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Research paper

A useful subdifferential in the Calculus of Variations

Piernicola Bettiol [a], Giuseppe De Marco [b], Carlo Mariconda [b,*]

[a] Univ Brest, UMR CNRS 6205, Laboratoire de Mathématiques de Bretagne Atlantique, 6 Avenue Victor Le Gorgeu, Brest, 29200-F, France
[b] Università degli Studi di Padova, Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy

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ABSTRACT

Consider the basic problem in the Calculus of Variations of minimizing an energy functional depending on absolutely continuous functions Under suitable assumptions on the Lagrangian, a well-known result establishes that the minimizers satisfy the Du Bois-Reymond equation. Recent work (cf. Bettiol and Mariconda, 2020 [1], 2023; Mariconda, 2023 [2], 2021, 2024) highlights not only that a Du Bois-Reymond condition for minimizers can be broadened to cover the case of nonsmooth extended valued Lagrangians, but also that a particular subdifferential (associated with the generalized Du Bois-Reymond condition) plays an important role in the approximation of the energy via its values along Lispchitz functions, no matter minimizers exist. A crucial point is establishing boundedness properties of this subdifferential, based on weak local boundedness properties of the Lagrangian. This is the main objective of this paper. Our approach is based on a refined analysis of the metric that can be employed to evaluate the distance from the complementary of the effective domain of the reference Lagrangian. As an application of our findings we show how it is possible to deduce the non-occurrence of the Lavrentiev phenomenon, providing a new general result.

1. Introduction

Consider the classical problem of the Calculus of Variations of minimizing an integral functional

$$F(y) = \int_a^b \Lambda(s, y(s), y'(s))\, ds$$

over absolutely continuous functions with, possibly, prescribed boundary values at a or b. If $\Lambda : (s, y, v) \in [a, b]\times\mathbb{R}^n\times\mathbb{R}^n \to \Lambda(s, y, v) \in \mathbb{R}$ is of class C^1 and Λ satisfies a suitable local Lipschitz condition (S) (stronger than condition (S$^+$) below) with respect to the variable s (see [3]), any minimizer y_*, even just local, satisfies the Du Bois-Reymond equation: the function

$$p(s) = \Lambda(s, y_*(s), y_*'(s)) - y_*'(s) \cdot \nabla_v \Lambda(s, y_*(s), y_*'(s)) \tag{1.1}$$

turns out to be absolutely continuous and

$$p'(s) = \Lambda_t(s, y_*(s), y_*'(s)) \text{ a.e. on } [a, b]. \tag{1.2}$$

Here Λ_t and $\nabla_v\Lambda$ denote, respectively, the derivative of Λ with respect to s and the gradient of Λ with respect to v. An extension of the Du Bois-Reymond equation to the nonsmooth setting was established in [4]: under a nonsmooth version of Condition (S) it

* Corresponding author.
E-mail addresses: piernicola.bettiol@univ-brest.fr (P. Bettiol), giuseppe.demarco@unipd.it (G. De Marco), carlo.mariconda@unipd.it (C. Mariconda).

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turns out that, for almost every $s \in [a,b]$, the function $0 < \mu \mapsto \Lambda\left(s, y_*(s), \frac{y_*'(s)}{\mu}\right)\mu$ is "convex at $\mu = 1$": we mean that there exists an real valued function $p(s)$ such that

$$\forall \mu > 0 \quad \Lambda\left(s, y_*(s), \frac{y_*'(s)}{\mu}\right)\mu - \Lambda(s, y_*(s), y_*'(s)) \geq p(s)(\mu - 1), \quad a.e.\ s. \tag{1.3}$$

In this situation the Du Bois-Reymond (in what follows DBR) condition takes the following form: if y_* is a (local) minimizer, then p is absolutely continuous and p' belongs to the Clarke's subdifferential of Λ with respect to s. (See also [1] and [2] for further discussion and developments.) If Λ is smooth, then from (1.3) and using the chain rule we deduce the classical (smooth) Du Bois-Reymond equation (1.1) – (1.2).

Another classical problem of interest in the Calculus of Variations is related to the possibility to investigate the value of the infimum of F by means of numerical methods: well-established numerical techniques allow to do that when the infimum of F over Lipschitz admissible arcs coincides with that one of F over the absolutely continuous admissible arcs (here, 'admissible' means that the arc satisfies some given boundary conditions). It is well-known that, even if Lipschitz functions on $[a,b]$ are dense in the set of absolutely continuous functions on $[a,b]$, a gap (also referred to as *Lavrentiev gap*) between the two infima of F might occur in some circumstances, as shown by the celebrated examples of Manià [5] and of Ball and Mizel [6], where the Lagrangians Λ are merely polynomials. In an attempt of trying, given an absolutely continuous admissible arc y, to approximate the energy $F(y)$ via the one of a Lipschitz function one can try to find it among the reparametrizations of the form $y \circ \varphi^{-1}$ of y, where $\varphi : [a,b] \to [a,+\infty[$ is absolutely continuous, strictly increasing. This approach was followed by Cellina in [7] in the autonomous case assuming continuity and convexity in the velocity variable, both weakened here. If $\overline{y}$ is such a reparametrization, then

$$F(\overline{y}) = \int_a^b \Lambda\left(t, y(t), \frac{y'(t)}{\varphi'(t)}\right)\varphi'(t)\, dt.$$

If $0 < r \mapsto \Lambda(s, y, rv)$ is convex, one can compare the value of $F(y)$ with that of the reparametrized energy, in terms of the subdifferential of the convex function $0 < \mu \mapsto \Lambda\left(s, y, \frac{v}{\mu}\right)\mu$ at $\mu = 1$, that we call DRB type subdifferential in view of its direct connection to the Du Bois-Reymond condition. To establish a comparison between the energy of the reference arc and the reparametrized one it is crucial to have some bounds of the DBR subdifferential $P(s, z, v)$ of Λ evaluated at points $(s, y(s), v)$ with v near to $y'(s)$, $s \in [a,b]$. The boundedness of P along the trajectory of a given minimizer y_* is trivial since, in this case, from (1.1)–(1.2), $s \mapsto P(s, y_*(s), y_*'(s))$ is absolutely continuous. A key point here is that y is an arbitrary trajectory, so that the Du Bois Reymond condition is not supposed to hold, in general.

The main objective of this paper is to provide some global boundedness properties of the DBR type subdifferential of Λ depending just on some local boundedness properties of Λ "inside" the effective domain

$$\mathrm{Dom}(\Lambda) := \{(s, y, v) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n : \Lambda(s, y, v) < +\infty\}.$$

The case of Lagrangians that may take the value $+\infty$ is of interest when one deals with state or velocity constraints. This case is more difficult to deal with: there are examples exhibiting the occurrence of the Lavrentiev gap in which the Lagrangian is just the indicator function of a particular domain; and recent work shows that the non-occurrence of the Lavrentiev gap may depend on the topology of the effective domain of the Lagrangian (see [8,9]).

The use of boundedness properties of a DBR type subgradient is not new; however, to our knowledge, these were previously derived either from the regularity of the Lagrangian as in [10], or were imposed a priori as in [11] and were not a consequence of just some local boundedness properties of the Lagrangian.

In this paper we develop the idea introduced in [12] that, making use of different (from the Euclidean one) metrics, we can obtain more precise information about important features of the reference Calculus of Variations problem. For instance, concerning the non-occurrence of the Lavrentiev phenomenon some sufficient conditions can be more easily verified employing a 'bigger' (w.r.t. some order) distance, but in some circumstances 'smaller' metrics can be more convenient (see Section 5).

More precisely, we characterize a class of 'admissible' (in a sense that will be made precise in Section 3 below) topologies in a metric space associated with particular equivalence relations families, which allows to deduce boundedness properties of the DBR type subgradient of Λ.

The first part of the paper is devoted to the study of metric spaces endowed with a distance built upon a suitable equivalence relation on a metric space. The central part of the paper is devoted to the main result (Theorem 4.3); it extends [12, Proposition 2.15] in a more precise way and employing a new topological framework. Though the main applications of Theorem 4.3 are thoroughly developed in a forthcoming paper [8], in the last section we give a simple, self-contained proof of the non-occurrence of the Lavrentiev phenomenon in the calculus of variations for problems with one initial endpoint (see Proposition 5.1). Finally, we formulate a general result (see Theorem 5.2) that shows how different admissible topologies can be useful to investigate more complex problems.

2. Notation and assumptions

2.1. Notation

We introduce some recurring notations:

- The Euclidean norm on $\mathbb{R}^n$ ($n = 1, 2, \ldots$) is denoted by $|\cdot|$, the Euclidean distance on $\mathbb{R}^n \times \mathbb{R}^n$ is written dist_e;

- The closed ball of $\mathbb{R}^n$ centered in the origin of radius $K \geq 0$ is denoted by B_K.

We shall consider the Calculus of Variations problem

$$\begin{cases} \min F(y) = \displaystyle\int_a^b \Lambda(s, y(s), y'(s))\, ds \\ \text{over admissible continuous arcs } y : [a, b] \to \mathbb{R}^n \text{ s.t.} \\ y'(s) \in \mathcal{V} \text{ a.e. } s \in [a, b], \\ y(s) \in \Delta \text{ for all } s \in [a, b], \end{cases} \tag{P}$$

where $[a, b]$ is a given closed bounded interval of $\mathbb{R}$, and $\Delta \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$ are given subsets. We say that an absolutely continuous arc $y : [a, b] \to \mathbb{R}^n$ is *admissible* for (P) if $y'(s) \in \mathcal{V}$ a.e. $s \in [a, b]$ and $y(s) \in \Delta$ for all $s \in [a, b]$. With the reference problem (P) we shall consider the following assumptions.

2.2. Basic assumptions

Basic Assumptions. We assume the following conditions.

(A1) $\mathcal{V}$ is a cone in $\mathbb{R}^n$ (i.e. if $v \in \mathcal{V}$ and $\lambda > 0$ then $\lambda v \in \mathcal{V}$);
(A2) $\Lambda : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a Borel measurable function.

Observe that (A2) guarantees, for instance, that if $y, v : [a, b] \to \mathbb{R}^n$ are (Lebesgue) measurable then $s \mapsto \Lambda(s, y(s), v(s))$ is (Lebesgue) measurable (cf. [13, Proposition 6.34]).

2.3. Structure and radial convexity assumptions.

The following additional conditions on Λ will be assumed throughout the paper.

Structure Assumption on Dom(Λ). When Λ is extended valued, we assume the following conditions on the effective domain, given by

$$\mathrm{Dom}(\Lambda) := \{(s, y, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n : \Lambda(s, y, v) < +\infty\}.$$

1. The effective domain of Λ is a product:
$$\mathrm{Dom}\,\Lambda = [a, b] \times D_\Lambda \text{ for some } D_\Lambda \subseteq \mathbb{R}^n \times \mathbb{R}^n. \tag{2.1}$$

2. For every $y \in \mathbb{R}^n$ let
$$D_\Lambda(y) := \{v \in \mathbb{R}^n : (y, v) \in \mathrm{Dom}(\Lambda)\}$$
be the y-section of D_Λ. We assume that $D_\Lambda(y)$ is strictly star-shaped on the variable v with respect to the origin, i.e.,
$$\forall (y, v) \in D_\Lambda,\ \forall r \in]0, 1] \qquad (y, rv) \in D_\Lambda. \tag{2.2}$$
Thus if $\Lambda(s, y, v) < +\infty$ then $\Lambda(s, y, rv) < +\infty$ for every $r \in]0, 1]$.
Fig. 1 illustrates an example of a domain that satisfies the Structure Assumption.

Remark 2.1. Condition (2.1) is fulfilled if Λ is a product of functions of the form $\Lambda(s, y, v) = a(s)L(y, v)$;

We recall here some important properties of the subgradient of a convex function, that will be used in the proof of the main results (see, for instance, [4,14] for more details).

Proposition 2.2 (*Convex Subgradient*). *Let $\phi :]0, +\infty[\to \mathbb{R} \cup \{+\infty\}$ be a convex function that is finite on $]0, 1]$ and has a **subgradient** $Q \in \mathbb{R}$ at* 1*:*

$$\forall r > 0 \qquad \phi(r) - \phi(1) \geq Q(r - 1).$$

*The subset of such elements $Q \in \mathbb{R}$ is called the (convex) **subdifferential of** ϕ **at** 1, and denoted by $\partial\phi(1)$. Then:*

1. *The function $0 < \mu \mapsto \Phi(\mu) := \phi\left(\frac{1}{\mu}\right)\mu$ is convex, finite on $[1, +\infty[$;*
2. *$P \in \partial\Phi(1)$ if and only if there is $Q \in \partial\phi(1)$ satisfying*
$$P = \phi(1) - Q.$$

In particular P is the ordinate of the intersection of the tangent line $z = Q(\mu - 1) + \phi(1)$ to the graph of ϕ at 1 *with the vertical axis.*

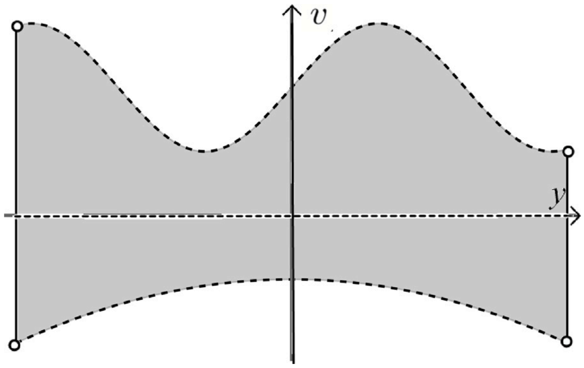

Fig. 1. An example of a subset D_Λ that satisfies the Structure Assumptions.

Radial Convexity Assumption. For a.e. $s \in [a,b]$, for all $(y,v) \in \mathbb{R}^n \times \mathcal{V}$ such that $(s,y,v) \in \mathrm{Dom}(\Lambda)$,

$$0 < r \mapsto \Lambda(s,y,rv) \text{ is convex,} \tag{RC}$$

and has a non-empty convex subdifferential at $r=1$, denoted by $\partial\big(\Lambda(s,y,rv)\big)_{r=1}$.

Remark 2.3.

1. Let $(s,y,v) \in \mathrm{Dom}(\Lambda)$. It follows from the Structure Assumption that the effective domain of $0 < r \mapsto \phi(r) = \Lambda(s,y,rv)$ is an interval $J_{y,v} =]0, r(y,v)[$ or $J_{y,v} =]0, r(y,v)]$ with $r(y,v) \in [1,+\infty[$. The convexity of $0 < r \mapsto \Lambda(s,y,rv)$ implies that has a non-empty subdifferential on the interior of $J_{y,v}$. The Radial Convexity assumption requires, in addition that, if $r(y,v)=1$, this is still true at $r=1$. This occurs, for instance, if Λ is identically equal to 0 on its effective domain, since in this case, $0 \in \partial\big(\Lambda(s,y,rv)\big)_{r=1}$.
2. If Λ is radially convex and $\mathcal{V} = \mathbb{R}^n$, then (2.2) is fulfilled if

$$\forall (s,y,v) \in [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \qquad \Lambda(s,y,v) < +\infty \Rightarrow \Lambda(s,y,0) < +\infty.$$

Notice, however, that it is not required, in general, that $(s,y,0) \in \mathrm{Dom}(\Lambda)$ for some $(s,y) \in [a,b]\times\mathbb{R}^n$.

2.4. The DBR type subdifferential and subgradient

For $(s,y,v) \in \mathrm{Dom}(\Lambda)$, we now apply Proposition 2.2 with $\phi(r) := \Lambda(s,y,rv)$. With the notation of Proposition 2.2, we have that

$$0 < \mu \mapsto \Phi(\mu) := \Lambda\Big(s,y,\frac{v}{\mu}\Big)\mu$$

is a convex function which is finite on $[1,+\infty[$; moreover it has a subgradient $P(s,y,v)$ at $\mu=1$. When Λ is smooth in the last variable we have that

$$P(s,y,v) = \Lambda(s,y,v) - v\cdot\nabla_v\Lambda(s,y,v).$$

The function P plays a special role in the calculus of variations: under suitable assumptions it turns out that a minimizer y_* of F satisfies the Du Bois-Reymond condition: the function $p(s) = P(s,y_*(s),y_*'(s))$ is absolutely continuous and

$$p'(s) = \Lambda_s(s,y_*(s),y_*'(s)) \text{ for a.e. } s \in [a,b].$$

This fact is well known in the smooth case, and was established in a general nonsmooth framework in [4,15].

Definition 2.4 (*The DBR Type Subgradient*)**.** Let $(s,y,v) \in \mathrm{Dom}(\Lambda)$. The set

$$\partial_\mu\left[\Lambda\Big(s,y,\frac{v}{\mu}\Big)\mu\right]_{\mu=1}$$

will be called the **DBR type subdifferential** of Λ at (s,y,v); any of its elements $P(s,y,v)$ will be called a **DBR type subgradient** of Λ at (s,y,v). It has the property that

$$\forall\mu>0 \quad \Lambda\Big(s,y,\frac{v}{\mu}\Big)\mu - \Lambda(s,y,v) \geq P(s,y,v)(\mu-1). \tag{2.3}$$

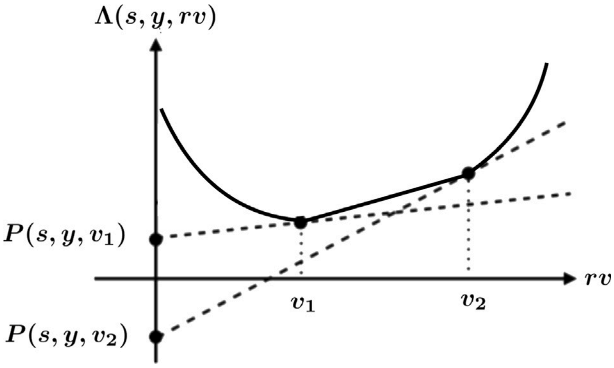

Fig. 2. Interpretation of $P(s,y,v) \in \partial_\mu \left[\Lambda\left(s,y,\frac{v}{\mu}\right)\mu\right]_{\mu=1}$ $(n=1)$.

Remark 2.5. It follows from Proposition 2.2 that $P(s,y,v)$ represents the intersection with the ordinate axis in the half-plane $\{rv,\ r>0\} \times \{w = \Lambda(s,y,rv)\ :\ r>0\}$ of the tangent line to $r \mapsto \Lambda(s,y,rv)$ at $r=1$ (see Fig. 2).

In many applications it is useful to evaluate a selection $P(s,y,v)$ of a DBR type subdifferential along an absolutely continuous arc y and guarantee that the map $s \to \Lambda(s,y(s),y'(s))$ is Lebesgue measurable. A key result for this purpose is, then, the following proposition.

Proposition 2.6 (*Existence of a Borel Selection of the DBR Type Subdifferential*)**.** *Assume that $\mathcal{V} \subset \mathbb{R}^n$ is a Borel set and that Λ is a Borel measurable function which satisfies the Radial Convexity condition (RC) and the Structure Assumption. Then there exists a Borel selection*

$$(s,y,v) \in \mathrm{Dom}(\Lambda) \cap ([a,b]\times\mathbb{R}^n\times\mathcal{V}) \mapsto P(s,y,v) \in \partial_\mu \left[\Lambda\left(s,y,\frac{v}{\mu}\right)\mu\right]_{\mu=1}.$$

Proof. Observe first that, since Λ is Borel measurable and $\mathcal{V}$ is a Borel set, $D := \mathrm{Dom}(\Lambda) \cap ([a,b]\times\mathbb{R}^n\times\mathcal{V})$ is a Borel set, and from the existence of $\partial\big(\Lambda(s,y,rv)\big)_{r=1}$ ensured by the Radial Convexity Assumption,

$$\mathrm{Dom}(\Lambda) \ni (s,y,v) \rightsquigarrow G(s,y,v) := \partial\big(\Lambda(s,y,rv)\big)_{r=1}$$

takes values nonempty (convex, closed) subsets of $\mathbb{R}$. Therefore, in view of Proposition 2.2, to confirm the proposition statement it is enough to prove that there exists a Borel selection $Q(s,y,v) \in G(s,y,v)$ for all $(s,y,v) \in D$. As a consequence $P(s,y,v) := \Lambda(s,y,v) - Q(s,y,v)$ will then be the required measurable selection. Let $(r_k)_{k\geq 0}$ be a dense sequence in $]0,+\infty[$. For each $k \geq 0$ we define the multifunction

$$G_k(s,y,v) := \{q \in \mathbb{R}\ :\ \Lambda\big(s,y,r_k v\big) - \Lambda(s,y,v) \geq q(r_k - 1)\} \quad \text{for all } (s,y,v) \in D.$$

We claim that

$$\mathrm{Graph}(G) = \bigcap_{k=0}^{\infty} \mathrm{Graph}(G_k). \tag{2.4}$$

Indeed, suppose first that $((s,y,v),q) \in \mathrm{Graph}(G)$. It means that

$$\Lambda(s,y,rv) - \Lambda(s,y,v) \geq q(r-1) \quad \text{for all } r>0, \tag{2.5}$$

and so, in particular, $q \in G_k(s,y,v)$ for all $k \geq 0$. Conversely, suppose that

$$((s,y,v),q) \in \bigcap_{k=0}^{\infty} \mathrm{Graph}(G_k).$$

Let $r>0$. Since the inequality (2.5) is trivially satisfied if $\Lambda(s,y,rv) = +\infty$, we can restrict attention to the case when $\Lambda(s,y,rv) < +\infty$. Then, from condition (2.2) ($D_\Lambda(y)$ is strictly star-shaped on v w.r.t. the origin) we deduce that there exists a subsequence $r_{k_m} \uparrow r$ such that

$$\Lambda\big(s,y,r_{k_m} v\big) - \Lambda(s,y,v) \geq q(r_{k_m} - 1), \quad \text{for all } m \geq 1\ . \tag{2.6}$$

Since Λ satisfies the Radial Convexity assumption (RC) and condition (2.2), the map $r' \mapsto \Lambda\big(s,y,r'v\big)$ is continuous on $]0,r]$. Taking the limit as $m \to \infty$ in (2.6), it follows that

$$\Lambda\big(s,y,rv\big) - \Lambda(s,y,v) = \lim_{m\to+\infty} \Lambda\big(s,y,r_{k_m} v\big) - \Lambda(s,y,v) \geq q(r-1).$$

Therefore, we obtain the validity of (2.5), and so $((s,y,v),q) \in \mathrm{Graph}(G)$. This confirms the claim. Observe now that the multivalued function G_k is Borel measurable, for all $k \geq 0$, and in view of (2.4) $\mathrm{Graph}(G)$ is Borel measurable. Owing to [16, Theorem III.22] we obtain that there exists a Borel measurable selection $Q : D \to \mathbb{R}$ of G. □

3. Admissible topologies for the basic problem of the calculus of variations

The projection D_Λ on $\mathbb{R}^n \times \mathbb{R}^n$ of the effective domain of the Lagrangian Λ can be endowed with different topologies, which give useful information about the reference Calculus of Variations problem (P), as we will show in the next section. We start providing some general considerations of topologies induced by equivalence relations in a given metric space.

3.1. *The topology induced by an equivalence relation in a metric space*

We will consider an equivalence relation in a metric space (Π, d) we allow here a metric to take the value $+\infty$. If $\mathcal{W} \subseteq \Pi \times \Pi$ is an equivalence relation on Π we will write $\omega \underset{\mathcal{W}}{\sim} \omega'$ whenever $(\omega, \omega') \in \mathcal{W}$.

Definition 3.1 (*The "metric"* $\mathrm{dist}_{\mathcal{W}}$)**.** Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d). We set

$$\forall \omega, \omega' \in \Pi \quad \mathrm{dist}_{\mathcal{W}}(\omega, \omega') := \begin{cases} d(\omega, \omega') & \text{if } \omega \underset{\mathcal{W}}{\sim} \omega', \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Proposition 3.2. *Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d). Then $\mathrm{dist}_{\mathcal{W}}$ is a distance on Π and*

$$\forall \omega, \omega' \in \Pi \qquad \mathrm{dist}_{\mathcal{W}}(\omega, \omega') \geq d(\omega, \omega').$$

Proof. Let $\omega \in \Pi$. Since $\omega \underset{\mathcal{W}}{\sim} \omega$, we have $\mathrm{dist}_{\mathcal{W}}(\omega, \omega) = 0$. Let $\omega, \omega' \in \Pi$. If $\mathrm{dist}_{\mathcal{W}}(\omega, \omega') = 0$ then, from (3.1), we deduce that $d(\omega, \omega') = 0$, whence $\omega = \omega'$. Clearly $\mathrm{dist}_{\mathcal{W}}(\omega, \omega') = \mathrm{dist}_{\mathcal{W}}(\omega', \omega)$. We check now the triangular inequality. Let $\omega'' \in \Pi$. If $\omega \underset{\mathcal{W}}{\sim} \omega''$ then, from (3.1),

$$\begin{aligned} \mathrm{dist}_{\mathcal{W}}(\omega, \omega'') = d(\omega, \omega'') &\leq d(\omega, \omega') + d(\omega', \omega'') \\ &\leq \mathrm{dist}_{\mathcal{W}}(\omega, \omega') + \mathrm{dist}_{\mathcal{W}}(\omega', \omega''). \end{aligned}$$

Otherwise $(\omega, \omega'') \notin \mathcal{W}$; then $(\omega, \omega') \notin \mathcal{W}$ or $(\omega', \omega'') \notin \mathcal{W}$: in both cases we have

$$+\infty = \mathrm{dist}_{\mathcal{W}}(\omega, \omega'') = \mathrm{dist}_{\mathcal{W}}(\omega, \omega') + \mathrm{dist}_{\mathcal{W}}(\omega', \omega''). \quad \square$$

Definition 3.3 (*The Topology* $\tau_{\mathcal{W}}$)**.** Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d). If $\omega \in \Pi$ and $r > 0$ we shall denote by $B_{\mathcal{W}}(\omega, r[$ the open ball of center $\omega \in \Pi$ and radius r with respect to $\mathrm{dist}_{\mathcal{W}}$:

$$\begin{aligned} B_{\mathcal{W}}(\omega, r[:&= \{\omega' \in \Pi \,:\, \mathrm{dist}_{\mathcal{W}}(\omega, \omega') < r\} \\ &= \{\omega' \in \Pi \,:\, (\omega, \omega') \in \mathcal{W},\, d(\omega, \omega') < r\}. \end{aligned}$$

The topology $\tau_{\mathcal{W}}$ is the topology induced on Π by the metric $\mathrm{dist}_{\mathcal{W}}$.

The next proposition tells us that the relations $\mathcal{W} \mapsto \tau_{\mathcal{W}}$ and $\mathcal{W} \mapsto \mathrm{dist}_{\mathcal{W}}$ are decreasing with respect to the natural orders.

Proposition 3.4 (*Inclusions Between Topologies and Metrics*)**.** *Let $\mathcal{W}_1 \subseteq \mathcal{W}_2$ be two equivalence relations in a metric space (Π, d). Then*

$$\mathrm{dist}_{\mathcal{W}_2} \leq \mathrm{dist}_{\mathcal{W}_1}, \quad \tau_{\mathcal{W}_2} \subseteq \tau_{\mathcal{W}_1}.$$

Proof. If $\omega, \omega' \in \Pi$ then

$$\mathrm{dist}_{\mathcal{W}_1}(\omega, \omega') = \begin{cases} d(\omega, \omega') = \mathrm{dist}_{\mathcal{W}_2}(\omega, \omega') & \text{if } \omega \underset{\mathcal{W}_1}{\sim} \omega', \\ +\infty \geq \mathrm{dist}_{\mathcal{W}_2}(\omega, \omega') & \text{otherwise.} \end{cases}$$

Thus in any case we obtain that $\mathrm{dist}_{\mathcal{W}_1}(\omega, \omega') \geq \mathrm{dist}_{\mathcal{W}_2}(\omega, \omega')$. The topological inclusion $\tau_{\mathcal{W}_2} \subseteq \tau_{\mathcal{W}_1}$ follows easily from the fact that, for all $\omega \in \Pi$ and $r > 0$, we have

$$B_{\mathcal{W}_1}(\omega, r[\subseteq B_{\mathcal{W}_2}(\omega, r[. \quad \square$$

Definition 3.5. Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d) and $A \subseteq \Pi$. If $\omega \in \Pi$ we set (using the convention that '$\inf \emptyset = +\infty$'):

$$\begin{aligned} \mathrm{dist}_{\mathcal{W}}(\omega, A) &= \inf\{\mathrm{dist}_{\mathcal{W}}(\omega, \omega') \,:\, \omega' \in A\} \\ &= \inf\{d(\omega, \omega') \,:\, \omega' \underset{\mathcal{W}}{\sim} \omega,\, \omega' \in A\}. \end{aligned}$$

Lemma 3.6. *Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d) and $A \subseteq \Pi$. Then, if $(\omega, \omega') \in \mathcal{W}$,*

$$\mathrm{dist}_{\mathcal{W}}(\omega, A) \le d(\omega, \omega') + \mathrm{dist}_{\mathcal{W}}(\omega', A). \tag{3.2}$$

Proof. Let $\omega'' \in A$. It follows from the triangular inequality that

$$\begin{aligned}\mathrm{dist}_{\mathcal{W}}(\omega, A) &\le \mathrm{dist}_{\mathcal{W}}(\omega, \omega'')\\ &\le \mathrm{dist}_{\mathcal{W}}(\omega, \omega') + \mathrm{dist}_{\mathcal{W}}(\omega', \omega'').\end{aligned}$$

Since $\omega \underset{\mathcal{W}}{\sim} \omega'$ we have $\mathrm{dist}_{\mathcal{W}}(\omega, \omega') = d(\omega, \omega')$. Therefore we obtain

$$\mathrm{dist}_{\mathcal{W}}(\omega, A) \le d(\omega, \omega') + \mathrm{dist}_{\mathcal{W}}(\omega', \omega'').$$

The conclusion follows. □

3.2. Admissible topologies on $\mathbb{R}^n \times \mathbb{R}^n$

To derive the main result of the paper we will consider different topologies on $\mathbb{R}^n \times \mathbb{R}^n$ built up from the Euclidean distance and a suitable equivalence relation subclasses.

Definition 3.7 (*Admissible Topologies*)**.** Let $\mathcal{W}$ be an equivalence relation on $\mathbb{R}^n \times \mathbb{R}^n$. The topology $\tau_{\mathcal{W}}$ will be called **admissible** whenever $\mathcal{W} \supseteq \mathcal{W}_0$, where

$$\mathcal{W}_0 := \{((y, \lambda_1 v), (y, \lambda_2 v)) : y \in \mathbb{R}^n, v \in \mathbb{R}^n, \lambda_1 > 0, \lambda_2 > 0\}.$$

We shall denote by $\mathcal{W}_e$ the maximal equivalence relation $(\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$, in which case we obtain the *Euclidean topology* τ_e: $\tau_{\mathcal{W}_e} = \tau_e$.

Remark 3.8. It follows from Proposition 3.4 that, if $\mathcal{W}$ is an equivalence relation in $\mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{W}_0 \subseteq \mathcal{W} \subseteq \mathcal{W}_e$, then it turns out that

$$\tau_e \subseteq \tau_{\mathcal{W}} \subseteq \tau_{\mathcal{W}_0}.$$

In particular any admissible (in the sense of Definition 3.7) topology $\tau_{\mathcal{W}}$ is finer than the Euclidean one.

Example 3.9 (*The topology of the y-sections*)**.** Among the admissible topologies on $\mathbb{R}^n \times \mathbb{R}^n$, by taking

$$\mathcal{W} = \mathcal{W}_* = \{((y, v), (y, v')) : y \in \mathbb{R}^n, v, v' \in \mathbb{R}^n\},$$

we obtain the one whose sets $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$ are open for $\tau_{\mathcal{W}}$ if and only if for all $y \in \mathbb{R}^n$ the sections

$$A(y) = \{v \in \mathbb{R}^n : (y, v) \in A\}$$

are open in $\mathbb{R}^n$.

The next result motivates the requirement that we consider just equivalence classes on $\mathbb{R}^n \times \mathbb{R}^n$ containing $\mathcal{W}_0$.

Proposition 3.10. *Let $\mathcal{W} \supseteq \mathcal{W}_0$ be an equivalence relation on $\mathbb{R}^n \times \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$, $(y, v) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\mathrm{dist}_{\mathcal{W}}((y, v), A) \ge \rho > 0$. Then, if $r \ge 0$,*

$$\mathrm{dist}_{\mathcal{W}}\big((y, v + rv), A\big) \ge \rho - r|v|.$$

Proof. Let $\omega = (y, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Notice that, if $\omega' \in \mathbb{R}^n \times \mathbb{R}^n$ is such that $\omega \underset{\mathcal{W}}{\sim} \omega'$ then, from (3.2) we deduce that

$$\mathrm{dist}_{\mathcal{W}}(\omega, A) \le |\omega - \omega'| + \mathrm{dist}_{\mathcal{W}}(\omega', A),$$

whence

$$\begin{aligned}\mathrm{dist}_{\mathcal{W}}(\omega', A) &\ge \mathrm{dist}_{\mathcal{W}}(\omega, A) - |\omega - \omega'|\\ &\ge \rho - |\omega - \omega'|.\end{aligned}$$

The conclusion follows immediately by setting $\omega' = (y, v + rv)$, recalling that $(y, v + rv) \underset{\mathcal{W}_0}{\sim} (y, v)$ and $\mathcal{W}_0 \subseteq \mathcal{W}$. □

3.3. Well-inside subsets

If D is a subset of a space Π we denote by D^c its complement in Π.

Definition 3.11 (*The Family $M_{\mathcal{W}}(D)$ and the Notation $\Subset_{\mathcal{W}}$*). Let $\mathcal{W}$ be an equivalence relation on a metric space (Π, d) and let $D \subseteq \Pi$. We say that a subset A of D is *well-inside* D with respect to $\mathrm{dist}_{\mathcal{W}}$ (briefly $A \Subset_{\mathcal{W}} D$) if there is $\rho > 0$ satisfying

$$A \subseteq \{\omega \in D : \mathrm{dist}_{\mathcal{W}}(\omega, D^c) \geq \rho\}. \tag{3.3}$$

We shall denote by $M_{\mathcal{W}}(D)$ the family of subsets of Π that are well-inside D for $\mathrm{dist}_{\mathcal{W}}$.

The family of sets that are well-inside a give subset $D \subseteq \Pi$ for $\mathrm{dist}_{\mathcal{W}}$ decreases as the equivalence relation set $\mathcal{W}$ increases.

Proposition 3.12. *Let (Π, d) be a metric space and $\mathcal{W}_1 \subseteq \mathcal{W}_2$ be two equivalence relations in Π. Then:*

1. *Let $A \subseteq \Pi$. For all $\omega \in \Pi$, $\mathrm{dist}_{\mathcal{W}_2}(\omega, A) \leq \mathrm{dist}_{\mathcal{W}_1}(\omega, A)$;*
2. *Let $D \subseteq \Pi$. Then $M_{\mathcal{W}_2}(D) \subseteq M_{\mathcal{W}_1}(D)$.*

Proof. Let $\omega' \in A$. It follows from Proposition 3.4 that

$$\mathrm{dist}_{\mathcal{W}_2}(\omega, A) \leq \mathrm{dist}_{\mathcal{W}_2}(\omega, \omega') \leq \mathrm{dist}_{\mathcal{W}_1}(\omega, \omega'),$$

whence Claim 1, from which it follows immediately also Claim 2. □

Remark 3.13. In view of Proposition 3.12, if $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{W} \supseteq \mathcal{W}_0$ is an equivalence relation on $\mathbb{R}^n \times \mathbb{R}^n$, then for any $\omega \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\mathrm{dist}_e(\omega, A) \leq \mathrm{dist}_{\mathcal{W}}(\omega, A) \leq \mathrm{dist}_{\mathcal{W}_0}(\omega, A), \tag{3.4}$$

and, for any given $D \subseteq \mathbb{R}^n$ we have

$$M_{\mathcal{W}_e}(D) \subseteq M_{\mathcal{W}}(D) \subseteq M_{\mathcal{W}_0}(D). \tag{3.5}$$

Observe that inequalities of (3.4) and the inclusions of (3.5) might be strict. This is illustrated by the following examples (see also Fig. 3).

Example 3.14. Suppose, for instance, that $n = 1$ and

$$A = (\{0\} \times \{1\}) \cup ((\mathbb{R} \setminus \{0\}) \times \{0\}).$$

Then $\mathrm{dist}_e((0,0), A) = 0$ whereas

$$\mathrm{dist}_{\mathcal{W}_0}((0,0), A) = \inf\{\lambda > 0 : (0, \lambda \times 0) \in A\} = +\infty.$$

Inclusion (3.5) may be strict, in general.

Example 3.15. We consider here $\mathbb{R}^n \times \mathbb{R}^n$ endowed with the Euclidean distance.

1. If $\mathcal{W} = \mathcal{W}_e$, then $\tau_{\mathcal{W}} = \tau_e$ and Condition (3.3) means that for every $(y, v) \in A$ we have

$$\forall (y', v') \in D^c \qquad |(y, v) - (y', v')| \geq \rho.$$

Therefore A is well-inside D for dist_e if A has a strictly positive Euclidean distance from D^c in $\mathbb{R}^n \times \mathbb{R}^n$. For instance, if $n = 1$ and $D =]-2, 2[\times]-2, 2[$ then $A := [-1, 1]\times[-1, 1]$ is well-inside D with respect to dist_e. Observe that $B :=]-2, 2[\times[-1, 1]$ is well-inside D w.r.t. $\mathrm{dist}_{\mathcal{W}_0}$, but it is not well-inside D for dist_e.

2. On the other hand, if $\mathcal{W} = \mathcal{W}_0$, Condition (3.3) means that for every $(y, \lambda_1 v) \in A \subset \mathbb{R}^n \times \mathbb{R}^n$ with $|v_1| = 1$, $\lambda_1 > 0$, we have

$$\forall (y, \lambda_2 v) \in D^c_\Lambda,\ \lambda_2 > 0 \qquad \lambda_2 \geq \lambda_1 + \rho \text{ or } \lambda_2 \leq \lambda_1 - \rho.$$

Consider a Lagrangian Λ whose domain D_Λ satisfies the Structure Assumption. Since $D_\Lambda(y)$ is strictly star-shaped with respect to 0, then $A \Subset_{\mathcal{W}_0} D_\Lambda$ if and only if there is $\rho > 0$ such that, for all $(y, \lambda v) \in A$ with $\lambda > 0$ and $|v| = 1$,

$$\sup\{\lambda_1 > 0 : (y, \lambda_1 v) \in A\} + \rho \leq \inf\{\lambda_2 > 0 : (y, \lambda_2 v) \in D^c_\Lambda\}.$$

If $n = 1$, the above means that, for all y in the first projection of A,

$$\sup(A(y) \cap \mathbb{R}^+) + \rho \leq \inf(D^c_\Lambda(y) \cap \mathbb{R}^+),$$

$$\inf(A(y) \cap \mathbb{R}^-) - \rho \leq \sup(D^c_\Lambda(y) \cap \mathbb{R}^-).$$

Example 3.16. Consider the equivalence relation $\mathcal{W}_*$ defined in Example 3.9. Let

$$D := \{(y, v) \in \mathbb{R}^2 : |v| \leq |y| \leq 1\}.$$

Then the origin is well-inside D for $\tau_{\mathcal{W}_0}$ but not for $\tau_{\mathcal{W}_*}$. Indeed, $\mathrm{dist}_{\mathcal{W}_0}((0,0), (y, v)) = +\infty$ for all $y, v \in \mathbb{R}$ with $v \neq 0$, however the origin is a limit point of D^c for $\tau_{\mathcal{W}_*}$ (which coincides with the Euclidean one along the v-axis).

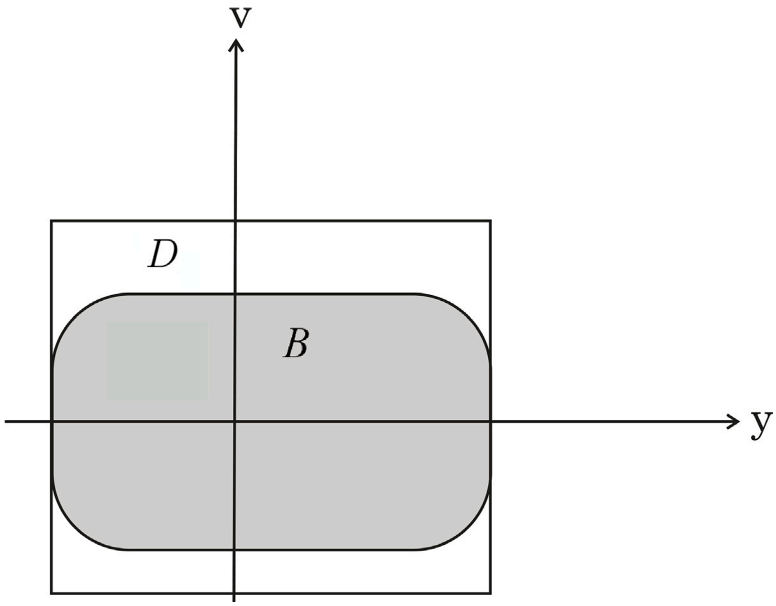

Fig. 3. The set B is well-inside D for $\mathrm{dist}_{\mathcal{W}_0}$ but not for dist_e.

4. Boundedness properties of the DBR type subgradient

Let $P(s,y,v)$ be a DBR type subgradient for Λ at $(s,y,v) \in \mathrm{Dom}(\Lambda)$, where Λ satisfies the Basic, Structure and Radial Convexity assumptions. The proof of some recent results on the non-occurrence of the Lavrentiev phenomenon (as [8, Theorem 5.4]) rely on Theorem 4.3 below, which is a refinement of [12, Lemma 3.17], [9, Proposition 4.24]. The monotonicity of the convex subgradients obviously implies that

$$\sup_{\substack{r\geq 1\\ (y,rv)\in D_\Lambda}} P(s,y,rv) \leq P(s,y,v) < +\infty, \tag{4.1}$$

$$\inf_{\substack{r\leq 1\\ (y,rv)\in D_\Lambda}} P(s,y,rv) \geq P(s,y,v) > -\infty. \tag{4.2}$$

Under some suitable local boundedness assumptions on Λ these estimates become somewhat uniform as v varies, respectively, out of a ball and inside a ball of given radii. One difficulty is that $P(s,y,v)$ may be unbounded, for instance it can tend to $-\infty$ when (s,y,v) approaches $\mathrm{Dom}(\Lambda)^c$. We shall prove that the validity of a uniform estimate as in (4.1)–(4.2) actually holds for points that are well-inside $\mathrm{Dom}(\Lambda)$.

Definition 4.1 (*Sets Enclosing the Origin*). We say that $A \subseteq \mathbb{R}^n$ encloses the origin if (see Fig. 4)

- There is $r_A > 0$ such that
  $$A \subseteq B_{r_A};$$
- For all $x \in \partial B_1$ there is $y \in A$ such that $x = \dfrac{y}{|y|}$ or, equivalently, every radius from the origin intersects A in at least one point.

Example 4.2. A typical set enclosing the origin is a sphere ∂B_{r_A} centered in the origin and of radius $r_A > 0$.

Theorem 4.3 (*Boundedness of the DBR Type Subgradient*). *Suppose that Λ satisfies (A1) of the Basic Assumptions, the Structure and Radial Convexity assumptions. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a bounded set and let $\mathcal{W} \supseteq \mathcal{W}_0$ be an equivalence relation on $\mathbb{R}^n \times \mathbb{R}^n$. Let, for any $(s,y,v) \in \mathrm{Dom}(\Lambda)$, $v \in \mathcal{V}$, $P(s,y,v)$ be a DBR type subgradient of Λ at (s,y,v), i.e.,*

$$P(s,y,v) \in \partial_\mu\Big(\Lambda\Big(s,y,\frac{v}{\mu}\Big)\mu\Big)_{\mu=1}.$$

(i) Assume that Λ is bounded from below and, moreover, that there exists a set $A_{\mathcal{K}}$ enclosing the origin such that Λ is bounded on

$$([a,b]\times\mathcal{K}\times(A_{\mathcal{K}}\cap\mathcal{V}))\cap\mathrm{Dom}(\Lambda).$$

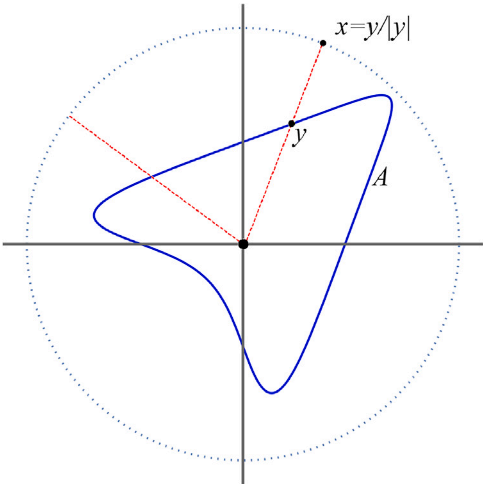

Fig. 4. A curve enclosing the origin in $\mathbb{R}^2$ (Definition 4.1).

Then, for all $r > r_{\mathcal{K}} := \sup\{|v| \,:\, v \in A_{\mathcal{K}} \cap \mathcal{U}\}$,

$$\sup_{\substack{s\in[a,b],y\in\mathcal{K},|v|\geq r\\(y,v)\in D_\Lambda\\ v\in\mathcal{U}}} P(s,y,v) < +\infty. \tag{4.3}$$

(ii) Suppose that there exists $\lambda_{\mathcal{K}} > 0$ *such that* Λ *is bounded on the subsets of the form* $[a,b]\times A$*, where* $A \subseteq \mathcal{K}\times(B_{\lambda_{\mathcal{K}}}\cap\mathcal{U})$, $A \Subset_{\mathcal{W}} D_\Lambda$*. Then, for every* $0<\lambda<\lambda_{\mathcal{K}}$*,*

$$\forall\rho>0 \qquad -\infty < \inf_{\substack{s\in[a,b],y\in\mathcal{K},|v|\leq\lambda\\ \mathrm{dist}_{\mathcal{W}}((y,v),D^c_\Lambda)\geq\rho\\ v\in\mathcal{U}}} P(s,y,v).$$

The proof of Theorem 4.3 is inspired by [17, Lemma 4.18, Proposition 4.24] and [12, Proposition 3.15]. The new distance depending on the equivalence relation $\mathcal{W}$ was not considered before, and some new arguments are presented here.

Proof. (i) Let $r > r_{\mathcal{K}}$. If the set

$$\{(s,y,v)\in\mathrm{Dom}(\Lambda)\,:\, s\in[a,b], y\in\mathcal{K},\, v\in\mathcal{U},\, |v|\geq r\}$$

is empty, then there is nothing to prove. Otherwise, let $(s,y,v)\in\mathrm{Dom}(\Lambda)$ with $s\in[a,b], y\in\mathcal{K},\, v\in\mathcal{U},\, |v|\geq r$. Since $A_{\mathcal{K}}$ encloses the origin there is $\mu_v \geq \frac{r}{r_{\mathcal{K}}}(>1)$ such that $\frac{v}{\mu_v}\in A_{\mathcal{K}}$. By applying (2.3),

$$\Lambda\Big(s,y,\frac{v}{\mu_v}\Big)\mu_v - \Lambda(s,y,v) \geq P(s,y,v)\left(\mu_v-1\right).$$

Assuming that $\beta\in\mathbb{R}$ minorizes Λ, we deduce that

$$|\beta| + \Lambda\Big(s,y,\frac{v}{\mu_v}\Big)\mu_v \geq P(s,y,v)\left(\mu_v-1\right)$$

whence

$$\begin{aligned} P(s,y,v) &\leq \Lambda\Big(s,y,\frac{v}{\mu_v}\Big)\frac{\mu_v}{\mu_v-1} + \frac{|\beta|}{\mu_v-1}\\ &\leq \Lambda\Big(s,y,\frac{v}{\mu_v}\Big)\frac{\mu_v}{\mu_v-1} + \frac{|\beta|\, r_{\mathcal{K}}}{r-r_{\mathcal{K}}}. \end{aligned}$$

Since $\frac{v}{\mu_v}\in A_{\mathcal{K}}$ the assumptions imply that $\Lambda\Big(s,y,\frac{v}{\mu_v}\Big)$ is bounded above by a constant depending only on $\mathcal{K}$. Moreover, $\mu_v \geq \frac{r}{r_{\mathcal{K}}}$ and $\sigma\mapsto\frac{\sigma}{\sigma-1}$ is bounded in $\Big[\frac{r}{r_{\mathcal{K}}},+\infty\Big[$; the required conclusion follows. (ii) Let $\rho>0$ and let A be the subset of D_Λ defined by

$$A := \{(y,v)\in D_\Lambda \,:\, y\in\mathcal{K},\, v\in\mathcal{U},\, |v|\leq\lambda, \mathrm{dist}_{\mathcal{W}}((y,v),D^c_\Lambda)\geq\rho\}.$$

It is not restrictive to assume that $\rho\leq 2(\lambda_{\mathcal{K}}-\lambda)$. If A is empty the claim is obvious. Otherwise, let $(y,v)\in A$ and $P(s,y,v)\in\partial_\mu\left(\Lambda\Big(s,y,\frac{v}{\mu}\Big)\mu\right)_{\mu=1}$. Notice that

$$\left|\frac{\rho}{2\lambda_{\mathcal{K}}}v\right| \leq \frac{\rho}{2}\leq\lambda_{\mathcal{K}}-\lambda, \qquad \left|v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right|\leq\lambda+(\lambda_{\mathcal{K}}-\lambda)\leq\lambda_{\mathcal{K}}.$$

It follows from Proposition 3.10 that

$$\mathrm{dist}_{\mathcal{W}}\left(\left(y, v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right), D^c_\Lambda\right) \geq \rho - \frac{\rho}{2\lambda_{\mathcal{K}}}|v| \geq \frac{\rho}{2}.$$

Therefore $\left(y, v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right) \in A' \Subset_{\mathcal{W}} D_\Lambda$, where

$$A' := \left\{(y,w)\in D_\Lambda \,:\, y\in\mathcal{K},\, w\in\mathcal{U},\, |w|\leq\lambda_{\mathcal{K}}, \mathrm{dist}_{\mathcal{W}}((y,w),D^c_\Lambda)\geq\frac{\rho}{2}\right\}.$$

The boundedness assumption implies that there exists $M\in\mathbb{R}$ such that

$$|\Lambda(s,y,w)|\leq M, \qquad \text{for all } (s,y,w)\in[a,b]\times(A'\cup A). \tag{4.4}$$

Now, from (2.3) we have

$$\Lambda\left(s,y,v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right)\frac{1}{1+\frac{\rho}{2\lambda_{\mathcal{K}}}} - \Lambda(s,y,v) \geq P(s,y,v)\left(\frac{1}{1+\frac{\rho}{2\lambda_{\mathcal{K}}}}-1\right),$$

so that

$$M + \Lambda\left(s,y,v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right)\frac{1}{1+\frac{\rho}{2\lambda_{\mathcal{K}}}} \geq P(s,y,v)\left(\frac{1}{1+\frac{\rho}{2\lambda_{\mathcal{K}}}}-1\right),$$

whence

$$\begin{aligned} P(s,y,v) &\geq -\frac{2\lambda_{\mathcal{K}}}{\rho}\Lambda\left(s,y,v+\frac{\rho}{2\lambda_{\mathcal{K}}}v\right) - M\frac{2\lambda_{\mathcal{K}}+\rho}{\rho} \\ &\geq -\frac{M}{\rho}\left(4\lambda_{\mathcal{K}}+\rho\right), \end{aligned}$$

recalling (4.4). We deduce that $P(s,y,v)$ is bounded from below on A by a constant depending only on β, M, $\lambda_{\mathcal{K}}$ and ρ. This confirms Claim *(*ii). □

Remark 4.4. Taking into account the geometric interpretation of the subdifferentials given in Remark 2.5:

- Condition *(i)* of Theorem 4.3 means that the intersection with the ordinate axis of the tangent lines to $r\mapsto\Lambda(s,y,rv)$ at $r=1$ are bounded above by a constant if $(s,y,v)\in\mathrm{Dom}(\Lambda)$, $v\in\mathcal{U}$, $|v|\geq r$. In the smooth case Condition (4.3) can be rewritten as

$$\sup_{\substack{s\in[a,b],y\in\mathcal{K},|v|\geq r\\(y,v)\in D_\Lambda\\ v\in\mathcal{U}}} \Lambda(s,y,v) - v\cdot\nabla_v\Lambda(s,y,v) < +\infty.$$

- Similarly, Condition *(*ii) in Theorem 4.3 geometrically means that the intersection with the ordinate axis of the tangent lines to $r\mapsto\Lambda(s,z,rv)$ at $r=1$ are bounded below by a constant when $\mathrm{dist}_{\mathcal{W}}((z,v),D^c_\Lambda)\geq\rho$, $z\in\mathcal{K}$, and $v\in\mathcal{U}$ with $|v|\leq\lambda$. In the smooth case Condition (ii) can be rewritten as

$$-\infty < \inf_{\substack{s\in[a,b],z\in\mathcal{K},|v|\leq\lambda\\ \mathrm{dist}_{\mathcal{W}}((z,v),D^c_\Lambda)\geq\rho\\ v\in\mathcal{U}}} \Lambda(s,z,v) - v\cdot\nabla_v\Lambda(s,z,v).$$

Remark 4.5. The topologies $\tau_{\mathcal{W}}$ involved here play a role in two conditions that, together, ensure the non-occurrence of the Lavrentiev phenomenon in [8]: the fact that D_Λ is open and the fact that Λ is bounded on the sets $[a,b]\times\Sigma$ where $\Sigma\Subset_{\mathcal{W}} D_\Lambda$. Now the first is more easily satisfied with a finer topology, whereas the opposite is true for the second: this motivates the need of statements, like Theorem 4.3, that are valid for admissible topologies on $\mathbb{R}^n\times\mathbb{R}^n$.

5. Non-ocurrence of the lavrentiev phenomenon

5.1. The case of one initial endpoint constraint

As an application of Theorem 4.3 we give a self-contained proof of the non-occurrence of the Lavrentiev gap for the initial endpoint problem, a particular case of [8, Theorem 4.1]. We denote by $W^{1,1}([a,b])$ the space of absolutely continuous functions on $[a,b]$, by $\mathrm{Lip}([a,b])$ the space of Lipschitz continuous functions on $[a,b]$.

Proposition 5.1 (*[8]*). *Let $\Lambda=\Lambda(y,v)\geq 0$ be autonomous, satisfies the Basic, Structure and Radial Convexity assumptions and $\mathcal{U}$ is Borel measurable. Let y be an admissible absolutely continuous arc for (P) such that $F(y)<+\infty$. Suppose that there is a subset $A\subset\mathbb{R}^n$ enclosing the origin such that Λ is bounded on*

$$(y([a,b])\times(A\cap\mathcal{U}))\cap\mathrm{Dom}(\Lambda).$$

Then there is ***no Lavrentiev gap*** *for F at y with a prescribed initial endpoint, i.e. we can find a sequence $(y_k)_k$ of Lipschitz functions such that:*

1. *For all* $k \in \mathbb{N}$, $y_k(a) = y(a)$;
2. $y_k \to y$ *in the* L^∞ *norm and* $y'_k \to y'$ *in the* L^1 *norm;*
3. $\limsup_k F(y_k) \leq F(y)$.

Moreover $y_k([a,b]) \subseteq y([a,b])$. *As a consequence, if* $\Delta \subseteq \mathbb{R}^n$, *there is* ***no Lavrentiev phenomenon*** *for* F ***with a prescribed initial endpoint and state constraint*** Δ, *i.e.,*

$$\inf\{F(y):\ y \in W^{1,1}([a,b]),\ y(a) = A \in \mathbb{R}^n,\ y([a,b]) \subseteq \Delta,\ y'(t) \in \mathcal{U} \text{ a. e. } t \in [a,b]\}$$
$$= \inf\{F(y):\ y \in \mathrm{Lip}([a,b]),\ y(a) = A \in \mathbb{R}^n,\ y([a,b]) \subseteq \Delta,\ y'(t) \in \mathcal{U} \text{ a. e. } t \in [a,b]\}.$$

Let us point out that the claim of Proposition 5.1 is by no far obvious, as shown by the following example, a modification of the one given in [18], where the Lagrangian could not be simpler since it equals to 0 in its effective domain.

Example 5.2. Let

$$\Lambda(y,v) = \begin{cases} 0 & \text{if } v = \dfrac{1}{2y}, y \neq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Let $y_*(s) = \sqrt{s}$, $s \in [0,1]$. Since $y'_* = \dfrac{1}{2y_*}$ a.e. in $[0,1]$ we have $F(y_*) = 0$ so that y_* minimizes F among the absolutely continuous functions y on $[0,1]$ satisfying $y(0) = 0$. Now let y be Lipschitz in $[0,1]$ such that $y(0) = 0$ and $F(y) < +\infty$. Then $\Lambda(y,y') < +\infty$ a.e. in $[0,1]$ so that

$$\begin{cases} y' = \dfrac{1}{2y} \text{ a.e. in } [0,1] \\ y(0) = 0. \end{cases}$$

It follows that $y(s) = \sqrt{s}$, a contradiction. Thus $F(y) = +\infty$ whenever y is Lipschitz and $y(0) = 0$. What fails here, among the assumptions of Proposition 5.1, is the fact that the projections of the effective domain onto the v variable are either empty or a non-zero singleton and, so, they are not in general star-shaped with respect to 0.

Proof of Proposition 5.1.

For each $k = 1, \dots,$ let

$$S_k := \{s \in [a,b]:\ |y'(s)| > k\}.$$

Define the absolutely continuous function $\varphi_k : [a,b] \to \mathbb{R}$ by

$$\varphi_k(a) = a, \quad \varphi'_k(s) = \begin{cases} \dfrac{|y'(s)|}{k} & s \in S_k, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\varphi'_k \geq 1$ a.e. on $[a,b]$ then $\varphi_k([a,b]) = [a,b_k]$ for some $b_k \geq b$. Let ψ_k be the inverse of φ_k restricted to $[a,b]$ (a Lipschitz function) and define

$$y_k := y \circ \psi_k.$$

Then $y_k(a) = y(\psi_k(a)) = y(a)$, $y_k([a,b]) \subseteq y([a,b])$ and y_k is Lipschitz: indeed

$$y'_k(\tau) = \begin{cases} k\dfrac{y'(\psi_k(\tau))}{|y'(\psi_k(\tau))|} & \tau \in \varphi_k(S_k) \cap [a,b], \\ y'(\psi_k(\tau)) & \text{otherwise.} \end{cases}$$

Moreover, $y'_k(\tau) \in \mathcal{U}$ a.e. $\tau \in [a,b]$. The convergence of y_k to y in the norm of the absolutely continuous functions follows from standard arguments. Let us check the limsup property 3. The change of variables $s = \psi_k(\tau)$ gives

$$F(y_k) = \int_a^b \Lambda\Big(y(s), \frac{y'(s)}{\varphi'_k(s)}\Big)\varphi'_k(s)\,ds.$$

Let $P(y,v)$ be a Borel selection of $\partial_\mu \left[\Lambda\Big(y, \dfrac{v}{\mu}\Big)\mu\right]_{\mu=1}$ on $\mathrm{Dom}(\Lambda)$ with $v \in \mathcal{U}$, whose existence is established in Proposition 2.6. Notice that the fact that the projections onto the v space are star-shaped and $\mathcal{U}$ is a cone, we have

$$\Big(y(s), \frac{y'(s)}{\varphi'_k(s)}\Big) \in \mathrm{Dom}(\Lambda) \quad \text{and} \quad \frac{y'(s)}{\varphi'_k(s)} \in \mathcal{U} \qquad \text{a. e. in } [a,b].$$

Let $s \in [a,b]$; the subgradient inequality (2.3) applied with $y = y(s)$, $v = \dfrac{y'(s)}{\varphi_k'(s)}$ and $\mu = \dfrac{1}{\varphi_k'(s)}$ gives

$$\Lambda(y(s), y'(s))\frac{1}{\varphi_k'(s)} - \Lambda\Big(y(s), \frac{y'(s)}{\varphi_k'(s)}\Big) \geq P\Big(y(s), \frac{y'(s)}{\varphi_k'(s)}\Big)\Big(\frac{1}{\varphi_k'(s)} - 1\Big),$$

from which we deduce that

$$\Lambda\Big(y(s), \frac{y'(s)}{\varphi_k'(s)}\Big)\varphi_k'(s) \leq \Lambda(y(s), y'(s)) + P\Big(y(s), \frac{y'(s)}{\varphi_k'(s)}\Big)(\varphi_k'(s) - 1). \tag{5.1}$$

Since $\varphi_k' \geq 1$ and $\varphi_k' = 1$ on $[a,b] \setminus S_k$, integrating over $[a,b]$ both sides of (5.1), we obtain

$$F(y_k) \leq F(y) + \int_{S_k} P\Big(y(s), k\frac{y'(s)}{|y'(s)|}\Big)\Big(\frac{|y'(s)|}{k} - 1\Big)\, ds. \tag{5.2}$$

It follows from (i) of Theorem 4.3 with $\mathcal{K} = y([a,b])$ that, for $k \geq r > \sup\{|v| \,:\, v \in A \cap \mathcal{U}\}$,

$$\sup_{\substack{z\in y([a,b]),|v|\geq k\\(z,v)\in \mathrm{Dom}(\Lambda)\\ v\in\mathcal{U}}} P(z,v) \leq \sup_{\substack{z\in y([a,b]),|v|\geq r\\(z,v)\in \mathrm{Dom}(\Lambda)\\ v\in\mathcal{U}}} P(z,v) \leq C < +\infty.$$

It follows from (5.2) that

$$F(y_k) \leq F(y) + C\int_{S_k}\Big(\frac{|y'(s)|}{k} - 1\Big)\, ds.$$

Now

$$0 \leq \int_{S_k}\Big(\frac{|y'(s)|}{k} - 1\Big)\, ds \leq \frac{1}{k}\int_a^b |y'(s)|\, ds \to 0 \qquad k \to +\infty,$$

whence the claim. □

Remark 5.3. The proof of Proposition 5.1 shows that the range of y_k is contained in the range of y, so that the sequence y_k takes values in $y([a,b])$. The conditions formulated there are not sufficient, in general, to derive the non-occurrence of the gap at y with two prescribed two endpoints at a and b (see [8]), i.e., to obtain a sequence of Lipschitz functions $(y_k)_k$ satisfying both $y_k(a) = y(a)$ and $y_k(b) = y(b)$: referring to the proof of Proposition 5.1, one has to introduce a subset where $\varphi_k' < 1$ and use Claim 2 of Theorem 4.3; we refer to [8, Theorem 4.1] for the details.

5.2. A general result on the avoidance of the Lavrentiev phenomenon

We fix $\Delta \subseteq \mathbb{R}^n$ and a cone $\mathcal{U} \subseteq \mathbb{R}^n$. We refer to [8] for a thorough discussion on the subject; the aim of this section is to show how the use of different topologies on $\mathbb{R}^n \times \mathbb{R}^n$ may be useful in the study of the non-occurrence of the Lavrentiev phenomenon. In the nonautonomous case, it seems essential that $\Lambda(s,y,v)$ satisfies a condition on the first variable, otherwise some well-known counterexamples show the phenomenon may occur (see Manià's example [5] or Ball–Mizel [6]).

Condition (S$^+$). For every $K \geq 0$ there exist $\gamma \in]0,1]$, $\varepsilon_* > 0$ and:

- a Lebesgue–Borel measurable function $h(s,y,v)$ in $(s,(y,v))$ such that
$$\forall y \in W^{1,1}([a,b]),\ F(y) < +\infty \Rightarrow h(s,y(s),y'(s)) \in L^1([a,b]);$$
- a Borel function $g(y,v)$ such that $g(y(s),y'(s)) \in L^\infty([a,b])$ whenever $y \in W^{1,1}([a,b])$ and $F(y) < +\infty$;
- a bounded variation function η on $[a,b]$ with values in $\mathbb{R}$

such that, for a.e. $s \in [a,b]$, for all $s' \in [s-\varepsilon_*, s+\varepsilon_*] \cap [a,b]$, $y \in B_K \cap \Delta$, $v \in \mathcal{U}$, $(s,y,v) \in \mathrm{Dom}(\Lambda)$,

$$|\Lambda(s',y,v) - \Lambda(s,y,v)| \leq |s'-s|^\gamma h(s,y,v) + |\eta(s') - \eta(s)| g(y,v).$$

Example 5.4. Functions h, g that satisfy the above conditions are, for instance,

$$h(s,y,v) = A(\Lambda(s,y,v) + |v| + \rho(s)), \quad g \text{ Borel and bounded}$$

with $A \geq 0, \rho \in L^1([a,b])$.

Definition 5.5. Suppose that $\mathrm{Dom}(\Lambda) = [a,b] \times D_\Lambda$ for some $D_\Lambda \subseteq \mathbb{R}^n \times \mathbb{R}^n$.

1. We say that Λ satisfies the conditions for the non occurrence of the Lavrentiev phenomenon for the *initial endpoint* problem on $[a,b]$ if Λ satisfies the Basic, Structure and Radial Convexity assumptions, Condition (S$^+$) and:

(B$^\sigma_\Lambda$) For every compact subset $\mathcal{K}$ of Δ there is a set $A_{\mathcal{K}} > 0$ enclosing the origin such that Λ is bounded on $([a,b] \times \mathcal{K} \times (A_{\mathcal{K}} \cap \mathcal{U})) \cap \mathrm{Dom}(\Lambda)$.

2. We say that Λ satisfies the conditions for the non-occurrence of the Lavrentiev phenomenon for the *initial and final endpoints* problem on $[a,b]$ if Λ satisfies the Basic and Structure assumptions, Condition (S^+) and, in addition to (B^{σ}_{Λ}), there is an equivalence relation $\mathcal{W} \supseteq \mathcal{W}_0$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that:

   (a) $D_\Lambda \cap (\Delta \times \mathcal{V})$ is contained in the interior of D_Λ w.r.t. the topology $\tau_{\mathcal{W}}$;
   (b) The following condition holds:

($B^{\Subset_{\mathcal{W}}}_{\Lambda}$) For every compact subset $\mathcal{K}$ of Δ, Λ is bounded on the subsets of the form $[a,b] \times A$ where $A \subseteq \mathcal{K} \times \mathcal{V}$, $A \Subset_{\mathcal{W}} D_\Lambda$.

Remark 5.6. Notice that the admissible topologies play an essential role in Point 2 of Definition 5.5. It follows from Proposition 3.4 that 2(a) is more easily satisfied when the equivalence class $\mathcal{W}$ gets smaller, whereas in order for 2(b) to hold it is more convenient, from Proposition 3.12, to deal with bigger equivalence classes on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem 5.7. *Let $\Lambda : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Suppose that there are a subdivision $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ of $[a,b]$ and subsets $D^0, \dots, D^m$ of $\mathbb{R}^n \times \mathbb{R}^n$ such that*

$$\forall i = 0, \dots, m \quad \mathrm{Dom}(\Lambda) \cap (]t_i, t_{i+1}[\times \mathbb{R}^n \times \mathbb{R}^n) =]t_i, t_{i+1}[\times D^i.$$

Suppose that there is $I \subseteq \{0, \dots, m\}$ such that:

- *For every $i \notin I$ the projection of D^i onto the v-space $\mathbb{R}^n$ is bounded;*
- *For all $i \in I$, $i < m$, Λ satisfies the conditions for the non-occurrence of the Lavrentiev phenomenon for initial and final endpoints problem on $[t_i, t_{i+1}]$.*

Then:

1. *If $m \notin I$ or Λ satisfies the conditions for the non-occurrence of the Lavrentiev phenomenon for the initial and final endpoints problem on $[t_m, b]$ there is no Lavrentiev phenomenon for the initial and final endpoints problem on $[a,b]$ with state constraint Δ;*
2. *otherwise there is no Lavrentiev phenomenon for the initial endpoint problem on $[a,b]$ with state constraint Δ.*

Proof. Fix $k \geq 1$ and let $y \in W^{1,1}([a,b])$ with $F(y) \leq \inf F + \frac{1}{k}$. If $i \notin I$, the restriction of y to $[t_i, t_{i+1}]$ is Lipschitz: in this case we set $y_{i,k} := y$ on $[t_i, t_{i+1}]$. For any $i \in I, i < m$, [8, Corollary 5.13] ensures the existence of $y_{i,k} \in \mathrm{Lip}([t_i, t_{i+1}])$ satisfying

$$y_{i,k}(t_i) = y(t_i), \quad y_{i,k}(t_{i+1}) = y(t_{i+1}), \quad y_{i,k}([t_i, t_{i+1}]) \subseteq y([t_i, t_{i+1}]),$$

$$\int_{t_i}^{t_{i+1}} \Lambda(s, y_{i,k}(s), (y_{i,k})'(s))\, ds \leq \int_{t_i}^{t_{i+1}} \Lambda(s, y(s), y'(s))\, ds + \frac{1}{k}.$$

Similarly, if $m \in I$, on $[t_m, b]$ there is $y_{m,k} \in \mathrm{Lip}([t_m, b])$ satisfying

$$y_{m,k}(t_m) = y(t_m), \quad y_{m,k}([t_m, b]) \subseteq y([t_m, b]).$$

Let y_k be the Lipschitz function defined to be equal to $y_{i,k}$ on $[t_i, t_{i+1}] (i = 0, \dots, m)$. Then y_k satisfies the desired boundary conditions and constraints, moreover

$$F(y_k) \leq F(y) + \frac{m+1}{k} \leq \inf F + \frac{m+2}{k}.$$

The conclusion follows. □

Example 5.8. Let

$$\ell(v) = \begin{cases} e^v \text{ if } v \neq 0, \\ 1 \text{ if } v = 0, \end{cases} \qquad q(y) = \frac{1}{1-y}, y \in [0,1[.$$

Let, for all $y, v \in \mathbb{R}$,

$$L_0(y,v) = \begin{cases} \frac{1}{|y|} & (y,v) \in D_0 := \{(y,v) \in \mathbb{R}^2 : y \neq 0, |v| \leq 1\} \\ +\infty & \text{otherwise}, \end{cases}$$

$$L_1(y,v) = \begin{cases} \frac{1}{|y|} + v^2 & (y,v) \in D_1 := \{(y,v) \in \mathbb{R}^2 : y \neq 0\} \\ +\infty & \text{otherwise}, \end{cases}$$

$$L_2(y,v) = \begin{cases} 0 & (y,v) \in D_2 := \{(y,v) \in \mathbb{R}^2 : v < \ell(y)\} \\ +\infty & \text{otherwise}, \end{cases}$$

$$L_3(y,v) = \begin{cases} 0 & (y,v) \in D_3 := \{(y,v) \in \mathbb{R}^2 : y \in [0,1[, v \leq q(y)\} \\ +\infty & \text{otherwise} \end{cases}.$$

Notice that:

- The projection of D_0 onto the v-axis is bounded;
- The y-sections of D_1 are empty (if $y = 0$) or equal to $\mathbb{R}$, D_1 is open for the Euclidean topology in $\mathbb{R}^2$, L_1 is continuous on its effective domain D_1 and $L_1(y,\cdot)$ is convex for all y. In particular Λ_1 is bounded on the compact subsets of D_1.
- The y-sections of D_2 are open intervals containing 0, D_2 is not open for the Euclidean topology in $\mathbb{R}^2$, L_2 is equal to 0 on its effective domain D_2 and $L_2(y,\cdot)$ is convex for all y.
- The y-sections of D_3 are empty (if $y \notin [0,1[$) or equal to $]-\infty, q(y)]$ on which L_3 equals 0 and $L_3(y,\cdot)$ is convex. Notice that D_3 being not open in $\mathbb{R}\times\mathbb{R}$ for $\tau_{\mathcal{W}_0}$, it is not open for any other admissible topology.

Define $\Lambda : [0,4]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ by

$$\forall (s,y,v)\in[0,4]\times\mathbb{R}\times\mathbb{R}\quad \Lambda(s,y,v)=\begin{cases} L_0(y,v) & s\in]0,1[,\\ L_1(y,v) & s\in]1,2[,\\ L_2(y,v) & s\in]2,3[,\\ L_3(y,v) & s\in]3,4[.\end{cases}$$

Then Λ satisfies the conditions of Theorem 5.7 with $I=\{1,2,3\}$, $\Delta=\mathbb{R}\setminus\{0\}, \mathcal{V}=\mathbb{R}$ if we choose $\mathcal{W}=\mathcal{W}_e$ on $[1,2]$ and $\mathcal{W}=\mathcal{W}_0$ on $[2,3]$ (since the vertical sections are open intervals but D_2 is not open in $\mathbb{R}^2$); notice that Λ satisfies the conditions just for the initial endpoint problem on $[3,4]$. It follows that there is no Lavrentiev phenomenon for the prescribed initial endpoint problem with $\Delta=\mathbb{R}\setminus\{0\}$, $\mathcal{V}=\mathbb{R}$.

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Data availability

No data was used for the research described in the article.

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