduce to the standard Euler equations for the motion of a rigid body with tensor of inertia $\tilde{\mathbb{I}}$. We recall this well-known result in the following proposition.

Proposition 1. *Denote by $\Omega(t)$ the angular velocity along a solution of (2.1) whose initial condition (M_0, γ_0) lies in \mathcal{V} . Then $\Omega(t)$ is a solution of the Euler equations*

$$\tilde{\mathbb{I}}\dot{\Omega} = (\tilde{\mathbb{I}}\Omega) \times \Omega. \tag{3.2}$$

Proof. Denote by $(M(t), \gamma(t))$ such a solution. Since \mathcal{V} is invariant, $M(t) = \lambda\gamma(t)$ for a scalar $\lambda \in \mathbb{R}$ that is necessarily constant since it satisfies $f_1(M_0, \gamma_0) = f_1(M(t), \gamma(t)) = \lambda^2$. As a consequence, we have $2H = \lambda\langle\Omega(t), \gamma(t)\rangle$, which shows that $\langle\Omega(t), \gamma(t)\rangle$ is also constant along the motion. Therefore, we have $\frac{d}{dt}\langle\Omega(t), \gamma(t)\rangle = \langle\dot{\Omega}(t), \gamma(t)\rangle = 0$ and (2.3) may be rewritten as (3.2). \square

3.2. The nD -case

The discussion above about the vertical and horizontal momentum spaces may be generalised to n -dimensions by defining \mathcal{H} and \mathcal{V} as the following submanifolds of the phase space $P = \mathfrak{so}(n) \times S^{n-1}$:

$$\mathcal{V} = \{(M, \gamma) \in P : M\gamma = 0\}, \quad \mathcal{H} = \{(M, \gamma) \in P : M - (M\gamma) \wedge \gamma = 0\}. \tag{3.3}$$

It may be easily checked using the equations of motion (2.5) that \mathcal{V} and \mathcal{H} are indeed invariant under the flow. The conditions to belong to \mathcal{H} and \mathcal{V} are that the *space* representation of the angular momentum about the contact point has the respective form:

$$\begin{pmatrix} \mathfrak{so}(n-1) & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{1 \times (n-1)} & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0}_{(n-1) \times (n-1)} & \mathbb{R}^{n-1} \\ -(\mathbb{R}^{n-1})^T & 0 \end{pmatrix},$$

where we have assumed that $e_n = (0, \dots, 0, 1)^t$ is the normal vector to the $(n-1)$ -dimensional hyperplane where the rolling takes place. Considering that $\gamma \in S^{n-1}$, it follows that the dimension of \mathcal{V} is $\frac{n(n-1)}{2}$, whereas that of \mathcal{H} is $2(n-1)$. In particular, and in contrast to the case $n = 3$, the set of vertical momentum is much bigger than that of horizontal momentum if n is large.

As mentioned in the introduction, the only known results of integrability for $n \geq 4$ were given by Jovanović [12]. His work is concerned only with initial conditions on \mathcal{H} , and assumes that the inertia tensor is of a very specific form (see [12, Eq. (49)]). If one requires this inertia tensor to be physical, i. e., satisfies condition (2.7), this leads to the condition that the sphere is axisymmetric, which we treat below and is the main topic of this paper.

Remark 1. The argument in the proof of Proposition 1, which shows that the 3D Chaplygin sphere evolves as the Euler top with inertia tensor $\tilde{\mathbb{I}}$ along the vertical space, depends crucially on the assumption that $n = 3$ and may not be extended for $n > 3$. In fact, at the end of Section 4 below, we show that such a simplification is not possible for $n > 3$.

4. THE AXISYMMETRIC CHAPLYGIN SPHERE

Suppose now that the mass distribution of the sphere is axisymmetric. We choose the body frame $\{E_1, \dots, E_n\}$ in such a way that E_n is aligned with the axis of symmetry. With the appropriate normalisation of units this leads to the following condition for the mass matrix:

$$\mathbb{J} = \text{Id}_n + aE_nE_n^t, \tag{4.1}$$

for a real parameter a that for physical reasons is assumed to satisfy $-1 \leq a \leq 1$. The case $a = 0$ corresponds to a spherical inertia tensor. We note that, in the 3D case, our assumptions imply that the 3×3 inertia matrix is the diagonal matrix with entries $(2 + a, 2 + a, 2)$.

With our assumption (4.1) we may write $\mathbb{I}(\Omega) = 2\Omega + a(\Omega E_n) \wedge E_n$, and (2.6) becomes

$$M = L_\gamma(\Omega) = 2\Omega + a(\Omega E_n) \wedge E_n + b(\Omega \gamma) \wedge \gamma. \tag{4.2}$$

We may also rewrite Eq. (2.8) as

$$2\dot{\Omega} + a(\dot{\Omega}E_n) \wedge E_n + b(\dot{\Omega}\gamma) \wedge \gamma = a[(\Omega E_n) \wedge E_n, \Omega].$$

This equation shows that, if the inertia tensor is spherical ($a = 0$), then $\dot{\Omega} = 0$ and the angular velocity is constant along the motion. This observation has already been made in [11].

For the rest of the paper we write $x := \gamma_n = \langle \gamma, E_n \rangle$. We also denote

$$\Delta(x) = (2 + a)(2 + b) - abx^2, \tag{4.3}$$

and we note that $\Delta(x) > 0$ due to the restrictions that $x^2 \leq 1$ and $a \in [-1, 1]$.

Proposition 2. *Under the axisymmetric assumption (4.1) the invariant measure of the multi-dimensional Chaplygin sphere is given by*

$$\mu = \frac{1}{\Delta(x)^{\frac{n-2}{2}}} dM d\gamma = \Delta(x)^{\frac{n-2}{2}} d\Omega d\gamma. \tag{4.4}$$

Proof. We will prove that $\det(L_\gamma)$ is proportional to $\Delta(x)^{n-2}$. The result then follows from the formula (2.10) for the invariant measure given by Fedorov and Kozlov [8].

Suppose that γ and E_n are linearly independent and let $\{\gamma, E_n, w_1, \dots, w_{n-2}\}$ be a basis of \mathbb{R}^n with the property that $\langle w_j, \gamma \rangle = \langle w_j, E_n \rangle = 0$ for all j . Using the expression (4.2) for the linear operator $L_\gamma : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$, one computes

$$\begin{aligned} L_\gamma(E_n \wedge w_j) &= (2 + a)E_n \wedge w_j + bx\gamma \wedge w_j, & L_\gamma(\gamma \wedge w_j) &= axE_n \wedge w_j + (2 + b)\gamma \wedge w_j, \\ L_\gamma(E_n \wedge \gamma) &= (2 + a + b)E_n \wedge \gamma, & L_\gamma(w_i \wedge w_j) &= 2w_i \wedge w_j. \end{aligned}$$

Therefore, the matrix representation of L_γ with respect to the ordered basis

$$E_n \wedge w_1, \gamma \wedge w_1, \dots, E_n \wedge w_{n-2}, \gamma \wedge w_{n-2}, E_n \wedge \gamma, w_i \wedge w_j,$$

of $\mathfrak{so}(n)$ is given in block-diagonal form as

$$\begin{pmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & \vdots \\ 0 & \dots & C_{n-2} & 0 & 0 \\ 0 & \dots & 0 & 2 + a + b & 0 \\ 0 & \dots & 0 & 0 & 2\text{Id}_{(n-2)(n-3)/2} \end{pmatrix},$$

where the 2×2 blocks C_j , $j = 1, \dots, n - 2$, are all identical and equal to

$$C = \begin{pmatrix} 2 + a & ax \\ bx & 2 + b \end{pmatrix}.$$

Given that $\det(C) = \Delta(x)$, it follows that, up to the constant factor $(2 + a + b)2^{(n-2)(n-3)/2}$, the determinant of L_γ equals $\Delta(x)^{n-2}$. The proof for configurations where γ and E_n are parallel follows by continuity. \square

The following proposition gives the explicit form of L_γ^{-1} under our axisymmetric assumption (4.1).

Proposition 3. Equation (4.2) may be inverted to express Ω in terms of M and γ as

$$\Omega = L_\gamma^{-1}(M) = \frac{1}{2}M + \frac{1}{2\Delta(x)}K, \tag{4.5}$$

where

$$K = abx(M\gamma) \wedge E_n - a(2 + b)(ME_n) \wedge E_n + abx(ME_n) \wedge \gamma - b(2 + a)(M\gamma) \wedge \gamma - \frac{ab(4 + a + b)\langle M\gamma, E_n \rangle}{2 + a + b} \gamma \wedge E_n. \tag{4.6}$$

Proof. We will obtain expressions for $(\Omega E_n) \wedge E_n$ and $(\Omega\gamma) \wedge \gamma$ in terms of M , and γ . Once this is done, the inversion of (4.2) is trivial.

First note that, thanks to the skew-symmetry of Ω , multiplying (4.2) on the left by γ^t and on the right by E_n leads to

$$\langle \Omega\gamma, E_n \rangle = \frac{\langle M\gamma, E_n \rangle}{2 + a + b}. \tag{4.7}$$

Next, multiplying (4.2) on the right by E_n (respectively γ) and taking the exterior product of the resulting expression with E_n leads to the system of two linear equations for the unknowns $U := (\Omega E_n) \wedge E_n$ and $Z := (\Omega\gamma) \wedge E_n$:

$$(2 + a)U + bxZ = (ME_n) \wedge E_n + b \frac{\langle M\gamma, E_n \rangle}{2 + a + b} \gamma \wedge E_n, \quad axU + (2 + b)Z = (M\gamma) \wedge E_n,$$

where we have made use of the expression for $(\Omega\gamma, E_n)$ given above. The determinant of this linear system is precisely $\Delta(x)$ in (4.3) and its unique solution gives

$$(\Omega E_n) \wedge E_n = U = \frac{1}{\Delta(x)} \left((2 + b)(ME_n) \wedge E_n - bx(M\gamma) \wedge E_n + \frac{b(2 + b)\langle M\gamma, E_n \rangle}{2 + a + b} \gamma \wedge E_n \right).$$

Proceeding in a completely analogous manner, multiplying (4.2) on the right by E_n (respectively γ) and taking the exterior product of the resulting expression with γ , one obtains a linear system of two equations for the unknowns $(\Omega\gamma) \wedge \gamma$ and $(\Omega E_n) \wedge \gamma$, whose solution gives

$$(\Omega\gamma) \wedge \gamma = \frac{1}{\Delta(x)} \left(-ax(ME_n) \wedge \gamma + (2 + a)(M\gamma) \wedge \gamma + \frac{a(2 + a)\langle M\gamma, E_n \rangle}{2 + a + b} \gamma \wedge E_n \right).$$

The proof of the proposition follows by inserting the above expressions in (4.2). \square

4.1. The Additional $SO(n - 1)$ Symmetry

Our assumption that the body is axisymmetric leads to a symmetry of the equations that we now describe. We shall write

$$SO(n - 1) = \{h \in SO(n) : hE_n = E_n\}. \tag{4.8}$$

Then $SO(n - 1)$ acts on the phase space $P = \mathfrak{so}(n) \times S^{n-1} \ni (M, \gamma)$ by

$$h \cdot (M, \gamma) = (hMh^{-1}, h\gamma). \tag{4.9}$$

Using the condition that $hE_n = E_n$, one may use Proposition 3 to check that Ω is mapped into $h\Omega h^{-1}$ by the action of $h \in SO(n - 1)$. Hence, taking into account the equivariance of the matrix commutator with respect to conjugation, it is straightforward to check that the system (2.5) is $SO(n - 1)$ -equivariant and the dynamics may be reduced to $P/SO(n - 1)$.

4.2. The Solution in the Case of Vertical Momentum

In this section we show that for an axisymmetric sphere of arbitrary dimension n , the system may be explicitly integrated for initial conditions on the space of vertical angular momentum \mathcal{V} defined by (3.3). These solutions are quasi-periodic in $P = \mathfrak{so}(n) \times S^{n-1}$ and are relative equilibria of the $\text{SO}(n - 1)$ action described above.

For our purposes it is convenient to note that the vertical space \mathcal{V} defined by (3.3) may be equivalently described in terms of γ and Ω as

$$\mathcal{V} = \{(\Omega, \gamma) \in \mathfrak{so}(n) \times S^{n-1} : (2 + b)\Omega\gamma + ax\Omega E_n = 0 \text{ and } \langle \Omega\gamma, E_n \rangle = 0\}. \tag{4.10}$$

This may be checked by multiplying Eq. (4.2) on the right by γ and using (4.7) to conclude that $\langle \Omega\gamma, E_n \rangle = 0$ along \mathcal{V} .

The 3D Case

Let us first consider the case $n = 3$. Under our assumptions on the inertia tensor, we have

$$\tilde{\mathbb{I}} = \text{diag}(2 + a + b, 2 + a + b, 2 + b),$$

and (3.2) yields

$$\dot{\Omega}_1 = \frac{a\Omega_2\Omega_3}{2 + a + b}, \quad \dot{\Omega}_2 = -\frac{a\Omega_1\Omega_3}{2 + a + b}, \quad \dot{\Omega}_3 = 0.$$

The solution of this system with initial condition $\Omega(0) = \Omega_0 \in \mathbb{R}^3$ is easily obtained in terms of sines and cosines. In order to compare with the nD case ahead, we write it in matrix form as

$$\hat{\Omega}(t) = \exp(\zeta t)\hat{\Omega}_0 \exp(-\zeta t),$$

where

$$\zeta = \frac{-a}{2 + a + b} \begin{pmatrix} 0 & -\bar{\Omega}_3 & 0 \\ \bar{\Omega}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{-a}{2 + a + b} \left(\hat{\Omega}_0 - (\hat{\Omega}_0 E_3) \wedge E_3 \right). \tag{4.11}$$

Here $\bar{\Omega}_3$ denotes the third component of Ω_0 and $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is the ‘‘hat map’’ (see, e. g., [14]) which is the well-known Lie algebra isomorphism determined by the condition that $\hat{u}\hat{v} = u \times v$ for all $u, v \in \mathbb{R}^3$.

We now claim that the solution of the Poisson equation $\dot{\gamma} = \gamma \times \Omega(t)$ is

$$\gamma(t) = \exp(\zeta t)\gamma_0,$$

where γ_0 is the initial condition. This is easy to prove after noting that, because our initial condition lies in \mathcal{V} , the matrix ζ in (4.11) satisfies

$$\zeta\gamma_0 = -\hat{\Omega}_0\gamma_0 = \gamma_0 \times \Omega_0.$$

Indeed, this may be checked by using the 3D version of (4.10). Finally, in view of (4.2) and since $\zeta E_3 = 0$, we have

$$\hat{M}(t) = \exp(\zeta t)\hat{M}_0 \exp(-\zeta t),$$

where M_0 is the initial condition.

The above expressions together with the definition of the $\text{SO}(2)$ action defined by (4.8) and (4.9), imply that the solutions along \mathcal{V} are contained in the orbits of the $\text{SO}(2)$ action. In other words, they are **relative equilibria**.

The nD Case

We now generalise the discussion above for general n by showing that the dynamics along \mathcal{V} consists of relative equilibria with respect to the $\text{SO}(n - 1)$ action defined by (4.8) and (4.9). As we shall see, for $n \geq 4$ the expression for the ‘‘velocity’’ ζ of the relative equilibria is more intricate than (4.11). Since $\text{SO}(n - 1)$ is compact, the corresponding solutions on P are quasi-periodic on tori whose generic dimension is $\text{rank}(\text{SO}(n - 1)) = \lfloor \frac{n-1}{2} \rfloor$.

Theorem 1. *Under the axisymmetric assumption (4.1) on the inertia tensor, the solution of the Chaplygin sphere equations (2.5) with initial condition $(M_0, \gamma_0) \in \mathcal{V}$ is quasi-periodic and given by*

$$M(t) = \exp(\zeta t)M_0 \exp(-\zeta t), \quad \gamma(t) = \exp(\zeta t)\gamma_0,$$

where

$$\zeta = -\frac{a(2 + b(1 - x_0^2))}{2\Delta(x_0)}M_0 + \frac{1}{2\Delta(x_0)}(a(2 + b)(M_0E_n) \wedge E_n - abx_0(M_0E_n) \wedge \gamma_0).$$

Here $x_0 = \langle E_n, \gamma_0 \rangle$ denotes the n^{th} component of γ_0 . In particular, $x(t)$ is constant and equal to x_0 along the motion and the dynamics along \mathcal{V} consists of relative equilibria with respect to the $\text{SO}(n - 1)$ action defined by (4.8) and (4.9).

Proof. That $x(t)$ is constant follows by direct differentiation of $x = \langle \gamma, E_n \rangle$ and the fact that $\langle \Omega\gamma, E_n \rangle = 0$ on \mathcal{V} .

Next note that a direct calculation gives $\zeta E_n = 0$, which implies that ζ belongs to the Lie algebra of the group $\text{SO}(n - 1)$ defined by (4.8). On the other hand, the condition $M_0\gamma_0 = 0$ together with Proposition 3 shows that the initial angular velocity Ω_0 satisfies

$$\Omega_0 = -\zeta + \frac{(2 + b)}{\Delta(x_0)}M_0.$$

Therefore,

$$[M_0, \Omega_0] = [\zeta, M_0], \quad \text{and} \quad \zeta\gamma_0 = -\Omega_0\gamma_0.$$

This shows that the velocity vector $(\dot{M}(0), \dot{\gamma}(0))$ of the curve $(M(t), \gamma(t))$ in the statement of the theorem coincides with the vector field defined by the Chaplygin sphere equations (2.5) under the initial condition (M_0, γ_0) . Considering that these equations are equivariant and that $(M(t), \gamma(t))$ coincides with the $\text{SO}(n - 1)$ action restricted to the 1-parameter subgroup $t \mapsto \exp(\zeta t)$ of $\text{SO}(n - 1)$, it follows that $(M(t), \gamma(t))$ is an integral curve of (2.5) (see, e. g., [5, Lemma 4.2.1.1]). \square

Corollary 1. *Under the axisymmetric assumption (4.1) on the inertia tensor and for initial conditions on \mathcal{V} , the angular velocity $\Omega(t)$ is also quasi-periodic and given by*

$$\Omega(t) = \exp(\zeta t)\Omega_0 \exp(-\zeta t),$$

where Ω_0 is the initial angular velocity.

Proof. This follows from the above theorem by noting that the action (4.9) maps $\Omega \mapsto h\Omega h^{-1}$. \square

The matrix ζ in the statement of Theorem 1 may be written in terms of the initial velocity Ω_0 and γ_0 . A direct calculation using (4.2) and the condition $0 = M_0\gamma_0 = \Omega_0((2 + b)\gamma_0 + ax_0E_n)$, which is valid on \mathcal{V} , gives

$$\zeta = -\frac{a(2 + b(1 - x_0^2))}{\Delta(x_0)}(\Omega_0 - (\Omega_0E_n) \wedge E_n) - \frac{abx_0}{\Delta(x_0)}(\Omega_0E_n) \wedge (\gamma_0 - x_0E_n). \quad (4.12)$$

The following proposition shows that our treatment of the n -dimensional case is consistent with the results that were obtained above for $n = 3$.

Proposition 4. *When $n = 3$, the expressions for ζ in Eq. (4.12) and Eq. (4.11) coincide.*

Proof. First suppose that the initial condition has $x_0 = 0$. Then, because of (4.10) we have $0 = \hat{\Omega}_0\gamma_0 = \Omega_0 \times \gamma_0$. In other words, the vectors Ω_0 and γ_0 are parallel. Considering that the last entry of γ_0 vanishes, the same is true of $\tilde{\Omega}_3$ in (4.11). Thus, in this case $\hat{\Omega}_0 - (\hat{\Omega}_0E_3) \wedge E_3 = 0$ and both expressions for ζ equal 0.

Now suppose that $\hat{\Omega}_0E_3 = 0$. Because of (4.10) we have again $0 = \hat{\Omega}_0\gamma_0 = \Omega_0 \times \gamma_0$ and the same reasoning leads to the conclusion that both expressions for ζ vanish.

Suppose now that $x_0 \neq 0$ and $\hat{\Omega}_0 E_3 \neq 0$, and note that both matrices, $\hat{\Omega}_0 - (\hat{\Omega}_0 E_3) \wedge E_3$ and $(\hat{\Omega}_0 E_3) \wedge (\gamma_0 - x_0 E_3)$, annihilate E_3 . Given that any two matrices in $\mathfrak{so}(3)$ with the same null-vector are proportional, we have

$$\mu(\hat{\Omega}_0 - (\hat{\Omega}_0 E_3) \wedge E_3) = (\hat{\Omega}_0 E_3) \wedge (\gamma_0 - x_0 E_3),$$

for a certain $\mu \in \mathbb{R}$. Multiplying the above equation by γ_0 on the right and using (4.10) gives

$$\left(-\mu x_0 \frac{a+b+2}{2+b} - 1 + x_0^2\right) \hat{\Omega}_0 E_3 = 0.$$

Because of our assumption that $x_0 \neq 0$ and $\hat{\Omega}_0 E_3 \neq 0$ we conclude that $\mu = -\frac{(1-x_0^2)(2+b)}{x_0(2+a+b)}$ and hence

$$(\hat{\Omega}_0 E_3) \wedge (\gamma_0 - x_0 E_3) = -\frac{(1-x_0^2)(2+b)}{x_0(2+a+b)} (\hat{\Omega}_0 - (\hat{\Omega}_0 E_3) \wedge E_3).$$

Substitution of the above expression in Eq. (4.12) simplifies to Eq. (4.11). □

Comparison with the Solutions of the n-dimensional Free Rigid Body

Proposition 1 showed that for the 3-dimensional Chaplygin sphere, the solutions along the invariant vertical space \mathcal{V} are also solutions of the Euler equations for a free rigid body with inertia tensor $\tilde{\mathbb{I}} = \mathbb{I} + b\text{Id}_3$. It is natural to ask if such a property is also valid in the multi-dimensional case. We shall prove that, although the vertical space \mathcal{V} is invariant by the the flow of both systems, their solutions are generically distinct.

In order to compare the multi-dimensional solutions of the two systems, consider the equations of motion of a multi-dimensional rigid body, accompanied with the evolution equation of the Poisson vector γ :

$$\tilde{\mathbb{I}}(\dot{\Omega}) = [\tilde{\mathbb{I}}(\Omega), \Omega], \quad \dot{\gamma} = -\Omega\gamma, \quad \Omega \in \mathfrak{so}(n), \quad \gamma \in S^{n-1}. \tag{4.13}$$

The inertia tensor $\tilde{\mathbb{I}} := \mathbb{I} + b\text{Id}_{\mathfrak{so}(n)}$ and we continue to assume that \mathbb{I} is defined in terms of the mass matrix in (4.1). We also continue to denote $x := \gamma_n = \langle \gamma, E_n \rangle$.

Proposition 5. ²⁾ *The flow of (4.13) leaves the vertical space \mathcal{V} defined by (4.10) invariant.*

Proof. It is a simple exercise to check that the set where $\tilde{\mathbb{I}}(\Omega)\gamma = 0$ is invariant by the flow of (4.13). But this set exactly coincides with \mathcal{V} given by (4.10). Indeed, due to our assumptions on the inertia tensor $\tilde{\mathbb{I}}$ we have

$$\tilde{\mathbb{I}}(\Omega) = (2+b)\Omega + a(\Omega E_n) \wedge E_n, \tag{4.14}$$

and it follows that

$$\tilde{\mathbb{I}}(\Omega)\gamma = (2+b)\Omega\gamma + ax\Omega E_n + a\langle \Omega\gamma, E_n \rangle E_n, \quad \langle \tilde{\mathbb{I}}(\Omega)\gamma, E_n \rangle = (2+a+b)\langle \Omega\gamma, E_n \rangle.$$

Therefore, $\tilde{\mathbb{I}}(\Omega)\gamma = 0$ if and only if $(\Omega, \gamma) \in \mathcal{V}$ as defined by (4.10). □

The solutions of (4.13) along \mathcal{V} are given by the following.

Proposition 6. *For the inertia tensor $\tilde{\mathbb{I}}$ given by (4.14), the solution of the multi-dimensional rigid body equations (4.13) with initial condition $(\Omega_0, \gamma_0) \in \mathcal{V}$ is quasi-periodic and given by*

$$\Omega(t) = \exp(\chi t)\Omega_0 \exp(-\chi t), \quad \gamma(t) = \exp(\chi t)\gamma_0,$$

where

$$\chi = \frac{-a}{2+a+b} (\Omega_0 - (\Omega_0 E_n) \wedge E_n). \tag{4.15}$$

Proof. Just like the axisymmetric Chaplygin sphere, the system (4.13) is equivariant with respect to the action $h \cdot (\Omega, \gamma) \mapsto (h\Omega h^{-1}, h\gamma)$ where $h \in \text{SO}(n-1)$ as defined by (4.8). Moreover, χ belongs to the Lie algebra of $\text{SO}(n-1)$ since $\chi E_n = 0$. Considering that

$$\Omega_0 = -\chi + \frac{1}{2+a+b} \tilde{\mathbb{I}}(\Omega_0),$$

²⁾The inclusion of this result in the final version of the paper was suggested by one of the anonymous referees.

we have $[\tilde{\mathbb{I}}(\Omega_0), \Omega_0] = [\chi, \tilde{\mathbb{I}}(\Omega_0)]$ for all $\Omega_0 \in \mathfrak{so}(n)$, and $-\Omega_0\gamma_0 = \chi\gamma_0$ for $(\Omega_0, \gamma_0) \in \mathcal{V}$. The rest of the proof proceeds as in the proof of Theorem 1. \square

For $n \geq 4$ the expression for ζ in (4.12) does not simplify to agree with χ given by (4.15) (compare with Proposition 4 for $n = 3$). Moreover, it is possible to find initial conditions $(\Omega_0, \gamma_0) \in \mathcal{V}$ with the property that $[\chi - \zeta, \Omega_0] \neq 0$. For these initial conditions we have

$$\exp(\zeta t)\Omega_0 \exp(-\zeta t) \neq \exp(\chi t)\Omega_0 \exp(-\chi t)$$

showing that the solutions for both systems are generally distinct.

5. THE 4-DIMENSIONAL CHAPLYGIN SPHERE

Now consider in more detail the special case $n = 4$. We briefly describe some facts that are valid for general inertia tensors and then go back to our discussion of the axisymmetric case. In our analysis we shall write

$$M = \begin{pmatrix} \hat{k} & u \\ -u^T & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \hat{\eta} & \xi \\ -\xi^T & 0 \end{pmatrix} \in \mathfrak{so}(4), \quad \gamma = (q, x)^T \in \mathbb{R}^4,$$

where $x \in [-1, 1]$, and $k, u, \xi, \eta, q \in \mathbb{R}^3$ are column vectors. In our notation, the equations of motion (2.5) can be rewritten as

$$\dot{k} = k \times \eta + u \times \xi, \quad \dot{u} = k \times \xi + u \times \eta, \quad \dot{q} = q \times \eta - x\xi, \quad \dot{x} = \langle q, \xi \rangle. \tag{5.1}$$

The geometric integral is $\|q\|^2 + x^2 = 1$ and the phase space $P = \mathbb{R}^3 \times \mathbb{R}^3 \times S^3 \ni (k, u, (q, x))$ is 9-dimensional.

The first integrals arising from the conservation of angular momentum about the contact point may be written down explicitly as

$$F_1 = \langle k, u \rangle, \quad F_2 = \|k\|^2 + \|u\|^2, \quad F_3 = \|u \times q\|^2 + x^2\|k\|^2 + \langle k, q \rangle^2 - 2x\langle u, k \times q \rangle.$$

Indeed, these functions satisfy

$$p(\lambda, \sigma) = \det(M + \sigma\gamma\gamma^t - \lambda\text{Id}_4) = \lambda^4 - \lambda^3\sigma\|\gamma\|^2 + \lambda^2F_2 + \lambda\sigma F_3 - F_1^2,$$

and the coefficients of this polynomial in (λ, σ) are first integrals because of the iso-spectral evolution (2.9). On the other hand, the energy integral can be written as

$$H = \frac{1}{2}(\langle u, \xi \rangle + \langle k, \eta \rangle).$$

The invariant sets (3.1) of horizontal \mathcal{H} and vertical \mathcal{V} momentum are 6-dimensional and may be represented in our notation as

$$\begin{aligned} \mathcal{H} &= \left\{ (k, u, (q, x)) \in P : x^2k + \langle k, q \rangle q - xq \times u = 0 \quad \text{and} \quad (1 - x^2)u - xk \times q - \langle u, q \rangle q = 0 \right\}, \\ \mathcal{V} &= \left\{ (k, u, (q, x)) \in P : \langle u, q \rangle = 0 \quad \text{and} \quad k \times q + xu = 0 \right\}. \end{aligned}$$

5.1. The Axisymmetric 4D Chaplygin Sphere

We now continue working with our assumption that the sphere is axisymmetric and the mass tensor has the form (4.1). In terms of the notation introduced above, Eq. (4.2) yields

$$k = 2\eta + bq \times (\eta \times q) - bx\xi \times q, \quad u = (2 + a + b)\xi - bq \times (\xi \times q) + bx\eta \times q.$$

The above relations may be inverted, e. g., using Proposition 3, to give

$$\begin{aligned} \xi &= \frac{b(2 + b)(1 - x^2) + \Delta(x)}{(a + b + 2)\Delta(x)}u - \frac{b(2 + b)\langle u, q \rangle}{(a + b + 2)\Delta(x)}q - \frac{bx}{\Delta(x)}k \times q, \\ \eta &= \frac{(a + 2 + bx^2)}{\Delta(x)}k + \frac{(a + 2)b\langle k, q \rangle}{2\Delta(x)}q + \frac{bx}{\Delta(x)}u \times q. \end{aligned} \tag{5.2}$$

Substitution of (5.2) into (5.1) gives the equations of motion written in explicit form. As follows from Proposition 2, the resulting equations possess the invariant measure

$$\mu = \frac{dk du dq dx}{\Delta(x)}.$$

5.2. Reduction by SO(3)

The SO(3) symmetry introduced in Section 4.1 takes the following form in our notation. The action of $h \in \text{SO}(3)$ on a point $(k, u, (q, x))$ in phase space P is

$$h \cdot (k, u, (q, x)) = (hk, hu, (hq, x)).$$

It is seen from (5.2) that (ξ, η) transform to $(h\xi, h\eta)$ and hence, as predicted by the discussion in Section 4.1, the equations of motion (5.1) are equivariant.

The action is not free since points in P where k, u and q are parallel have a one-dimensional isotropy subgroup isomorphic to SO(2). As a consequence, the reduced space $P/\text{SO}(3)$ is not smooth but is rather a stratified space. To describe it we recall that the ring of invariants of the action is generated by the pairwise inner products of the vectors k, u, q and the triple vector product $\delta = \langle q, k \times u \rangle$, see, e. g., [13]. Let us define:

$$A = \|k\|^2, \quad B = \|u\|^2, \quad E = \langle k, q \rangle, \quad G = \langle u, q \rangle.$$

We also recall that $\|q\|^2 = 1 - x^2$ and $F_1 = \langle k, u \rangle$. These invariants are not independent but satisfy

$$\delta^2 = \det(\Lambda), \quad \text{where} \quad \Lambda = \begin{pmatrix} A & F_1 & E \\ F_1 & B & G \\ E & G & 1 - x^2 \end{pmatrix}.$$

Moreover, they satisfy the inequalities

$$AB - F_1^2 \geq 0, \quad B(1 - x^2) - G^2 \geq 0, \quad A(1 - x^2) - E^2 \geq 0. \tag{5.3}$$

The reduced space $P/\text{SO}(3)$ is isomorphic as a stratified space to the 6-dimensional semi-algebraic variety \mathcal{R} imbedded in \mathbb{R}^7 as

$$\mathcal{R} = \{(A, B, F_1, E, G, x, \delta) \in \mathbb{R}^7 : A, B \geq 0, -1 \leq x \leq 1, \delta^2 - \det(\Lambda) = 0 \text{ and (5.3) hold}\}.$$

As may be verified directly using Eqs. (5.1) and (5.2), the reduced equations of motion are the restriction of the following vector field on \mathbb{R}^7 to \mathcal{R} :

$$\begin{aligned} \dot{A} &= -\frac{2b(2+b)G\delta}{(a+b+2)\Delta(x)} + \frac{2bx}{\Delta(x)}(F_1E - GA), \\ \dot{B} &= \frac{2b(2+b)G\delta}{(a+b+2)\Delta(x)} - \frac{2bx}{\Delta(x)}(F_1E - GA), \\ \dot{E} &= -\frac{abx}{(a+b+2)\Delta(x)}GE - \frac{2x}{\Delta(x)}F_1, \\ \dot{G} &= \frac{bx}{\Delta(x)}\left(A(1-x^2) - E^2 + \frac{(2+b)}{2+a+b}G^2\right) - x\left(\frac{b(2+b)(1-x^2) + \Delta(x)}{(a+b+2)\Delta(x)}\right)B + \frac{2+b-2bx^2}{\Delta(x)}\delta, \\ \dot{F}_1 &= 0, \\ \dot{x} &= \frac{G}{a+b+2}, \\ \dot{\delta} &= \left(\frac{bx^2(a+b+2) - \Delta(x)}{(a+b+2)\Delta(x)}\right)AG - \left(\frac{b(2+b)(1-x^2)}{(a+b+2)\Delta(x)}\right)BG + \left(\frac{2+b-2bx^2}{\Delta(x)}\right)F_1E \\ &\quad + \frac{b(2+b)}{(a+b+2)\Delta(x)}G(G^2 - E^2) - \frac{abx}{(a+b+2)\Delta(x)}G\delta. \end{aligned} \tag{5.4}$$

The 4, generically independent, integrals of the system are invariant under the action and descend to

$$F_1, \quad F_2 = A + B, \quad F_3 = x^2 A + B(1 - x^2) + E^2 - G^2 + 2x\delta,$$

$$H = \frac{a + 2 + bx^2}{2\Delta(x)} A + \frac{2 + b(1 - x^2)}{2\Delta(x)} B - \frac{b(2 + b)}{2(a + b + 2)\Delta(x)} G^2 + \frac{b(2 + a)}{4\Delta(x)} E^2 + \frac{bx}{\Delta} \delta.$$

The invariant measure also passes to the quotient. It is the restriction of the volume form

$$\mu_r = \frac{dA dB dE dG dF_1 dx d\delta}{\Delta(x)^2}$$

on the ambient space \mathbb{R}^7 to \mathcal{R} .

Considering that the generic dimension of $P/\text{SO}(3)$ is 6, and there exist 4 independent integrals and an invariant measure, the Euler–Jacobi theorem (see, e.g., [1, 2]) implies that the reduced system is integrable. In particular, the regular compact level sets of the integrals which have no equilibrium points are 2-tori where the flow is quasi-periodic after a time reparametrisation. Determining the nature of the reconstruction of this type of dynamics to P is a difficult problem for which little is known [6, 16].

On the other hand, the dynamics on the subsets \mathcal{V} and \mathcal{H} of P may be completely described in the light of Theorem 1 and the work of Jovanović [12], respectively. For completeness, we show how these results may be deduced from the reduced system (5.4). First note that \mathcal{V} and \mathcal{H} are $\text{SO}(3)$ -invariant and, under the symmetry reduction, project to subsets $\mathcal{V}/\text{SO}(3)$ and $\mathcal{H}/\text{SO}(3)$ of \mathcal{R} which are invariant by the flow of (5.4). These are given by

$$\mathcal{H}/\text{SO}(3) = \{(A, B, F_1, E, G, x, \delta) \in \mathcal{R} : F_1 = 0, \quad E = 0, \quad xA = -\delta, \quad (1 - x^2)B - G^2 = -x\delta\},$$

and

$$\mathcal{V}/\text{SO}(3) = \{(A, B, F_1, E, G, x, \delta) \in \mathcal{R} : G = 0, \quad F_1 = 0, \quad xB = \delta, \quad (1 - x^2)A - E^2 = x\delta\}.$$

The Dynamics in the Case of Vertical Momentum

Theorem 1 guarantees that the set $\mathcal{V}/\text{SO}(3)$ consists of equilibrium points. This may be verified by substituting the relations in the description of $\mathcal{V}/\text{SO}(3)$ into the reduced Eqs. (5.4) and checking that the right-hand side of the equations vanishes. Considering that $\text{SO}(3)$ has rank 1, we conclude that the \mathcal{V} is foliated by periodic orbits.

The Dynamics in the Case of Horizontal Momentum

This is the case considered by Jovanović [12], whose work implies integrability for arbitrary n .

We note that along this set the integrals F_1 and F_3 vanish. Now we restrict the flow to the level set within $\mathcal{H}/\text{SO}(3)$ of the other two integrals. Suppose that $H = h \geq 0$ and $F_2 = f_2 \geq 0$. We may write

$$A = \frac{(2h(1 - x^2)(a + b + 2) - G^2)\Delta(x)}{(a + b + 2)(2 + a(1 - x^2))} = (1 - x^2)f_2 - G^2. \tag{5.5}$$

The evolution equation for G in (5.4) may be simplified using the identities that define $\mathcal{H}/\text{SO}(3)$ together with (5.5) to eliminate the dependence on δ, B, A and G^2 . Together with the equation for x one obtains the uncoupled 2×2 linear system

$$\dot{G} = -\frac{f_2 - 2bh}{2 + a}x, \quad \dot{x} = \frac{G}{a + b + 2}.$$

The solution of this system with initial condition $x(0) = x_0, G(0) = G_0$ is

$$x(t) = x_0 \cos \omega t, \quad G(t) = -(a + b + 2)x_0 \omega \sin \omega t,$$

where

$$\omega^2 = \frac{f_2 - 2bh}{2 + a} = \frac{aG_0^2 + 4h(a + b + 2)}{(a + b + 2)(2 + a(1 - x_0^2))} \geq 0.$$

The evolution of A , B and δ is also periodic as may be seen, respectively, from (5.5), and the relations $B = f_2 - A$, and $\delta = -xA$. Therefore, $\mathcal{H}/\text{SO}(3)$ is foliated by periodic orbits. Since the rank of $\text{SO}(3)$ is 1, the reconstructed motion on \mathcal{H} consists of quasi-periodic motion on 2-dimensional tori (see, e. g., [9, 10]). This is in agreement with [12, Theorem 9].

5.3. A Family of Steady Rotations

Finally, we describe another class of solutions of Eqs. (5.1) and (5.2) which have constant angular velocity and lead to quasi-periodic dynamics on $P = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3$. For this purpose, it is convenient to write Eqs. (5.1) and (5.2) in terms of ξ , η , q and x , and without involving k or u . After a long, but straightforward calculation, one finds:

$$\begin{aligned} \dot{\xi} &= \frac{a(2 + b(1 - x^2))}{\Delta(x)}\xi \times \eta - \frac{ab(b + 2)\langle q, \xi \times \eta \rangle}{(2 + a + b)\Delta(x)}q, & \dot{\eta} &= -\frac{abx}{\Delta(x)}q \times (\xi \times \eta), \\ \dot{q} &= q \times \eta - x\xi, & \dot{x} &= \langle q, \xi \rangle. \end{aligned} \tag{5.6}$$

Proposition 7. *If the initial angular velocity*

$$\Omega_0 = \begin{pmatrix} \hat{\eta}_0 & \xi_0 \\ -\xi_0^T & 0 \end{pmatrix} \in \mathfrak{so}(4)$$

of the 4D axisymmetric Chaplygin sphere satisfies $\eta_0 \times \xi_0 = 0$, then the angular velocity remains constant throughout the motion and the solution for $M(t)$ and $\gamma(t)$ with respective initial conditions M_0, γ_0 is given by

$$M(t) = \exp(-\Omega_0 t)M_0 \exp(\Omega_0 t), \quad \gamma(t) = \exp(-\Omega_0 t)\gamma_0.$$

Proof. It is readily seen from (5.6) that, if ξ and η are parallel at some time, then they remain constant throughout the motion. So, for these initial conditions the angular velocity $\Omega(t) = \Omega_0$ and we have a steady rotation. It is then straightforward to check that $M(t)$ and $\gamma(t)$ as defined above satisfy (2.5). \square

Recall that points in P with non-trivial $\text{SO}(3)$ isotropy are those for which k, u and q are collinear. In view of Eq. (5.2) at these points the vectors ξ , η and q are also parallel. Therefore, Proposition 7 describes the dynamics for this type of initial conditions.

We finish our discussion by indicating that the initial conditions in Proposition 7 do not generically belong to \mathcal{H} nor \mathcal{V} .

CONCLUSIONS AND FUTURE WORK

We have considered the dynamics of the axisymmetric n -dimensional Chaplygin sphere. Our main contribution is to show that the dynamics is quasi-periodic when the angular momentum about the contact point is vertical. Also, to indicate how a further reduction of the system by the additional $\text{SO}(n - 1)$ symmetry may be useful to understand the dynamics of the system for generic initial conditions. Indeed, in the case $n = 4$ the dynamics on the reduced system $P/\text{SO}(3)$ was shown to be integrable by the Euler–Jacobi theorem. The following problems remain open:

- Determine if Eqs. (2.5) on $P = \mathfrak{so}(n) \times \mathbb{S}^{n-1}$ allow a Hamiltonisation. Such a Hamiltonisation is known to exist only in the case $n = 3$ [3] and for initial conditions on \mathcal{H} for $n \geq 4$ [12].

The existence of a Hamiltonian structure for the equations on P would be useful to reconstruct the dynamics from the reduced space $P/\text{SO}(n - 1)$. In particular, for $n = 4$ the (reparametrised) quasi-periodic dynamics on $P/\text{SO}(3)$ predicted by the Euler–Jacobi theorem would be guaranteed to be quasi-periodic on P (see [16]).

- Obtain the reduced equations on $P/\text{SO}(n - 1)$ for $n \geq 5$ and determine if they are integrable.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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