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Optimal reinsurance via BSDEs in a partially observable model with jump clusters

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Abstract
We investigate an optimal reinsurance problem when the loss process exhibits jump clustering features and the insurance company has restricted information about the loss process. We maximise expected exponential utility of terminal wealth and show that an optimal strategy exists. By exploiting both the Kushner–Stratonovich and Zakai approaches, we provide the equation governing the dynamics of the (infinite-dimensional) filter and characterise the solution of the stochastic optimisation problem in terms of a BSDE, for which we prove existence and uniqueness of a solution. After discussing the optimal strategy for a general reinsurance premium, we provide more explicit results in some relevant cases.

Keywords Optimal reinsurance · Partial information · Hawkes processes · Cox processes with shot noise · BSDEs

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1 Introduction

Optimal reinsurance problems have attracted special attention during the past few years and have been investigated in many different model settings. Insurance companies can hardly deal with all the different sources of risk in the real world; so they hedge against at least part of them by reinsuring with other institutions. A reinsurance agreement allows the primary insurer to transfer part of the risk to another company, and it is well known that this is an effective tool in risk management. Moreover, the subscription of such contracts is required by some financial regulators; see e.g. the Directive Solvency II in the European Union. A large part of the existing literature focuses mainly on classical reinsurance contracts such as proportional and excess-of-loss, which were extensively investigated under a variety of optimisation criteria, e.g. ruin probability minimisation, dividend optimisation or expected utility maximisation. Here we are interested in the latter approach (see Irgens and Paulsen [18], Mania and Santacroce [24], Brachetta and Ceci [4] and references therein). Some of the classical papers devoted to the subject assume a diffusive dynamics for the surplus process, while the more recent literature considers surplus processes including jumps, see Schmidli [26].

The pioneering risk model with jumps in non-life insurance is given by the classical Cramér–Lundberg model, where the claim arrival process is a Poisson process with constant intensity. This assumption implies that the instantaneous probability that an accident occurs is always constant, which is too restrictive in the real world as already motivated by Grandell [16, Chap. 2]. In recent years, many authors made a great effort to go beyond the classical model formulation. For example, Cox processes were employed to introduce a stochastic intensity for the claim arrival process; see e.g. Albrecher and Asmussen [1], Björk and Grandell [3], Embrechts et al. [15]. Moreover, other authors introduced Hawkes processes in order to capture the self-exciting property of the insurance risk model in the presence of catastrophic events. Hawkes processes were introduced by Hawkes [17] to describe geological phenomena with clustering features like earthquakes. Hawkes processes with general kernels are not Markov processes; they can include long-range dependence, while Hawkes processes with an exponential kernel exhibit the appealing property that the process–intensity pair is Markovian. Moreover, they are affine processes according to the definition provided by Duffie et al. [14]. For the latter literature strand, we mention here Stabile and Torrisi [28] and Swishchuk et al. [29].

Dassios and Zhao [13] proposed a model which combines the two approaches by introducing a Cox process with shot noise intensity and a Hawkes process with an exponential kernel for describing the claim arrival dynamics. Recently Cao et al. [7] investigated the optimal reinsurance–investment problem in the model setting proposed by Dassios and Zhao [13] with a reward function of mean–variance type.

A different line of research related to the optimal reinsurance–investment problem focuses on the possibility that the insurer does not have access to all the information when choosing the reinsurance strategy. As a matter of fact, only the claim arrivals and the corresponding disbursements are observable. In this case, we need to solve a stochastic optimisation problem under partial information. Liang and Bayraktar [22] were the first to introduce a partial information framework in optimal reinsurance problems. They consider the optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution are not known, but have to be inferred from the observations of claim arrivals. Ceci et al. [10] derive risk-minimising investment strategies when the information available to investors is restricted and they provide optimal hedging strategies for unit-linked life insurance contracts. Jang et al. [20] present a systematic comparison between optimal reinsurance strategies in complete and partial information frameworks and quantify the information value in a diffusion setting.

More recently, Brachetta and Ceci [5] investigate the optimal reinsurance problem under the criterion of maximising the expected exponential utility of terminal wealth when the insurance company has restricted information on the loss process in a model with claim arrival intensity and claim size distribution affected by an unobservable environmental stochastic factor.

In the present paper, we investigate the optimal reinsurance strategy for a risk model with jump clustering properties in a partial information setting. The risk model is similar to that proposed by Dassios and Zhao [13] and includes two different jump processes driving the claims arrivals: one process with constant intensity describing the exogenous jumps, and another with stochastic intensity representing the endogenous jumps which exhibits self-exciting features. The externally-excited component represents catastrophic events which generate claims clustering increasing the claim arrival intensity. The endogenous part allows us to capture the clustering effect due to self-exciting features. That is, when an accident occurs, it increases the likelihood of such events. The insurance company has only partial information at its disposal; more precisely, the insurer can only observe the cumulative claim process. The externally-excited component of the intensity is not observable and the insurer needs to estimate the stochastic intensity by solving a filtering problem. Our approach is substantially different from that of Cao et al. [7] in several respects. Firstly, we work in a partial information setting; secondly, the intensity of the self-excited claim arrival exhibits a slight more general dependence on the claim severity; finally, we maximise an exponential utility function instead of following a mean–variance criterion. In a partially observable framework, our goal is to characterise the value process and the optimal strategy. The optimal stochastic control problem in our case turns out to be infinite-dimensional, and the characterisation of the optimal strategy cannot be performed by solving a Hamilton–Jacobi–Bellman equation, but via a backward stochastic differential equation (BSDE) approach.

A difficulty naturally arises when dealing with Hawkes processes: the intensity of the jumps is not bounded a priori, although a non-explosivity condition holds. Hence we are not able to exploit some relevant bounds which are usually required to prove a verification theorem and results on the existence and uniqueness of the solution for the related BSDE. Nevertheless, we are going to show that the optimal stochastic control problem has a solution, which admits a characterisation in terms of a unique solution to a suitable BSDE.

Our paper aims to contribute in different directions to the literature on optimal reinsurance problems. First, we provide a rigorous and formal construction of the dynamic contagion model. Second, we study the filtering problem associated to our model, providing a characterisation of the filter process in terms of the Kushner–Stratonovich equation and the Zakai equation as well. To the best of our knowledge, this problem has not been addressed so far in the existing literature. We refer to Dassios and Jang [11] for a similar problem without the self-exciting component. Third, we solve the optimal reinsurance problem under an expected utility criterion.

We remark that our study differs from Brachetta and Ceci [5] in many key aspects. The risk model is substantially different, in that it requires a strong effort to be rigorously constructed and leads to the study of a new filtering problem. What is more, a crucial assumption in Brachetta and Ceci [5] is the boundedness of the claim arrival intensity which is not satisfied in our case, thus leading to additional technicalities in most of the proofs. This happens for example when one needs to prove existence and uniqueness of the solution of the BSDE. Moreover, we perform the optimisation over a class of admissible contracts instead of maximising over the retention level. This feature allows us to cover a larger class of problems. Finally, we do not require the existence of an optimal control for the derivation of the BSDE; hence the general presentation turns out to be different.

The paper is organised as follows. In Sect. 2, we introduce the risk model and specify what information is available to the insurer. A rigorous mathematical construction is provided, based on a measure change approach which is necessary to develop the following analysis in full detail. In Sect. 3, the filtering problem is investigated in order to reduce the optimal stochastic control problem to a complete information setting. The stochastic differential equation satisfied by the filter is obtained by exploiting both the Kushner–Stratonovich and the Zakai approaches. In Sect. 4, the optimal stochastic control problem is formulated, while in Sect. 5, a characterisation of the value process associated with the optimal stochastic control problem is illustrated. Due to the infinite dimension of the filter, the approach based on the Hamilton–Jacobi–Bellman equation cannot be exploited; so the value process is characterised as the unique solution of a BSDE. In Sect. 6, the optimal reinsurance strategy is investigated under general assumptions, and some relevant cases are discussed. Some proofs and useful computations are collected in Appendices A–C.

2 The mathematical model

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual hypotheses. The time $T > 0$ is a finite time horizon that represents the maturity of a reinsurance contract. Here we start by giving an overview of the optimal reinsurance problem from the primary insurer's point of view; then in Sect. 2.1, we provide a rigorous construction of our model setting.

Our aim is to introduce a dynamic contagion process generalising the Hawkes and Cox processes with shot noise intensity introduced e.g. by Dassios and Zhao [13]. More precisely, the claims counting process $N^{(1)}$ has the stochastic $(\mathbb{P}, \mathbb{F})$-intensity, for $t \in [0, T]$,

$$\lambda_t = \beta + (\lambda_0 - \beta)e^{-\alpha t} + \sum_{j=1}^{N_t^{(1)}} e^{-\alpha(t-T_j^{(1)})}\ell(Z_j^{(1)}) + \sum_{j=1}^{N_t^{(2)}} e^{-\alpha(t-T_j^{(2)})} Z_j^{(2)}, \qquad (2.1)$$

where

– $\beta > 0$ is the constant reversion level;
– $\lambda_0 > 0$ is the initial value;
– $\alpha > 0$ is the constant rate of exponential decay;
– $N^{(2)}$ is a Poisson process with constant intensity $\rho > 0$;
– $(T_n^{(1)})_{n\geq 1}$ are the jump times of $N^{(1)}$, i.e., the time instants when claims are reported;
– $(T_n^{(2)})_{n\geq 1}$ are the jump times of $N^{(2)}$, i.e., the time instants when exogenous/external factors make the intensity jump;
– $(Z_n^{(1)})_{n\geq 1}$ represent the claim size and are modelled as a sequence of i.i.d. $\mathbb{R}_+$-valued random variables with distribution function $F^{(1)} : (0, \infty) \to [0, 1]$ such that $\mathbb{E}[Z^{(1)}] < \infty$;
– $\ell : [0, \infty) \to [0, \infty)$ is a measurable function (for instance we could take $\ell(z) = az$, $a > 0$, and the self-exciting jumps would be proportional to claim sizes) such that $\mathbb{E}[\ell(Z^{(1)})] < \infty$;
– $(Z_n^{(2)})_{n\geq 1}$ are the externally-excited jumps and are modelled as a sequence of i.i.d. $\mathbb{R}_+$-valued random variables with distribution function $F^{(2)} : (0, \infty) \to [0, 1]$ such that $\mathbb{E}[Z^{(2)}] < \infty$.

Notice that the counting process $N^{(1)}$ is defined via its intensity λ in (2.1), which in turn depends on the history of $N^{(1)}$. So an apparent logical loop seems to arise about the existence of λ. We postpone this issue to Sect. 2.1 where we give a rigorous construction of the model based on an equivalent change of probability measure.

The following assumption will hold **from now on**.

Assumption 2.1 We assume $N^{(2)}$, $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ to be independent of each other under $\mathbb{P}$.

We define the cumulative claim process $C = (C_t)_{t\in[0,T]}$ at time t as

$$C_t = \sum_{j=1}^{N_t^{(1)}} Z_j^{(1)}, \qquad t \in [0, T]. \tag{2.2}$$

Remark 2.2 Our model includes many meaningful properties of risk models. The claim arrival process has stochastic intensity, reflecting random changes in the instantaneous probability that accidents occur. Most importantly, our framework captures both self-exciting (endogenous) and externally-exciting (exogenous) factors via, respectively, the claim arrival times and sizes $(T_n^{(1)}, Z_n^{(1)})_{n\geq 1}$ and $(T_n^{(2)}, Z_n^{(2)})_{n\geq 1}$. For this reason, it is well suited to describe for instance catastrophic events; see Cao et al. [7] where self-exciting jump sizes are independent from the claim severity. In contrast, in our model, they depend on claim sizes via $\ell(Z_j^{(1)})$. Moreover, the decay coefficient α is considered because catastrophic events typically exhibit this behaviour.

The insurance company is allowed to subscribe a reinsurance contract with a retention function $\Phi(z, u)$ parametrised by a dynamic reinsurance strategy (the control) taking values in U. That is, under a dynamic strategy $u = (u_t)_{t\in[0,T]}$, the aggregate losses covered by the insurer, denoted by $C^u = (C^u_t)_{t\in[0,T]}$, read

$$C^u_t = \sum_{j=1}^{N^{(1)}_t} \Phi(Z^{(1)}_j, u_{T^{(1)}_j}), \qquad t \in [0, T],$$

so that the remaining losses $C - C^u$ are covered by the reinsurer. We highlight that in our setting, the insurer can choose the optimal reinsurance arrangement over a class of admissible contracts; see Sect. 4 for details. For this service, a reinsurance premium rate $q^u = (q^u_t)_{t\in[0,T]}$ must be paid. Hence the primary insurer receives the insurance premium rate c, pays the reinsurance premium rate q^u and bears the aggregate losses C^u so that the surplus process R^u follows the stochastic differential equation (SDE)

$$dR^u_t = (c_t - q^u_t)\mathrm{d}t - dC^u_t, \qquad R^u_0 = R_0 \in \mathbb{R}_+,$$

where R_0 denotes the initial capital. Investing the surplus in a risk-free asset with interest rate $r > 0$, the total wealth X^u of the primary insurer is

$$dX^u_t = dR^u_t + rX^u_t\mathrm{d}t, \qquad X^u_0 = R_0 \in \mathbb{R}_+.$$

We assume that the information at disposal is limited: the insurer only observes the cumulative claim process C in (2.2). Let us denote by $\mathbb{H}$ the natural filtration generated by C,

$$\mathbb{H} = \mathbb{F}^C = (\mathcal{F}^C_t)_{t\in[0,T]} \subseteq \mathbb{F}, \qquad \mathcal{F}^C_t = \sigma(C_s, 0 \le s \le t). \tag{2.3}$$

Notice that the filtration $\mathbb{H}$ is right-continuous (see e.g. Brémaud [6, Theorem T25 in Appendix A2]). We assume that the insurer and the reinsurer have the same information represented by $\mathbb{H}$. Therefore, the insurance and the reinsurance premium have to be $\mathbb{H}$-predictable. The same applies to the insurer's control u. The insurer aims at maximising the expected exponential utility of terminal wealth over a suitable class $\mathcal{U}$ of $\mathbb{H}$-predictable strategies (which will be made precise later in Definition 4.4), i.e.,

$$\sup_{u\in\mathcal{U}} \mathbb{E}[1 - e^{-\eta X^u_T}],$$

where $\eta > 0$ denotes the insurer's risk aversion. More mathematical details on the control problem to be solved are given in Sect. 4.

Remark 2.3 Notice that the stochastic wealth X^u can possibly take negative values, due to the possibility of borrowing money from the bank account.

This setting leads to investigate a stochastic control problem under partial information. Due to the presence of the externally-excited component, the claim arrival intensity in (2.1) is $\mathbb{F}$-adapted rather than $\mathbb{H}$-adapted; hence it is not observable by the insurance and reinsurance companies. We reduce the original problem to a stochastic control problem under complete information by solving a filtering problem in Sect. 3.

The knowledge of the filter process allows to compute the $\mathbb{H}$-predictable intensity of the claim arrival process $N^{(1)}$, which represents the best estimate of the stochastic intensity λ based on the available information.

The next subsection provides a formal and rigorous construction of our model.

2.1 Model construction

We introduce the dynamic contagion model by a suitable measure change, starting from two Poisson processes with constant intensity on a given probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$; $N^{(1)}$ is standard and $N^{(2)}$ has constant intensity $\rho > 0$. Moreover, we take two sequences $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ of i.i.d. positive random variables with distribution functions $F^{(1)}$ and $F^{(2)}$, respectively, and such that $\mathbb{E}^{\mathbb{Q}}[\ell(Z^{(1)})] < \infty$ and $\mathbb{E}^{\mathbb{Q}}[Z^{(2)}] < \infty$. We assume $N^{(1)}$, $N^{(2)}$, $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ to be independent of each other under $\mathbb{Q}$.

The key idea behind our construction is to introduce a new measure $\mathbb{P}$, equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}, \mathbb{F})$, such that under $\mathbb{P}$, the intensity of $N^{(2)}$ and the distributions of $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ do not change and $N^{(1)}$ is a counting process with stochastic intensity λ given by (2.1). Notice that under $\mathbb{P}$, $N^{(1)}$, $N^{(2)}$, $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ are not independent anymore.

Let us introduce the integer-valued random measures $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$, by

$$m^{(i)}(\mathrm{d}t, \mathrm{d}z) = \sum_{n\geq 1} \delta_{(T_n^{(i)}, Z_n^{(i)})}(\mathrm{d}t, \mathrm{d}z) \mathbf{1}_{\{T_n^{(i)} < \infty\}}, \tag{2.4}$$

where $\delta_{(t,z)}$ denotes the Dirac measure in (t, z). Under $\mathbb{Q}$, $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$, are independent Poisson measures with compensator measures given respectively by

$$\nu^{(1),\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) = F^{(1)}(\mathrm{d}z)\mathrm{d}t, \qquad \nu^{(2),\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t.$$

The measure change from $(\mathbb{Q}, \mathbb{F})$ to $(\mathbb{P}, \mathbb{F})$ is performed via the stochastic process $L = (L_t)_{t\in[0,T]}$ defined by

$$\begin{aligned} L_t &= \mathcal{E}\left(\int_0^t \int_0^\infty (\lambda_{s-} - 1)\big(m^{(1)}(\mathrm{d}s, \mathrm{d}z) - F^{(1)}(\mathrm{d}z)\mathrm{d}s\big)\right) \\ &= \mathcal{E}\left(\int_0^t (\lambda_{s-} - 1)(dN_s^{(1)} - \mathrm{d}s)\right), \end{aligned}$$

where $\mathcal{E}(M)$ denotes the Doléans-Dade exponential of a martingale M and λ under $\mathbb{Q}$ is defined by (2.1). This process will be proved to be a $(\mathbb{Q}, \mathbb{F})$-martingale under

Assumption 2.4 We assume that there exists $\varepsilon > 0$ such that

$$\mathbb{E}^{\mathbb{Q}}[e^{\varepsilon \ell(Z^{(1)})}] < \infty, \qquad \mathbb{E}^{\mathbb{Q}}[e^{\varepsilon Z^{(2)}}] < \infty.$$

Before proving the martingale property, we notice the following.

Remark 2.5 We observe that $(\int_0^t (\lambda_{s-} - 1)(dN_s^{(1)} - \mathrm{d}s)_{t\in[0,T]}$ is a $(\mathbb{Q}, \mathbb{F})$-martingale since $\mathbb{E}^{\mathbb{Q}}[\int_0^t \lambda_s \mathrm{d}s] < \infty$ for all $t \in [0, T]$. In fact, by (2.1),

$$\lambda_t \leq \max\{\lambda_0, \beta\} + \sum_{j=1}^{N_t^{(1)}} \ell(Z_j^{(1)}) + \sum_{j=1}^{N_t^{(2)}} Z_j^{(2)} \tag{2.5}$$

and

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\lambda_t] &\leq \max\{\lambda_0, \beta\} + \mathbb{E}^{\mathbb{Q}}[N_t^{(1)}]\mathbb{E}^{\mathbb{Q}}[\ell(Z^{(1)})] + \mathbb{E}^{\mathbb{Q}}[N_t^{(2)}]\mathbb{E}^{\mathbb{Q}}[Z^{(2)}] \\ &= \max\{\lambda_0, \beta\} + \big(\mathbb{E}^{\mathbb{Q}}[\ell(Z^{(1)})] + \rho\mathbb{E}^{\mathbb{Q}}[Z^{(2)}]\big)t.\end{aligned}$$

We then have for L_t the explicit expression

$$L_t = e^{-\int_0^t (\lambda_s - 1)\mathrm{d}s + \int_0^t \ln(\lambda_{s-}) dN_s^{(1)}}, \qquad t \in [0, T], \tag{2.6}$$

and we define the equivalent measure $\mathbb{P}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_T} = L_T.$$

Proposition 2.6 *Under Assumption 2.4, the Radon–Nikodým density process L given in (2.6) is a* $(\mathbb{Q}, \mathbb{F})$*-martingale.*

Proof This proof is based on Sokol and Hansen [27, Corollary 2.5]. We observe that the left-limit process $(\lambda_{t-})_{t\in[0,T]}$, where λ is defined in (2.1), is nonnegative, predictable and locally bounded. Hence [27, Corollary 2.5] can be straightforwardly applied after we prove that condition (2.6) therein holds: there exists $\varepsilon > 0$ such that whenever $0 \leq u \leq t, t - u \leq \varepsilon$,

$$\mathbb{E}^{\mathbb{Q}}[e^{\int_u^t \log^+(\lambda_{s-}) dN_s^{(1)}}] < \infty, \tag{2.7}$$

where $\log^+ x := \max\{0, \log x\}$. Applying Lemma A.1 under the measure $\mathbb{Q}$, we obtain that

$$\mathbb{E}^{\mathbb{Q}}[e^{\int_u^t \log^+(\lambda_{s-})\, dN_s^{(1)}}] \leq \mathbb{E}^{\mathbb{Q}}[e^{\int_u^t (\lambda_{s-} - 1)\mathrm{d}s}].$$

Hence condition (2.7) is fulfilled if the expectation $\mathbb{E}^{\mathbb{Q}}[e^{\int_u^t \lambda_{s-} \mathrm{d}s}]$ is finite. By applying (2.5) to λ_s and noticing that the right-hand side of (2.5) is increasing with respect to time, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[e^{\int_u^t \lambda_{s-} \mathrm{d}s}] &\leq e^{\varepsilon(\lambda_0 \vee \beta)}\mathbb{E}^{\mathbb{Q}}\big[e^{\varepsilon \sum_{j=1}^{N_t^{(1)}} \ell(Z_j^{(1)})} e^{\varepsilon \sum_{j=1}^{N_t^{(2)}} Z_j^{(2)}}\big] \\ &\leq e^{\varepsilon(\lambda_0 \vee \beta)}\mathbb{E}^{\mathbb{Q}}\big[e^{\varepsilon \sum_{j=1}^{N_t^{(1)}} \ell(Z_j^{(1)})}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{\varepsilon \sum_{j=1}^{N_t^{(2)}} Z_j^{(2)}}\big],\end{aligned}$$

where we used the mutual independence of $N^{(1)}$, $N^{(2)}$, $(Z_n^{(1)})_{n\geq 1}$ and $(Z_n^{(2)})_{n\geq 1}$ under $\mathbb{Q}$. By exploiting Lemma A.2, we immediately find

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{\varepsilon \sum_{j=1}^{N_t^{(1)}} \ell(Z_j^{(1)})}\Big] = e^{t(\mathbb{E}^{\mathbb{Q}}[e^{\varepsilon \ell(Z^{(1)})}]-1)} < \infty.$$

Analogously, one shows that $\mathbb{E}^{\mathbb{Q}}[e^{\varepsilon \sum_{j=1}^{N_t^{(2)}} Z_j^{(2)}}] = e^{\rho t(\mathbb{E}^{\mathbb{Q}}[e^{\varepsilon Z^{(2)}}]-1)} < \infty.$ □

Now that the change of measure has been rigorously introduced, we can safely introduce the $(\mathbb{P}, \mathbb{F})$-compensator measures of $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$.

Remark 2.7 By the Girsanov theorem, the $(\mathbb{P}, \mathbb{F})$-predictable projections (the so-called compensator measures) of $m^{(1)}(\mathrm{d}t, \mathrm{d}z)$ and $m^{(2)}(\mathrm{d}t, \mathrm{d}z)$ from (2.4) are given respectively by

$$\nu^{(1)}(\mathrm{d}t, \mathrm{d}z) = \lambda_{t-} F^{(1)}(\mathrm{d}z)\mathrm{d}t, \qquad \nu^{(2)}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t. \tag{2.8}$$

In particular, $N^{(1)}$ is a point process with predictable $(\mathbb{P}, \mathbb{F})$-intensity $(\lambda_{s-})_{s\in[0,T]}$, while $N^{(2)}$ remains a point process with constant $(\mathbb{P}, \mathbb{F})$-intensity $\rho > 0$.

It turns out that for $i = 1, 2$ and any $\mathbb{F}$-predictable and nonnegative random field $(H(t, z))_{t\in[0,T], z\in[0,\infty)}$, we have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t \int_0^\infty H(s, z) m^{(i)}(\mathrm{d}s, \mathrm{d}z)\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_0^\infty H(s, z) \nu^{(i)}(\mathrm{d}s, \mathrm{d}z)\Big], \qquad t \in [0, T],\end{aligned}$$

where $\nu^{(i)}(\mathrm{d}s, \mathrm{d}z)$, $i = 1, 2$, are defined in (2.8). Moreover, under the condition

$$\mathbb{E}\Big[\int_0^T \int_0^\infty |H(s, z)| \nu^{(i)}(\mathrm{d}s, \mathrm{d}z)\Big] < \infty,$$

the process

$$\int_0^t \int_0^\infty H(s, z)\big(m^{(i)}(\mathrm{d}s, \mathrm{d}z) - \nu^{(i)}(\mathrm{d}s, \mathrm{d}z)\big), \qquad t \in [0, T],$$

is a $(\mathbb{P}, \mathbb{F})$-martingale.

2.2 Markov property

In this subsection, we discuss and characterise the Markovian structure of the intensity, working on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Equation (2.1) reads

$$\mathrm{d}\lambda_t = \alpha(\beta - \lambda_t)\mathrm{d}t + \int_0^\infty \ell(z) m^{(1)}(\mathrm{d}t, \mathrm{d}z) + \int_0^\infty z m^{(2)}(\mathrm{d}t, \mathrm{d}z).$$

Proposition 2.8 *The process* λ *is a* $(\mathbb{P}, \mathbb{F})$*-Markov process with generator*

$$\mathcal{L}f(\lambda) = \alpha(\beta - \lambda)f'(\lambda) + \int_0^\infty \Big(f\big(\lambda + \ell(z)\big) - f(\lambda)\Big)\lambda F^{(1)}(\mathrm{d}z) + \int_0^\infty \big(f(\lambda + z) - f(\lambda)\big)\rho F^{(2)}(\mathrm{d}z).$$

The domain $\mathcal{D}(\mathcal{L})$ *of the generator* $\mathcal{L}$ *contains the class of functions* $f \in C^1(0,\infty)$ *such that*

$$\mathbb{E}\bigg[\int_0^t \int_0^\infty \big|f\big(\lambda_s + \ell(z)\big) - f(\lambda_s)\big|\lambda_s F^{(1)}(\mathrm{d}z)\mathrm{d}s\bigg] < \infty,$$
$$\mathbb{E}\bigg[\int_0^t \int_0^\infty |f(\lambda_s + z) - f(\lambda_s)| F^{(2)}(\mathrm{d}z)\mathrm{d}s\bigg] < \infty$$

and

$$\mathbb{E}\bigg[\int_0^t \lambda_s |f'(\lambda_s)|\mathrm{d}s\bigg] < \infty.$$

Proof This is a direct application of Itô's formula. □

In what follows, we need the following, which will be crucial to prove Proposition 2.10.

Assumption 2.9

$$\mathbb{E}\big[\big(\ell(Z^{(1)})^k\big)\big] < \infty, \quad \mathbb{E}[(Z^{(2)})^k] < \infty, \qquad k = 1, 2, \dots$$

Proposition 2.10 *Under Assumption 2.9, for any* $t \in [0, T]$,

$$\mathbb{E}\bigg[\int_0^t \lambda_s^k \mathrm{d}s\bigg] < \infty, \qquad k = 1, 2, \dots$$

Proof We proceed by induction on k. We first prove that $\mathbb{E}[\lambda_t] \le h_1(t)$, $t \ge 0$, with a measurable, nonnegative function h_1 such that $\int_0^T h_1(t)\mathrm{d}t < \infty$. Let us observe that (2.1) reads

$$\lambda_t = \beta + (\lambda_0 - \beta)e^{-\alpha t} + \int_0^t \int_0^\infty e^{-\alpha(t-s)}\ell(z)m^{(1)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_0^\infty e^{-\alpha(t-s)} z m^{(2)}(\mathrm{d}s, \mathrm{d}z).$$

Hence by Remark 2.7,

$$\mathbb{E}[\lambda_t] = \beta + \left(\lambda_0 - \beta - \frac{\rho\mathbb{E}[Z^{(2)}]}{\alpha}\right)e^{-\alpha t} + \frac{1}{\alpha}\rho\mathbb{E}[Z^{(2)}] + \mathbb{E}[\ell(Z^{(1)})]\int_0^t e^{-\alpha(t-s)}\mathbb{E}[\lambda_s]\mathrm{d}s.$$

By applying Gronwall's lemma, we obtain

$$\mathbb{E}[\lambda_t] \leq \left(\beta + \left(\lambda_0 - \beta - \frac{\rho\mathbb{E}[Z^{(2)}]}{\alpha}\right)e^{-\alpha t} + \frac{1}{\alpha}\rho\mathbb{E}[Z^{(2)}]\right)e^{\mathbb{E}[\ell(Z^{(1)})]\frac{1-e^{-\alpha t}}{\alpha}}$$
$$= h_1(t), \qquad t \in [0, T].$$

It is immediate to verify from Assumption 2.9 that $h_1(t) \geq 0$ is continuous and $\int_0^T h_1(t)\mathrm{d}t < \infty$. Let us assume that $\mathbb{E}[\lambda_t^i] \leq h_i(t)$ with a measurable, nonnegative function h_i such that $\int_0^T h_i(t)\mathrm{d}t < \infty$ for any $i = 1, 2, \ldots, k-1$. By Itô's formula, we get

$$\lambda_t^k = \lambda_0^k + \int_0^t \alpha(\beta - \lambda_s)k\lambda_s^{k-1}\mathrm{d}s + \int_0^t \int_0^\infty \left(\left(\lambda_{s-} + \ell(z)\right)^k - (\lambda_{s-})^k\right)m^{(1)}(\mathrm{d}s, \mathrm{d}z)$$
$$+ \int_0^t \int_0^\infty \left((\lambda_{s-} + z)^k - (\lambda_{s-})^k\right)m^{(2)}(\mathrm{d}s, \mathrm{d}z)$$
$$= \lambda_0^k + \int_0^t \alpha(\beta - \lambda_s)k\lambda_s^{k-1}\mathrm{d}s + \int_0^t \int_0^\infty \sum_{i=0}^{k-1} \binom{k}{i} (\lambda_{s-})^i \ell(z)^{k-i} m^{(1)}(\mathrm{d}s, \mathrm{d}z)$$
$$+ \int_0^t \int_0^\infty \sum_{i=0}^{k-1} \binom{k}{i} (\lambda_{s-})^i z^{k-i} m^{(2)}(\mathrm{d}s, \mathrm{d}z).$$

Then there exist $c_i > 0$, $i = 1, 2, \ldots, k$, such that

$$\mathbb{E}[\lambda_t^k] = \lambda_0^k + \int_0^t \alpha k(\beta\mathbb{E}[\lambda_s^{k-1}] - \mathbb{E}[\lambda_s^k])\mathrm{d}s + \int_0^t \sum_{i=0}^{k-1} \binom{k}{i} \mathbb{E}[\lambda_s^{i+1}]\mathbb{E}[\ell(Z^{(1)})^{k-i}]\mathrm{d}s$$
$$+ \int_0^t \sum_{i=0}^{k-1} \binom{k}{i} \mathbb{E}[\lambda_s^i]\rho\mathbb{E}[(Z^{(2)})^{k-i}]\mathrm{d}s$$
$$\leq \lambda_0^k + \int_0^t \sum_{i=0}^{k-1} c_i h_i(s)\mathrm{d}s + \int_0^t c_k\mathbb{E}[\lambda_s^k]\mathrm{d}s,$$

and again by Gronwall's lemma, it follows that $\mathbb{E}[\lambda_t^k] \leq h_k(t)$ with a measurable, integrable and nonnegative function h_k on $[0, T]$. This concludes the proof. □

Proposition 2.11 *Under Assumption 2.9, the functions* $f_k(\lambda) := \lambda^k$, $k = 1, 2, \ldots,$ *belong to* $\mathcal{D}(\mathcal{L})$.

Proof By computations similar to those in the proof of Proposition 2.10, we get the claim. □

3 The filtering problem

We assume that the insurance company has partial information because the externally-exciting component in the intensity process λ introduced in (2.1) is not observable. For filtering of Cox processes with shot noise intensity, that is, without the self-exciting component in (2.1), we refer to Dassios and Jang [11], where the estimation of the intensity λ given the observations of the claim arrival process $N^{(1)}$ reduces to the use of the classical Kalman–Bucy filter after a Gaussian approximation of the intensity is performed. This result applies in the case where the intensity ρ of the externally-exciting component is sufficiently large. Their setting can be seen as a particular case of our contagion model, and their results can then be obtained as special cases, with no assumption on ρ needed (see also Remark 3.7).

The insurance company aims at estimating the intensity λ by observing the cumulative claim process C in (2.2), that is, by observing the sequence $(T_n^{(1)}, Z_n^{(1)})_{n\geq 1}$ of arrival times and claim sizes. This leads to a filtering problem with marked point process observations.

Let us recall that $\mathbb{H} = \mathbb{F}^C$, defined in (2.3), is the observation flow representing the information at the disposal of the insurance company. So the estimate of the intensity λ can be described through the filter process $\pi = (\pi_t)_{t\in[0,T]}$ which provides the conditional distribution of λ_t given $\mathcal{H}_t$ for any time $t \in [0, T]$. More precisely, the filter is the $\mathbb{H}$-adapted and càdlàg (right-continuous with left limits) process taking values in the space of probability measures on $[0, \infty)$ such that

$$\pi_t(f) = \mathbb{E}[f(\lambda_t)|\mathcal{H}_t]$$

for any function f satisfying $\mathbb{E}[\int_0^t |f(\lambda_s)|\mathrm{d}s] < \infty$, $t \in [0, T]$. It is easy to verify that $(\pi_{t-}(\lambda))_{t\in[0,T]}$, where $\pi_t(\lambda) = \mathbb{E}[\lambda_t|\mathcal{H}_t]$ and $\pi_{t-}(\lambda) = \lim_{s\nearrow t^-} \pi_s(\lambda)$, provides the $\mathbb{H}$-predictable intensity of $N^{(1)}$.

Remark 3.1 For any function f satisfying $\mathbb{E}[\int_0^t |f(\lambda_s)|\mathrm{d}s] < \infty$ for any $t \in [0, T]$, we have $\mathbb{E}[\int_0^t \pi_s(f)\mathrm{d}s] = \mathbb{E}[\int_0^t f(\lambda_s)\mathrm{d}s]$, and Jensen's inequality implies

$$\begin{aligned}\mathbb{E}\left[\int_0^t |\pi_s(f)|\mathrm{d}s\right] &\leq \mathbb{E}\left[\int_0^t \pi_s(|f|)\mathrm{d}s\right]\\ &= \mathbb{E}\left[\int_0^t |f(\lambda_s)|\mathrm{d}s\right] < \infty, \qquad t \in [0, T].\end{aligned}$$

By applying the innovation method (see for instance Brémaud [6, Chap. IV]), we characterise the filter in terms of the so called *Kushner–Stratonovich* (KS) *equation*.

Theorem 3.2 *For any $f \in \mathcal{D}(\mathcal{L})$, the filter is the unique strong solution to the filtering equation, for $t \in [0, T]$,*

$$
\begin{aligned}
\pi_t(f) = f(\lambda_0) &+ \int_0^t \pi_s(\mathcal{L}f)\mathrm{d}s \\
&+ \int_0^t \int_0^\infty \left(\frac{\pi_{s-}(f(\lambda + \ell(z))\lambda)}{\pi_{s-}(\lambda)} - \pi_{s-}(f) \right) \\
&\qquad \big(m^{(1)}(\mathrm{d}s, \mathrm{d}z) - \pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\big),
\end{aligned}
\tag{3.1}
$$

where $\mathcal{L}$ and $\mathcal{D}(\mathcal{L})$ are given in Proposition 2.8.

Proof We denote by $\widehat{R}$ the $(\mathbb{P}, \mathbb{H})$-optional projection of an $\mathbb{F}$-progressively measurable process R such that $\mathbb{E}[|R_t|] < \infty$, $t \in [0, T]$. We use two well-known facts:

– For any $(\mathbb{P}, \mathbb{F})$-martingale m, the $(\mathbb{P}, \mathbb{H})$-optional projection $\widehat{m}$ is a $(\mathbb{P}, \mathbb{H})$-martingale.

– The process given by $(\widehat{\int_0^t \Psi_s \mathrm{d}s} - \int_0^t \widehat{\Psi}_s \mathrm{d}s)_{t\in[0,T]}$ is a $(\mathbb{P}, \mathbb{H})$-martingale for any $\mathbb{F}$-progressively measurable process Ψ.

By Itô's formula, for any $f \in \mathcal{D}(\mathcal{L})$, we have

$$
f(\lambda_t) = f(\lambda_0) + \int_0^t \mathcal{L}f(\lambda_s)\mathrm{d}s + m_t^f, \qquad t \in [0, T],
$$

where m^f is a $(\mathbb{P}, \mathbb{F})$-martingale. Taking the $(\mathbb{P}, \mathbb{H})$-optional projection, we get

$$
\widehat{f(\lambda_t)} = f(\lambda_0) + \int_0^t \widehat{\mathcal{L}f(\lambda_s)}\mathrm{d}s + M_t^f, \qquad t \in [0, T], \tag{3.2}
$$

where M^f is a $(\mathbb{P}, \mathbb{H})$-martingale. By the martingale representation theorem (see Jacod and Shiryaev [19, Theorems III.4.34 and III.4.36]), there exists an $\mathbb{H}$-predictable random field $h^f = (h_t^f(z))_{t\in[0,T], z\in[0,\infty)}$ such that for any $t \in [0, T]$,

$$
M_t^f = \int_0^t \int_0^\infty h_s^f(z)\big(m^{(1)}(\mathrm{d}s, \mathrm{d}z) - \pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\big) \tag{3.3}
$$

and $\mathbb{E}[\int_0^t \int_0^\infty |h_s^f(z)|\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s] < \infty$. To derive the expression of h^f, we consider an $\mathbb{H}$-adapted and bounded process

$$
\Gamma_t = \int_0^t \int_0^\infty U_s(z) m^{(1)}(\mathrm{d}s, \mathrm{d}z)
$$

with an $\mathbb{H}$-predictable bounded random field U. Since Γ is $\mathbb{H}$-adapted, we have the equality

$$
\widehat{\Gamma_t f(\lambda_t)} = \Gamma_t \widehat{f(\lambda_t)}, \qquad t \in [0, T], \mathbb{P}\text{-a.s.} \tag{3.4}
$$

By applying the product rule, we get

$$
\begin{aligned}
\mathrm{d}\big(\Gamma_t f(\lambda_t)\big) &= \Gamma_{t-}\mathrm{d}f(\lambda_t) + f(\lambda_{t-})\mathrm{d}\Gamma_t + \mathrm{d}[\Gamma, f(\lambda)]_t \\
&= \Gamma_{t-}\mathcal{L}f(\lambda_t)\mathrm{d}t + \Gamma_{t-}\mathrm{d}m_t^f + \int_0^\infty f(\lambda_{t-})U_t(z)m^{(1)}(\mathrm{d}t,\mathrm{d}z) \\
&\quad + \int_0^\infty U_t(z)\Big(f\big(\lambda_{t-}+\ell(z)\big) - f(\lambda_{t-})\Big)m^{(1)}(\mathrm{d}t,\mathrm{d}z) \\
&= \Gamma_{t-}\mathcal{L}f(\lambda_t)\mathrm{d}t + \int_0^\infty U_t(z)f\big(\lambda_{t-}+\ell(z)\big)\lambda_t F^{(1)}(\mathrm{d}z)\mathrm{d}t + \mathrm{d}\overline{m}_t^f,
\end{aligned}
$$

where $\overline{m}^f$ is a $(\mathbb{P},\mathbb{F})$-martingale. Taking the $(\mathbb{P},\mathbb{H})$-optional projection, we obtain

$$
\begin{aligned}
\mathrm{d}\big(\widehat{\Gamma_t f(\lambda_t)}\big) &= \left(\Gamma_{t-}\,\widehat{\mathcal{L}f(\lambda_t)} + \int_0^\infty U_t(z)\Big(\widehat{\lambda_t f\big(\lambda_t+\ell(z)\big)}\Big)F^{(1)}(\mathrm{d}z)\right)\mathrm{d}t \\
&\quad + \mathrm{d}\mathcal{M}_t^f,
\end{aligned} \tag{3.5}
$$

where $\mathcal{M}^f$ is a $(\mathbb{P},\mathbb{H})$-martingale. On the other hand, we have

$$
\begin{aligned}
\mathrm{d}\big(\Gamma_t\,\widehat{f(\lambda_t)}\big) &= \Gamma_{t-}\mathrm{d}\widehat{f(\lambda_t)} + \widehat{f(\lambda_{t-})}\mathrm{d}\Gamma_t + \mathrm{d}[\Gamma,\widehat{f(\lambda)}]_t \\
&= \left(\Gamma_{t-}\widehat{\mathcal{L}f(\lambda_t)} + \int_0^\infty U_t(z)\big(h_t^f(z) + \widehat{f(\lambda_t)}\big)\widehat{\lambda_t}F^{(1)}(\mathrm{d}z)\right)\mathrm{d}t \\
&\quad + \mathrm{d}\overline{\mathcal{M}}_t^f,
\end{aligned} \tag{3.6}
$$

where $\overline{\mathcal{M}}^f$ is a $(\mathbb{P},\mathbb{H})$-martingale. By (3.4), the finite-variation parts in (3.5) and (3.6) have to coincide; so for any $t\in[0,T]$,

$$
\begin{aligned}
&\int_0^t\int_0^\infty U_s(z)h_s^f(z)\widehat{\lambda_s}F^{(1)}(\mathrm{d}z) \\
&= \int_0^t\int_0^\infty U_s(z)\left(\Big(\widehat{\lambda_s f\big(\lambda_s+\ell(z)\big)}\Big) - \widehat{f(\lambda_s)}\widehat{\lambda_s}\right)F^{(1)}(\mathrm{d}z).
\end{aligned}
$$

We select U of the form $U_t(z) = Y_t\mathbf{1}_A(z)\mathbf{1}_{\{t\le T_n^{(1)}\}}$ with $Y=(Y_t)_{t\in[0,T]}$ any bounded $\mathbb{H}$-predictable nonnegative process and $A\in\mathcal{B}([0,\infty))$. With this choice, we get that Γ is bounded and for all $A\in\mathcal{B}([0,\infty))$ and $t\le T_n^{(1)}\wedge T$,

$$
\int_A h_t^f(z)\widehat{\lambda_t}F^{(1)}(\mathrm{d}z) = \int_A\left(\Big(\widehat{\lambda_t f\big(\lambda_t+\ell(z)\big)}\Big) - \widehat{f(\lambda_t)}\widehat{\lambda_t}\right)F^{(1)}(\mathrm{d}z).
$$

Recalling that $\lambda_t > 0$ for all $t\in[0,T]$ (which implies $\widehat{\lambda_t} = \pi_t(\lambda) > 0$ for all $t\in[0,T]$), we obtain that

$$
h_t^f(z) = \frac{\pi_{t-}(f(\lambda+\ell(z))\lambda)}{\pi_{t-}(\lambda)} - \pi_{t-}\big(f(\lambda)\big), \qquad t\le T_n^{(1)}\wedge T.
$$

Finally, since the counting process $N^{(1)}$ is not explosive, we have $T_n^{(1)} \to \infty$ as $n \to \infty$, and by (3.2) and (3.3), we obtain that the filter is a solution to (3.1).

It remains to prove uniqueness for this equation. As in Ceci and Colaneri [8, Theorem 3.3], strong uniqueness of the solution to the KS equation follows by uniqueness of the filtered martingale problem (FMP($\bar{\mathcal{L}}, \lambda_0, C_0$)) associated to the generator $\bar{\mathcal{L}}$ of the pair $(\lambda_t, C_t)_{t\in[0,T]}$ for any initial condition $(\lambda_0, C_0) \in (0,\infty) \times [0,\infty)$. For details on FMPs, we refer to Kurtz and Ocone [21]. The operator $\bar{\mathcal{L}}$ is given by

$$\begin{aligned}\bar{\mathcal{L}} f(\lambda, C) &= \alpha(\beta-\lambda)\frac{\partial f}{\partial \lambda}(\lambda, C) + \int_0^\infty \Big(f\big(\lambda+\ell(z), C+z\big) - f(\lambda, C)\Big)\lambda F^{(1)}(\mathrm{d}z)\\ &\quad + \int_0^\infty \big(f(\lambda+z, C) - f(\lambda, C)\big)\rho F^{(2)}(\mathrm{d}z). \qquad (3.7)\end{aligned}$$

Next, in order to prove that FMP($\bar{\mathcal{L}}, \lambda_0, C_0$) has a unique solution, we apply Kurtz and Ocone [21, Theorem 3.3] after checking that the required hypotheses are fulfilled. By Itô's formula, we get that

$$f(\lambda_t, C_t) = f(\lambda_0, 0) + \int_0^t \bar{\mathcal{L}} f(\lambda_s, C_s)\mathrm{d}s + M_t,$$

where $\bar{\mathcal{L}}$ is given in (3.7) and

$$\begin{aligned}M_t^f &= \int_0^t\int_0^\infty \Big(f\big(\lambda_{s-}+\ell(z), C_{s-}+z\big) - f(\lambda_{s-}, C_{s-})\Big)\widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z)\\ &\quad + \int_0^t\int_0^\infty \big(f(\lambda_{s-}+z, C_{s-}) - f(\lambda_{s-}, C_{s-})\big)\widetilde{m}^{(2)}(\mathrm{d}s, \mathrm{d}z).\end{aligned}$$

Thus the domain of the operator $\bar{\mathcal{L}}$ contains the set $\mathcal{C}$ of all those functions f which are in $C((0,\infty)\times[0,\infty))$, have compact support and are C^1 with respect to λ, because for this class of functions, the process M^f turns out to be a $(\mathbb{P}, \mathbb{F})$-martingale. This implies that the martingale problem for the operator $\bar{\mathcal{L}}$ is well posed on the space of càdlàg $(0,\infty)\times[0,\infty)$-valued paths. Then for any $f \in \mathcal{C}$, there exists $R_f > 0$ such that if (λ, C) belongs to the ball centered at zero with radius R_f and f is null outside this ball, then with $\|f\|_\infty$ denoting the sup-norm, we have

$$|\bar{\mathcal{L}} f(\lambda, C)| \le \left(\alpha(\beta+\lambda)\left|\frac{\partial f}{\partial \lambda}(\lambda, C)\right| + 2\|f\|(\lambda+\rho)\right) \le K$$

with a positive constant K. Thus $\bar{\mathcal{L}} f(\lambda, C)$ is in $C_b((0,\infty)\times[0,\infty))$. Finally, $\mathcal{C}$ is dense in the space of continuous functions which vanish at infinity, and so all hypotheses of [21, Theorem 3.3] are satisfied. This concludes the proof. □

The filtering equation (3.1) has a natural recursive structure in terms of the sequence $(T_n^{(1)})_{n\ge1}$. Indeed, for $t \in [T_n^{(1)} \wedge T, T_{n+1}^{(1)} \wedge T)$ (between two consecutive jump times), (3.1) reads

$$\mathrm{d}\pi_t(f) = \pi_t(\widetilde{\mathcal{L}} f)\mathrm{d}t - \big(\pi_t(\lambda f) - \pi_t(\lambda)\pi_t(f)\big)\mathrm{d}t, \qquad (3.8)$$

where

$$\widetilde{\mathcal{L}}f(\lambda) = \alpha(\beta-\lambda)f'(\lambda) + \int_0^{\infty} \big(f(\lambda+z) - f(\lambda)\big)\rho F^{(2)}(\mathrm{d}z). \tag{3.9}$$

At a jump time $T_n^{(1)} \leq T$, the value of the filter is completely determined by the knowledge of the filter π_t, with $t \in (T_{n-1}^{(1)} \wedge T, T_n^{(1)} \wedge T)$ and the observed data $(T_n^{(1)}, Z_n^{(1)})$; more precisely,

$$\pi_{T_n^{(1)}}(f) = \frac{\pi_{T_n^{(1)}-}(\lambda f(\lambda + \ell(Z_n^{(1)})))}{\pi_{T_n^{(1)}-}(\lambda)}. \tag{3.10}$$

Notice that $\widetilde{\mathcal{L}}$ is the Markov generator of a shot noise Cox process, obtained by taking $\ell(z) = 0$ in (2.1).

Remark 3.3 Let us consider $f_k(\lambda) = \lambda^k$, $k = 1, 2, \dots$ Since

$$\widetilde{\mathcal{L}}f_k(\lambda) = \alpha(\beta-\lambda)kf_{k-1}(\lambda) + \int_0^{\infty} \big((\lambda+z)^k - \lambda^k\big)\rho F^{(2)}(\mathrm{d}z),$$

(3.8) and (3.10) yield for any $k = 1, 2, \dots$ that between two consecutive jump times,

$$\begin{aligned}\mathrm{d}\pi_t(f_k) &= \alpha\big(\beta\pi_t(f_{k-1}) - \pi_t(f_k)\big)k\mathrm{d}t + \sum_{i=0}^{k-1}\binom{k}{i}\pi_t(f_i)\rho\mathbb{E}[(Z^{(2)})^{k-i}]\mathrm{d}t \\ &\quad - \big(\pi_t(f_{k+1}) - \pi_t(f_1)\pi_t(f_k)\big)\mathrm{d}t,\end{aligned} \tag{3.11}$$

and at a jump time $T_n^{(1)} \leq T$,

$$\pi_{T_n^{(1)}}(f_k) = \frac{\pi_{T_n^{(1)}-}(\lambda(\lambda + \ell(Z_n^{(1)}))^k)}{\pi_{T_n^{(1)}-}(f_1)} = \frac{\sum_{i=0}^k \binom{k}{i}\pi_{T_n^{(1)}-}(f_{i+1})\ell(Z_n^{(1)})^{k-i}}{\pi_{T_n^{(1)}-}(f_1)}. \tag{3.12}$$

In particular, for $k = 1$, we have that $\pi_t(f_1) = \pi_t(\lambda)$ provides the $(\mathbb{P}, \mathbb{H})$-intensity of $N^{(1)}$, and the KS equation reads

$$\begin{aligned}&\pi_t(\lambda) \\ &= \lambda_0 + \int_0^t \pi_s(\mathcal{L}f_1)\mathrm{d}s \\ &\quad + \int_0^t \int_0^{\infty} \left(\frac{\pi_{s-}((\lambda + \ell(z))\lambda)}{\pi_{s-}(\lambda)} - \pi_{s-}(\lambda)\right)\big(m^{(1)}(\mathrm{d}s, \mathrm{d}z) - \pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\big) \\ &= \lambda_0 + \int_0^t \Big(\alpha\big(\beta - \pi_s(\lambda)\big) + \rho\mathbb{E}[Z^{(2)}] - \big(\pi_s(\lambda^2) - \pi_s(\lambda)^2\big)\Big)\mathrm{d}s \\ &\quad + \int_0^t \int_0^{\infty} \left(\ell(z) + \frac{\pi_{s-}(\lambda^2) - \pi_{s-}(\lambda)^2}{\pi_{s-}(\lambda)}\right)m^{(1)}(\mathrm{d}s, \mathrm{d}z),\end{aligned}$$

i.e.,

$$\mathrm{d}\pi_t(\lambda) = \alpha\left(\beta + \rho\frac{\mathbb{E}[Z^{(2)}]}{\alpha} - \pi_t(\lambda)\right)\mathrm{d}t - \left(\pi_t(\lambda^2) - \pi_t(\lambda)^2\right)\mathrm{d}t$$
$$+ \int_0^\infty \ell(z)m^{(1)}(\mathrm{d}s, \mathrm{d}z) + \frac{\pi_{t-}(\lambda^2) - \pi_{t-}(\lambda)^2}{\pi_{t-}(\lambda)}\mathrm{d}N_t^{(1)}. \qquad (3.13)$$

Notice that the equations for $\pi_t(f_k)$ depend on $\pi_t(f_1), \ldots, \pi_t(f_{k+1})$ for $k = 1, 2, \ldots$ Thus the predictable $(\mathbb{P}, \mathbb{H})$-intensity $\pi_{t-}(\lambda) = \pi_{t-}(f_1)$ of $N^{(1)}$ is completely characterised by a countable system of equations given in (3.11) and (3.12). Moreover, (3.13) involves the quantities $\pi_t(\lambda)$ and

$$\pi_t(\lambda^2) - \pi_t(\lambda)^2 = \mathbb{E}\left[\left(\lambda_t - \pi_t(\lambda)\right)^2 \middle| \mathcal{H}_t\right] = \mathrm{Var}[\lambda_t|\mathcal{H}_t].$$

Remark 3.4 By Jensen's inequality, since $\pi_t(\lambda^2) \geq \pi_t(\lambda)^2$, we get by (3.13) and a comparison result that

$$\pi_t(\lambda) \leq Y_t \qquad \mathbb{P}\text{-a.s.}, t \in [0, T],$$

where the process Y has the same jumps as $\pi(\lambda)$ and between two consecutive jumps solves the SDE $\mathrm{d}Y_t = \alpha(\widetilde{\beta} - Y_t)\mathrm{d}t$, where $\widetilde{\beta} = \beta + \frac{\rho\mathbb{E}[Z^{(2)}]}{\alpha}$. More precisely, for $t \in [T_n^{(1)} \wedge T, T_{n+1}^{(1)} \wedge T)$, we have $Y_t = \widetilde{\beta} + (\pi_{T_n^{(1)}}(\lambda) - \widetilde{\beta})e^{-\alpha(t - T_n^{(1)})}$. Hence the filter is dominated by a process with exponential decay behaviour between consecutive jump times.

Thanks to Theorem 3.2, we have characterised the filter in terms of a nonlinear stochastic equation. In our framework, it is possible to describe the filter also in terms of the unnormalised filter as a solution of the so-called *Zakai equation*, which has the advantage of being linear.

By the Kallianpur–Striebel formula, we get for any $t \in [0, T]$ that

$$\pi_t(f) = \frac{\mathbb{E}^{\mathbb{Q}}[L_t f(\lambda_t)|\mathcal{H}_t]}{\mathbb{E}^{\mathbb{Q}}[L_t|\mathcal{H}_t]} = \frac{\sigma_t(f)}{\sigma_t(1)},$$

where $\mathbb{Q}$ is the equivalent probability measure introduced in Sect. 2.1 and L is given in (2.6). The process $\sigma_t(f) = \mathbb{E}^{\mathbb{Q}}[L_t f(\lambda_t)|\mathcal{H}_t]$, $t \in [0, T]$, denotes the unnormalised filter and is a finite-measure-valued $\mathbb{H}$-adapted and càdlàg process.

Proposition 3.5 *For any $f \in \mathcal{D}(\mathcal{L})$, the unnormalised filter is the unique strong solution to the* Zakai equation, *for any $t \in [0, T]$,*

$$\sigma_t(f) = f(\lambda_0) + \int_0^t \sigma_s(\mathcal{L}f)\mathrm{d}s$$
$$+ \int_0^t \int_0^\infty \left(\sigma_{s-}\left(\lambda f\left(\lambda + \ell(z)\right)\right) - \sigma_{s-}(f)\right)$$
$$\left(m^{(1)}(\mathrm{d}s, \mathrm{d}z) - F^{(1)}(\mathrm{d}z)\mathrm{d}s\right). \qquad (3.14)$$

Proof First observe that $\sigma_t(1) = \mathbb{E}^{\mathbb{Q}}[L_t|\mathcal{H}_t] = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}|_{\mathcal{H}_t}$, $t \in [0, T]$. Thus the dynamics of $\sigma(1)$ can be easily obtained by considering the effect of the Girsanov change of measure. Indeed, $\sigma(1)$ is the Doléans-Dade exponential of the $(\mathbb{Q}, \mathbb{H})$-martingale $(\int_0^t (\pi_{s-}(\lambda) - 1)(dN_s^{(1)} - \mathrm{d}s))$, i.e.,

$$\sigma_t(1) = \mathcal{E}\bigg(\int_0^{\cdot} \big(\pi_{s-}(\lambda) - 1\big)(dN_s^{(1)} - \mathrm{d}s)\bigg)_t.$$

Hence it solves

$$\mathrm{d}\sigma_t(1) = \sigma_{t-}(1)\big(\pi_{t-}(\lambda) - 1\big)(\mathrm{d}N_t^{(1)} - \mathrm{d}t). \tag{3.15}$$

By Itô's formula, we get

$$d\sigma_t(f) = \pi_{t-}(f)\mathrm{d}\sigma_t(1) + \sigma_{t-}(1)\mathrm{d}\pi_t(f) + d\bigg(\sum_{0<s\le t} \Delta\pi_s(f)\Delta\sigma_s(1)\bigg).$$

Taking into account (3.1) and (3.15) and that

$$\begin{aligned} &d\bigg(\sum_{0<s\le t} \Delta\pi_s(f)\Delta\sigma_s(1)\bigg)\\ &= \int_0^\infty \sigma_{t-}(1)\big(\pi_{t-}(\lambda) - 1\big)\bigg(\frac{\pi_{t-}(\lambda f(\lambda + \ell(z)))}{\pi_{t-}(\lambda)} - \pi_{t-}(f)\bigg)m^{(1)}(\mathrm{d}t, \mathrm{d}z), \end{aligned}$$

we get (3.14). Finally, as in Ceci and Colaneri [9, Theorem 4.7], we can prove strong uniqueness for the Zakai equation by the strong uniqueness of the KS equation. □

The Zakai equation can also be written as

$$\begin{aligned} \mathrm{d}\sigma_t(f) = &\Big(\sigma_t(\widetilde{\mathcal{L}} f) - \sigma_t\big((\lambda - 1)f\big)\Big)\mathrm{d}t\\ &+ \int_0^\infty \Big(\sigma_{t-}\Big(\lambda f\big(\lambda + \ell(z)\big)\Big) - \sigma_{t-}(f)\Big)m^{(1)}(\mathrm{d}t, \mathrm{d}z), \end{aligned}$$

where the operator $\widetilde{\mathcal{L}}$ is defined in (3.9). Like the KS equation, the above equation has a natural recursive structure in terms of the sequence $(T_n^{(1)})_{n\ge1}$. Indeed, for $t \in [T_n^{(1)} \wedge T, T_{n+1}^{(1)} \wedge T)$ (between two consecutive jump times), it reads

$$\mathrm{d}\sigma_t(f) = \Big(\sigma_t(\widetilde{\mathcal{L}} f) - \sigma_t\big((\lambda - 1)f\big)\Big)\mathrm{d}t, \tag{3.16}$$

and at a jump time $T_n^{(1)} \le T$,

$$\sigma_{T_n^{(1)}}(f) = \sigma_{T_n^{(1)}-}\Big(\lambda f\big(\lambda + \ell(Z_n^{(1)})\big)\Big). \tag{3.17}$$

By the linear structure of the Zakai equation between consecutive jumps, we get a convenient expression of the filter.

Proposition 3.6 *For any $f \in \mathcal{D}(\mathcal{L})$ and $n = 1, 2, \ldots$, we have the representation*

$$\pi_t(f) = \frac{\mathbb{E}[f(\widetilde{\lambda}^n_t)e^{-\int_s^t(\widetilde{\lambda}^n_u - 1)\mathrm{d}u}]|_{s=T^{(1)}_{n-1}}}{\mathbb{E}[e^{-\int_s^t(\widetilde{\lambda}^n_u - 1)\mathrm{d}u}]|_{s=T^{(1)}_{n-1}}}, \qquad t \in (T^{(1)}_{n-1} \wedge T, T^{(1)}_n \wedge T),$$

where $\widetilde{\lambda}^n$ is the shot noise Cox process, i.e., the solution for $t \in (T^{(1)}_{n-1} \wedge T, T^{(1)}_n \wedge T)$ of the SDE

$$\mathrm{d}\widetilde{\lambda}^n_t = \alpha(\beta - \widetilde{\lambda}^n_t)\mathrm{d}t + \int_0^\infty z m^{(2)}(\mathrm{d}t, \mathrm{d}z) \tag{3.18}$$

with initial law $\pi_{T^{(1)}_{n-1}}$.

Proof Denote the solution to (3.18) with initial condition $(s, x) \in [0, T) \times (0, \infty)$ by $\widetilde{\lambda}^{s,x}$. By Itô's formula, for $s < t \leq T$,

$$f(\widetilde{\lambda}^{s,x}_t) = f(x) + \int_s^t \widetilde{\mathcal{L}} f(\widetilde{\lambda}^{s,x}_u)\mathrm{d}u + M_t - M_s$$

with a $(\mathbb{P}, \mathbb{F})$-martingale M. With $\gamma_t = e^{-\int_s^t(\widetilde{\lambda}^{s,x}_u - 1)\mathrm{d}u}$, the product rule gives

$$f(\widetilde{\lambda}^{s,x}_t)\gamma_t = f(x) + \int_s^t \widetilde{\mathcal{L}} f(\widetilde{\lambda}^{s,x}_u)\gamma_u \mathrm{d}u - \int_s^t f(\widetilde{\lambda}^{s,x}_u)(\widetilde{\lambda}^{s,x}_u - 1)\gamma_u \mathrm{d}u + \int_s^t \gamma_u \mathrm{d}M_u,$$

and taking expectations, we obtain

$$\mathbb{E}[f(\widetilde{\lambda}^{s,x}_t)\gamma_t] = f(x) + \int_s^t \mathbb{E}[\widetilde{\mathcal{L}} f(\widetilde{\lambda}^{s,x}_u)\gamma_u]\mathrm{d}u - \int_s^t \mathbb{E}[f(\widetilde{\lambda}^{s,x}_u)(\widetilde{\lambda}^{s,x}_u - 1)\gamma_u]\mathrm{d}u.$$

Thus for any $f \in \mathcal{D}(\mathcal{L})$, the function $\Psi_t(s, x)(f) := \mathbb{E}[f(\widetilde{\lambda}^{s,x}_t)\gamma_t]$ solves (3.16), and so $\frac{\Psi_t(s,x)(f)}{\Psi_t(s,x)(1)}$ solves between two consecutive jump times the KS equation in (3.8). Finally, the statement follows by uniqueness of the KS equation observing that

$$\frac{\int_0^\infty \Psi_t(T^{(1)}_{n-1}, x)(f)\pi_{T^{(1)}_{n-1}}(\mathrm{d}x)}{\int_0^\infty \Psi_t(T^{(1)}_{n-1}, x)(1)\pi_{T^{(1)}_{n-1}}(\mathrm{d}x)}$$

coincides with the filter at the jump times $T^{(1)}_{n-1}$. □

In the following, we focus on the special case of filtering for shot noise Cox processes.

Remark 3.7 If we take $\beta = 0$ and $\ell(z) \equiv 0$ in (2.1), the claim arrival process $N^{(1)}$ reduces to the Cox process with shot noise intensity considered in Dassios and Jang [12]. Denoting by $\mathcal{L}^{SN}$ the Markov generator given by

$$\mathcal{L}^{SN} f(\lambda) = -\alpha\lambda f'(\lambda) + \int_0^\infty \big(f(\lambda + z) - f(\lambda)\big)\rho F^{(2)}(\mathrm{d}z),$$

the KS and Zakai equations in this special case are driven by $N^{(1)}$ and given by

$$d\pi_t(f) = \pi_t(\mathcal{L}^{SN} f)ds + \int_0^\infty \left(\frac{\pi_{t-}(\lambda f)}{\pi_{t-}(\lambda)} - \pi_{t-}(f)\right)\big(dN_t^{(1)} - \pi_{t-}(\lambda)dt\big),$$

$$d\sigma_t(f) = \sigma_t(\mathcal{L}^{SN} f)dt + \big(\sigma_{t-}(\lambda f) - \sigma_{t-}(f)\big)(dN_t^{(1)} - dt),$$

respectively. In particular, the KS equation between two consecutive jump times coincides with that in the general case in (3.8) (with $\widetilde{\mathcal{L}}$ replaced by $\mathcal{L}^{SN}$), while the update at a jump time $T_n^{(1)}$ from (3.10) is given by

$$\pi_{T_n^{(1)}}(f) = \frac{\pi_{T_n^{(1)}-}(\lambda f)}{\pi_{T_n^{(1)}-}(\lambda)}.$$

Analogously, the Zakai equation between two consecutive jump times coincides with that in the general case in (3.16) (with $\widetilde{\mathcal{L}}$ replaced by $\mathcal{L}^{SN}$), while the update at a jump time $T_n^{(1)}$ from (3.17) is given by $\sigma_{T_n^{(1)}}(f) = \sigma_{T_n^{(1)}-}(\lambda f)$.

4 The reduced optimal control problem under complete information

By the filtering techniques developed in Sect. 3, the original problem under partial information is now reduced to a complete observation stochastic control problem under $\mathbb{P}$ which involves only processes adapted to or predictable with respect to the filtration $\mathbb{H}$. The $(\mathbb{P}, \mathbb{H})$-predictable projection of $m^{(1)}(dt, dz)$ (see (2.4)) associated with the loss process C can be written in terms of the filter π as $\pi_{t-}(\lambda)F^{(1)}(dz)dt$. In the sequel, we denote by $\widetilde{m}^{(1)}(dt, dz)$ the $(\mathbb{P}, \mathbb{H})$-compensated jump measure, i.e.,

$$\widetilde{m}^{(1)}(dt, dz) = m^{(1)}(dt, dz) - \pi_{t-}(\lambda)F^{(1)}(dz)dt. \tag{4.1}$$

We are now ready to state the analogue of Remark 2.7 in $(\mathbb{P}, \mathbb{H})$.

Remark 4.1 For $i = 1, 2$ and for any $\mathbb{H}$-predictable and nonnegative random field $(H(t, z))_{t\in[0,T],z\in[0,\infty)}$, we have for $t \in [0, T]$ that

$$\mathbb{E}\left[\int_0^t \int_0^\infty H(s, z)m^{(i)}(ds, dz)\right] = \mathbb{E}\left[\int_0^t \int_0^\infty H(s, z)\pi_{s-}(\lambda)F^{(i)}(dz)ds\right].$$

Moreover, if $\mathbb{E}[\int_0^T \int_0^\infty |H(s, z)|\pi_{s-}(\lambda)F^{(i)}(dz)ds] < \infty$, the process

$$\int_0^t \int_0^\infty H(s, z)\widetilde{m}^{(i)}(ds, dz), \qquad t \in [0, T],$$

is a $(\mathbb{P}, \mathbb{H})$-martingale.

The primary insurer wishes to subscribe a reinsurance contract to optimally control her wealth. The surplus process without reinsurance evolves according to the equation

$$dR_t = c_t dt - \int_0^\infty z\, m^{(1)}(dt, dz), \qquad R_0 = R_0 \in \mathbb{R}_+,$$

where $(c_t)_{t\in[0,T]}$ denotes the insurance premium which is assumed to be $\mathbb{H}$-predictable and such that $\mathbb{E}[\int_0^T c_t \mathrm{d}t] < \infty$, and R_0 is the initial capital. The primary insurer subscribes a generic reinsurance contract which is characterised by the retention function Φ, which is in general an $\mathbb{H}$-predictable random field. We assume that the insurer can choose any reinsurance arrangement in a given class of admissible contracts, which is a family of functions of $z \in [0, \infty)$ representing the retained loss. For practical applications, we suppose that the contracts are parametrised by an n-tuple u (the control) taking values in a closed set $U \subseteq \overline{\mathbb{R}}^n$, with $n \in \mathbb{N}$ and $\overline{\mathbb{R}}$ denoting the compactification of $\mathbb{R}$. Under an admissible strategy $u \in \mathcal{U}$ (the definition of $\mathcal{U}$ is given in Definition 4.4 below), she retains the amount $\Phi(Z_j^{(1)}, u_{T_j^{(1)}})$ of the jth claim, while the remainder $Z_j^{(1)} - \Phi(Z_j^{(1)}, u_{T_j^{(1)}})$ is paid by the reinsurer.

We suppose that $\Phi(z, u)$ is continuous in u and there exist at least two points $u_N, u_M \in U$ such that

$$0 \le \Phi(z, u_M) \le \Phi(z, u) \le \Phi(z, u_N) = z, \qquad (z, u) \in [0, \infty) \times U,$$

so that $u = u_N$ corresponds to null reinsurance while $u = u_M$ represents the maximum reinsurance protection. Notice that u_M corresponds to full reinsurance when applicable.

Example 4.2 We can show how standard reinsurance contracts fit into our model formulation.

1) Under *proportional reinsurance*, the insurer transfers a percentage $1 - u$ of any future loss to the reinsurer; so we set

$$\Phi(z, u) = uz, \qquad u \in [0, 1] =: U.$$

Selecting the scalar u is equivalent to choosing the retention level of the contract. Notice that here $u_N = 1$ means no reinsurance and $u_M = 0$ is full reinsurance.

2) Under an *excess-of-loss reinsurance* policy, the reinsurer covers all the losses exceeding a retention level u; hence we fix the class of all functions of the form

$$\Phi(z, u) = u \wedge z, \qquad u \in [0, \infty] =: U.$$

So here, $u_N = \infty$ and $u_M = 0$, corresponding to full reinsurance.

3) Under a *limited excess-of-loss reinsurance*, the reinsurer covers for any claim the losses exceeding a threshold u_1 up to a maximum level $u_2 > u_1$, so that the maximum loss is limited to $u_2 - u_1$ on the reinsurer's side. In this case,

$$\Phi(z, u) = z - (z - u_1)^+ + (z - u_2)^+$$

with $U := \{(u_1, u_2) : u_1 \ge 0, u_2 \in [u_1, \infty]\}$ and $u = (u_1, u_2)$. Clearly, we have that $u_M = (u_{M,1}, u_{M,2}) = (0, \infty)$ and u_N can be any point on the line $u_1 = u_2$. A particular case is the so-called *limited excess-of-loss with fixed reinsurance coverage*, in which $u_2 = u_1 + \beta$, $\beta > 0$. Here $U = [0, \infty]$, $u_N = \infty$ and $u_M = 0$, in which case the maximum reinsurance coverage here is β.

Clearly, the insurer has to pay a reinsurance premium $q^u = (q^u_t)_{t\in[0,T]}$ which depends on the strategy u. We assume that the reinsurance premium admits the representation

$$q^u_t(\omega) = q(t,\omega,u), \qquad (t,\omega,u) \in [0,T]\times\Omega\times U, \tag{4.2}$$

for a function $q(t,\omega,u)\colon [0,T]\times\Omega\times U \to [0,\infty)$ continuous in u, $\mathbb{H}$-predictable for fixed u and with continuous partial derivatives $\frac{\partial q(t,\omega,u)}{\partial u_i}$, $i=1,\dots,n$. We assume that for any $(t,\omega)\in[0,T]\times\Omega$,

$$q(t,\omega,u_N)=0, \quad q(t,\omega,u)\le q(t,\omega,u_M), \qquad u\in U,$$

since a null protection is not expensive and the maximum reinsurance is the most expensive. In the following, q^u denotes the reinsurance premium associated with the dynamic reinsurance strategy $(u_t)_{t\in[0,T]}$. Notice that both insurance and reinsurance premia are assumed to be $\mathbb{H}$-predictable since insurer and reinsurer share the same information. Finally, we require the integrability condition

$$\mathbb{E}\Big[\int_0^T q^{u_M}_t\,\mathrm{d}t\Big] < \infty,$$

which ensures that $\mathbb{E}[\int_0^T q^u_s \mathrm{d}s] < \infty$ for any $u \in \mathcal{U}$.

Example 4.3 Under any admissible reinsurance strategy $u\in\mathcal{U}$, the expected cumulative losses covered by the reinsurer in the interval $[0,t]$ are given by

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t\int_0^\infty \big(z-\Phi(z,u_s)\big)\,m^{(1)}(\mathrm{d}s,\mathrm{d}z)\Big]\\ &= \mathbb{E}\Big[\int_0^t\int_0^\infty \big(z-\Phi(z,u_s)\big)\,\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\Big].\end{aligned}$$

If we use the *expected value principle*, the premium q^u applied by the reinsurer has to satisfy for all $u\in\mathcal{U}$ and $t\in[0,T]$ that

$$\mathbb{E}\Big[\int_0^t q^u_s\,\mathrm{d}s\Big] = (1+\theta_R)\mathbb{E}\Big[\int_0^t\int_0^\infty \big(z-\Phi(z,u_s)\big)\,\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\Big],$$

where $\theta_R>0$ denotes the safety loading applied by the reinsurer. Thus

$$q^u_t = (1+\theta_R)\pi_{t-}(\lambda)\int_0^\infty \big(z-\Phi(z,u_t)\big)F^{(1)}(\mathrm{d}z). \tag{4.3}$$

In general, the surplus process with reinsurance evolves according to

$$dR^u_t = (c_t - q^u_t)\mathrm{d}t - \int_0^\infty \Phi(z,u_t)\,m^{(1)}(\mathrm{d}t,\mathrm{d}z), \qquad R^u_0 = R_0\in\mathbb{R}_+.$$

Let us observe that

$$\int_0^t \int_0^\infty \Phi(z, u_s)\widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z), \qquad t \in [0, T],$$

turns out to be a $(\mathbb{P}, \mathbb{H})$-martingale because

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_0^\infty \Phi(z, u_s)\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\right] &\leq \mathbb{E}\left[\int_0^T \int_0^\infty z\,\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\right] \\ &= \mathbb{E}[Z^{(1)}]\mathbb{E}\left[\int_0^T \lambda_s \mathrm{d}s\right]\end{aligned}$$

is finite since Proposition 2.10 holds, and so Remarks 3.1 and 4.1 apply.

The insurance company invests its surplus in a risk-free asset with constant interest rate $r > 0$. So for any reinsurance strategy $u \in \mathcal{U}$, the wealth dynamics is

$$dX_t^u = dR_t^u + rX_t^u \mathrm{d}t, \qquad X_0^u = R_0 \in \mathbb{R}_+,$$

whose solution is given by

$$X_t^u = R_0 e^{rt} + \int_0^t e^{r(t-s)}(c_s - q_s^u)\mathrm{d}s - \int_0^t \int_0^\infty e^{r(t-s)}\Phi(z, u_s)\, m^{(1)}(\mathrm{d}s, \mathrm{d}z).$$

As announced before, the insurer aims at optimally controlling her wealth using reinsurance. More formally, she aims at maximising the expected exponential utility of terminal wealth, that is,

$$\sup_{u\in\mathcal{U}} \mathbb{E}[1 - e^{-\eta X_T^u}],$$

which is trivially equivalent to the minimisation problem

$$\inf_{u\in\mathcal{U}} \mathbb{E}[e^{-\eta X_T^u}], \tag{4.4}$$

where $\eta > 0$ denotes the insurer's risk aversion.

Definition 4.4 The class $\mathcal{U}$ of admissible strategies consists of all U-valued and $\mathbb{H}$-predictable processes $(u_t)_{t\in[0,T]}$ such that $\mathbb{E}[e^{-\eta X_T^u}] < \infty$. Given $t \in [0, T]$, we denote by $\mathcal{U}_t$ the class $\mathcal{U}$ restricted to the time interval $[t, T]$.

Clearly, the admissible strategies must be $\mathbb{H}$-predictable since they are based on the information at our disposal. The next assumptions are required in the sequel.

Assumption 4.5 We assume that for every $a > 0$,

i) $\mathbb{E}[e^{a\ell(Z^{(1)})}] < \infty, \quad \mathbb{E}[e^{aZ^{(1)}}] < \infty, \quad \mathbb{E}[e^{aZ^{(2)}}] < \infty;$
ii) $\mathbb{E}[e^{a\int_0^T q_t^{u^M}\mathrm{d}t}] < \infty.$

Lemma 4.6 *Under Assumption 4.5 i), we have* $\mathbb{E}[e^{aC_T}] < \infty$ *for every* $a > 0$*, where* C *is defined in* (2.2).

Proof See Appendix B. □

Remark 4.7 Usually insurance companies apply a maximum policy $D > 0$, i.e., they only repay claims up to the amount D to the policyholders. In this setting, claim sizes are of the form $\min\{Z_n^{(1)}, D\} \le D$; hence the condition $\mathbb{E}[e^{aZ^{(1)}}] < \infty$ in Assumption 4.5 is trivially satisfied.

The class of admissible strategies is non-empty, as shown by the next result.

Proposition 4.8 *Under Assumption 4.5, every* $\mathbb{H}$*-predictable process* $(u_t)_{t\in[0,T]}$ *with values in* U *is admissible.*

Proof Thanks to Lemma 4.6, the proof is basically the same as in Brachetta and Ceci [5, Proposition 2.2]. □

5 The value process and its BSDE characterisation

In this section, we study the value process associated to the problem in (4.4). Let us introduce for any $u \in \mathcal{U}$ the Snell envelope

$$W_t^u = \operatorname*{ess\,inf}_{\bar{u}\in\mathcal{U}(t,u)} \mathbb{E}[e^{-\eta X_T^{\bar{u}}} | \mathcal{H}_t], \qquad t \in [0,T], \tag{5.1}$$

with $\mathcal{U}(t,u)$ for an arbitrary control $u \in \mathcal{U}$ defined as the class of controls almost surely equal to u over $[0,t]$,

$$\mathcal{U}(t,u) := \{\bar{u} \in \mathcal{U} : \bar{u}_s = u_s \text{ a.s. for all } s \le t\}.$$

Denoting by $\bar{X}_t^u = e^{-rt} X_t^u$ the discounted wealth, we get

$$\bar{X}_t^u = R_0 + \int_0^t e^{-rs}(c_s - q_s^u)\mathrm{d}s - \int_0^t \int_0^\infty e^{-rs}\Phi(z,u_s)\, m^{(1)}(\mathrm{d}s,\mathrm{d}z), \tag{5.2}$$

and introducing the value process as

$$V_t = \operatorname*{ess\,inf}_{\bar{u}\in\mathcal{U}_t} \mathbb{E}[e^{-\eta e^{rT}(\bar{X}_T^{\bar{u}} - \bar{X}_t^{\bar{u}})} | \mathcal{H}_t], \qquad t \in [0,T], \tag{5.3}$$

(where $\mathcal{U}_t$ is introduced in Definition 4.4), we can show that for all $u \in \mathcal{U}$,

$$W_t^u = e^{-\eta \bar{X}_t^u e^{rT}} V_t.$$

In turn, choosing null reinsurance $u_t = u_N$ for any $t \in [0,T]$, we get

$$V_t = e^{\eta \bar{X}_t^N e^{rT}} W_t^N, \qquad t \in [0,T], \tag{5.4}$$

where $\bar{X}^N$ and W^N denote the discounted wealth and the Snell envelope in (5.2) and (5.1), respectively, associated to null reinsurance. Our aim is to develop a BSDE characterisation for the process $(W_t^N)_{t\in[0,T]}$ which also provides a complete description of the value process $(V_t)_{t\in[0,T]}$ in (5.3).

The following definitions play a key role for our BSDE characterisation and its solution.

Definition 5.1 We define the following three classes of stochastic processes:

– $\mathcal{S}^2$ denotes the space of all càdlàg $\mathbb{H}$-adapted processes Y such that

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]}|Y_t|\Big)^2\Big] < \infty.$$

– $\mathcal{L}^2$ denotes the space of all càdlàg $\mathbb{H}$-adapted processes Y such that

$$\mathbb{E}\Big[\int_0^T |Y_t|^2 \mathrm{d}t\Big] < \infty.$$

– $\widehat{\mathcal{L}}^2$ denotes the space of all $[0,\infty)$-indexed and $\mathbb{H}$-predictable random fields $\Theta = (\Theta_t(z))_{t\in[0,T],z\in[0,\infty)}$ such that

$$\mathbb{E}\Big[\int_0^T \int_0^\infty \Theta_t^2(z)\pi_{t-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big] < \infty.$$

Definition 5.2 We define

$$\mathbb{M} = \big\{\big(t,\omega,y,\theta(\cdot)\big) : (t,\omega,y) \in [0,T]\times\Omega\times[0,\infty) \text{ and } \theta : [0,\infty)\to\mathbb{R} \text{ is measurable}\big\},$$

and similarly, we denote by $\mathbb{M}^u$ the same set augmented with the variable $u \in U$, i.e.,

$$\mathbb{M}^u = \big\{\big(t,\omega,y,\theta(\cdot),u\big) : (t,\omega,y,u) \in [0,T]\times\Omega\times[0,\infty)\times U \text{ and } \theta : [0,\infty)\to\mathbb{R} \text{ is measurable}\big\}.$$

Definition 5.3 Let ξ be an $\mathcal{H}_T$-measurable random variable. A *solution to a BSDE* driven by the compensated random measure $\widetilde{m}^{(1)}(\mathrm{d}t,\mathrm{d}z)$ given in (4.1) and generator g is a pair $(Y,\Theta^Y) \in \mathcal{L}^2\times\widehat{\mathcal{L}}^2$ such that for all $t\in[0,T]$, $\mathbb{P}$-a.s., we have

$$Y_t = \xi + \int_t^T g\big(s,Y_s,\Theta_s^Y(\cdot)\big)\mathrm{d}s - \int_t^T\int_0^\infty \Theta_s^Y(z)\widetilde{m}^{(1)}(\mathrm{d}s,\mathrm{d}z),$$

where $g(t,\omega,y,\theta(\cdot))$ is a real-valued function on $\mathbb{M}$ which is $\mathbb{H}$-predictable with respect to $(t,\omega)\in[0,T]\times\Omega$.

We first give some preliminary results.

Proposition 5.4 *Under Assumption 4.5* i), *we have that*

$$0 < M_t^{(1)} \leq W_t^N \leq M_t^{(2)}, \qquad t \in [0, T], \tag{5.5}$$

where $M^{(i)}$, $i = 1, 2$, *are the* $(\mathbb{P}, \mathbb{H})$*-martingales*

$$M_t^{(1)} = e^{-\eta R_0 e^{rT}} \mathbb{E}\big[e^{-\eta \int_0^T e^{r(T-s)} c_s \mathrm{d}s} \big| \mathcal{H}_t\big], \qquad t \in [0, T],$$
$$M_t^{(2)} = \mathbb{E}[e^{\eta e^{rT} C_T} | \mathcal{H}_t], \qquad t \in [0, T].$$

Moreover,

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} W_t^N\Big)^2\Big] < \infty. \tag{5.6}$$

Proof The discounted wealth in (5.2) for $u = u_N$ becomes

$$\bar{X}_t^N = R_0 + \int_0^t e^{-rs} c_s \mathrm{d}s - \int_0^t \int_0^\infty e^{-rs} z\, m^{(1)}(\mathrm{d}s, \mathrm{d}z);$$

hence (5.3) implies that

$$0 \leq V_t \leq \mathbb{E}[e^{-\eta e^{rT}(\bar{X}_T^N - \bar{X}_t^N)} | \mathcal{H}_t] \leq \mathbb{E}[e^{\eta e^{rT}(C_T - C_t)} | \mathcal{H}_t], \qquad t \in [0, T], \mathbb{P}\text{-a.s.}$$

By (5.4) and observing that

$$\bar{X}^{N_t} \geq -\int_0^t \int_0^\infty e^{-rs} z m^{(1)}(\mathrm{d}s, \mathrm{d}z) \geq -\int_0^t \int_0^\infty z m^{(1)}(\mathrm{d}z) = -C_t,$$

we get for any $t \in [0, T]$ that

$$W_t^N \leq e^{-\eta \bar{X}_t^N e^{rT}} \mathbb{E}[e^{\eta e^{rT}(C_T - C_t)} | \mathcal{H}_t] \leq \mathbb{E}[e^{\eta e^{rT} C_T} | \mathcal{H}_t] = M_t^{(2)} \qquad \mathbb{P}\text{-a.s.}$$

Moreover, we have for all $t \in [0, T]$ that

$$W_t^N = \operatorname*{ess\,inf}_{\bar{u}\in\mathcal{U}(t,u_N)} \mathbb{E}[e^{-\eta X_T^{\bar{u}}} | \mathcal{H}_t] \geq \mathbb{E}[e^{-\eta X_T^N} | \mathcal{H}_t]$$
$$\geq e^{-\eta R_0 e^{rT}} \mathbb{E}[e^{-\eta \int_0^T e^{r(T-s)} c_s \mathrm{d}s} | \mathcal{H}_t] = M_t^{(1)} > 0.$$

To complete the proof, we observe that Doob's martingale inequality implies that

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} W_t^N\Big)^2\Big] \leq \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} M_t^{(2)}\Big)^2] \leq 4\mathbb{E}[(M_T^{(2)})^2] = 4\mathbb{E}[e^{2\eta e^{rT} C_T}]$$

which is finite according to Lemma 4.6. □

The next result is *Bellman's optimality principle* in our setting.

Proposition 5.5 *Under Assumption 4.5,*
i) $(W_t^u)_{t\in[0,T]}$ *is a* $(\mathbb{P}, \mathbb{H})$*-submartingale for all* $u \in \mathcal{U}$*;*
ii) $(W_t^{u^*})_{t\in[0,T]}$ *is a* $(\mathbb{P}, \mathbb{H})$*-martingale if and only if* $u^* \in \mathcal{U}$ *is an optimal control.*

Proof The proof follows the lines of Brachetta and Ceci [5, Proposition 3.2]. □

Remark 5.6 We highlight some interesting points that are useful in the sequel:
– The filtration $\mathbb{H}$ is right-continuous (see e.g. Brémaud [6, Theorem T25 in Appendix A2]).
– Using the same arguments as in Lim and Quenez [23, Proposition 4.2], we can safely state that there exists a càdlàg version of W^N, which we use **henceforth**.
– Proposition 5.4, in particular (5.5) and (5.6), implies that W^N is bounded from above and from below by two $(\mathbb{P}, \mathbb{H})$-martingales, and hence W^N is of class D.

Remark 5.7 By Proposition 5.5, since $u = u_N \in \mathcal{U}$, $(W_t^N)_{t\in[0,T]}$ is a $(\mathbb{P}, \mathbb{H})$-submartingale and $W^N \in \mathcal{S}^2 \subseteq \mathcal{L}^2$ by Proposition 5.4. As a consequence, by the Doob–Meyer decomposition and the $(\mathbb{P}, \mathbb{H})$-martingale representation theorem (see Jacod and Shiryaev [19, Theorems III.4.34 and III.4.36]), it admits the expression

$$W_t^N = \int_0^t \int_0^\infty \Theta_s^{W^N}(z)\, \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) + A_t,$$

where $\Theta^{W^N} \in \widehat{\mathcal{L}}^2$ and $(A_t)_{t\in[0,T]}$ in an increasing $\mathbb{H}$-predictable process. Moreover, $W_T^N = e^{-\eta X_T^N} =: \xi$, and since the wealth associated to null reinsurance $u = u_N$ is given by

$$X_T^N = R_0 e^{rT} + \int_0^T e^{r(T-t)} c_t \mathrm{d}t - \int_0^T \int_0^\infty e^{r(T-t)} z m^{(1)}(\mathrm{d}t, \mathrm{d}z),$$

we get the inequality $\xi \leq e^{\eta e^{rT} C_T}$. Thus Lemma 4.6 guarantees that ξ is a random variable with finite moments of any order. Summarising, we obtain that

$$W_t^N = \xi - \int_t^T \int_0^\infty \Theta_s^{W^N}(z)\, \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) + \int_t^T \mathrm{d}A_s.$$

The next step provides an explicit expression for the process A and characterises W^N and the optimal control via a BSDE approach.

We now give the main result of this section.

Theorem 5.8 *Under Assumption 4.5,* $(W^N, \Theta^{W^N}) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ *is the unique solution to the BSDE*

$$\begin{aligned} W_t^N = \xi &- \int_t^T \int_0^\infty \Theta_s^{W^N}(z)\, \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) \\ &- \int_t^T \operatorname*{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}\big(s, W_s^N, \Theta_s^{W^N}(\cdot), u_s\big)\mathrm{d}s \end{aligned} \tag{5.7}$$

with terminal condition $\xi = e^{-\eta X_T^N}$, *where*

$$
\begin{aligned}
&\widetilde{f}\big(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t\big) \\
&= -W_{t-}^N \eta e^{r(T-t)} q_t^u \\
&\quad - \int_0^\infty \big(W_{t-}^N + \Theta_t^{W^N}(z)\big)\big(e^{-\eta e^{r(T-t)}(z-\Phi(z,u_t))} - 1\big)\pi_{t-}(\lambda) F^{(1)}(\mathrm{d}z). \qquad (5.8)
\end{aligned}
$$

Moreover, the process $u^* \in \mathcal{U}$ *which satisfies*

$$
\widetilde{f}\big(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t^*\big) = \operatorname*{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}\big(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t\big), \qquad t \in [0,T], \quad (5.9)
$$

is an optimal control.

Proof Theorem 5.8 follows directly by an existence result for a solution to the BSDE (5.7) (see Theorem 5.10 below) and a verification result (see Theorem 5.12 below), which imply that any solution to the BSDE (5.7) coincides with the process (W^N, Θ^{W^N}). □

Remark 5.9 Let us notice the following points:

i) The driver of the BSDE (5.7) is always nonnegative since by using (5.8) and the property that $q^{u_N} = 0$ and $\Phi(z, u_N) = z$, we get

$$
\operatorname*{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}\big(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t\big) \geq \widetilde{f}\big(t, W_t^N, \Theta_t^{W^N}(\cdot), u_N\big) = 0.
$$

ii) There exists $u^* \in \mathcal{U}$ which satisfies (5.9). Indeed, by hypothesis, q_t^u and $\Phi(z,u)$ are continuous in $u \in U$ and U is compact; hence measurable selection results (see e.g. Beneš [2]) ensure that the maximiser is an $\mathbb{H}$-predictable process and we can use Proposition 4.8 to obtain $u^* \in \mathcal{U}$.

Theorem 5.10 *Under Assumption 4.5, there exists a unique solution* (Y, Θ^Y) *in the space* $\mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ *to the BSDE (5.7), i.e.,*

$$
Y_t = \xi - \int_t^T \int_0^\infty \Theta_s^Y(z) \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) + \int_t^T f\big(s, Y_s, \Theta_s^Y(\cdot)\big)\mathrm{d}s \qquad (5.10)
$$

with generator $f : \mathbb{M} \to [0, \infty)$ *given by*

$$
\begin{aligned}
f\big(s, y, \theta(\cdot)\big) &= -\operatorname*{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}\big(s, y, \theta(\cdot), u_s\big) \\
&= -\operatorname*{ess\,sup}_{u\in\mathcal{U}} \Bigg(- y\eta e^{r(T-s)} q_s^u \\
&\qquad - \int_0^\infty \big(y + \theta(z)\big)\big(e^{-\eta e^{r(T-s)}(z-\Phi(z,u_s))} - 1\big) \\
&\qquad \times \pi_{s-}(\lambda) F^{(1)}(\mathrm{d}z)\Bigg), \qquad (5.11)
\end{aligned}
$$

where $\mathbb{M}$ *is given in Definition 5.2, and with terminal condition* $\xi = e^{-\eta X_T^N}$.

Proof The proof is postponed to Appendix C. □

We now wish to provide a verification result. To this end, we recall the following result in Brachetta and Ceci [5, Proposition 3.4].

Proposition 5.11 *Suppose there exists an $\mathbb{H}$-adapted process D such that*

(i) $(D_t e^{-\eta \bar{X}^u_t e^{rT}})_{t\in[0,T]}$ *is for any* $u \in \mathcal{U}$ *a* $(\mathbb{P}, \mathbb{H})$*-submartingale and for some* $u^* \in \mathcal{U}$ *a* $(\mathbb{P}, \mathbb{H})$*-martingale*;

(ii) $D_T = 1$.

Then $D \equiv V$ *and* u^* *is an optimal control.*

The next result is a verification theorem.

Theorem 5.12 *Under Assumption 4.5, let* $(Y, \Theta^Y) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ *be the solution to the BSDE (5.7) and* $u^* \in \mathcal{U}$ *a process satisfying (5.9). Then* Y *coincides with* W^N,

$$V_t = e^{\eta \bar{X}^N_t e^{rT}} Y_t, \qquad t \in [0, T],$$

and u^* *is an optimal control.*

Proof Let $(Y, \Theta^Y) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ be the solution to the BSDE (5.7) and $u^* \in \mathcal{U}$ a process satisfying (5.9) (see ii) in Remark 5.9). Define $D_t := e^{\eta \bar{X}^N_t e^{rT}} Y_t$, $t \in [0, T]$, and observe that $D_T = e^{\eta X^N_T} \xi = 1$. We now prove that $(D_t e^{-\eta \bar{X}^u_t e^{rT}})_{t\in[0,T]}$ is a $(\mathbb{P}, \mathbb{H})$-submartingale for any $u \in \mathcal{U}$ and a $(\mathbb{P}, \mathbb{H})$-martingale for u^*. Then the statement will follow by Proposition 5.11.

By the product rule, for any $u \in \mathcal{U}$,

$$\begin{aligned} \mathrm{d}(D_t\, e^{-\eta \bar{X}^u_t e^{rT}}) &= \mathrm{d}(e^{\eta(\bar{X}^N_t - \bar{X}^u_t)e^{rT}} Y_t) \\ &= e^{\eta(\bar{X}^N_{t-} - \bar{X}^u_{t-})e^{rT}}\, \mathrm{d}Y_t + Y_{t-}\, \mathrm{d}(e^{\eta(\bar{X}^N_t - \bar{X}^u_t)e^{rT}}) \\ &\quad + \mathrm{d}\Big(\sum_{0<s\le t} \Delta Y_s\, \Delta(e^{\eta(\bar{X}^N_s - \bar{X}^u_s)e^{rT}}) \Big). \end{aligned}$$

Recalling (5.2), we notice that

$$\bar{X}^N_t - \bar{X}^u_t = \int_0^t e^{-rs} q^u_s \mathrm{d}s - \int_0^t \int_0^\infty e^{-rs} \big(z - \Phi(z, u_s)\big)\, m^{(1)}(\mathrm{d}s, \mathrm{d}z), \tag{5.12}$$

and applying Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}(e^{\eta(\bar{X}^N_t - \bar{X}^u_t)e^{rT}}) &= \eta e^{rT} e^{\eta(\bar{X}^N_t - \bar{X}^u_t)e^{rT}}\, e^{-rt}\, q^u_t\, \mathrm{d}t \\ &\quad + e^{\eta(\bar{X}^N_{t-} - \bar{X}^u_{t-})e^{rT}} \int_0^\infty (e^{-\eta e^{r(T-t)}(z - \Phi(z,u_t))} - 1) m^{(1)}(\mathrm{d}t, \mathrm{d}z). \end{aligned}$$

Recall now that Y solves the BSDE (5.10) with f given by (5.11). Inserting the dynamics of Y above, we get after some calculations for any $u \in \mathcal{U}$ that

$$\begin{aligned}
&\mathrm{d}(D_t e^{-\eta \bar{X}_t^u e^{rT}}) \\
&= \mathrm{d}M_t^u + e^{\eta(\bar{X}_t^N - \bar{X}_t^u)e^{rT}} \Big(\operatorname*{ess\,sup}_{w \in \mathcal{U}} \widetilde{f}\big(t, Y_t, \Theta_t^Y(\cdot), w_t\big) - \widetilde{f}\big(t, Y_t, \Theta_t^Y(\cdot), u_t\big)\Big),
\end{aligned}$$

where

$$\begin{aligned}
M_t^u &= \int_0^t \int_0^\infty e^{\eta(\bar{X}_{s-}^N - \bar{X}_{s-}^u)e^{rT}} \Theta_s^Y(z) e^{-\eta e^{r(T-s)}(z - \Phi(z,u_s))} \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t \int_0^\infty Y_{s-} e^{\eta(\bar{X}_{s-}^N - \bar{X}_{s-}^u)e^{rT}} (e^{-\eta e^{r(T-s)}(z - \Phi(z,u_s))} - 1) \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

It remains to verify that for any $u \in \mathcal{U}$, the process $(M_t^u)_{t \in [0,T]}$ is a $(\mathbb{P}, \mathbb{H})$-martingale. To this end, it is sufficient to prove that

$$\mathbb{E}\Big[\int_0^T \int_0^\infty e^{\eta(\bar{X}_t^N - \bar{X}_t^u)e^{rT}} |\Theta_t^Y(z)| e^{-\eta e^{r(T-t)}(z - \Phi(z,u_t))} \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big] < \infty,$$

$$\mathbb{E}\Big[\int_0^T \int_0^\infty e^{\eta(\bar{X}_t^N - \bar{X}_t^u)e^{rT}} |Y_t| |e^{-\eta e^{r(T-t)}(z - \Phi(z,u_t))} - 1| \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big] < \infty.$$

Using (5.12), $\Phi(z, u_t) \le z$, the elementary inequality $2ab \le a^2 + b^2$ and Jensen's inequality, the first expectation above is dominated by

$$\begin{aligned}
&\mathbb{E}\Big[e^{\eta e^{rT} \int_0^T e^{-rt} q_t^{uM} \mathrm{d}t} \int_0^T \int_0^\infty |\Theta_t^Y(z)| \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big] \\
&\le \frac{1}{2}\Big(\mathbb{E}\Big[e^{2\eta e^{rT} \int_0^T e^{-rt} q_t^{uM} \mathrm{d}t} \int_0^T \pi_t(\lambda)\mathrm{d}t\Big] \\
&\qquad + \mathbb{E}\Big[\int_0^T \int_0^\infty |\Theta_t^Y(z)|^2 \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big]\Big) \\
&\le \frac{1}{4}\mathbb{E}[e^{4\eta e^{rT} \int_0^T e^{-rt} q_t^{uM} \mathrm{d}t}]T + \frac{1}{4}\mathbb{E}\Big[\int_0^T \pi_t^2(\lambda)\mathrm{d}t\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_0^T \int_0^\infty |\Theta_t^Y(z)|^2 \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big] \\
&\le \frac{1}{4}\mathbb{E}[e^{4\eta e^{rT} \int_0^T e^{-rt} q_t^{uM} \mathrm{d}t}]T + \frac{1}{4}\mathbb{E}\Big[\int_0^T \pi_t(\lambda^2)\mathrm{d}t\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_0^T \int_0^\infty |\Theta_t^Y(z)|^2 \pi_t(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t\Big]
\end{aligned}$$

which is finite because of Assumption 4.5 ii), Remark 3.1, Proposition 2.10 and recalling that $\Theta^Y \in \widehat{\mathcal{L}}^2$. The second expectation is less than

$$\mathbb{E}\left[e^{\eta e^{rT} \int_0^T e^{-rt} q_t^{u_M} \mathrm{d}t} \int_0^T |Y_t| \, \pi_t(\lambda)\mathrm{d}t\right]$$
$$\leq \frac{1}{2}\mathbb{E}\left[\int_0^T |Y_t|^2 \mathrm{d}t\right] + \frac{1}{4}\mathbb{E}\left[e^{4\eta e^{rT} \int_0^T e^{-rt} q_t^{u_M} \mathrm{d}t}\right]T + \frac{1}{4}\mathbb{E}\left[\int_0^T \pi_t^4(\lambda)\mathrm{d}t\right],$$

where the first term is finite because $Y \in \mathcal{L}^2$, the second is finite by Assumption 4.5 ii) and the third by Remark 3.1 and Proposition 2.10. □

6 The optimal reinsurance strategy

The aim of this section is to provide more insight into the structure of the optimal reinsurance strategy and investigate some special cases.

By Theorem 5.8, $(W^N, \Theta^{W^N}) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ is the unique solution to the BSDE (5.7) and any maximiser in (5.9) provides an optimal control. Hence, exploiting the expression in (4.2), we look over $u \in \mathcal{U}$ for a maximiser of the function $\widetilde{f}: \mathbb{M}^u \to \mathbb{R}$ given by

$$\begin{aligned}\widetilde{f}\big(t, \omega, w, \theta(\cdot), u\big) &= -w\eta e^{r(T-t)} q(t, \omega, u) \\ &\quad - \int_0^\infty \big(w + \theta(z)\big)\big(e^{-\eta e^{r(T-t)}(z-\Phi(z,u))} - 1\big)\pi_{t-}(\lambda)(\omega) F^{(1)}(\mathrm{d}z). \end{aligned} \tag{6.1}$$

The following general result provides a characterisation of the optimal reinsurance strategy in the one-dimensional case, where $u \in [u_M, u_N] \subseteq \overline{\mathbb{R}}$ and $\Phi(z, u)$ is increasing in u. In order to obtain some definite results, we need to introduce a concavity hypothesis for the function $\widetilde{f}$ with respect to to the variable $u \in [u_M, u_N]$.

Proposition 6.1 *Under Assumption 4.5, suppose $\Phi(z, u)$ is differentiable in u for almost every $z \in (0, \infty)$ and $\widetilde{f}$ in (6.1) is strictly concave in u, both on $[u_M, u_N]$. Then the optimal reinsurance strategy is $u^* = (\hat{u}(t, W_{t-}^N, \Theta_t^{W^N}(\cdot)))_{t\in[0,T]}$, where $\hat{u}$ is given by*

$$\hat{u}\big(t, \omega, w, \theta(\cdot)\big) = \begin{cases} u_M & \text{for } (t, \omega, w, \theta(\cdot)) \in R_0, \\ \bar{u}(t, \omega, w, \theta(\cdot)) & \text{on } \mathbb{M}\backslash(R_0 \cup R_1), \\ u_N & \text{for } (t, \omega, w, \theta(\cdot)) \in R_1, \end{cases} \tag{6.2}$$

where we define the two regions

$$R_0 = \left\{ \big(t, \omega, w, \theta(\cdot)\big) \in \mathbb{M} : \frac{\partial \widetilde{f}(t, \omega, w, \theta(\cdot), u_M)}{\partial u} < 0 \right\},$$

$$R_1 = \left\{ \big(t, \omega, w, \theta(\cdot)\big) \in \mathbb{M} : \frac{\partial \widetilde{f}(t, \omega, w, \theta(\cdot), u_N)}{\partial u} > 0 \right\},$$

and $\bar{u}(t, \omega, w, \theta(\cdot))$ *is the solution* $u \in (u_M, u_N)$ *to the equation*

$$-w \frac{\partial q(t, \omega, u)}{\partial u} = \int_0^\infty \big(w + \theta(z)\big) z e^{-\eta e^{r(T-t)(z - \Phi(z,u))}} \times \frac{\partial \Phi(z, u)}{\partial u} \pi_{t-}(\lambda)(\omega) F^{(1)}(\mathrm{d}z). \tag{6.3}$$

Proof We observe that $\widetilde{f}$ given in (6.1) is continuous and strictly concave in u on $[u_M, u_N]$ by hypothesis. Hence the first order condition, which reads as (6.3), admits a unique solution which is a measurable function $\bar{u}(t, \omega, w, \theta(\cdot))$ on $\mathbb{M}$. If we extend the function $\widetilde{f}$ to the whole real line, i.e., allow $u \in \mathbb{R}$, then $\widetilde{f}$ is in u decreasing for $u < \bar{u}$ and increasing for $u > \bar{u}$; hence the maximiser on $[u_M, u_N]$ must be given by

$$\hat{u}\big(t, \omega, w, \theta(\cdot)\big) = \max \Big\{ u_M, \min \big\{ \bar{u}\big(t, \omega, w, \theta(\cdot)\big), u_N \big\} \Big\},$$

which is equivalent to (6.2). □

Remark 6.2 If $q(t, \omega, u)$ and $\Phi(z, u)$ are linear or convex in u on $[u_M, u_N]$, then $\widetilde{f}$ is strictly concave in u on $[u_M, u_N]$ and Proposition 6.1 applies.

We now consider a few examples under the expected value principle for the reinsurance premium (see Remark 4.3).

6.1 Proportional reinsurance

In this subsection, we take $\Phi(z, u) = zu$ and $u \in [0, 1]$. According to (4.3), the reinsurance premium reads

$$q_t^u = (1 + \theta_R)\mathbb{E}[Z^{(1)}]\pi_{t-}(\lambda)(1 - u_t), \qquad u \in \mathcal{U}.$$

Notice that Assumption 4.5 ii) is automatically satisfied since for every $a > 0$,

$$\mathbb{E}\big[e^{a \int_0^T \pi_t(\lambda) \mathrm{d}t}\big] < \infty$$

(see Appendix B).

Proposition 6.3 *Under Assumption* 4.5 i), *there exist two stochastic threshold processes* $\theta^F < \theta^N$ *such that*

$$u_t^*(\omega) = \begin{cases} 0 & \textit{if } \theta_R < \theta_t^F(\omega), \\ 1 & \textit{if } \theta_R > \theta_t^N(\omega), \\ \bar{u}(t, \omega, W_{t-}^N(\omega), \Theta_t^{W^N}(\cdot)(\omega)) & \textit{otherwise.} \end{cases}$$

They are given by

$$\theta_t^F = \frac{1}{\mathbb{E}[Z^{(1)}]} \int_0^\infty \frac{W_{t-}^N + \Theta_t^{W^N}(z)}{W_{t-}^N} z e^{-\eta e^{r(T-t)z}} F^{(1)}(\mathrm{d}z) - 1,$$

$$\theta_t^N = \frac{1}{\mathbb{E}[Z^{(1)}]} \int_0^\infty \frac{W_{t-}^N + \Theta_t^{W^N}(z)}{W_{t-}^N} z F^{(1)}(\mathrm{d}z) - 1,$$

and $\bar{u}(t, w, \theta(\cdot))$ *is the solution* $u \in (0, 1)$ *to the equation*

$$(1+\theta_R)\mathbb{E}[Z^{(1)}] = \int_0^\infty \frac{w+\theta(z)}{w} z e^{-\eta e^{r(T-t)z(1-u)}} F^{(1)}(\mathrm{d}z).$$

Proof This follows immediately from Proposition 6.1. □

Let us briefly comment on the previous result. We can distinguish three cases, depending on the stochastic conditions (in particular, depending on the solution of the BSDE (5.7)):

– If the reinsurer's safety loading θ_R is smaller than θ_t^F, then full reinsurance is optimal.

– If θ_R is larger than θ_t^N, then null reinsurance is optimal and the contract is not subscribed.

– Finally, if $\theta_t^F < \theta_R < \theta_t^N$, then the optimal retention level takes values in (0, 1), that is, the ceding company transfers to the reinsurance a non-null percentage of risk, but not the full risk.

In other words, if the reinsurance contract is inexpensive, the full reinsurance is purchased. On the other hand, when the reinsurance cost is excessive, the primary insurer will retain all the risk. In the intermediate case $\theta_t^F < \theta_R < \theta_t^N$, the retention level takes values in the interval (0, 1). In any case, the concepts of inexpensive and expensive must be related to the underlying risk through the stochastic processes W^N and Θ^{W^N}; hence the thresholds are stochastic.

6.2 Limited excess-of-loss reinsurance

The reinsurer's loss function is (see Example 4.2, 3))

$$\begin{aligned} z - \Phi(z,u) &= z - \Phi\big(z,(u_1,u_2)\big) \\ &= (z-u_1)^+ - (z-u_2)^+ = \begin{cases} 0 & \text{if } z \le u_1, \\ z-u_1 & \text{if } z \in (u_1,u_2), \\ u_2-u_1 & \text{if } z \ge u_2, \end{cases} \end{aligned}$$

with $u_1 < u_2$, so that the retention function is $\Phi(z,u) = z - (z-u_1)^+ + (z-u_2)^+$.

To obtain explicit results, we reduce our analysis to the case where the control is $u = u_1$, while $u_2 = u_1 + \beta$ is unequivocally determined with $\beta > 0$ being the fixed maximum reinsurance coverage.

According to (4.3), the expected value principle becomes

$$q_t^u = (1+\theta_R)\pi_{t-}(\lambda) \int_{u_t}^{u_t+\beta} S_Z(z)\mathrm{d}z, \qquad u \in \mathcal{U}, \tag{6.4}$$

where S_Z is the survival function $S_Z(z) = 1 - F^{(1)}(z)$.

We observe that Assumption 4.5 ii) is automatically satisfied by Lemma B.1.

Proposition 6.4 *Under Assumption* 4.5 i), *there exists a stochastic threshold* θ_t^L *such that*

$$u_t^*(\omega) = \begin{cases} 0 & \text{if } \theta_R < \theta_t^L(\omega), \\ \bar{u}(t, \omega, W_{t-}^N(\omega), \Theta_t^{W^N}(\cdot)(\omega)) & \text{otherwise}, \end{cases}$$

where

$$\theta_t^L = \frac{1}{F^{(1)}(\beta)} \int_0^\beta \frac{W_{t-}^N + \Theta_t^{W^N}(z)}{W_{t-}^N} e^{-\eta e^{r(T-t)z}} F^{(1)}(\mathrm{d}z) - 1$$

and $\bar{u}(t, w, \theta(\cdot))$ *is the solution* $u \in (0, \infty)$ *to the equation*

$$(1+\theta_R)\big(F^{(1)}(u+\beta) - F^{(1)}(u)\big) = \int_u^{u+\beta} \frac{w+\theta(z)}{w} e^{-\eta e^{r(T-t)(z-u)}} F^{(1)}(\mathrm{d}z). \tag{6.5}$$

Proof It is immediate to verify that $\widetilde{f}$ in (6.1) is strictly concave in $u \in [0, \infty)$ because the premium in (6.4) is convex in u and $\frac{\partial \Phi(z,u)}{\partial u} = 1$ for $z \in [u, u+\beta)$, while it is null elsewhere. The first order derivative is

$$\begin{aligned} &\frac{\partial \widetilde{f}(t, \omega, w, \theta(\cdot), u)}{\partial u} \\ &= w\eta e^{r(T-t)}(1+\theta_R)\pi_{t-}(\lambda)(\omega)\big(F^{(1)}(u+\beta) - F^{(1)}(u)\big) \\ &\quad - \int_u^{u+\beta} \big(w+\theta(z)\big)\eta e^{r(T-t)} e^{-\eta e^{r(T-t)}(z-u)} \pi_{t-}(\lambda)(\omega) F^{(1)}(\mathrm{d}z). \end{aligned}$$

The maximiser is always finite (we can rule out the possibility of having null reinsurance, $u^* = \infty$), while it is null if and only if $\frac{\partial \widetilde{f}(t,\omega,w,\theta(\cdot),0)}{\partial u} < 0$, i.e., when $\theta_R < \theta_t^L(\omega)$. Conversely, if $\theta_R \geq \theta_t^L(\omega)$, the maximiser coincides with the unique stationary point satisfying $\frac{\partial \widetilde{f}(t,\omega,w,\theta(\cdot),u)}{\partial u} = 0$, which can be written as (6.5). □

Let us briefly comment on the previous result. Differently from the proportional reinsurance, null reinsurance is never optimal and we can distinguish two cases, depending on the maximum coverage β and the solution of the BSDE (5.7):

– If the reinsurer's safety loading θ_R is smaller than θ^L, then the maximum reinsurance coverage β is optimal.

– If θ_R is larger than θ^L, then it is optimal to purchase reinsurance, but not with maximum coverage.

6.3 Excess-of-loss reinsurance

The excess-of-loss contract, i.e., $z - \Phi(z, u) = (z - u)^+$ (see Example 4.2, 2)) can be easily obtained from the previous case by letting $\beta \to \infty$. The optimal reinsurance strategy under Assumption 4.5 i) then becomes

$$u_t^*(\omega) = \begin{cases} 0 & \text{if } \theta_R < \theta_t^L(\omega), \\ \bar{u}(t, \omega, W_{t-}^N(\omega), \Theta_t^{W^N}(\,\cdot\,)(\omega)) & \text{otherwise,} \end{cases}$$

where

$$\theta_t^L = \int_0^\infty \frac{W_{t-}^N + \Theta_t^{W^N}(z)}{W_{t-}^N} e^{-\eta e^{r(T-t)z}} F^{(1)}(\mathrm{d}z) - 1$$

and $\bar{u}(t, w, \theta(\,\cdot\,))$ is the solution $u \in (0, \infty)$ to the equation

$$(1 + \theta_R) S_Z(u) = \int_u^\infty \frac{w + \theta(z)}{w} e^{-\eta e^{r(T-t)(z-u)}} F^{(1)}(\mathrm{d}z).$$

As in the limited excess-of-loss reinsurance case, null reinsurance is never optimal and two cases are possible, depending on the solution of the BSDE (5.7):

- When $\theta_R < \theta^L$, the full reinsurance is optimal.
- Otherwise, it becomes optimal to purchase an intermediate protection level.

Appendix A: Proofs of auxiliary results

Lemma A.1 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual hypotheses. Let N be a standard Poisson process with $\mathbb{F}$-intensity $\lambda > 0$ and $(b_t)_{t\in[0,T]}$ an $\mathbb{F}$-predictable process. Then*

$$\mathbb{E}\big[e^{\int_0^T b_t\, dN_t}\big] = \mathbb{E}\big[e^{\int_0^T (e^{b_t} - 1)\lambda \mathrm{d}t}\big],$$

provided that the last expectation is finite.

Proof To show that the statement is valid for any bounded $\mathbb{F}$-predictable process, see Brémaud [6, Appendix A1, Theorem T4], it is sufficient to prove it for any process of the form

$$b_t = \mathbb{1}_{(t_1, t_2]}(t)\mathbb{1}_A, \qquad 0 \le t_1 < t_2 \le T, A \in \mathcal{F}_{t_1}.$$

For such a process, we have

$$\begin{aligned} \mathbb{E}\big[e^{\int_0^T b_t\, dN_t}\big] &= \mathbb{E}\big[e^{\int_{t_1}^{t_2} \mathbb{1}_A\, dN_t}\big] \\ &= \mathbb{E}[e^{(N_{t_2} - N_{t_1})\mathbb{1}_A}(\mathbb{1}_A + \mathbb{1}_{A^c})] \\ &= \mathbb{E}\big[\mathbb{E}[e^{(N_{t_2} - N_{t_1})}|\mathcal{F}_{t_1}]\mathbb{1}_A + \mathbb{1}_{A^c}\big] \\ &= \mathbb{E}\big[\mathbb{E}[e^{(N_{t_2} - N_{t_1})}]\mathbb{1}_A + \mathbb{1}_{A^c}\big]. \end{aligned}$$

Because $N_{t_2} - N_{t_1} \sim \text{Po}\,(\lambda(t_2 - t_1))$, we get $\mathbb{E}[e^{(N_{t_2} - N_{t_1})}] = e^{(e-1)\lambda(t_2 - t_1)}$. Substituting and rearranging the terms then gives

$$\mathbb{E}\big[e^{\int_0^T b_t\, dN_t}\big] = \mathbb{E}[e^{(e-1)\lambda(t_2-t_1)\mathbb{1}_A}].$$

On the other hand, we notice that

$$e^{b_t} - 1 = e^{\mathbb{1}_{(t_1,t_2]}(t)\mathbb{1}_A} - 1 = e\mathbb{1}_{(t_1,t_2]}(t)\mathbb{1}_A - \mathbb{1}_{(t_1,t_2]}(t)\mathbb{1}_A = (e-1)\mathbb{1}_{(t_1,t_2]}(t)\mathbb{1}_A,$$

and so

$$\mathbb{E}\big[e^{\int_0^T (e^{b_t}-1)\lambda \mathrm{d}t}\big] = \mathbb{E}\big[e^{\int_0^T (e-1)\mathbb{1}_{(t_1,t_2]}(t)\mathbb{1}_A \lambda \mathrm{d}t}\big] = \mathbb{E}[e^{(e-1)\lambda(t_2-t_1)\mathbb{1}_A}].$$

This proves the statement for any bounded $\mathbb{F}$-predictable process. To extend the result to unbounded processes, assume that $(b_t)_{t\geq 0}$ is an arbitrary $\mathbb{F}$-predictable process and define the $\mathbb{F}$-stopping times $\tau_n = \inf\{t \geq 0 : b_t > n\}$, $n \geq 1$. Clearly, $\tau_n \to \infty$ as $n \to \infty$. By the first part of the proof, we know that

$$\mathbb{E}\big[e^{\int_0^{T\wedge\tau_n} b_t\, dN_t}\big] = \mathbb{E}\big[e^{\int_0^{T\wedge\tau_n} (e^{b_t}-1)\lambda \mathrm{d}t}\big]$$

so that it remains to pass to the limit $n \to \infty$. We can apply monotone convergence theorem to the family of random variables $X_n := e^{\int_0^{T\wedge\tau_n} b_t\, dN_t}$, $n \geq 1$, if b is positive, or to $\overline{X}_n := \frac{e^{\int_0^{T\wedge\tau_n} b_t^+\, dN_t}}{e^{\int_0^{T\wedge\tau_n} b_t^-\, dN_t}}$, $n \geq 1$, for general b. □

Lemma A.2 *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space and assume that the filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ *satisfies the usual hypotheses. Let* $N(\mathrm{d}t, \mathrm{d}z)$ *be a Poisson random measure on* $[0, T] \times [0, \infty)$ *with* $\mathbb{F}$*-intensity kernel* $\lambda F(\mathrm{d}z)\mathrm{d}t$*. Then for any* $\mathbb{F}$*-predictable and* $[0, \infty)$*-indexed process* $(H(t, z))_{t\in[0,T], z\in[0,\infty)}$*, we have that*

$$\mathbb{E}\big[e^{\int_0^T \int_0^\infty H(t,z)\, N(\mathrm{d}t,\mathrm{d}z)}\big] = \mathbb{E}\big[e^{\int_0^T \int_0^\infty (e^{H(t,z)}-1)\lambda F(\mathrm{d}z)\mathrm{d}t}\big],$$

provided that the last expectation is finite.

Proof It is sufficient to prove the result for any process H of the form

$$H(t, z) = b_t \mathbb{1}_A, \qquad t \geq 0,\ A \in \mathcal{B}\big([0, \infty)\big),$$

where b is $\mathbb{F}$-predictable. By Lemma A.1, we readily obtain that

$$\begin{aligned}
\mathbb{E}\big[e^{\int_0^T H(t,z)\, N(\mathrm{d}t,\mathrm{d}z)}\big] &= \mathbb{E}\big[e^{\int_0^T b_t N(\mathrm{d}t, A)}\big] \\
&= \mathbb{E}\big[e^{\int_0^T (e^{b_t}-1) \int_A F(\mathrm{d}z)\, \lambda \mathrm{d}t}\big] \\
&= \mathbb{E}\big[e^{\int_0^T \int_0^\infty (e^{b_t}-1)\mathbb{1}_A(z)\, F(\mathrm{d}z)\, \lambda \mathrm{d}t}\big] \\
&= \mathbb{E}\big[e^{\int_0^T \int_0^\infty (e^{b_t \mathbb{1}_A(z)}-1)\, F(\mathrm{d}z)\, \lambda \mathrm{d}t}\big] \\
&= \mathbb{E}\big[e^{\int_0^T \int_0^\infty (e^{H(t,z)}-1)\, F(\mathrm{d}z)\, \lambda \mathrm{d}t}\big],
\end{aligned}$$

where we use that $(N((0, t] \times A))$ is a Poisson process with intensity $\int_A F(\mathrm{d}z)\, \lambda$. □

Appendix B: Proof of key lemmas

We focus here on the finiteness of the four expectations $\mathbb{E}[e^{aN_T^{(1)}}]$, $\mathbb{E}[e^{a\int_0^T \lambda_s \mathrm{d}s}]$, $\mathbb{E}[e^{a\int_0^T \pi_s(\lambda)\mathrm{d}s}]$ and $\mathbb{E}[e^{aC_T}]$ computed under $\mathbb{P}$ for an arbitrary real constant $a > 0$. Here $N^{(1)}$ is a standard Poisson process under $(\mathbb{Q}, \mathbb{F})$ and a counting process with $(\mathbb{P}, \mathbb{F})$-intensity λ given in (2.1). We exploit the measure change introduced in detail in Sect. 2 and work under Assumption 4.5 i). We prove the following result.

Lemma B.1 *Under Assumption* 4.5 i), *we have*

$$\mathbb{E}[e^{aN_T^{(1)}}] < \infty, \qquad \mathbb{E}\big[e^{a\int_0^T \lambda_s \mathrm{d}s}\big] < \infty, \qquad \mathbb{E}\big[e^{a\int_0^T \pi_s(\lambda)\mathrm{d}s}\big] < \infty.$$

Proof First of all, we show that under Assumption 4.5 i), we have

$$\mathbb{E}^{\mathbb{Q}}\big[e^{a\int_0^T \lambda_s \mathrm{d}s}\big] < \infty. \tag{B.1}$$

Recalling (2.5), for a suitable $c_1 > 0$ and for $c_2 = aT$, we find that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[e^{a\int_0^T \lambda_s \mathrm{d}s}\big] &\le \mathbb{E}^{\mathbb{Q}}\big[e^{aT(\max\{\lambda_0,\beta\}+\sum_{j=1}^{N_T^{(1)}} \ell(Z_j^{(1)})+\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)})}\big] \\
&\le c_1\mathbb{E}^{\mathbb{Q}}\big[e^{c_2(\sum_{j=1}^{N_T^{(1)}} \ell(Z_j^{(1)})+\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)})}\big] \\
&= c_1 e^{T(\mathbb{E}^{\mathbb{Q}}[e^{c_2\ell(Z^{(1)})}]-1)} e^{T(\mathbb{E}^{\mathbb{Q}}[e^{c_2 Z^{(2)}}]-1)} < \infty,
\end{aligned}$$

where we used the independence of $N^{(1)}$, $N^{(2)}$, $(Z_n^{(1)})_{n\ge1}$, $(Z_n^{(2)})_{n\ge1}$ under $\mathbb{Q}$, and in the last equality, we followed the path traced in the proof of Proposition 2.6. Finally, Assumption 4.5 i) gives the finiteness of the expectation under $\mathbb{Q}$.

To prove that $\mathbb{E}[e^{aN_T^{(1)}}]$ is finite, we exploit the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ via $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}|_{\mathcal{F}_T} = L_T$, with L_T given in (2.6), so that

$$\begin{aligned}
\mathbb{E}[e^{aN_T^{(1)}}] &= \mathbb{E}^{\mathbb{Q}}[L_T e^{aN_T^{(1)}}] \\
&= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^T(\lambda_s-1)\mathrm{d}s+\int_0^T(\ln\lambda_{s-}+a)dN_s^{(1)}}\big] \\
&\le C\ \mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T(\ln\lambda_{s-}+a)dN_s^{(1)}}\big]
\end{aligned}$$

for a suitable constant $C > 0$. Now we recall that under $\mathbb{Q}$, the Poisson process $N^{(1)}$ has unit intensity and take $b_s = \ln\lambda_{s-} + a$ in Lemma A.1 to obtain

$$\mathbb{E}[e^{aN_T^{(1)}}] \le C\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T(\ln\lambda_{s-}+a)dN_s^{(1)}}\big] = C\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T(e^a\lambda_s-1)\mathrm{d}s}\big],$$

which is finite because of (B.1).

We now show that $\mathbb{E}[e^{a\int_0^T \lambda_s \mathrm{d}s}] < \infty$ for all $a > 0$. We proceed as above: passing to $\mathbb{Q}$ via L_T, recalling (2.5) and introducing the integer-valued random measure $m^{(1)}(\mathrm{d}t, \mathrm{d}z)$, we find

$$\begin{aligned}\mathbb{E}\big[e^{a\int_0^T \lambda_s \mathrm{d}s}\big] &= \mathbb{E}^{\mathbb{Q}}\big[L_T e^{a\int_0^T \lambda_s \mathrm{d}s}\big]\\ &= \mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T ((a-1)\lambda_s+1)\mathrm{d}s+\int_0^T \ln\lambda_{s-} dN_s^{(1)}}\big]\\ &\le C_1\mathbb{E}^{\mathbb{Q}}\big[e^{C_2(\sum_{j=1}^{N_T^{(1)}} \ell(Z_j^{(1)})+\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)})}e^{\int_0^T \ln\lambda_{s-} dN_s^{(1)}}\big]\\ &= C_1\mathbb{E}^{\mathbb{Q}}\big[e^{C_2\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)}}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T\int_0^\infty (C_2\ell(z)+\ln\lambda_{s-})m^{(1)}(\mathrm{d}s,\mathrm{d}z)}\big]\end{aligned}$$

for a suitable constant $C_1 > 0$. We now apply Lemma A.2 under $\mathbb{Q}$ and for the process $H(t, z) = (C_2\ell(z) + \ln\lambda_{t-})$ and with $\nu^{(1),\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) = F^{(1)}(\mathrm{d}z)\mathrm{d}t$, and we get

$$\begin{aligned}\mathbb{E}\big[e^{a\int_0^T \lambda_s \mathrm{d}s}\big] &\le C_1\ \mathbb{E}^{\mathbb{Q}}\big[e^{C_2\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)}}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T\int_0^\infty (C_2\ell(z)+\ln\lambda_{s-})m^{(1)}(\mathrm{d}s,\mathrm{d}z)}\big]\\ &= C_1\mathbb{E}^{\mathbb{Q}}\big[e^{C_2\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)}}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T\int_0^\infty (\lambda_s e^{C_2\ell(z)}-1)F^{(1)}(\mathrm{d}z)\mathrm{d}s}\big]\\ &= C_1\mathbb{E}^{\mathbb{Q}}\big[e^{C_2\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)}}\big]\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T (\lambda_s\mathbb{E}^{\mathbb{Q}}[e^{C_2\ell(Z_1^{(1)})}]-1)\mathrm{d}s}\big],\end{aligned}$$

which is finite under Assumption 4.5 i).

It remains to prove that $\mathbb{E}[e^{a\int_0^T \pi_s(\lambda)\mathrm{d}s}] < \infty$ for all $a > 0$. The structure of the filtering equation implies that over $[0, T]$, the filter attains its maximum value at a jump time. More precisely, we showed in Remark 3.4 that the filter is dominated by a process with exponential decay behaviour between two consecutive jumps; hence the maximum over $[0, T]$ is attained at a jump time $\tau \le T$ such that

$$\pi_\tau(\lambda) = \max\left\{\pi_{T_1^{(1)}}(\lambda), \dots, \pi_{T^{(1)}_{N_T^{(1)}}}(\lambda)\right\}.$$

Notice that the maximum is taken over a finite number of elements because the jump process $N^{(1)}$ is non-explosive. Then using Jensen's inequality, we have that

$$\mathbb{E}\big[e^{a\int_0^T \pi_t(\lambda)\mathrm{d}t}\big] \le \mathbb{E}[e^{aT\pi_\tau(\lambda)}] \le \mathbb{E}[\pi_\tau(e^{aT\lambda})] = \mathbb{E}[e^{aT\lambda_\tau}] < \infty.$$

The last inequality is due to the fact that $\tau \le T$, and so we have the inequalities

$$\begin{aligned}\mathbb{E}[e^{aT\lambda_\tau}] &= \mathbb{E}^{\mathbb{Q}}[L_T e^{aT\lambda_\tau}]\\ &\le C_1\mathbb{E}^{\mathbb{Q}}\big[e^{aT\lambda_\tau}e^{\int_0^T \ln\lambda_{s-} dN_s^{(1)}}\big]\\ &\le C_1\mathbb{E}^{\mathbb{Q}}\big[e^{C_2(\sum_{j=1}^{N_T^{(1)}} \ell(Z_j^{(1)})+\sum_{j=1}^{N_T^{(2)}} Z_j^{(2)})}e^{\int_0^T \ln\lambda_{s-} dN_s^{(1)}}\big]\end{aligned}$$

for suitable constants $C_i > 0$, $i = 1, 2$, and we can prove the finiteness by doing the same computations as for proving that $\mathbb{E}[e^{a\int_0^T \lambda_s \mathrm{d}s}] < \infty$. □

Based on Lemma B.1, we conclude this section proving the useful result given in Lemma 4.6, i.e., that for every $a > 0$, we have

$$\mathbb{E}[e^{aC_T}] < \infty.$$

Proof of Lemma 4.6 For a suitable constant $\kappa > 0$, passing to $\mathbb{Q}$ via the Radon–Nikodým derivative L_T given in (2.6) and using Lemma A.2, we get

$$\begin{aligned}
\mathbb{E}[e^{aC_T}] &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^T(\lambda_t-1)\mathrm{d}t+\int_0^T \ln\lambda_{t-}\, dN_t^{(1)}} e^{\int_0^T\int_0^\infty az\, m^{(1)}(\mathrm{d}t,\mathrm{d}z)}\big]\\
&\le \kappa\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T\int_0^\infty(\ln\lambda_{t-}+az)\, m^{(1)}(\mathrm{d}t,\mathrm{d}z)}\big]\\
&= \kappa\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T\int_0^\infty(e^{\ln\lambda_{t-}+az}-1)\, F^{(1)}(\mathrm{d}z)\mathrm{d}t}\big]\\
&= \kappa\mathbb{E}^{\mathbb{Q}}\big[e^{\int_0^T \lambda_{t-}(\mathbb{E}[e^{aZ^{(1)}}]-1)\mathrm{d}t}\big] < \infty,
\end{aligned}$$

where the finiteness comes from (B.1) and Assumption 4.5 i). □

Appendix C: Proof of Theorem 5.10

Proof In order to apply Papapantoleon et al. [25, Theorem 3.5], we start by verifying that the BSDE data are standard under $\widehat{\beta}$, i.e., that assumptions (**F1**)–(**F5**) in [25] are satisfied for a $\widehat{\beta} > 0$. We show that in our setting, any $\widehat{\beta} > 0$ works (see (**F4**) below).

(**F1**) The process $(\widetilde{C}_t)_{t\in[0,T]}$ with $\widetilde{C}_t = \int_0^t\int_0^\infty z\widetilde{m}^{(1)}(\mathrm{d}s,\mathrm{d}z)$ is a $(\mathbb{P},\mathbb{H})$-martingale because of Remark 4.1. Notice that $\widetilde{C}$ is a pure-jump martingale since the Brownian part is absent. Moreover,

$$\mathbb{E}[\widetilde{C}_t^2] = \mathbb{E}\bigg[\int_0^t\int_0^\infty z^2\pi_{s-}(\lambda)F^{(1)}(\mathrm{d}z)\mathrm{d}s\bigg] = \mathbb{E}[(Z^{(1)})^2]\mathbb{E}\bigg[\int_0^t \pi_{s-}(\lambda)\mathrm{d}s\bigg]$$

is finite for every $t \in [0, T]$ by Remark 3.1. Hence $\sup_{t\in[0,T]}\mathbb{E}[\widetilde{C}_t^2] < \infty$ and [25, Assumption 2.10] is satisfied. In particular, the disintegration property is fulfilled with the transition kernel K^ω on $(\Omega\times[0,T], \mathcal{P}(\mathbb{H}))$ given by

$$K_t^\omega(\mathrm{d}z) = \pi_{t-}(\lambda)F^{(1)}(\mathrm{d}z).$$

(**F2**) Lemma 4.6 guarantees that the terminal condition $\xi = e^{-\eta X_T^N}$ has finite moments of any order. See also (**F4**) below for additional details.

(**F3**) We need to prove that the generator f satisfies a stochastic Lipschitz condition, i.e., there exist two positive $\mathbb{H}$-predictable processes γ, $\bar{\gamma}$ such that on $\mathbb{M}$,

$$\begin{aligned}
&\big|f\big(t,\omega,y,\theta(\cdot)\big) - f\big(t,\omega,y',\theta'(\cdot)\big)\big|^2\\
&\le \gamma_t(\omega)|y-y'|^2 + \bar{\gamma}_t(\omega)\big(|||\theta(\cdot)-\theta'(\cdot)|||_t(\omega)\big)^2, \qquad \text{(C.1)}
\end{aligned}$$

where

$$\big(|||\theta(\cdot)|||_t(\omega)\big)^2 = \int_0^\infty \theta^2(z) K_t^\omega(\mathrm{d}z) \geq 0.$$

Exploiting the definition of f in (5.11), we first need to deal with the ess sup via

$$\begin{aligned}
&\big|f\big(t,\omega,y,\theta(\cdot)\big) - f\big(t,\omega,y',\theta'(\cdot)\big)\big|^2 \\
&\leq \Big(\operatorname*{ess\,sup}_{u\in U}\big|\widetilde{f}\big(t,\omega,y,\theta(\cdot),u\big) - \widetilde{f}\big(t,\omega,y',\theta'(\cdot),u\big)\big|\Big)^2,
\end{aligned}$$

and we first work on the absolute value difference involving $\widetilde{f}$ by

$$\begin{aligned}
&\big|\widetilde{f}\big(t,\omega,y,\theta(\cdot),u\big) - \widetilde{f}\big(t,\omega,y',\theta'(\cdot),u\big)\big| \\
&= \Bigg|(y-y')\eta e^{r(T-t)} q_t^u(\omega) \\
&\quad + \int_0^\infty \big(y-y'+\theta(z)-\theta'(z)\big)\big(e^{-\eta e^{r(T-t)}(z-\Phi(z,u))} - 1\big) K_t^\omega(\mathrm{d}z)\Bigg| \\
&\leq |y-y'|\eta e^{r(T-t)} q_t^{u_M}(\omega) + \int_0^\infty |y-y'| K_t^\omega(\mathrm{d}z) \\
&\quad + \int_0^\infty |\theta(z)-\theta'(z)| K_t^\omega(\mathrm{d}z),
\end{aligned}$$

where we have used the boundedness of $|e^{-\eta e^{R(T-t)}(z-\Phi(z,u))} - 1|$ and that $q_t^u \leq q_t^{u_M}$ for any $u \in U$. Now since the inequality above does not depend on u, the $\operatorname{ess\,sup}_{u\in U}$ also satisfies it and we can take its square to find

$$\begin{aligned}
&\Big(\operatorname*{ess\,sup}_{u\in U}\big|\widetilde{f}\big(t,\omega,y,\theta(\cdot),u\big) - \widetilde{f}\big(t,\omega,y',\theta'(\cdot),u\big)\big|\Big)^2 \\
&\leq 3|y-y'|^2\eta^2 e^{2r(T-t)}\big(q_t^{u_M}(\omega)\big)^2 + 3\bigg(\int_0^\infty \big|y-y'\big| K_t^\omega(\mathrm{d}z)\bigg)^2 \\
&\quad + 3\bigg(\int_0^\infty |\theta(z)-\theta'(z)| K_t^\omega(\mathrm{d}z)\bigg)^2.
\end{aligned}$$

Recalling now that the transition kernel reads $K_t^\omega(\mathrm{d}z) = \pi_{t-}(\lambda) F^{(1)}(\mathrm{d}z)$ and $F^{(1)}$ is a distribution function, we use Jensen's inequality for an integrable function ϑ to get

$$\bigg(\int_0^\infty |\vartheta(\omega,z)|\pi_{t-}(\lambda) F^{(1)}(\mathrm{d}z)\bigg)^2 \leq \int_0^\infty |\vartheta(\omega,z)|^2 \pi_{t-}^2(\lambda) F^{(1)}(\mathrm{d}z).$$

So we finally find that

$$
\begin{aligned}
&\Big(\operatorname*{ess\,sup}_{u\in U}\big|\widetilde{f}\big(t,\omega,y,\theta(\cdot),u\big)-\widetilde{f}\big(t,\omega,y',\theta'(\cdot),u\big)\big|\Big)^2\\
&\le 3|y-y'|^2\eta^2e^{2r(T-t)}\big(q_t^{u_M}(\omega)\big)^2+3|y-y'|^2\pi_{t-}^2(\lambda)\\
&\quad+3\int_0^{\infty}|\theta(z)-\theta'(z)|^2\pi_{t-}(\lambda)K_t^{\omega}(\mathrm{d}z)\\
&=3|y-y'|^2\Big(\eta^2e^{2r(T-t)}\big(q_t^{u_M}(\omega)\big)^2+\pi_{t-}^2(\lambda)\Big)\\
&\quad+3\int_0^{+\infty}|\theta(z)-\theta'(z)|^2\pi_{t-}(\lambda)K_t^{\omega}(\mathrm{d}z).
\end{aligned}
$$

This gives (C.1), and we get for the stochastic Lipschitz coefficients γ_t, $\bar{\gamma}_t$ the values

$$
\begin{aligned}
\gamma_t&=3\eta^2e^{2r(T-t)}(q_t^{u_M})^2+3\pi_{t-}^2(\lambda),\\
\bar{\gamma}_t&=3\pi_{t-}(\lambda),
\end{aligned}
$$

which are independent of the control u.

(**F4**) Since by definition $\alpha_\cdot^2=\max\{\sqrt{\gamma_\cdot},\bar{\gamma}_\cdot\}$, we find that

$$
\alpha_s^2=\max\Big\{\sqrt{3\eta^2e^{2r(T-s)}(q_s^{u_M})^2+3\pi_{s-}^2(\lambda)},\,3\pi_{s-}(\lambda)\Big\}.
$$

As $A_t=\int_0^t\alpha_s^2\mathrm{d}s$, the inequality $\Delta A_t\le\Phi$ $\mathbb{P}$-a.s. holds true for any $\Phi>0$ since A has no jumps. Notice that (**F2**) requires that the terminal condition $\xi=e^{-\eta X_N^T}$ belongs to the set of $\mathcal{H}_T$-measurable random variables such that $\mathbb{E}[e^{\widehat{\beta}A_T}e^{-2\eta X_T^N}]<\infty$ for some $\widehat{\beta}>0$. This is true for any $\widehat{\beta}>0$ since $\alpha_s^2\le\sqrt{3}\eta e^{r(T-s)}q_s^{u_M}+3\pi_{s-}(\lambda)$ and so

$$
\mathbb{E}[e^{\widehat{\beta}A_T}e^{-2\eta X_T^N}]\le\mathbb{E}\big[e^{\widehat{\beta}\sqrt{3}\eta\int_0^T e^{r(T-s)}q_s^{u_M}\mathrm{d}s}e^{3\widehat{\beta}\int_0^T\pi_{s-}(\lambda)\mathrm{d}s}e^{-2\eta X_T^N}\big],
$$

which is finite for any $\widehat{\beta}>0$ thanks to Assumption 4.5 (ii) (see also Lemma B.1).

(**F5**) Finally, by using the same $\widehat{\beta}>0$ and A introduced to prove (**F4**), we obtain

$$
\mathbb{E}\Big[\int_0^T e^{\widehat{\beta}A_t}\frac{|f(t,0,0,0)|^2}{\alpha_t^2}\mathrm{d}t\Big]<\infty,
$$

since here $f(t,0,0,0)=-\operatorname{ess\,sup}_{u\in\mathcal{U}}\widetilde{f}(t,0,0,u_t)=0$. It now remains to prove that the quantity

$$
M^{\Phi}(\widehat{\beta})=\frac{9}{\widehat{\beta}}+\frac{\Phi^2(2+9\widehat{\beta})}{\sqrt{\widehat{\beta}^2\Phi^2+4}-2}\exp\left(\frac{\widehat{\beta}\Phi+2-\sqrt{\widehat{\beta}^2\Phi^2+4}}{2}\right),
$$

with $\Phi>0$ introduced in (**F4**) and $\widehat{\beta}>0$, satisfies $M^{\Phi}(\widehat{\beta})<\frac{1}{2}$. Thanks to [25, Lemma 3.4], it suffices to take $\Phi<\frac{1}{18e}$ and $\widehat{\beta}$ sufficiently large because we have $\lim_{\widehat{\beta}\to\infty}M^{\Phi}(\widehat{\beta})=9e\Phi$.

It remains to show that $(Y, \Theta^Y) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$. According to [25, Theorem 3.5], we have

$$\mathbb{E}\left[\int_0^T e^{\widehat{\beta} A_t} \alpha_t^2 |Y_t|^2 \mathrm{d}t\right] < \infty.$$

Moreover, $\alpha_t^2 \geq 3\pi_{t-}(\lambda) \geq 3\min\{\lambda_0, \beta\}$ implies $\mathbb{E}[\int_0^T e^{\widehat{\beta} A_t} |Y_t|^2 \mathrm{d}t] < \infty$ and therefore $Y \in \mathcal{L}^2$. The same argument can be used to prove that $\Theta^Y \in \widehat{\mathcal{L}}^2$. □

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Declarations

Competing Interests The authors declare that they have no conflict of interest.

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