GEOMETRIC BOUNDS FOR THE MAGNETIC NEUMANN EIGENVALUES IN THE PLANE

BRUNO COLBOIS, CORENTIN LÉNA, LUIGI PROVENZANO, AND ALESSANDRO SAVO

ABSTRACT. We consider the eigenvalues of the magnetic Laplacian on a bounded domain Ω of $\mathbb{R}^2$ with uniform magnetic field $\beta > 0$ and magnetic Neumann boundary conditions. We find upper and lower bounds for the ground state energy λ_1 and we provide semiclassical estimates in the spirit of Kröger for the first Riesz mean of the eigenvalues. We also discuss upper bounds for the first eigenvalue for non-constant magnetic fields $\beta = \beta(x)$ on a simply connected domain in a Riemannian surface.

In particular: we prove the upper bound $\lambda_1 < \beta$ for a general plane domain for a constant magnetic field, and the upper bound $\lambda_1 < \max_{x\in\overline{\Omega}} |\beta(x)|$ for a variable magnetic field when Ω is simply connected.

For smooth domains, we prove a lower bound of λ_1 depending only on the intensity of the magnetic field β and the rolling radius of the domain.

The estimates on the Riesz mean imply an upper bound for the averages of the first k eigenvalues which is sharp when $k \to \infty$ and consists of the semiclassical limit $\dfrac{2\pi k}{|\Omega|}$ plus an oscillating term.

We also construct several examples, showing the importance of the topology: in particular we show that an arbitrarily small tubular neighborhood of a generic simple closed curve has lowest eigenvalue bounded away from zero, contrary to the case of a simply connected domain of small area, for which λ_1 is always small.

Contents

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1. Introduction

The main scope of this paper is to derive upper and lower bounds for the eigenvalues of the magnetic Laplacian with constant magnetic field $\beta > 0$ and magnetic Neumann boundary conditions on domains of $\mathbb{R}^2$. Specifically, let

$$A_1 = \frac{1}{2}(-x_2 dx_1 + x_1 dx_2). \tag{1.1}$$

We consider the magnetic Laplacian associated with the potential 1-form

$$\beta A_1 = \frac{\beta}{2}(-x_2 dx_1 + x_1 dx_2) \tag{1.2}$$

which generates the magnetic field of constant strength β, in the sense that $d(\beta A_1) = \beta dv$ where $dv = dx_1 \wedge dx_2$ is the volume 2-form. Note that replacing β by $-\beta$ does not change the spectrum, therefore it is not restrictive to consider $\beta > 0$, see Subsection 2.7. The eigenvalues correspond to the energy levels of a quantum charged particle in a two-dimensional region subject to a transversal magnetic field of constant strength β. It is clear that the interest in the study of the corresponding spectrum originates in Quantum Mechanics and Mathematical Physics. We refer to the books [19, 35] for more detailed discussions on the topic.

Nevertheless the subject has attracted a lot of attention in the last decades also in Analysis and Geometry. Relevant questions which are usually posed in these contexts include geometric bounds for the eigenvalues and isoperimetric inequalities. In the present paper we will focus on eigenvalue bounds, and in particular on how the geometry of the domain influences the eigenvalues, with particular attention to the ground state energy $\lambda_1(\Omega, \beta A_1)$, which turns out to be positive. Concerning previous results on eigenvalue bounds for the magnetic Neumann problem, we refer to [9, 10, 13, 14, 20, 29, 31].

In this paper the notation $\lambda_j(\Omega, \beta A_1)$ refers to the j-th eigenvalue of the magnetic Laplacian with Neumann conditions and potential βA_1 as in (1.2) (when Ω is not simply connected, the choice of the potential form generating the magnetic field β may affect the spectrum, see Subsection 2.2.)

If we set $\beta = 0$ in (1.2) we fall back into the case of the Neumann Laplacian, for which a huge literature on eigenvalue bounds is available. In our notation, $\lambda_1(\Omega, 0) = 0$, while $\lambda_2(\Omega, 0) > 0$ is the first positive eigenvalue of the (non-magnetic) Neumann Laplacian. On the other hand, when $\beta > 0$, the magnetic spectrum, in particular $\lambda_1(\Omega, \beta A_1)$, displays a peculiar behavior when compared to the usual Laplacian spectrum. We will list here just a few instances in order to give a glimpse of this fact.

– First, the behavior of the first eigenvalue under homotheties involves the strength of the magnetic field: for any $\alpha > 0$ one has:

$$\lambda_j(\alpha\Omega, \beta A_1) = \frac{1}{\alpha^2}\lambda_j(\Omega, \alpha^2\beta A_1).$$

– This and the upper bound (2.16) imply that, for $\beta > 0$ fixed, $\lambda_1(\alpha\Omega, \beta A_1) \to 0$ as $\alpha \to 0^+$: the first eigenvalue tends to 0 when the domain shrinks homothetically to a point.

– However, the first eigenvalue does not necessarily go to 0 when $|\Omega| \to 0^+$: there exist domains with arbitrarily small area and first eigenvalue bounded away from zero (as a matter of fact, a small tubular neighborhood of a "generic" simple closed curve has first eigenvalue bounded away from zero, see Example C.3).

– Still concerning homotheties, given any domain Ω, for any fixed $\beta > 0$ we have $\lambda_1(\alpha\Omega, \beta A_1) \to \Theta_0\beta$ as $\alpha \to +\infty$, where $\Theta_0 \approx 0.590106$ is a universal constant (de Gennes constant, see [19, Chapter 3]). This implies that an arbitrarily large volume does not imply a small first eigenvalue. Moreover, note that the function $\alpha \mapsto \lambda_1(\alpha\Omega, \beta A_1)$ is not generally increasing (see Figure 3 when Ω is a disk).

– There exist convex domains with inradius bounded below by a positive constant and first eigenvalue arbitrarily small (see Example 5.12).

– There are striking differences between the magnetic Neumann and the magnetic Dirichlet eigenvalues.

Let us briefly comment on that. Let $\lambda_1^D(B_R, \beta A_1)$ denote the first magnetic Dirichlet eigenvalue on a disk of radius R. It is quite standard to prove that $\lambda_1^D(B_R, \beta A_1)$ is decreasing from $+\infty$ to β (which is a strict and sharp lower bound) as a function of $R \in (0, +\infty)$, and that the first eigenfunction is real and radial for any R (see e.g., [38]). Moreover, the Faber-Krahn inequality holds for $\lambda_1^D(\Omega, \beta A_1)$, see [15]. On the other hand, the behavior of the first Neumann eigenvalue $\lambda_1(B_R, \beta A_1)$ on disks as a function of R is very complicated: the first eigenfunction has angular momentum which increases with R; the eigenvalue

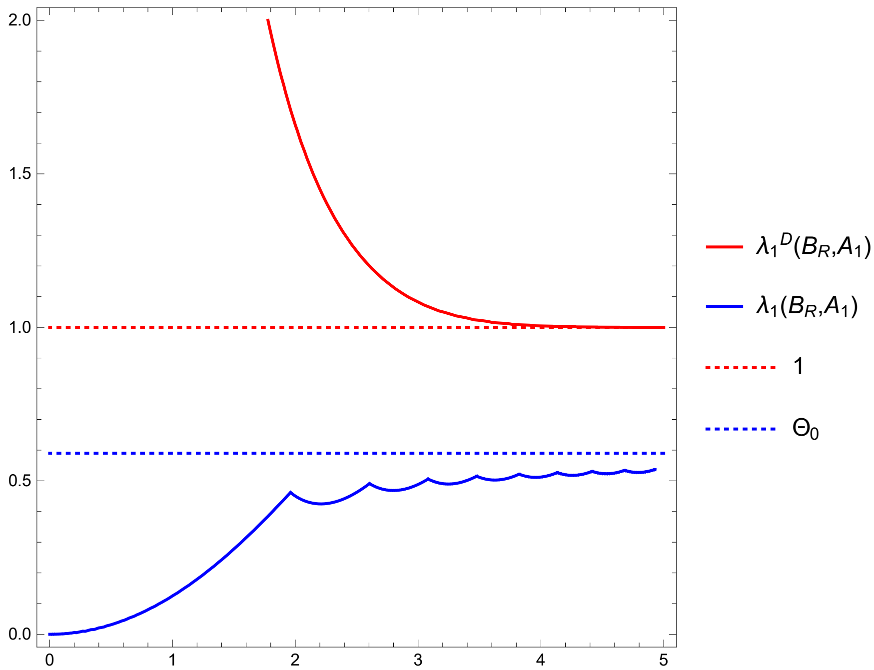

FIGURE 1. First magnetic Dirichlet eigenvalue (in red) of a disk B_R as a function of R; first magnetic Neumann eigenvalue (in blue) of a disk B_R as a function of R. Here $\beta = 1$.

is uniformly bounded, vanishes as $R \to 0^+$ and presents an oscillating (i.e., non-monotonic) behavior as a function of R. This phenomenon is known as the Little-Parks effect. It is described for example in [21], see in particular paragraphs 5 and 6. In the introduction of [21], there are also some physical explanations of this phenomenon. It could be interesting to compare these facts with the discussion of [18], where the authors prove the monotonicity of $\lambda_1(\Omega, \beta A_1)$ with respect to β for large β (see also [19, §5]). From numerical studies it seems that $\lambda_1(B_R, \beta A_1) < \Theta_0\beta$ for all R, but we have no proof of this fact at the moment. See Figure 1 for a plot of $\lambda_1^D(B_R, A_1)$ and $\lambda_1(B_R, A_1)$ as functions of R. We refer to Appendix B for more details on the Neumann problem for disks.

– Finally, we note that the reverse Faber-Krahn inequality for the first magnetic Neumann eigenvalue is still an open problem, and it definitely does not hold for multiply connected domains. In fact, in [20] the authors show that, given an annulus Ω, there exists $\beta_0 = \beta_0(\Omega)$ such that, for any $\beta > \beta_0$, $\lambda_1(\Omega^*, \beta A_1) < \lambda_1(\Omega, \beta A_1)$, where Ω^* is a disk with $|\Omega^*| = |\Omega|$. Note that this is an asymptotic counterexample. From our Example C.3 we see that, for any $\beta > 0$, there exist plenty of non-simply connected domains which have first eigenvalue larger than that of the disk of the same area. In fact, as already mentioned, a small tubular neighborhood (of small area) of a "generic" simple closed curve has first eigenvalue uniformly bounded away from zero, while a disk with the same (small) area has small first eigenvalue. What we describe in Example C.3 was also observed in [27, Theorem 3]. In our Appendix A and Example C.3, we give a simple and self-contained proof of these facts.

These few examples show that understanding the geometric conditions which imply upper or lower bounds on $\lambda_1(\Omega, \beta A_1)$ is not trivial.

In the present paper we improve the known bounds in different ways. First, we focus on the ground state energy $\lambda_1(\Omega, \beta A_1)$, which is strictly positive for any value of $\beta > 0$. We prove a universal upper bound, valid for any domain, which is strict and is given by β, the intensity of the magnetic field (Theorem 2.1). For certain classes of domains, which include sub-graphs and polygonal rep-tiles, we prove that an upper bound is given by $\Theta_0\beta$ (Theorem 2.2), which is optimal in view of the asymptotic behavior $\lambda_1(\Omega, \beta A_1) = \Theta_0\beta + o(\beta)$ as $\beta \to +\infty$ (see [19, Chapter 5]). We also prove a general upper bound for the first eigenvalue when the magnetic field is non constant, on simply connected Riemannian surfaces, in terms of the sup-norm of the magnetic field (see Theorem 2.3 and also Section 4 for a discussion on the non simply connected case).

We continue by considering lower bounds for $\lambda_1(\Omega, \beta A_1)$. Our starting point are the lower bounds proved in [14] for simply connected domains, which we use to produce a new lower bound for arbitrary smooth domains in terms of β or β^2, depending only on the rolling radius δ of the domain, see Theorem 2.5.

We then consider the whole spectrum and we prove semiclassical estimates on eigenvalues averages in the spirit of Kröger [28], which are asymptotically sharp (Theorem 2.6). These estimates imply upper bounds on single eigenvalues of any order. Note that bounds on eigenvalue averages turn out to be equivalent to bounds on the first Riesz mean $R_1(z)$, in the tradition of Berezin and Li-Yau [4, 32]. Lower bounds for Riesz-means in case of variable magnetic field have been also obtained in [9]. Upper bounds for Riesz means already exist for the Dirichlet magnetic eigenvalues in [16] (see also [22]). We note that the behavior of our lower bounds on R_1 is significantly different from the behavior of the upper bounds for Dirichlet Riesz means [16], and reflects the interplay between the area of the domain, the strength of the magnetic field and the eigenvalue index.

We include three appendices, where we discuss results which are related to eigenvalue bounds, but have an interest on their own. Namely, we consider the magnetic Laplacian on embedded curves, establishing in this setting a sort of "reverse Faber-Krahn" inequality. We also discuss the case of disks and collect a few other examples, which are instructive in order to understand some of the difficulties in establishing precise bounds. As already mentioned, there is some intersection between the results of these appendices and results contained in [21] and [27]. However, the proof we give here are rather simple and self-contained.

The paper is organized as follows. In Section 2 we introduce the mathematical problem, fix the notation, and state our main results. In Sections 3 and 5 we prove, respectively, the upper and the lower bounds for $\lambda_1(\Omega, \beta A_1)$. In Section 4 we prove upper bounds for the first eigenvalue in the case of variable magnetic field on a Riemannian surface. In Section 6 we prove the asymptotically sharp, semiclassical estimates for Riesz means and averages. In Appendix A we study the eigenvalue problem obtained by restricting the magnetic potential to embedded curves, and prove an isoperimetric result. In Appendix B we collect a few properties of magnetic eigenvalues on disks. In Appendix C we provide further examples which help to clarify the difficulties in finding good bounds.

2. Notation and statement of results

2.1. Generalities on the magnetic Laplacian. Let Ω be a bounded domain in $\mathbb{R}^2$ and let A be a smooth real 1-form. We define the *magnetic differential* of a smooth complex valued function u as the complex 1-form defined as follows:

$$d^A u = du - iuA.$$

The adjoint of d^A is the operator δ^A acting on a 1-form ω as $\delta^A\omega = \delta\omega + i\omega(A^\sharp)$, where $A^\sharp$ is the dual vector field of A and δ is the adjoint of d (note that $\delta = -\mathrm{div}$). The *magnetic Laplacian* associated to the potential A is then defined as $\Delta_A u = \delta^A d^A u$. A standard calculation shows that

$$\Delta_A u = \Delta u + |A|^2 u + 2i\langle du, A\rangle + iu \,\mathrm{div}\, A,$$

where divA $= -\delta A$ is the usual divergence of the 1-form A. Note that in this paper Δ denote the positive-definite operator $\Delta := -\partial^2_{x_1x_1} - \partial^2_{x_2x_2}$. By x we denote a point in $\mathbb{R}^2$ with Cartesian coordinates (x_1, x_2). We will often use polar coordinates $(r, t) \in [0, +\infty) \times [0, 2\pi]$. In particular $r = |x| = \sqrt{x_1^2 + x_2^2}$.

We will often identify, by abuse of language (and when this will not create confusion) the form A with its dual vector field $A^\sharp$ (*vector potential*); the magnetic Laplacian can then be written in the following form, often found in the literature:

$$\Delta_A = -(\nabla - iA)^2.$$

Dually, we define the *magnetic gradient* of a complex function u as

$$\nabla^A u := \nabla u - iuA$$

where A is thought as a vector potential. We consider the eigenvalue problem for the magnetic Laplacian with *magnetic Neumann conditions* in Ω, namely

$$\begin{cases} \Delta_A u = \lambda u\,, & \text{in } \Omega, \\ \langle \nabla^A u, N\rangle = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where N is the outer unit normal to $\partial\Omega$. In the second line, one can see $\langle \nabla^A u, N\rangle$ as being the magnetic normal derivative of u.

Problem (2.1) is understood in the weak sense as follows: find $u \in H^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\int_\Omega \langle \nabla^A u, \overline{\nabla^A \phi}\rangle = \lambda \int_\Omega u\bar{\phi}\,, \quad \forall \phi \in H^1(\Omega). \tag{2.2}$$

Here $H^1(\Omega)$ is the standard Sobolev space of complex valued functions in $L^2(\Omega)$ with weak first derivatives in $L^2(\Omega)$.

It is standard to prove that, under reasonable assumptions on Ω (e.g., Ω Lipschitz) problem (2.2) admits an increasing sequence of positive eigenvalues of finite multiplicity diverging to $+\infty$

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots \nearrow +\infty.$$

Remark on notation. In this paper the notation $\lambda_j(\Omega, A)$ refers to the j-th eigenvalue of the Neumann problem on Ω for the magnetic Laplacian associated to the potential 1-form A, that is, problem (2.1).

Through the rest of the paper we shall implicitly assume that Ω is a bounded domain for which the spectrum of (2.2) is discrete. The eigenvalues are variationally characterized as follows

$$\lambda_j(\Omega, A) = \min_{\substack{U \subset H^1(\Omega)\\ \dim U = j}} \max_{0 \ne u \in U} \frac{\int_\Omega |\nabla^A u|^2}{\int_\Omega |u|^2}. \tag{2.3}$$

Note that $\lambda_j(\Omega, A)$ normally depends on both the domain and the potential 1-form; however we observe the well-known *gauge invariance* of the spectrum, according to which if we replace A by $A + df$, for any function f such that e^{if} is smooth, the spectrum remains unchanged:

$$\lambda_j(\Omega, A) = \lambda_j(\Omega, A + df). \tag{2.4}$$

2.2. Constant magnetic field. In the present paper we will mainly consider the following potential 1-form A in (2.1):

$$A = \beta A_1 = \frac{\beta}{2}(-x_2 dx_1 + x_1 dx_2),$$

which we will often call *standard potential*; here β is a positive constant. Note that $dA = \beta dv$ is a constant magnetic field of strength β, where $dv = dx_1 \wedge dx_2$ is the usual area element of $\mathbb{R}^2$. Also observe that $\operatorname{div} A = 0$. The corresponding eigenvalues will be denoted by $\lambda_j(\Omega, \beta A_1)$.

A remark is perhaps in order here. Observe that if Ω is simply connected, then the spectrum of (2.1) depends only on β and not on the potential A. In fact, if A, A' are two potentials such that $dA = dA' = \beta dv$, then they differ by a closed 1-form, which is exact on Ω since Ω is simply connected: in that case the spectra of (2.1) with A and A' coincide by gauge invariance (2.4). The situation is completely different if the domain is not simply connected, in which case the spectra corresponding to A, A' differing by a closed 1-form in general may not coincide. In this case, as we have already declared, we are considering the spectrum of (2.1) with A defined by (1.2), i.e., $A = \beta A_1$.

2.3. Upper bounds for λ_1. We list here the main results concerning upper bounds on the ground state energy $\lambda_1(\Omega, \beta A_1)$.

The first result is an upper bound for $\lambda_1(\Omega, \beta A_1)$, valid for any bounded domain in $\mathbb{R}^2$. We present its proof in Section 3 (Theorem 3.1).

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then*

$$\lambda_1(\Omega, \beta A_1) < \beta. \tag{2.5}$$

Actually, the upper bound (2.5) is a consequence of a more precise bound that we establish in Theorem 3.1, namely

$$\lambda_1(\Omega, \beta A_1) \le \begin{cases} \beta - \frac{1}{2R_\Omega^2}\,, & \text{if } R_\Omega > \frac{1}{\sqrt{\beta}}\,, \\ \frac{R_\Omega^2 \beta^2}{2}\,, & \text{if } R_\Omega \le \frac{1}{\sqrt{\beta}}, \end{cases}$$

where R_Ω denotes the circumradius of Ω, namely, the radius of the smallest disk containing Ω. Alternatively, for simply connected domains the upper bound (2.5) follows from

$$\lambda_1(\Omega, \beta A_1) \le \beta(1 - e^{-\frac{\beta|\Omega|}{2\pi}})$$

which we prove in Theorem 4.2 (see also Theorem 2.3 here below). Note that this latter bound implies that for simply connected domains, $\lambda_1(\Omega, \beta A_1) \to 0$ as $|\Omega| \to 0^+$. This is no longer true if Ω is not simply connected, see Example C.3.

For certain classes of domains we prove an asymptotically sharp upper bound. This bound depends on a universal positive constant, the *de Gennes constant* $\Theta_0 \approx 0.590106$, which we discuss in more detail in the next subsection (see (2.13)). Namely, we prove the following

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^2$. Assume that, up to isometries, one of the following holds:*

1) Ω is a sub-graph, namely $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : a < x_1 < b, 0 < x_2 < g(x_1)\}$ for some smooth $g : (a, b) \to [0, +\infty)$;

2) Ω is contained in some strip $(a, b) \times (0, +\infty)$ and contains $(a, b) \times (0, 2\sqrt{\Theta_0/\beta})$.

Then

$$\lambda_1(\Omega, \beta A_1) < \Theta_0 \beta. \tag{2.6}$$

Alternatively, assume that

3) Ω is a polygonal rep-tile, namely that $\partial\Omega$ is a polygon and that $\Omega = \mathrm{Int}\left(\bigcup_{i=1}^m \overline{\omega}_i\right)$, $m \geq 2$, where each ω_i is open and is isometric to $\frac{1}{\sqrt{m}}\Omega$.

Then

$$\lambda_1(\Omega, \beta A_1) \leq \Theta_0 \beta. \tag{2.7}$$

The constant Θ_0 is related with the asymptotic behavior of $\lambda_1(\Omega, \beta A_1)$ as $\beta \to +\infty$, in fact, for any smooth Ω it holds $\Theta_0 = \lim_{\beta \to +\infty} \lambda_1(\Omega, \beta A_1)/\beta$ (see (2.12)). Hence the bounds (2.6)-(2.7) are optimal in this sense. We can pose then the following questions.

Open problem 1. Prove that $\lambda_1(\Omega, \beta A_1) < \Theta_0 \beta$ for all Ω.

Note that our Theorem 2.2 does not cover the case of an arbitrary disk (disks of small radius are covered by Theorems 2.1 and 2.3). However, numerical evidences (see Appendix B) suggest that $\Theta_0 \beta$ is a strict upper bound for all disks. Hence we ask:

Open problem 2. Prove the weaker statement: $\lambda_1(B, \beta A_1) < \Theta_0 \beta$ for any disk B.

Points 1) and 2) of Theorem 2.2 are proved in Theorem 3.3, while point 3) is proved in Theorem 3.7. In the case 3) we actually prove in Theorem 3.7 that $\lambda_1(\Omega, \beta A_1) \leq \Lambda(\Omega)\beta$, where $\Lambda(\Omega) \leq \Theta_0$ is given by $\lim_{\beta \to +\infty} \lambda_1(\Omega, \beta A_1)/\beta$. For certain domains with convex corners it is possible to show that $\Lambda(\Omega) < \Theta_0$, see [5]. Polygonal rep-tiles with this property are, for example, triangles and parallelograms.

Concerning λ_1 we have also considered the case of a variable magnetic field $\beta = \beta(x)$ on simply connected Riemannian surfaces. We have established a general upper bound which depends only on the field β, and not on the magnetic potential.

Theorem 2.3. *Let Ω be a smooth, bounded, simply connected domain in an orientable Riemannian surface M, let A be a smooth 1-form, and let $\beta : \Omega \to \mathbb{R}$ be defined by $dA = \beta dv$, where dv is the volume 2-form. Let $\lambda_1(\Omega, A)$ denote the first eigenvalue of (2.1) with magnetic potential A. Let $\phi : \Omega \to \mathbb{R}$ be the unique solution to*

$$\begin{cases} \Delta\phi = \beta\,, & \text{in } \Omega, \\ \phi = 0\,, & \text{on } \partial\Omega. \end{cases}$$

and let $\beta^* := \max_{\overline{\Omega}} |\beta|$, $\phi^* := \max_{\overline{\Omega}} |\phi|$. The following inequalities hold:

1) $\lambda_1(\Omega, A) < \beta^*$.

2) If $\beta(x) \geq 0$ for all $x \in \Omega$, then $\lambda_1(\Omega, A) \leq \beta^*(1 - e^{-2\phi^*})$.

3) If $\beta(x) \geq 0$ for all $x \in \Omega$ and Ω is a domain of $\mathbb{R}^2$, then $\lambda_1(\Omega, A) \leq \beta^*(1 - e^{-\frac{\beta^*|\Omega|}{2\pi}})$.

4) If Ω is a domain of $\mathbb{R}^2$, $\beta > 0$ is constant and $A = \beta A_1$ is the standard potential (1.2), then

$$\lambda_1(\Omega, \beta A_1) \leq \beta(1 - e^{-\frac{\beta|\Omega|}{2\pi}}).$$

Theorem 2.3 is a consequence of Theorem 4.2 and Corollary 4.4. In Section 4 we also discuss bounds in the case of non-simply connected domains.

Remark 2.4. We remark that 3) cannot hold if Ω is not simply connected. In fact, as we show in Example C.3, there exist non-simply connected domains ω_h with $|\omega_h| \to 0$ and $\lambda_1(\omega_h, \beta A_1) \to c > 0$ as $h \to 0^+$. On the other hand, if Ω is simply connected, 3) implies that $\lambda_1(\Omega, A) \to 0$ as $|\Omega| \to 0^+$. In the case of $\beta > 0$ constant, this can be also deduced from [20, Theorem 1.2], which implies that $\lambda_1(\Omega, \beta A_1) \leq \frac{\beta^2|\Omega|}{8\pi}$.

2.4. Lower bounds for λ_1. The next result concerns lower bounds for the first eigenvalue when β is constant. In order to state the result, we recall that a domain is said to satisfy the δ-interior ball condition with $\delta > 0$ if for any $x \in \partial\Omega$ there exists a disk of radius $\delta > 0$ tangent to $\partial\Omega$ at x and entirely contained in Ω. In more technical terms, this condition can be also expressed by saying that the injectivity radius of the boundary is bounded below by δ.

Theorem 2.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ satisfying the δ-interior ball condition. Then there exists a universal constant $C > 0$ such that $\lambda_1(\Omega, \beta A_1) \geq C\beta \min\{\beta\delta^2, 1\}$. That is:*
1) $\lambda_1(\Omega, \beta A_1) \geq C\beta^2\delta^2$ if $\beta\delta^2 \leq 1$;
2) $\lambda_1(\Omega, \beta A_1) \geq C\beta$ if $\beta\delta^2 \geq 1$.

Theorem 2.5 is proved in Section 5 (Theorem 5.10). Its proof relies on a combination of the lower bounds for magnetic eigenvalues in [14] (see Theorem 5.1) and for the Laplacian eigenvalues in [8] (see Theorem 5.4). Note that the behavior of our lower bounds in δ and β is consistent with the upper bounds of Theorem 2.1. It is also consistent with the asymptotic behavior of the first eigenvalue with respect to β as $\beta \to 0^+$ and $\beta \to +\infty$. We refer to Remark 5.11 for more discussions on the sharp behavior in β and δ of the lower bounds of Theorem 2.5. A series of examples show that in many situations the bounds given by Theorem 2.5 are good in capturing the behavior of the first eigenvalue (see Examples 5.2, C.2, C.5). In Section 5 we also prove lower bounds for star-shaped domains in terms of the radii $0 < R < R_0$ of two disks $B(p, R), B(p, R_0)$ such that $B(p, R) \subset \Omega \subset B(p, R_0)$ (Proposition 5.6).

2.5. Upper bounds for higher eigenvalues and averages. The next results involve upper bounds for all the eigenvalues when β is constant. By means of the so-called averaged variational principle (Theorem 6.6) we obtain asymptotically sharp lower bounds on the first Riesz mean R_1 of magnetic eigenvalues $\lambda_j(\Omega, \beta A_1)$, which is defined by $R_1(z) := \sum_{j=1}^{\infty}(z - \lambda_j(\Omega, \beta A_1))_+$, where $a_+ := \max\{0, a\}$. Lower bounds on $R_1(z)$ are equivalent to upper bounds for eigenvalues averages.

Theorem 2.6. *For all $z \geq 0$ we have*

$$R_1(z) \geq \frac{|\Omega|}{8\pi} z^2 - \frac{|\Omega|\beta^2}{2\pi} \psi^2 \left(\frac{z}{2\beta} + \frac{1}{2} \right), \tag{2.8}$$

where $\psi(a) = a - [a] - \frac{1}{2}$ denotes the fluctuation function of $a \in \mathbb{R}$, and $[a]$ denotes the integer part of $a \in \mathbb{R}$. Moreover, for any $k \in \mathbb{N}$, $k \geq 1$ we have

$$\begin{cases} \frac{1}{k}\sum_{j=1}^{k} \lambda_j(\Omega, \beta A_1) \leq \beta\,, & \text{if } k \leq \frac{\beta|\Omega|}{2\pi} \\ \frac{1}{k}\sum_{j=1}^{k} \lambda_j(\Omega, \beta A_1) \leq \frac{2\pi k}{|\Omega|} + R\left(\frac{2\pi k}{\beta|\Omega|}\right)\,, & \text{if } k > \frac{\beta|\Omega|}{2\pi}, \end{cases} \tag{2.9}$$

where $R(a) = \frac{\beta}{a}(a - [a])([a] - a + 1) \in [0, \beta/4a]$.

Theorem 2.6 is proved in Section 6 (Theorem 6.1). We note that our upper bounds are asymptotically sharp, in fact Weyl's law for magnetic eigenvalues implies that $R_1(z) = \frac{|\Omega|}{8\pi}z^2 + o(z^2)$ as $z \to +\infty$, or, equivalently, $\frac{1}{k}\sum_{j=1}^{k} \lambda_j(\Omega, \beta A_1) = \frac{2\pi k}{|\Omega|} + o(k)$ as $k \to +\infty$. The bounds given in Theorem 2.6 are the analogue of the Kröger upper bounds for the averages of Laplacian eigenvalues [28]. Note that the Weyl term $\frac{2\pi k}{|\Omega|}$ appears in the estimates (2.9) only for large k, and this is natural, since magnetic eigenvalues do not scale as Laplacian eigenvalues. The first inequality of (2.9) tells us that for small k (depending on β and $|\Omega|$) the average of the first k eigenvalues is smaller than β. This is somehow sharp, as this behavior can be observed in the case of disks, see Appendix B. However, as $k \to +\infty$, the upper bound is given by the semiclassical limit $\frac{2\pi k}{|\Omega|}$, plus a remainder term which is oscillating, bounded, and of $o(1/k)$ as $k \to +\infty$.

As a corollary of Theorem 2.6 we get upper bounds on single eigenvalues (see Corollary 6.3).

Corollary 2.7. *For all $k \in \mathbb{N}$ we have*

$$\lambda_{k+1}(\Omega, \beta A_1) \leq \frac{8\pi k}{|\Omega|} + \beta. \tag{2.10}$$

Note that this agrees with the fact that the second eigenvalue might go to $+\infty$ as $|\Omega| \to 0^+$ (contrarily to the first eigenvalue).

2.6. A reverse Faber-Krahn inequality for the first eigenvalue of embedded curves. We consider in Appendix A a one-dimensional eigenvalue problem related to the magnetic Laplacian. Namely, let Γ be a simple closed curve bounding some connected domain Ω. We consider the standard potential 1-form $A = \frac{\beta}{2}(-x_2 dx_1 + x_1 dx_2)$ on $\mathbb{R}^2$ and restrict it to Γ, thus obtaining a potential 1-form $\hat{A}_\Gamma$ on Γ which is simply defined by $\hat{A}_\Gamma(X) = A(X)$ for all tangent vectors X to Γ. If Γ is parametrized by arc-length as $(x(t), y(t))$, $t \in [0, |\Gamma|]$, then $\hat{A}_\Gamma(t) = \frac{\beta}{2}(-y(t)x'(t) + x(t)y'(t))dt$. We can then define the magnetic differential of a smooth function $u : \Gamma \to \mathbb{C}$ by $d^{\hat{A}_\Gamma} u = du - iu\hat{A}_\Gamma$. Repeating the construction of Section 2.1, we can finally associate with $d^{\hat{A}_\Gamma}$ the magnetic Laplacian $\Delta_{\hat{A}_\Gamma}$, which is a second-order differential operator on Γ, seen as a one-dimensional Riemannian manifold. We consider the first eigenvalue of $\Delta_{\hat{A}_\Gamma}$ on Γ which we denote by $\lambda_1(\Gamma, \hat{A}_\Gamma)$.

It turns out (see Theorem A.1) that

$$\lambda_1(\Gamma, \hat{A}_\Gamma) = \frac{4\pi^2}{|\Gamma|^2} \min_{n \in \mathbb{Z}} \left(n - \frac{\beta|\Omega|}{2\pi} \right)^2 .$$

Note that $\lambda_1(\Gamma, \hat{A}_\Gamma) = 0$ if and only if $\frac{\beta|\Omega|}{2\pi} \in \mathbb{N}$. We deduce then the following isoperimetric inequality (see Theorem A.2)

Theorem 2.8. *Let Ω be a bounded, simply connected domain with boundary Γ, and let Ω^* be a disk with $|\Omega| = |\Omega^*|$ and boundary Γ^*. Then*

$$\lambda_1(\Gamma, \hat{A}_\Gamma) \le \lambda_1(\Gamma^*, \hat{A}_{\Gamma^*}). \tag{2.11}$$

If $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$, then equality holds if and only if $\Omega = \Omega^*$.

2.7. A few properties of magnetic eigenvalues. We collect in this subsection a few properties of the eigenvalues of (2.1) which will be useful in the sequel.

First, we recall that $\lambda_j(\Omega, \beta A_1)$ are invariant under isometries, namely, if M is an isometry of $\mathbb{R}^2$, then

$$\lambda_j(\Omega, \beta A_1) = \lambda_j(M(\Omega), \beta A_1).$$

For the proof we refer to [30, Appendix A].

For any $\beta \in \mathbb{R}$ it is not difficult to show that

$$\lambda_j(\Omega, \beta A_1) = \lambda_j(\Omega, -\beta A_1).$$

The proof can be performed by observing that u is an eigenfunction corresponding to $\lambda_j(\Omega, \beta A_1)$ if and only if $\bar{u}$ is an eigenfunction corresponding to $\lambda_j(\Omega, -\beta A_1)$. Therefore it is not restrictive to consider only positive values of β.

The asymptotics of $\lambda_1(\Omega, \beta A_1)$ for large magnetic field have been investigated in depth (see e.g., [19, §8]). It turns out that if Ω is smooth, then

$$\lim_{\beta \to +\infty} \frac{\lambda_1(\Omega, \beta A_1)}{\beta} = \Theta_0 \tag{2.12}$$

where $\Theta_0 > 0$ is a universal constant (de Gennes constant) defined as

$$\Theta_0 = \min_{\xi \in \mathbb{R}} \mu_1(\xi) = \mu_1(\xi_0), \tag{2.13}$$

with $\mu_1(\xi)$ being the first eigenvalue of the following one-dimensional problem:

$$\begin{cases} -f''(t) + (\xi + t)^2 f(t) = \mu(\xi) f(t) , & t \in (0, +\infty) \\ f'(0) = 0. \end{cases} \tag{2.14}$$

For any $\xi \in \mathbb{R}$ it is standard to show that problem (2.14) admits a discrete spectrum made of a sequence of simple, non-negative eigenvalues diverging to $+\infty$. It is known (see [19]) that

$$\Theta_0 = \xi_0^2 \approx 0.590106,$$

and that $\xi_0 < 0$. We may refer e.g., to [6] for the numerical approximation of Θ_0 and for an estimate of the remainder.

The limit (2.12) has a surprising consequence for the first eigenvalue of a family of homothetic domains. It follows from (2.3) (see also [19]) that for all $\alpha > 0$ and $\beta > 0$,

$$\lambda_j(\Omega, \beta) = \alpha^2 \lambda_j\left(\alpha\Omega, \frac{\beta}{\alpha^2} A_1\right). \tag{2.15}$$

From (2.15) we see that, for fixed $\beta > 0$,

$$\lambda_1(\alpha\Omega, \beta A_1) = \frac{1}{\alpha^2}\lambda_1(\Omega, \alpha^2\beta A_1) = \Theta_0\beta + o(1)\,, \quad \text{as } \alpha \to +\infty.$$

In particular, the asymptotic limit is strictly positive and does not depend on the measure of the domain.

On the other hand we have that for all fixed $\beta > 0$, $\lim_{\alpha\to 0^+} \lambda_1(\alpha\Omega, \beta A_1) = 0$. In fact, let us take f to be the solution of

$$\begin{cases} \Delta f = 0\,, & \text{in } \Omega, \\ \langle \nabla f, N\rangle = -A_1(N)\,, & \text{on } \partial\Omega. \end{cases}$$

Then the 1-form $A' = A_1 + df$ satisfies

$$\begin{cases} dA' = dv, \\ \delta A' = 0 \\ A'(N) = 0, \end{cases}$$

and it follows from gauge invariance (see (2.4)) that $\lambda_1(\Omega, \alpha^2\beta A') = \lambda_1(\Omega, \alpha^2\beta A_1)$. Taking the test function $u = 1$ in (2.3) we get

$$\lambda_1(\alpha\Omega, \beta A_1) = \frac{1}{\alpha^2}\lambda_1(\Omega, \alpha^2\beta A_1) \le \frac{\alpha^2\beta^2}{|\Omega|}\int_\Omega |A'|^2$$

Actually, by [19, Proposition 1.5.2] we see that the inequality is exact as $\alpha \to 0$ in the sense that

$$\lambda_1(\alpha\Omega, \beta A_1) = \frac{\alpha^2\beta^2}{|\Omega|}\int_\Omega |A'|^2 + O(\alpha^4\beta^3). \tag{2.16}$$

as $\alpha \to 0^+$. If Ω is simply connected, $\beta A' = A_{can}$, where A_{can} is a distinguished potential 1-form which differs from A by an exact 1-form (see Section 4 for the precise definition of A_{can}).

The peculiar behavior of magnetic Neumann eigenvalues is clearly highlighted in Figure 3, were we have represented the analytic branches of the eigenvalues of the magnetic Laplacian with $\beta = 1$ on the disk $B_R := B(0, R)$ as functions of R. The first eigenvalue is singled out just by taking the minimum among all analytic branches. It vanishes as $R \to 0^+$ with quadratic speed, and shows an oscillating behavior as R grows. It remains bounded and converges to Θ_0 as $R \to +\infty$. For the disk B_R, from (2.16) (see also [20]) we can make the asymptotic behavior at $R = 0$ more precise:

$$\lambda_1(B_R, \beta A_1) = \frac{\beta^2 R^2}{8} + o(R^2)\,, \quad \text{as R} \to 0^+. \tag{2.17}$$

3. Upper bounds for λ_1: proofs

In this section we establish upper bounds for $\lambda_1(\Omega, \beta A_1)$ ($\beta > 0$ constant).

In order to state our first result, we recall the definition of circumradius R_Ω of a domain Ω:

$$R_\Omega := \inf\left\{R > 0 : \text{there exists } x_R \in \mathbb{R}^2 \text{ such that } \Omega \subset B(x_R, R)\right\}.$$

We have the following theorem, which implies Theorem 2.1.

Theorem 3.1. *For any bounded domain Ω with circumradius R_Ω we have*

$$\lambda_1(\Omega, \beta A_1) \le \begin{cases} \beta - \frac{1}{2R_\Omega^2}\,, & \text{if } R_\Omega > \frac{1}{\sqrt{\beta}}, \\ \frac{R_\Omega^2\beta^2}{2}\,, & \text{if } R_\Omega \le \frac{1}{\sqrt{\beta}} \end{cases} \tag{3.1}$$

In particular, if $R_\Omega \le 1/\sqrt{\beta}$, then $\lambda_1(\Omega, \beta A_1) \le \beta/2$. It follows that, for all $\beta > 0$

$$\lambda_1(\Omega, \beta A_1) < \beta.$$

Proof. Through the proof we shall denote $\lambda_1(\Omega, \beta A_1)$ simply by λ_1, while we shall write A for βA_1. Let (r,t) denote the standard polar coordinates in $\mathbb{R}^2$, where $r = |x|$ and t is the angular variable. We define the family of functions $\{u_n(r,t)\}_{n\in\mathbb{N}}$, expressed in polar coordinates, by setting $u_n(r,t) := r^n e^{int} e^{-\frac{\beta r^2}{4}}$. Recalling that in polar coordinates

$$\Delta_A u = -\partial^2_{rr} u - \frac{\partial_r u}{r} - \frac{\partial^2_{tt} u}{r^2} + \frac{\beta^2 r^2}{4} u + i\beta\partial_t u, \tag{3.2}$$

it is standard to prove that $\Delta_A u_n = \beta u_n$. A standard computation (see also [3]) shows that

$$|\nabla^A u_n|^2 = \beta |u_n|^2 - \frac{1}{2}\Delta |u_n|^2.$$

Hence, from the min-max principle (2.3) we find that for all $n \in \mathbb{N}$

$$\lambda_1 \int_\Omega |u_n|^2 \le \int_\Omega |\nabla^A u_n|^2 = \beta \int_\Omega |u_n|^2 - \frac{1}{2}\int_\Omega \Delta |u_n|^2. \tag{3.3}$$

Now, if the last term of (3.3) has a negative sign, this would immediately imply that β is a strict upper bound, but this is not in general the case.

We start by proving the second inequality of (3.1). Assume that, up to translations, $\Omega \subset B(0, R_\Omega)$, and consider (3.3) with $n = 0$. We have

$$\lambda_1 \int_\Omega e^{-\frac{\beta r^2}{2}} \le \beta \int_\Omega e^{-\frac{\beta r^2}{2}} + \frac{1}{2}\int_\Omega \beta(r^2\beta - 2) e^{-\frac{\beta r^2}{2}} = \frac{\beta^2}{2}\int_\Omega r^2 e^{-\frac{\beta r^2}{2}} \le \frac{R_\Omega^2 \beta^2}{2}\int_\Omega e^{-\frac{\beta r^2}{2}}.$$

This proves the second inequality in (3.1). Note that this inequality is valid for any β, however it implies a strict upper bound by β only for $R_\Omega < \sqrt{\frac{2}{\beta}}$.

We want to improve the upper bound for large R_Ω and to conclude the proof of (3.1). The main idea behind the proof of the first inequality of (3.1) is to *average* inequality (3.3) with respect to n. Namely, we multiply both sides of (3.3) by some $a_n > 0$, and sum the resulting inequalities over n, where n ranges in some subset of $\mathbb{N}$. Choosing the weights a_n in a suitable way, we will be able to make the sum of the terms involving $\Delta |u_n|^2$ at the right-hand side of (3.3) negative, in a controlled way. Let then $a_n > 0$, $n = 0, ..., N$. From (3.3) we get

$$\lambda_1 \int_\Omega \sum_{n=0}^N a_n |u_n|^2 \le \beta \int_\Omega \sum_{n=0}^N a_n |u_n|^2 - \frac{1}{2}\int_\Omega \Delta\left(\sum_{n=0}^N a_n |u_n|^2\right),$$

which implies

$$\lambda_1 \le \beta - \frac{1}{2}\frac{\int_\Omega \Delta\left(\sum_{n=0}^N a_n |u_n|^2\right)}{\int_\Omega \sum_{n=0}^N a_n |u_n|^2}. \tag{3.4}$$

Now, we note that

$$|u_n|^2 = e^{-\frac{\beta r^2}{2}} r^{2n},$$

hence

$$\sum_{n=0}^N a_n |u_n|^2 = e^{-\frac{\beta r^2}{2}} \sum_{n=0}^N a_n r^{2n}.$$

The scope is now to choose suitable $a_n > 0$ and take the limit as $N \to \infty$. This is done by noting that

$$e^{\frac{\beta r^2}{2}} = \sum_{n=0}^\infty \frac{\beta^n}{2^n n!} r^{2n},$$

and the convergence is uniform on any compact interval of $\mathbb{R}_+$. Then we choose

$$a_n = \frac{\beta^n}{c^n n!}$$

with $c > 0$. Hence, on any compact set of $\mathbb{R}^2$, we have

$$\lim_{N\to+\infty}\sum_{n=0}^{N} a_n|u_n|^2 = e^{\frac{\beta r^2}{2c}(2-c)} \tag{3.5}$$

and

$$\lim_{N\to+\infty} -\Delta\left(\sum_{n=0}^{N} a_n|u_n|^2\right) = e^{\frac{\beta r^2}{2c}(2-c)}\beta\frac{(2c+(2-c)r^2\beta)(2-c)}{c^2}$$

Now, we assume that $\Omega \subset B(0, R_\Omega)$. Then we have, for $r \le R_\Omega$

$$\frac{(2c+(2-c)r^2\beta)(2-c)}{c^2} = \frac{2(2-c)}{c} + \frac{(2-c)^2r^2\beta}{c^2} \le \frac{2(2-c)}{c} + \frac{(2-c)^2R_\Omega^2\beta}{c^2}.$$

Consider now the function

$$g(c) = \frac{2(2-c)}{c} + \frac{(2-c)^2R_\Omega^2\beta}{c^2}.$$

We have that

$$g'(c) = -\frac{4}{c^3}(c+\beta(2-c)R_\Omega^2)$$

and

$$\lim_{c\to0^+} g(c) = +\infty\,, \qquad \lim_{c\to+\infty} g(c) = \beta R_\Omega^2 - 2.$$

We see that if $R_\Omega \ne \frac{1}{\sqrt{\beta}}$

$$g'(c) = 0 \quad\Longleftrightarrow\quad c = \frac{2R_\Omega^2\beta}{R_\Omega^2\beta-1}.$$

Now, if $R_\Omega > \frac{1}{\sqrt{\beta}}$, we choose this c, and with this choice $g\left(\frac{2R_\Omega^2\beta}{R_\Omega^2\beta-1}\right) = -\frac{1}{\beta R_\Omega^2}$. We conclude that, when $c = \frac{2R_\Omega^2\beta}{R_\Omega^2\beta-1}$,

$$\lim_{N\to+\infty} -\Delta\left(\sum_{n=0}^{N} a_n|u_n|^2\right) = e^{\frac{\beta r^2}{2c}(2-c)}\beta\frac{(2c+(2-c)r^2\beta)(2-c)}{c^2} \le -\frac{1}{R_\Omega^2}e^{\frac{\beta r^2}{2c}(2-c)}. \tag{3.6}$$

Using (3.5) and (3.6) in (3.4) we deduce the first inequality of (3.1).

□

Remark 3.2. The upper bound for $R_\Omega \le \frac{1}{\sqrt{\beta}}$ shows a correct behavior with respect to β, which is quadratic, in view of (2.15), (2.16) and (2.17).

In view of the asymptotic behavior (2.12) the natural question is whether $\Theta_0\beta$ is an upper bound for $\lambda_1(\Omega, \beta A_1)$, for any domain Ω. We prove this result for certain classes of domains.

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^2$ satisfying (up to isometries) one of the following two conditions:*

1) Ω is a sub-graph, namely

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : a < x_1 < b\,, 0 < x_2 < g(x_1)\}$$

for some smooth $g : (a, b) \to [0, +\infty)$, $-\infty < a < b < +\infty$.

2) Ω is contained in some strip $(a, b) \times (0, +\infty)$ and contains $(a, b) \times (0, -2\xi_0/\sqrt{\beta})$, where $\xi_0 < 0$ is the constant defined in (2.13).

Then

$$\lambda_1(\Omega, \beta A_1) < \Theta_0\beta. \tag{3.7}$$

Proof. We first remark that it is sufficient to prove the result for $\beta = 1$. In fact, from (2.15) we have that $\lambda_1(\Omega, \beta A_1) = \beta\lambda_1(\Omega', A_1)$, where $\Omega' = \sqrt{\beta}\Omega$. Now, Ω is a sub-graph of the form 1) if and only if Ω' is; Ω satisfies condition 2) if and only if Ω' does with $\beta = 1$.

Let f be a first eigenfunction of (2.14) with $\xi = \xi_0 < 0$ and hence first eigenvalue Θ_0, defined in (2.13). Recall that $\Theta_0 = \xi_0^2$. We can choose $f > 0$ on $[0, +\infty)$. We recall that f satisfies

$$-f''(t) + (\xi_0 + t)^2 f(t) = \Theta_0 f(t)\,,\ t \in (0, +\infty)\,, \quad f'(0) = 0. \tag{3.8}$$

We prove that $f' < 0$ on $(0,+\infty)$ and that $\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} f'(t) = 0$. Equation (3.8) implies that f'' has only one zero in $(0,+\infty)$, namely $-2\xi_0$ (moreover, $f''(0) = 0$). This implies that f' is monotone on $(-2\xi_0,+\infty)$, and since $f \in H^1((0,+\infty))$, necessarily f' is increasing to 0 on $(-2\xi_0,+\infty)$ and $\lim_{t\to\infty} f'(t) = 0$. On the other hand, f' is decreasing on $(0,-2\xi_0)$. Moreover, $f'(0) = 0$ and $f(0) > 0$. Suppose by contradiction that f is not decreasing on $(0,+\infty)$. This would imply the existence of $t_0 \in (0,+\infty)$ such that $f'(t_0) = 0$. Since $f'(0) = 0$ and $\lim_{t\to+\infty} f'(t) = 0$, this would imply the existence of two distinct points $t_1, t_2 \in (0,+\infty)$ such that $f''(t_1) = f''(t_2) = 0$, but this is impossible since $f''(t) = 0$ on $(0,+\infty)$ if and only if $t = -2\xi_0$. This proves that $f' < 0$ on $(0,+\infty)$. In particular then, $\lim_{t\to+\infty} f(t) = 0$, since $f \in L^2((0,+\infty))$.

We consider the magnetic Laplacian $\Delta_{A'}$, where $A' = -(\xi_0 + x_2)dx_1$. By gauge invariance, the Neumann spectrum of $\Delta_{A'}$ coincides with the Neumann spectrum of Δ_{A_1} on Ω (the two forms differ by an exact 1-form). From now on we shall denote $\lambda_1(\Omega, A_1)$ simply by λ_1

We are ready to prove 1) for $\beta = 1$. Using $f(x_2)$ as test function in (2.3) we get

$$\lambda_1 \int_\Omega f^2 \le \int_\Omega |\nabla f|^2 + (\xi_0 + x_2)^2 f^2 = \int_a^b \int_0^{g(x_1)} ((f'(x_2))^2 + (\xi_0 + x_2)^2 f^2(x_2)) dx_2 dx_1$$
$$= \Theta_0 \int_\Omega f^2 + \int_a^b f(g(x_1)) f'(g(x_1)) dx_1 < \Theta_0 \int_\Omega f^2.$$

We have used, in the integration by parts, the fact that $f'(0) = 0$; the last inequality follows since f is positive and strictly decreasing.

In order to prove 2) (and re-prove 1)), we use $f(x_2)$ as test function in (2.3), and use the identity $f'^2 = -ff'' + \frac{1}{2}(f^2)''$. We obtain

$$\lambda_1 \int_\Omega f^2 \le \int_\Omega |\nabla f|^2 + (\xi_0 + x_2)^2 f^2 = \int_\Omega (-ff'' + (\xi_0 + x_2) f^2) + \frac{1}{2}\int_\Omega (f^2)'' \tag{3.9}$$
$$= \Theta_0 \int_\Omega f^2 + \frac{1}{2}\int_\Omega (f^2)''$$

We are left with the study $\frac{1}{2}\int_\Omega (f^2)''$. In fact the upper bound $\lambda_1 < \Theta_0$ holds provided $\frac{1}{2}\int_\Omega (f^2)'' < 0$.

We see that

$$\frac{1}{2}\int_0^{+\infty} (f^2)'' = \frac{1}{2}\lim_{s\to+\infty} \int_0^s (f^2)'' = \lim_{s\to+\infty} \frac{1}{2}(f^2)'(s) = \lim_{s\to+\infty} f(s) f'(s) = 0$$

since $f'(0) = 0$. Moreover, using the identity $f'^2 = -ff'' + \frac{1}{2}(f^2)''$ and the differential equation satisfied by f, we find that

$$\frac{1}{2}(f^2)''(t) = (f')^2(t) + t f^2(t)(t + 2\xi_0)$$

and this quantity is non-negative for $t \ge -2\xi_0$.

We easily deduce two facts:

a) $\frac{1}{2}\int_0^L (f^2)'' < 0$ for all $L > 0$.
b) $\frac{1}{2}\int_I (f^2)'' < 0$ for all $I \subset \mathbb{R}^+$ such that $(0,-2\xi_0) \subset I$.

Either a) or b) imply the inequality (3.7). We show now that 1) and 2) in the statement of Theorem 3.3 imply a) and b), respectively. In fact, we can re-write the last term of (3.9) as

$$\frac{1}{2}\int_\Omega (f^2)'' = \int_{P(\Omega)} \left(\frac{1}{2}\int_{E(x_1)} (f^2)''(x_2) dx_2 \right) dx_1$$

where $P(\Omega) = \{x_1 \in \mathbb{R} : (x_1, x_2) \in \Omega\} \subset \mathbb{R}$ and, for any $x_1 \in P(\Omega)$, $E(x_1) = \{x_2 \in (0,+\infty) : (x_1, x_2) \in \Omega\}$. Recall that we have assumed that Ω is in the half-plane $x_2 > 0$. Assume we are in the hypothesis 1) or 2):

1) Ω is a sub-graph. Then we have $E(x_1) = (0, L(x_1))$ for all x_1, $L(x_1) > 0$, which is a).
2) Ω is contained in some strip $(a,b) \times (0,+\infty)$ and Ω contains $(a,b) \times (0,-2\xi_0)$, which is b).

The proof is now concluded.

∎

Remark 3.4. Note that the case 2) of Theorem 3.3 implies an upper bound for $\lambda_1(\Omega, \beta A_1)$ with $\Theta_0\beta$ for a class of domains containing also non-simply connected domains. We just require that a suitable rectangle is contained in the domain. Note that as $\beta \to +\infty$ the size of the rectangle becomes small, hence more domains are allowed for the upper bound. As $\beta \to 0^+$, less domains are allowed. However, as $\beta \to 0^+$ we have in general better upper bounds than $\Theta_0\beta$, in fact upper bounds behave like $C\beta^2$ as $\beta \to 0^+$ (see (2.16), see also [19]). Note also that suitable unions of domains of the form 1) and 2) still enjoy the upper bound $\Theta_0\beta$.

We prove now a similar upper bound for *polygonal rep-tiles.*

Definition 3.5. The interior Ω of a polygon in $\mathbb{R}^2$ is called a *rep-tile* if there exists an integer $m \geq 2$ such that

$$\Omega = \mathrm{Int}\left(\bigcup_{i=1}^{m} \overline{\omega_i}\right),$$

where each ω_i is isometric to $\frac{1}{\sqrt{m}}\Omega$ (Ω is then said to be *rep-m*).

Remark 3.6. The above terminology was introduced by S. W. Golomb as an abbreviation of *replicating figure* (see [23], where many examples are given). He also considered non-polygonal rep-tiles, which are however too irregular to be domains in our sense. Although the definition is restrictive, all triangles and all parallelograms are polygonal rep-tiles, more precisely all are rep-4. Let us note that not all polygonal rep-tiles are covered by the previous Theorem 3.3, e.g., parallelograms are not.

We have already recalled that for any smooth bounded domain, the limit (2.12) holds. We now assume that the boundary of Ω is a polygon, with N vertices and with corresponding angles $\alpha_1, \dots, \alpha_N$. Under this assumption, it follows from [5, Corollary 1.3] that there exists a constant $\Lambda(\Omega) \leq \Theta_0$, depending only on the angles α_s, such that

$$\lim_{\beta \to +\infty} \frac{\lambda_1(\Omega, \beta A_1)}{\beta} = \Lambda(\Omega).$$

More specifically,

$$\Lambda(\Omega) = \min_{1 \leq s \leq N} \mu(\alpha_s),$$

where, for $\alpha \in]0, 2\pi[$, $\mu(\alpha)$ is the bottom of the spectrum of the magnetic Laplacian with unit magnetic field in the infinite sector of angular opening α (see e.g. [19, Section 4.4] for more details). It is proved in [5, Remarks 2.6 and 4.3] that $\mu(\alpha) < \Theta_0$ for all $\alpha \in]0, \frac{\pi}{2}]$, which implies that $\Lambda(\Omega) < \Theta_0$ whenever $\min_{1 \leq s \leq N} \alpha_s \leq \frac{\pi}{2}$. This is in particular the case when Ω is a triangle or a parallelogram. It has been conjectured that $\mu(\alpha) < \Theta_0$ for all $\alpha \in]0, \pi[$, and progress in this direction is made in [7, 17]. The result corresponds to the existence of an eigenvalue below the essential spectrum. It would immediately follow from the stronger conjecture that $\alpha \mapsto \mu(\alpha)$ is increasing in the interval $]0, \pi[$ (see [5, Remark 3.1] and [19, Conjecture 4.4.2]).

We are now ready to state the next theorem.

Theorem 3.7. *Let Ω be a polygonal rep-tile. Then*

$$\lambda_1(\Omega, \beta A_1) \leq \Lambda(\Omega)\, \beta. \tag{3.10}$$

Proof. The proof is an adaptation of the argument from Pólya (see [34]). We fix one of the pieces in the decomposition of Ω, say ω_1. For all $\beta \in \mathbb{R}$,

$$\lambda_1(\omega_1, \beta A_1) \leq \lambda_1(\Omega, \beta A_1). \tag{3.11}$$

Indeed, the Sobolev space $H^1(\Omega)$ can be seen as a subspace of $H^1(\Omega')$, with

$$\Omega' := \bigcup_{i=1}^{m} \omega_i. \quad \text{(the union is disjoint)}$$

The Hilbert space $H^1(\Omega')$ can itself be seen as the direct sum

$$\bigoplus_{i=1}^{m} H^1(\omega_i).$$

This last identification tells us that $\lambda_1(\Omega', \beta A_1) = \lambda_1(\omega_1, \beta A_1)$, and the inclusion $H^1(\Omega) \subset H^1(\Omega')$ implies, by the variational definition of eigenvalues, that

$$\lambda_1(\Omega', \beta A_1) \le \lambda_1(\Omega, \beta A_1).$$

This yields (3.11). In particular, replacing β by $m\beta$ we have

$$\lambda_1(\omega_1, m\beta A_1) \le \lambda_1(\Omega, m\beta A_1)$$

for all $m > 0$. On the other hand, the scaling property of the magnetic eigenvalues tells us that

$$\lambda_1(\omega_1, m\beta A_1) = \lambda_1\left(\frac{1}{\sqrt{m}}\Omega, m\beta A_1\right) = m\,\lambda_1\left(\Omega, \beta A_1\right). \tag{3.12}$$

We get

$$\lambda_1(\Omega, \beta A_1) \le \frac{1}{m}\lambda_1(\Omega, m\beta A_1). \tag{3.13}$$

Iterating (3.13), we obtain that for all positive integers k,

$$\lambda_1(\Omega, \beta A_1) \le \frac{1}{m^k}\lambda_1\left(\Omega, m^k\beta A_1\right). \tag{3.14}$$

Taking $k \to +\infty$, we find

$$\frac{\lambda_1(\Omega, \beta A_1)}{\beta} \le \liminf_{k\to+\infty} \frac{\lambda_1\left(\Omega, m^k\beta A_1\right)}{m^k\,\beta} \le \Lambda(\Omega). \qquad \square$$

This concludes the proof.

4. Upper bounds of λ_1 for variable magnetic fields: proofs

In this section we consider the first eigenvalue of problem (2.1) when Ω is a planar domain or more generally a domain in an orientable compact surface M, and A is an arbitrary smooth magnetic potential (a smooth 1-form) giving rise to a (variable) magnetic field $dA = \beta dv$, where dv is the Riemannian volume form of M. Through this section we shall denote the first eigenvalue of (2.1) by $\lambda_1(\Omega, A)$. As we have already discussed in Subsection 2.7, if Ω is simply connected, $\lambda_1(\Omega, A)$ depends only on β. In the case that Ω is not simply connected, given A, A' with $dA = dA' = \beta dv$, we have in general $\lambda_1(\Omega, A) \neq \lambda_1(\Omega, A')$. In this case we shall choose a distinguished *primitive* of βdv, which we will call A_{can}. In order to define A_{can}, we need a few preliminaries.

Let Ω be a smooth, bounded domain in an orientable Riemannian surface M, and let $\beta : \Omega \to \mathbb{R}$ be a given smooth function. Consider problem

$$\begin{cases} \Delta\phi = \beta\,, & \text{in } \Omega,\\ \phi = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Problem (4.1) admits a unique solution which reduces to the torsion function when $\beta = 1$. We denote A_{can} the 1-form defined by $A_{can} = -\star d\phi$, where $\star$ is the Hodge-star operator acting on differential forms, for a chosen orientation of Ω. Recall that the Hodge-star operator is defined by the following relation: for any pair of 1-forms ψ_1, ψ_2 we have

$$\psi_1 \wedge \star\psi_2 := \langle \psi_1, \psi_2\rangle dv,$$

and $\star 1 = dv$, $\star dv = 1$. For example, in $\mathbb{R}^2$ $\langle\cdot,\cdot\rangle$ stands for the standard scalar product and, with Cartesian coordinates (x_1, x_2) and positive orthonormal basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, one has $\star 1 = dx_1 \wedge dx_2$, $\star dx_1 = dx_2$, $\star dx_2 = -dx_1$, $\star(dx_1 \wedge dx_2) = 1$. We denote by δ the co-differential (on 1-forms we have $\delta = -\mathrm{div}$). We prove the following lemma.

Lemma 4.1. *The form A_{can} is a primitive of βdv, i.e., $dA_{can} = \beta dv$.*

Proof. We recall that $\delta = -\star d\star$, which implies that, for 1-forms, $\star\delta = -d\star$ and $\delta\star = -\star d$. Consider the 1-form $A_{can} = -\star d\phi$, where ϕ solves (4.1). Then

$$dA_{can} = -d\star d\phi = \star\delta d\phi = \star\Delta\phi = \star\beta = \beta dv.$$

$\square$

We are ready to state the main result of this section.

Theorem 4.2. *Let Ω be a smooth, bounded, simply connected domain in an orientable Riemannian surface M. Let A be a smooth 1-form and let $\beta : \Omega \to \mathbb{R}$ be defined by $dA = \beta dv$, where dv is the volume 2-form. Let $\lambda_1(\Omega, A)$ denote the first eigenvalue of (2.1) with magnetic potential A. Let $\phi : \Omega \to \mathbb{R}$ be the unique solution to (4.1). Then*

$$\lambda_1(\Omega, A) \leq \frac{\int_\Omega \beta(e^{2\phi} - 1)dv}{\int_\Omega e^{2\phi}dv}. \tag{4.2}$$

If Ω is not simply connected, the inequality holds for $\lambda_1(\Omega, A_{can})$.

Proof. Clearly, it is enough to show the assertion for $A = A_{can} = -\star d\phi$. Note that then, since the Hodge star operator is an isometry, we have $|A|^2 = |d\phi|^2$. If u is a real valued smooth function then we have from (2.3)

$$\lambda_1(\Omega, A)\int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 + |A|^2 u^2 = \int_\Omega |\nabla u|^2 + |\nabla\phi|^2 u^2 \tag{4.3}$$

We take $u = e^\phi$ so that $\Delta u = -u|\nabla\phi|^2 + u\Delta\phi = -u|\nabla\phi|^2 + u\beta$. Integrating by parts, taking into account that on $\partial\Omega$ one has $u = 1$ and $\frac{\partial u}{\partial N} = \frac{\partial\phi}{\partial N}$, and using the Green formula on the last boundary integral, we get:

$$\int_\Omega |\nabla u|^2 = \int_\Omega u\Delta u + \int_{\partial\Omega} u\frac{\partial u}{\partial N} = -\int_\Omega |\nabla\phi|^2 u^2 + \int_\Omega \beta(u^2 - 1).$$

Inserting this identity in the right-hand side of (4.3) we obtain the assertion. □

Remark 4.3. It can be shown that A_{can} has the least L^2-norm among all primitives of β. The following natural question arises: is $\lambda_1(\Omega, A_{can}) = \inf_{dA=\beta dv}\lambda_1(\Omega, A)$? At the moment we don't have an answer. We thank the Referee for posing the question.

Let us set $\beta^* = \max_{\overline{\Omega}}|\beta|$, $\phi^* = \max_{\overline{\Omega}}|\phi|$. From Theorem 4.2 we deduce the following

Corollary 4.4. *Let Ω be a smooth, bounded, simply connected domain in an orientable Riemannian surface M, and let A be any smooth 1-form such that $dA = \beta dv$, with $\beta : \Omega \to \mathbb{R}$ smooth. Then*

1)

$$\lambda_1(\Omega, A) < \beta^*. \tag{4.4}$$

2) If $\beta(x) \geq 0$ for all $x \in \Omega$, then

$$\lambda_1(\Omega, A) \leq \beta^*(1 - e^{-2\phi^*}). \tag{4.5}$$

3) If $\beta(x) \geq 0$ for all $x \in \Omega$ and Ω is a domain in $\mathbb{R}^2$, then

$$\lambda_1(\Omega, A) \leq \beta^*(1 - e^{-\frac{\beta^*|\Omega|}{2\pi}}). \tag{4.6}$$

4) If Ω is a domain in $\mathbb{R}^2$, $\beta > 0$ is constant and $A = \beta A_1$ is the standard potential (1.2) *(hence $\beta(x) \equiv \beta > 0$), then*

$$\lambda_1(\Omega, \beta A_1) \leq \beta(1 - e^{-\frac{\beta|\Omega|}{2\pi}}). \tag{4.7}$$

5) If Ω is not simply connected, all the inequalities above hold with $A = A_{can}$.

Proof. We start by proving 1). We note that replacing A by $-A$, the eigenfunctions of problem (2.1) are changed to their conjugates, but the spectrum remains the same. From Theorem 4.2 we have

$$\lambda_1(\Omega, A)\int_\Omega e^{2\phi} \leq \int_\Omega \beta(e^{2\phi} - 1)$$

and, at the same time, changing β to $-\beta$ and ϕ to $-\phi$:

$$\lambda_1(\Omega, A)\int_\Omega e^{-2\phi} \leq \int_\Omega -\beta(e^{-2\phi} - 1).$$

Summing up the two inequalities we obtain

$$\lambda_1(\Omega, A)\int_\Omega \cosh(2\phi) \leq \int_\Omega \beta\sinh(2\phi) \leq \beta^*\int_\Omega |\sinh(2\phi)| = \beta^*\int_\Omega \sinh(|2\phi|) < \beta^*\int_\Omega \cosh(2\phi).$$

This proves (4.4).

We pass to 2). Inequality (4.5) follows immediately from (4.2): from the maximum principle, since $\beta \geq 0$, we have that $\phi \geq 0$, and we conclude by rough estimates.

We prove 3). Inequality (4.6), is a consequence of the well-known isoperimetric inequality

$$\max_{\overline{\Omega}} \phi_\Omega \leq \max_{\overline{B_R}} \phi_{B_R},$$

where ϕ_Ω, ϕ_{B_R} are the solutions of (4.1) with $\beta = 1$ on Ω and B_R, respectively, see [39]. Here $\Omega \subset \mathbb{R}^2$, and $B_R \subset \mathbb{R}^2$ is a disk of radius R centered at 0 with $|B_R| = |\Omega|$. In fact, we have the explicit expression $\phi_R(x) = \frac{1}{4}(R^2 - |x|^2)$, and therefore $\phi^*_{B_R} := \max_{\overline{B_R}} \phi_{B_R} = \frac{|\Omega|}{4\pi}$. Now, observe that $\Delta(\beta^*\phi_\Omega) = \beta^* \geq \beta$ on Ω and $\beta^*\phi_\Omega = 0$ on $\partial\Omega$. Hence, by the maximum principle we have $\phi \leq \beta^*\phi_\Omega$ on Ω and hence

$$\phi^* = \max_{\overline{\Omega}} \phi \leq \beta^* \max_{\overline{\Omega}} \phi_\Omega \leq \frac{\beta^*|\Omega|}{4\pi}.$$

Next we consider 4): inequality (4.7) follows immediately from (4.6).

The assertion 5) is straightforward. This concludes the proof. □

5. Lower bounds for λ_1: proofs

The study of lower bounds for $\lambda_1(\Omega, \beta A_1)$ ($\beta > 0$ constant) is rather challenging. It is easy to produce small eigenvalues by perturbing a domain Ω with a local perturbation near the boundary, namely, attaching to Ω a small Cheeger dumbbell, as for the usual Neumann problem for the Laplacian (see the counterexample in [12, p. 420]). On the other hand, one can produce examples of convex domains of large diameter and any area with first eigenvalue either arbitrarily small or bounded away from zero. Also, there exist thin domains with first eigenvalue arbitrarily small or uniformly bounded away from zero as the thickness goes to zero. We have collected a series of examples in Appendix C.

The starting point of our analysis is [14, Theorem 5.1], which provides lower bounds for $\lambda_1(\Omega, \beta A_1)$ in terms of $\beta, |\Omega|$, $\lambda_2^N(\Omega)$ (the second Neumann eigenvalue of the Laplacian on Ω), and the inradius ρ_Ω of Ω, defined by

$$\rho_\Omega := \sup_{x \in \Omega} \inf_{y \in \partial\Omega} |x - y|. \tag{5.1}$$

We recall it here for the reader's convenience.

Theorem 5.1 ([14, Theorem 5.1])**.** *Let Ω be bounded, simply connected domain in $\mathbb{R}^2$. Then*

$$\lambda_1(\Omega, \beta A_1) \geq \frac{\pi}{4|\Omega|} \cdot \frac{\beta^2 \rho_\Omega^4 \lambda_2^N(\Omega)}{\beta^2 \rho_\Omega^2 + 6\lambda_2^N(\Omega)}, \quad \text{if } \beta \leq \rho_\Omega^{-2} \tag{5.2}$$

and

$$\lambda_1(\Omega, \beta A_1) \geq \frac{\pi}{4|\Omega|} \cdot \frac{\beta \rho_\Omega^2 \lambda_2^N(\Omega)}{\beta + 24\lambda_2^N(\Omega)}, \quad \text{if } \beta \geq \rho_\Omega^{-2}, \tag{5.3}$$

where $\lambda_2^N(\Omega)$ is the first positive eigenvalue of the Neumann Laplacian on Ω and ρ_Ω is the inradius of Ω.

Note that for some domains the lower bounds given by Theorem 5.1 are not optimal. The following example clarifies this.

Example 5.2. *Let $\Omega =]-k, k[\times]-1/2, 1/2[$, with $k \in \mathbb{N}$. Then $\rho_\Omega = 1/2$. As $k \to +\infty$, $\lambda_2^N(\Omega) \to 0$, and therefore the lower bound given by (5.2)-(5.3) goes to 0 as well. On the other hand, we have*

$$\Omega = \text{Int} \bigcup_{l=-k}^{k} \overline{\Omega_l}$$

where

$$\Omega_l =]l, (l+1)[\times] - 1/2, 1/2[.$$

Clearly, bounds (5.2)-(5.3) hold for $\lambda_1(\Omega_l, \beta A_1)$ with $\rho_{\Omega_l} = 1/2$, $\lambda_2^N(\Omega_l) = \pi^2$ and $|\Omega_l| = 1$. Therefore the same lower bound holds for Ω (see Theorem 5.3), and this lower bound does not depend on k, therefore is uniformly bounded away from 0 as $k \to +\infty$.

The direct application of Theorem 5.1 to the previous example does not yield a good lower bound since the behavior of $\lambda_2^N(\Omega)$ and $\lambda_1(\Omega,\beta A_1)$ drastically diverge as $k\to+\infty$: $\lambda_2^N(\Omega)$ tends to 0 (the area goes to $+\infty$) while $\lambda_1(\Omega,\beta A_1)$ stays uniformly bounded away from zero (large area does not imply small eigenvalue). However, the use of a suitable covering of Ω and the application of Theorem 5.1 on each piece of the covering, allow to improve the lower bound. This is the main idea behind the main result of this section. Before stating it, we need some preliminary results.

Theorem 5.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ such that*

$$\Omega = \mathrm{Int}\bigcup_{i=1}^N \overline{\Omega_i},$$

where $\Omega_i \subset \Omega$ are subdomains such that a point $p\in\Omega$ is contained in at most K such subdomains. Then

$$\lambda_1(\Omega,\beta A_1) \geq \frac{1}{K}\min_{i=1,\dots,N}\lambda_1(\Omega_i,\beta A_1). \tag{5.4}$$

Proof. Through the proof we shall denote βA_1 simply by A. Let u be an eigenfunction associated with $\lambda_1(\Omega,\beta A_1)$. Then its restriction to Ω_i is a suitable test function for the min-max principle (2.3) for $\lambda_1(\Omega_i,\beta A_1)$, which means:

$$\lambda_1(\Omega_i,A)\int_{\Omega_i}|u|^2 \leq \int_{\Omega_i}|\nabla^A u|^2.$$

Summing over $i=1,...,N$ we get

$$\sum_{i=1}^N \lambda_1(\Omega_i,A)\int_{\Omega_i}|u|^2 \leq \sum_{i=1}^N\int_{\Omega_i}|\nabla^A u|^2. \tag{5.5}$$

For the left-hand side of (5.5) we have

$$\sum_{i=1}^N \lambda_1(\Omega_i,A)\int_{\Omega_i}|u|^2 \geq \min_{j=1,\dots,N}\lambda_1(\Omega_j,A)\sum_{i=1}^N\int_{\Omega_i}|u|^2 \geq \min_{j=1,\dots,N}\lambda_1(\Omega_j,A)\int_{\Omega}|u|^2, \tag{5.6}$$

while for the right-hand side of (5.5) we have

$$\sum_{i=1}^N\int_{\Omega_i}|\nabla^A u|^2 \leq K\int_\Omega|\nabla^A u|^2. \tag{5.7}$$

Thanks to (5.6), (5.7) and (5.5) we get

$$\min_{j=1,\dots,N}\lambda_1(\Omega_j,A)\int_\Omega|u|^2 \leq K\int_\Omega|\nabla^A u|^2.$$

It follows that

$$\frac{\int_\Omega|\nabla^A u|^2}{\int_\Omega|u|^2} \geq \frac{1}{K}\min_{j=1,\dots,N}\lambda_1(\Omega_j,A).$$

□

We will look for coverings $\Omega=\mathrm{Int}\bigcup_{i=1}^N\overline{\Omega_i}$, where each Ω_i is star-shaped with respect to some of its points. In order to combine Theorems 5.1 and 5.3 we need to have good estimates for $\lambda_2^N(\Omega_i)$. We can get it as a special case of a result of [8]:

Theorem 5.4 ([8, Theorem 1])**.** *Let $\Omega\subset\mathbb{R}^2$ be a smooth, bounded domain. Assume that Ω is star-shaped with respect to a point $p\in\Omega$. Let R be the radius of the largest ball centered at p contained in Ω and R_0 be the radius of the smallest ball centered at p containing Ω. There exists a universal constant $C_1>0$ such that the first nonzero Neumann eigenvalue $\lambda_2^N(\Omega)$ has a lower bound given by*

$$\lambda_2^N(\Omega) \geq C_1\frac{R^2}{R_0^4}. \tag{5.8}$$

Remark 5.5. The result of [8] is stated for compact manifold with smooth boundary. In particular it holds for smooth Euclidean domains. However, it is valid also for piecewise smooth, Lipschitz domains by approximation with smooth domains. In fact, since Ω is star-shaped, we can consider a family of smooth domains $\Omega_\varepsilon \subset \Omega$ with boundaries living in an ε-tubular neighborhood of $\partial\Omega$, which uniformly approximate Ω and which are star-shaped with respect to p. Clearly, the radii $R(\varepsilon), R_0(\varepsilon)$ of Ω_ε (as in Theorem 5.4) converge to R, R_0, respectively. Moreover, we have spectral convergence of $\lambda_j^N(\Omega_\varepsilon)$ to $\lambda_j^N(\Omega)$ (see e.g., [2, §5]).

A combination of Theorems 5.1 and 5.4 allows to prove a lower bound for $\lambda_1(\Omega, \beta A_1)$ for star-shaped domains.

Proposition 5.6. *Let Ω be a domain which is star-shaped with respect to a point p, and let $R, R_0 > 0$ be defined as in Theorem 5.4. Then*

$$\lambda_1(\Omega, \beta A_1) \geq c\beta^2 \frac{R^8}{R_0^6}, \quad \text{if } \beta \leq \rho_\Omega^{-2}$$

and

$$\lambda_1(\Omega, \beta A_1) \geq c\frac{R^6\beta}{R_0^6(R^2\beta+1)}, \quad \text{if } \beta \geq \rho_\Omega^{-2},$$

for some universal constant $c > 0$.

Proof. By Theorem 5.4 we have

$$\lambda_2^N(\Omega) \geq C_1 \frac{R^2}{R_0^4}.$$

Moreover

$$\lambda_2^N(\Omega) \leq \lambda_1^D(\Omega) \leq \frac{\gamma}{R^2}, \quad \frac{\pi}{4|\Omega|} \geq \frac{1}{4R_0^2}, \quad \rho_\Omega \geq R,$$

where $\gamma := \lambda_1^D(B(0,1))$ denotes the first Dirichlet eigenvalue of the disk of radius 1. Recall that $\lambda_2^N(\Omega) \leq \lambda_1^D(\Omega)$ is the Friedlander inequality.

We use these inequalities in (5.2) to get

$$\lambda_1(\Omega, \beta A_1) \geq \frac{C_1\beta^2 R^8}{4R_0^6(\beta^2\rho_\Omega^2 R^2 + 6\gamma)}$$

when $\beta \leq \rho_\Omega^{-2}$. We note that in this hypothesis, $\beta^2\rho_\Omega^2 R^2 \leq \beta^2\rho_\Omega^4 \leq 1$. Hence we deduce that for $\beta \leq \rho_\Omega^{-2}$

$$\lambda_1(\Omega, \beta A_1) \geq \frac{C_1\beta^2 R^8}{4R_0^6(1+6\gamma)}.$$

Analogously, considering (5.3), we get that

$$\lambda_1(\Omega, \beta A_1) \geq \frac{C_1\beta R^6}{4R_0^6(R^2\beta + 24\gamma)}$$

when $\beta \geq \rho_\Omega^{-2}$. The proof is concluded by observing that a suitable constant c in the proposition is given by $c = \frac{C_1}{96\gamma}$. □

We recall the following

Definition 5.7. A domain Ω satisfies the δ-interior ball condition if, for any $x \in \partial\Omega$, there exists a ball of radius δ tangent to $\partial\Omega$ at x and entirely contained in Ω.

For smooth domains, this is equivalent to saying that the injectivity radius of the normal exponential map is at least δ. Therefore any point of a segment hitting the boundary orthogonally at $p \in \partial\Omega$ minimizes the distance to the boundary up to distance δ to p.

Definition 5.8. Let Ω be a bounded domain with smooth boundary, and let $\varepsilon > 0$. A maximal collection of points $\mathcal{P}_\varepsilon = \{p_1, ..., p_n\}$ with the following properties:

- $\mathrm{dist}(p_j, p_k) \geq \varepsilon$ for all $j \neq k$,
- $\mathrm{dist}(p_j, \partial\Omega) \geq \varepsilon$ for all j,

is called a *maximal ε-net*.

The goal is to produce a general lower bound for the first eigenvalue $\lambda_1(\Omega, \beta A_1)$ of domains Ω with the δ-interior ball condition depending only on δ (and β). To this aim we cover Ω by star-shaped subdomains Ω_i, and use Proposition 5.6 to control $\lambda_1(\Omega_i, \beta A_1)$ and then Theorem 5.3 to control $\lambda_1(\Omega, \beta A_1)$. In the case of convex domains with the δ-interior ball condition, a suitable covering is proved in [11, Lemma 11]. We extend here this last result dropping the convexity assumption.

Lemma 5.9. *Let Ω be a bounded domain with smooth boundary and let $\mathcal{P}_\varepsilon = \{p_1, ..., p_n\}$ be a maximal ε-net in Ω. Assume that Ω satisfies the δ-interior ball condition with $\delta \geq \varepsilon$. Then Ω admits an open covering $\{\Omega_1, ..., \Omega_n\}$, $\Omega = \bigcup_{i=1}^n \Omega_i$, with the following properties:*

- *every Ω_i is star-shaped with respect to some point $p_i \in \Omega_i$ and has piecewise smooth boundary.*
- *For each $i = 1, ..., n$ one has $B(p_i, \varepsilon/2) \subseteq \Omega_i \subseteq B(p_i, 2\varepsilon)$.*
- *There exists a universal constant $M \in \mathbb{N}$ (not depending on Ω) such that a point $x \in \Omega$ can be contained in at most M of the domains Ω_i.*

Proof. We remark that the difficulty of covering Ω by star-shaped domains Ω_i lies in the region near the boundary. Indeed, far from the boundary it is easy to find a nice covering. We will proceed in two steps. The family $\{\Omega_i\}$ will be the union of a family $\{\Omega_{i,B}\}$ of domains having a non empty intersection with the boundary $\partial\Omega$ and a family of domains $\{\Omega_{i,I}\}$ which do not intersect the boundary. We will prove the result for $\varepsilon = \delta$; it clearly holds for $0 < \varepsilon < \delta$ since the δ-interior ball condition implies the ε-interior ball condition for all $0 < \varepsilon \leq \delta$.

Step 1: construction of the domains $\{\Omega_{i,B}\}$. Let $\partial\Omega_\delta$ be the equidistant set to $\partial\Omega$:

$$\partial\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \delta\}.$$

By definition, if $x \in \partial\Omega_\delta$, then the ball $B(x, \delta) \subset \Omega$ is tangent to $\partial\Omega$ at at least one point.

Let $\mathcal{P}_1 = \{x_1, ..., x_n\}$ be a maximal $\frac{\delta}{2}$-net in $\partial\Omega_\delta$, that is $\operatorname{dist}(x_i, x_j) \geq \frac{\delta}{2}$ if $i \neq j$.

Then, we define

$$\Omega_{i,B} = \left\{ \bigcup B(x, \delta) : x \in \partial\Omega_\delta \text{ and } \operatorname{dist}(x, x_i) \leq \frac{\delta}{2} \right\}.$$

In particular, $x_i \in B(x, \delta)$ for each $x \in \partial\Omega_\delta$ with $\operatorname{dist}(x, x_i) \leq \frac{\delta}{2}$ so that $\Omega_{i,B}$ is star-shaped with respect to the point x_i and

$$B(x_i, \delta) \subset \Omega_{i,B} \subset B(x_i, 2\delta).$$

We have:

$$\left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \frac{3\delta}{2} \right\} \subset \cup_{i=1}^n \Omega_{i,B}.$$

To see this, let $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) \leq \frac{3\delta}{2}$ and $y' \in \partial\Omega$ be such that $\operatorname{dist}(y, y') = \operatorname{dist}(y, \partial\Omega)$. The line segment determined by y' and y cuts $\partial\Omega_\delta$ at a point x. By maximality of the net $\mathcal{P}_1$, there exists $x_i \in \mathcal{P}_1$ with $\operatorname{dist}(x, x_i) \leq \frac{\delta}{2}$ and $B(x, \delta) \subset \Omega_{i,B}$. As $\operatorname{dist}(x, y) \leq \delta$, we have $y \in \Omega_{i,B}$.

Step 2: construction of the domains $\{\Omega_{i,I}\}$. Let $\mathcal{P}_2 = \{y_1, ..., y_m\}$ be a maximal $\frac{\delta}{2}$-net of the domain $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{3\delta}{2}\}$. Then, we choose the domain $\Omega_{i,I}$ to be the ball $B(y_i, \frac{\delta}{2})$. As the intersection with $\partial\Omega$ is empty, $\Omega_{i,I}$ is convex, and star-shaped with respect to y_i. By maximality, the domain $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{3\delta}{2}\}$ is covered by $\cup_{i=1}^m B(y_i, \frac{\delta}{2})$.

It follows that the domain Ω is covered by the union of the domains $\{\Omega_{i,B}\}$ and $\{\Omega_{i,I}\}$, which we denote by $\{\Omega_i\}_{i=1}^n$. From Step 1 and Step 2 it follows that for each $i = 1, ..., n$ there exists $p_i \in \Omega_i$ such that $B(p_i, \frac{\delta}{2}) \subset \Omega_i \subset B(p_i, 2\delta)$, and $\operatorname{dist}(p_i, p_j) \geq \frac{\delta}{2}$ for all $i \neq j$. It is then easy to deduce that there is a universal constant $M \in \mathbb{N}$ (not depending on Ω) such that a point can be contained in at most M of these domains (see for example Step 2 of the proof of Theorem 1 in [11]). □

We introduce the following class of domains:

$$\mathcal{A}_\delta = \{\Omega \subset \mathbb{R}^2 : \Omega \text{ is smooth, with } \delta\text{-interior ball condition}\}. \tag{5.9}$$

We define the *rolling radius* of Ω as

$$\delta(\Omega) := \sup\{\delta \,:\, \Omega \in \mathcal{A}_\delta\}.$$

We are now ready to state the main result of this section

Theorem 5.10. *There exist a universal constant $C>0$ such that, for all $\Omega\in\mathcal{A}_\delta$ we have $\lambda_1(\Omega,\beta A_1)\geq C\beta\min\{\beta\delta^2,1\}$. That is:*

i) $\lambda_1(\Omega,\beta A_1)\geq C\beta^2\delta^2$ *if* $\beta\delta^2\leq 1$;
ii) $\lambda_1(\Omega,\beta A_1)\geq C\beta$ *if* $\beta\delta^2\geq 1$.

Proof. We consider first the case $\beta=1$. Let $\Omega\in\mathcal{A}_\delta$ for some $\delta>0$.

Assume first $\delta\leq 1$. From Lemma 5.9 we find that Ω admits an open covering $\{\Omega_1,...,\Omega_N\}$ with each Ω_j star-shaped with respect to some $p_j\in\Omega_j$, and with $B(p_j,\delta/2)\subseteq\Omega_j\subseteq B(p_j,2\delta)$, $j=1,...,N$. We apply Proposition 5.6 to each Ω_j, and find that

$$\lambda_1(\Omega_j,A_1)\geq c_1\delta^2\,,\quad \text{if } 1\leq\rho_{\Omega_j}^{-2} \tag{5.10}$$

and

$$\lambda_1(\Omega_j,A_1)\geq c_1\,,\quad \text{if } 1\geq\rho_{\Omega_j}^{-2}, \tag{5.11}$$

where ρ_{Ω_j} is the inradius of Ω_j and c_1 is a universal constant given by $2^{-14}c$. From Theorem 5.3, Lemma 5.9, and the fact that $\delta\leq 1$, we deduce that

$$\lambda_1(\Omega,A_1)\geq\frac{c_1}{M}\delta^2. \tag{5.12}$$

Let now $\delta>1$. Since $\Omega\in\mathcal{A}_\delta$ we see that $\Omega\in\mathcal{A}_1$, and with the same arguments above we deduce that

$$\lambda_1(\Omega,A_1)\geq\frac{c_1}{M}. \tag{5.13}$$

In conclusion, from (5.12) and (5.13) we obtain, for all $\delta>0$,

$$\lambda_1(\Omega,A_1)\geq\frac{c_1}{M}\min\{\delta^2,1\}. \tag{5.14}$$

Finally, by applying (5.14) to the domain $\sqrt{\beta}\Omega$, which belongs to $\mathcal{A}_{\sqrt{\beta}\delta}$, we get that for any $\beta>0$

$$\lambda_1(\Omega,\beta A_1)=\beta\lambda_1(\sqrt{\beta}\Omega,A_1)\geq\frac{c_1}{M}\min\{\beta^2\delta^2,\beta\}. \tag{5.15}$$

This yields the final result once we set $C:=c_1/M$. □

Remark 5.11. Let us discuss the bounds of Theorem 5.10. We note that the lower bound behaves like $\beta^2\delta^2$ as $\delta\to 0^+$. This is consistent with the examples of domains with small width (for which $\lambda_1(\Omega,\beta A_1)$ behaves like $\beta^2\delta^2$), see Theorem C.1, see also Appendices A and C. Concerning specifically the lower bound *i*), we note that the behavior is quadratic in δ and in β. We have seen in Theorem 3.1 that this quadratic behavior in β for small β is correct, in particular when $R_\Omega\sqrt{\beta}\leq 1$, where R_Ω denotes the circumradius of Ω. As for the linear behavior in β of the lower bound *ii*), this is correct in view of the asymptotic behavior of $\lambda_1(\Omega,\beta A_1)$ as $\beta\to+\infty$, see (2.12). In any case, we remark that the relevant quantity in our bounds is the behavior of the product $\sqrt{\beta}\delta$ (and not of δ or β alone).

One may wonder if the hypothesis on the δ-interior ball condition is too restrictive, and if just having a large inradius would imply a large lower bound. This is not the case as we can see in the following example of a convex domain with large inradius and small first eigenvalue.

Example 5.12. *Let T be a triangle with base $\{(x_1,0):x_1\in(-1/2,1/2)\}$ and height of length L (namely, the segment $(0,x_2)$ with $x_2\in(0,L)$). Then, the inradius is uniformly bounded from below and the first eigenvalue vanishes as $L\to+\infty$. To see this, consider the subsets $T'=\{(x_1,x_2)\in T: x_2\in(L-2\sqrt{L},L)\}$ and $T''=\{(x_1,x_2)\in T: x_2\in(L-2\sqrt{L},L-\sqrt{L})\}$. We can build a function u supported on T' with arbitrarily small Rayleigh quotient as follows. Take $u(x_1,x_2)=e^{-\frac{i\beta}{2}x_1x_2}\phi(x_2)$ where $\phi(x_2)\equiv 1$ on $T'\setminus T''$, $\phi(L-2\sqrt{L})=0$ and $\phi(x_2)$ is linear in T''. Standard computations (see also Example C.2) show that then $\lambda_1(T,\beta A_1)\leq CL^{-1}$.*

However, Theorem 5.10 does not apply to the case of a triangle as in the previous example, since it is not smooth (its rolling radius is $\delta=0$). For non-smooth domains we would rather use Proposition 5.6, which gives a good lower bound for the domain in Example 5.12.

6. Semiclassical estimates for averages of eigenvalues: proofs

In this section we prove asymptotically sharp lower bounds for the first Riesz mean of magnetic eigenvalues (upper bounds for averages) in the spirit of Kröger [28]. This will imply upper bounds on single eigenvalues. We recall that the first Riesz mean R_1 of magnetic eigenvalues is defined by $R_1(z) = \sum_{j=1}^{\infty}(z-\lambda_j(\Omega,\beta A_1))_+$, where $a_+ = \max\{0,a\}$. Through all this section we shall drop the dependence of $\lambda_j(\Omega,\beta A_1)$ on A_1,β,Ω and simply write λ_j. Also, through all this section, A will denote the standard potential βA_1. We also denote by $[a]$ the integer part of $a\in\mathbb{R}$ and by $\psi(a) := a-[a]-\frac{1}{2}$, $a\in\mathbb{R}$, the fluctuation function.

Theorem 6.1. *Let Ω be a bounded domain of $\mathbb{R}^2$. For all $z\geq 0$ we have*

$$R_1(z) \geq \frac{|\Omega|}{8\pi}z^2 - \frac{\beta^2|\Omega|}{2\pi}\psi^2\left(\frac{z}{2\beta}+\frac{1}{2}\right). \tag{6.1}$$

Equivalently, for all $k\in\mathbb{N}$, $k\geq 1$ we have

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \leq \frac{2\pi k}{|\Omega|} + R\left(\frac{2\pi k}{\beta|\Omega|}\right), \tag{6.2}$$

where

$$R(X) = \frac{\beta}{X}(X-[X])([X]-X+1) \tag{6.3}$$

Before proving the theorem, we state a few remarks and consequences.

Corollary 6.2. *For all $k\in\mathbb{N}$, $k\leq\frac{\beta|\Omega|}{2\pi}$ we have*

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \leq \beta. \tag{6.4}$$

Proof. We give an estimate for the remainder function

$$R(X) = \frac{\beta}{X}(X-[X])([X]-X+1).$$

First of all, note that the function

$$G(X) = (X-[X])([X]-X+1)$$

is such that $G(n)=0$ for all $n\in\mathbb{N}$, $G(x)\geq 0$, and on each interval $(n,n+1)$ has a unique maximum which is $\frac{1}{4}$ and is reached when $X=n+\frac{1}{2}$. Hence $0\leq G(X)\leq\frac{1}{4}$. Therefore

$$0\leq R(X)\leq\frac{\beta}{4X}.$$

If $0\leq X\leq 1$ one immediately checks that $R(X)=\beta(1-X)$. If $k\leq\frac{\beta|\Omega|}{2\pi}$ then $0\leq X\leq 1$ and from (6.16) we immediately get (6.4). ☐

We observe that Theorem 6.1 implies bounds on single eigenvalues, as in [28].

Corollary 6.3. *For any $k\in\mathbb{N}$ we have*

$$\lambda_{k+1}\leq\frac{8\pi k}{|\Omega|}+\beta. \tag{6.5}$$

Proof. For $k=0$ we have already proved that $\lambda_1<\beta$ in Theorem 3.1. We consider (6.1) with $z=\lambda_{k+1}$. Then for the left-hand side of (6.1) we have

$$R_1(\lambda_{k+1}) = \sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j) = k\lambda_{k+1}-\sum_{j=1}^{k}\lambda_j \leq k\lambda_{k+1}.$$

Hence we obtain the inequality

$$k\lambda_{k+1}\geq\frac{\lambda_{k+1}^2|\Omega|}{8\pi}-\frac{\beta^2|\Omega|}{2\pi}\psi^2\left(\frac{\lambda_{k+1}}{2\beta}+\frac{1}{2}\right)$$

which yields

$$\lambda_{k+1} \leq \frac{4\pi}{|\Omega|}\left(k+\sqrt{k^2+\frac{\beta^2|\Omega|^2}{4\pi^2}\psi^2\left(\frac{\lambda_{k+1}}{2\beta}+\frac{1}{2}\right)}\right).$$

Since $0 \leq \psi^2(a) \leq \frac{1}{4}$ for all $a \in \mathbb{R}$, and using $\sqrt{a^2+b^2} \leq a+b$ for all $a, b \geq 0$, we obtain (6.5). □

Remark 6.4. Let us compare the bounds given by Theorem 6.1 with the corresponding bounds for the Neumann Laplacian in two dimensions proved in [28]:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j^N \leq \frac{2\pi k}{|\Omega|}. \tag{6.6}$$

Here by λ_j^N we denote the Neumann eigenvalues of the Laplacian on Ω. Clearly, in our situation, bounds of the form (6.6) cannot hold for any k and any value of $|\Omega|$: the inequality is clearly violated when $k=1$ and $|\Omega| \to +\infty$. Hence it is natural to distinguish the regime $|\Omega| \geq \frac{2\pi k}{\beta}$ and $|\Omega| \leq \frac{2\pi k}{\beta}$. Also, the appearing of oscillations in the remainders of the estimates of Theorem 6.1 seems to be natural for this operator (see Appendices A and B). The semiclassical estimates of Theorem 6.1 should be compared with those for the magnetic Dirichlet Laplacian proved in [16]. In particular, for magnetic Dirichlet eigenvalues a lower bound on eigenvalues averages is given by the Weyl term, as for the Laplacian.

Another Corollary of Theorem 6.1 is the following lower bound on the trace of the magnetic heat kernel, which is asymptotically sharp as $t \to 0^+$.

Corollary 6.5. *For all $t > 0$ we have*

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \geq \frac{\beta|\Omega|}{4\pi\sinh(\beta t)}$$

Proof. The inequality follows by Laplace transforming inequality (6.1) (see e.g., [26, §2.1]). □

The proof of Theorem 6.1 relies on the so-called *averaged variational principle* of Harrell-Stubbe [25, Theorem 3.1], which is an efficient way of recovering Kröger's result [28], and can be easily applied in various situations. We recall it here for the reader's convenience.

Theorem 6.6. *Let H be a self-adjoint operator in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ with discrete spectrum, made of eigenvalues denoted by*

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_j \leq \cdots$$

with corresponding orthonormalized eigenvectors $\{g_j\}_{j\in\mathbb{N}\setminus\{0\}}$. The closed quadratic form corresponding to H is denoted $Q(\varphi,\varphi)$ for any φ in the quadratic form domain $\mathcal{Q}(H) \subset \mathcal{H}$. Let $f_p \in \mathcal{Q}(H)$ be a family of vectors indexed by a variable p ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a measurable subset of $\mathfrak{M}$. Then for any $z \in \mathbb{R}$,

$$\sum_{j=1}^{\infty}(z-\omega_j)_+\int_{\mathfrak{M}}\left|\langle g_j, f_p\rangle_{\mathcal{H}}\right|^2 d\sigma \geq \int_{\mathfrak{M}_0}\left(z\|f_p\|_{\mathcal{H}}^2 - Q(f_p,f_p)\right) d\sigma, \tag{6.7}$$

provided that the integrals converge. Here a_+ denotes the positive part of a real number a.

We state the following Lemma which contains some known facts on eigenfunctions of Δ_A on $\mathbb{R}^2$.

Lemma 6.7. *Let $u_{n,l} \in C^\infty(\mathbb{R}^2)$ be defined in polar coordinates (r,t) by*

$$u_{n,l}(r,t) := e^{-\frac{\beta r^2}{4}} r^n L_l^n\left(\frac{r^2\beta}{2}\right)e^{int},$$

where by $L_l^n(y)$ we denote the associated Laguerre polynomial, namely

$$L_l^n(y) = \sum_{i=0}^{l}(-1)^i\binom{l+n}{l-i}\frac{y^i}{i!} = y^{-n}\frac{e^y}{l!}\frac{d^l}{dy^l}(y^{l+n}e^{-y}) \tag{6.8}$$

with $l, n \in \mathbb{N}$ (in particular, $L_0^n(y) = 1$). Then *i)* $\Delta_A u_{n,l} = \beta(1+2l)u_{n,l}$ *on* $\mathbb{R}^2$*: the functions* $u_{n,l}$ *are eigenfunctions of* Δ_A *on* $\mathbb{R}^2$ *with eigenvalue* $\beta(1+2l)$*. Each eigenspace has infinite dimension.*

ii) $|\nabla^A u_{n,l}|^2 = \beta(1+2l)|u_{n,l}|^2 - \frac{1}{2}\Delta|u_{n,l}|^2$.

iii) $\int_{\mathbb{R}^2} u_{n,l}\overline{u_{m,k}} = 0$ *if* $m \neq n$ *or* $l \neq k$.

iv) $\int_{\mathbb{R}^2} |u_{n,l}|^2 = \pi\left(\frac{2}{\beta}\right)^{n+1}\frac{(l+n)!}{l!} =: c_{n,l}^2$.

v) Let $v_{n,l} := \frac{u_{n,l}}{c_{n,l}}$*; then* $\int_{\mathbb{R}^2} v_{n,l}\overline{v_{m,k}} = \delta_{mn}\delta_{lk}$*, hence* $\{v_{n,l}\}_{n,l\in\mathbb{N}}$ *is an orthonormal system in* $L^2(\mathbb{R}^2)$.

The proof follows from standard computations, using the expression of Δ_A in polar coordinates (3.2) (see also [3]).

The next lemma establishes a basic inequality for $R_1(z)$ which is the cornerstone of the proof of Theorem 6.1.

Lemma 6.8. *For all* $z \geq 0$ *we have*

$$\sum_{j=1}^{\infty}(z-\lambda_j)_+ \geq \frac{\beta|\Omega|}{2\pi}\sum_{l=1}^{\infty}(z-\beta(2l-1))_+. \tag{6.9}$$

Proof. We apply the *averaged variational principle* (6.7) with $\mathcal{H} = L^2(\Omega)$, $H = \Delta_A$, $Q(f,f) = \int_\Omega |\nabla^A f|^2$, $\mathcal{Q}(H) = H^1(\Omega)$, $\omega_j = \lambda_j$, $g_j = u_j$, where u_j are the the $L^2(\Omega)$-normalized eigenfunctions associated with the eigenvalues λ_j of Δ_A on Ω, $\mathfrak{M} = \mathbb{N}\times\mathbb{N}$ with the counting measure σ, and $\mathfrak{M}_0 = \mathbb{N}\times\{0,...,L\}$ for some $L \in \mathbb{N}$, and f_p which is, for $p = (n,l)$, the restriction to Ω of $v_{n,l}$. Then (6.7) reads

$$\sum_{j=1}^{\infty}\left[(z-\lambda_j)_+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left|\int_\Omega v_{n,l}\overline{u_j}\right|^2\right] \geq \sum_{l=0}^{L}\sum_{n=0}^{\infty}\int_\Omega\left(z|v_{n,l}|^2 - |\nabla^A v_{n,l}|^2\right). \tag{6.10}$$

Now, since $\{v_{n,l}\}_{n,l\in\mathbb{N}}$ is an orthonormal family in $L^2(\mathbb{R}^2)$, we have that

$$\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left|\int_\Omega v_{n,l}\overline{u_j}\right|^2 \leq \int_\Omega |u_j|^2 = 1$$

hence, by Lemma 6.7, ii)

$$\begin{aligned}\sum_{j=1}^{\infty}(z-\lambda_j)_+ &\geq \int_\Omega\left(z\sum_{l=0}^{L}\sum_{n=0}^{\infty}|v_{n,l}|^2 - \sum_{l=0}^{L}\sum_{n=0}^{\infty}|\nabla^A v_{n,l}|^2\right)\\ &= \int_\Omega\left(\sum_{l=0}^{L}\sum_{n=0}^{\infty}(z-\beta(2l+1))|v_{l,n}|^2 + \frac{1}{2}\sum_{l=0}^{L}\sum_{n=0}^{\infty}\Delta|v_{n,l}|^2\right)\\ &= \sum_{l=0}^{L}(z-\beta(2l+1))\sum_{n=0}^{\infty}\int_\Omega|v_{l,n}|^2 + \frac{1}{2}\sum_{l=0}^{L}\sum_{n=0}^{\infty}\int_\Omega\Delta|v_{n,l}|^2.\end{aligned} \tag{6.11}$$

We will prove in Lemma 6.9 here below that

$$\sum_{n=0}^{\infty}|v_{n,l}(x)|^2 = \frac{\beta}{2\pi}\left(1+e^{-\frac{\beta|x|^2}{2}}P_l\left(\frac{\beta|x|^2}{2}\right)\right), \tag{6.12}$$

where $P_l(y)$ is a polynomial of degree $2l-1$ in the variable y. The convergence of the series (6.12) is uniform on any compact set. In particular, $\sum_{n=0}^{\infty}\Delta|v_{n,l}(x)|^2 = \frac{\beta}{2\pi}\Delta\left(e^{-\frac{\beta|x|^2}{2}}P_l(\beta|x|^2/2)\right)$.

Then, from (6.11) we get that, for all $L\in\mathbb{N}$ and $z \geq 0$,

$$\begin{aligned}\sum_{j=1}^{\infty}(z-\lambda_j)_+ &\geq \int_\Omega\frac{\beta}{2\pi}\sum_{l=0}^{L}\left(1+e^{-\frac{\beta|x|^2}{2}}P_l\left(\frac{\beta|x|^2}{2}\right)\right)(z-\beta(2l+1))\\ &\quad + \frac{\beta}{4\pi}\sum_{l=0}^{L}\int_\Omega\Delta\left(e^{-\frac{\beta|x|^2}{2}}P_l\left(\frac{\beta|x|^2}{2}\right)\right).\end{aligned} \tag{6.13}$$

Inequality (6.13) holds for any fixed L, and it is clearly valid if we replace $|x|$ by $|x+x_0|$, $x_0 \in \mathbb{R}^2$ (this amounts to choosing $v_{n,l}(r,t)$ where (r,t) are polar coordinates centered at x_0). Then, for any $L \in \mathbb{N}$, taking $|x_0| \to +\infty$, we deduce that the last term of (6.13) goes to 0, and hence

$$\sum_{j=1}^{\infty}(z-\lambda_j)_+ \geq \frac{\beta|\Omega|}{2\pi}\sum_{l=0}^{L}(z-\beta(2l+1)) = \frac{\beta|\Omega|}{2\pi}\sum_{l=1}^{L+1}(z-\beta(2l-1)). \tag{6.14}$$

This implies immediately (6.9). □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Inequality (6.9) is the cornerstone of this proof. The statements of Theorem 6.1 are consequences of this inequality. We prove first (6.1). We consider the right-hand side of (6.9) and re-write it as

$$\frac{\beta^2|\Omega|}{2\pi}\sum_{l=1}^{\infty}(\frac{z}{\beta}-(2l-1))_+ = \frac{\beta^2|\Omega|}{2\pi}\sum_{l=1}^{\infty}((2w-1)-(2l-1))_+$$

where $w=\frac{z}{2\beta}+\frac{1}{2}$. Then, some algebraic manipulations yield

$$\begin{aligned}\frac{\beta^2|\Omega|}{2\pi}\sum_{l=1}^{\infty}((2w-1)-(2l-1))_+ &= \frac{\beta^2|\Omega|}{2\pi}\sum_{l=1}^{[w]}((2w-1)-(2l-1)) = \frac{\beta^2|\Omega|}{2\pi}[w](2w-[w]-1)\\ &= \frac{\beta^2|\Omega|}{2\pi}\left(w-\frac{1}{2}-\psi(w)\right)\left(w-\frac{1}{2}+\psi(w)\right) = \frac{\beta^2|\Omega|}{2\pi}\left(w-\frac{1}{2}\right)^2-\frac{\beta^2|\Omega|}{2\pi}\psi^2(w)\\ &= \frac{\beta^2|\Omega|}{2\pi}\cdot\frac{z^2}{4\beta^2}-\frac{\beta^2|\Omega|}{2\pi}\psi^2\left(\frac{z}{2\beta}+\frac{1}{2}\right).\end{aligned}$$

This proves (6.1).

Before proving (6.2), we note that, choosing $z=\lambda_1$, inequality (6.9) reads

$$0 \leq \frac{\beta|\Omega|}{2\pi}\sum_{l=1}^{\infty}(\lambda_1-\beta(2l-1))_+ \leq 0$$

which implies $\lambda_1 \leq \beta(2l-1)$ for all $l \geq 1$, and in particular, $\lambda_1 \leq \beta$. This is an alternative way of recovering (2.1) (however the inequality is not strict).

Now we prove (6.2). Consider now the two functions

$$f(z)=\sum_{j=1}^{\infty}(z-\lambda_j)_+$$

and

$$g(z)=\frac{\beta|\Omega|}{2\pi}\sum_{l=1}^{\infty}(z-\beta(2l-1))_+$$

These two functions are convex. Let us define, for any $w \geq 0$, the functions

$$\mathcal{L}[f](w) := \sup_{z\geq 0}(zw-f(z)), \quad \mathcal{L}[g](w) := \sup_{z\geq 0}(zw-g(z)).$$

These two functions are the Legendre transforms of f and g. Since f,g are convex, we have, for all $w \geq 0$, that $f(z) \geq g(z) \iff \mathcal{L}[f](w) \leq \mathcal{L}[g](w)$.

A standard computation shows that

$$\mathcal{L}[f](w)=\lambda_{[w]+1}(w-[w])+\sum_{j=1}^{[w]}\lambda_j$$

and

$$\mathcal{L}[g](w) = \frac{\beta^2|\Omega|}{2\pi}\left(2\left[\frac{2\pi w}{\beta|\Omega|}\right]+1\right)\left(\frac{2\pi w}{\beta|\Omega|}-\left[\frac{2\pi w}{\beta|\Omega|}\right]\right)+\sum_{j=1}^{\left[\frac{2\pi w}{\beta|\Omega|}\right]}\frac{\beta^2|\Omega|(2j-1)}{2\pi}$$
$$= \frac{\beta^2|\Omega|}{2\pi}\left(2\left[\frac{2\pi w}{\beta|\Omega|}\right]+1\right)\left(\frac{2\pi w}{\beta|\Omega|}-\left[\frac{2\pi w}{\beta|\Omega|}\right]\right)+\frac{\beta^2|\Omega|}{2\pi}\left[\frac{2\pi w}{\beta|\Omega|}\right]^2$$

We choose now $w = k$, so that the inequality $\mathcal{L}[f](k) \leq L[g](k)$ reads

$$\sum_{j=1}^{k}\lambda_j \leq \frac{\beta^2|\Omega|}{2\pi}\left(2\left[\frac{2\pi k}{\beta|\Omega|}\right]+1\right)\left(\frac{2\pi k}{\beta|\Omega|}-\left[\frac{2\pi k}{\beta|\Omega|}\right]\right)+\frac{\beta^2|\Omega|}{2\pi}\left[\frac{2\pi k}{\beta|\Omega|}\right]^2 \tag{6.15}$$

Setting $X = \frac{2\pi k}{\beta|\Omega|}$ in (6.15), we see that

$$\sum_{j=1}^{k}\lambda_j \leq \frac{\beta^2|\Omega|}{2\pi}(2[X]+1)(X-[X])+\frac{\beta^2|\Omega|}{2\pi}[X]^2$$
$$= \frac{\beta^2|\Omega|}{2\pi}(2[X]+1)(X-[X])+\frac{\beta^2|\Omega|}{2\pi}(X^2+[X]^2-X^2)$$
$$= \frac{\beta^2|\Omega|}{2\pi}X^2+\frac{\beta^2|\Omega|}{2\pi}(2[X]+1)(X-[X])+\frac{\beta^2|\Omega|}{2\pi}([X]^2-X^2)$$
$$= \frac{2\pi k^2}{|\Omega|}+\frac{\beta^2|\Omega|}{2\pi}(2[X]+1)(X-[X])+\frac{\beta^2|\Omega|}{2\pi}([X]^2-X^2),$$

where we have used the fact that $\frac{\beta^2|\Omega|}{2\pi}X^2 = \frac{2\pi k^2}{|\Omega|}$. Dividing both sides by k we find

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \leq \frac{2\pi k}{|\Omega|}+\frac{\beta}{X}(X-[X])([X]-X+1) \tag{6.16}$$

where we recall that $X = \frac{2\pi k}{\beta|\Omega|}$. This concludes the proof.

□

We prove the following lemma on sums of eigenfunctions of Δ_A on $\mathbb{R}^2$.

Lemma 6.9. *We have*

$$\sum_{n=0}^{\infty}|v_{n,l}(x)|^2 = \frac{\beta}{2\pi}\left(1+e^{-\frac{\beta|x|^2}{2}}P_l\left(\frac{\beta|x|^2}{2}\right)\right)$$

where $P_l(y)$ is a polynomial of degree $2l-1$ in the variable y. If $l = 0$, then $P_0(y) = 0$.

Proof. We note that

$$|v_{n,l}(x)|^2 = \frac{\beta}{2\pi}e^{-y}\frac{l!}{(l+n)!}y^nL_l^n(y)^2$$

with $y = \frac{\beta|x|^2}{2}$, so that

$$\sum_{n=0}^{\infty}|v_{n,l}(x)|^2 = \frac{\beta}{2\pi}e^{-y}\sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)^2. \tag{6.17}$$

Therefore, we need to study

$$\sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)^2. \tag{6.18}$$

We use (6.8) to expand one factor $L_l^n(y)$ in (6.18)

$$\sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)^2 = \sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)\sum_{i=0}^{l}(-1)^i\binom{l+n}{l-i}\frac{y^i}{i!}$$

and change the order of summation:

$$\sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)^2=\sum_{i=0}^{l}\frac{(-1)^iy^i}{(l-i)!i!}\sum_{n=0}^{\infty}\frac{y^nL_l^n(y)(l+n)!l!}{(n+i)!(l+n)!}.$$

Using the second identity (Rodrigues formula) in (6.8) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{l!}{(l+n)!}y^nL_l^n(y)^2&=\sum_{i=0}^{l}\frac{(-1)^iy^i}{(l-i)!i!}\sum_{n=0}^{\infty}\frac{e^y}{(n+i)!}\frac{d^l}{dy^l}(y^{l+n}e^{-y})\\&=e^y\sum_{i=0}^{l}\frac{(-1)^iy^i}{(l-i)!i!}\frac{d^l}{dy^l}\left(e^{-y}y^{l-i}\sum_{n=0}^{\infty}\frac{y^{n+i}}{(n+i)!}\right)\\&=e^y\sum_{i=0}^{l}\frac{(-1)^iy^i}{(l-i)!i!}\frac{d^l}{dy^l}\left(e^{-y}y^{l-i}\left(e^y-\sum_{j=0}^{i-1}\frac{y^j}{j!}\right)\right)\\&=e^y\sum_{i=0}^{l}\binom{l}{i}\frac{(-1)^iy^i}{l!}\frac{d^l}{dy^l}\left(y^{l-i}-y^{l-i}e^{-y}\sum_{j=0}^{i-1}\frac{y^j}{j!}\right)\\&=e^y+e^y\sum_{i=1}^{l}\binom{l}{i}\frac{(-1)^iy^i}{l!}\frac{d^l}{dy^l}\left(y^{l-i}-y^{l-i}e^{-y}\sum_{j=0}^{i-1}\frac{y^j}{j!}\right).\end{aligned}\tag{6.19}$$

The proof follows now inserting (6.19) in (6.17), and observing that the second summand in the last line of (6.19) is just a polynomial of degree $2l-1$. □

Appendix A. Eigenvalues of embedded curves and thin tubular neighborhoods

In this section we consider the magnetic Laplacian on embedded curves. Throughout this section, by Γ we denote a simple, closed curve, which is the boundary of some simply connected domain Ω in $\mathbb{R}^2$ (namely, $\Gamma=\partial\Omega$).

As potential one-form, we consider the restriction $\hat{A}_\Gamma$ of the standard magnetic potential $A=\frac{\beta}{2}(-x_2dx_1+x_1dx_2)$ to Γ and study the magnetic Laplacian associated to $\hat{A}_\Gamma$ (see also the discussion in Subsection 2.6).

It should be noted that $\hat{A}_\Gamma$ is closed for dimensional reasons (i.e., $d\hat{A}_\Gamma=0$), hence it generates a vanishing magnetic field on Γ. We denote by $\lambda_j(\Gamma,\hat{A}_\Gamma)$ the corresponding eigenvalues, which can be explicitly computed.

Theorem A.1. *Let Γ be an embedded curve, which is the boundary of a simply connected domain $\Omega\subset\mathbb{R}^2$, and consider the magnetic Laplacian associated with the potential $\hat{A}_\Gamma$ as above. Its spectrum is then given by the collection*

$$\frac{4\pi^2}{|\Gamma|^2}\left(n-\frac{\beta|\Omega|}{2\pi}\right)^2,\quad n\in\mathbb{Z}.$$

In particular

$$\lambda_1(\Gamma,\hat{A}_\Gamma)=\frac{4\pi^2}{|\Gamma|^2}\min_{n\in\mathbb{Z}}\left(n-\frac{\beta|\Omega|}{2\pi}\right)^2\tag{A.1}$$

hence $\lambda_1(\Gamma,\hat{A}_\Gamma)=0$ if and only if $|\Omega|=\frac{2n\pi}{\beta}$ for some $n\in\mathbb{N}$.

Proof. Note that, being a compact one-dimensional Riemannian manifold, Γ is isometric to the circle with the same length; from [10, Proposition 7], we know that the spectrum is given by:

$$\lambda_n(\Gamma,\hat{A}_\Gamma)=\frac{4\pi^2}{|\Gamma|^2}(n-\Phi^{\hat{A}_\Gamma})^2,\quad n\in\mathbb{Z}$$

where $\Phi^{\hat{A}_\Gamma}=\dfrac{1}{2\pi}\int_\Gamma\hat{A}_\Gamma$ is the flux of $\hat{A}_\Gamma$ around Γ oriented counter-clockwise (however the spectrum does not depend on the orientation). We compute the flux knowing that $\Gamma=\partial\Omega$ and get, by the Stokes formula:

$$\Phi^{\hat{A}_\Gamma}=\frac{1}{2\pi}\int_\Omega d(\beta A_1)=\frac{\beta\,|\Omega|}{2\pi}.$$

The conclusion follows. □

As a corollary, the classical isoperimetric inequality implies the following fact which, by abuse of language, can be interpreted as a "reverse Faber-Krahn inequality" for the first magnetic eigenvalue of the boundary of simply connected domains.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain with boundary Γ, and let Ω^* be a disk with $|\Omega| = |\Omega^*|$ and boundary Γ^*. Then*

$$\lambda_1(\Gamma, \hat{A}_\Gamma) \leq \lambda_1(\Gamma^*, \hat{A}_{\Gamma^*}).$$

If $|\Omega| \neq \frac{2n\pi}{\beta}$ for all $n \in \mathbb{N}$, then equality holds if and only if $\Omega = \Omega^*$. In particular, we have

$$\lambda_1(\Gamma, \hat{A}_\Gamma) \leq \frac{\beta}{4} \tag{A.2}$$

with equality if and only if Ω is a disk of radius $R = \frac{1}{\sqrt{\beta}}$.

Proof. The first assertion is an immediate consequence of (A.1) and the isoperimetric inequality. We prove the second assertion. Assume first that $\frac{\beta|\Omega|}{2\pi} \leq \frac{1}{2}$ so that we have $\min_{n\in\mathbb{Z}} \left(n - \frac{\beta|\Omega|}{2\pi}\right)^2 = \left(\frac{\beta|\Omega|}{2\pi}\right)^2 \leq \frac{1}{2}\frac{\beta|\Omega|}{2\pi}$ hence

$$\lambda_1(\Gamma, \hat{A}_\Gamma) \leq \frac{4\pi^2}{|\Gamma|^2} \cdot \frac{1}{2}\frac{\beta\,|\Omega|}{2\pi} \leq \frac{\beta}{4}$$

by the isoperimetric inequality $|\Gamma|^2 \geq 4\pi\,|\Omega|$. Note that the equality holds if and only if Ω is a disk of area $\frac{\pi}{\beta}$. Then, we assume $\frac{\beta|\Omega|}{2\pi} > \frac{1}{2}$, and observe that $\min_{n\in\mathbb{Z}} \left(n - \frac{\beta|\Omega|}{2\pi}\right)^2 \leq \frac{1}{4}$. It follows that

$$\lambda_1(\Gamma, \hat{A}_\Gamma) \leq \frac{4\pi^2}{|\Gamma|^2} \cdot \frac{1}{4} \leq \frac{\pi}{4\,|\Omega|} < \frac{\beta}{4}.$$

Finally, one checks easily that for a disk of radius $R = \frac{1}{\sqrt{\beta}}$ we have equality in (A.2). The proof is complete. □

Remark A.3. Note that, in the case of a circle Γ_R of radius R, for a fixed β, we always have a sequence of radii R_n such that $\lambda_1(\Gamma_{R_n}, \hat{A}_{\Gamma_{R_n}}) = 0$. This amounts to $R_n = \sqrt{\frac{2n}{\beta}}$, $n \in \mathbb{N}$.

Note also, that for these values of R_n, we have that β is an eigenvalue of the magnetic Laplacian on B_{R_n} (see Appendix B).

Moreover, $\lambda_1(\Gamma_R, \hat{A}_{\Gamma_R}) \to 0$ as $R \to +\infty$, but the convergence is not monotonic. There is a subsequence, as we have said, where it is equal to zero. See Figure 2 below and Figure 1 in [21]. We note again the oscillating behaviour of the first eigenvalue as a function of the radius (Little-Parks effect). We observed an analogous behavior in the remainder of the lower bound for $R_1(z)$ in Theorem 6.1. Also, an oscillating behavior is evident numerically for the magnetic eigenvalues of disks, (see Figure 3).

Appendix B. Eigenvalues of the disk

In this section we consider the eigenvalues of the disk $B_R := B(0, R)$ when $\beta > 0$ is constant. We shall denote them by $\lambda_j(B_R, \beta A_1)$. Let $A = \beta A_1$. Writing $\Delta_A u = \lambda u$ in B_R in polar coordinates (r, t) (see (3.2)), for the ansatz $u = v(r)e^{int}$, $n \in \mathbb{Z}$, we see that v solves

$$\begin{cases} -v''(r) - \frac{v'(r)}{r} + \left(\frac{\beta r}{2} - \frac{n}{r}\right)^2 v(r) = \lambda v(r) \\ v'(R) = 0. \end{cases} \tag{B.1}$$

A bounded solution to the differential equation in (B.1) is given by

$$v_n(r) = e^{-\frac{\beta r^2}{4}} r^n L^n_{\frac{1}{2}(\frac{\lambda}{\beta}-1)}\left(\frac{r^2\beta}{2}\right), \tag{B.2}$$

where $L^a_\gamma(x)$ denotes the generalized Laguerre function (see e.g., [1, §22] for precise definitions and properties). The eigenvalues are determined by imposing $v_n'(R) = 0$. For each $n \in \mathbb{Z}$ we have a sequence

$$0 < \lambda_1(n, \beta, R) < \lambda_2(n, \beta, R) \leq \cdots \nearrow +\infty.$$

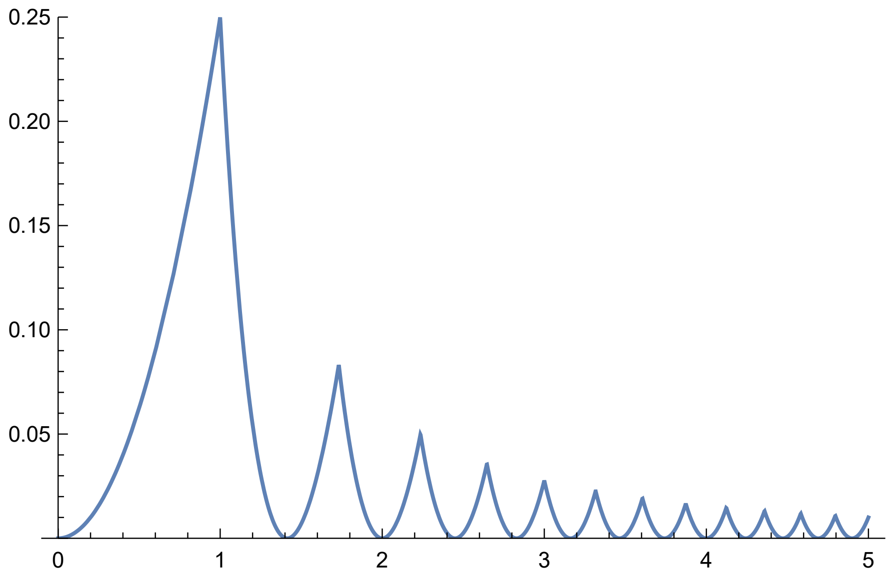

FIGURE 2. First eigenvalue on a circle of radius R as a function of R, for $\beta = 1$.

Clearly,

$$\lambda_1(B_R, \beta A_1) = \min_{n \in \mathbb{Z}} \lambda_1(n, \beta, R).$$

This minimum depends on β and R, but it is rather complicated to identify which $n \in \mathbb{Z}$ realizes it, for given β, R. It is easy to see that the minimum, in any case, is achieved for some $n \in \mathbb{N}$. The analytic branches (i.e., $\lambda_1(n, \beta, R)$) show a very intriguing behavior. A first numerical study can be found e.g., in [37]. Due to the rescaling properties of $\lambda_1(B_R, \beta A_1)$, it is not restrictive to fix $\beta = 1$ and study the behavior of the first eigenvalue as function of R.

Note that the analogous study for the magnetic Laplacian with Dirichlet conditions is simpler (even if the computations are not trivial), and a complete picture is available (see [38]). In particular, the first eigenfunction is always radial, i.e., $n = 0$.

The peculiar behavior of the eigenvalue $\lambda_1(B_R, \beta A_1)$ as a function of R is well illustrated in Figure 3 (see also Figure 1 in [21]). Recall that the eigenvalues of B_R are implicitly characterized by $v_n'(R) = 0$, where v_n is defined in (B.2). In Figure 3 we have plotted the zero level sets of the function

$$F(R, \lambda) = v_n'(R) = \frac{d}{dr}\left(e^{-\frac{\beta r^2}{4}} r^n L^n_{\frac{1}{2}(\frac{\lambda}{\beta}-1)}\left(\frac{r^2\beta}{2}\right)\right)_{|r=R},$$

for the choice $\beta = 1$, in the region $(R, \lambda) \in [0, 6] \times [0, 1]$ for $n = 0, ..., 10$. The first eigenvalue is the minimum of all the analytic branches of eigenvalues $\lambda_1(n, 1, R)$ for $n = 0, ..., 10$.

We note that the first eigenvalue has an oscillating behavior as a function of R. The black dashed line in Figure 3 corresponds to $\Theta_0 \approx 0.590106$. This suggests that **open problem 2** should have a positive answer. In Figure 3 we recognize a precise order. In fact, the first eigenvalue is given by $\lambda_1(0, 1, R)$ for $R \in (0, R_1)$, then by $\lambda_1(1, 1, R)$ for $R \in (R_1, R_2)$, and so on. This was highlighted in the numerical study of [37].

Even though we cannot give precise information on the eigenvalues of disks, explicit computations allow to have more insight on Theorem 6.1. We have observed in Corollary 6.2 that if $|\Omega| \geq \frac{2\pi n}{\beta}$, then

$$\frac{1}{n}\sum_{j=1}^{n} \lambda_j(\Omega, \beta A_1) \leq \beta. \tag{B.3}$$

For the disk B_R the condition reads $R \geq \sqrt{\frac{2n}{\beta}}$. We show now that when $R = \sqrt{\frac{2n}{\beta}}$, the first n eigenvalues are strictly smaller than β, which clearly implies (B.3). Moreover, in this case β is an eigenvalue, and, in particular, it is at least the $n+1$-th eigenvalue.

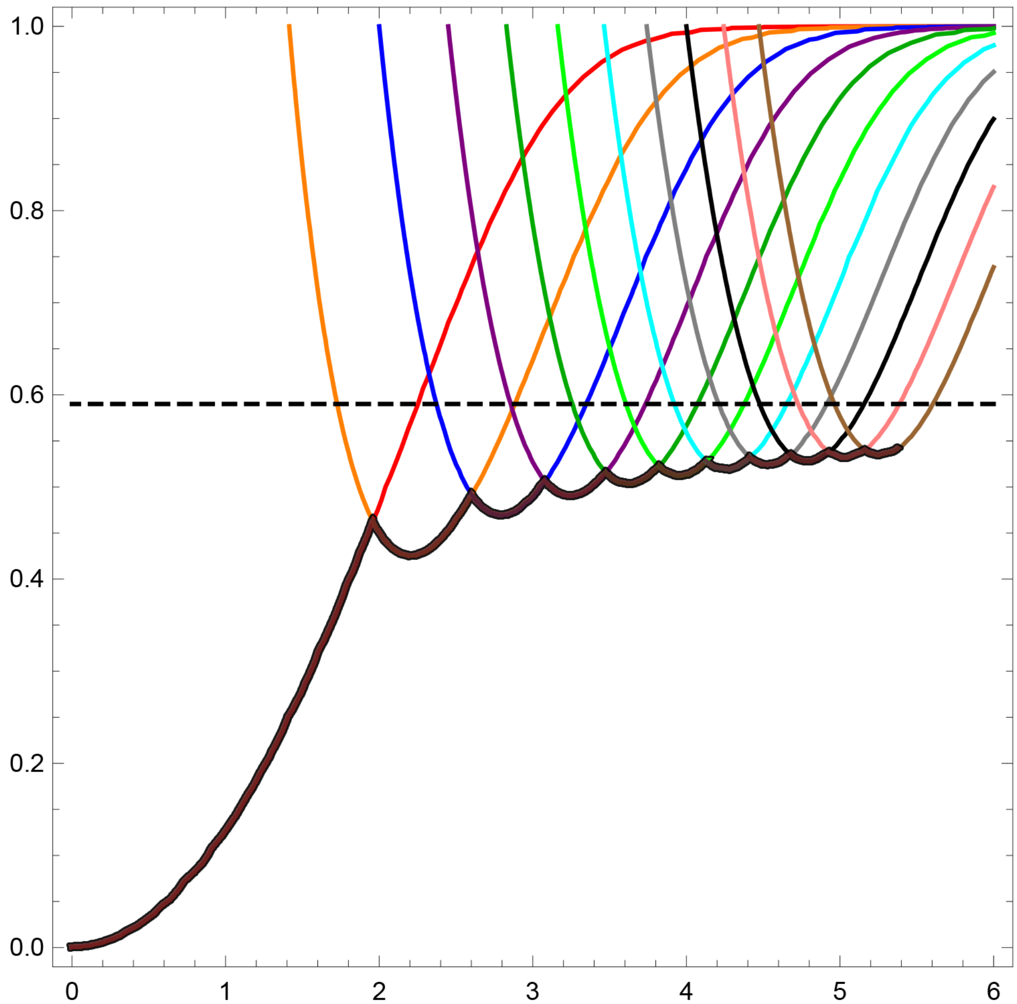

FIGURE 3. Analytic branches $\lambda_1(n,1,R)$, $n = 0, ..., 10$. In particular $n = 0$ (red), $n = 1$ (orange), $n = 2$ (blue), $n = 3$ (purple), $n = 4$ (dark green), $n = 5$ (light green), $n = 6$ (cyan), $n = 7$ (gray), $n = 8$ (black), $n = 9$ (pink), $n = 10$ (brown). Here the variable on the abscissae axis is the radius R. The first eigenvalue is given by the minimum of all analytic branches, and is highlighted in dark brown.

Proposition B.1. *Let $R = \sqrt{\frac{2n}{\beta}}$ for some $n \in \mathbb{N}$. Then $\lambda_1(B_R, \beta A_1) \leq \cdots \leq \lambda_n(B_R, \beta A_1) < \beta$ and $\beta = \lambda_k(B_R, \beta A_1)$ for some $k \geq n+1$.*

Proof. It is easy to show that the function (in polar coordinates (r,t))

$$f_n(r,t) = e^{-\frac{\beta r^2}{4}} r^n e^{int}$$

is an eigenfunction corresponding to the eigenvalue β (see (3.2)). Consider now the functions $f_m(r,t) = e^{-\frac{\beta r^2}{4}} r^m e^{imt}$ for $m \in \mathbb{N}$. We prove that

$$\frac{\int_{B(0,R)} |\nabla^A f_m|^2}{\int_{B(0,R)} |f_m|^2} < \beta$$

if and only if $m < n$. Since $\{f_m\}_{m=0}^{n-1}$ is an orthogonal family in $L^2(B_R)$, this will imply from the min-max principle (2.3) that there are at least n eigenvalues strictly below β, and, as a consequence, that β is at least the $n+1$-th eigenvalue. A standard computation shows that

$$\frac{\int_{B(0,R)} |\nabla^A f_m|^2}{\int_{B(0,R)} |f_m|^2} = \beta \frac{m! - m^2\Gamma(m,n) + 2m\Gamma(m+1,n) - \Gamma(m+2,n)}{m! - \Gamma(m+1,n)}$$

where Γ denotes here the incomplete Gamma function. Now, for $m = n$ this gives exactly β. Now, let

$$G(m,n) := \frac{m! - m^2\Gamma(m,n) + 2m\Gamma(m+1,n) - \Gamma(m+2,n)}{m! - \Gamma(m+1,n)}.$$

We have $G(n,n) = 1$. Moreover,

$$\lim_{n\to+\infty} G(m,n) = 1.$$

Now $G(m,n) < 1$ if and only if

$$-m^2\Gamma(m,n) + (2m+1)\Gamma(m+1,n) - \Gamma(m+2,n) < 0$$

and using the properties of the Gamma function we see that

$$-m^2\Gamma(m,n) + (2m+1)\Gamma(m+1,n) - \Gamma(m+2,n) = n^m e^{-n}(m-n)$$

which means that $G(m,n) < 1$ for all $m < n$. The proof is concluded. □

From Figure 3 it seems evident that when $R = \sqrt{\frac{2n}{\beta}}$, β is exactly the $n+1$-th eigenvalue (recall that $\beta = 1$ in Figure 3): the analytic branch corresponding to $n \in \mathbb{N}$ intersects the horizontal line $\lambda = \beta = 1$ at $R = \sqrt{2n}$. We are left with the following

Open problem 3 Prove that for $R \geq \sqrt{\frac{2n}{\beta}}$ the first $n+1$ eigenvalues of B_R lie below β; prove that for any domain with $|\Omega| \geq \frac{2\pi n}{\beta}$ the first $n+1$ eigenvalues lie below β; improve (if true) inequality (6.4) as follows: for all $|\Omega| \geq \frac{2\pi n}{\beta}$

$$\frac{1}{n+1}\sum_{j=1}^{n+1} \lambda_j(\Omega, \beta A_1) \leq \beta.$$

Appendix C. Further examples

In this Appendix we provide further examples which highlight the difficulties in finding lower bounds for $\lambda_1(\Omega, \beta A_1)$.

We first show that domains with small *width* have small first eigenvalue. Recall that the width ϵ of a domain Ω is defined as the infimum of the numbers $h > 0$ such that, up to isometries, Ω is contained in a strip $]-\infty,\infty[\times]-h/2,h/2[$.

Theorem C.1. *Let Ω be a bounded domain of width ϵ. Then*

$$\lambda_1(\Omega, \beta A_1) \leq \frac{\epsilon^2\beta^2}{4}.$$

Proof. Suppose that $\Omega \subset]-\infty,\infty[\times]-\epsilon/2,\epsilon/2[$. We consider the test function $u(x_1,x_2) = e^{i\frac{\beta}{2}x_1x_2}$ in (2.3). A standard computation shows that

$$|\nabla^A u|^2 = \beta^2 x_2^2.$$

Since $|u| = 1$, we deduce

$$\lambda_1(\Omega, \beta A_1) \leq \frac{\int_\Omega |\nabla^A u|^2}{\int_\Omega |u|^2} = \frac{\beta^2\epsilon^2}{4}.$$

□

Of course, this is only relevant if the width is small enough.

Thanks to Theorem C.1 we show that there exist convex domains of any area with small first eigenvalue.

Example C.2. *Let $\Omega_\epsilon =]-L,L[\times]-\epsilon/2,\epsilon/2[$, $L,\epsilon > 0$. For any $L > 0$,*

$$\lambda_1(\Omega_\epsilon, \beta A_1) \leq \frac{\beta^2\epsilon^2}{4}. \tag{C.1}$$

In particular, $\lambda_1(\Omega_\epsilon, \beta A_1) \to 0$ as $\epsilon \to 0^+$.

Note that the upper bound (C.1) does not depend on L, so that $|\Omega_\epsilon|$ could be as large (or small) as one wishes. The Rayleigh quotient goes to 0 proportionally to $\beta^2\epsilon^2$. We recall the Payne-Weinberger inequality [33] for the first positive eigenvalue of the Neumann Laplacian λ_2^N:

$$\lambda_2^N \geq \frac{\pi^2}{d_\Omega^2}, \tag{C.2}$$

valid for convex domains. Here d_Ω is the diameter of Ω. Example C.2 shows that (C.2) does not extend to the first magnetic eigenvalue.

One may be tempted to conclude that thin domains, or domains with small area, have small first eigenvalue. To this regard, we remark that topology plays a role: if Ω is simply connected with small area, it is true that the first eigenvalue is small, see Remark 2.4. A bit surprisingly, this is not true if the domain is not simply connected. We show examples of thin domains, of arbitrarily small area and first eigenvalue uniformly bounded away from zero.

Example C.3. *Let Γ be a closed simple curve which is the boundary of a smooth bounded domain Ω. Let $\omega_h := \{x \in \Omega : \operatorname{dist}(x, \Gamma) < h\}$ be a small tubular neighborhood of Γ. Then*

1) if $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$,

$$\lambda_1(\omega_h, \beta A_1) \geq \frac{4\pi^2}{|\Gamma|^2} \min_{n\in\mathbb{Z}} \left(n - \frac{\beta|\Omega|}{2\pi}\right)^2 - \sqrt{\beta}h,$$

for all $h \in (0, \epsilon_0)$, for some $\epsilon_0 > 0$ depending only on Ω and β;

2) if $\frac{\beta|\Omega|}{2\pi} \in \mathbb{N}$, then

$$\lambda_1(\omega_h, \beta A_1) \to 0$$

as $h \to 0^+$

In particular, $\lambda_1(\omega_h, \beta A_1) \to 0$ as $h \to 0^+$ if and only if $\frac{\beta|\Omega|}{2\pi} \in \mathbb{N}$.

Proof. Point 1) follows from Theorem C.7, while point 2) follows from Proposition C.9. We postpone the corresponding proofs at the end of this section.

Alternatively, it is proved in [36, §9] that

$$\lambda_j(\omega_h, \beta A_1) \to \lambda_j(\Gamma, \hat{A}_\Gamma)$$

as $h \to 0^+$, where $\lambda_j(\Gamma, \hat{A}_\Gamma)$ denote the magnetic eigenvalues on Γ endowed with the restriction of the standard potential βA_1. In particular, $\lambda_1(\omega_h, \beta A_1) \to \lambda_1(\Gamma, \hat{A}_\Gamma)$ as $h \to 0^+$, and we have seen in Theorem A.1 that $\lambda_1(\Gamma, \hat{A}_\Gamma) = 0$ if and only if $\frac{\beta|\Omega|}{2\pi} \in \mathbb{N}$. ☐

Remark C.4. 1) If the curve $\Gamma = \partial\Omega$ is *generic*, in the sense that $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$, then for small h the domain ω_h has arbitrarily small area and first eigenvalue bounded away from zero.

2) If Γ is an arbitrary curve then we know from A.2 that $\lambda_1(\Gamma, \hat{A}_\Gamma) \leq \frac{\beta}{4}$; as a consequence, when h is sufficiently small, one has $\lambda_1(\omega_h, \beta A_1) < \Theta_0\beta$, and we have another family of domains for which $\lambda_1(\Omega, \beta A_1) < \Theta_0\beta$.

In conclusion we have plenty of domains with arbitrarily small volume and thickness, and first eigenvalue either arbitrarily close to zero, or bounded away from zero, and this depends on the area enclosed by Γ. Note that the domains of Example C.3 are thin, but not simply connected. If we consider tubes around open curves the first eigenvalue always tends to zero as the tube shrinks to the curve.

Example C.5. *Let Γ be an open simple curve and let $\omega_h := \{x \in \Omega : \operatorname{dist}(x, \Gamma) < h\}$. Then*

$$\lambda_1(\omega_h, \beta A_1) \to 0$$

as $h \to 0^+$.

Proof. Through all the proof, we will denote by A the standard potential βA_1. Let $\lambda_j(\Gamma, \hat{A}_\Gamma)$ denote the magnetic eigenvalues on Γ endowed with the restriction of A to the curve, and magnetic Neumann boundary conditions at the endpoints (see Subsection 2.6). Since the restriction of A to Γ is exact, we conclude that $\lambda_j(\Gamma, \hat{A}_\Gamma)$ are just the Neumann eigenvalues on Γ, and in particular, $\lambda_1(\Gamma\hat{A}_\Gamma) = 0$. Let ω_h be a tube of size h around Γ. Then, by [36, §9] we have that $\lambda_1(\omega_h, \beta A_1) \to \lambda_1(\Gamma, \hat{A}_\Gamma) = 0$ as $h \to 0^+$. Note that the limit

$\lambda_1(\omega_h, \beta A_1) \to 0$ follows also by the fact that ω_h is simply connected and its area goes to zero, see Corollary 4.4, point 4). □

We show now a final example of annuli with large area, small thickness (i.e., small rolling radius at each point of the boundary), and first eigenvalue close to zero.

Example C.6. *Let $\mathcal{C}_n := \{x \in \mathbb{R}^2 : \sqrt{Cn} \leq r \leq \sqrt{Cn + Dn^\alpha}\}$, where (r,t) are the polar coordinates in $\mathbb{R}^2$, $C, D > 0$ and $0 \leq \alpha < \frac{1}{2}$. We have that $|\mathcal{C}_n| \to +\infty$ and $\lambda_1(\mathcal{C}_n, \beta A_1) \to 0$ as $n \to +\infty$.*

Proof. We have $|\mathcal{C}_n| = \pi D n^\alpha$, while the thickness of the annulus (the difference between the two radii) behaves like $n^{\alpha - 1/2}$ as $n \to +\infty$. Taking $C = \frac{2}{\beta}$ and $u(r,t) = e^{int}$ as test functions in (2.3), a standard computation shows that $\lambda_1(\mathcal{C}_n, \beta A_1) \to 0$ as $n \to +\infty$. With a bit of more work it is possible to deduce the same result for any $C > 0$. □

These examples show that finding good lower bounds for $\lambda_1(\Omega, \beta A_1)$ is a difficult task. It seems that the condition on the rolling radius of Theorem 5.10 is quite natural in many situations.

We conclude this section with the proofs of Theorem C.7 and Proposition C.9 which we have used to discuss Example C.3.

Theorem C.7. *Let Γ be a closed simple curve which is the boundary of a smooth bounded domain Ω, and let $\beta > 0$ be such that $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$. Let $\omega_h := \{x \in \Omega : \operatorname{dist}(x, \Gamma) < h\}$ be a small tubular neighborhood of Γ. There exists $\epsilon_0 > 0$ depending only on Ω and β such that, for all $h \in (0, \epsilon_0)$ one has:*

$$\lambda_1(\omega_h, \beta A_1) \geq \lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} h,$$

Proof. Through all the proof, we will denote by A the standard potential βA_1. Let δ denote the injectivity radius of the normal exponential map, which is positive being Ω smooth. For $r \leq \delta$, let $\Gamma_r = \{x \in \Omega : \operatorname{dist}(x, \Gamma) = r)$ be the equidistant at distance r to the boundary. In Lemma C.8 here below we prove that

$$\lambda_1(\Gamma_r, \hat{A}_{\Gamma_r}) \geq \lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} r, \tag{C.3}$$

for all $r \in [0, \epsilon_0)$, where $\lambda_1(\Gamma_r, \hat{A}_{\Gamma_r})$ is the lowest eigenvalue of the curve Γ_r with potential $\hat{A}_{\Gamma_r}$ (the restriction of the standard potential to the curve). The constant ϵ_0 will be defined in the proof of Lemma C.8. Inequality (C.3) is the main ingredient of the proof of Theorem C.7.

Take a first eigenfunction u of ω_h with $\|u\|_{L^2(\omega_h)} = 1$. By the coarea formula

$$\int_{\omega_h} |\nabla^A u|^2 = \int_0^h \int_{\Gamma_r} |\nabla^A u|^2 ds dr.$$

Fix a point $p \in \Gamma_r$ and consider an orthonormal frame (T, N) at p, where T is tangent to Γ_r and N is normal to it. At p we have:

$$|\nabla^A u|^2 = |\langle \nabla^A u, T\rangle|^2 + |\langle \nabla^A u, N\rangle|^2 \geq |\langle \nabla^A u, T\rangle|^2 = |\nabla^{\hat{A}_{\Gamma_r}} u|^2$$

We can then use the restriction of u as a test-function for the magnetic Laplacian associated to the pair $(\Gamma_r, \hat{A}_{\Gamma_r})$. This gives, using (C.3)

$$\int_{\Gamma_r} |\nabla^{\hat{A}_{\Gamma_r}} u|^2 ds \geq \lambda_1(\Gamma_r, \hat{A}_{\Gamma_r}) \int_{\Gamma_r} |u|^2\, ds \geq (\lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} r) \int_{\Gamma_r} |u|^2 ds \geq (\lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} h) \int_{\Gamma_r} |u|^2 ds$$

for all $r \in [0, h]$. Integrating on $[0, h]$ we obtain

$$\int_{\omega_h} |\nabla^A u|^2 \geq (\lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} h) \int_{\omega_h} |u|^2 = \lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} h.$$

The proof is complete by observing that $\int_{\omega_h} |\nabla^A u|^2 = \lambda_1(\omega_h, \beta A_1)$.

□

Lemma C.8. *Let Γ be a closed simple curve which is the boundary of a smooth bounded domain Ω, and let $\beta > 0$ be such that $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$. Let $\Gamma_r := \{x \in \Omega : \operatorname{dist}(x, \Gamma) = r\}$ be the equidistant at distance r to the boundary. There exists $\epsilon_0 > 0$ depending only on Ω and β such that, for all $r \in (0, \epsilon_0)$ one has:*

$$\lambda_1(\Gamma_r, \hat{A}_{\Gamma_r}) \geq \lambda_1(\Gamma, \hat{A}_\Gamma) - \sqrt{\beta} r.$$

Proof. Let δ be the injectivity radius of the normal exponential map; hence the distance function $\rho(x) := \operatorname{dist}(x, \Gamma)$ is smooth for $\rho \leq \delta$.

Let $\Omega_r = \{x \in \Omega : \rho(x) > r\}$: for $r \in [0, \delta)$, the curve $\Gamma_r = \partial\Omega_r$ is smooth. We set for brevity:

$$A(r) = |\Omega_r|\,, \quad L(r) = |\Gamma_r|\,,$$

so that $A(0) = |\Omega|\,, L(0) = |\Gamma|$. Theorem A.1 gives, for all $r \in [0, \delta)$:

$$\lambda_1(\Gamma_r, \hat{A}_\Gamma) = \frac{4\pi^2}{L(r)^2} \min_{n \in \mathbb{Z}} \Big(n - \frac{\beta A(r)}{2\pi}\Big)^2. \tag{C.4}$$

We recall the well-known facts that, on the interval $[0, \delta)$, $A(r)$ is smooth and decreasing, $A'(r) = -L(r)$ and $L'(r) = -2\pi$, so that $A(r) = A(0) - L(0)r + \pi r^2$ (see e.g., [24, §1.2]). We will use the inequalities:

$$A(0) - L(0)r \leq A(r) \leq A(0) \quad \text{and} \quad L(r) \leq L(0). \tag{C.5}$$

Since $\frac{\beta|\Omega|}{2\pi} \notin \mathbb{N}$, we have from Theorem A.1 that $\lambda_1(\Gamma, \hat{A}_\Gamma) > 0$. In particular, there is a unique $n \in \mathbb{N}$ such that

$$\frac{\beta\,|\Omega|}{2\pi} \in (n - \frac{1}{2}, n) \quad \text{or} \quad \frac{\beta\,|\Omega|}{2\pi} \in (n, n + \frac{1}{2}]. \tag{C.6}$$

Since $|\Omega_r| = A(r)$ is continuous and decreasing, there exists a positive $\epsilon_0 \leq \delta$ for which the inequalities in (C.6) continue to hold for the domain Ω_r for all $r \in [0, \epsilon_0)$, that is:

$$\frac{\beta A(r)}{2\pi} \in (n - \frac{1}{2}, n) \quad \text{or} \quad \frac{\beta A(r)}{2\pi} \in (n, n + \frac{1}{2}] \tag{C.7}$$

and (C.4) gives:

$$\sqrt{\lambda_1(\Gamma_r, \hat{A}_{\Gamma_r})} = \begin{cases} \dfrac{2\pi n - \beta A(r)}{L(r)} & \text{if} \quad \dfrac{\beta A(r)}{2\pi} \in (n - \frac{1}{2}, n), \\ \dfrac{\beta A(r) - 2\pi n}{L(r)} & \text{if} \quad \dfrac{\beta A(r)}{2\pi} \in (n, n + \frac{1}{2}] \end{cases} \tag{C.8}$$

Set for brevity $f(r) = \sqrt{\lambda_1(\Gamma_r, \hat{A}_{\Gamma_r})}$. We use (C.5) to see that, in the first case of (C.8) we have immediately $f(r) \geq f(0)$, while in the second we obtain $f(r) \geq f(0) - r$. Squaring both sides of this last inequality we get

$$f(r)^2 > f(0)^2 - 2f(0)r,$$

From Theorem A.2 we see that $f(0) = \sqrt{\lambda_1(\Gamma, \hat{A}_\Gamma)} \leq \frac{\sqrt{\beta}}{2}$ so that $f(r)^2 > f(0)^2 - \sqrt{\beta}r$, which is the assertion.

□

Proposition C.9. *Let Γ be a closed simple curve which is the boundary of a smooth bounded domain Ω for which $\frac{\beta|\Omega|}{2\pi} \in \mathbb{N}$. Let $\omega_h := \{x \in \Omega : \operatorname{dist}(x, \Gamma) < h\}$ be a small tubular neighborhood of Γ. Then*

$$\lim_{h \to 0^+} \lambda_1(\omega_h, \beta A_1) = 0.$$

Proof. Let δ be the injectivity radius of the normal exponential map. From now on we shall assume $h \in (0, \delta)$.

Let $f : \Omega \to \mathbb{R}$ be the solution of

$$\begin{cases} \Delta f = 0\,, & \text{in } \Omega, \\ \langle \nabla f, N \rangle = -A_1(N)\,, & \text{on } \partial\Omega. \end{cases}$$

Define $A' = A_1 + df$, where A_1 is defined in (1.1). We note that $dA' = dA_1 = dv$, $\operatorname{div} A' = 0$, and $A'(N) = 0$ on $\partial\Omega$. On Ω, and on any ω_h, A_1 and A' differ by an exact one-form, hence $\lambda_1(\omega_h, \beta A_1) = \lambda_1(\omega_h, \beta A')$ for all $h \in (0, \delta)$. Consider now the function $v : \Gamma \to \mathbb{C}$

$$v(s) := e^{i \int_0^s \hat{A}'_\Gamma},$$

where s is the arc-length variable on Γ and $\hat{A}'_\Gamma$ is the restriction of $\beta A'$ to Γ. We define a test function u on ω_h extending v constantly in the normal direction to Γ. Namely, for $x \in \omega_h$, we set $u(x) := v(s(x))$, where $s(x)$ is the arc-length coordinate of the (unique) nearest point to x on Γ. By construction u is smooth on ω_h, since $v(|\Gamma|) = e^{i \int_0^s \hat{A}'_\Gamma} = e^{i\beta|\Omega|} = e^{i2\pi n}$ for some $n \in \mathbb{N}$. Moreover, it does not depend on h. Let $p \in \Gamma$, and let (T, N) be an orthonormal frame, where T is tangent to Γ at p, and N is a unit normal to Γ at p. Then, at any $p \in \Gamma$ we have $du(T) = iu\hat{A}'_\Gamma(T)$, so that $d^{\beta A'}u(T) = 0$. Moreover $d^{\beta A'}u(N) = 0$ since $A'(N) = 0$. We conclude that $d^{\beta A'}u = 0$ on Γ, or, equivalently, $\nabla^{\beta A'}u = 0$ on Γ.

This implies that $\|\nabla^{\beta A'}u\|_{L^\infty(\omega_h)} \to 0$ as $h \to 0^+$. From the min-max principle (2.3) we have

$$\lambda_1(\omega_h, \beta A_1) = \lambda_1(\omega_h, \beta A') \leq \frac{\int_{\omega_h} |\nabla^{\beta A'}u|^2}{\int_{\omega_h} |u|^2} \leq \frac{\|\nabla^{\beta A'}u\|^2_{L^\infty(\omega_h)}|\omega_h|}{|\omega_h|}$$

since $\int_{\omega_h} |u|^2 = \int_{\omega_h} 1 = |\omega_h|$. The proof is complete. ∎

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Bruno Colbois, Université de Neuchâtel, Institute de Mathémathiques, Rue Emile Argand 11, 2000 Neuchâtel, Switzerland

Email address: bruno.colbois@unine.ch

Corentin Léna, Università degli Studi di Padova, Dipartimento di Tecnica e Gestione dei Sistemi Industriali (DTG), Stradella S. Nicola 3, 36100 Vicenza and Dipartimento di Matematica "Tullio Levi-Civita", via Trieste 63, 35121 Padova, Italy

Email address: corentin.lena@unipd.it

Luigi Provenzano, Sapienza Università di Roma, Dipartimento di Scienze di Base e Applicate per l'Ingegneria (SBAI), Via Antonio Scarpa 16, 00161 Roma, Italy

Email address: luigi.provenzano@uniroma1.it

Alessandro Savo, Sapienza Università di Roma, Dipartimento di Scienze di Base e Applicate per l'Ingegneria (SBAI), Via Antonio Scarpa 16, 00161 Roma, Italy

Email address: alessandro.savo@uniroma1.it