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Exact in-plane stress field solution for isotropic plates with circular holes reinforced with cylindrically orthotropic rings

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ABSTRACT

In this paper an analytical solution for the stress distribution in an infinite plate with a concentrically reinforced hole under a general loading condition is derived. The developed solution explicitly accounts for the elastic properties of the plate and of the reinforcing annulus, for the hole radius to reinforcing ring thickness ratio, as well as the loading conditions, including loads applied onto the hole boundary and on the plate, far away from the hole. A careful validation of the proposed framework is carried out by comparing the newly developed solution with the results from a number of finite element analyses, documenting a very satisfactory agreement.

The solution derived represents a useful tool toward the understanding of the stress fields in polymeric components with reinforced holes and made with fused deposition modelling technologies.

1. Introduction

Geometrical variations can severely weaken the mechanical properties of industrial components, independently of the loading conditions and materials used. However, for design reasons, holes and notches are often unavoidable in real parts and, accordingly, explicitly accounting for stress concentrators in the design process is vital to ensure appropriate mechanical performances.

For this reason, many authors devoted efforts to derive stress distributions in the proximity of notches and holes under several loading conditions, both for isotropic and anisotropic materials (see, for example, [1–4] and references reported therein).

A possible strategy to improve the strength of mechanical components with holes is to reinforce them with stiff rings or inserts. The problem of finding the stress distributions around reinforced holes is that of considerable difficulty. Within this context, and in relation to isotropic plates, worth of mentioning is the work carried out by Savin, who developed an analytical solution for an elliptic hole reinforced by two infinitely rigid arc-shaped plates connected by two symmetrical braces [5], and Chao et al. [6], deriving the stress fields around an elliptic hole with an elliptic isotropic annulus as reinforcement. Moving to orthotropic solids, Lekhnitskii [7] studied the problem of the stress field in holes with infinitely rigid reinforcements, and many years later his framework was validated experimentally by Toubal et al. [8] taking advantage of ESPI technologies.

Belfield et al. [9] further focused on this problem and provided an analytical solution for concentrically highly anisotropic reinforcements around holes in infinite isotropic plates under certain loading conditions. Starting from these bases, several works were carried out to investigate the stress fields around reinforced geometrical variations in solids subjected to mechanical or thermal loads (see, among the others, Aluko and Whitworth [10], Pan et al. [11], Jafari and Jafari [12], Mohan and Kumar [13], and references reported therein).

On parallel tracks, the development of modern additive manufacturing technologies for long fiber composites gave rise to the possibility to generate geometries, and to place reinforcement material, without the constraints imposed by traditional technologies and opened new design possibilities worth to be investigated. For this reason, in the recent years much efforts have been devoted to the study and optimization of the fused deposition modelling technologies for long fiber composites (see Khosravani et al. [14] and references quoted therein), as well as to the fracture behavior of 3D printed specimens around geometrical discontinuities (Khosravan et al. [15] and reference quoted therein).

The possibility of strengthening and stress relieving the regions of composite components close to geometrical variations was also studied in various works. Among the others, Dave and Sharma [16] studied the effect of functionally graded laminas on the stress concentrations around rectangular holes in composite laminates, whilst Yang et al. [17] derived the stress fields in an isotropic plate with an elliptic hole reinforced with

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a functionally graded annulus.

Different from previous studies, in this work the attention is focused on the problem of the stress and displacement fields arising in an isotropic infinite plate with a circular hole reinforced with a polarly orthotropic material. The plate is supposed to be subjected to a generic loading condition, including far applied biaxial tension and shear loadings, as well as a general radial or shear stress applied onto the hole boundary.

In order to address this problem, initially, the analytical statement is formulated using, in combination, the Muskhelishvili complex potential approach for isotropic plates [18] and the solution provided by Spencer [19] for a polarly orthotropic annulus. Subsequently, the exact solution for the problem is obtained in closed form by applying appropriate boundary conditions and analytically solving the corresponding system. The derived explicit expressions for the displacement and stress fields account for the geometrical features and the elastic properties of the plate and of the reinforcing region. Relevant examples are, eventually, discussed, comparing the analytical solution with the results from a bulk of finite element analyses carried out on finite geometries under various loading conditions.

The solution developed in this work represents a useful tool toward the understanding of the stress fields in polymeric components with reinforced holes and made with fused deposition modelling technologies.

2. Analytical solution

2.1. Statement of the problem

Consider an infinite isotropic plate with a circular hole of radius r_0, reinforced with a concentric polarly orthotropic annulus of thickness s and outer radius r_1, with $s = r_1 - r_0$. The plate is subjected to radial and shear stresses in the hole boundary, generally given in the following Fourier series form:

$$\begin{aligned}\overline{\sigma}_{rr}(\theta) &= S_0 + \sum_{n=1}^{N}\left(S_{n,1}\cos(n\theta) + S_{n,2}\sin(n\theta)\right)\\ \overline{\sigma}_{r\theta}(\theta) &= T_0 + \sum_{n=1}^{N}\left(T_{n,1}\cos(n\theta) + T_{n,2}\sin(n\theta)\right)\end{aligned} \tag{1}$$

where $S_0, T_0, S_{ij}, T_{ij} \in \mathbb{R}\ \forall i : 1 \leqslant i \leqslant N \in \mathbb{N}; j \in \{1,2\}$.

Being based on a Fourier series expansion, Eq. (1) allows any loading condition to be described (see also Ref. [9]).

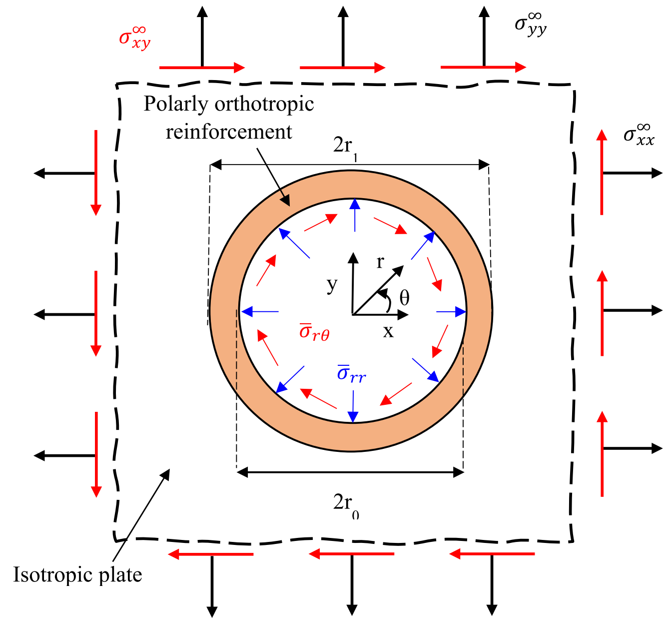

Fig. 1. Infinite plate with a reinforced circular hole under generic loading conditions.

Moreover, external far-field constant stresses σ_{xx}^{∞}, σ_{yy}^{∞}, σ_{xy}^{∞} are applied to the plate (see Fig. 1).

2.2. General form for the displacement and stress fields in the reinforcing region

Consider again the schematic in Fig. 1. In the (x,y) plane, the reinforcing region is modelled as a polarly orthotropic ring for which the radial and circumferential directions are principal directions of elasticity.

Accordingly, the stress and displacements fields in the reinforcing region can be sought in the same form used by Belfield et al. [9] for the problem of a concentric annulus. Thus, the following expressions for displacements and stresses can be invoked (see, again, the reference system shown in Fig. 1):

$$\begin{aligned}u_r^{(1)} = {} & A_0\left(\frac{r_0}{r}\right)^{\alpha_0} + a_0\left(\frac{r}{r_1}\right)^{\alpha_0} - E_1\left(\frac{1+d^{-2}}{1+\varsigma^{-2}}\right)cos\theta + (E_1cos\theta + F_1sin\theta)log\left(\frac{r}{r_0}\right) + \\ & + \sum_{n=1}^{N}\left((A_n\cos n\theta + B_n\sin n\theta)\left(\frac{r_0}{r}\right)^{\alpha_n} + (a_n\cos n\theta + b_n\sin n\theta)\left(\frac{r}{r_1}\right)^{\alpha_n}\right) + \\ & + \sum_{n=2}^{N}\left((E_n\cos n\theta + F_n\sin n\theta)\left(\frac{r_0}{r}\right)^{\beta_n} + (e_n\cos n\theta + f_n\sin n\theta)\left(\frac{r}{r_1}\right)^{\beta_n}\right)\end{aligned} \tag{2a}$$

$$\begin{aligned}u_\theta^{(1)} = {} & F_1\left(\frac{1+d^{-2}}{1+\varsigma^{-2}}\right)cos\theta - H_0\left(\frac{r_0}{r}\right) + (F_1cos\theta - E_1sin\theta)log\left(\frac{r}{r_0}\right) + \\ & + \sum_{n=1}^{N}\left((C_n\sin n\theta - D_n\cos n\theta)\left(\frac{r_0}{r}\right)^{\alpha_n} + (c_n\sin n\theta - d_n\cos n\theta)\left(\frac{r}{r_1}\right)^{\alpha_n}\right) + \\ & + \sum_{n=2}^{N}\left((G_n\sin n\theta - H_n\cos n\theta)\left(\frac{r_0}{r}\right)^{\beta_n} + (g_n\sin n\theta - h_n\cos n\theta)\left(\frac{r}{r_1}\right)^{\beta_n}\right)\end{aligned} \tag{2b}$$

$$
\begin{aligned}
\sigma_{rr}^{(1)} = \frac{G_{\theta r}^{(1)}}{rc^2d_1^2}\Bigg\{ & \left(\frac{r_0}{r}\right)^{\alpha_0} A_0\left(c^2 - d^2\alpha_0\right) + \left(\frac{r}{r_1}\right)^{\alpha_0} a_0\left(c^2 + d^2\alpha_0\right) + \\
& + E_1 \frac{d^4 - \left(c^2\left(1 + d^2\right) - d^4\right)\varsigma^2}{d^2\left(1 + \varsigma^2\right)} cos\theta + F_1 \frac{d^4 - \left(c^2\left(1 + d^2\right) - d^4\right)\varsigma^2}{d^2\left(1 + \varsigma^2\right)} sin\theta + \\
& + \sum_{n=1}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\alpha_n} \left[(A_n\cos(n\theta) + B_n\sin(n\theta))\left(c^2 - d^2\alpha_n\right) + (C_n\cos(n\theta) + D_n\sin(n\theta))nc^2 \right] + \\
& + \left(\frac{r}{r_1}\right)^{\alpha_n} \left[(a_n cos(n\theta) + b_n sin(n\theta))\left(c^2 + d^2\alpha_n\right) + (c_n cos(n\theta) + d_n sin(n\theta))nc^2 \right] \Bigg] + \\
& + \sum_{n=2}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\beta_n} \left[(E_n\cos(n\theta) + F_n\sin(n\theta))\left(c^2 - d^2\beta_n\right) + (G_n\cos(n\theta) + H_n\sin(n\theta))nc^2 \right] + \\
& + \left(\frac{r}{r_1}\right)^{\beta_n} \left[(e_n cos(n\theta) + f_n sin(n\theta))\left(c^2 + d^2\beta_n\right) + (g_n cos(n\theta) + h_n sin(n\theta))nc^2 \right] \Bigg] \Bigg\}
\end{aligned} \tag{3a}
$$

$$
\begin{aligned}
\sigma_{\theta\theta}^{(1)} = \frac{G_{\theta r}^{(1)}}{rd^2\varsigma^2}\Bigg\{ & \left(\frac{r_0}{r}\right)^{\alpha_0} A_0\left(d^2 - \varsigma^2\alpha_0\right) + \left(\frac{r}{r_1}\right)^{\alpha_0} a_0\left(d^2 + \varsigma^2\alpha_0\right) + \\
& + E_1 \frac{\varsigma^2\left(\varsigma^2 - d^2\right)}{1 + \varsigma^2} \cos\theta + F_1 \frac{\varsigma^2\left(\varsigma^2 - d^2\right)}{1 + \varsigma^2} \sin\theta + \\
& + \sum_{n=1}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\alpha_n} \left[(A_n\cos(n\theta) + B_n\sin(n\theta))\left(d^2 - \varsigma^2\alpha_n\right) + (C_n\cos(n\theta) + D_n\sin(n\theta))nd^2 \right] + \\
& + \left(\frac{r}{r_1}\right)^{\alpha_n} \left[(a_n\cos(n\theta) + b_n\sin(n\theta))\left(d^2 + \varsigma^2\alpha_n\right) + (c_n\cos(n\theta) + d_n\sin(n\theta))nd^2 \right] \Bigg] + \\
& + \sum_{n=2}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\beta_n} \left[(E_n\cos(n\theta) + F_n\sin(n\theta))\left(d^2 - \varsigma^2\beta_n\right) + (G_n\cos(n\theta) + H_n\sin(n\theta))nd^2 \right] + \\
& + \left(\frac{r}{r_1}\right)^{\beta_n} \left[(e_n\cos(n\theta) + f_n\sin(n\theta))\left(d^2 + \varsigma^2\beta_n\right) + (g_n\cos(n\theta) + h_n\sin(n\theta))nd^2 \right] \Bigg] \Bigg\}
\end{aligned} \tag{3b}
$$

$$
\begin{aligned}
\sigma_{r\theta}^{(1)} = \frac{G_{\theta r}^{(1)}}{r}\Bigg\{ & \frac{2r_0H_0}{r} + F_1 \frac{d^2 - \varsigma^2}{d^2\left(1 + \varsigma^2\right)} \cos\theta + E_1 \frac{d^2 - \varsigma^2}{d^2\left(1 + \varsigma^2\right)} \sin\theta + \\
& + \sum_{n=1}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\alpha_n} [n(B_n\cos(n\theta) - A_n\sin(n\theta)) + (D_n\cos(n\theta) - C_n\sin(n\theta))(\alpha_n + 1)] + \\
& + \left(\frac{r}{r_1}\right)^{\alpha_n} [n(b_n\cos(n\theta) - a_n\sin(n\theta)) + (c_n\sin(n\theta) - d_n\cos(n\theta))(\alpha_n - 1)] \Bigg] + \\
& + \sum_{n=2}^{N} \Bigg[\left(\frac{r_0}{r}\right)^{\beta_n} [n(F_n\cos(n\theta) - E_n\sin(n\theta)) + (H_n\cos(n\theta) - G_n\sin(n\theta))(\beta_n + 1)] + \\
& + \left(\frac{r}{r_1}\right)^{\beta_n} [n(f_n\cos(n\theta) - e_n\sin(n\theta)) + (g_n\sin(n\theta) - h_n\cos(n\theta))(\beta_n - 1)] \Bigg] \Bigg\}
\end{aligned} \tag{3c}
$$

In Eqs. (2) and (3a-c), the superscript (1) refers to the elastic constants, stresses and displacements of the reinforcing region.

Unknowns A_n, a_n, B_n, b_n, C_n, c_n, D_n, d_n, E_n, e_n, F_n, f_n, G_n, g_n, H_n, h_n $\forall n \leqslant N \in \mathbb{N}$ are linearly dependent constants to determine by applying proper boundary conditions and linked through the following relations [9]:

$$
\begin{array}{lll}
A_n = C_n\phi_{\alpha,n} & a_n = c_n\phi_{-\alpha,n} & n \geqslant 1 \\
B_n = D_n\phi_{\alpha,n} & b_n = d_n\phi_{-\alpha,n} & n \geqslant 1 \\
E_n = G_n\phi_{\beta,n} & e_n = g_n\phi_{-\beta,n} & n \geqslant 2 \\
F_n = H_n\phi_{\beta,n} & f_n = h_n\phi_{-\beta,n} & n \geqslant 2
\end{array} \tag{4a}
$$

In Eq. (4a):

$$
\phi_{\gamma,n} = \frac{\gamma_n^2 - \varsigma^{-2}n^2 - 1}{n\left(\left(1 + \varsigma^{-2}\right) - \gamma_n\left(1 + d^{-2}\right)\right)} \tag{4b}
$$

whilst $\pm\alpha_n$ and $\pm\beta_n$ $\forall n \in \mathbb{N}$ are the roots of the following characteristic equations:

$$
f(\gamma_n) = \gamma_n^4 - \gamma_n^2\left(\frac{\left(n^2 + c^2\right)}{\varsigma^2} - \frac{n^2c^2}{d^2}\left(2 + d^{-2}\right) + 1\right) + \frac{c^2}{\varsigma^2}\left(n^2 - 1\right)^2 = 0 \tag{5}
$$

where:

$$\varsigma^2 = \frac{G^{(1)}_{\theta r}}{L} \qquad d^2 = \frac{G^{(1)}_{\theta r}}{M} \qquad c^2 = \frac{G^{(1)}_{\theta r}}{N} \tag{6}$$

and:

$$L = \frac{\frac{E^{(1)}_\theta}{\nu^{(1)}_{r\theta}}}{\frac{1}{\nu^{(1)}_{r\theta}} - \frac{E^{(1)}_r}{E^{(1)}_\theta}\nu^{(1)}_{r\theta}} \qquad M = \frac{E^{(1)}_\theta}{\frac{1}{\nu^{(1)}_{r\theta}}\frac{E^{(1)}_\theta}{E^{(1)}_r} - \nu^{(1)}_{r\theta}} \qquad N = \frac{\frac{E^{(1)}_\theta}{\nu^{(1)}_{r\theta}}}{\frac{1}{\nu^{(1)}_{r\theta}}\frac{E^{(1)}_\theta}{E^{(1)}_r} - \nu^{(1)}_{r\theta}} \tag{7a}$$

for plain stress.

Differently, the plain strain case can be treated considering a transversely isotropic behaviour in the through the thickness direction (as usually done for Fibre Reinforced Polymers), according to which $E^{(1)}_z = E^{(1)}_r$ and $G^{(1)}_{rz} = \frac{E^{(1)}_r}{2\left(1+\nu^{(1)}_{rz}\right)}$. Thus, in this last-mentioned case, the following expressions for constants L, M and N hold valid:

$$\begin{aligned} L &= \frac{E^{(1)}_\theta \frac{1-\nu^{(1)}_{rz}}{\nu^{(1)}_{r\theta}}}{\frac{1-\nu^{(1)}_{rz}}{\nu^{(1)}_{r\theta}} - 2\frac{E^{(1)}_r}{E^{(1)}_\theta}\nu^{(1)}_{r\theta}} \qquad M = \frac{E^{(1)}_\theta}{\frac{1-\nu^{(1)}_{rz}}{\nu^{(1)}_{r\theta}}\frac{E^{(1)}_\theta}{E^{(1)}_r} - 2\nu^{(1)}_{r\theta}} \\ N &= \frac{E^{(1)}_\theta}{\nu^{(1)}_{rz}+1}\,\frac{\frac{1}{\nu^{(1)}_{r\theta}} - \nu^{(1)}_{r\theta}\frac{E^{(1)}_r}{E^{(1)}_\theta}}{\frac{E^{(1)}_\theta}{E^{(1)}_r}\frac{1-\nu^{(1)}_{rz}}{\nu^{(1)}_{r\theta}} - 2\nu^{(1)}_{r\theta}} \end{aligned} \tag{7b}$$

where $E^{(1)}_\theta$, $E^{(1)}_r$, $E^{(1)}_z$ are Young moduli, $\nu^{(1)}_{r\theta}$, $\nu^{(1)}_{rz}$ Poisson ratios and $G^{(1)}_{\theta r}$, $G^{(1)}_{rz}$ shear moduli.

2.3. General form for the displacement and stress fields in the reinforced plate (outside the reinforcing region)

The general form for the displacement and stress fields in the plate outside the reinforcing region can be determined using Muskhelishvili complex function [18]:

$$\begin{aligned} &\sigma^{(2)}_{rr} + \sigma^{(2)}_{\theta\theta} = 4Re\left[\frac{d\Phi(z)}{dz}\right] \\ &\sigma^{(2)}_{\theta\theta} - \sigma^{(2)}_{rr} + 2i\sigma^{(2)}_{r\theta} = 2e^{2i\theta}\left(\overline{z}\frac{d^2\Phi(z)}{dz^2} + \frac{dX(z)}{dz}\right) \\ &u^{(2)}_r + iu^{(2)}_\theta = \frac{1+\nu^{(2)}}{E^{(2)}}\left(\kappa\Phi\left(z\right) - z\overline{\frac{d\Phi(z)}{dz}} - \overline{X(z)}\right)e^{-i\theta} \end{aligned} \tag{8}$$

where superscript (2) identifies the elastic constants, stresses and displacements in the isotropic plate and $E^{(2)}$, $\nu^{(2)}$ are the Young modulus and the Poisson ratio, respectively.

In order to properly address the problem under investigation, the following two potential functions are used:

$$\begin{aligned} &\Phi(z) = (\overline{A}_0 + i\overline{B}_0)log(z) + (\overline{A}_1 + i\overline{B}_1)z + (\overline{A}_c + i\overline{B}_c) + \sum_{n=1}^{N-1}(\overline{A}_{-n} + i\overline{B}_{-n})z^{-n} \\ &X(z) = -\kappa(\overline{A}_0 - i\overline{B}_0)log(z) + (\overline{C}_1 + i\overline{D}_1)z + \sum_{n=1}^{N+1}(\overline{C}_{-n} + i\overline{D}_{-n})z^{-n} \end{aligned} \tag{9}$$

which allow to apply in exact form all the boundary conditions relevant to the problem under investigation.

Substituting Eq. (9) into Eq. (8) results in the following displacement and stress fields (according to the reference system shown in Fig. 1):

$$\begin{aligned} u^{(2)}_r = \frac{\left(\nu^{(2)}+1\right)}{E^{(2)}}\Bigg\{&\frac{\left(r^2(\kappa-1)\overline{A}_1 - \overline{C}_{-1}\right)}{r} + \\ &+\frac{\left(r^2\left(\left(\kappa\log\left(r^2\right)-1\right)\overline{A}_0 + \kappa\overline{A}_c\right) - \overline{C}_{-2}\right)}{r^2}\cos(\theta) + \frac{\left(r^2\left(\left(\kappa\log\left(r^2\right)-1\right)\overline{B}_0 + \kappa\overline{B}_c\right) - \overline{D}_{-2}\right)}{r^2}\sin(\theta) + \\ &+\frac{\left(r^2(1+\kappa)\overline{A}_{-1} - \overline{C}_{-3} - r^4\overline{C}_1\right)}{r^3}\cos(2\theta) + \frac{\left(r^2(1+\kappa)\overline{B}_{-1} - \overline{D}_{-3} + r^4\overline{D}_1\right)}{E^{(2)}r^3}\sin(2\theta) + \\ &+\sum_{n=3}^{N}\left[\frac{\left(r^2((n-1)+\kappa)\overline{A}_{-(n-1)} - \overline{C}_{-(n+1)}\right)}{r^{n+1}}\cos(n\theta) + \frac{\left(r^2((n-1)+\kappa)\overline{B}_{-(n-1)} - \overline{D}_{-(n+1)}\right)}{r^{n+1}}\sin(n\theta)\right]\Bigg\} \end{aligned} \tag{10a}$$

$$\begin{aligned} u^{(2)}_\theta = \frac{\left(\nu^{(2)}+1\right)}{E^{(2)}}\Bigg\{&\frac{\left(r^2(1+\kappa)\overline{B}_1 + \overline{D}_{-1}\right)}{r} + \\ &+\frac{\left(r^2\left(\left(1+\kappa\log\left(r^2\right)\right)\overline{B}_0 + \kappa\overline{B}_c\right) + \overline{D}_{-2}\right)}{r^2}\cos(\theta) - \frac{\left(r^2\left(\left(1+\kappa\log\left(r^2\right)\right)\overline{A}_0 + \kappa\overline{A}_c\right) + \overline{C}_{-2}\right)}{r^2}\sin(\theta) + \\ &+\frac{\left(r^2(\kappa-1)\overline{B}_{-1} + \overline{D}_{-3} + r^4\overline{D}_1\right)}{r^3}\cos(2\theta) + \frac{\left(r^2(1-\kappa)\overline{A}_{-1} - \overline{C}_{-3} + r^4\overline{C}_1\right)}{E^{(2)}r^3}\sin(2\theta) + \\ &+\sum_{n=3}^{N}\left[\frac{\left(r^2(\kappa-(n-1))\overline{B}_{-(n-1)} + \overline{D}_{-(n+1)}\right)}{r^{n+1}}\cos(n\theta) + \frac{\left(r^2((n-1)-\kappa)\overline{A}_{-(n-1)} - \overline{C}_{-(n+1)}\right)}{r^{n+1}}\sin(n\theta)\right]\Bigg\} \end{aligned} \tag{10b}$$

$$\begin{aligned}\sigma_{rr}^{(2)} = \frac{1}{(\widetilde{\nu}-1)}\Bigg\{ & \frac{\left(\overline{C}_{-1}(\widetilde{\nu}-1) - r^2(\kappa-1)(\widetilde{\nu}+1)\overline{A}_1\right)}{r^2} + (\widetilde{\nu}-1)(D_1\sin(2\theta) - C_1\cos(2\theta)) + \\ & + \frac{2(r^2(\widetilde{\nu}-\kappa)\overline{A}_0 + (\widetilde{\nu}-1)\overline{C}_{-2})}{r^3}\cos(\theta) + \frac{2(r^2(\widetilde{\nu}-\kappa)\overline{B}_0 + (\widetilde{\nu}-1)\overline{D}_{-2})}{r^3}\sin(\theta) + \\ & + \sum_{n=2}^{N}\Bigg[\frac{((n-1)r^2((n-1)-(n+1)\widetilde{\nu}+\kappa(\widetilde{\nu}+1))\overline{A}_{-(n-1)} + (n+1)(\widetilde{\nu}-1)\overline{C}_{-(n+1)})}{r^{n+1}}\cos(n\theta) + \\ & \frac{((n-1)r^2((n-1)-(n+1)\widetilde{\nu}+\kappa(\widetilde{\nu}+1))\overline{B}_{-(n-1)} + (n+1)(\widetilde{\nu}-1)\overline{D}_{-(n+1)})}{r^{n+1}}\sin(n\theta)\Bigg]\Bigg\}\end{aligned} \tag{11a}$$

$$\begin{aligned}\sigma_{\theta\theta}^{(2)} = \frac{1}{(\widetilde{\nu}-1)}\Bigg\{ & \frac{\left(\overline{C}_{-1}(1-\widetilde{\nu}) - r^2(\kappa-1)(\widetilde{\nu}+1)\overline{A}_1\right)}{r^2} + (\widetilde{\nu}-1)(C_1\cos(2\theta) - D_1\sin(2\theta)) + \\ & + \frac{2(r^2(1-\widetilde{\nu}\kappa)\overline{A}_0 - (\widetilde{\nu}-1)\overline{C}_{-2})}{r^3}\cos(\theta) + \frac{2(r^2(1-\widetilde{\nu}\kappa)\overline{B}_0 - (\widetilde{\nu}-1)\overline{D}_{-2})}{r^3}\sin(\theta) + \\ & + \sum_{n=2}^{N}\Bigg[\frac{((n-1)r^2((n-1)\widetilde{\nu}-(n+1)+\kappa(\widetilde{\nu}+1))\overline{A}_{-(n-1)} - (n+1)(\widetilde{\nu}-1)\overline{C}_{-(n+1)})}{r^{n+1}}\cos(n\theta) + \\ & + \frac{((n-1)r^2((n-1)\widetilde{\nu}-(n+1)+\kappa(\widetilde{\nu}+1))\overline{B}_{-(n-1)} - (n+1)(\widetilde{\nu}-1)\overline{D}_{-(n+1)})}{r^{n+1}}\sin(n\theta)\Bigg]\Bigg\}\end{aligned} \tag{11b}$$

$$\begin{aligned}\sigma_{r\theta}^{(2)} = & -\frac{\overline{D}_{-1}}{r^2} + D_1\cos(2\theta) + C_1\sin(2\theta) + \\ & + \frac{(r^2(\kappa-1)\overline{B}_0 - 2\overline{D}_{-2})}{r^3}\cos(\theta) + \frac{(r^2(1-\kappa)\overline{A}_0 + 2\overline{C}_{-2})}{r^3}\sin(\theta) + \\ & + \sum_{n=2}^{N}\Bigg[\frac{((n+1)(n-1)\overline{B}_{-(n-1)} - (n+1)\overline{D}_{-(n+1)})}{r^{n+1}}\cos(n\theta) + \\ & + \frac{((n+1)\overline{C}_{-(n+1)} - (n+1)(n-1)\overline{A}_{-(n-1)})}{r^{n+1}}\sin(n\theta)\Bigg]\end{aligned} \tag{11c}$$

$\overline{A}_c, \overline{B}_c, \overline{A}_n, \overline{B}_n, \overline{C}_n, \overline{D}_n\ \forall n \leqslant N \in \mathbb{N}$ are integration constants to determine by applying proper boundary conditions, and $\kappa = \frac{3-\nu^{(2)}}{1+\nu^{(2)}}$, $\widetilde{\nu} = \frac{\nu^{(2)}}{1-\nu^{(2)}}$ for plane stress conditions, whilst for plane strain $\kappa = 3-4\nu^{(2)}$, $\widetilde{\nu} = \nu^{(2)}$.

2.4. Boundary conditions and explicit solution for unknown coefficients

The unknowns in the expressions derived in the previous sections can be determined by applying the following boundary conditions:

i. Far away from the hole ($r\rightarrow\infty$) the following conditions must be verified:

$$\begin{aligned}&\lim_{r\to\infty}\sigma_{xx}^{(2)}(r,\theta) = \lim_{r\to\infty}\left(\frac{\sigma_{rr}^{(2)}+\sigma_{\theta\theta}^{(2)}}{2} + \frac{\sigma_{rr}^{(2)}-\sigma_{\theta\theta}^{(2)}}{2}\cos2\theta - \sigma_{r\theta}^{(2)}\sin2\theta\right) = \sigma_{xx}^{\infty}\\ &\lim_{r\to\infty}\sigma_{yy}^{(2)}(r,\theta) = \lim_{r\to\infty}\left(\frac{\sigma_{rr}^{(2)}+\sigma_{\theta\theta}^{(2)}}{2} - \frac{\sigma_{rr}^{(2)}-\sigma_{\theta\theta}^{(2)}}{2}\cos2\theta - \sigma_{r\theta}^{(2)}\sin2\theta\right) = \sigma_{yy}^{\infty}\\ &\lim_{r\to\infty}\sigma_{xy}^{(2)}(r,\theta) = \lim_{r\to\infty}\left(-\frac{\sigma_{rr}^{(2)}-\sigma_{\theta\theta}^{(2)}}{2}\cos2\theta - \sigma_{r\theta}^{(2)}\sin2\theta\right) = \sigma_{xy}^{\infty}\end{aligned} \tag{12}$$

Explicitly substituting Eq. (11a-c) into Eqs. (12) results in:

$$\overline{A}_1 = \frac{\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}}{4} \qquad \overline{C}_1 = \frac{\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}}{2} \qquad \overline{D}_1 = \sigma_{xy}^{\infty} \tag{13}$$

ii. Along the boundary of the reinforcing annulus, equilibrium and compatibility conditions should be guaranteed, namely:

$$\begin{aligned}&\sigma_{rr}^{(1)}(r=r_0,\theta) = \overline{\sigma}_{rr}(\theta) && \sigma_{r\theta}^{(1)}(r=r_1,\theta) = \sigma_{r\theta}^{(2)}(r=r_1,\theta)\\ &\sigma_{r\theta}^{(1)}(r=r_0,\theta) = \overline{\sigma}_{r\theta}(\theta) && u_r^{(1)}(r=r_1,\theta) = u_r^{(2)}(r=r_1,\theta)\\ &\sigma_{rr}^{(1)}(r=r_1,\theta) = \sigma_{rr}^{(2)}(r=r_1,\theta) && u_\theta^{(1)}(r=r_1,\theta) = u_\theta^{(2)}(r=r_1,\theta)\end{aligned} \tag{14}$$

with, $r_1 = r_0 + s$.

Eqs. (14) gives a total of $12N+6$ linearly independent equations, enough to determine all the remaining unknowns, where N is the number of terms used to express internal loads, according to Eqs. (1).

To simplify the expressions for the determined constants, the following auxiliary parameters are introduced:

$$\begin{aligned}&\xi = \frac{d^2-\varsigma^2}{d^2(1+\varsigma^2)} \qquad \zeta = \frac{(1+d^2)\varsigma^2}{d^2(1+\varsigma^2)} \qquad \delta = \frac{1}{c^2} - \frac{(1+d^2)\varsigma^2}{d^4(1+\varsigma^2)}\\ &\eta = \frac{1}{d^2} - \frac{\alpha_0}{c^2} \qquad \partial = \frac{1}{d^2} + \frac{\alpha_0}{c^2}\end{aligned} \tag{15}$$

Under the condition of plane stress, the following close form expressions for the unknowns can be determined, for the reinforcing annulus:

$$E_1 = \frac{\Xi_{E_1}{}^*S_{1,1} + \Upsilon_{E_1}{}^*T_{1,2}}{\Omega_{E_1}} \quad F_1 = \frac{\Xi_{F_1}{}^*S_{1,2} + \Upsilon_{F_1}{}^*T_{1,1}}{\Omega_{F_1}} \quad H_0 = \frac{r_0 T_0}{2G_{\theta r}^{(2)}}$$

$$A_0 = \frac{r_1^{\alpha_0}\left(-r_0^{\alpha_0} r_1 G_{\theta r}^{(1)} \varrho\left(\sigma_{xx}^\infty + \sigma_{yy}^\infty\right) + r_0 r_1^{\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\varrho\right)S_0\right)}{G_{\theta r}^{(1)}\left(-r_0^{2\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\eta\right)\varrho + r_1^{2\alpha_0}\eta\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\varrho\right)\right)}$$

$$a_0 = \frac{r_1^{\alpha_0}\left(r_1^{1+\alpha_0} G_{\theta r}^{(1)} \eta\left(\sigma_{xx}^\infty + \sigma_{yy}^\infty\right) - r_0^{1+\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\eta\right)S_0\right)}{G_{\theta r}^{(1)}\left(-r_0^{2\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\eta\right)\varrho + r_1^{2\alpha_0}\eta\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\varrho\right)\right)} \tag{16}$$

$$C_n = \frac{\Xi_{C_n}{}^*S_{n,1} + \Upsilon_{C_n}{}^*T_{n,2} + \Theta_{C_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{C_n}}$$

$$D_n = \frac{\Xi_{D_n}{}^*S_{n,2} + \Upsilon_{D_n}{}^*T_{n,1} + \Theta_{D_n}{}^*\sigma_{xy}^\infty}{\Omega_{D_n}} \tag{17a}$$

$$G_n = \frac{\Xi_{G_n}{}^*S_{n,1} + \Upsilon_{G_n}{}^*T_{n,2} + \Theta_{G_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{G_n}}$$

$$H_n = \frac{\Xi_{H_n}{}^*S_{n,2} + \Upsilon_{H_n}{}^*T_{n,1} + \Theta_{H_n}{}^*\sigma_{xy}^\infty}{\Omega_{H_n}}$$

$$c_n = \frac{\Xi_{c_n}^* S_{n,1} + \Upsilon_{c_n}{}^*T_{n,2} + \Theta_{c_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{c_n}}$$

$$d_n = \frac{\Xi_{c_n}^* S_{n,2} + \Upsilon_{d_n}{}^*T_{n,1} + \Theta_{d_n}{}^*\sigma_{xy}^\infty}{\Omega_{d_n}} \tag{17b}$$

$$g_n = \frac{\Xi_{g_n}{}^*S_{n,1} + \Upsilon_{g_n}{}^*T_{n,2} + \Theta_{g_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{g_n}}$$

$$h_n = \frac{\Xi_{h_n}{}^*S_{n,2} + \Upsilon_{h_n}{}^*T_{n,1} + \Theta_{h_n}{}^*\sigma_{xy}^\infty}{\Omega_{h_n}}$$

whereas for the isotropic reinforced plate the constants are:

$$\overline{C}_{-1} = \frac{r_1^2\left(-r_0^{2\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(\nu^{(2)}-1\right)\eta\right)\varrho + r_1^{2\alpha_0}\eta\left(E^{(2)} + G_{\theta r}^{(1)}\left(\nu^{(2)}-1\right)\varrho\right)\right)\left(\sigma_{xx}^\infty + \sigma_{yy}^\infty\right)}{2\left(r_0^{2\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\eta\right)\varrho - r_1^{2\alpha_0}\eta\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\varrho\right)\right)} + \\ + \frac{r_1\left(-2r_0^{1+\alpha_0} r_1^{\alpha_0} E^{(2)}(\eta - \varrho)S_0\right)}{2\left(r_0^{2\alpha_0}\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\eta\right)\varrho - r_1^{2\alpha_0}\eta\left(E^{(2)} + G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\varrho\right)\right)}$$

$$\overline{D}_{-1} = -r_0^2 T_0 \qquad \overline{B}_1 = \frac{r_0^2\left(-E^{(2)} + 2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\right)T_0}{8r_1^2 G_{\theta r}^{(1)}} \tag{18}$$

$$\overline{A}_n = \frac{\Xi_{\overline{A}_n}{}^*S_{n,1} + \Upsilon_{\overline{A}_n}{}^*T_{n,2} + \Theta_{C_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{\overline{A}_n}}$$

$$\overline{B}_n = \frac{\Xi_{\overline{B}_n}{}^*S_{n,2} + \Upsilon_{\overline{B}_n}{}^*T_{n,1} + \Theta_{D_n}{}^*\sigma_{xy}^\infty}{\Omega_{\overline{B}_n}}$$

$$\overline{C}_n = \frac{\Xi_{\overline{C}_n}{}^*S_{n,1} + \Upsilon_{\overline{\Upsilon_{\overline{C}_n n}}}{}^*T_{n,2} + \Theta_{C_n}{}^*\left(\sigma_{xx}^\infty - \sigma_{yy}^\infty\right)}{\Omega_{\overline{C}_n}} \tag{19}$$

$$\overline{D}_n = \frac{\Xi_{\overline{D}_n}{}^*S_{n,2} + \Upsilon_{d_n}{}^*T_{n,1} + \Theta_{D_n}{}^*\sigma_{xy}^\infty}{\Omega_{\overline{D}_n}}$$

$$\overline{A}_c = \frac{\Xi_{\overline{A}_c}{}^*S_{n,1} + \Upsilon_{\overline{A}_c}{}^*T_{n,2}}{\Omega_{\overline{A}_c}}$$

$$\overline{B}_c = \frac{\Xi_{\overline{B}_c}{}^*S_{n,2} + \Upsilon_{\overline{B}_c}{}^*T_{n,1}}{\Omega_{\overline{B}_c}}$$

where Ξ_i, Υ_i, Θ_i, Ω_i are real terms explicitly reported in Appendix A and B and satisfying the following equalities:

$$\begin{array}{llll} \Xi_{E_1} = \Xi_{F_1} & \Upsilon_{E_1} = -\Upsilon_{F_1} & \Omega_{E_1} = \Omega_{F_1} & \\ \Xi_{C_n} = \Xi_{D_n} & \Xi_{c_n} = \Xi_{d_n} & \Xi_{G_n} = \Xi_{H_n} & \Xi_{g_n} = \Xi_{h_n} \\ \Upsilon_{C_n} = -\Upsilon_{D_n} & \Upsilon_{c_n} = -\Upsilon_{d_n} & \Upsilon_{G_n} = -\Upsilon_{H_n} & \Upsilon_{g_n} = -\Upsilon_{h_n} \\ \multicolumn{4}{l}{\Omega_{C_n} = \Omega_{c_n} = \Omega_{D_n} = \Omega_{d_n} = \Omega_{G_n} = \Omega_{g_n} = \Omega_{H_n} = \Omega_{h_n}} \\ \multicolumn{4}{l}{\Theta_{C_n} = \Theta_{c_n} = \Theta_{D_n} = \Theta_{d_n} = \Theta_{G_n} = \Theta_{g_n} = \Theta_{H_n} = \Theta_{h_n} = 0 \quad \forall n \neq 2} \\ 2\Theta_{D_2} = \Theta_{C_n} & 2\Theta_{d_2} = \Theta_{c_n} & 2\Theta_{G_2} = \Theta_{H_n} & 2\Theta_{g_2} = \Theta_{h_n} \end{array} \tag{20}$$

$$\begin{array}{lll} \Xi_{\overline{A}_n} = \Xi_{\overline{B}_n} & \Xi_{\overline{C}_n} = \Xi_{\overline{D}_n} & \Xi_{\overline{A}_c} = \Xi_{\overline{B}_c} \\ \Upsilon_{\overline{A}_n} = -\Upsilon_{\overline{B}_n} & \Upsilon_{\overline{C}_n} = -\Upsilon_{\overline{D}_n} & \Upsilon_{\overline{A}_c} = -\Upsilon_{\overline{B}_c} \\ \multicolumn{2}{l}{\Omega_{\overline{A}_n} = \Omega_{\overline{B}_n} = \Omega_{\overline{C}_{n+2}} = \Omega_{\overline{D}_{n+2}}} & \Omega_{\overline{A}_c} = \Omega_{\overline{B}_c} \\ \Theta_{\overline{A}_n} = \Theta_{\overline{B}_n} = 0 \quad \forall n \neq -1 & \multicolumn{2}{l}{\Theta_{\overline{C}_n} = \Theta_{\overline{D}_n} = 0 \quad \forall n \neq -3} \\ 2\Theta_{\overline{A}_{-1}} = \Theta_{\overline{B}_{-1}} & 2\Theta_{\overline{C}_{-3}} = \Theta_{\overline{D}_{-3}} & \end{array} \tag{21}$$

The constants under plane strain conditions have similar expressions, and can be easily obtained:

Table 1
Results of the mesh convergence analysis. FE nodes are those of one-quarter of a plate.

e_0 [mm]	FE Nodes	$\sigma_{\theta\theta}\big\vert_{\substack{\theta=0\\ r=r_0}} / \sigma^g_{yy}$
1	$4.348{\cdot}10^3$	4.11
0.5	$11.176{\cdot}10^3$	4.23
0.1	$167.374{\cdot}10^3$	4.24
0.05	$618.748{\cdot}10^3$	4.24

i. changing $E^{(2)}$ for $\frac{E^{(2)}}{1-[\nu^{(2)}]^2}$;
ii. changing $\nu^{(2)}$ for $\frac{\nu^{(2)}}{1-\nu^{(2)}}$;
iii. Using Eqs. (7b) instead of Eq. (7a) for parameters L, M and N.

3. Discussion and comparison with numerical results

Even if the solution obtained in this work is mathematically exact for infinite plates, the stress fields resulting from the developed framework were compared with several numerical analyses carried out on finite plates with reinforced holes, in order to check their accuracy also in the presence of finite bodies.

To this end, in the numerical investigations, finite plates with height and width equal to 250 mm, having a central hole of radius $r_0 = 10$ mm and reinforced with annuli of thickness equal to s = 2, 5 and 10 mm were considered.

Numerical analyses were carried out with Ansys® version 21 software package, using PLANE183 quadrilateral plane elements with plane stress condition and pure displacement formulation. The mesh is created in three steps: as a first step, the perimeter of the plate hole is divided into elements of a given edge-length, e_0, then, up to a radius of 40 mm from the hole center, a mapped mesh is created imposing the length of the quadrilateral elements in the radial direction equal to e_0; finally, for the rest of the plate the mesh remains mapped, but imposing 100 elements in the radial direction with spacing ratio 100. In order to find a value for e_0 that guarantees the accuracy of displacement and stress values, a convergence analysis was carried out. To this end, a 250x250 mm plate made of quasi-isotropic GFRP with a hole with radius $r_0 = 10$ mm, reinforced with a ring of width s = 2 mm ($s/r_0 = 0.2$) was considered. The plate was subjected to uniaxial tension, with a far applied nominal stress, σ^g_{yy}. With reference to this particular case, the size of e_0 was progressively reduced, and the value of the normalized hoop stress at the hole pole was determined, $\sigma_{\theta\theta}\big\vert_{\substack{\theta=0\\ r=r_0}} / \sigma^g_{yy}$. The results are presented in Table 1, where it is evident that a value $e_0 = 0.05$ mm guarantees the convergence of results.

Accordingly, a value for e_0 of 0.05 mm was chosen for all the analyses, resulting in meshes of about 620k nodes and 220k elements for each plate quadrant.

In this way very fine and regular mesh patterns were obtained, especially in the region close to the reinforced hole, guaranteeing in all cases a great accuracy of the numerical results.

Whenever possible, symmetry or skew-symmetry boundary conditions were used, thus modelling one quarter of the plate; whenever this was not possible, due to the loading conditions the plate was subjected to, the whole plate was modelled. As a representative example, Fig. 2a-d reports contour plots for $\sigma_{\theta\theta}$, σ_{rr}, u_θ and u_r in the case of a plate under biaxial tension and internal pressure.

According to the schematic shown in Fig. 3, for each analysed case, stresses were evaluated along three main straight paths starting from the centre of the hole and inclined along the $\theta = 0^\circ$, 45° and 90° directions (paths 1, 2 and 3, respectively, in Fig. 3). In addition, stresses were assessed along the hole boundary (path 4) and along the boundary of the reinforcing region (path 5). Several loading conditions were considered in the analyses, including uniaxial tension, biaxial tension, pure shear, and uniform and non-uniform internal pressure.

In all the analyses an annular reinforcement made of Carbon Fiber Reinforced Polymer (CFRP) was considered, with the following properties: $E_\theta = 147$ GPa, $E_r = 10.3$ GPa, $G_{\theta r} = 7$ GPa, $\nu_{r\theta} = 0.02$.

Differently, for the reinforced plates, two different materials were considered, i.e.:

i. Polyamide (PA) with E = 3 GPa and $\nu = 0.35$;
ii. Quasi-isotropic laminate made of Glass Fiber Reinforced Polymer (GFRP), with $E_x = E_y = 54$ GPa, and $\nu_{12} = \nu_{21} = 0.31$.

Comparisons between the analytical solution and some of the results from numerical analyses are shown in Figs. 4-12 for different combinations of materials, geometries and loading conditions. It is evident that, in all the cases the agreement is very satisfactory. Accordingly, the proposed solution, mathematically exact in the case of an infinite plate, can be successfully used also to characterize the stress and displacement fields in finite plates.

In more details, the hoop stress component evaluated along the boundary of the hole for plates under uniaxial tension is reported in Fig. 4, where it is noteworthy that the stress field magnitude increases while decreasing the $\frac{s}{r_0}$ ratio.

The stress components, as evaluated along the boundary of the reinforcing annulus are reported in Fig. 5 for a PA plate, much less stiff than the CFRP ring, documenting that also for this case the proposed framework is very precise.

The accuracy of the analytical solution varying the loading condition is demonstrated in Figs. 6-11 for different materials and size of the reinforcing ring. In particular, the following loading conditions were considered:

i. biaxial tension combined with a constant internal radial pressure (Fig. 6);
ii. pure in-plane shear (Fig. 7);
iii. in-plane shear combined with a constant internal radial pressure (Fig. 8);
iv. in-plane shear combined with a non-uniform internal pressure, simulating the pressure exerted by a pin. In this case, the internal pressure was described taking advantage of the following equation (see also Fig. 9a):

$$\overline{\sigma}_{rr}(\theta) = \Phi(\theta) = \frac{100}{\pi} + 50\sin[\theta] - \frac{200\cos[2\theta]}{3\pi} - \frac{40\cos[4\theta]}{3\pi} - \frac{40\cos[6\theta]}{7\pi} - \frac{200\cos[8\theta]}{63\pi} - \frac{200\cos[10\theta]}{99\pi} \tag{22}$$

The results related to this last-mentioned case are presented in Fig. 9b. Moreover, examples of the displacement distributions are reported in Figs. 10 and 11.

As evident, in all the cases above discussed, the agreement is extremely satisfactory due to the fact that the hole size ($r_0 = 10$ mm) is small compared to the plate size (plate ligament: 250 mm).

It is also worth of noting that in Figs. 6, 8 and 9 the hoop stress component is discontinuous; such a behavior is due to the nature of the problem under investigation and is due to elastic mismatch between the involved materials; different from stresses, the displacement field is continuous, as evident from Figs. 10 and 11.

Eventually, in Fig. 12, the attention is focused into a 100x100 mm plate with a hole of radius 10 mm and a 5 mm thick reinforcement, under a complex loading condition (namely: biaxial tension, shear stress and a non-uniform internal pressure). In this latest case, the plate size is comparable with the hole radius and, as a consequence, the accuracy of the proposed solution is lower with respect to the previous cases, but still acceptable, with a maximum difference between numerical results and theoretical solution within 10%.

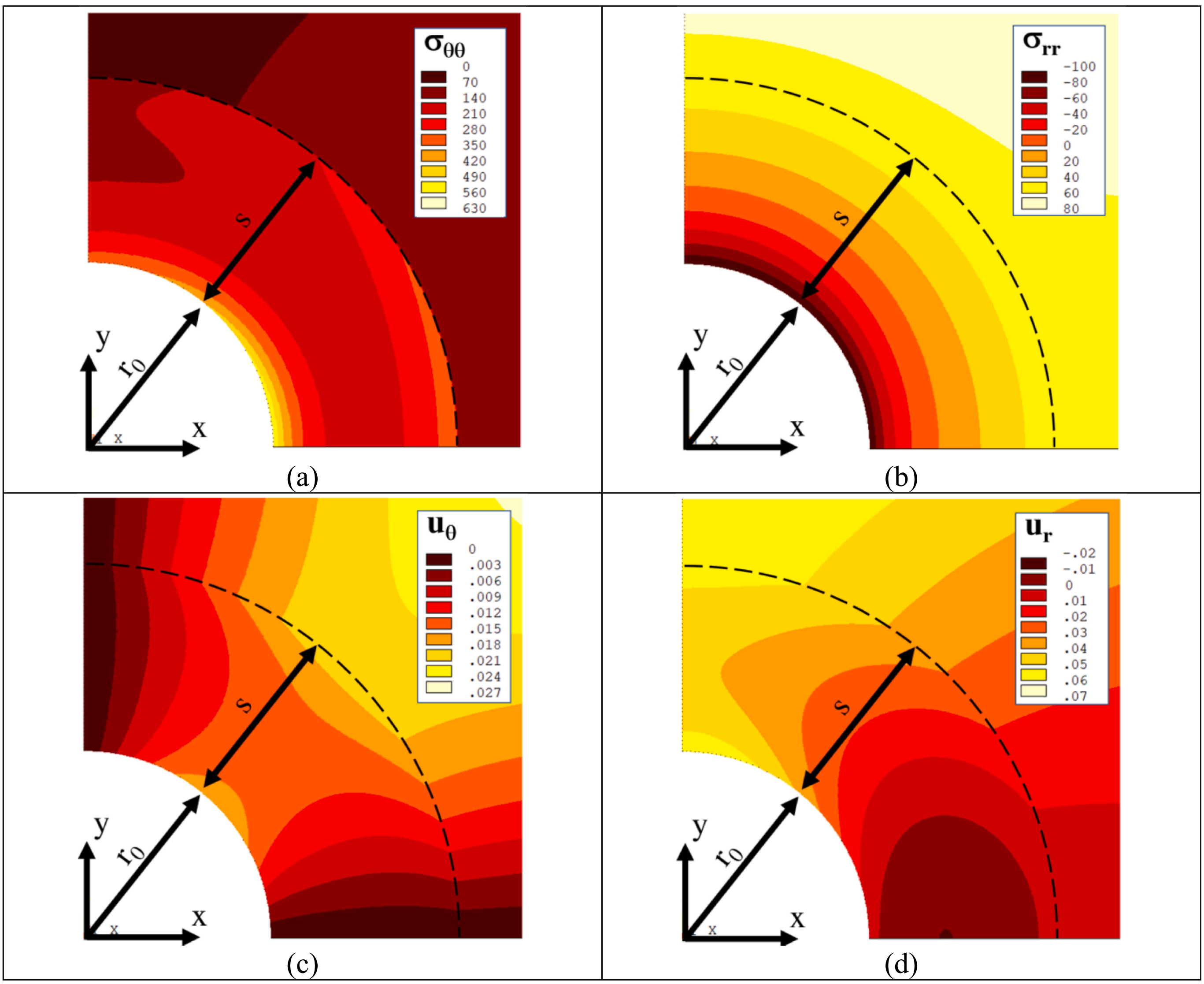

Fig. 2. Example of contour plots in 250x250 mm plates made of quasi-isotropic GFRP; hole radius $r_0 = 10$ mm reinforced with a ring of width s = 10 mm. Applied loads: biaxial tension ($\sigma_{yy}^{g} = 100\,MPa$, $\sigma_{xx}^{g} = 50\,MPa$) and internal pressure ($\overline{\sigma}_{rr} = 100\,MPa$). Stresses in MPa; displacements in mm.

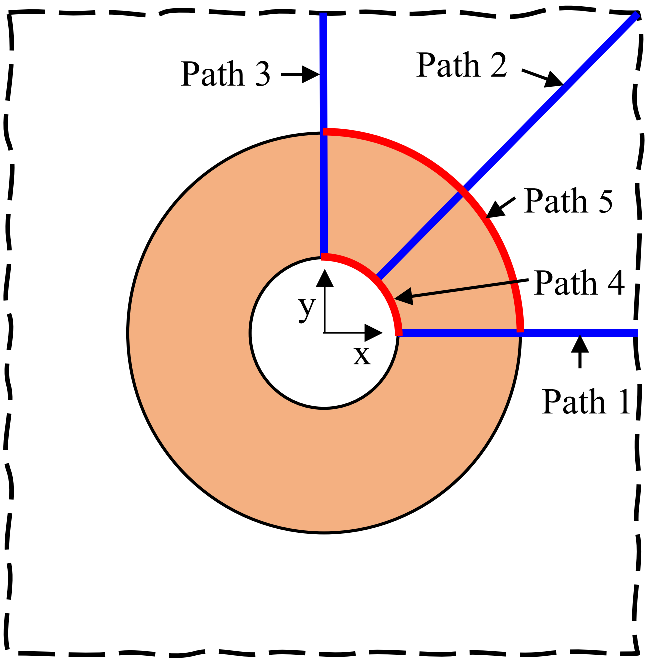

Fig. 3. Schematics of the paths used for comparing the analytical solution with numerical (FE) results.

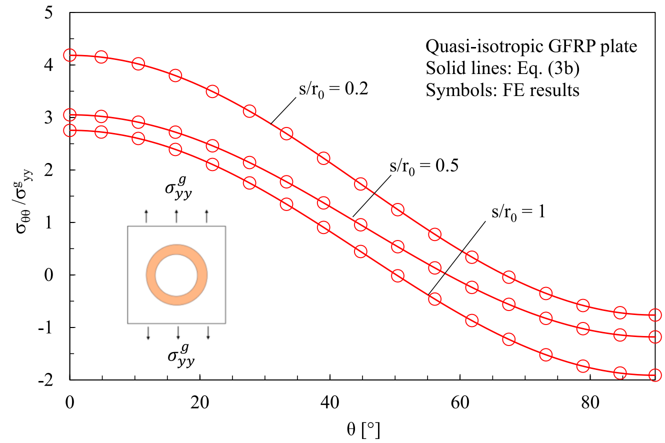

Fig. 4. Stress component $\sigma_{\theta\theta}$ evaluated along the boundary of the hole (path 4 in Fig. 3). 250x250 mm isotropic plates made of quasi-isotropic GFRP; hole radius $r_0 = 10$ mm reinforced with rings of width s = 2, 5 and 10 mm. Uniaxial tension $\sigma_{yy}^{g} = 100\,MPa$. Analytical solution compared with the results from FE analyses.

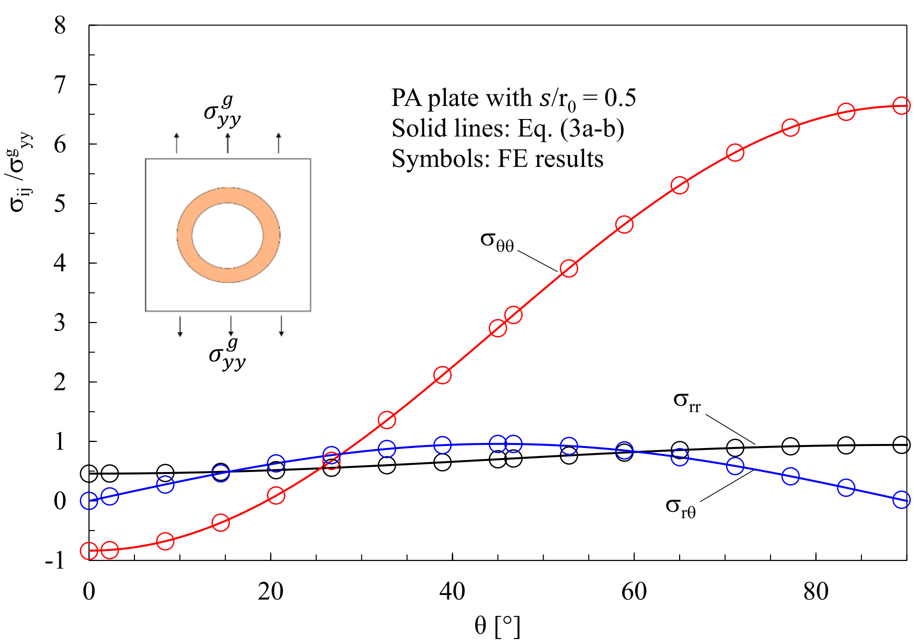

Fig. 5. Stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ evaluated along the edge of the reinforcing ring (path 5 in Fig. 3). 250x250 mm plates made of PA; hole radius $r_0 = 10$ mm reinforced with a ring of width s = 5 mm. Uniaxial tension, $\sigma_{yy}^{g} = 100\,MPa$. Analytical solution compared with the results from FE analyses.

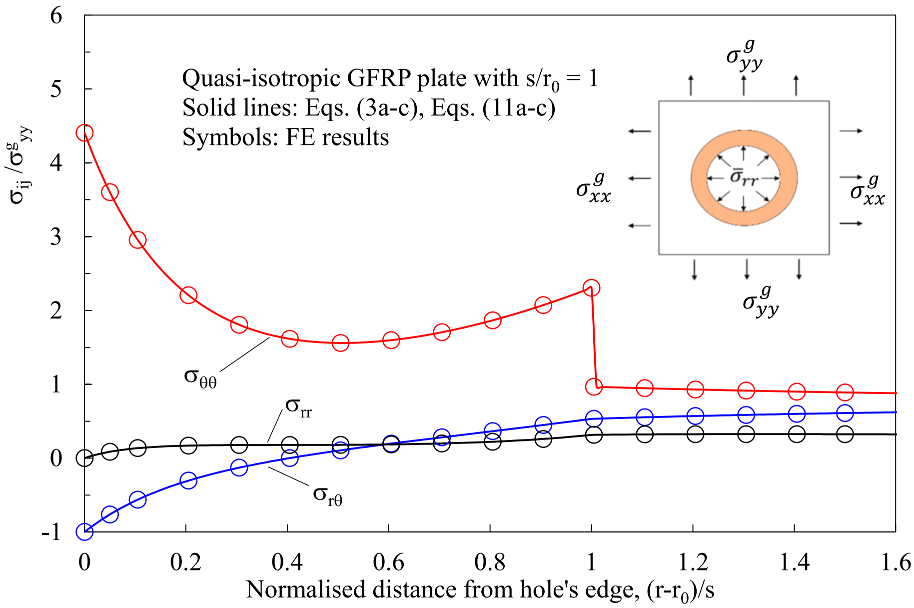

Fig. 6. Stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ evaluated along a straight path with θ = 45° (path 2 in Fig. 3). 250x250 mm plates made of quasi-isotropic GFRP; hole radius r_0 = 10 mm reinforced with a ring of width s = 10 mm. Applied loads: biaxial tension ($\sigma^g_{yy} = 100\,MPa$, $\sigma^g_{xx} = 50\,MPa$) and internal pressure ($\overline{\sigma}_{rr} = 100\,MPa$). Analytical solution compared with the results from FE analyses.

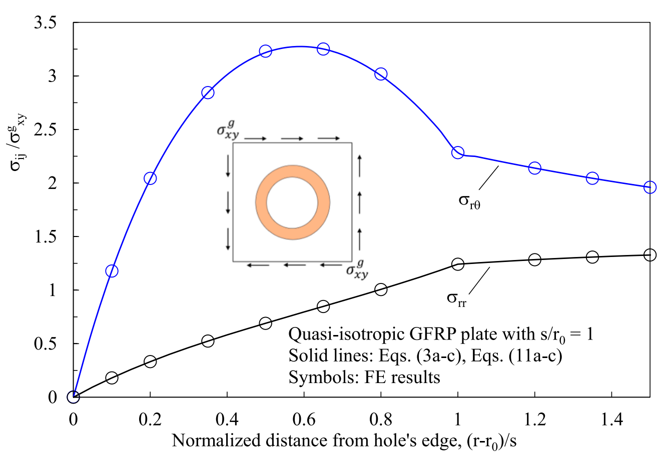

Fig. 7. Stress components σ_{rr}, $\sigma_{r\theta}$ evaluated along a straight path with θ = 90° (path 3 in Fig. 3). 250x250 mm plates made of quasi-isotropic GFRP; hole radius r_0 = 10 mm reinforced with a ring of width s = 10 mm. Pure shear loadings, $\sigma^g_{xy} = 100\,MPa$. Analytical solution compared with the results from FE analyses.

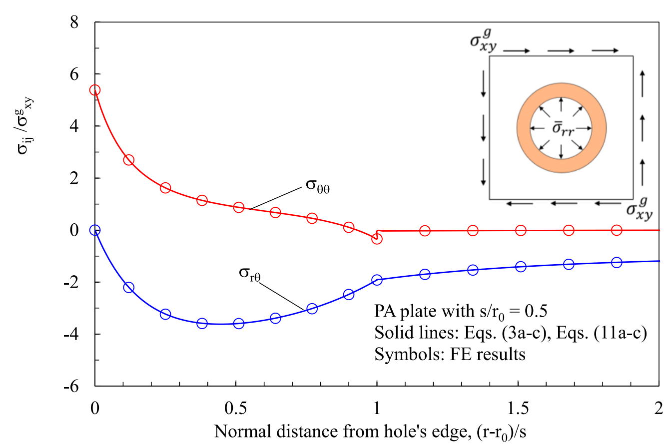

Fig. 8. Stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ evaluated along a straight path with θ = 0° (path 1 in Fig. 3). 250x250 mm plates made of PA; hole radius r_0 = 10 mm reinforced with a ring of width s = 5 mm. Applied loads: pure shear loadings $\sigma^g_{xy} = 100\,MPa$ and internal pressure ($\overline{\sigma}_{rr} = 100\,MPa$). Analytical solution compared with the results from FE analyses.

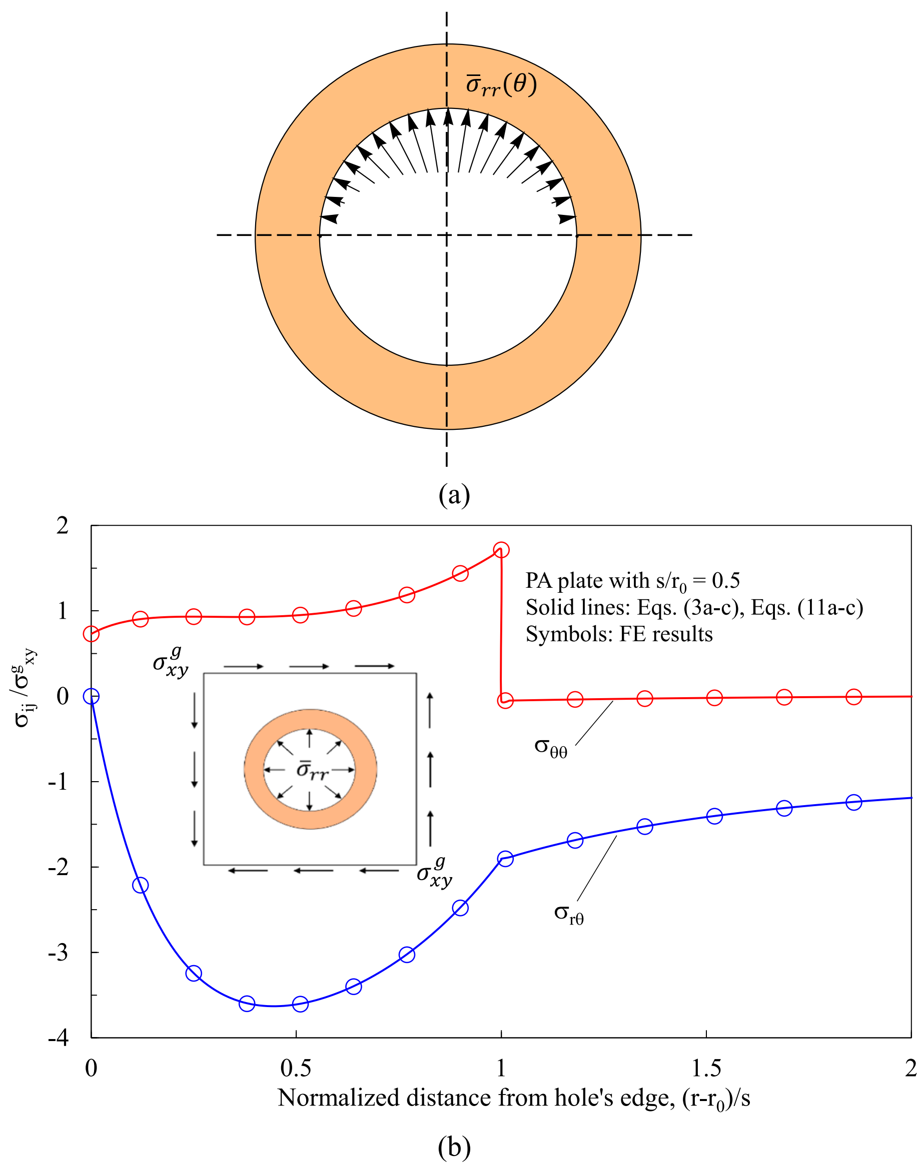

Fig. 9. (a) Schematic of the internal pressure exerted by a pin, according to the Eq. (22). (b) Stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ evaluated along a straight path with θ = 90° (path 3 in Fig. 3). 250x250 mm plates made of PA; hole radius $r_0 = 10$ mm reinforced with a ring of width s = 5 mm. Applied loads: pure shear loadings $\sigma_{xy}^{g} = 100\,MPa$ and non-uniform internal pressure according to Eq. (22) (see also Fig. 9a). Analytical solution compared with the results from FE analyses.

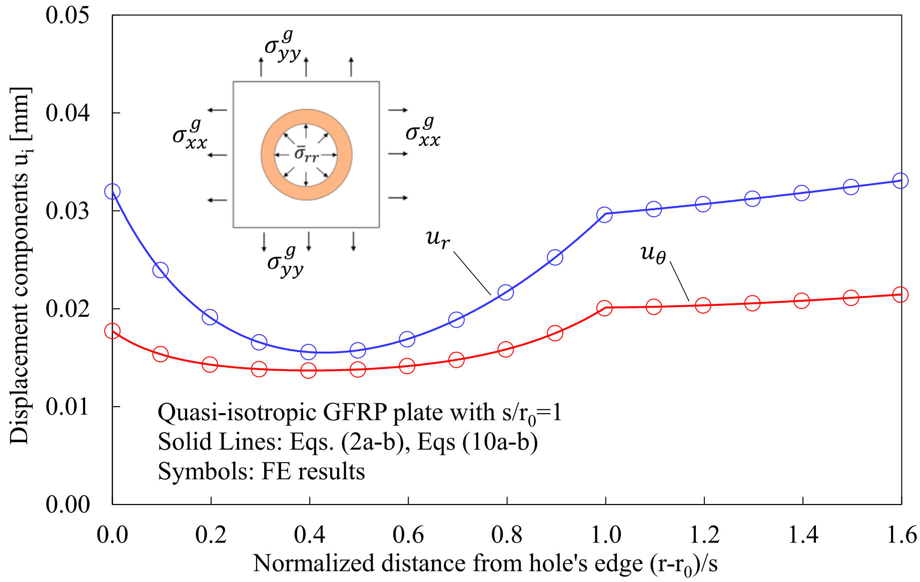

Fig. 10. Displacement components u_r and u_θ evaluated along a straight path with θ = 45° (path 2 in Fig. 2). 250x250 mm plates made of quasi-isotropic GFRP; hole radius $r_0 = 10$ mm reinforced with a ring of width s = 10 mm. Applied loads: biaxial tension ($\sigma^g_{yy} = 100\,MPa$, $\sigma^g_{xx} = 50\,MPa$) and internal pressure ($\overline{\sigma}_{rr} = 100\,MPa$). Analytical solution compared with the results from FE analyses.

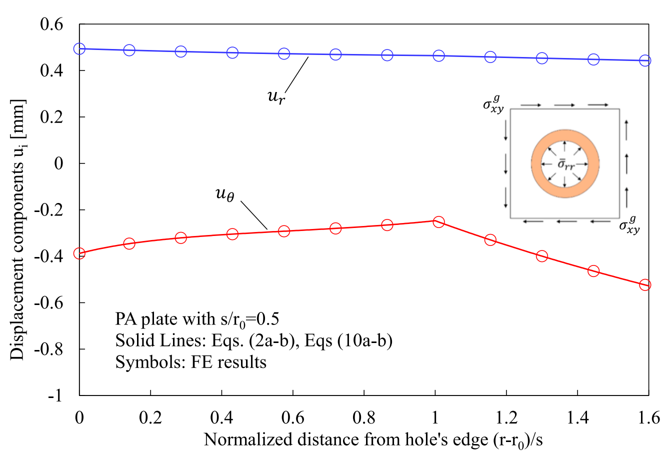

Fig. 11. Displacement components u_r and u_θ evaluated along a straight path with θ = 90° (path 3 in Fig. 2). 250x250 mm plates made of PA; hole radius $r_0 = 10$ mm reinforced with a ring of width s = 5 mm. Applied loads: pure shear loadings $\sigma^g_{xy} = 100\,MPa$ and non-uniform internal pressure according to Eq. (22) (see also Fig. 9a). Analytical solution compared with the results from FE analyses.

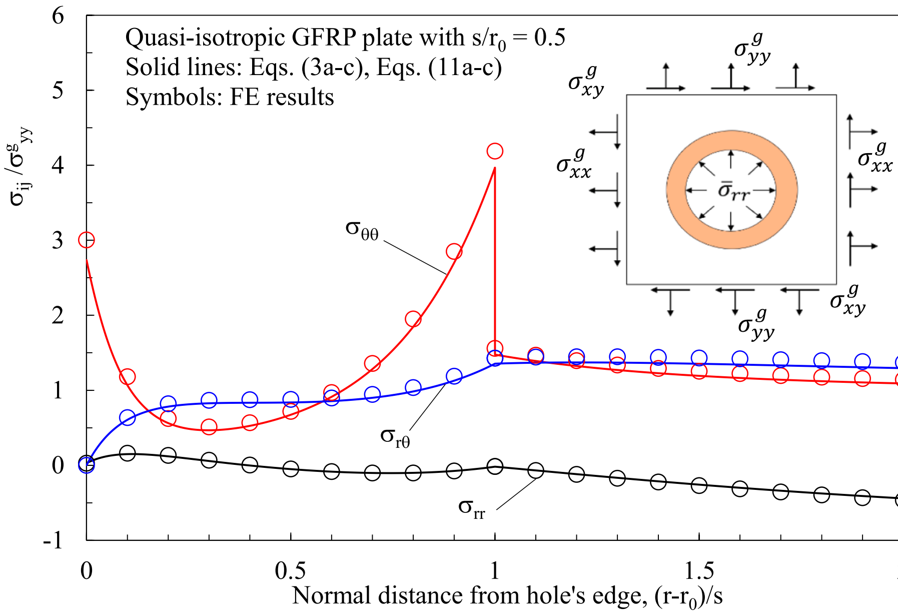

Fig. 12. Stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ evaluated along a straight path with θ = 0° (path 1 in Fig. 3). 100x100 mm plates made of quasi-isotropic GFRP; hole radius $r_0 = 10$ mm reinforced with a ring of width $s = 5$ mm. Applied loads: biaxial tension–compression ($\sigma^g_{yy} = 100\,MPa$ $\sigma^g_{xx} = -50MPa$) shear stress ($\sigma^g_{xy} = 100\,MPa$) and non-uniform internal pressure according to Eq. (22) (see also Fig. 9a). Analytical solution compared with the results from FE analyses.

4. Conclusions

In this work a new and exact analytical solution was derived for an isotropic infinite plate with a circular hole reinforced with a polarly orthotropic material and a generic loading condition.

The proposed close form solution for the displacement and stress fields is able to properly account for the geometrical features and the elastic properties of the plate and of the reinforcing region.

The accuracy of the proposed framework, exact in the case of infinite bodies, was checked against the results from a bulk of finite element analyses carried out on relevant cases related to plates with a finite size, documenting a very satisfactory agreement.

The solution developed in this work represents a useful tool toward the understanding of the stress fields in polymeric components with reinforced holes and made with fused deposition modelling technologies.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

We have attached supplmentary material

Appendix A

Let's introduce the following constants:

$$
\begin{aligned}
& g_{\gamma,n} = \frac{n+\phi_{\gamma,n}}{d^2} - \frac{\gamma\phi_{\gamma,n}}{c^2}; \qquad f_{\gamma,n} = 1+\gamma+n\phi_{\gamma,n}; \\
& m_{\gamma,n} = (n+2) + (2n+1)\phi_{\gamma,n} - \nu^{(2)}\left(n+\phi_{\gamma,n}\right) \qquad h_{\gamma,n} = \nu^{(2)} - (2n+1) + \left(n\nu^{(2)} - (n+2)\right)\phi_{\gamma,n} \\
& p_{\gamma,n} = \phi_{\gamma,n} - (n+2) + \nu^{(2)}\left(-(n-2)+n\phi_{\gamma,n}\right)
\end{aligned}
\tag{A.1}
$$

Using relations (21) it is possible to explicitly derive parameters Ξ_i, Υ_i, Ω_i to be used in Eq. (20):

$$
\begin{aligned}
\Xi_{E_1} = {} & r_1^{2\alpha_1}\left(-r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right) f_{\alpha,1}\left(f_{-\alpha,1}-g_{-\alpha,1}\right) + 2r_0E^{(2)} f_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)\right) + \\
& + r_0^{2\alpha_1}\left(r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right) f_{-\alpha,1}\left(f_{\alpha,1}-g_{\alpha,1}\right) - 2r_0E^{(2)} f_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)\right) \\
\Upsilon_{E_1} = {} & r_1^{2\alpha_1}\left(-r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,1}-g_{-\alpha,1}\right) g_{\alpha,1} + 2r_0E^{(2)} g_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)\right) + \\
& + r_0^{2\alpha_1}\left(r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right) g_{-\alpha,1}\left(f_{\alpha,1}-g_{\alpha,1}\right) - 2r_0E^{(2)} g_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)\right) \\
\Omega_{E_1} = {} & G_{\theta r}^{(1)}\Big[r_0^{\alpha_1} r_1^{\alpha_1}\left(2E^{(2)}\zeta - G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)(\delta+\xi)\right)\left(f_{\alpha,1}g_{-\alpha,1} - f_{-\alpha,1}g_{\alpha,1}\right) + \\
& + r_1^{2\alpha_1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)\left(-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,1}-g_{-\alpha,1}\right) + 2E^{(2)}\left(1+\phi_{-\alpha,1}\right)\right) + \\
& + r_0^{2\alpha_1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)\left(G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{\alpha,1}-g_{\alpha,1}\right) - 2E^{(2)}\left(1+\phi_{\alpha,1}\right)\right)\Big]
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\Xi_{C_1} &= r_0^{\alpha_1} r_1^{\alpha_1}\Big(2r_0E^{(2)}\zeta f_{-\alpha,1} - r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)(\delta+\xi)f_{-\alpha,1}\Big)+\\
&\quad + r_1^{2\alpha_1}\Big(r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\xi\big(f_{-\alpha,1}-g_{-\alpha,1}\big) - 2r_0E^{(2)}\xi\big(1+\phi_{-\alpha,1}\big)\Big)\\
\Upsilon_{C_1} &= r_0^{\alpha_1} r_1^{\alpha_1}\Big(2r_0E^{(2)}\zeta g_{-\alpha,1} - r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)(\delta+\xi)g_{-\alpha,1}\Big)+\\
&\quad + r_1^{2\alpha_1}\Big(-r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\delta\big(f_{-\alpha,1}-g_{-\alpha,1}\big) + 2r_0E^{(2)}\delta\big(1+\phi_{-\alpha,1}\big)\Big)\\
\Xi_{c_1} &= r_1^{2\alpha_1}\Big(2r_0E^{(2)}\zeta f_{\alpha,1} - r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)(\delta+\xi)f_{\alpha,1}\Big)+\\
&\quad + r_0^{\alpha_1} r_1^{\alpha_1}\Big(r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\xi\big(f_{\alpha,1}-g_{\alpha,1}\big) - 2r_0E^{(2)}\xi\big(1+\phi_{\alpha,1}\big)\Big)\\
\Upsilon_{c_1} &= r_1^{2\alpha_1}\Big(2r_0E^{(2)}\zeta g_{\alpha,1} - r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)(\delta+\xi)g_{\alpha,1}\Big)+\\
&\quad + r_0^{\alpha_1} r_1^{\alpha_1}\Big(-r_0G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\delta\big(f_{\alpha,1}-g_{\alpha,1}\big) + 2r_0E^{(2)}\delta\big(1+\phi_{\alpha,1}\big)\Big)\\
\Omega_{C_1} &= G_{\theta r}^{(1)}\Big[r_0^{\alpha_1} r_1^{\alpha_1}\Big(-2E^{(2)}\zeta + G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)(\delta+\xi)\Big)\big(f_{\alpha,1}g_{-\alpha,1} - f_{-\alpha,1}g_{\alpha,1}\big)+\\
&\quad + r_1^{2\alpha_1}\big(\delta f_{\alpha,1}+\xi g_{\alpha,1}\big)\Big(G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\big(f_{-\alpha,1}-g_{-\alpha,1}\big) - 2E^{(2)}\big(1+\phi_{-\alpha,1}\big)\Big)+\\
&\quad + r_0^{2\alpha_1}\big(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\big)\Big(-G_{\theta r}^{(1)}\big(1+\nu^{(2)}\big)\big(f_{\alpha,1}-g_{\alpha,1}\big) + 2E^{(2)}\big(1+\phi_{\alpha,1}\big)\Big)\Big]
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\Theta_{C_2} &= r_0^{\beta_2} r_1^{2\alpha_2+\beta_2}\Big[2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{-\alpha,2}-g_{-\alpha,2}\big)\big(f_{\beta,2}g_{-\beta,2} - f_{-\beta,2}g_{\beta,2}\big)-\\
&\quad - 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{\beta,2}g_{-\beta,2} - f_{-\beta,2}g_{\beta,2}\big)\big(1+\phi_{-\alpha,2}\big)\Big]+\\
&\quad + r_0^{\alpha_2} r_1^{\alpha_2}\Big\{r_1^{2\beta_2}\Big[-2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{-\beta,2}-g_{-\beta,2}\big)\big(f_{\beta,2}g_{-\alpha,2} - f_{-\alpha,2}g_{\beta,2}\big)+\\
&\quad + 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{\beta,2}g_{-\alpha,2} - f_{-\alpha,2}g_{\beta,2}\big)\big(1+\phi_{-\beta,2}\big)\Big]+\\
&\quad + r_0^{2\beta_2}\Big[2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{-\beta,2}g_{-\alpha,2} - f_{-\alpha,2}g_{-\beta,2}\big)\big(f_{\beta,2}-g_{\beta,2}\big)-\\
&\quad - 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{-\beta,2}g_{-\alpha,2} - f_{-\alpha,2}g_{-\beta,2}\big)\big(1+\phi_{\beta,2}\big)\Big]\Big\}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\Xi_{C_n} &= r_0^{\alpha_n+\beta_n} r_1^{\alpha_n+\beta_n}\big(r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)f_{-\alpha,n}\big(f_{\beta,n}g_{-\beta,n} - f_{-\beta,n}g_{\beta,n}\big)+\\
&\quad + r_0G_{12}^{(1)}E^{(2)}f_{-\alpha,n}\big(g_{\beta,n}h_{n,-\beta} - g_{-\beta,n}h_{n,\beta} - f_{\beta,n}m_{n,-\beta} + f_{-\beta,n}m_{n,\beta}\big)+\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}f_{-\alpha,n}\big(\phi_{-\beta,n}-\phi_{\beta,n}\big)\big)+\\
&\quad + r_1^{2\alpha_n}\Big(r_1^{2\beta_n}\Big(r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)f_{\beta,n}\big(f_{-\beta,n}g_{-\alpha,n} - f_{-\alpha,n}g_{-\beta,n}\big)+\\
&\quad + r_0G_{\theta r}^{(1)}E^{(2)}f_{\beta,n}\big(g_{-\beta,n}h_{n,-\alpha} - g_{-\alpha,n}h_{n,-\beta} - f_{-\beta,n}m_{n,-\alpha} + f_{-\alpha,n}m_{n,-\beta}\big)\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}f_{\beta,n}\big(\phi_{-\alpha,n}-\phi_{-\beta,n}\big)\big)+\\
&\quad + r_0^{2\beta_n}\big(-r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)f_{-\beta,n}\big(f_{\beta,n}g_{-\alpha,n} - f_{-\alpha,n}g_{\beta,n}\big)+\\
&\quad + r_0G_{\theta r}^{(1)}E^{(2)}f_{-\beta,n}\big(-g_{\beta,n}h_{n,-\alpha} + g_{-\alpha,n}h_{n,\beta} + f_{\beta,n}m_{n,-\alpha} - f_{-\alpha,n}m_{n,\beta}\big)+\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}f_{-\beta,n}\big(-\phi_{-\alpha,n}+\phi_{\beta,n}\big)\big)\big)\big)
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\Upsilon_{C_n} &= r_0^{\alpha_n+\beta_n} r_1^{\alpha_n+\beta_n}\big(r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)g_{-\alpha,n}\big(f_{\beta,n}g_{-\beta,n} - f_{-\beta,n}g_{\beta,n}\big)+\\
&\quad + r_0G_{\theta r}^{(1)}E^{(2)}g_{-\alpha,n}\big(g_{\beta,n}h_{n,-\beta} - g_{-\beta,n}h_{n,\beta} - f_{\beta,n}m_{n,-\beta} + f_{-\beta,n}m_{n,\beta}\big)+\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}g_{-\alpha,n}\big(\phi_{-\beta,n}-\phi_{\beta,n}\big)\big)+\\
&\quad + r_1^{2\alpha_n}\Big(r_1^{2\beta_n}\Big(r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)\big(f_{-\beta,n}g_{-\alpha,n} - f_{-\alpha,n}g_{-\beta,n}\big)g_{\beta,n}+\\
&\quad + r_0G_{\theta r}^{(1)}E^{(2)}g_{\beta,n}\big(g_{-\beta,n}h_{n,-\alpha} - g_{-\alpha,n}h_{n,-\beta} - f_{-\beta,n}m_{n,-\alpha} + f_{-\alpha,n}m_{n,-\beta}\big)+\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}g_{\beta,n}\big(\phi_{-\alpha,n}-\phi_{-\beta,n}\big)\big)+\\
&\quad + r_0^{2\beta_n}\big(-r_0G_{\theta r}^{(1)2}\big(-3+\nu^{(2)}\big)\big(1+\nu^{(2)}\big)g_{-\beta,n}\big(f_{\beta,n}g_{-\alpha,n} - f_{-\alpha,n}g_{\beta,n}\big)+\\
&\quad + r_0G_{\theta r}^{(1)}E^{(2)}g_{-\beta,n}\big(-g_{\beta,n}h_{n,-\alpha} + g_{-\alpha,n}h_{n,\beta} + f_{\beta,n}m_{n,-\alpha} - f_{-\alpha,n}m_{n,\beta}\big)+\\
&\quad + \big(n^2-1\big)r_0E^{(2)2}g_{-\beta,n}\big(-\phi_{-\alpha,n}+\phi_{\beta,n}\big)\big)\big)\big)
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\Theta_{c_2} &= r_0^{\alpha_2+\beta_2} r_1^{\alpha_2+\beta_2}\Big[-2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{\alpha,2}-g_{\alpha,2}\big)\big(f_{\beta,2}g_{-\beta,2} - f_{-\beta,2}g_{\beta,2}\big)+\\
&\quad + 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{\beta,2}g_{-\beta,2} - f_{-\beta,2}g_{\beta,2}\big)\big(1+\phi_{\alpha,2}\big)\Big]+\\
&\quad + r_1^{2\alpha_2}\Big\{r_1^{2\beta_2}\Big[2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{-\beta,2}-g_{-\beta,2}\big)\big(f_{\beta,2}g_{\alpha,2} - f_{\alpha,2}g_{\beta,2}\big)-\\
&\quad - 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{\beta,2}g_{\alpha,2} - f_{\alpha,2}g_{\beta,2}\big)\big(1+\phi_{-\beta,2}\big)\Big]+\\
&\quad + r_0^{2\beta_2}\Big[-2r_1G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)\big(f_{-\beta,2}g_{\alpha,2} - f_{\alpha,2}g_{-\beta,2}\big)\big(f_{\beta,2}-g_{\beta,2}\big)+\\
&\quad + 6r_1G_{\theta r}^{(1)}E^{(2)}\big(f_{-\beta,2}g_{\alpha,2} - f_{\alpha,2}g_{-\beta,2}\big)\big(1+\phi_{\beta,2}\big)\Big]\Big\}
\end{aligned}
\tag{A.7}
$$

$$\begin{aligned}
\Xi_{c_n} = & r_0^{\alpha_n} r_1^{\alpha_n} \Big(r_1^{2\beta_n} \Big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{\beta,n} \big(f_{-\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{-\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{\beta,n} \big(-g_{-\beta,n} h_{n,\alpha} + g_{\alpha,n} h_{n,-\beta} + f_{-\beta,n} m_{n,\alpha} - f_{\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{\beta,n} \big(-\phi_{\alpha,n} + \phi_{-\beta,n} \big) \Big) + \\
& + r_0^{2\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{-\beta,n} \big(f_{\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{-\beta,n} \big(g_{\beta,n} h_{n,\alpha} - g_{\alpha,n} h_{n,\beta} - f_{\beta,n} m_{n,\alpha} + f_{\alpha,n} m_{n,\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{-\beta,n} \big(\phi_{\alpha,n} - \phi_{\beta,n} \big) \big) \Big) + \\
& + r_0^{\beta_n} r_1^{2\alpha_n+\beta_n} \big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{\alpha,n} \big(f_{\beta,n} g_{-\beta,n} - f_{-\beta,n} g_{\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{\alpha,n} \big(-g_{\beta,n} h_{n,-\beta} + g_{-\beta,n} h_{n,\beta} + f_{\beta,n} m_{n,-\beta} - f_{-\beta,n} m_{n,\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{\alpha,n} \big(-\phi_{-\beta,n} + \phi_{\beta,n} \big) \big)
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
\Upsilon_{c_n} = & r_0^{\alpha_n} r_1^{\alpha_n} \Big(r_1^{2\beta_n} \Big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) \big(f_{-\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{-\beta,n} \big) g_{\beta,n} + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} g_{\beta,n} \big(-g_{-\beta,n} h_{n,\alpha} + g_{\alpha,n} h_{n,-\beta} + f_{-\beta,n} m_{n,\alpha} - f_{\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{\beta,n} \big(-\phi_{\alpha,n} + \phi_{-\beta,n} \big) \Big) + \\
& + r_0^{2\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) g_{-\beta,n} \big(f_{\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} g_{-\beta,n} \big(g_{\beta,n} h_{n,\alpha} - g_{\alpha,n} h_{n,\beta} - f_{\beta,n} m_{n,\alpha} + f_{\alpha,n} m_{n,\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{-\beta,n} \big(\phi_{\alpha,n} - \phi_{\beta,n} \big) \big) \Big) + \\
& + r_0^{\beta_n} r_1^{2\alpha_n+\beta_n} \big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) g_{\alpha,n} \big(f_{\beta,n} g_{-\beta,n} - f_{-\beta,n} g_{\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} g_{\alpha,n} \big(-g_{\beta,n} h_{n,-\beta} + g_{-\beta,n} h_{n,\beta} + f_{\beta,n} m_{n,-\beta} - f_{-\beta,n} m_{n,\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{\alpha,n} \big(-\phi_{-\beta,n} + \phi_{\beta,n} \big) \big)
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
\Theta_{G_2} = & r_0^{\beta_2} r_1^{2\alpha_2+\beta_2} \Big[2 r_1 G_{\theta r}^{(1)2} \big(1+\nu^{(2)} \big) \big(f_{-\alpha,2} - g_{-\alpha,2} \big) \big(f_{-\beta,2} g_{\alpha,2} - f_{\alpha,2} g_{-\beta,2} \big) - \\
& - 6 r_1 G_{\theta r}^{(1)} E^{(2)} \big(f_{-\beta,2} g_{\alpha,2} - f_{\alpha,2} g_{-\beta,2} \big) \big(1+\phi_{-\alpha,2} \big) \Big] + \\
& + r_0^{\alpha_2} r_1^{\alpha_2} \Big\{ r_0^{\alpha_2+\beta_2} \Big[-2 r_1 G_{\theta r}^{(1)2} \big(1+\nu^{(2)} \big) \big(f_{\alpha,2} - g_{\alpha,2} \big) \big(f_{-\beta,2} g_{-\alpha,2} - f_{-\alpha,2} g_{-\beta,2} \big) + \\
& + 6 r_1 G_{\theta r}^{(1)} E^{(2)} \big(f_{-\beta,2} g_{-\alpha,2} - f_{-\alpha,2} g_{-\beta,2} \big) \big(1+\phi_{\alpha,2} \big) \Big] + \\
& + r_1^{2\beta_2} \Big[2 r_1 G_{\theta r}^{(1)2} \big(1+\nu^{(2)} \big) \big(f_{\alpha,2} g_{-\alpha,2} - f_{-\alpha,2} g_{\alpha,2} \big) \big(f_{-\beta,2} - g_{-\beta,2} \big) - \\
& - 6 r_1 G_{\theta r}^{(1)} E^{(2)} \big(f_{\alpha,2} g_{-\alpha,2} - f_{-\alpha,2} g_{\alpha,2} \big) \big(1+\phi_{-\beta,2} \big) \Big] \Big\}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\Xi_{G_n} = & r_0^{\alpha_n+\beta_n} r_1^{\alpha_n+\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{-\beta,n} \big(f_{\alpha,n} g_{-\alpha,n} - f_{-\alpha,n} g_{\alpha,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{-\beta,n} \big(g_{\alpha,n} h_{n,-\alpha} - g_{-\alpha,n} h_{n,\alpha} - f_{\alpha,n} m_{n,-\alpha} + f_{-\alpha,n} m_{n,\alpha} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{-\beta,n} \big(\phi_{-\alpha,n} - \phi_{\alpha,n} \big) \big) + \\
& + r_0^{2\alpha_n} r_1^{2\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{-\alpha,n} \big(f_{-\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{-\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{-\alpha,n} \big(g_{-\beta,n} h_{n,\alpha} - g_{\alpha,n} h_{n,-\beta} - f_{-\beta,n} m_{n,\alpha} + f_{\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{-\alpha,n} \big(\phi_{\alpha,n} - \phi_{-\beta,n} \big) \big) + \\
& + r_1^{2\alpha_n+2\beta_n} \big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) f_{\alpha,n} \big(f_{-\beta,n} g_{-\alpha,n} - f_{-\alpha,n} g_{-\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} f_{\alpha,n} \big(-g_{-\beta,n} h_{n,-\alpha} + g_{-\alpha,n} h_{n,-\beta} + f_{-\beta,n} m_{n,-\alpha} - f_{-\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} f_{\alpha,n} \big(-\phi_{-\alpha,n} + \phi_{-\beta,n} \big) \big)
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\Upsilon_{G_n} = & r_0^{\alpha_n+\beta_n} r_1^{\alpha_n+\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) \big(f_{\alpha,n} g_{-\alpha,n} - f_{-\alpha,n} g_{\alpha,n} \big) g_{-\beta,n} + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} g_{-\beta,n} \big(g_{\alpha,n} h_{n,-\alpha} - g_{-\alpha,n} h_{n,\alpha} - f_{\alpha,n} m_{n,-\alpha} + f_{-\alpha,n} m_{n,\alpha} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{-\beta,n} \big(\phi_{-\alpha,n} - \phi_{\alpha,n} \big) \big) + \\
& + r_0^{2\alpha_n} r_1^{2\beta_n} \big(r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) g_{-\alpha,n} \big(f_{-\beta,n} g_{\alpha,n} - f_{\alpha,n} g_{-\beta,n} \big) + \\
& + r_0 G_{\theta r}^{(1)} E^{(2)} g_{-\alpha,n} \big(g_{-\beta,n} h_{n,\alpha} - g_{\alpha,n} h_{n,-\beta} - f_{-\beta,n} m_{n,\alpha} + f_{\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{-\alpha,n} \big(\phi_{\alpha,n} - \phi_{-\beta,n} \big) \big) + \\
& + r_1^{2\alpha_n+2\beta_n} \big(-r_0 G_{\theta r}^{(1)2} \big(-3+\nu^{(2)} \big) \big(1+\nu^{(2)} \big) g_{\alpha,n} \big(f_{-\beta,n} g_{-\alpha,n} - f_{-\alpha,n} g_{-\beta,n} \big) + \\
& + r_0 G_{12}^{(1)} E^{(2)} g_{\alpha,n} \big(-g_{-\beta,n} h_{n,-\alpha} + g_{-\alpha,n} h_{n,-\beta} + f_{-\beta,n} m_{n,-\alpha} - f_{-\alpha,n} m_{n,-\beta} \big) + \\
& + \big(n^2-1 \big) r_0 E^{(2)2} g_{\alpha,n} \big(-\phi_{-\alpha,n} + \phi_{-\beta,n} \big) \big)
\end{aligned} \tag{A.12}$$

$$\begin{aligned}\Theta_{g_2} = & r_1^{2\alpha_2+2\beta_2}\Big[-2r_1G_{\theta r}^{(1)2}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,2}-g_{-\alpha,2}\right)\left(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\right)+\\ & +6r_1G_{\theta r}^{(1)}E^{(2)}\left(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\right)\left(1+\phi_{-\alpha,2}\right)\Big]+\\ & +r_0^{\alpha_2}r_1^{\beta_2}\Big\{r_0^{\alpha_2}r_1^{\beta_2}\Big[2r_1G_{\theta r}^{(1)2}\left(1+\nu^{(2)}\right)\left(f_{\alpha,2}-g_{\alpha,2}\right)\left(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\right)-\\ & -6r_1G_{\theta r}^{(1)}E^{(2)}\left(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\right)\left(1+\phi_{\alpha,2}\right)\Big]+\\ & +r_1^{\alpha_2}r_0^{\beta_2}\Big[-2r_1G_{\theta r}^{(1)2}\left(1+\nu^{(2)}\right)\left(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\right)\left(f_{\beta,2}-g_{\beta,2}\right)+\\ & +6r_1G_{\theta r}^{(1)}E^{(2)}\left(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\right)\left(1+\phi_{\beta,2}\right)\Big]\Big\}\end{aligned}\tag{A.13}$$

$$\begin{aligned}\Xi_{g_n} = & r_0^{\alpha_n}r_1^{\alpha_n+2\beta_n}(r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)f_{\beta,n}\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}f_{\beta,n}\left(g_{\alpha,n}h_{n,-\alpha}-g_{-\alpha,n}h_{n,\alpha}-f_{\alpha,n}m_{n,-\alpha}+f_{-\alpha,n}m_{n,\alpha}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}f_{\beta,n}\left(\phi_{-\alpha,n}-\phi_{\alpha,n}\right))+\\ & +r_0^{2\alpha_n+\beta_n}r_1^{\beta_n}(r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)f_{-\alpha,n}\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}f_{-\alpha,n}\left(g_{\beta,n}h_{n,\alpha}-g_{\alpha,n}h_{n,\beta}-f_{\beta,n}m_{n,\alpha}+f_{\alpha,n}m_{n,\beta}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}f_{-\alpha,n}\left(\phi_{\alpha,n}-\phi_{\beta,n}\right))+\\ & +r_0^{\beta_n}r_1^{2\alpha_n+\beta_n}(-r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)f_{\alpha,n}\left(f_{\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\beta,n}\right)+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}f_{\alpha,n}\left(-g_{\beta,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,\beta}+f_{\beta,n}m_{n,-\alpha}-f_{-\alpha,n}m_{n,\beta}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}f_{\alpha,n}\left(-\phi_{-\alpha,n}+\phi_{\beta,n}\right))\end{aligned}\tag{A.14}$$

$$\begin{aligned}\Upsilon_{g_n} = & r_0^{\alpha_n}r_1^{\alpha_n+2\beta_n}(r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)g_{\beta,n}+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}g_{\beta,n}\left(g_{\alpha,n}h_{n,-\alpha}-g_{-\alpha,n}h_{n,\alpha}-f_{\alpha,n}m_{n,-\alpha}+f_{-\alpha,n}m_{n,\alpha}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}g_{\beta,n}\left(\phi_{-\alpha,n}-\phi_{\alpha,n}\right))+\\ & +r_0^{2\alpha_n+\beta_n}r_1^{\beta_n}(r_0G_{\theta r}^{(1)}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)g_{-\alpha,n}\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}g_{-\alpha,n}\left(g_{\beta,n}h_{n,\alpha}-g_{\alpha,n}h_{n,\beta}-f_{\beta,n}m_{n,\alpha}+f_{\alpha,n}m_{n,\beta}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}g_{-\alpha,n}\left(\phi_{\alpha,n}-\phi_{\beta,n}\right))+\\ & +r_0^{\beta_n}r_1^{2\alpha_n+\beta_n}(-r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)g_{\alpha,n}\left(f_{\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\beta,n}\right)+\\ & +r_0G_{\theta r}^{(1)}E^{(2)}g_{\alpha,n}\left(-g_{\beta,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,\beta}+f_{\beta,n}m_{n,-\alpha}-f_{-\alpha,n}m_{n,\beta}\right)+\\ & +\left(n^2-1\right)r_0E^{(2)2}g_{\alpha,n}\left(-\phi_{-\alpha,n}+\phi_{\beta,n}\right))\end{aligned}\tag{A.15}$$

$$\begin{aligned}\Omega_{C_n} = & G_{\theta r}^{(1)}(r_1^{2\alpha_n}\Big(r_1^{2\beta_n}\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)\Big(G_{\theta r}^{(1)}f_{-\beta,n}\Big(-G_{\theta r}^{(1)}\left(\nu^{(2)}-3\right)\left(1+\nu^{(2)}\right)g_{-\alpha,n}+E^{(2)}m_{n,-\alpha}\Big)+\\ & +G_{\theta r}^{(1)}f_{-\alpha,n}\Big(G_{\theta r}^{(1)}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)g_{-\beta,n}-E^{(2)}m_{n,-\beta}\Big)+\\ & +E^{(2)}\Big(G_{\theta r}^{(1)}\left(-g_{-\beta,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,-\beta}\right)+\left(n^2-1\right)E^{(2)}\left(-\phi_{-\alpha,n}+\phi_{-\beta,n}\right)\Big))+\\ & +r_0^{2\beta_n}\left(f_{-\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{-\beta,n}\right)(G_{\theta r}^{(1)}f_{\beta,n}\Big(G_{\theta r}^{(1)}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)g_{-\alpha,n}-E^{(2)}m_{n,-\alpha}\Big)+\\ & +G_{\theta r}^{(1)}f_{-\alpha,n}\Big(-G_{\theta r}^{(1)}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)g_{\beta,n}+E^{(2)}m_{n,\beta}\Big)+\\ & +E^{(2)}\Big(G_{\theta r}^{(1)}\left(g_{\beta,n}h_{n,-\alpha}-g_{-\alpha,n}h_{n,\beta}\right)+\left(n^2-1\right)E^{(2)}\left(\phi_{-\alpha,n}-\phi_{\beta,n}\right)\Big)))+\\ & +r_0^{\alpha_n+\beta_n}r_1^{\alpha_n+\beta_n}(2G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)\left(f_{\beta,n}g_{-\beta,n}-f_{-\beta,n}g_{\beta,n}\right)+\\ & +G_{\theta r}^{(1)}E^{(2)}(-(\left(f_{\beta,n}g_{-\beta,n}-f_{-\beta,n}g_{\beta,n}\right)\left(-g_{\alpha,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,\alpha}\right))+\\ & +f_{\alpha,n}(g_{-\alpha,n}\left(g_{\beta,n}h_{n,-\beta}-g_{-\beta,n}h_{n,\beta}\right)+f_{\beta,n}\left(-g_{-\beta,n}m_{n,-\alpha}-g_{-\alpha,n}m_{n,-\beta}\right)+\\ & +f_{-\beta,n}\left(g_{\beta,n}m_{n,-\alpha}+g_{-\alpha,n}m_{n,\beta}\right))+f_{-\alpha,n}\left(g_{\alpha,n}\left(-g_{\beta,n}h_{n,-\beta}+g_{-\beta,n}h_{n,\beta}\right)+\right.\\ & +f_{\beta,n}\left(g_{-\beta,n}m_{n,\alpha}+g_{\alpha,n}m_{n,-\beta}\right)+f_{-\beta,n}\left(-g_{\beta,n}m_{n,\alpha}-g_{\alpha,n}m_{n,\beta}\right)))+\left(n^2-1\right)E^{(2)2}\\ & (\left(f_{\beta,n}g_{-\beta,n}-f_{-\beta,n}g_{\beta,n}\right)\left(\phi_{-\alpha,n}-\phi_{\alpha,n}\right)+\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)\left(\phi_{-\beta,n}-\phi_{\beta,n}\right)))+\\ & +r_0^{2\alpha_n}\Big(r_1^{2\beta_n}\left(f_{\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\beta,n}\right)\Big(G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{-\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{-\beta,n}\right)+\\ & +\left(n^2-1\right)E^{(2)2}\left(\phi_{\alpha,n}-\phi_{-\beta,n}\right)-G_{\theta r}^{(1)}E^{(2)}(-(2n+1)g_{\alpha,n}+\nu^{(2)}g_{\alpha,n}+\\ & +(2n+1)g_{-\beta,n}-\nu^{(2)}g_{-\beta,n}+f_{-\beta,n}m_{n,\alpha}-f_{\alpha,n}m_{n,-\beta}+(n+2)g_{-\beta,n}\phi_{\alpha,n}-\\ & -n\nu^{(2)}g_{-\beta,n}\phi_{\alpha,n}+\left(-(n+2)+n\nu^{(2)}\right)g_{\alpha,n}\phi_{-\beta,n}))+\\ & +r_0^{2\beta_n}\left(f_{-\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{-\beta,n}\right)(-G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)+\\ & +\left(n^2-1\right)E^{(2)2}\left(-\phi_{\alpha,n}+\phi_{\beta,n}\right)+G_{\theta r}^{(1)}E^{(2)}(-(2n+1)g_{\alpha,n}+\nu^{(2)}g_{\alpha,n}+(2n+1)g_{\beta,n}-\\ & -\nu^{(2)}g_{\beta,n}+f_{\beta,n}m_{n,\alpha}-f_{\alpha,n}m_{n,\beta}+(n+2)g_{\beta,n}\phi_{\alpha,n}-\\ & -n\nu^{(2)}g_{\beta,n}\phi_{\alpha,n}+\left(n\nu^{(2)}-(n+2)\right)g_{\alpha,n}\phi_{\beta,n}))))\end{aligned}\tag{A.16}$$

Appendix B

Using the auxiliary functions (A.1) and Eqs. (23), it is possible to further derive parameters Ξ_i, Υ_i, Θ_i, Ω_i used in Eq. (22):

$$
\begin{aligned}
\Xi_{\overline{A}_c} = {} & {r_1}^{2\alpha_1}\Big(r_0 G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2 log\left[{r_1}^2\right]f_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)+\\
& +8r_0E^{(2)2}f_{\alpha,1}\left(\zeta - log\left[\frac{r_1}{r_0}\right]-log\left[\frac{r_1}{r_0}\right]\phi_{-\alpha,1}\right)-\\
& -r_0G_{\theta r}^{(1)}E^{(2)}\left(1+\nu^{(2)}\right)f_{\alpha,1}(2(\delta+\xi)+\nu^{(2)}(\delta-\xi)log\left[{r_1}^2\right]+3(-\delta+\xi)log\left[{r_1}^2\right]+\\
& +\left(2\zeta-4log\left[\frac{r_1}{r_0}\right]-3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right)f_{-\alpha,1}+\\
& +\left(-2\zeta+4log\left[\frac{r_1}{r_0}\right]-3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right)g_{-\alpha,1}+\\
& +\left(-2(\delta+\xi)+\nu^{(2)}(\delta-\xi)log\left[{r_1}^2\right]+3(-\delta+\xi)log\left[{r_1}^2\right]\right)\phi_{-\alpha,1}))+\\
& +{r_0}^{2\alpha_1}(-r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2 log\left[{r_1}^2\right]f_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)+\\
& +8r_0E^{(2)2}f_{-\alpha,1}\left(-\zeta+log\left[\frac{r_1}{r_0}\right]+log\left[\frac{r_1}{r_0}\right]\phi_{\alpha,1}\right)+\\
& +r_0G_{\theta r}^{(1)}E^{(2)}\left(1+\nu^{(2)}\right)f_{-\alpha,1}(2(\delta+\xi)+\nu^{(2)}(\delta-\xi)log\left[{r_1}^2\right]+3(-\delta+\xi)log\left[{r_1}^2\right]+\\
& +\left(2\zeta-4log\left[\frac{r_1}{r_0}\right]-3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right)f_{\alpha,1}+\\
& +\left(-2\zeta+4log\left[\frac{r_1}{r_0}\right]-3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right)g_{\alpha,1}+\\
& +\left(-2(\delta+\xi)+\nu^{(2)}(\delta-\xi)log\left[{r_1}^2\right]+3(-\delta+\xi)log\left[{r_1}^2\right]\right)\phi_{\alpha,1}))+\\
& +{r_0}^{\alpha_1}{r_1}^{\alpha_1}(-r_0G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2\xi log\left[{r_1}^2\right]\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)+\\
& +8r_0E^{(2)2}\xi\left(\phi_{-\alpha,1}-\phi_{\alpha,1}\right)+r_0G_{\theta r}^{(1)}E^{(2)}\left(1+\nu^{(2)}\right)\xi(2g_{-\alpha,1}-\\
& -3log\left[{r_1}^2\right]g_{-\alpha,1}+\nu^{(2)}log\left[{r_1}^2\right]g_{-\alpha,1}-2g_{\alpha,1}+3log\left[{r_1}^2\right]g_{\alpha,1}-\\
& -\nu^{(2)}log\left[{r_1}^2\right]g_{\alpha,1}+2g_{\alpha,1}\phi_{-\alpha,1}+3log\left[{r_1}^2\right]g_{\alpha,1}\phi_{-\alpha,1}-\nu^{(2)}log\left[{r_1}^2\right]g_{\alpha,1}\phi_{-\alpha,1}+\\
& +f_{\alpha,1}\left(2+3log\left[{r_1}^2\right]+\left(-2+3log\left[{r_1}^2\right]\right)\phi_{-\alpha,1}-\nu^{(2)}log\left[{r_1}^2\right]\left(1+\phi_{-\alpha,1}\right)\right)+\\
& +\left(-2-3log\left[{r_1}^2\right]+\nu^{(2)}log\left[{r_1}^2\right]\right)g_{-\alpha,1}\phi_{\alpha,1}+f_{-\alpha,1}(-2-3log\left[{r_1}^2\right]+\\
& +\left(2-3log\left[{r_1}^2\right]\right)\phi_{\alpha,1}+\phi_{\alpha,1}+\nu^{(2)}log\left[{r_1}^2\right]\left(1+\phi_{\alpha,1}\right))))
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
Y_{\overline{A}_c} = {} & {r_1}^{2\alpha_1} r_0 G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2 log\left[{r_1}^2\right] g_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)+ \\
& +8 r_0 E^{(2)2} g_{\alpha,1}\left(\zeta - log\left[\frac{r_1}{r_0}\right] - log\left[\frac{r_1}{r_0}\right]\phi_{-\alpha,1}\right)- \\
& -r_0 G_{\theta r}^{(1)} E^{(2)}\left(1+\nu^{(2)}\right) g_{\alpha,1}(2(\delta+\xi)+\nu^{(2)}(\delta-\xi) log\left[{r_1}^2\right]+3(-\delta+\xi) log\left[{r_1}^2\right]+ \\
& +\left(2\zeta - 4 log\left[\frac{r_1}{r_0}\right] - 3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right) f_{-\alpha,1}+ \\
& +\left(-2\zeta + 4 log\left[\frac{r_1}{r_0}\right] - 3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right) g_{-\alpha,1}+ \\
& +\left(-2(\delta+\xi)+\nu^{(2)}(\delta-\xi) log\left[{r_1}^2\right]+3(-\delta+\xi) log\left[{r_1}^2\right]\right)\phi_{-\alpha,1}))+ \\
& +{r_0}^{2\alpha_1}(-r_0 G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2 log\left[{r_1}^2\right] g_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)+ \\
& +8 r_0 E^{(2)2} g_{-\alpha,1}\left(-\zeta + log\left[\frac{r_1}{r_0}\right] + log\left[\frac{r_1}{r_0}\right]\phi_{\alpha,1}\right)+ \\
& +r_0 G_{\theta r}^{(1)} E^{(2)}\left(1+\nu^{(2)}\right) g_{-\alpha,1}(2(\delta+\xi)+\nu^{(2)}(\delta-\xi) log\left[{r_1}^2\right]+3(-\delta+\xi) log\left[{r_1}^2\right]+ \\
& +\left(2\zeta - 4 log\left[\frac{r_1}{r_0}\right] - 3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right) f_{\alpha,1}+ \\
& +\left(-2\zeta + 4 log\left[\frac{r_1}{r_0}\right] - 3\zeta log\left[{r_1}^2\right]+\nu^{(2)}\zeta log\left[{r_1}^2\right]\right) g_{\alpha,1}+ \\
& +\left(-2(\delta+\xi)+\nu^{(2)}(\delta-\xi) log\left[{r_1}^2\right]+3(-\delta+\xi) log\left[{r_1}^2\right]\right)\phi_{\alpha,1}))+ \\
& +{r_0}^{\alpha_1}{r_1}^{\alpha_1}(-r_0 G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)^2 \delta log\left[{r_1}^2\right]\left(f_{\alpha,1} g_{-\alpha,1} - f_{-\alpha,1} g_{\alpha,1}\right)+ \\
& +8 r_0 E^{(2)2}\delta\left(\phi_{\alpha,1}-\phi_{-\alpha,1}\right) - r_0 G_{\theta r}^{(1)} E^{(2)}\left(1+\nu^{(2)}\right)\delta(2 g_{-\alpha,1}- \\
& -3 log\left[{r_1}^2\right] g_{-\alpha,1}+\nu^{(2)} log\left[{r_1}^2\right] g_{-\alpha,1} - 2 g_{\alpha,1} + 3 log\left[{r_1}^2\right] g_{\alpha,1}- \\
& -\nu^{(2)} log\left[{r_1}^2\right] g_{\alpha,1} + 2 g_{\alpha,1}\phi_{-\alpha,1} + 3 log\left[{r_1}^2\right] g_{\alpha,1}\phi_{-\alpha,1} - \nu^{(2)} log\left[{r_1}^2\right] g_{\alpha,1}\phi_{-\alpha,1}+ \\
& +f_{\alpha,1}\left(2+3 log\left[{r_1}^2\right]+\left(-2+3 log\left[{r_1}^2\right]\right)\phi_{-\alpha,1} - \nu^{(2)} log\left[{r_1}^2\right]\left(1+\phi_{-\alpha,1}\right)\right)+ \\
& +\left(-2-3 log\left[{r_1}^2\right]+\nu^{(2)} log\left[{r_1}^2\right]\right) g_{-\alpha,1}\phi_{\alpha,1} + f_{-\alpha,1}\left(-2-3 log\left[{r_1}^2\right]\right]+ \\
& +\left(2-3 log\left[{r_1}^2\right]\right)\phi_{\alpha,1}+\phi_{\alpha,1}+\nu^{(2)} log\left[{r_1}^2\right]\left(1+\phi_{\alpha,1}\right))))
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\Omega_{\overline{A}_c} = {} & 4 G_{\theta r}^{(1)}\left(-3+\nu^{(2)}\right)(r_0^{\alpha_1} r_1^{\alpha_1}\left(2E^{(2)}\zeta - G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)(\delta+\xi)\right)\left(f_{\alpha,1} g_{-\alpha,1} - f_{-\alpha,1} g_{\alpha,1}\right)+ \\
& +r_1^{2\alpha_1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)\left(-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,1}-g_{-\alpha,1}\right)+2E^{(2)}\left(1+\phi_{-\alpha,1}\right)\right)+ \\
& +r_0^{2\alpha_1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)\left(G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{\alpha,1}-g_{\alpha,1}\right)-2E^{(2)}\left(1+\phi_{\alpha,1}\right)\right))
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\Xi_{\overline{A}_0} = {} & r_1^{2\alpha_1}(-r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 f_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)+ \\
& +r_0 E^{(2)}\left(1+\nu^{(2)}\right) f_{\alpha,1}\left(\zeta\left(f_{-\alpha,1}+g_{-\alpha,1}\right)+(\delta-\xi)\left(1+\phi_{-\alpha,1}\right)\right))+ \\
& +r_0^{2\alpha_1}(r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 f_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)- \\
& -r_0 E^{(2)}\left(1+\nu^{(2)}\right) f_{-\alpha,1}\left(\zeta\left(f_{\alpha,1}+g_{\alpha,1}\right)+(\delta-\xi)\left(1+\phi_{\alpha,1}\right)\right))+ \\
& +r_0^{\alpha_1} r_1^{\alpha_1}(r_0 G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2\xi\left(f_{\alpha,1} g_{-\alpha,1} - f_{-\alpha,1} g_{\alpha,1}\right)+ \\
& +r_0 E^{(2)}\left(1+\nu^{(2)}\right)\xi(f_{\alpha,1}+g_{\alpha,1}+\left(f_{\alpha,1}+g_{\alpha,1}\right)\phi_{-\alpha,1}- \\
& -f_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)-g_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)))
\end{aligned}
\tag{B.4}
$$

$$\begin{aligned}\Upsilon_{\overline{A}_0} = & r_1^{2\alpha_1}(-r_0G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 g_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)+\\ & +r_0E^{(2)}\left(1+\nu^{(2)}\right)g_{\alpha,1}\left(\zeta\left(f_{-\alpha,1}+g_{-\alpha,1}\right)+(\delta-\xi)\left(1+\phi_{-\alpha,1}\right)\right))+\\ & +r_0^{2\alpha_1}(r_0G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 g_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)-\\ & -r_0E^{(2)}\left(1+\nu^{(2)}\right)g_{-\alpha,1}\left(\zeta\left(f_{\alpha,1}+g_{\alpha,1}\right)+(\delta-\xi)\left(1+\phi_{\alpha,1}\right)\right))+\\ & +r_0^{\alpha_1}r_1^{\alpha_1}(-r_0G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2\delta\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)+\\ & +r_0E^{(2)}\left(1+\nu^{(2)}\right)\delta(-f_{\alpha,1}-g_{\alpha,1}-\left(f_{\alpha,1}+g_{\alpha,1}\right)\phi_{-\alpha,1}+\\ & +f_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)+g_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)))\end{aligned} \tag{B.5}$$

$$\begin{aligned}\Omega_{\overline{A}_0} = & 4(r_0^{\alpha_1}r_1^{\alpha_1}\left(2E^{(2)}\zeta-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)(\delta+\xi)\right)\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)+\\ & +r_1^{2\alpha_1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)\left(-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,1}-g_{-\alpha,1}\right)+2E^{(2)}\left(1+\phi_{-\alpha,1}\right)\right)+\\ & +r_0^{2\alpha_1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)\left(G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{\alpha,1}-g_{\alpha,1}\right)-2E^{(2)}\left(1+\phi_{\alpha,1}\right)\right))\end{aligned} \tag{B.6}$$

$$\begin{aligned}\Xi_{\overline{C}_{-2}} = & r_1^{2\alpha_1}(r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 f_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)-\\ & -r_0r_1^2E^{(2)}f_{\alpha,1}(\left(3+\nu^{(2)}\right)\zeta f_{-\alpha,1}+\left(-1+\nu^{(2)}\right)\zeta g_{-\alpha,1}+\\ & +\left(-\delta+\nu^{(2)}(\delta-\xi)-3\xi\right)\left(1+\phi_{-\alpha,1}\right)))+r_0^{2\alpha_1}(-r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 f_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)-\\ & -r_0r_1^2E^{(2)}f_{-\alpha,1}(\left(\zeta-\nu^{(2)}\zeta\right)g_{\alpha,1}-\left(\left(3+\nu^{(2)}\right)\zeta f_{\alpha,1}\right)-\left(\nu^{(2)}(\delta-\xi)-\delta-3\xi\right)\left(1+\phi_{\alpha,1}\right)))+\\ & +r_0^{\alpha_1}r_1^{\alpha_1}(-r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2\xi\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)+\\ & +r_0r_1^2E^{(2)}\xi(-\left(\left(3+\nu^{(2)}\right)f_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)\right)+\left(3+\nu^{(2)}\right)f_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)+\\ & +\left(-1+\nu^{(2)}\right)\left(-g_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)+g_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)\right)))\end{aligned} \tag{B7}$$

$$\begin{aligned}\Upsilon_{\overline{C}_{-2}} = & r_1^{2\alpha_1}(r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 g_{\alpha,1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)-\\ & -r_0r_1^2E^{(2)}f_{\alpha,1}(\left(3+\nu^{(2)}\right)\zeta g_{-\alpha,1}+\left(-1+\nu^{(2)}\right)\zeta g_{-\alpha,1}+\\ & +\left(-\delta+\nu^{(2)}(\delta-\xi)-3\xi\right)\left(1+\phi_{-\alpha,1}\right)))+r_0^{2\alpha_1}(-r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2 g_{-\alpha,1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)-\\ & -r_0r_1^2E^{(2)}g_{-\alpha,1}(\left(\zeta-\nu^{(2)}\zeta\right)g_{\alpha,1}-\left(\left(3+\nu^{(2)}\right)\zeta f_{\alpha,1}\right)-\left(\nu^{(2)}(\delta-\xi)-\delta-3\xi\right)\left(1+\phi_{\alpha,1}\right)))+\\ & +r_0^{\alpha_1}r_1^{\alpha_1}(-r_0r_1^2G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)^2\delta\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)-\\ & -r_0r_1^2E^{(2)}\delta(-\left(\left(3+\nu^{(2)}\right)f_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)\right)+\left(3+\nu^{(2)}\right)f_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)+\\ & +\left(-1+\nu^{(2)}\right)\left(-g_{\alpha,1}\left(1+\phi_{-\alpha,1}\right)+g_{-\alpha,1}\left(1+\phi_{\alpha,1}\right)\right)))\end{aligned} \tag{B.8}$$

$$\begin{aligned}\Omega_{\overline{C}_{-2}} = & 4(r_0^{\alpha_1}r_1^{\alpha_1}\left(2E^{(2)}\zeta-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)(\delta+\xi)\right)\left(f_{\alpha,1}g_{-\alpha,1}-f_{-\alpha,1}g_{\alpha,1}\right)+\\ & +r_1^{2\alpha_1}\left(\delta f_{\alpha,1}+\xi g_{\alpha,1}\right)\left(-G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{-\alpha,1}-g_{-\alpha,1}\right)+2E^{(2)}\left(1+\phi_{-\alpha,1}\right)\right)+\\ & +r_0^{2\alpha_1}\left(\delta f_{-\alpha,1}+\xi g_{-\alpha,1}\right)\left(G_{\theta r}^{(1)}\left(1+\nu^{(2)}\right)\left(f_{\alpha,1}-g_{\alpha,1}\right)-2E^{(2)}\left(1+\phi_{\alpha,1}\right)\right))\end{aligned} \tag{B.9}$$

$$
\begin{aligned}
\Theta_{\overline{A}_{-1}} = {} & r_1{}^2(r_0{}^{2\alpha_2}\big(r_1{}^{2\beta_2}\big(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\big)\big(G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{-\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{-\beta,2}\big)\\
& +3E^{(2)2}\big(\phi_{\alpha,2}-\phi_{-\beta,2}\big)-G_{\theta r}^{(1)}E^{(2)}\big(1+\nu^{(2)}\big)(2f_{\alpha,2}+g_{\alpha,2}-f_{-\beta,2}\big(2+\phi_{\alpha,2}\big)-\\
& -g_{-\beta,2}\big(1+2\phi_{\alpha,2}\big)+\big(f_{\alpha,2}+2g_{\alpha,2}\big)\phi_{-\beta,2}))+\\
& +r_0{}^{2\beta_2}\big(f_{-\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{-\beta,2}\big)(-G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(-\phi_{\alpha,2}+\phi_{\beta,2}\big)+G_{\theta r}^{(1)}E^{(2)}\big(1+\nu^{(2)}\big)(2f_{\alpha,2}+g_{\alpha,2}-\\
& -f_{\beta,2}\big(2+\phi_{\alpha,2}\big)-g_{\beta,2}\big(1+2\phi_{\alpha,2}\big)+\big(f_{\alpha,2}+2g_{\alpha,2}\big)\phi_{\beta,2})))+\\
& +r_1{}^{2\alpha_2}\big(r_1{}^{2\beta_2}\big(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\big)\big(-G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{-\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{-\beta,2}\big)+\\
& +3E^{(2)2}\big(-\phi_{-\alpha,2}+\phi_{-\beta,2}\big)+G_{\theta r}^{(1)}E^{(2)}\big(1+\nu^{(2)}\big)(-f_{-\beta,2}\big(2+\phi_{-\alpha,2}\big)-g_{-\beta,2}\big(1+2\phi_{-\alpha,2}\big)+\\
& +f_{-\alpha,2}\big(2+\phi_{-\beta,2}\big)+g_{-\alpha,2}\big(1+2\phi_{-\beta,2}\big)))+\\
& +r_0{}^{2\beta_2}\big(f_{-\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{-\beta,2}\big)(G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(\phi_{-\alpha,2}-\phi_{\beta,2}\big)-G_{\theta r}^{(1)}E^{(2)}\big(1+\nu^{(2)}\big)\big(-2f_{\beta,2}-g_{\beta,2}-\\
& -\big(f_{\beta,2}+2g_{\beta,2}\big)\phi_{-\alpha,2}+f_{-\alpha,2}\big(2+\phi_{\beta,2}\big)+g_{-\alpha,2}\big(1+2\phi_{\beta,2}\big))))+\\
& +r_0{}^{\alpha_2+\beta_2}r_1{}^{\alpha_2+\beta_2}(2G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\big)\big(f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(\big(f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)\big(\phi_{-\alpha,2}-\phi_{\alpha,2}\big)+\big(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\big)\big(\phi_{-\beta,2}-\phi_{\beta,2}\big)\big)-\\
& -G_{\theta r}^{(1)}E^{(2)}\big(1+\nu^{(2)}\big)(\big(f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)\big(-g_{\alpha,2}\big(1+2\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(1+2\phi_{\alpha,2}\big)\big)+\\
& +f_{-\alpha,2}(f_{\beta,2}\big(g_{-\beta,2}\big(2+\phi_{\alpha,2}\big)+g_{\alpha,2}\big(2+\phi_{-\beta,2}\big)\big)-f_{-\beta,2}\big(g_{\beta,2}\big(2+\phi_{\alpha,2}\big)+g_{\alpha,2}\big(2+\phi_{\beta,2}\big)\big)+\\
& +g_{\alpha,2}\big(g_{\beta,2}\big(1+2\phi_{-\beta,2}\big)-g_{-\beta,2}\big(1+2\phi_{\beta,2}\big)))+\\
& +f_{\alpha,2}(-f_{\beta,2}\big(g_{-\beta,2}\big(2+\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(2+\phi_{-\beta,2}\big)\big)+\\
& +f_{-\beta,2}\big(g_{\beta,2}\big(2+\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(2+\phi_{\beta,2}\big)\big)+\\
& +g_{-\alpha,2}\big(-g_{\beta,2}\big(1+2\phi_{-\beta,2}\big)+g_{-\beta,2}\big(1+2\phi_{\beta,2}\big))))))
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\Theta_{\overline{C}_{-3}} = {} & r_1{}^4(r_1{}^{2\alpha_2}\big(r_1{}^{2\beta_2}\big(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\big)\big(-G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{-\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{-\beta,2}\big)+\\
& +3E^{(2)2}\big(-\phi_{-\alpha,2}+\phi_{-\beta,2}\big)+G_{\theta r}^{(1)}E^{(2)}\big(-g_{-\alpha,2}+g_{-\beta,2}+f_{-\beta,2}\big(-2\big(2+\nu^{(2)}\big)-\\
& -\big(3+\nu^{(2)}\big)\phi_{-\alpha,2})+f_{-\alpha,2}\big(2\big(2+\nu^{(2)}\big)+\big(3+\nu^{(2)}\big)\phi_{-\beta,2}\big)+\\
& +\nu^{(2)}\big(-g_{-\beta,2}\big(1+2\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(1+2\phi_{-\beta,2}\big)))))+\\
& +r_0{}^{2\beta_2}\big(f_{-\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{-\beta,2}\big)(G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(\phi_{-\alpha,2}-\phi_{\beta,2}\big)-G_{\theta r}^{(1)}E^{(2)}(-g_{-\alpha,2}+g_{\beta,2}+f_{\beta,2}\big(-2\big(2+\nu^{(2)}\big)-\big(3+\nu^{(2)}\big)\phi_{-\alpha,2}\big)+\\
& +f_{-\alpha,2}\big(4+3\phi_{\beta,2}+\nu^{(2)}\big(2+\phi_{\beta,2}\big)\big)+\nu^{(2)}\big(-g_{\beta,2}\big(1+2\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(1+2\phi_{\beta,2}\big)))))+\\
& +r_0{}^{2\alpha_2}\big(r_1{}^{2\beta_2}\big(f_{\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\beta,2}\big)\big(G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{-\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{-\beta,2}\big)+\\
& +3E^{(2)2}\big(\phi_{\alpha,2}-\phi_{-\beta,2}\big)-G_{\theta r}^{(1)}E^{(2)}(-g_{\alpha,2}+g_{-\beta,2}-f_{-\beta,2}\big(4+3\phi_{\alpha,2}+\nu^{(2)}\big(2+\phi_{\alpha,2}\big)\big)+\\
& +f_{\alpha,2}\big(2\big(2+\nu^{(2)}\big)+\big(3+\nu^{(2)}\big)\phi_{-\beta,2}\big)+\nu^{(2)}\big(-g_{-\beta,2}\big(1+2\phi_{\alpha,2}\big)+g_{\alpha,2}\big(1+2\phi_{-\beta,2}\big))))+\\
& +r_0{}^{2\beta_2}\big(f_{-\beta,2}g_{-\alpha,2}-f_{-\alpha,2}g_{-\beta,2}\big)(-G_{12}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\beta,2}g_{\alpha,2}-f_{\alpha,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(-\phi_{\alpha,2}+\phi_{\beta,2}\big)+G_{\theta r}^{(1)}E^{(2)}(-g_{\alpha,2}+g_{\beta,2}+\\
& +f_{\beta,2}\big(-4-3\phi_{\alpha,2}-\nu^{(2)}\big(2+\phi_{\alpha,2}\big)\big)+f_{\alpha,2}\big(4+3\phi_{\beta,2}+\nu^{(2)}\big(2+\phi_{\beta,2}\big)\big)+\\
& +\nu^{(2)}\big(-g_{\beta,2}\big(1+2\phi_{\alpha,2}\big)+g_{\alpha,2}\big(1+2\phi_{\beta,2}\big)))))+\\
& +r_0{}^{\alpha_2+\beta_2}r_1{}^{\alpha_2+\beta_2}(2G_{\theta r}^{(1)2}\big(1+\nu^{(2)}\big)^2\big(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\big)\big(f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)+\\
& +3E^{(2)2}\big(\big(f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)\big(\phi_{-\alpha,2}-\phi_{\alpha,2}\big)+\big(f_{\alpha,2}g_{-\alpha,2}-f_{-\alpha,2}g_{\alpha,2}\big)\big(\phi_{-\beta,2}-\phi_{\beta,2}\big)\big)+\\
& +G_{\theta r}^{(1)}E^{(2)}\big(-((f_{\beta,2}g_{-\beta,2}-f_{-\beta,2}g_{\beta,2}\big)(g_{\alpha,2}-\nu^{(2)}g_{\alpha,2}\big(1+2\phi_{-\alpha,2}\big)+\\
& +g_{-\alpha,2}\big(-1+\nu^{(2)}\big(1+2\phi_{\alpha,2}\big)))))+\\
& +f_{\alpha,2}(f_{\beta,2}\big(g_{-\beta,2}\big(2\big(2+\nu^{(2)}\big)+\big(3+\nu^{(2)}\big)\phi_{-\alpha,2}\big)+g_{-\alpha,2}\big(2\big(2+\nu^{(2)}\big)+\\
& +\big(3+\nu^{(2)}\big)\phi_{-\beta,2}))-f_{-\beta,2}(g_{\beta,2}\big(2\big(2+\nu^{(2)}\big)+\big(3+\nu^{(2)}\big)\phi_{-\alpha,2}\big)+\\
& +g_{-\alpha,2}\big(4+3\phi_{\beta,2}+\nu^{(2)}\big(2+\phi_{\beta,2}\big)))+\\
& +g_{-\alpha,2}\big(g_{\beta,2}\big(-1+\nu^{(2)}\big(1+2\phi_{-\beta,2}\big)\big)-g_{-\beta,2}\big(-1+\nu^{(2)}\big(1+2\phi_{\beta,2}\big))))+\\
& +f_{-\alpha,2}(-f_{\beta,2}\big(g_{-\beta,2}\big(4+3\phi_{\alpha,2}+\nu^{(2)}\big(2+\phi_{\alpha,2}\big)\big)+g_{\alpha,2}\big(2\big(2+\nu^{(2)}\big)+\big(3+\nu^{(2)}\big)\phi_{-\beta,2}\big)\big)+\\
& +f_{-\beta,2}\big(g_{\beta,2}\big(4+3\phi_{\alpha,2}+\nu^{(2)}\big(2+\phi_{\alpha,2}\big)\big)+g_{\alpha,2}\big(4+3\phi_{\beta,2}+\nu^{(2)}\big(2+\phi_{\beta,2}\big)))+\\
& +g_{\alpha,2}\big(g_{\beta,2}-\nu^{(2)}g_{\beta,2}\big(1+2\phi_{-\beta,2}\big)+g_{-\beta,2}\big(-1+\nu^{(2)}\big(1+2\phi_{\beta,2}\big)))))))
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
\Xi_{\overline{A}_{-(n-1)}} = {} & r_0 {r_1}^{n-1} E^{(2)} ({r_0}^{\beta_n} {r_1}^{2\alpha_n+\beta_n} f_{\alpha,n} \Big(- G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{-\beta,n} (\phi_{-\alpha,n} - 1) - g_{-\alpha,n} (\phi_{-\beta,n} - 1)) + \\
& + f_{-\beta,n} (g_{-\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{\beta,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{\beta,n} - 1))) + \\
& + (n+1) E^{(2)} (g_{-\beta,n} \phi_{-\alpha,n} - g_{\beta,n} \phi_{-\alpha,n} + g_{\beta,n} \phi_{-\beta,n} + f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + \\
& + f_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}) - g_{-\beta,n} \phi_{\beta,n})) + \\
& + {r_0}^{2\alpha_n+\beta_n} {r_1}^{\beta_n} f_{-\alpha,n} \Big(G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{-\beta,n} (- 1 + \phi_{\alpha,n}) - g_{\alpha,n} (- 1 + \phi_{-\beta,n})) + \\
& + f_{-\beta,n} (g_{\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{\alpha,n} - 1)) + f_{\alpha,n} (g_{\beta,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{\beta,n} - 1))) - \\
& - (n+1) E^{(2)} (g_{-\beta,n} \phi_{\alpha,n} - g_{\beta,n} \phi_{\alpha,n} + g_{\beta,n} \phi_{-\beta,n} + f_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + \\
& + f_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) - (f_{\alpha,n} + g_{\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}) - g_{-\beta,n} \phi_{\beta,n})) + \\
& + {r_0}^{\alpha_n} {r_1}^{\alpha_n} ({r_1}^{2\beta_n} f_{\beta,n} \Big(- G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{-\beta,n} (g_{\alpha,n} (- 1 + \phi_{-\alpha,n}) - g_{-\alpha,n} (- 1 + \phi_{\alpha,n})) + \\
& + f_{\alpha,n} (g_{-\alpha,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{-\beta,n} (\phi_{\alpha,n} - 1) - g_{\alpha,n} (\phi_{-\beta,n} - 1))) + \\
& + (n+1) E^{(2)} (g_{\alpha,n} \phi_{-\alpha,n} - g_{-\beta,n} \phi_{-\alpha,n} + g_{-\beta,n} \phi_{\alpha,n} + f_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{-\beta,n}) - g_{\alpha,n} \phi_{-\beta,n})) + \\
& + {r_0}^{2\beta_n} f_{-\beta,n} \Big(G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{\alpha,n} (- 1 + \phi_{-\alpha,n}) - g_{-\alpha,n} (- 1 + \phi_{\alpha,n})) + \\
& + f_{\alpha,n} (g_{-\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{\beta,n} (\phi_{\alpha,n} - 1 +) - g_{\alpha,n} (\phi_{\beta,n} - 1 +))) - \\
& - (n+1) E^{(2)} (g_{\alpha,n} \phi_{-\alpha,n} - g_{\beta,n} \phi_{-\alpha,n} + g_{\beta,n} \phi_{\alpha,n} + f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{\beta,n}) - g_{\alpha,n} \phi_{\beta,n}))))
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
\Upsilon_{\overline{A}_{-(n-1)}} = {} & r_0 {r_1}^{n-1} E^{(2)} ({r_0}^{\beta_n} {r_1}^{2\alpha_n+\beta_n} g_{\alpha,n} \Big(- G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{-\beta,n} (\phi_{-\alpha,n} - 1) - g_{-\alpha,n} (\phi_{-\beta,n} - 1)) + \\
& + f_{-\beta,n} (g_{-\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{\beta,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{\beta,n} - 1))) + \\
& + (n+1) E^{(2)} (g_{-\beta,n} \phi_{-\alpha,n} - g_{\beta,n} \phi_{-\alpha,n} + g_{\beta,n} \phi_{-\beta,n} + f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + \\
& + f_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}) - g_{-\beta,n} \phi_{\beta,n})) + \\
& + {r_0}^{2\alpha_n+\beta_n} {r_1}^{\beta_n} g_{-\alpha,n} \Big(G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{-\beta,n} (- 1 + \phi_{\alpha,n}) - g_{\alpha,n} (- 1 + \phi_{-\beta,n})) + \\
& + f_{-\beta,n} (g_{\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{\alpha,n} - 1)) + f_{\alpha,n} (g_{\beta,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{\beta,n} - 1))) - \\
& - (n+1) E^{(2)} (g_{-\beta,n} \phi_{\alpha,n} - g_{\beta,n} \phi_{\alpha,n} + g_{\beta,n} \phi_{-\beta,n} + f_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + \\
& + f_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) - (f_{\alpha,n} + g_{\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}) - g_{-\beta,n} \phi_{\beta,n})) + \\
& + {r_0}^{\alpha_n} {r_1}^{\alpha_n} ({r_1}^{2\beta_n} g_{\beta,n} \Big(- G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{-\beta,n} (g_{\alpha,n} (- 1 + \phi_{-\alpha,n}) - g_{-\alpha,n} (- 1 + \phi_{\alpha,n})) + \\
& + f_{\alpha,n} (g_{-\alpha,n} (\phi_{-\beta,n} - 1) - g_{-\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{-\beta,n} (\phi_{\alpha,n} - 1) - g_{\alpha,n} (\phi_{-\beta,n} - 1))) + \\
& + (n+1) E^{(2)} (g_{\alpha,n} \phi_{-\alpha,n} - g_{-\beta,n} \phi_{-\alpha,n} + g_{-\beta,n} \phi_{\alpha,n} + f_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{-\beta,n}) - g_{\alpha,n} \phi_{-\beta,n})) + \\
& + {r_0}^{2\beta_n} g_{-\beta,n} \Big(G^{(1)}_{\theta r} (1+\nu^{(2)}) \Big(f_{\beta,n} (g_{\alpha,n} (- 1 + \phi_{-\alpha,n}) - g_{-\alpha,n} (- 1 + \phi_{\alpha,n})) + \\
& + f_{\alpha,n} (g_{-\alpha,n} (\phi_{\beta,n} - 1) - g_{\beta,n} (\phi_{-\alpha,n} - 1)) + f_{-\alpha,n} (g_{\beta,n} (\phi_{\alpha,n} - 1 +) - g_{\alpha,n} (\phi_{\beta,n} - 1 +))) - \\
& - (n+1) E^{(2)} (g_{\alpha,n} \phi_{-\alpha,n} - g_{\beta,n} \phi_{-\alpha,n} + g_{\beta,n} \phi_{\alpha,n} + f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - (f_{-\alpha,n} + g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{\beta,n}) - g_{\alpha,n} \phi_{\beta,n}))))
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
\Xi_{\overline{C}_{-(n+1)}} = {} & r_0 {r_1}^{n+1} E^{(2)} ({r_0}^{\alpha_n} {r_1}^{\alpha_n} ({r_1}^{2\beta_n} f_{\beta,n} (G_{\theta r}^{(1)} (g_{-\beta,n} (f_{\alpha,n} p_{-\alpha,n} - f_{-\alpha,n} p_{\alpha,n}) + \\
& + f_{-\beta,n} (- g_{\alpha,n} p_{-\alpha,n} + g_{-\alpha,n} p_{\alpha,n}) + (- f_{\alpha,n} g_{-\alpha,n} + f_{-\alpha,n} g_{\alpha,n}) p_{-\beta,n}) + \\
& + 4E^{(2)} ((n+2) f_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + n g_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + (n+2) f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) + n g_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{-\beta,n}))) + \\
& + {r_0}^{2\beta_n} f_{-\beta,n} (G_{\theta r}^{(1)} (f_{\beta,n} (g_{\alpha,n} p_{-\alpha,n} - g_{-\alpha,n} p_{\alpha,n}) + f_{\alpha,n} (- g_{\beta,n} p_{-\alpha,n} + g_{-\alpha,n} p_{\beta,n}) + \\
& + f_{-\alpha,n} (g_{\beta,n} p_{\alpha,n} - g_{\alpha,n} p_{\beta,n})) - 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + n g_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + (n+2) f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) + \\
& + n g_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{\beta,n})))) + \\
& + {r_0}^{\beta_n} {r_1}^{2\alpha_n+\beta_n} f_{\alpha,n} (G_{\theta r}^{(1)} (f_{\beta,n} (- g_{-\beta,n} p_{-\alpha,n} + g_{-\alpha,n} p_{-\beta,n}) + \\
& + f_{-\beta,n} (g_{\beta,n} p_{-\alpha,n} - g_{-\alpha,n} p_{\beta,n}) + f_{-\alpha,n} (- g_{\beta,n} p_{-\beta,n} + g_{-\beta,n} p_{\beta,n})) + \\
& + 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + n g_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + \\
& + (n+2) f_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) + n g_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}))) + \\
& + {r_0}^{2\alpha_n+\beta_n} {r_1}^{\beta_n} f_{-\alpha,n} (G_{\theta r}^{(1)} (f_{\beta,n} (g_{-\beta,n} p_{\alpha,n} - g_{\alpha,n} p_{-\beta,n}) + \\
& + f_{-\beta,n} (- g_{\beta,n} p_{\alpha,n} + g_{\alpha,n} p_{\beta,n}) + f_{\alpha,n} (g_{\beta,n} p_{-\beta,n} - g_{-\beta,n} p_{\beta,n})) - \\
& - 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + n g_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + (n+2) f_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) + \\
& + n g_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{\alpha,n} + n g_{\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}))))
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
\Upsilon_{\overline{C}_{-(n+1)}} = {} & r_0 {r_1}^{n+1} E^{(2)} ({r_0}^{\alpha_n} {r_1}^{\alpha_n} ({r_1}^{2\beta_n} g_{\beta,n} (G_{\theta r}^{(1)} (g_{-\beta,n} (f_{\alpha,n} p_{-\alpha,n} - f_{-\alpha,n} p_{\alpha,n}) + \\
& + f_{-\beta,n} (- g_{\alpha,n} p_{-\alpha,n} + g_{-\alpha,n} p_{\alpha,n}) + (- f_{\alpha,n} g_{-\alpha,n} + f_{-\alpha,n} g_{\alpha,n}) p_{-\beta,n}) + \\
& + 4E^{(2)} ((n+2) f_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + n g_{-\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + (n+2) f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) + n g_{\alpha,n} (\phi_{-\alpha,n} - \phi_{-\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{-\beta,n}))) + \\
& + {r_0}^{2\beta_n} g_{-\beta,n} (G_{\theta r}^{(1)} (f_{\beta,n} (g_{\alpha,n} p_{-\alpha,n} - g_{-\alpha,n} p_{\alpha,n}) + f_{\alpha,n} (- g_{\beta,n} p_{-\alpha,n} + g_{-\alpha,n} p_{\beta,n}) + \\
& + f_{-\alpha,n} (g_{\beta,n} p_{\alpha,n} - g_{\alpha,n} p_{\beta,n})) - 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + \\
& + n g_{\beta,n} (- \phi_{-\alpha,n} + \phi_{\alpha,n}) + (n+2) f_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) + \\
& + n g_{\alpha,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{\alpha,n} - \phi_{\beta,n})))) + \\
& + {r_0}^{\beta_n} {r_1}^{2\alpha_n+\beta_n} g_{\alpha,n} (G_{\theta r}^{(1)} (f_{\beta,n} (- g_{-\beta,n} p_{-\alpha,n} + g_{-\alpha,n} p_{-\beta,n}) + \\
& + f_{-\beta,n} (g_{\beta,n} p_{-\alpha,n} - g_{-\alpha,n} p_{\beta,n}) + f_{-\alpha,n} (- g_{\beta,n} p_{-\beta,n} + g_{-\beta,n} p_{\beta,n})) + \\
& + 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + n g_{\beta,n} (- \phi_{-\alpha,n} + \phi_{-\beta,n}) + \\
& + (n+2) f_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) + n g_{-\beta,n} (\phi_{-\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{-\alpha,n} + n g_{-\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}))) + \\
& + {r_0}^{2\alpha_n+\beta_n} {r_1}^{\beta_n} g_{-\alpha,n} (G_{\theta r}^{(1)} (f_{\beta,n} (g_{-\beta,n} p_{\alpha,n} - g_{\alpha,n} p_{-\beta,n}) + \\
& + f_{-\beta,n} (- g_{\beta,n} p_{\alpha,n} + g_{\alpha,n} p_{\beta,n}) + f_{\alpha,n} (g_{\beta,n} p_{-\beta,n} - g_{-\beta,n} p_{\beta,n})) - \\
& - 4E^{(2)} ((n+2) f_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + n g_{\beta,n} (- \phi_{\alpha,n} + \phi_{-\beta,n}) + (n+2) f_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) + \\
& + n g_{-\beta,n} (\phi_{\alpha,n} - \phi_{\beta,n}) - ((n+2) f_{\alpha,n} + n g_{\alpha,n}) (\phi_{-\beta,n} - \phi_{\beta,n}))))
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
\Omega_{\overline{A}_{n-1}} = {} & r_1^{2\alpha_n}\left(2r_1^{2\beta_n}\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)\left(-G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{-\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{-\beta,n}\right)+\right.\right.\\
& +G_{\theta r}^{(1)}E^{(2)}\left(-g_{-\beta,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,-\beta}+f_{-\beta,n}m_{n,-\alpha}-f_{-\alpha,n}m_{n,-\beta}\right)+\\
& +\left(n^2-1\right)E^{(2)2}\left(-\phi_{-\alpha,n}+\phi_{-\beta,n}\right))+2r_0^{2\beta_n}\left(f_{-\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{-\beta,n}\right)\\
& \left(G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\beta,n}\right)+\right.\\
& +G_{\theta r}^{(1)}E^{(2)}\left(g_{\beta,n}h_{n,-\alpha}-g_{-\alpha,n}h_{n,\beta}-f_{\beta,n}m_{n,-\alpha}+f_{-\alpha,n}m_{n,\beta}\right)+\left(n^2-1\right)E^{(2)2}\left(\phi_{-\alpha,n}-\phi_{\beta,n}\right)))+\\
& +r_0^{\alpha_n+\beta_n}r_1^{\alpha_n+\beta_n}(4G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)\left(f_{\beta,n}g_{-\beta,n}-f_{-\beta,n}g_{\beta,n}\right)-\\
& -2G_{\theta r}^{(1)}E^{(2)}(-(\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)\left(g_{\beta,n}h_{n,-\beta}-g_{-\beta,n}h_{n,\beta}\right))+\\
& +f_{\beta,n}\left(g_{-\beta,n}\left(-g_{\alpha,n}h_{n,-\alpha}+g_{-\alpha,n}h_{n,\alpha}+f_{\alpha,n}m_{n,-\alpha}-f_{-\alpha,n}m_{n,\alpha}\right)+\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)m_{n,-\beta}\right)+\\
& +f_{-\beta,n}\left(g_{\beta,n}\left(g_{\alpha,n}h_{n,-\alpha}-g_{-\alpha,n}h_{n,\alpha}-f_{\alpha,n}m_{n,-\alpha}+f_{-\alpha,n}m_{n,\alpha}\right)+\left(-f_{\alpha,n}g_{-\alpha,n}+f_{-\alpha,n}g_{\alpha,n}\right)m_{n,\beta}\right))+\\
& +2\left(n^2-1\right)E^{(2)2}\left(\left(f_{\beta,n}g_{-\beta,n}-f_{-\beta,n}g_{\beta,n}\right)\left(\phi_{-\alpha,n}-\phi_{\alpha,n}\right)+\left(f_{\alpha,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\alpha,n}\right)\left(\phi_{-\beta,n}-\phi_{\beta,n}\right)\right))+\\
& +r_0^{2\alpha_n}\left(2r_1^{2\beta_n}\left(f_{\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{\beta,n}\right)\left(G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{-\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{-\beta,n}\right)+\right.\right.\\
& +G_{\theta r}^{(1)}E^{(2)}\left(g_{-\beta,n}h_{n,\alpha}-g_{\alpha,n}h_{n,-\beta}-f_{-\beta,n}m_{n,\alpha}+f_{\alpha,n}m_{n,-\beta}\right)+\\
& +\left(n^2-1\right)E^{(2)2}\left(\phi_{\alpha,n}-\phi_{-\beta,n}\right))+2r_0^{2\beta_n}\left(f_{-\beta,n}g_{-\alpha,n}-f_{-\alpha,n}g_{-\beta,n}\right)\\
& \left(-G_{\theta r}^{(1)2}\left(-3+\nu^{(2)}\right)\left(1+\nu^{(2)}\right)\left(f_{\beta,n}g_{\alpha,n}-f_{\alpha,n}g_{\beta,n}\right)+\right.\\
& \left.\left.+G_{\theta r}^{(1)}E^{(2)}\left(-g_{\beta,n}h_{n,\alpha}+g_{\alpha,n}h_{n,\beta}+f_{\beta,n}m_{n,\alpha}-f_{\alpha,n}m_{n,\beta}\right)+\left(n^2-1\right)E^{(2)2}\left(-\phi_{\alpha,n}+\phi_{\beta,n}\right)\right)\right))
\end{aligned}
\tag{B.16}
$$

Appendix C. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.tafmec.2023.103821.

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