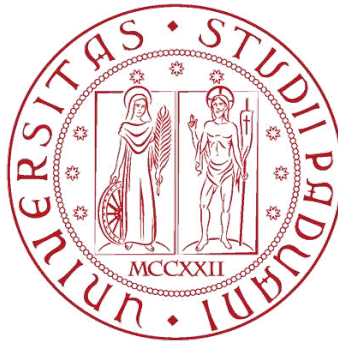


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Sustainability of common pool resources

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Thesis Summary

The sustainable management of renewable resources is a central problem in the study of social–ecological systems, where the dynamics of human behavior and environmental processes are tightly interdependent. In such contexts, individual decisions may collectively lead either to long-term coexistence between a community and its resource base, or to collapse through over-exploitation. Understanding how ecological feedbacks, behavioral adaptation, and social structure shape these outcomes is therefore essential to characterize the conditions under which sustainability can emerge. In particular, common pool resources are goods that can be accessed without regulation by any member of a community of consumers. However, precisely because access is open, each individual may be encouraged to maximize personal benefit, even when doing so contributes to collective harm. As argued by Hardin (Hardin, 1968), this misalignment between individual and collective interest can drive the progressive exhaustion of the resource, a phenomenon he famously termed the *tragedy of the commons*. It is then extremely relevant to determine under which conditions the tragedy can be avoided, namely the consumers adopt an approach towards the consumption of the resource that allows it to remain sustainable.

From a physics perspective, these questions can be addressed by viewing social–ecological systems as complex dynamical systems composed of many interacting components. Indeed, methods originally developed in statistical physics and nonlinear dynamics provide a natural framework to connect microscopic mechanisms — such as individual decisions, interactions among agents, and local environmental constraints — to emergent macroscopic outcomes, including col-

lective cooperation, critical transitions, and collapse. In particular, mathematical modeling makes it possible to explore how feedback loops between human behavior and ecological dynamics shape the stability of the system, and to identify critical parameters separating sustainable and unsustainable regimes. Moreover, tools from evolutionary game theory allow one to represent strategic interactions in a quantitative way, capturing how incentives, adaptation, and learning processes can drive behavioral shifts over time. This approach is especially useful when sustainability depends on the balance between individual gain and collective cost, a tension that naturally gives rise to social dilemmas.

In this thesis, we address this problem by building on previous work on the coupled dynamics of renewable resource pools and communities of consuming agents (Tilman et al., 2020; Tu et al., 2023). Within this framework, the resource evolves through ecological processes while being simultaneously affected by extraction and consumption, and the collective outcomes emerge from the interaction between individual decision-making and shared ecological constraints.

We first develop a stochastic formulation of the model, which enables the analysis of the system from a probabilistic viewpoint and allows us to assess the impact of intrinsic fluctuations on the long-term behavior of the coupled dynamics. This stochastic perspective complements deterministic approaches and provides additional insight into regimes where noise may play a relevant role, especially near critical transitions between sustainable and depleted states.

We then extend the ecological component of the model by incorporating an Allee effect, a phenomenon observed in many biological populations for which growth can be suppressed at low densities due to reduced reproduction rates, cooperative interactions, or other density-dependent mechanisms. By including this effect, we relax the standard assumption of logistic growth and explore how more realistic ecological constraints modify the feedback between consumption and regeneration. When social dynamics and interactions are included, we observe a richer phenomenology. We first introduce a dynamical description for agents' strategies, allowing harvesting behaviors to evolve in time in response to the global state of the resource, thereby introducing an explicit feedback from ecological conditions to collective decision-making.

Finally, as an additional extension, we move beyond a fully mixed population and

study the model in structured communities, where agents interact through a non-trivial network topology. This framework captures the fact that real populations are often heterogeneous and organized in interaction patterns that can significantly influence macroscopic outcomes.

Our results demonstrate that the specific form of the ecological dynamics crucially affects the long-term state of the coupled human–resource system, particularly in terms of resource sustainability. Remarkably, the introduction of an Allee effect tends to increase the likelihood of sustaining the resource compared with the baseline logistic case. Moreover, coupling the evolution of strategies to the global resource level further promotes sustainable outcomes, as adaptive responses can mitigate over-exploitation and stabilize the resource dynamics.

Overall, this thesis work provides a unified modeling framework linking behavioral adaptation, ecological feedbacks, and social structure, and highlights their joint role in shaping sustainability in common pool resource systems.

Introduction

Common-pool resources (CPR - commons, for short) are goods that are simultaneously non-excludable and subtractable: they are non-excludable in the sense that it is exceedingly hard to regulate access to them, and they are subtractable in the sense that their usage decreases their availability or diminishes their quality for other consumers. Classic examples include drinkable water, timber, fisheries, grazing lands, and even air quality. Because of these structural properties, commons are inherently vulnerable to over-exploitation: when individuals act solely in their short-term self-interest, the resource can be depleted faster than it regenerates, potentially collapsing entirely. This scenario is widely known as the tragedy of the commons (TOC), following the seminal work by Hardin (Hardin, 1968). For instance, consider the case of northern cod fishery off the coast of the region of Newfoundland, Canada. In 1992 the landings dropped abruptly due to overfishing, as shown in figure I.1.

Yet, despite the intuitive appeal of Hardin's parable, decades of empirical work —most notably by Elinor Ostrom — have shown that communities across the world have devised institutional arrangements, monitoring norms, and cooperative harvesting practices that enable sustainable long-term use of shared resources (Ostrom, 1990; Ostrom, 2003). Ostrom's framework emphasizes that commons dilemmas are not merely ecological or economic problems, but socio-ecological systems in which human behavior, institutions, and environmental dynamics co-evolve.

In recent years, this perspective has been greatly enriched by the emergence of eco-evolutionary game theory — a field that explicitly couples strategic behavior with environmental feedbacks. Traditional evolutionary game theory assumes that



Figure I.1.: Trend of the landings of northern cod stocks in the region of Newfoundland, Canada. Figure taken from (Millennium Ecosystem Assessment, 2005).

the payoff structure is fixed. However, in many real-world systems the environment itself changes in response to the strategies adopted by individuals, and these environmental changes feed back to alter incentives. Examples range from fisheries, where effort levels shape stock biomass and thereby future profitability, to climate change, where emissions accumulate in the atmosphere and shift the strategic landscape of mitigation, to psychological-economical systems where deliberative versus automatic decision-making shapes and is shaped by shared social norms. In such coupled dynamics, strategic decisions affect the environment, and the environment in turn reshapes strategic incentives. Tilman et al. (Tilman et al., 2020) show that this feedback can generate a rich repertoire of dynamical behaviors, including bistability, interior fixed points with coexistence of strategies, and even sustained oscillations (“oscillating tragedies of the commons”). These oscillations typically arise when there are incentives both to lead and follow strategic shifts, and when environmental feedback occurs on sufficiently slow timescales relative to behavioral change. This theoretical insight has influenced recent modeling efforts on resource harvesting, climate-economy interactions, species coexistence, behavioral feedbacks, and cultural evolution.

In this direction, Tu et al. (Tu et al., 2023) combine controlled experiments and theory to study how shared goals and feedback between decisions and resource dynamics affect the governance of common-pool resources. They design the Systemic Sustainability Game, an online experiment where N players harvest from the same renewing resource $R(t)$, whose dynamics follows a logistic growth with extraction by cooperators and defectors. When rewards are purely proportional to each individual's own extraction (type-1 games), players tend to behave selfishly, and the common resource is systematically overexploited and eventually depleted, reproducing a tragedy-of-the-commons outcome. In contrast, when a collective goal is induced by rewarding players based on the group's total payoff (type-2 games), a high level of self-organized cooperation emerges, resource levels remain high, and long-term cumulative rewards are substantially larger, demonstrating how human environmental systems can be used as theoretical ground to test Ostrom empirically driven theory on how shared goals can sustain commons through collective action (Tu et al., 2023).

Because of the urgency of contemporary sustainability challenges — including climate change, biodiversity loss, and the depletion of fisheries, soil nutrients, and groundwater — the study of socio-ecological feedbacks is more important than ever. The problem is so broad and touches so many interacting processes that it must be addressed using methods from multiple disciplines.

In this thesis, I explore these issues from a physicist's perspective, employing tools from theoretical ecology, evolutionary game theory, dynamical systems, and stochastic processes to develop models that capture the co-evolution of human behavior and environmental states.

In the first chapter, I will deal with the model introduced by Tu et al. a few years ago (Tu et al., 2023) and the new related developments.

In the second chapter, I will talk about the introduction in the previous model of a richer dynamics of the resource and the effect of knowledge of the global state of the resource (*knowledge feedback*) on the sustainability of the resource. The project culminated in the following publication:

Chengyi Tu et al. (2025). « The role of the Allee effect in common pool resource games with environmental feedback ». In: *Chaos, Solitons &*

Fractals 197.C, None. DOI: 10.1016/j.chaos.2025.116497

In the third chapter, I will talk about an additional twist on the basic model, including again the *knowledge feedback* effect and, secondly, a more complex social dynamics. The project culminated in the following publication:

Meghdad Saeedian et al. (2024). « Modelling co-evolution of resource feedback and social network dynamics in human-environmental systems ». In: *New Journal of Physics* 26. DOI: 10.1088/1367-2630/ad67fe

I eventually discuss some preliminary insights on a new development of the model, specifically regarding stochastic amplification.

The basic sustainability model

In this chapter, after giving some background, we introduce the basic version of the model, as described in (Tu et al., 2023), and show more recent advances. Specifically, sections 1.1 and 1.2.1 draw almost entirely from the cited work and provide the necessary foundation on which the later models build upon. Section 1.2.2, on the other hand, is original. In later chapters, we discuss more substantial variations.

1.1 Background: games with environmental feedbacks

The field that investigates quantitatively the behavior of agents in the context of diverse challenges consisting in clashes with other agents is *game theory*; some seminal works are (Neumann et al., 1944), (Nash, 1950), (Nash, 1951). Every agent has a certain payoff, depending on its choice in these clashes.

When the field first started, such payoffs were given as parameters, as the focus was primarily the economical aspects of the models. However, in the context of biology, the matter of how populations adopt strategies that prove to be successful and thus directly impact their fitness and are selected by evolution becomes of paramount importance. This development of the field is called *evolutionary game theory* (see (Smith, 1982) for an extensive introduction by the pioneer of this branch, or (Hofbauer and Sigmund, 1998) for another complete treatment).

Additionally, in such a context, naturally, the game payoffs are linked with the surrounding environment: its state *feeds back* on the payoff available to the agents, as pointed out by (Tilman et al., 2020) in diverse settings.

In the context of CPRs, intuitively, the level of resource available will influence

the harvesting strategy of the agents: if the resources are abundant, they will be more prone to adopt an aggressive strategy, namely extracting a large amount of resources as they guess they are far from exhaustion; on the other hand, if the resources are already scarce, they will tend to be more conservative (assuming that the agents are rational). This is the basic idea behind the work by (Tu et al., 2023), which we now explain.

1.2 Analytical derivation of the model

In this section, we first derive the model from deterministic considerations, then we re-derive it from microscopic processes.

1.2.1 Deterministic formulation

For the sake of analytical tractability, let us consider the simplest case of one CPR and two harvesting strategies: *cooperation* and *defection*. The resource is *biotic*, meaning that it reproduces, hence it has a birth-death dynamics; the simplest model for such a dynamics is the *logistic growth* (see (Verhulst, 1838) as the historic reference, (Murray, 2007) for a more modern and pedagogical treatment) where we use the symbols $R = R(t)$ for the level of resource, b for the growth rate and K for the carrying capacity; hence, the *growth term* $G(R)$ reads:

$$G(R) = bR \left(1 - \frac{R}{K}\right) \quad (1.1)$$

We assume that we have a fixed number N of consumers; therefore, if N_c of them are cooperators, the other $N_d = N - N_c$ will be defectors or, equivalently, the composition of the community will be completely defined by N_c ; we can also define the *fraction of cooperators*: $x = \frac{N_c}{N}$. The strategies are defined according to the following assumptions: the cooperators harvest the resource at a rate such that, if the community was completely made up by cooperators, it is able to renew itself, without getting exhausted; defectors, instead, do not pay attention to this issue and harvest as much as possible in the short term, implying that the harvesting by a

community of defectors would inevitably lead to the TOC. Therefore, we describe the two strategy by means of two *extraction rates*, e_C for cooperation and e_D for defection, and we translate quantitatively the previous considerations with the constraints:

$$0 < Ne_C < b < Ne_D \quad (1.2)$$

Hence, the *extraction term* $E(R, x)$ reads:

$$\begin{aligned} E(R, x) &= R(N_C e_C + N_D e_D) \\ &= RN(xe_C + (1 - x)e_D) \end{aligned} \quad (1.3)$$

Putting together equations (1.1) for the growth and (1.3) for the harvesting we get the equation for the dynamics of the resource:

$$\begin{aligned} \frac{dR}{dt} &= G(R) - E(R, x) \\ &= bR\left(1 - \frac{R}{K}\right) - RN(xe_C + (1 - x)e_D) \\ &= bR\left[\left(1 - \frac{R}{K}\right) - (x\hat{e}_C + (1 - x)\hat{e}_D)\right] \end{aligned} \quad (1.4)$$

where, in the last passage, we defined some normalized extraction rates as $\hat{e}_Y = \frac{Ne_Y}{b}$, $Y = C, D$; the constraint given in equation (1.2), in terms of these vnormalized rates, becomes:

$$0 < \hat{e}_C < 1 < \hat{e}_D \quad (1.5)$$

Regarding the dynamics of the population, we assume that the agents satisfy a *replicator dynamics* (see (Taylor and Jonker, 1978) or again (Hofbauer and Sigmund, 1998), section 7.2): assuming that each encounter is a 1 versus 1 clash, agent i , $i = 1, \dots, N$, will be tempted to imitate the strategy of a competitor, agent j , if his/her payoff U_j is larger than its own, U_i , where the payoff of agent i is denoted by U_i . Because there are only two strategies, there are only two possible values for the individual payoff as well, which we define precisely as the resource harvested by any individual per unit time: $U_C(t) = R(t) e_C$ for cooperators and $U_D(t) = R(t) e_D$ for defectors¹. Notice that, because R clearly depends on time, U does as well. We

¹Notice that $U_D > U_C$ always. Then, defectors would always dominate unless some optimization principle is introduced so that agents need to satisfy a trade-off between such an optimal

define the maximum payoff difference simply as:

$$\begin{aligned}\Delta U_{max} &= K e_D - K e_C \\ &= K(e_D - e_C)\end{aligned}\tag{1.6}$$

keeping in mind that K is the maximum level attainable by the resource.

Then, the replicator equation in the case of 2 species (that is, two strategies) reads (Hofbauer and Sigmund, 1998)²:

$$\frac{dx}{dt} = -x(1-x)\frac{R}{K}\tag{1.7}$$

We introduced an additional pressure *pressure parameter* $w \in [-1, 1]$, representing an external push towards a particular game-theoretical strategy. This enters the equation for the strategies (1.7) as follows:

$$\frac{dx}{dt} = -wx(1-x)\frac{R}{K}\tag{1.8}$$

Evidently, $|w|$ regulates the speed of the harvesting process, while its sign regulates its direction altogether.

Thus, the full *human-environment system* describing the coupled dynamics of the resource and the community is:

$$\begin{cases} \frac{dR}{dt} = bR\left[\left(1 - \frac{R}{K}\right) - (x\hat{e}_C + (1-x)\hat{e}_D)\right] \\ \frac{dx}{dt} = -wx(1-x)\frac{R}{K} \end{cases}\tag{1.9}$$

A simple cartoon representing our formulation is shown in figure 1.1.

This model was originally formulated in the work by (Tu et al., 2023). Typical time series are shown in figure 1.2. We can see that choosing a value of w such that $w > 0$, exemplified in the figure by $w = 1$, leads the community towards defection. A fraction of cooperators actually remains because the dynamics on the community

condition and their instantaneous gain. This was done in (Tu et al., 2023) but was not the focus of the project discussed in this thesis.

²In the next section, we will present a more thorough description starting from the microscopic dynamics.

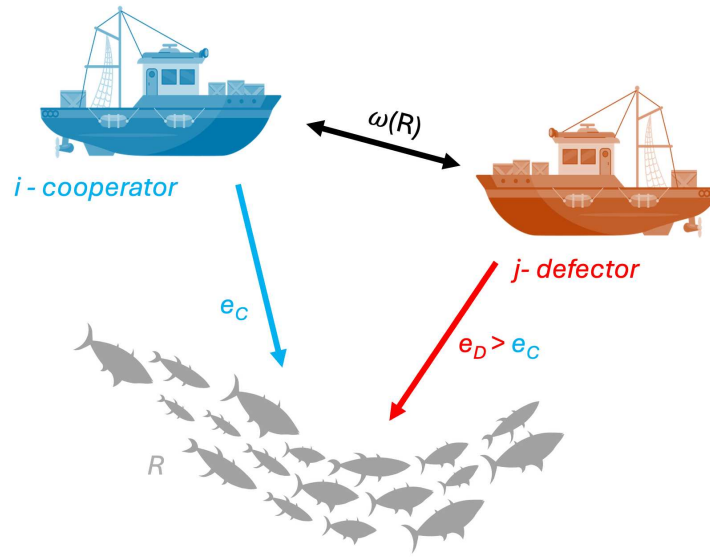


Figure 1.1.: Cartoon of the basic CPR sustainability model. A biotic resource (grey) is subjected to harvesting of a community that can take up two strategies: cooperation (blue) or defection (red); the two strategies are distinguished by their extraction parameters. The composition of the community is dynamic, meaning that the consumers can switch to the other strategy with a probability, which we call ω in this figure, in order to remain general, which depends on R , namely on the state of the environment (black).

slows down with R and stops altogether when $R = 0$. Therefore, the final value of x , in such a case, is not significant, because it depends on at which point the dynamic stops, which in turn is determined by the initial conditions. If we choose some value $w < 0$, exemplified in the figure by $w = -1$, the community converges to cooperation and the resource remains sustainable at stationarity, as a consequence of our choice of extraction rates.

Quite naturally, we can identify an *instantaneous payoff* for an individual consumer with the amount of resource it is able to collect in a single extraction event:

$$U(R) = R e_X \quad (1.10)$$

where $X = \{C, D\}$; therefore, an individual's instantaneous payoff depends solely on its strategy. Hence, given that $e_D > e_C$ by definition, the strategy of defection is always advantageous in the immediate term. However, we define a different payoff

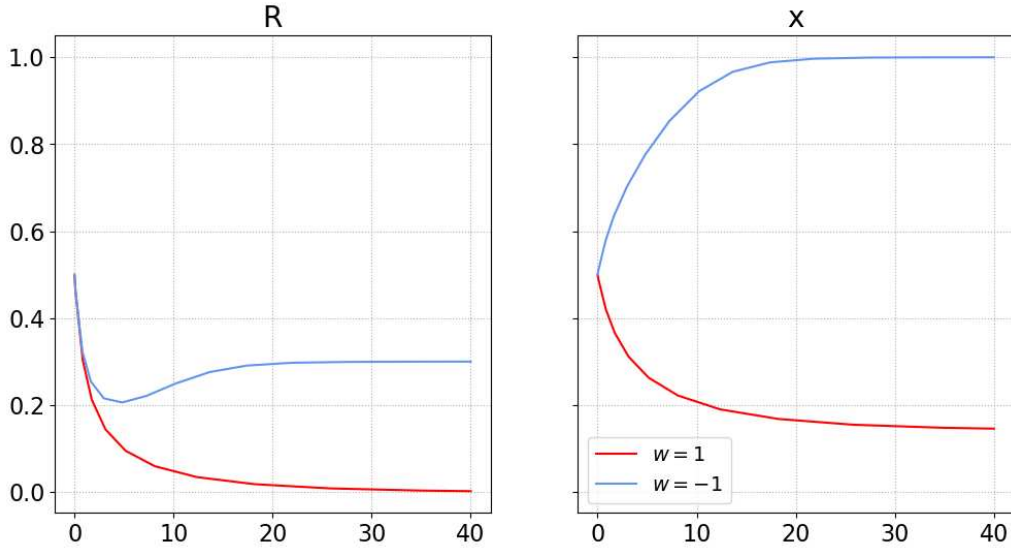


Figure 1.2.: Typical time series of R (left panel) and x (right panel) for the basic CPR sustainability model, both in the case of promotion of defection ($w = 1$, curves in red) and of cooperation ($w = -1$, curves in blue). Common parameters: $N = 50$, $b = 2$, $\hat{e}_C = 0.7$, $\hat{e}_D = 1.1$.

functional, integrating the amount of resource collected over a finite window of time:

$$\mathcal{P}(t_f) = \int_0^{t_f} R(t) e_X dt \quad (1.11)$$

Figure 1.3 shows the cumulative payoffs for both types of agents. For small lengths of the time window, the payoff of defectors is greater than the one of cooperators, as was immediately seen comparing the instantaneous payoffs. However, there is a certain time threshold over which the cumulative payoff of cooperators surpasses the one of defectors, meaning that, in the long term, cooperation is advantageous, because it allows consumers to collect the resource for longer, avoiding the TOC, instead of exhausting it immediately aiming for a short term gain.

The dynamics we devised is based on the replicator dynamics, in which agents compare their instantaneous payoffs. Given the observation that the sign of w promotes either cooperation or defection, it is evident that, unless we insert an external push towards cooperation (which may represent policies), defection will always be favored. In the following, we aim to find out whether cooperation becomes viable if we include stochastic effects.

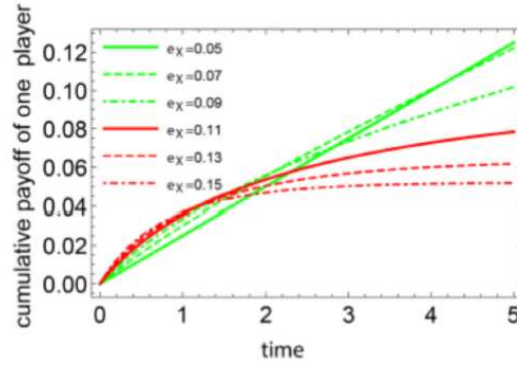


Figure 1.3.: Cumulative payoff for cooperators (green curves) and defectors (red curves) as a function of the length of the time window. The curves with the same line styles pertain to the same iteration of the dynamics.

Common parameters: $b = 2$, $N = 20$.

Figure taken from (Tu et al., 2023) (figure S2).

1.2.2 Stochastic formulation

We also derived the model from microscopic principles. We did so in what we call an *hybrid* way and in a *fully stochastic* way. The first approach amounts to regard x only as a stochastic variable, while the second approach regards both x and R as stochastic.

Hybrid approach

Initially, we treat x only as stochastic, leaving R deterministic. This assumes a separation of time scales between the dynamics of the two variables: the dynamics of R is much slower than the one on x , allowing to treat its evolution as deterministic; in other words, a macroscopic change in the level of resource happens only if the community evolved for a long period of time. Therefore, the coupled evolution of R and x is given by the first line of equation (1.9) coupled with a Fokker-Planck equation for the community only, which was derived in (Tu et al., 2023) and is

given in equation (S2) therein, which we report here:

$$\frac{\partial P(x; t)}{\partial t} = -\frac{\partial}{\partial x} (a(x|R; t)P(x; t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b^2(x|R; t)P(x; t)) \quad (1.12)$$

$$a(x|R; t) = -R(t) x(t)(1 - x(t)) \quad (1.13)$$

$$b(x|R; t) = \sqrt{\frac{1}{N} x(t)(1 - x(t))} \quad (1.14)$$

where the writing $f(x|R; t)$ means « f at time t as a function of x , where R is a parameter ». Writing equation (1.12) at stationarity and substituting the expressions of the coefficients:

$$0 = R^* \frac{\partial Q(x)}{\partial x} + \frac{1}{2N} \frac{\partial^2 Q(x)}{\partial x^2} \quad (1.15)$$

where $Q(x) = x(1 - x)P^*(x)$.

Integrating once, assuming that $R^* \neq 0$:

$$c = Q(x) + \frac{1}{2NR^*} \frac{\partial Q(x)}{\partial x} \quad (1.16)$$

Given the definition of Q , the above equation evaluated in $x = 0$ gives $c = 0$, but c is a constant and so it is $0 \forall x$. Then, we can integrate again by separation of variables, obtaining:

$$P^*(x) = \frac{C}{x(1 - x)} e^{-2NR^*x} \quad (1.17)$$

where C is another constant which can be fixed, in principle, by normalization. However, the function (1.17) is not normalizable and diverges for x going towards the borders of its domain; in particular, it diverges much more rapidly for $x \rightarrow 0$ than for $x \rightarrow 1$, meaning that x values close to 1 are more common; this is confirmed by numerical simulations, as shown in figure 1.4. Another case of interest which also enables a more thorough analytical treatment is inserting reflecting boundary conditions at the borders of the domain of x , namely setting $T^+(x = 0) = \mu_C$ and $T^-(x = 1) = \mu_D$ where μ_C and μ_D are some finite values, whereas, in the previous implementation, $T^+(x = 0) = T^-(x = 1) = 0$ because $T^\pm(x|R; t) \propto x(1 - x)$. This may describe a system in which the consumers are in contact with other communities which represent an external influence that allows them to change strategy even if the community they are part of saturated

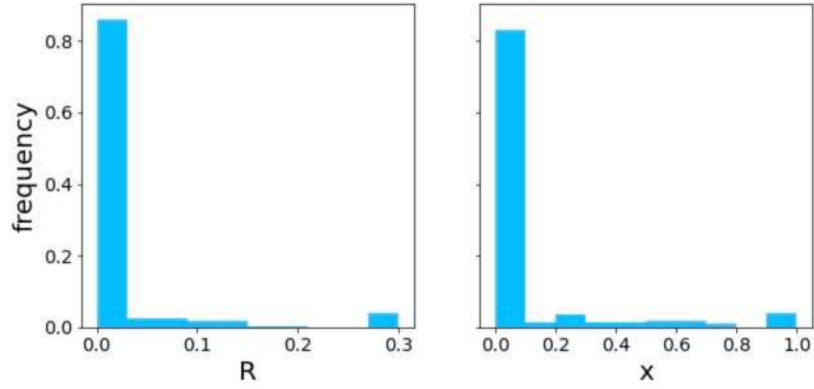


Figure 1.4.: Histogram of the stationary values of R and x for the sustainability model. The results are obtained from 200 Langevin simulations of the model with $N = 20$.

to cooperation or defection. In this sense, we say that this version of the model allows for *invasion* of the strategies. In this case as well we can derive the stationary distribution of N_C , namely the total number of cooperators in the system, exploiting a recursive argument. Let us consider the master equation for N_C , where time τ is a discrete index and not yet approximated by a continuous variable, as usually done:

$$P(N_C, \tau + 1) - P(N_C, \tau) = T^+(N_C - 1|R; \tau)P(N_C - 1, \tau) + T^-(N_C + 1|R; \tau)P(N_C + 1, \tau) - (T^+(N_C|R; \tau) + T^-(N_C|R; \tau)) P(N_C, \tau) \quad (1.18)$$

At stationarity, the above equation becomes:

$$0 = T^{*+}(n-1)P^*(n-1) + T^{*-}(n+1)P^*(n+1) - P^*(n) (T^{*+}(n) + T^{*-}(n)) \quad (1.19)$$

where, for simplicity, we write $n = N_C$ and $T^{*\pm}(n) = T^{\pm}(n|R^*x)$.

We now observe the following facts:

- For $n = 0$, $T^{*+}(-1) = T^{*-}(0) = 0$, therefore equation (1.19) becomes:

$$\begin{aligned} 0 &= T^{*-}(1)P^*(1) - P^*(0)T^{*+}(0) \\ \Rightarrow P^*(1) &= \frac{\mu_C}{T^{*-}(1)} P^*(0) \end{aligned} \quad (1.20)$$

- For $n = 1$, equation (1.19) becomes:

$$0 = T^{*+}(0)P^*(0) + T^{*-}(2)P^*(2) - P^*(1)(T^{*+}(1) + T^{*-}(1)) \quad (1.21)$$

Solving for $P^*(2)$ and using equation (1.20) for $P^*(1)$, one gets:

$$P^*(2) = \frac{T^{*+}(1)\mu_C}{T^{*-}(2)T^{*-}(1)} P^*(0) \quad (1.22)$$

- Iterating, we can write an expression for $P^*(n)$, $0 < n \leq N$:

$$P^*(n) = \prod_{k=0}^{n-1} \frac{T^{*+}(k)}{T^{*-}(k+1)} P^*(0) \quad (1.23)$$

From equations (1.38) and (1.39) (see next paragraph for a full explanation of these expressions), the transition rates are:

$$T^{\pm}(n|R;t) = \left(\frac{1}{2} \mp \frac{R}{2}\right) \frac{n}{N} \frac{N-n}{N} \quad (1.24)$$

Therefore, plugging the above expression into our recursive relation for $P^*(n)$, equation (1.24), for $n < N$:

$$\begin{aligned} P^*(n) &= \prod_{k=1}^{n-1} \frac{\left(\frac{1}{2} - \frac{R^*}{2}\right) \frac{k}{N} \frac{N-k}{N}}{\left(\frac{1}{2} + \frac{R^*}{2}\right) \frac{k+1}{N} \frac{N-k-1}{N}} \underbrace{\frac{\mu_C}{\left(\frac{1}{2} + \frac{R^*}{2}\right) \frac{1}{N} \frac{N-1}{N}}}_{k=0 \text{ term}} P^*(0) \\ &= \frac{N^2 \mu_C}{N-1} P^*(0) \frac{\left(\frac{1}{2} - \frac{R^*}{2}\right)^{n-1}}{\left(\frac{1}{2} + \frac{R^*}{2}\right)^n} \cdot \frac{1(N-1)2(N-2) \dots (n-1)(N-n+1)}{2(N-2) \dots (n-1)(N-n+1)n(N-n)} \\ &= \frac{N^2 \mu_C}{N(N-n)} P^*(0) \frac{\left(\frac{1}{2} - \frac{R^*}{2}\right)^{n-1}}{\left(\frac{1}{2} + \frac{R^*}{2}\right)^n} \\ &= \frac{2\mu_C}{\frac{n}{N} \left(1 - \frac{n}{N}\right)} \frac{1}{1 - R^*} \left(\frac{1 - R^*}{1 + R^*}\right)^n P^*(0) \end{aligned} \quad (1.25)$$

For $n = N$, instead, there is another « special » term in the product, namely

the one that contains $T^{*-}(N) = \mu_D$:

$$\begin{aligned}
P^*(N) &= \underbrace{\left(\frac{1}{2} - \frac{R^*}{2}\right) \frac{N-1}{N} \frac{1}{N}}_{\substack{\mu_D \\ k=N-1 \text{ term}}} \cdot \prod_{k=1}^{N-2} \frac{\left(\frac{1}{2} - \frac{R^*}{2}\right) \frac{k}{N} \frac{N-k}{N}}{\left(\frac{1}{2} + \frac{R^*}{2}\right) \frac{k+1}{N} \frac{N-k-1}{N}} \underbrace{\frac{\mu_C}{\left(\frac{1}{2} + \frac{R^*}{2}\right) \frac{1}{N} \frac{N-1}{N}}}_{k=0 \text{ term}} P^*(0) \\
&= \frac{\mu_C}{\mu_D} P^*(0) \left(\frac{1 - R^*}{1 + R^*}\right)^{N-1} \cdot \frac{1(N-1)2(N-2)\dots(N-2)2}{2(N-2)\dots(N-2)2(N-1)1} \\
&= \frac{\mu_C}{\mu_D} P^*(0) \left(\frac{1 - R^*}{1 + R^*}\right)^{N-1} \tag{1.26}
\end{aligned}$$

Having an expression for all n , $P^*(0)$ can be found by normalization:

$$\begin{aligned}
\sum_{n=0}^N P^*(n) &= 1 \\
P^*(0) + \sum_{n=1}^N P^*(n) &= 1 \\
P^*(0) + \sum_{n=1}^N \frac{P^*(n)}{P^*(0)} P^*(0) &= 1 \\
\Rightarrow P^*(0) &= \frac{1}{1 + \sum_{n=1}^N \frac{P^*(n)}{P^*(0)}} \tag{1.27}
\end{aligned}$$

where we exploit the fact that, as per equation (1.24), $P^*(n)$ are proportional to $P^*(0)$ and thus their ratio is independent of $P^*(0)$.

As shown in figure 1.5, resulting $P^*(n)$ now exhibits a peak close to the $x = 0$ boundary, but it is not diverging. This is also confirmed by numerical simulations.

Fully stochastic approach

Let us now derive again equation (1.9) from a microscopic picture. In line with the concluding remarks of section 1.2.1, we eliminate the pressure parameter w ,

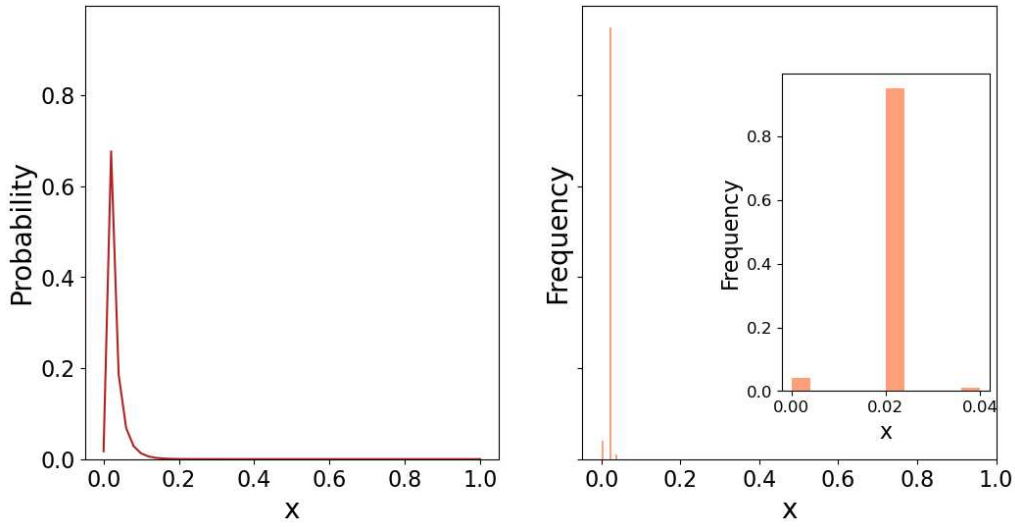


Figure 1.5.: Analytical (left panel) and numerical (right panel) stationary distribution of x for the CPR sustainability model. The numerical histogram has been obtained from 100 realizations of the dynamics, employing the Gillespie algorithm.

namely we set it to 1, removing any artificial push to cooperation.

We consider a bi-layer system of N and K sites, representing the community and the pool of resource, respectively: in the first layer, each site, corresponding to an individual, can take the cooperative or defecting state, namely we say that it can be in either the c or d state; in the second layer, each site can be occupied by a particle of resource, in which case it takes the state r , but it can also be empty as the resource isn't conserved, in which case we say that it takes the e state. By definition, we have the following constraint on the populations of particles of resource and empty sites, N_r, N_e : $K = N_r + N_e$.

In the language of chemical kinetics (see (Gardiner, 2009), chapter 7), we say that, in the community layer, each individual can belong to either the c or d species, while, in the resource layer, each individual can belong to either the r or e species. Regarding the behavior of the consumers, let us consider a 1-on-1 confrontation between two agents, i and j , where i is the active player, meaning the one that can potentially switch strategy, while the strategy of j remains fixed. Agent i compares his/her instantaneous payoff, given in our case by equation (1.10), with agent j 's payoff and will switch to his/her strategy with probability depending on the

difference between their payoffs, as stated in section 1.2.1:

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2} \frac{U_j - U_i}{\Delta U_{max}} \\ &= \frac{1}{2} + \frac{1}{2} \frac{U_j - U_i}{K(e_D - e_C)} \end{aligned} \quad (1.28)$$

Therefore, if i and j have the same strategy, then $p = \frac{1}{2}$ and dynamics is completely random; if they have different strategies, we can distinguish the case in which i is a defector and j is a cooperator:

$$p^+ = \frac{1}{2} - \frac{1}{2} \frac{R}{K} \quad (1.29)$$

and i is a cooperator and j is a defector:

$$p^- = \frac{1}{2} + \frac{1}{2} \frac{R}{K} \quad (1.30)$$

Each individual in the community layer interacts with the resource layer by harvesting it. Notice that we are assuming that everything is well-mixed, meaning that every consumer has the same probability of encountering any of his competitors and is also in contact with every site of the resource layer.

Finally, the dynamics described in section 1.2.1 is captured microscopically by the following processes:

1. Offspring of resource:

$$r + e \xrightarrow{b} r + r \quad (1.31)$$

Notice that the process requires an e species: the resource is able to reproduce only if its population is sufficiently far from the carrying capacity; in other words, although there is no spatial description in the model, intuitively, the resource is able to reproduce provided that there is room for other individuals. This encapsulates the *overcrowding* effect of the logistic growth.

2. Death of resource due to maintenance of a cooperator:

$$r \xrightarrow{\frac{N_C}{N} e_C} e \quad (1.32)$$

3. Death of resource due to maintenance of a defector:



4. Replicator dynamics, defector imitates a cooperator:



5. Replicator dynamics, cooperator imitates a defector:



We can derive a *master equation* for the probability that the system is in a given state, defined by (N_c, N_r) , at time t : $P(N_c, N_r, t)$; then, assuming the thermodynamic limit, namely $N, K \gg 1$, we pass to the continuous variables x and $R = \frac{N_r}{K}$ and derive an equation of motion for $P(x, \bar{R}, t)$, that is the well known technique of the *Kramers-Moyal expansion* (see, again (Gardiner, 2009), chapter 7, or section A.1 of appendix A³). In the following, we make the procedure explicit, following the notation by (McKane and Newman, 2005).

From the microscopic processes, we can write the following *propensity rates*:

- Coming from process 1:

$$\begin{aligned} T^{+0}(N_r, N_c) &= T[(N_r, N_c) \rightarrow (N_r + 1, N_c)] \\ &= bN_r \frac{K - N_r}{K} \end{aligned} \quad (1.36)$$

³Notice that there, in order to keep the notation compact while remaining general, we write $a_j(\mathbf{x})$ for the propensity rate of process j of a generic set of processes, depending on the collection of variables $\mathbf{x}$. Here, instead, we adopt a more explicit notation, suitable for the present model.

- Coming from processes 2,3:

$$\begin{aligned} T^{-0}(N_r, N_c) &= T[(N_r, N_c) \rightarrow (N_r + 1, N_c)] \\ &= N_r e_C \frac{N_c}{N} + N_r e_D \frac{N - N_c}{N} \end{aligned} \quad (1.37)$$

- Coming from process 4:

$$\begin{aligned} T^{0+}(N_r, N_c) &= T[(N_r, N_c) \rightarrow (N_r, N_c + 1)] \\ &= p^+ N_c \frac{N - N_c}{N} \end{aligned} \quad (1.38)$$

- Coming from process 5:

$$\begin{aligned} T^{0-}(N_r, N_c) &= T[(N_r, N_c) \rightarrow (N_r, N_c - 1)] \\ &= p^- N_c \frac{N - N_c}{N} \end{aligned} \quad (1.39)$$

Then, we can write the master equation as a balance of a *gain term* (the positive terms) and a *loss term* (the negative terms):

$$\begin{aligned} \frac{dP(N_r, N_c, t)}{dt} &= T^{+0}(N_r - 1, N_c)P(N_r - 1, N_c, t) + T^{-0}(N_r + 1, N_c)P(N_r + 1, N_c, t) + \\ &\quad + T^{0+}(N_r, N_c - 1)P(N_r, N_c - 1, t) + T^{0-}(N_r, N_c + 1)P(N_r, N_c + 1, t) + \\ &\quad - P(N_r, N_c, t)[T^{-0}(N_r, N_c) + T^{+0}(N_r, N_c) + T^{0+}(N_r, N_c) + T^{0-}(N_r, N_c)] \end{aligned} \quad (1.40)$$

We introduce annihilation and creation operators, $\mathcal{E}^\pm$, describing the increment or decrement of an integer variable within a function f :

$$\mathcal{E}_i^\pm[f(m_1, \dots, m_i, \dots, m_N)] = f(m_1, \dots, m_i \pm 1, \dots, m_N) \quad (1.41)$$

Using these, the ME (1.40) becomes:

$$\begin{aligned} \frac{dP(N_r, N_c, t)}{dt} = & (\mathcal{E}_r^- - \mathbb{1})[T^{+0}(N_r, N_c)P(N_r, N_c, t)] + \\ & + (\mathcal{E}_r^+ - \mathbb{1})[T^{-0}(N_r, N_c)P(N_r, N_c, t)] + \\ & + (\mathcal{E}_n^- - \mathbb{1})[T^{0+}(N_r, N_c)P(N_r, N_c, t)] + \\ & + (\mathcal{E}_n^+ - \mathbb{1})[T^{0-}(N_r, N_c)P(N_r, N_c, t)] \end{aligned} \quad (1.42)$$

As anticipated, we can then pass to continuous variables $x = \frac{N_c}{N}$ and $R = \frac{N_r}{K}$, in terms of which the rates, from equations (1.36)-(1.39), become:

$$T^{+0}(N_r, N_c) = KT^{+0}(R, x), \quad T^{+0}(R, x) = bR(1 - R) \quad (1.43)$$

$$T^{-0}(N_r, N_c) = KT^{-0}(R, x), \quad T^{-0}(R, x) = NRe_Cx + NRe_D(1 - x) \quad (1.44)$$

$$T^{0+}(N_r, N_c) = NT^{0+}(R, x), \quad T^{0+}(R, x) = p^+x(1 - x) \quad (1.45)$$

$$T^{0-}(N_r, N_c) = NT^{0-}(R, x), \quad T^{0-}(R, x) = p^-x(1 - x) \quad (1.46)$$

Additionally, the probability densities of the discrete and continuous variables must satisfy the constraint that the total probabilities in the same regions of probability space in the two formulations are the same:

$$\begin{aligned} P(N_r, N_c, t) \underbrace{\Delta r}_1 \underbrace{\Delta n}_1 &= P(R, x, t) \underbrace{\Delta R}_{\frac{1}{K}} \underbrace{\Delta x}_{\frac{1}{N}} \\ P(N_r, N_c, t) &= \frac{1}{NK} P(R, x, t) \end{aligned} \quad (1.47)$$

In terms of continuous variables, the master equation becomes:

$$\begin{aligned} \frac{dP(R, x, t)}{dt} = & (\mathcal{E}_R^- - \mathbb{1})[KT^{+0}(R, x)P(R, x, t)] + \\ & + (\mathcal{E}_R^+ - \mathbb{1})[KT^{-0}(R, x)P(R, x, t)] + \\ & + (\mathcal{E}_x^- - \mathbb{1})[NT^{0+}(R, x)P(R, x, t)] + \\ & + (\mathcal{E}_x^+ - \mathbb{1})[NT^{0-}(R, x)P(R, x, t)] \end{aligned} \quad (1.48)$$

where the $\mathcal{E}_R^\pm$ and $\mathcal{E}_x^\pm$ operators are the analogous to those defined for the discrete variables, but increment or decrement the variables by an amount $\Delta R = 1/K$ or

$\Delta x = 1/N$, respectively.

In the infinite-size limit we can derive a *Fokker-Planck* equation for the process. Assuming the thermodynamic limit, namely $N \gg 1$ and $K \gg 1$, which imply $\Delta R \ll 1$ and $\Delta x \ll 1$, we can expand the creation and annihilation operators around $\mathbb{1}$:

$$(\mathcal{E}_y^\pm - \mathbb{1})[f] = f(y) \pm \Delta y \frac{\partial f}{\partial y} + \frac{\Delta y^2}{2} \frac{\partial^2 f}{\partial y^2} + o(\Delta y^3) - f(y) \quad (1.49)$$

where $y = \{R, x\}$. Performing this approximation in the master equation (1.48), we get:

$$\begin{aligned} \frac{dP(R, x, t)}{dt} = & -\frac{\partial}{\partial R} \left((T^{+0}(R, x) - T^{-0}(R, x)) P(R, x, t) \right) + \\ & + \frac{1}{2K} \frac{\partial^2}{\partial R^2} \left((T^{+0}(R, x) + T^{-0}(R, x)) P(R, x, t) \right) + \\ & - \frac{\partial}{\partial x} \left((T^{0+}(R, x) - T^{0-}(R, x)) P(R, x, t) \right) + \\ & + \frac{1}{2N} \frac{\partial^2}{\partial x^2} \left((T^{0+}(R, x) + T^{0-}(R, x)) P(R, x, t) \right) \end{aligned} \quad (1.50)$$

From this, as usual (see the reference cited before), we can identify the *drift* and *diffusion* terms to write an equivalent *Langevin equation* for the process:

$$\begin{cases} \frac{dR}{dt} = T^{+0}(R, x) - T^{-0}(R, x) + \sqrt{\frac{T^{+0}(R, x) + T^{-0}(R, x)}{K}} \eta_R(t) \\ \frac{dx}{dt} = T^{0+}(R, x) - T^{0-}(R, x) + \sqrt{\frac{T^{0+}(R, x) + T^{0-}(R, x)}{N}} \eta_x(t) \end{cases} \quad (1.51)$$

where $\eta_i(t)$, $i = \{R, x\}$, are uncorrelated Gaussian white noises satisfying:

$$\begin{aligned} \langle \eta_i(t) \rangle &= 0 \quad \forall t \\ \langle \eta_i(t) \eta_j(t') \rangle &= \delta_{ij} \delta(t - t') \quad \forall t, t' \\ \eta_i(t) &\sim \mathcal{N}(0, 1) \end{aligned} \quad (1.52)$$

Notice that equation (1.51) can be written in this rather simple form because the diffusion matrix B is diagonal, as can be seen from the fact that the right hand side of equation (1.50) doesn't contain mixed derivatives.

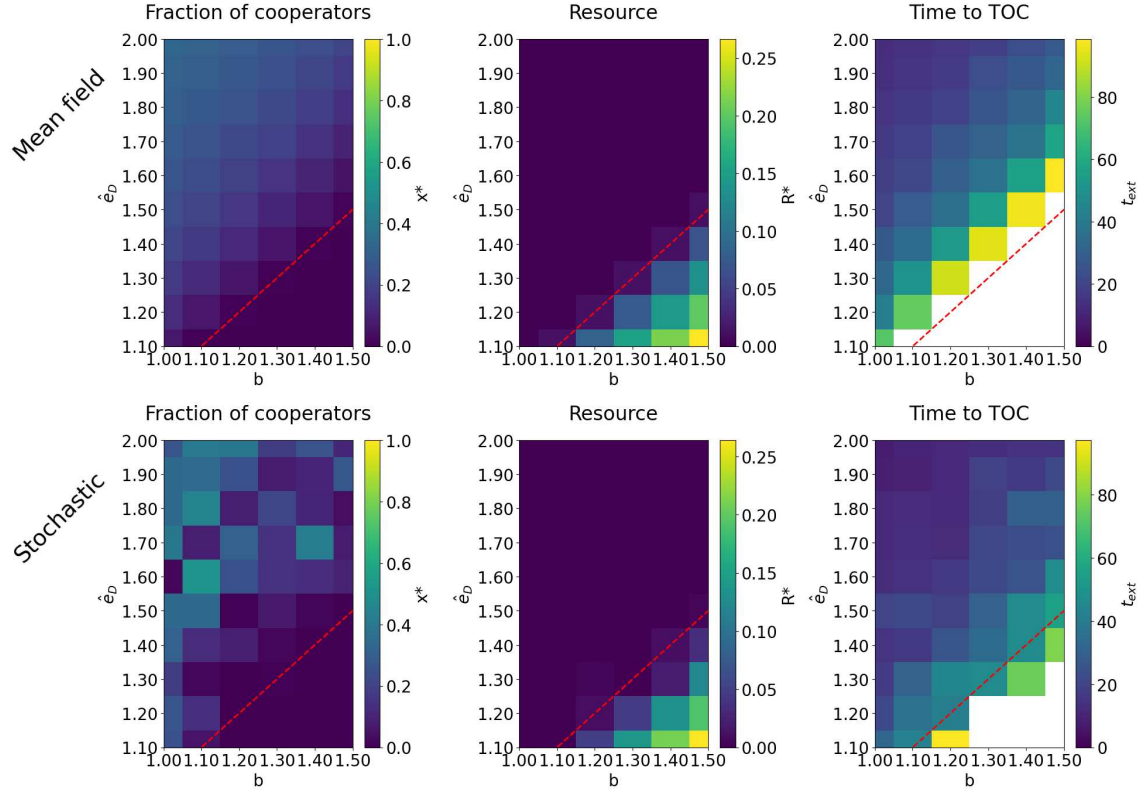


Figure 1.6.: Comparison of the phase diagrams on the plane $(b, \hat{e}_D)$ for the CPR sustainability model obtained by solving the deterministic HES (1.9) with $w = 1$ (upper row) and from microscopic processes (lower row). From left to right, in both rows, are represented the stationary fraction of cooperators, the stationary fraction of resource and the time to the TOC, namely the time at which the resource is extinguished. The red dashed line represents the limit of the feasibility region, defined by relation (1.2).

Parameters: $N = 200$, $K = 1000$, $\hat{e}_C = 0.7$; initial conditions $R_0/K = x_0 = 0.5$. Each dot in the plots relative to the stochastic was obtained from 4 repetition of the dynamics.

Finally, substituting the expression of the rates T , equations (1.43)-(1.46), and taking further advantage of the infinite-size limit, we can neglect the stochastic effects, that is the noise terms, retrieving thus the deterministic HES, equation (1.9).

We carried out individual-based simulations of the model starting from the microscopic processes by means of the Gillespie algorithm, (Gillespie, 1976), and compared them with the results of the integration of the deterministic equation with $w = 1$. The resulting phase diagrams on the plane $(b, \hat{e}_D)$ are shown in figure 1.6.

Notice that we have the constraint (1.2) on the extraction rates, which identifies a feasibility region within the possible values of the parameters; this region is the one above the red dashed line in the plots of figure 1.6.

In both implementations of the model, deterministic and stochastic, the TOC isn't avoided for values of the parameters satisfying the constraint (1.2). The resource remains sustainable in the unfeasible region, implying that a possible method to avoid the TOC is, quite obviously, imposing from above that all agents adopt an extraction rate that allows the resource to recover. The results of the two methods are in agreement; the stochastic implementation shows a larger variability in the stationary values of the variables, but the presence of these fluctuations is inherent in the fact that we are, indeed, implementing a stochastic dynamics.

1.3 Conclusions and outlook

Proceeding from what was done in the work by (Tu et al., 2023), we re-derived the CPR sustainability model from a fully stochastic picture, defining the microscopic processes and recovering the HES (1.9) by means of a Kramers-Moyal expansion. We compared the results of the deterministic and the stochastic models: we find perfect agreement between the two, but this also means that they aren't able to avert the TOC. This is ultimately due to the fact that, due to our choice of extraction rates and because in the replicator dynamics agents compare their instantaneous payoff, defectors are always favored. Thus, the community converges to defection and the resource is exhausted.

However, we can identify several weak points of our formulation. For one, the dynamics of the resource is the simplest one for describing a biotic resource and doesn't take into account potentially important effects; we will address this point in the next chapter, chapter 2.

Secondly, our definition of the transition rates takes care only weakly of the state of the environment: although $p^\pm$ do contain R , our environment variable, the defectors are always favored, as we remarked many times. A less trivial dynamics could arise if we replace the replicator scheme with one that truly takes into account the state of the environment: this is the mechanism of *knowledge feedback*, which

we introduce in 2.2.2 (again in chapter 2) and will resurface in chapter 3.

Thirdly, we dealt with a homogeneous population of consumers which harvest a single resource. Future research could include multiple resources and investigate more than two harvesting strategies. Moreover, we want to investigate the effects of inhomogeneities, such as a network structure, in the community of consumers, representing, for instance, their tendency to form subgroups, economical associations or similar; we tackle this issue in chapter 3.

Finally, we could include spatial effects on the resource itself as well, reflecting the fact that, in general, resources aren't distributed homogeneously, but are more abundant in certain hotspots.

As an additional advancement, which is still in progress, we started to develop an online platform to implement the model and let people actually play it as an online game. This project follows from the initiative of the « moral machine » by (Awad et al., 2018). In this work, the authors developed an online platform for collecting the feedback of people on their desired behavior of self-driven cars. At each round of the game, players were shown two everyday-life scenarios, which involved some type of social dilemma related to road accidents, which differed because of the behavior of the self-driving car involved and were asked to choose the more acceptable of two. The players were also asked to share some information regarding themselves and their background: country of origin, age, level of instruction, etc. The study unveiled clusters of players with similar preferences distinguished by geographical origin, that is, in other words, culture. Our aim is to collect data regarding the attitudes towards resource usage and potentially uncover similar patterns. This objective expands from what was done in the original work by (Tu et al., 2023) on the CPR sustainability model, in which games were implemented locally, involving 40 students at University of California in Berkeley, CA, U.S.A., and 40 at Yunnan University, Kunming, Yunnan, China. However, the aim is to make the game playable in single player versus a virtual community of agents, so as to remove the need of local experiments and enabling to collect much more data, coming from players from all around the world. I started the coding of such platform as part of an internship of 6 months at the software house Piratech, as expected by my kind of PON scholarship. In that period, I developed at the backend level the part of the platform devoted to user registration and a log of their

performance in each game; basically, I created a database ready to store the data we wish to analyze. In the time devoted to the internship, I couldn't develop the actual dynamical part of the game.

The Allee effect

In this chapter, we introduce the *Allee effect*, an important ecological effect, and we show how it affects the model introduced before.

2.1 An ecological introduction to the Allee effect

The Allee effect, introduced by W.C. Allee (see (Allee, 1931) for the original source, (Méndez et al., 2019) for a more recent, mathematical treatment) denotes the positive correlation between the growth rate of a population and its abundance. In other words: individuals of a given population will be able to reproduce, which implies finding a mate or being able to form groups that are sufficiently big to efficiently resist predation or otherwise cooperate, if they are abundant (accounting for the overcrowding effect encapsulated by the logistic growth, clearly); on the other hand, they will have a hard time doing so if they are few. This means that, at low population levels, its growth will be slow; at intermediate population levels, growth is promoted as mating is easier; then, when the population approaches the carrying capacity, the overpopulation effect kicks in.

Quantitatively, a way of modeling the growth dynamics of a population affected by Allee effect is given by the following equation:

$$\frac{dR}{dt} = bR\left(\frac{R}{A} - 1\right)\left(1 - \frac{R}{K}\right) \quad (2.1)$$

that is the usual logistic equation with the addition of a cubic term. A , called *Allee parameter* is a new critical population level: if $R/A < 1$ the right hand side of the equation is negative, meaning that the population is doomed to extinction; on the other hand, if $R/A > 1$, the right hand side of the equation is positive, meaning that the population is able to grow and reach the carrying capacity. As we anticipated,

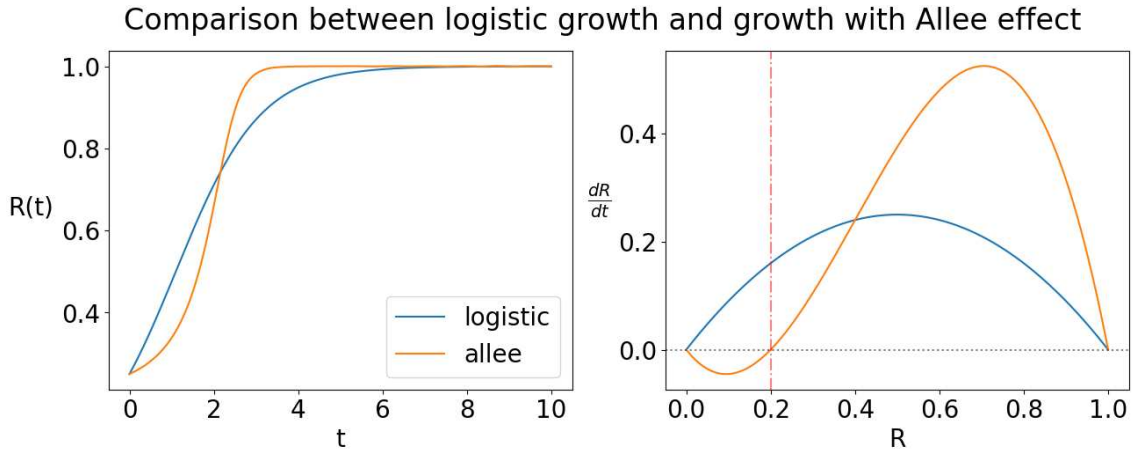


Figure 2.1.: Comparison between logistic growth curve and the growth curve including the Allee effect. Left panel: population density; right panel: derivative of the population density.
Parameters: $K = 1$, $A = 0.2$, $b = 1$.

this is one of the possible ways of describing the Allee effect, which appears with different mechanisms and intensities in nature; however, here we focus on the cubic formulation described by equation (2.1), which gives rise to the so-called *strong* Allee effect (as opposed to the weak Allee effect, characterized qualitatively by the fact that the derivative of the population level is never negative).

Figure 2.1 shows the comparison between the customary logistic growth and the growth curve including the Allee effect.

2.2 The Allee effect in the sustainability model

In (Tu et al., 2025), we integrated the Allee effect in the CPR sustainability model. We did so both in the *replicator dynamics* framework, that is the one in which the consuming agents behave as assumed up to now, and in what we call *knowledge feedback* framework, in which, instead that on individual payoffs, players decide their strategy on the basis of the global level of resource; see the relevant section 2.2.2 for a more precise explanation.

2.2.1 Replicator dynamics

First, we include the Allee effect in the growth rate of the CPR in a naive way, simply replacing the logistic term in the equation for the resource in the HES (1.9), obtaining a new HES:

$$\begin{cases} \frac{dR}{dt} = bR \left[\left(\frac{R}{A} - 1 \right) \left(1 - \frac{R}{K} \right) - (x\hat{e}_C + (1-x)\hat{e}_D) \right] \\ \frac{dx}{dt} = -x(1-x) \frac{R}{K} \end{cases} \quad (2.2)$$

More precisely, applying the microscopic description presented in section 1.2.2, the HES 2.2 is generated by the processes:

1. Offspring of resource:



This represents the ability of the population to reproduce at high densities, when the individuals are able to form groups; that's why we have the term $r + r$ at the left hand side. On the other hand, to take into account the usual overcrowding effect, we also include the e term.

2. Death of resource due to isolation:



Contrary to the standard logistic case, we now introduce explicitly a death process for the resource, in order to model the difficulty in reproducing at low population levels, in agreement with the remarks of the previous section; this is why we have $r + e$ at the left hand side.

Then, we also have the four additional processes dealing with the composition of the community and its interaction with the resources, equations (1.32)-(1.35). As we said, these microscopic processes generate the HES (2.2), where $A = K d/b$.

We now determine the stationary solutions of equation (2.2) and evaluate their

stability, following from the general concepts of linear stability theory of dynamical systems. For simplicity, without loss of generality, we set $K = 1$.

A first solution, which can be easily verified by substitution in the system, is $R = 0, \forall x$. The Jacobian matrix evaluated at $R = 0$ reads:

$$J(R, x)|_{R=0} = \begin{pmatrix} -b(1 + x\hat{e}_C + (1-x)\hat{e}_D) & 0 \\ -x(1-x) & 0 \end{pmatrix} \quad (2.5)$$

which is a lower triangular matrix. From the properties of such matrices, we know that its eigenvalues are simply their diagonal entries; therefore, this solution has an associated null eigenvalue and we must conclude that it's *neutrally stable*: practically, considering a time series resulting from our dynamics that reaches this solution, when R reaches 0 the dynamics on the community stops at whatever value of x was attained in the previous time frame, while the resource was finite. Then, if we perturb the system, poised in such state, adding a quantity of resource, the dynamics resumes, but stops again at a different x when the resource is exhausted. In more formal terms, every solution $R = 0, \forall x$, is a neutrally stable absorbing state.

We next look for the solutions such that $R \neq 0$. From the equation for x in the HES (2.2) we immediately deduce that the stationary values for x are either $x^D = 0$ or $x^C = 1$, namely there is either monodominance of the cooperating or of the defecting strategy. Therefore, we have the following pairs of solutions:

$$R_+^C = \frac{1}{2} \left(1 + A + \sqrt{(1-A)^2 - 4A\hat{e}_C} \right), \quad x^C = 1 \quad (2.6)$$

$$R_-^C = \frac{1}{2} \left(1 + A - \sqrt{(1-A)^2 - 4A\hat{e}_C} \right), \quad x^C = 1 \quad (2.7)$$

$$R_+^D = \frac{1}{2} \left(1 + A + \sqrt{(1-A)^2 - 4A\hat{e}_D} \right), \quad x^D = 0 \quad (2.8)$$

$$R_-^D = \frac{1}{2} \left(1 + A - \sqrt{(1-A)^2 - 4A\hat{e}_D} \right), \quad x^D = 0 \quad (2.9)$$

These solutions exist if:

$$\hat{e}_X \leq \frac{(1-A)^2}{4A} \quad (2.10)$$

where $X = \{C, D\}$, or equivalently, visually, the extraction rates must lie in the region below the curve given by the right hand side of the previous inequality (see

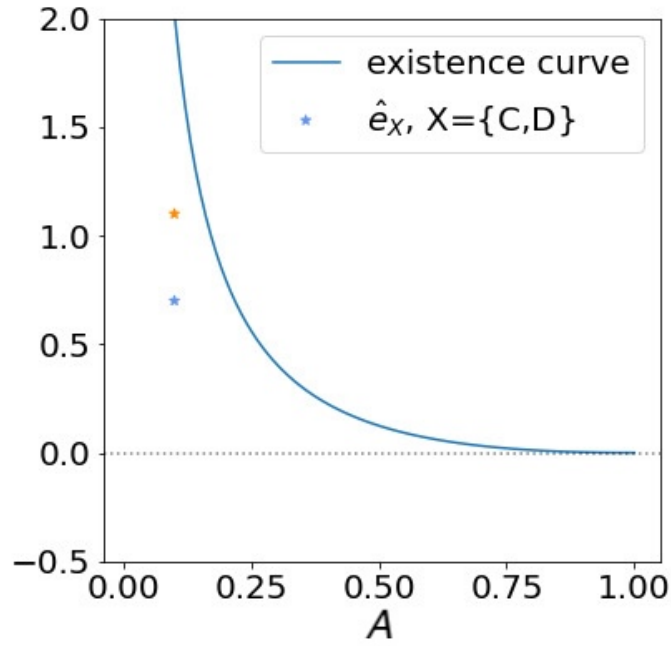


Figure 2.2.: Curve defining the existence condition 2.10, with an example of two extraction rates which both satisfy it (blue=cooperation, orange=defection).

figure 2.2). Notice that the curve is decreasing, that is, for increasing A , we have more stringent conditions on the extraction rates for the resources to remain finite at stationarity, which is agreement with intuition; moreover, because, by definition, $\hat{e}_C < \hat{e}_D$, solutions $R_C^\pm$ have a larger domain of existence with respect to $R_D^\pm$. Remember also that $\hat{e}_D > 1$, as assumed in the previous chapter; hence, there is a critical value of A such that $R_D^\pm$ doesn't exist:

$$\frac{(1 - A_c)^2}{4A_c} = 1 \quad \Rightarrow \quad A_c = 3 - 2\sqrt{2} \simeq 0.17 \quad (2.11)$$

Let us study the stability of such solutions.

As we anticipated, when $R^* = 0, \forall x$, the jacobian associated to the HES has a zero eigenvalue, therefore the solution is only neutrally stable.

An important thing to underline is that there is the possibility of a sustainable solution (in terms of the resource) even when there is mono-dominance of defectors. This is made possible because the resource dynamics embedding the Allee effect, if the resource level is sufficiently high, is faster than the standard logistic dynamics, allowing the resource to be replenished more quickly.

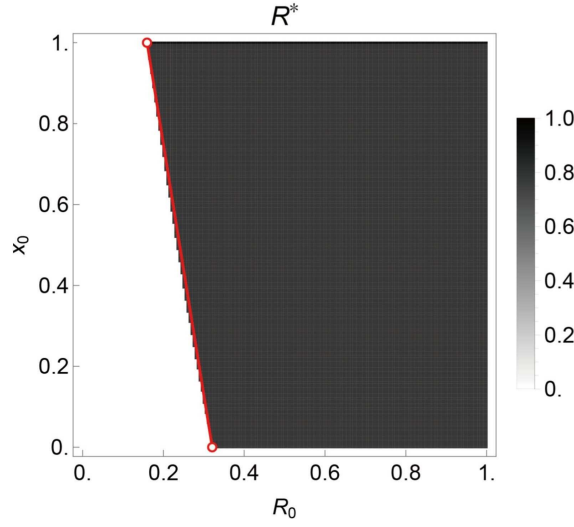


Figure 2.3.: Stationary value of the resource of the HES with Allee effect under resource dynamics versus the initial conditions in x and R . The red curve is our approximating line separating the white and black regions.

When $R^* \neq 0$, $(R_-^C, 1)$, $(R_+^C, 1)$ and $(R_-^D, 0)$ are unstable while $(R_+^D, 0)$ is the only stable solution. Hence, the system shows bistability between the $(R^* = 0, \forall x)$ and the $(R_+^D, 0)$ solutions, depending on the initial conditions (R_0, x_0) .

Figure 2.3 shows the basins of attraction of the two solutions for a sample configuration of parameters satisfying the feasibility condition (2.10). Qualitatively, we see that, if the initial level of resource and the initial fraction of cooperators are sufficiently large, then the system attains the sustainable stationary state; otherwise, the resource is exhausted; in other words, the system exhibits a line of *tipping points* between sustainability and extinction of the resource. In agreement with intuition, there is a tradeoff between the initial level of resource and fraction of cooperators regulating the boundary between the two basins of attraction: the larger the initial resource level, the lower the initial fraction of cooperation ought to be to obtain sustainability.

We now give an estimate of the curve of tipping points separating the basins of attraction. If $x_0 = 0$, then the community doesn't evolve and the stationary value of the resource is R_+^D ; on the other hand, if $x_0 = 1$, the stationary value of the resource R_+^C ; these are the two red dots in figure 2.3. The curve connecting them is simply

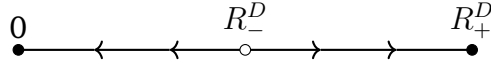


Figure 2.4.: Flow of the HES system with Allee effect in the R direction, given that x is fixed to its absorbing values $x = \{0, 1\}$.

given by the line passing through the points, given by the relation:

$$x_0 = \frac{R_0 - R_+^D}{R_+^C - R_+^D} \quad (2.12)$$

Notice that we were not able to determine the actual boundary curve analytically, hence the above line is simply an approximation; however, we conclude that it is meaningful by visual inspection.

Consider again the stationary solutions and their stability. Let us begin with the case $x_0 = 0$. Since $x = 0$ is absorbing in regards of the composition of the community, there is only a dynamics for the resource. As we already pointed out, the possible stationary values for the resource are either $R^* = 0$ or $R^* = R_{\pm}^D$. Let us also make a step back to the linear stability analysis: another relevant result is $(\lambda_0)_R < 0$, $(\lambda_-^D)_R > 0$ and $(\lambda_+^D)_R < 0$, where the notation indicates the R -components of the eigenvalues associated to the stationary solutions. Hence, the $(R_-^D, 0)$ solution delimits the basins of attraction of the other two, attracting solutions; visually, the flow in the R direction, with x fixed to 1, is shown in figure 2.4. Similar results are obtained in the case $x_0 = 1$, that is we find that the stationary values for R are 0 and $R^* = R_{\pm}^C$ and their associated eigenvalues have R -components such that $(\lambda_0)_R < 0$, $(\lambda_-^C)_R > 0$ and $(\lambda_+^C)_R < 0$. Again, the $(R_-^C, 1)$ is another point that distinguishes the basins of attraction of the other two solutions.

As an approximation for the line separating the basins of attraction of the stationary values of the resource we give the equation of the line passing through the two previous points:

$$x_0 = \frac{R_0 - R_-^D}{R_-^C - R_-^D} \quad (2.13)$$

2.2.2 Knowledge feedback

We now introduce the concept of *knowledge feedback*, which will come back in the following chapter and has been a focus of our research on the topic, in general.

In the *knowledge feedback* setting, consumers don't adopt replicator dynamics for deciding their harvesting strategy, namely they don't compare their individual payoffs and don't adopt with larger probability the strategy of the fittest individual; instead, the probability of switching the strategy is determined solely by the global level of resource R , for all consumers. For simplicity, we assume that the consumers satisfy perfectly this assumption, that is they adopt cooperation with probability 1 if the resource is below the Allee threshold, $R < A$. When $R > A$, we assume that the probability of switching from cooperation to defection is simply proportional to $R - A$, namely we simply care for the fact that it grows with $R - A$ and we choose the simplest form possible.

Practically, this amounts to replace the probabilities of the processes of birth and death of a cooperator, equations (1.34) and (1.35), respectively, with new ones, given by:

$$p^- = \vartheta(R - A) \left(\frac{R - A}{K - A} \right) \quad (2.14)$$

$$p^+ = 1 - p^- \quad (2.15)$$

where, as per the usual notation, $\vartheta(x)$ is the Heaviside step function. Notice that $p^\pm$ is already normalized to 1, as the maximum value of the numerator of p^- is attained when R is maximum, that is when it is equal to the carrying capacity K .

We can obtain the HES with the usual machinery:

$$\begin{cases} \frac{dR}{dt} = bR \left[\left(\frac{R}{A} - 1 \right) \left(1 - \frac{R}{K} \right) - (x\hat{e}_C + (1-x)\hat{e}_D) \right] \\ \frac{dx}{dt} = 1 - x - \vartheta(R - A) \left(\frac{R - A}{K - A} \right) \end{cases} \quad (2.16)$$

As in the replicator dynamics case, there is the possibility of bistability between the sustainable and unsustainable solution, depending on the initial conditions. However, contrary to the previous case, the system doesn't admit such easy-to-

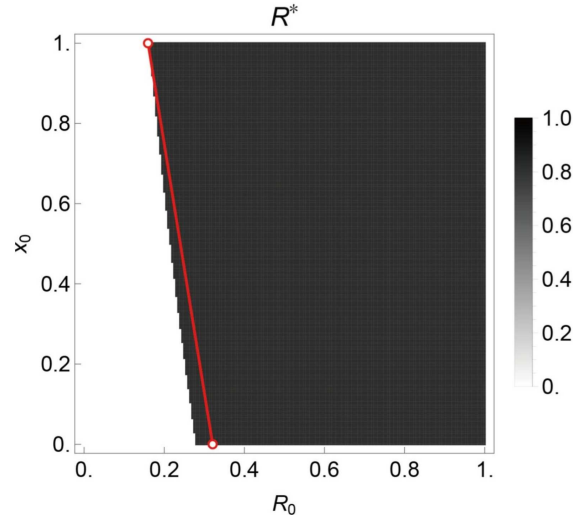


Figure 2.5.: Stationary value of the resource of the HES with Allee effect under knowledge feedback resource dynamics versus the initial conditions in x and R . Notice that the red line is our estimate of the curve separating the basins of attraction of the sustainable and unsustainable solutions for the case of the HES with Allee effect and **replicator dynamics**, while the dots are obtained from the system with **knowledge feedback**.

handle solutions and the condition for the existence of the sustainable solution is quite convoluted; in particular, it reads:

$$\begin{cases} 0 < A < 3 - 2\sqrt{2} \\ 2\sqrt{\frac{(A-1)^2\hat{e}_C}{A}} - A - \frac{1}{A} - \hat{e}_C + \hat{e}_D + 2 < 0 \end{cases}$$

or

$$\begin{cases} 3 - 2\sqrt{2} < A < \frac{3 - \sqrt{5}}{2} \\ 2\sqrt{\frac{(A-1)^2\hat{e}_C}{A}} - A - \frac{1}{A} - \hat{e}_C + \hat{e}_D + 2 < 0 \\ 2\sqrt{\frac{(A-1)^2}{A}} - A - \frac{1}{A} + \hat{e}_C + 1 < 0 \end{cases} \quad (2.17)$$

In figure 2.5, for some value of $\hat{e}_D$ satisfying the previous conditions, we compare the basins of attraction of the sustainable and unsustainable solution, comparing them to the equivalent plots obtained in the case of replicator dynamics. We can see that, $\hat{e}_C$ being the same in the replicator dynamics case and in the knowledge

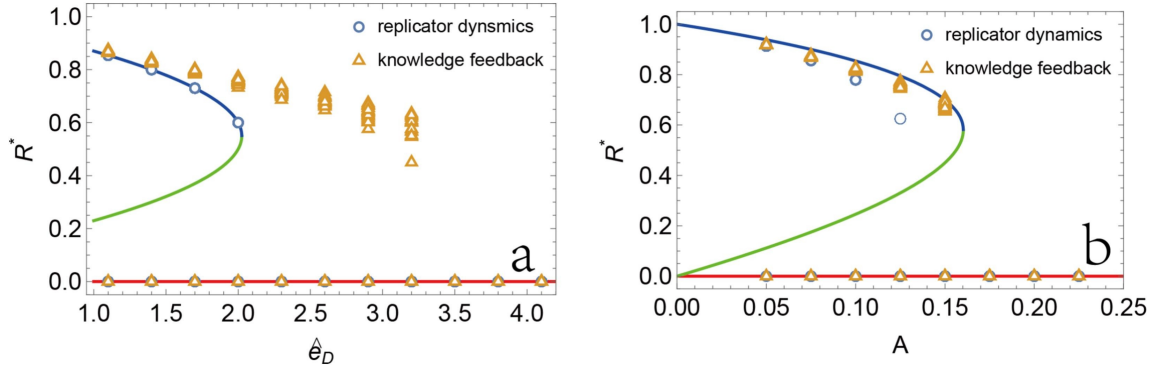


Figure 2.6.: Left panel: stationary value of the resource as a function of $\hat{e}_D$, where $A = 0.1$. Right panel: stationary value of the resource as a function of A , where $\hat{e}_D = 1.5$. The open markers represent numerical simulation results for the model with replicator dynamics and knowledge feedback. The blue, green, red line represent the stable sustainable state, the unstable state, and the stable unsustainable state from analytical derivation of the stationary solutions of the model with replicator dynamics, respectively.

feedback case, the basin of attraction of the sustainable solution is consistently larger in the latter case, in line with our expectations.

The same conclusion can be drawn from figure 2.6. In these figures, we plot the stationary value of the resource for 50 independent simulations of the system, each with random initial conditions. In both panels, $\hat{e}_C$ is fixed. It is evident that knowledge feedback improves the resilience of the sustainable solution. Furthermore, these plots highlight the bistability which characterize the model with both mechanisms: depending on the initial conditions while fixing the other parameters, both the sustainable and unsustainable states can be attained.

2.3 Conclusions and outlook

Our investigation represents a starting point for the study of the influence of the Allee effect in the context of the harvesting of CPRs; however, it is possible to include many modifications to account for more complex phenomena.

Firstly, the definition of the « knowledge feedback » mechanism assumes perfect availability of the information on the global resource level R and perfect rationality on the consumers' side, who never tend to defection if the resource is below the

Allee threshold. We can relax these assumptions: for one, quite simply, the global resource level may not be known exactly; instead, the agents may have a noisy estimate of this quantity, for example because it is estimated by censuses separated by long periods of time. Assuming that it is always known is a simplification made for the sake of analytical tractability. We could also allow for a percentage of « cheaters », namely consumers that declare that they are cooperative, concerning the replicator dynamics, but they actually behave as defectors; or agents that don't precisely follow the desirable behavior of turning to cooperation when the level of resource drops below the Allee threshold. Finally, we could simply assume a less idealized game-theoretical dynamics.

Further developments are essentially the same we pointed out at the end of chapter 1: introduce spatial effects in the distribution of resource, introduce structure in the community of consumers, investigate the case of more resources.

Knowledge feedback and social network dynamics

In this chapter we get back to the topic of *knowledge feedback*, which we introduced in section 2.2.2 in the previous chapter. As we anticipated there, we put much effort in the investigation of such a phenomenon.

We also leave the basic approximation of a well-mixed community of consumers, which we adopted up to now, and we introduce social structure, accounting for well-known societal phenomena such as the formation of *echo chambers* (namely clusters of agents with agreeing opinions that tend to cut interactions with individuals with opposing views), involving it in the dynamics. First, however, as a benchmark, we still study a well-mixed scenario.

First, we introduce the ingredients of the models presented in the rest of the chapter in a qualitative way. Then we define models with increasing complexity which incorporate them and show our findings.

3.1 The effects of knowledge feedback and social structure: a qualitative introduction

As we said in the previous chapter, *knowledge feedback* means that the consumers perform their choice of harvesting strategy on the basis of the global resource level: they will tend to be more aggressive if the resource is abundant, namely the probability with which they adopt defection grows with R ; on the other hand, they will be more cautious if the resource is scarce; see panel (a) of figure 3.1. We are again assuming that there are two strategies only. Quantitatively, these behavioral assumptions are encapsulated in the rates of the processes equivalent to those of equations (1.34) and (1.35), namely in $p^\pm$: $p^- \propto R$ (p^+ is, of course, its complementary: $p^+ = 1 - p^-$). Because we use the same definition of $p^\pm$ in the rest of the chapter,

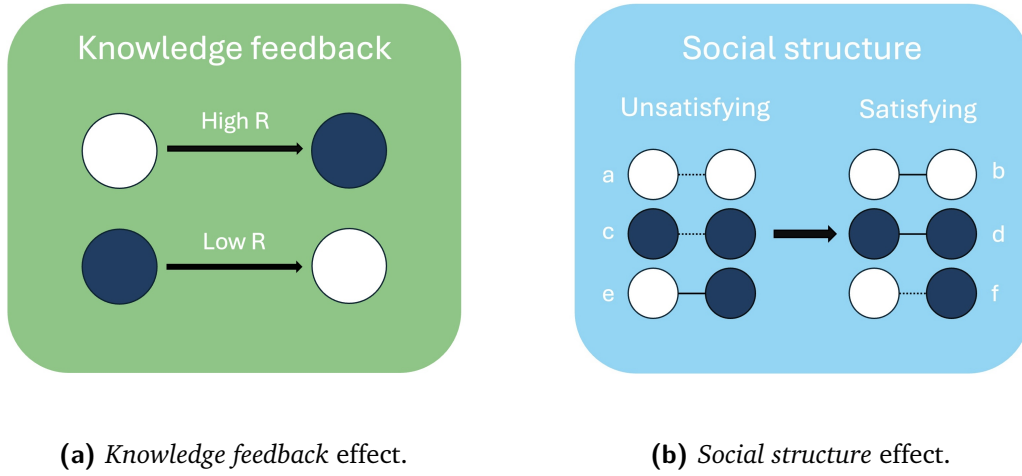


Figure 3.1.: Cartoons for introducing qualitatively the ideas of *knowledge feedback* (left panel) and *social structure* (right panel). We assume that there are two harvesting strategies: cooperators (defectors) are represented by white (blue) circles. A positive (negative) relationship between consumers is represented by a solid (dashed) line.

we write their definitions here:

$$p^- = R' \quad (3.1)$$

$$p^+ = 1 - R' \quad (3.2)$$

where:

$$R' = \frac{R}{K(1 - \hat{e}_C)} \quad (3.3)$$

where the denominator of R' is an effective carrying capacity for the dynamics of the coupled dynamics of resource and consumers. Indeed, consider the usual equation for the resource in our HES-type system, e.g. (1.9); we repeat it here for simplicity:

$$\frac{dR}{dt} = bR \left[\left(1 - \frac{R}{K} \right) - (x\hat{e}_C + (1-x)\hat{e}_D) \right]$$

We can cast in the usual form of the logistic equation by defining an effective birth rate, $b'(t) = (1 - (x(t)\hat{e}_C + (1-x(t))\hat{e}_D))b$, and an effective carrying capacity $K'(t) = (1 - (x(t)\hat{e}_C + (1-x(t))\hat{e}_D))K$, where recall that $\hat{e}_C = \frac{Ne_C}{b}$. Notice that we made explicit the dependence of these effective quantities on time; suppose now,

however, that the community of consumers is entirely dominated by cooperators: in symbols, $x(t) = \bar{x} = 1$, where we highlighted that in this case, namely mono-dominance of a strategy, then the dynamics on the community stops. Then, the effective carrying capacity becomes $K' = K(1 - \hat{e}_C)$; this is the maximum level of resource attainable by the human-environment system, since we assumed that all consumers adopt the more conservative strategy. It follows that $R' \in [0, 1]$.

The second effect consists in introducing some kind of *structure* in the community of consumers. In particular, we regard the consumers as the nodes of a network. The links model the fact that, within a more realistic community, each consumer is in contact with a subset of the other individuals; furthermore, such links can have an associated weight, quantifying a degree of support or friendship between two extractors, which can be thought of as, for instance, companies involved in the extraction of the CPRs of relevance. For simplicity, we dealt with the case of binary links; we can think of the link state as labeling a positive/friendly/supportive relationship or a negative/unfriendly/boycotting one. Assuming again that there are two harvesting strategies, we distinguish *satisfying* and *unsatisfying* pair of nodes from the following ideas:

- Consumers with different strategies tend to develop negative relationships, meaning that they tend to avoid consumers with different views;
- Consumers with the same strategies tend to develop positive relationships, meaning that they tend to form groups with agreeing ideas.

These are behavioral assumptions based on an observed societal phenomenon known as *echo chambers* formation.

Broadly speaking, the community will try to achieve a state in which only satisfying node pairs are present; if it reaches such a state, it would stop evolving. Therefore, satisfying pairs don't evolve anymore; in other words, they are absorbing states for the local dynamics. On the other hand, the community will try to get away from a state in which there are unsatisfying pairs. Such pair connections do have a finite probability of undergoing evolution and become satisfying, therefore they are not absorbing.

Since both the states of the nodes and the links are binary, we have six possible configurations of pairs of nodes, which we label by lower-case letters as in panel (b) of figure 3.1. We now describe individually:

- *Pair type a.* Two cooperators are connected by a negative link. We regard this pair as unsatisfying because the consumers will tend to either develop a positive relationship, having the same attitude in regards of the resource, or one of them will « cede » to temptation and switch to defection.
- *Pair type b.* Two cooperators are connected by a positive link. We regard this pair as satisfying because both consumers adopt the same harvesting strategy and interact positively.
- *Pair type c.* Two defectors are connected by a negative link. Similarly to the case of the pair of type *a*, we regard this pair as unsatisfying because the consumers will tend to either develop a positive relationship, having the same attitude in regards of the resource, or one of them will switch to defection.
- *Pair type d.* Two defectors are connected by a positive link. As in the case of pair type *b*, we regard this pair as satisfying because both consumers adopt the same harvesting strategy and interact positively.
- *Pair type e.* A cooperator and a defector are connected by a positive link. We regard this pair as unsatisfying because the consumers interact positively but have opposite approaches regarding the harvesting of the resource. Therefore, for the pair to become satisfying, either one of them switches to the opposite strategy, or the state of the relationship becomes negative.
- *Pair type f.* A cooperator and a defector are connected by a negative link. We regard this pair as satisfying because the consumers have opposite harvesting strategies and interact negatively, that is, in our interpretation, they are unfriendly towards one another.

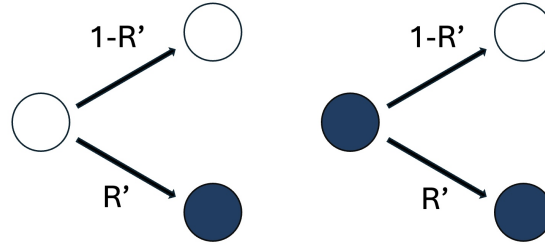


Figure 3.2.: Update rules for the harvesting strategies of the consumers in the case of the « no social structure » model.

Panel (b) of figure 3.1 represents these ideas qualitatively.

We now deal with the specific models we devised for studying these effects.

3.2 Definition of the models

Null model: no social structure

First of all, as a benchmark, we deal with a version of the model without one of the two ingredients, that is without social structure; we refer to this model as the *no social structure* (NS) model. We assume that the resource dynamics obeys the logistic growth, as in chapter 1. The microscopic rates regarding the birth and death of the resource are the usual ones of equations (1.31)-(1.33), while the rates regarding the switch of the harvesting strategies, following the assumptions made in the previous section, are:

$$c \xrightarrow{R'} d \quad (3.4)$$

$$d \xrightarrow{1-R'} c \quad (3.5)$$

Figure 3.2 shows a graphical representation of such processes.

The HES can be derived phenomenologically: the equation for the resource is the usual one, while the equation for the composition of the community can be obtained simply as a balance between gain and loss of cooperators due to the

previous processes:

$$\begin{cases} \frac{dR}{dt} = R(1 - R) - R(x\hat{e}_C + (1 - x)\hat{e}_D) \\ \frac{dx}{dt} = \underbrace{(1 - x)(1 - R')}_{\text{Gain}} - \underbrace{xR'}_{\text{Loss}} = 1 - x - R' \end{cases} \quad (3.6)$$

where we assumed, for the sake of readability, $K = 1$ and $b = 1$. The gain term is simply given by the product of the population of defectors and the probability with which they adopt cooperation, while the loss term is the population of cooperators times the probability with which they adopt defection. The system can also be derived from microscopic grounds with the usual Van Kampen expansion.

Introducing social structure: common remarks for the *quenched* and *evolving* models

As anticipated, we introduce structure in the community. We do this both in a *quenched* way, in which the structure is assigned a priori and doesn't participate in the dynamics, and, conversely, in an annealed way, which we call fully *evolving*. Before dealing with the two individual cases, we introduce a formalism that we used to describe both models; basically, as usual, after defining the microscopic processes characterizing the system, we want to derive an equivalent set of deterministic *rate equations* via the Van Kampen expansion. Since the present models are more intricate than the basic one, which we dealt with in chapter 1, and require a more careful use of the notation, it is worth to retrace the procedure step by step.

Let us consider an homogeneous random network (see (Barabási and Pósfai, 2016), chapter 3) with node degree μ , implying that the total number of links in the network is $L = N\mu/2$ ¹. The state of the system is given by the number of the pair connections of each type, L_i , where $i \in \{a, b, c, d, e, f\}$; we define the densities of pair connections as $\rho_i = L_i/L$ and the state of the system is then given by the vector $\boldsymbol{\rho} = \{\rho_a, \rho_b, \rho_c, \rho_d, \rho_e, \rho_f\}$. A minimal update of the system amounts to change a

¹The following procedure works for Erdős-Renyi random graphs as well in the thermodynamic limit, namely when $\mu \ll N$, though the degree is only fixed on average, implying that L is not known exactly a priori.

component of this vector by one, namely we define the update vectors:

$$\begin{aligned} \mathbf{r}_i^\pm &= \pm\{\delta_{a,i}, \delta_{b,i}, \delta_{c,i}, \delta_{d,i}, \delta_{e,i}, \delta_{f,i}\} \\ \delta_{j,i} &= \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3.7)$$

that is, $\delta_{j,i}$ is the usual Krönecker delta. In terms of the densities we have the state vector $\boldsymbol{\rho} = \mathbf{L}/L$ and the related update vectors $\boldsymbol{\epsilon}_i^\pm = \epsilon \mathbf{r}_i^\pm$, $\epsilon = 1/L$.

Using this notation, we can write the master equation for the probability that the system is in state $\mathbf{L}$ at time t , $P(\mathbf{L}, t)$, as a balance of the probability currents going in and out of such a state:

$$\frac{\partial P(\mathbf{L}, t)}{\partial t} = \sum_i \left(\sum_{\mathbf{r}_i^\pm} \left(W(\mathbf{L} - \mathbf{r}_i^\pm, \mathbf{r}_i^\pm, t) P(\mathbf{L} - \mathbf{r}_i^\pm, t) - W(\mathbf{L}, \mathbf{r}_i^\pm, t) P(\mathbf{L}, t) \right) \right) \quad (3.8)$$

where $W(\mathbf{L} - \mathbf{r}_i^\pm, \mathbf{r}_i^\pm, t)$ is the probability per unit time of jumping from state $\mathbf{L} - \mathbf{r}_i^\pm$ to state $\mathbf{L}$. These, in many cases, are tied to the transition rates for the microscopic variables by a scaling law of the form (see (Hänggi, 1978)):

$$W(\mathbf{L} - \mathbf{r}_i^\pm, \mathbf{r}_i^\pm, t) = f(L) L \left(\psi_0(\boldsymbol{\rho} - \boldsymbol{\epsilon}_i^\pm, \boldsymbol{\epsilon}_i^\pm, t) + \epsilon \psi_1(\boldsymbol{\rho} - \boldsymbol{\epsilon}_i^\pm, \boldsymbol{\epsilon}_i^\pm, t) + \dots \right) \quad (3.9)$$

$f(L)$ is actually an innocuous factor, since it can be absorbed in the time variable, therefore, for simplicity, we assume $f(L) = 1$.

We pass to intensive variables exploiting the constraint that the probabilities of being in a certain state must be the same regardless of whether we describe it in terms of L_i or ρ_i variables: $P(\mathbf{L}, t) = P(\boldsymbol{\rho}, t)$; therefore, the master equation (3.8) becomes:

$$\begin{aligned} \epsilon \frac{\partial P(\boldsymbol{\rho}, t)}{\partial t} &= \sum_i \left(\sum_{\boldsymbol{\epsilon}_i^\pm} \left(\left(\psi_0(\boldsymbol{\rho} - \boldsymbol{\epsilon}_i^\pm, \boldsymbol{\epsilon}_i^\pm, t) + \epsilon \psi_1(\boldsymbol{\rho} - \boldsymbol{\epsilon}_i^\pm, \boldsymbol{\epsilon}_i^\pm, t) + \dots \right) P(\boldsymbol{\rho} - \boldsymbol{\epsilon}_i^\pm, t) + \right. \right. \\ &\quad \left. \left. - \left(\psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^\pm, t) + \epsilon \psi_1(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^\pm, t) + \dots \right) P(\boldsymbol{\rho}, t) \right) \right) \end{aligned} \quad (3.10)$$

Let us assume the function W admits a Taylor expansion in powers of ϵ with respect to the first argument; then, collecting the terms, from equation (3.10) we get:

$$\frac{\partial P(\boldsymbol{\rho}, t)}{\partial t} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \epsilon^{m-1} \sum_{\{j_1\}} \dots \sum_{\{j_m\}} \frac{\partial^m}{\partial \rho_{j_1} \dots \partial \rho_{j_m}} A_{j_1 \dots j_m}^{(m)}(\boldsymbol{\rho}, t) P(\boldsymbol{\rho}, t) \quad (3.11)$$

where $j_n \in \{a, b, c, d, e, f\} \forall n, n = 1, \dots, m$, $\sum_{\{j_n\}}$ means « sum over all possible values of j_n » and:

$$A_{j_1 \dots j_m}^{(m)}(\boldsymbol{\rho}, t) = \sum_i \left(\sum_{\epsilon_i^{\pm}} (r_{j_1} \dots r_{j_m}) (\psi_0(\boldsymbol{\rho}, \epsilon_i^{\pm}, t) + \epsilon \psi_1(\boldsymbol{\rho}, \epsilon_i^{\pm}, t) + \dots) \right) \quad (3.12)$$

Since $\mathbf{r}_i^{\pm}$ are orthogonal unit vectors, the product $r_{j_1} \dots r_{j_m}$ is non-zero only if all the indices are the same; therefore:

$$\sum_{\{j_1\}} \dots \sum_{\{j_m\}} \frac{\partial^m}{\partial \rho_{j_1} \dots \partial \rho_{j_m}} \sum_i \sum_{\epsilon_i^{\pm}} (r_{j_1} \dots r_{j_m}) \langle \cdot \rangle \longrightarrow \sum_{\{j\}} \left(\frac{\partial}{\partial \rho_j} \right)^m \sum_{\epsilon_j^{\pm}} (\pm 1)^m \langle \cdot \rangle \quad (3.13)$$

Therefore, the master equation (3.11) simplifies to:

$$\frac{\partial P(\boldsymbol{\rho}, t)}{\partial t} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \epsilon^{m-1} \sum_{\{j\}} A_j^{(m)}(\boldsymbol{\rho}, t) P(\boldsymbol{\rho}, t) \quad (3.14)$$

where:

$$A_j^{(m)}(\boldsymbol{\rho}, t) = \sum_{\epsilon_j^{\pm}} (\pm 1)^m (\psi_0(\boldsymbol{\rho}, \epsilon_j^{\pm}, t) + \epsilon \psi_1(\boldsymbol{\rho}, \epsilon_j^{\pm}, t) + \dots) \quad (3.15)$$

Truncating the expansion at the $m = 2$ order corresponds to stop at first order in $\epsilon = 1/L$; therefore, this amounts to perform the usual Van Kampen system-size expansion and, from the master equation (3.14), we derive a Fokker-Planck equation:

$$\frac{\partial P(\boldsymbol{\rho}, t)}{\partial t} = - \sum_{\{j\}} \frac{\partial}{\partial \rho_j} A_j^{(1)}(\boldsymbol{\rho}, t) P(\boldsymbol{\rho}, t) + \frac{\epsilon}{2} \sum_{\{j\}} \left(\frac{\partial}{\partial \rho_j} \right)^2 A_j^{(2)}(\boldsymbol{\rho}, t) P(\boldsymbol{\rho}, t) \quad (3.16)$$

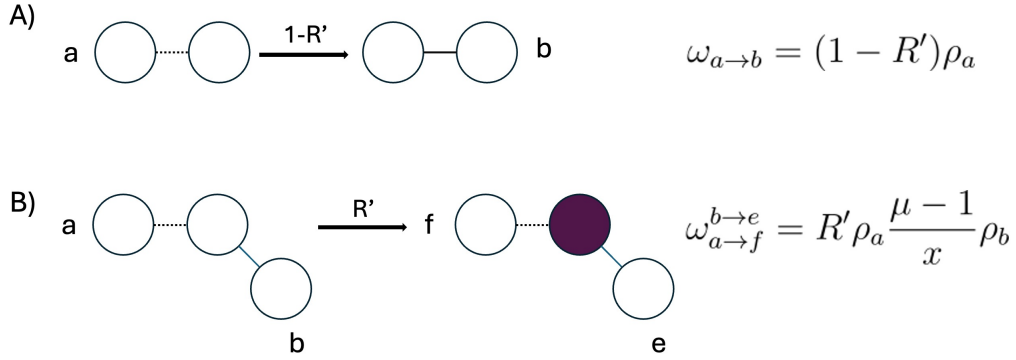


Figure 3.3.: Panel A: probability per unit time of the direct update of pair connection of type a to pair connection of type b .
 Panel B: probability per unit time of the collateral update of pair connection of type b to pair connection of type e following from the direct update of pair connection of type a to pair connection of type f .

where:

$$A_j^{(1)}(\boldsymbol{\rho}, t) = \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^+, t) - \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^-, t) \quad (3.17)$$

$$A_j^{(2)}(\boldsymbol{\rho}, t) = \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^+, t) + \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^-, t) \quad (3.18)$$

As usual, from the Fokker-Planck equation, we can write the equivalent Langevin equation and then, discarding the stochastic term, which is less relevant the more we approach the thermodynamic limit, we end up with deterministic rate equations for the densities of pair types:

$$\frac{d\rho_j}{dt} = \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^+, t) - \psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^-, t) \quad (3.19)$$

The functions $\psi_0(\boldsymbol{\rho}, \boldsymbol{\epsilon}_i^\pm, t)$ are the probabilities per unit time that the system jumps out of the $\boldsymbol{\rho}$ state at time t , where the jump involves the single i -th component of $\boldsymbol{\rho}$ which is changed by the « minimal » possible quantity, ϵ . Such jumps can be either *direct* or *collateral*: we say that the direct updates for a given ρ_j are the changes that actually change a pair connection of type j ; such changes can happen only if

the pair connection is unsatisfying, by definition. As an example, consider the link update from a pair connection of type a to type b (shown in panel A of figure 3.3); such process appears as a loss term in the equation for ρ_a and a gain term for ρ_b . We denote its probability per unit time as $\omega_{a \rightarrow b}$.

As for the writing of such quantity, it is simply given by the product of the rate of the $a \rightarrow b$ transition times the probability of picking a pair connection of type a :

$$\omega_{a \rightarrow b} = (1 - R')\rho_a \quad (3.20)$$

Collateral updates, instead, follow from the direct update of a node delimiting another another, adjacent pair connection. For instance, consider a triplet composed by a pair connection of type a and a pair connection of type b . Since the pair of type b is satisfying, it is absorbing. However, the pair of type a can change; in particular, one of the allowed exit processes consists in a node update from cooperator to defector, namely in changing from type a to type f . Suppose that such node is the one that is connected to the pair connection of type b ; then, this one will also be updated, becoming a pair connection of type e (this event is shown in panel B of figure 3.3). Such process constitutes a loss for ρ_a and ρ_b and a gain for ρ_e and ρ_f . We denote its probability per unit time as $\omega_{a \rightarrow f}^{b \rightarrow e}$: the subscript indicates the direct update, while the superscript indicated the collateral one.

As for the writing of this quantity, notice the following relationships between the densities of pair connections and the density of cooperating agents x in a homogeneous random graph:

- The pair connections of type a and b are the only ones involving two cooperators; if n is the number of cooperating nodes, the number of links between them is $n(n-1)p/2$, where $p = \mu/N$ is the connection probability defining the topology of the random network. The total number of links, instead, is $N(N-1)p/2$. Of course, these estimates are approximate because k is fixed *on average*; however, they become more accurate the closer we get to the thermodynamic limit, in which $n, N \gg 1$. Finally, we have the relationship:

$$\rho_a + \rho_b \simeq \frac{n(n-1)}{N(N-1)} \xrightarrow{n, N \gg 1} x^2 \quad (3.21)$$

- The pair connections of type c and d are the only ones involving two defectors; the number of links between them is $(N - n)(N - n - 1)p/2$. We have:

$$\rho_c + \rho_d \simeq \frac{(N - n)(N - n - 1)}{N(N - 1)} \xrightarrow{n, N \gg 1} (1 - x)^2 \quad (3.22)$$

- The remaining links must make up the remaining two types of pair connections, e and f :

$$\rho_e + \rho_f \simeq \frac{N(N - 1) - (N - n)(N - n - 1) - n(n - 1)}{N(N - 1)} \xrightarrow{n, N \gg 1} 2x(1 - x) \quad (3.23)$$

Hence, putting relationships (3.21)-(3.23) together, we get equations for the density of cooperating agents x and its complementary, which we call ϕ for brevity:

$$x = \rho_a + \rho_b + \frac{\rho_e + \rho_f}{2} \quad (3.24)$$

$$\phi = \rho_c + \rho_d + \frac{\rho_e + \rho_f}{2} \quad (3.25)$$

Now, the probability per unit time $\omega_{a \rightarrow f}^{b \rightarrow e}$ is given by the rate of the $a \rightarrow f$ process times the probability of picking an a pair times the probability that this is connected to a b pair; the latter quantity is proportional to the number of neighbors of the relevant node, minus one because one of them is part of the starting a pair, over the total fraction of cooperating nodes x , times the density of having a positive link between two cooperators, ρ_b ; in symbols:

$$\omega_{a \rightarrow f}^{b \rightarrow e} = R' \rho_a \frac{\mu - 1}{x} \rho_b \quad (3.26)$$

One can write each process and write the associated probability per unit time as just illustrated.

Quenched social structure

We now introduce structure in the community; as a first step, we do so in a *quenched* way, namely we assign a priori the network realization both regarding its

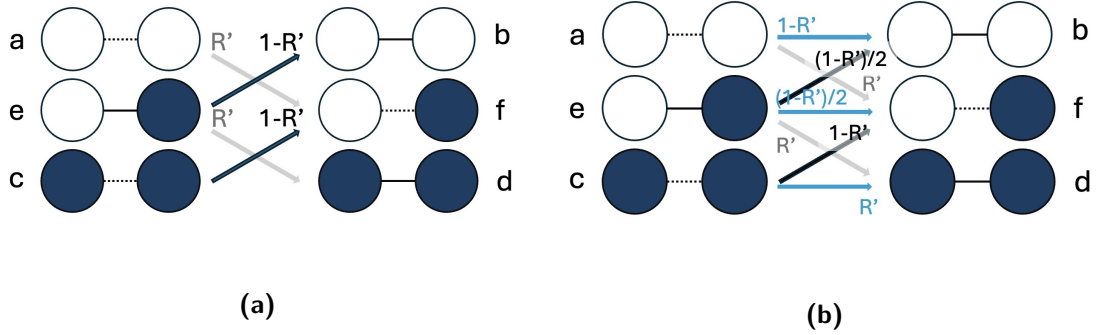


Figure 3.4.: Update rules for the *quenched* (left) and *fully evolving* (right) models with knowledge feedback and social structure. The arrows and the relevant rates are written in different colors only for the sake of readability.

topology and the configuration of positive/negative link states, and this remains fixed in the dynamics; we refer to this model as the *quenched binary state* (QBS) model. The update rules are shown in panel (a) of figure 3.4. Putting together the concepts introduced in the previous section, we get the following system of deterministic rate equations:

$$\begin{aligned}
 \frac{d\rho_a}{dt} &= -R'\rho_a + (\mu - 1) \left[-\frac{R'}{x}(\rho_a + \rho_e)\rho_a + \frac{1 - R'}{2\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_f \right] \\
 \frac{d\rho_b}{dt} &= \frac{1 - R'}{2}\rho_e + (\mu - 1) \left[-\frac{R'}{x}(\rho_a + \rho_e)\rho_b + \frac{1 - R'}{2\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_e \right] \\
 \frac{d\rho_c}{dt} &= (1 - R')\rho_c + (\mu - 1) \left[\frac{R'}{2x}(\rho_a + \rho_e)\rho_f - \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_c \right] \\
 \frac{d\rho_d}{dt} &= R'\rho_e + (\mu - 1) \left[\frac{R'}{2x}(\rho_a + \rho_e)\rho_e - \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_d \right] \\
 \frac{d\rho_e}{dt} &= \frac{1 + R'}{2}\rho_e + (\mu - 1) \left[\frac{R'}{x}(\rho_a + \rho_e) \left(\rho_b - \frac{\rho_e}{2} \right) + \right. \\
 &\quad \left. + \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \left(\rho_d - \frac{\rho_e}{2} \right) \right] \\
 \frac{d\rho_f}{dt} &= R'\rho_a + (1 - R')\rho_c + (\mu - 1) \left[\frac{R'}{x}(\rho_a + \rho_e) \left(\rho_a - \frac{\rho_f}{2} \right) + \right. \\
 &\quad \left. + \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \left(\rho_c - \frac{\rho_f}{2} \right) \right] \quad (3.27)
 \end{aligned}$$

where x and ϕ are given by equations (3.24) and (3.25), respectively and the dynamics of R is given, as usual, by the first equation of the system (3.6), recalling that the normalized level of R , R' , is defined as in equation (3.3).

Evolving social structure

We then allow the society of consumers to fully evolve according to the rules shown in figure (b), obtaining what we call the *evolving binary strategies* (EBS) model. The rate equations become:

$$\begin{aligned}
\frac{d\rho_a}{dt} &= -\rho_a + (\mu - 1) \left[-\frac{R'}{x}(\rho_a + \rho_e)\rho_a + \frac{1 - R'}{2\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_f \right] \\
\frac{d\rho_b}{dt} &= (1 - R') \left(\rho_a + \frac{\rho_e}{2} \right) + (\mu - 1) \left[-\frac{R'}{x}(\rho_a + \rho_e)\rho_b + \frac{1 - R'}{2\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_e \right] \\
\frac{d\rho_c}{dt} &= -\rho_c + (\mu - 1) \left[\frac{R'}{2x}(\rho_a + \rho_e)\rho_f - \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_c \right] \\
\frac{d\rho_d}{dt} &= R'(\rho_c + \rho_e) + (\mu - 1) \left[\frac{R'}{2x}(\rho_a + \rho_e)\rho_e - \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \rho_d \right] \\
\frac{d\rho_e}{dt} &= -\rho_e + (\mu - 1) \left[\frac{R'}{x}(\rho_a + \rho_e) \left(\rho_b - \frac{\rho_e}{2} \right) + \right. \\
&\quad \left. + \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \left(\rho_d - \frac{\rho_e}{2} \right) \right] \\
\frac{d\rho_f}{dt} &= R'\rho_a + (1 - R') \left(\rho_c + \frac{\rho_e}{2} \right) + (\mu - 1) \left[\frac{R'}{x}(\rho_a + \rho_e) \left(\rho_a - \frac{\rho_f}{2} \right) + \right. \\
&\quad \left. + \frac{1 - R'}{\phi} \left(\rho_c + \frac{\rho_e}{2} \right) \left(\rho_c - \frac{\rho_f}{2} \right) \right]
\end{aligned} \tag{3.28}$$

Again, the dynamics of R follows the first equation of the system (3.6) and the normalized level of R , R' , is defined as in equation (3.3).

We investigated the models both analytically, starting from the deterministic equations, and computationally. In this latter case, we adopted the following procedure:

1. **Update the community.** A dynamical unit of the community, namely a node

in the NS model or a link (identifying a pair connection) in the QBS and EBS models, is chosen randomly. It is updated according to the relevant update rules as per the Gillespie algorithm (Gillespie, 1976) (see section A.2 of appendix A).

2. **Update the resource level.** After the composition of the community is updated, the new level of resource is calculated via the simple Euler algorithm (see (Butcher, 2016), chapter 2):

$$R(t + \Delta t) = R(t) + \Delta t \left(R(t)(1 - R(t)) - R(t)(\hat{e}_C x(t) + \hat{e}_D(1 - x(t))) \right) \quad (3.29)$$

where Δt is the inverse of the number of elementary units of the community; simply put, $\Delta t = 1/N$ for the NS model and $\Delta t = 1/L$ for the QBS and EBS models. These choices follow from the fact that the time unit must be the time interval required for an update of the whole community.

3.3 Results

No social structure model

The stationary solution of the system (3.6) are:

$$R'^* = 0 \quad x^* = 1 \quad (3.30)$$

$$R'^* = \frac{1 - \hat{e}_C}{1 - \hat{e}_C + \hat{e}_D} \quad x^* = \frac{\hat{e}_D}{1 - \hat{e}_C + \hat{e}_D} \quad (3.31)$$

The first solution is trivial: if there is no resource, the community converges to cooperation, since the probability of each consumers to adopt defection is 0. Linear stability analysis also shows that such solution is unstable, because one of the eigenvalues of the associated jacobian is always positive.

On the other hand, the second solution is stable: therefore, unless the system is initialized in the pathological $R = 0$ state, the resource is always sustainable at stationarity. Furthermore, we find that $x^* \geq 0$ for any value of $\hat{e}_C$ and $\hat{e}_D$ satisfying the usual constraints (1.2): in this model, at stationarity, cooperators

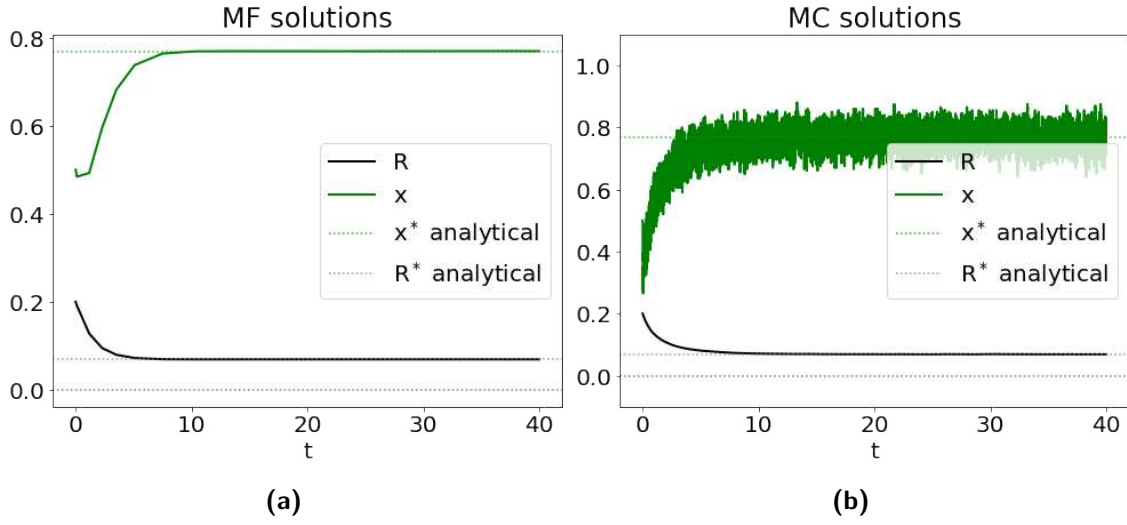


Figure 3.5.: Mean field (left panel) and stochastic (right panel) time series for x and R in the NS model. The model parameters are common for both approaches. The plot for the stochastic simulation shows a single repetition.

always constitute the largest part of the community. In figure 3.5 we show the mean field solution and the stochastic simulation of the model. There is perfect agreement between the two methods.

Quenched end evolving social structure

We treat together these cases, because the main results are common to both. We simulated the system for three different sizes, $N = 40, 400, 4000$, each time initialized on an independent realization of an Erdős-Renyi graph, defining as initial conditions x_0 and l_0 , the initial fraction of cooperating nodes and the initial fraction of positive links respectively, the specific node and link states assigned randomly. Typical time series are shown in figure 3.6 (they are obtained for the EBS model but are completely analogous to those coming from the QBS model). We also plot the stationary values of the fraction of cooperators and of the normalized resource level versus the average degree of the random network, see figure 3.7. We find that, for increasing μ , $\langle x \rangle^{st}$ increases and, most notably, $\langle R' \rangle^{st}$ transitions from a zero to a non-zero value, namely from a *depleted* to a *repleted* phase. The main difference between the two cases is the nature of the transition, which seems continuous in the QBS case and 1st-order in the EBS case; however, we leave the

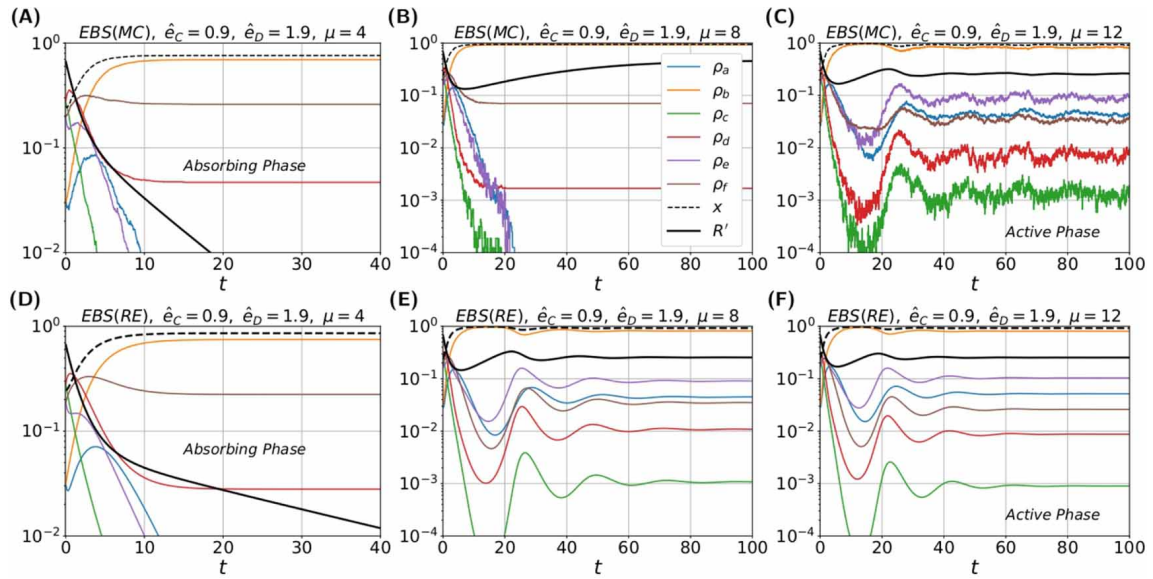


Figure 3.6.: Time series for the EBS model: stochastic (panels A, B, C in the upper row) and deterministic (panels D, E, F in the lower row). From left to right, the mean degree of the network increases; the other model parameters are fixed. Panels B and E show the discrepancy between the two approaches near the phase transition: even if all model parameters are the same, the deterministic model reaches the active stationary state, while the stochastic one reaches an absorbing but sustainable steady state. Parameters are indicated in the title of each panel.

characterization of the transitions for future work. The deterministic results are recovered well for small and large values of μ , especially for increasing size; close to the phase transition, instead, the results are only qualitatively similar, which is typical of numerical simulations of phase transitions. See, for instance, panels B and E of figure 3.6: even if the two plots are obtained with the same model parameters, the deterministic model reaches the active stationary state, while the stochastic one reaches an absorbing but sustainable steady state attains an absorbing, yet sustainable state, which is never found deterministically.

We also tested the model for different network topologies, namely *scale-free* Barabási-Albert (Barabási and Albert, 1999) and *small-world* Watts-Strogatz (Watts and Strogatz, 1998) networks. As shown in figure 3.8, all topologies exhibit the depletion-repletion phase transition with network degree; the critical value of μ varies. In particular, in the case of Barabási-Albert graphs it is shifted to larger values: such graphs are also the most heterogeneous ones, therefore we argue that

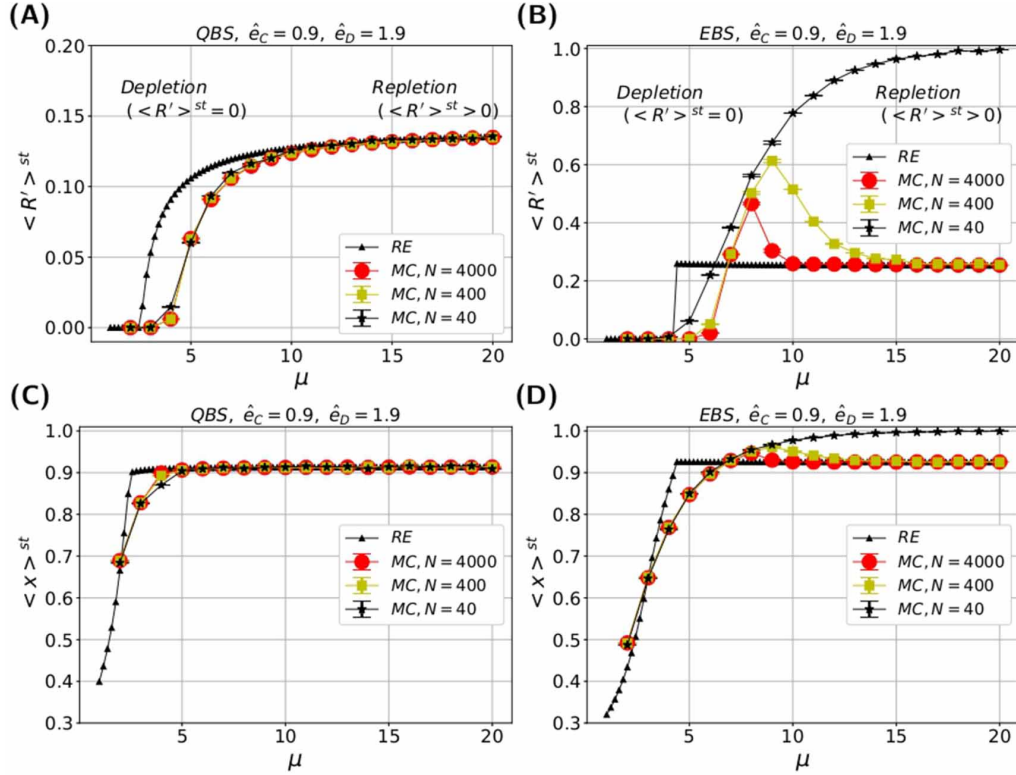


Figure 3.7.: Stationary values of $\langle x \rangle^{(st)}$ and $\langle R' \rangle^{(st)}$ for the QBS (panels A and C in the left column) and EBS models (panels B and D in the right column). We show the results for the deterministic integration of the equations (black line with triangular markers) and for numerical simulations with different system sizes. Both models show a transition between a depleted and repleted phase (panels A, B). Parameters are indicated in the title of each panel.

the hubs in the network facilitate the depletion of resource.

Another finding is that the depleted-repleted phase transition is accompanied by a dynamical phase transition. We mean that, for low μ , the community reaches an absorbing state, namely, as we said in the previous section, one in which all the pair connections are satisfying. Instead, for large μ , the community remains in an active or frustrated state: though on average its related observables don't change, we do have microscopic updates because we have finite stationary values of the densities of unsatisfying pair connections. Again, the integration of the analytical rate equations and the stochastic simulations agree well for values of μ far from the phase transition, whereas, close to the critical value of μ , there is some discrepancy; we again argue that this is a manifestation of strong fluctuations close

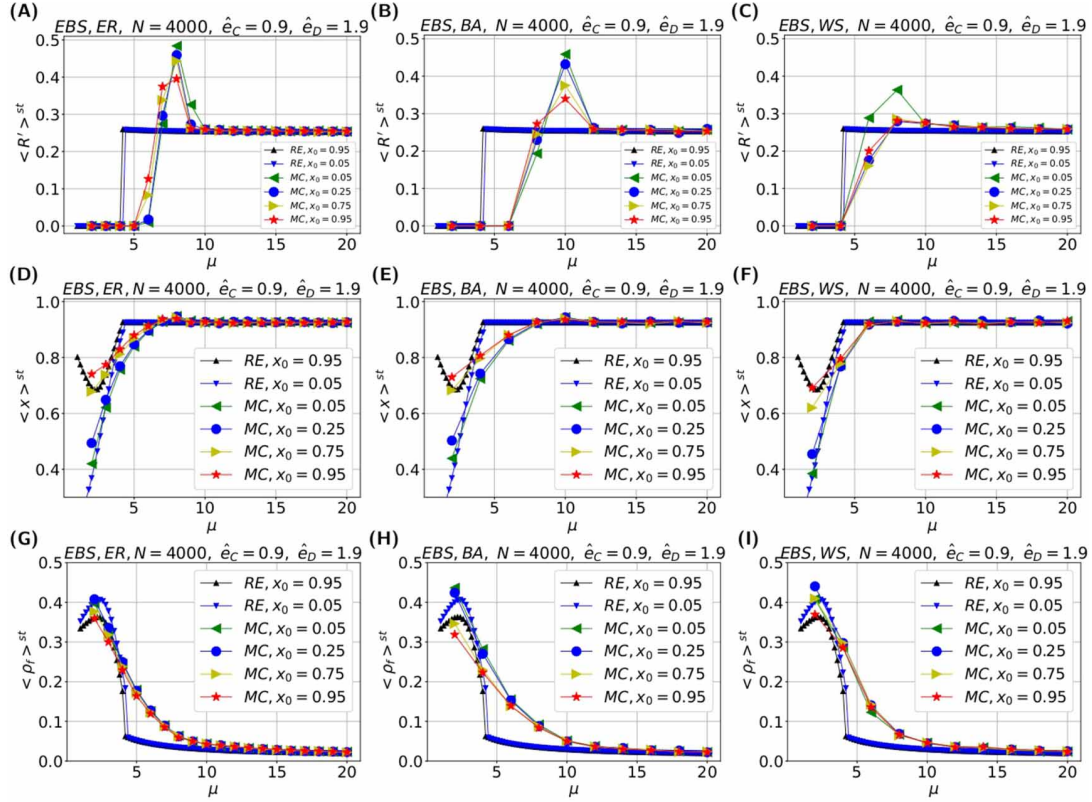


Figure 3.8.: Effects of the network topology on the stationary values of fraction of cooperators, resource level and ρ_f . Shown are: $\langle x \rangle^{(st)}$ as a function of the network degree for the three tested network topologies: Erdős-Rényi, Barabási-Albert and Watts-Strogatz (panels A, B and C in the first row); $\langle R' \rangle^{(st)}$ as a function of the network degree for the three tested network topologies (panels D, E and F in the second row); $\langle \rho_f \rangle^{(st)}$ as a function of the network degree for the three tested network topologies (panels G, H and I in the second row). Parameters are indicated in the title of each panel.

to criticality, but we leave a more precise investigation for future work. The repleted stationary state is robust with respect to the initial conditions; on the other hand, the depleted stationary state isn't, in the sense that it is defined by $R^{st} = 0$, all densities of unsatisfying pair connections are 0, $\rho_a^{st} = \rho_c^{st} = \rho_e^{st} = 0$, but the densities of satisfying pair connections are in whatever combination; this is analogous to the unsustainable neutrally stable state which emerges in all iterations of the model. In order to characterize this transition, we define a measure of polarization. We argue that polarization is not only tied to the fact that the consumers have different harvesting strategies, forming sub-populations of cooperators and defectors, but should also take into account the kind of relationship that exists between antagonist

agents: not only agents should separate in different sub-populations, but they should also develop negative relationships with the components of the opposing sub-population. Hence, we measure the polarization with the product of the density of consumers connected to other agents with different strategy, $2(\rho_e + \rho_f)^2$, times the fraction of negative links among the ones that connect consumers with opposing strategies, $\rho_f/(\rho_e + \rho_f)$:

$$r = 2\rho_f \quad (3.32)$$

The bottom row of figure 3.8 (panels G, H and I) shows ρ_f versus the network degree μ . It shows a decreasing trend, as expected. Importantly, the critical degree signaling this transition is the same characterizing the depleted-repleted one; the equivalence is more precise in the mean field case while it is blurred in the stochastic case. Again, this result is consistent with respect to the topology of the network. We also find that the stationary value of ρ_f depends on the initial conditions in the depleted phase (error bars are not shown in figure 3.8), in line with the fact that the depleted phase corresponds to a dynamical absorbing phase where the final values of the densities of pair connections depends on the initial conditions. Instead, in the repleted/active phase, ρ_f approaches consistently a finite value.

3.4 Conclusions and outlook

The main conclusions we are able to draw are that knowledge feedback alone allows resource sustainability, as shown by the NS model. In this picture, consumers self-organize to adopt cooperation, that is reduce the overall extraction rate, when the global level of resource is low, preventing the tragedy of the commons.

The effect of social structure is more complex: in both the QBS and EBS models there is a transition between a depleted and repleted phase driven by the connectivity of the society. This transition also corresponds to an absorbing-active one in the society of the consumers.

We can trace a parallelism between our findings and the well-known *complexity-stability paradox* (May, 2001), which states, collecting many empirical and analyt-

²The factor 2 comes in because the ρ_i s can be regarded as densities of links, so, in order to pass to densities of nodes, we have to include it.

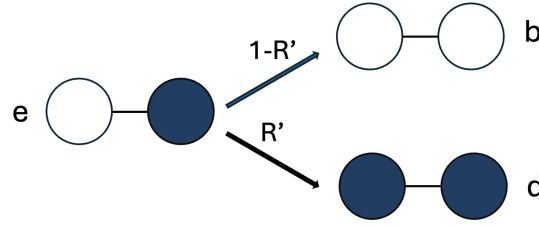


Figure 3.9.: Dynamical update rules for the SLS model with knowledge feedback and social structure.

ical conclusions, that the more an ecosystem is complex, the less it is capable to sustain a stable state. In our case, the NS model always allows sustainability, while this is not granted in the QBS and EBS models.

Future work should address some of the points we haven't investigated thoroughly yet, which we pointed out in the discussion, namely the characterization of the phase transitions in the QBS and EBS models with the non-trivial numerical issues it implies. Other lines of investigation are essentially the same as those we pointed out in section 2.3, regarding the sustainability model with Allee effect: we could replace the knowledge feedback-driven game-theoretical dynamics with a less idealized one, allowing for individualistic behaviors; we could generalize the model to more resources; we could introduce spatial effects in the population of the resource as well.

Another idea is to devise different types of social interactions. Actually, we are currently working in this direction and we show here some of our preliminary observations; we put this in the present section because the results haven't been collected in a publication yet and some of them have to be investigated more thoroughly.

As an intermediate step between the NS and QBS and EBS models, in terms of complexity, we devise the *single link state* (SLS) model. In this model there is the social network as well, but the links have no quality. Therefore, in comparison with the QBS and EBS models, only three types of pair connection exists, namely *b*, *d* and *e*, *b* and *d* being satisfying and *e* being the only unsatisfying one, and all microscopic processes involving link updates are removed. The dynamical update

rules are shown in figure 3.9. The system of rate equations analogous to equations (3.27) or (3.28) is:

$$\begin{aligned}
\frac{d\rho_b}{dt} &= (1 - R')\rho_e + (\mu - 1) \left[-\frac{R'}{x}\rho_e\rho_b + \frac{1 - R'}{\phi}\frac{\rho_e^2}{2} \right] \\
\frac{d\rho_d}{dt} &= R'\rho_e + (\mu - 1) \left[\frac{R'}{2x}\rho_e^2 - \frac{1 - R'}{\phi}\rho_e\rho_d \right] \\
\frac{d\rho_e}{dt} &= -\rho_e + (\mu - 1) \left[\frac{R'}{x}\rho_e \left(\rho_b - \frac{\rho_e}{2} \right) + \frac{1 - R'}{\phi}\rho_e \left(\rho_d - \frac{\rho_e}{2} \right) \right] \\
x &= \rho_b + \frac{\rho_e}{2} \\
\phi &= \rho_d + \frac{\rho_e}{2} = 1 - x
\end{aligned} \tag{3.33}$$

Again, the dynamics of R follows the first equation of the system (3.6) and the normalized level of R , R' , is defined as in equation (3.3). Figure 3.10 shows some typical time series for the model. As in the case of the QBS and EBS models, the deterministic and stochastic approaches give the same results. The phenomenon of the depleted-repleted phase transition, however, is not that trivial. Numerically, for small degree, we obtain both depleted and replenished solutions, as shown in figure 3.11. Considering the depleted ones, we found that the critical degree is as small as $\mu_c < 2$, smaller than the one of the QBS and EBS given the same extraction rates, namely the same pressure of the individual consumers on the resource. Therefore, networks leading to the depleted phase are sparse. This implies that the role of isolated nodes, which will inevitably be present when we generate a large enough random network with such low average degree, is impactful, because they do not participate in the dynamics, according to the rules of figure 3.9. This also means that they retain their initial strategy, which is assigned randomly as in the QBS and EBS model, forever. This also explains why, performing different simulations of the model with such sparse networks, we see that the two absorbing solutions coexist, in the sense that both can be attained depending on the initial conditions. An idea to tackle the issue of disconnected nodes is to integrate the rules of the SLS model with other ones specific for isolated nodes, for instance those of the NS model; however, this was only briefly touched upon and needs more careful study. The most interesting feature of this model, in our opinion, is that it exhibits the phenomenon of *stochastic amplification*: while in the deterministic model, the

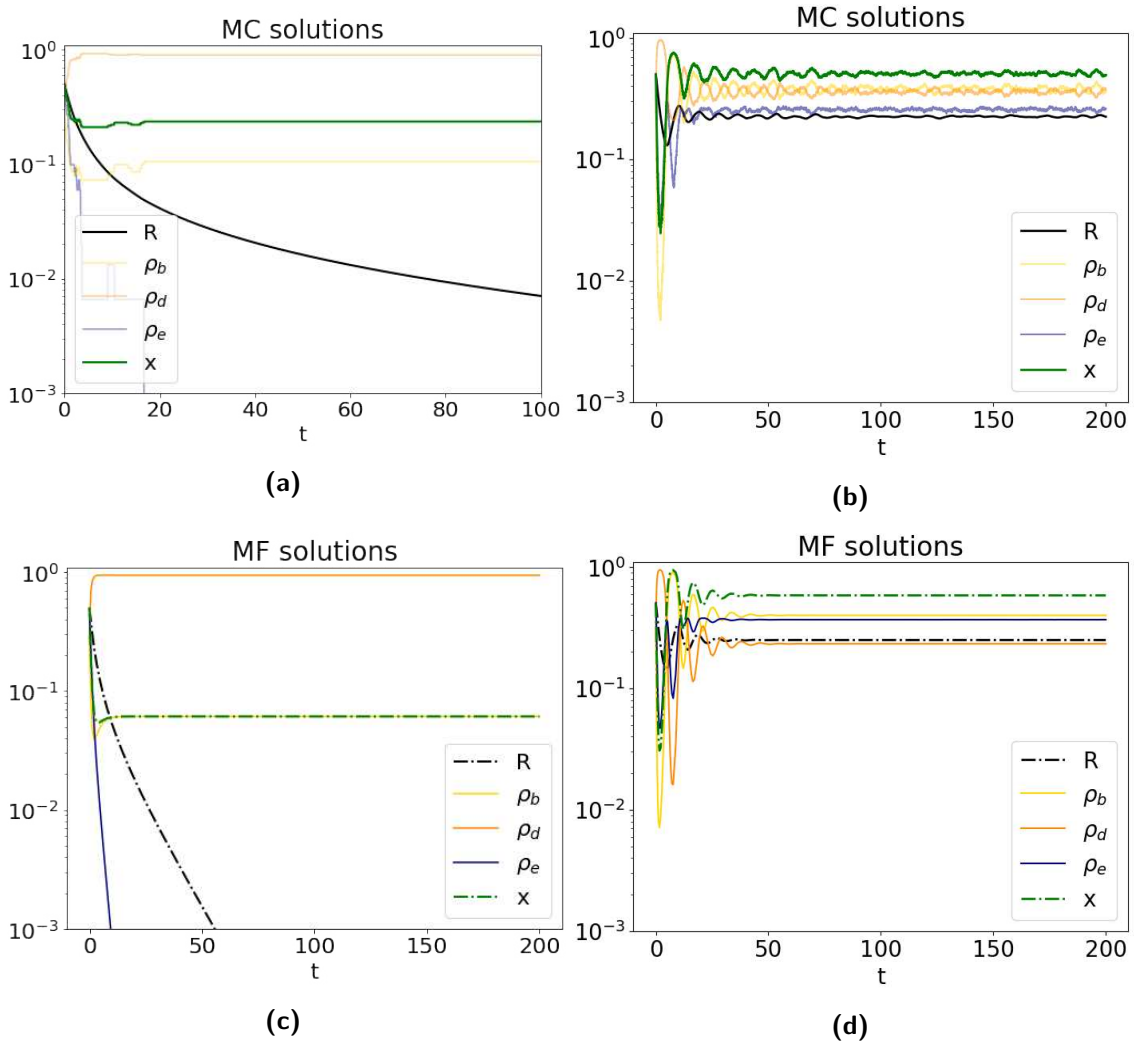


Figure 3.10.: Time series for the EBS model: stochastic (upper row) and deterministic (lower row). In the left column, $\mu = 1.5$; in the right column, $\mu = 5$; the other model parameters are fixed.

time series of the variables relax with transient, damped oscillations to internal, meaning far from the boundaries, stationary points, those oscillations are sustained in the stochastic realizations, as shown in (McKane and Newman, 2005) in the context of the predator-prey Lotka-Volterra model. The goal is then to evaluate the power spectrum in order to characterize the oscillations. Of course, this can be done starting from the numerical simulations, but we may do so analytically as

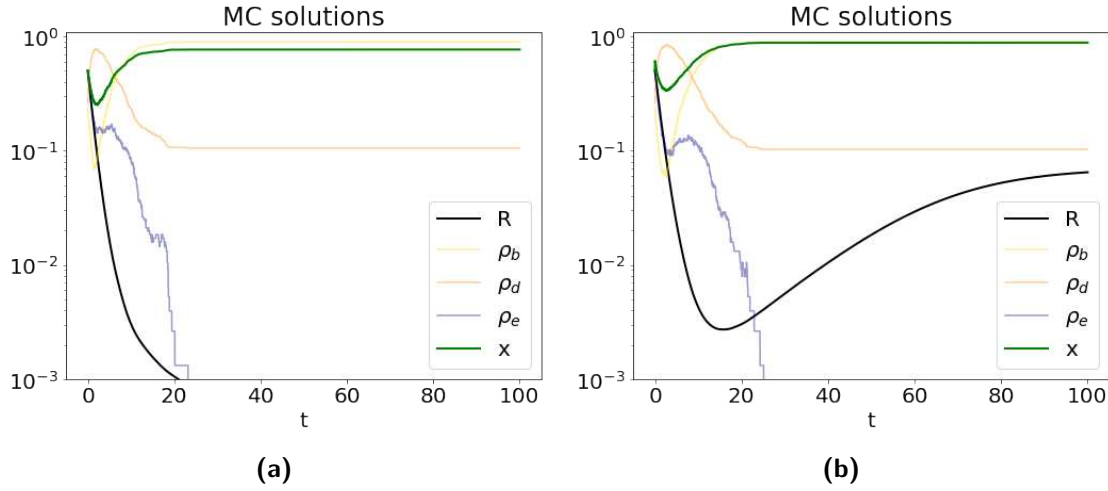


Figure 3.11.: Role of the disconnected nodes in the low degree limit for the SLS model. The two panel show two stochastic simulations of the model with the same parameters, but in panel (a) the strategy of cooperating isolated nodes is left random, leading to a depleted state, while in panel (b) all cooperating isolated nodes are given the cooperating strategy, leading to a repleted state.

well, exploiting the following relation:

$$\langle \tilde{z}(\omega) \tilde{z}^\dagger(\omega') \rangle = 2\pi\delta(\omega + \omega') \underbrace{(J(\mathbf{z}^*) + i\omega\mathbb{1})^{-1} B(\mathbf{z}^*) (J(\mathbf{z}^*)^T + i\omega\mathbb{1})^{-1}}_{P(\omega)} \quad (3.34)$$

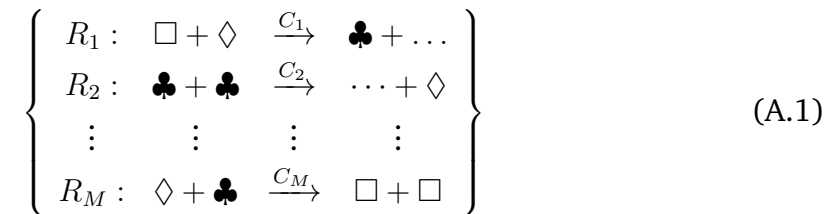
where $\mathbf{z} = (R, \rho_b, \rho_d, \rho_e)$ is the vector collecting the variables of the model, $\tilde{z}$ is its Fourier transform, $B(\mathbf{z})$ is the diffusion matrix of the underlying stochastic process. Having an analytical writing of the power spectrum would also provide a value of the resonant frequency, to which we could compare the numerical estimate. The step that we still have doubts on is the writing of the set of microscopic processes, which would enable us to write the diffusion matrix. Specifically, the issue is that, in order to employ the usual Kramers-Moyal expansion, the microscopic processes need to be direct, as opposed to being collateral, where we use the language of paragraph 3.2. In the case of this model, however, we couldn't achieve this because, as we pointed out, any process involving a node modifies all the pair connections that have such node as an ending. Therefore, it would be necessary to develop a novel approach altogether to derive the deterministic rate equations from microscopic processes with this characteristic, which we didn't tackle within

this PhD.

The Gillespie algorithm

A.1 From reactions to master equations

Let us consider a solution of different chemical species. Suppose that $\square$, $\diamond$ and $\clubsuit$ are some of N different types of molecules (species). These components can give rise to M different reactions $R_{j=1\dots M}$:



where, each reaction is described by the corresponding *rate* C_j , which defines how fast the reaction itself happens, from reagents to products. Properly, C_j is the probability per unit time and per particle of the j^{th} reaction. Basically, our goal is to determine, at a given time $t = 1 \dots T$, the abundances of each species $k = 1 \dots N$, where at the initial time t_0 (i.e. before the reaction) they are supposed to be known; with a vector notation we can write the set of initial abundances as follows

$$\{x_k(t_0)\}_{k=1\dots N} = \mathbf{x}(t_0) \quad (\text{A.2})$$

At this point, for each reaction $j = 1 \dots M$, we introduce a *state-change vector*

$$\mathbf{v}_j = (v_j^1 \dots v_j^N) \quad (\text{A.3})$$

which quantifies the differences in abundances of the species between the beginning and the end of the j^{th} reaction.

Examples

1. Consider a single chemical species, A . We define a reaction and the corresponding state-change vector as:

$$\{R_1 : A \longrightarrow \emptyset\} \implies \mathbf{v}_1 = (-1, 0) \implies \mathbf{x}(1) = \mathbf{x}(0) + \mathbf{v}_1 \quad (\text{A.4})$$

2. Consider two chemical species, A and B . We define a reaction and the corresponding state-change vector as:

$$\left\{ \begin{array}{l} R_1 : B \longrightarrow A + A \\ R_2 : A + A \longrightarrow B \end{array} \right\} \implies \left\{ \begin{array}{l} \mathbf{v}_1 = (+2, -1) \\ \mathbf{v}_2 = (-2, +1) \end{array} \right\} \implies \mathbf{x}(1) = \begin{cases} \mathbf{x}(0) + \mathbf{v}_1 \\ \mathbf{x}(0) + \mathbf{v}_2 \end{cases} \quad (\text{A.5})$$

depending on which process, 1 or 2, occurs.

We can then define the *propensity rate* $a_j(\mathbf{x})$, which is defined in such that $a_j(\mathbf{x})\Delta t$ gives the probability for j^{th} reaction of occurring within the interval Δt . We have for instance

- **Uni-molecular reactions** ($A \longrightarrow \text{products}$): $a_j(\mathbf{x})\Delta t = x_A C_j \Delta t$, where x_A is the abundance of the species A , while C_j is the probability per unit of time and per molecule that the reaction will occur.
- **Bimolecular reactions** ($A + B \longrightarrow \text{products}$): $a_j(\mathbf{x})\Delta t = \frac{x_A x_B}{\Omega} C_j \Delta t$, where Ω is the total number of individuals (it scales like the volume of the solution itself) and x_B/Ω gives the density of the species B . From the above expression, we see that, the larger the volume of the solution, the smaller the probability of reacting; this happens because, practically, the particles are less likely to meet and initialize the reaction.
- **M-molecular reactions**: generalizing the previous case, we write

$$a_j(\mathbf{x})\Delta t = x_1 \frac{C_j \Delta t}{\Omega^{M-1}} \prod_{k=2}^M x_k \quad (\text{A.6})$$

Notice that $a_j(\mathbf{x})$ is an *extensive quantity*, which means that the probability of the reaction j scales with Ω for non-unimolecular reactions.

Given the state of the system $\mathbf{x}(t)$, we can define two different fluxes, one *outgoing*:

$$\bullet \xleftarrow[\mathbf{x}(t)]{R_j} \bullet \xrightarrow[\mathbf{x}(t)+\mathbf{v}_j]{} \bullet \implies \sum_{j=1}^M a_j(\mathbf{x}) P(\mathbf{x}, t | \mathbf{x}_0, t_0) \quad (\text{A.7})$$

and another one *incoming*:

$$\bullet \xrightarrow{\mathbf{x}(t)} \bullet \xleftarrow[\mathbf{x}(t)-\mathbf{v}_i]{R_i} \bullet \implies \sum_{i=1}^M a_i(\mathbf{x} - \mathbf{v}_i) P(\mathbf{x} - \mathbf{v}_i, t | \mathbf{x}_0, t_0) \quad (\text{A.8})$$

where $P(\mathbf{x}, t | \mathbf{x}_0, t_0)$ is the probability of finding the system in the configuration $\mathbf{x}$ at the time t , given the initial set of abundances $\mathbf{x}_0$ at the time t_0 . Together, the two fluxes give rise to the following **Master equation**:

$$\begin{aligned} \frac{\partial P(\mathbf{x}, t)}{\partial t} &= \sum_{j=1}^M a_j(\mathbf{x} - \mathbf{v}_j) P(\mathbf{x} - \mathbf{v}_j, t | \mathbf{x}_0, t_0) - \sum_{i=1}^M a_i(\mathbf{x}) P(\mathbf{x}, t | \mathbf{x}_0, t_0) \\ &= \sum_{j=1}^M \left[a_j(\mathbf{x} - \mathbf{v}_j) P(\mathbf{x} - \mathbf{v}_j, t | \mathbf{x}_0, t_0) - a_j(\mathbf{x}) P(\mathbf{x}, t | \mathbf{x}_0, t_0) \right]. \end{aligned} \quad (\text{A.9})$$

A.2 The algorithm

The Master Equation can be then simulated using the **Gillespie algorithm** (Gillespie, 1976).

In general, Master Equations cannot be solved analytically. The Gillespie algorithm is a useful way to simulate stochastic discrete trajectories $\mathbf{x}(t)$ in such a way that their statistical properties obey the corresponding Master equation.

We call $P(\tau_j | \mathbf{x}(t))$ the probability that, given the state of the system $\mathbf{x}(t)$, the next reaction will be the j^{th} one and will occur between the time steps $t + \tau$ and $t + \tau + d\tau$. In order to visualize the situation, let us divide the time section $[t, t + \tau]$ in k equal intervals of length $d\tau = \tau/k$ (remember that we want the j^{th} reaction to occur in the $k + 1^{\text{th}}$ interval starting from the initial state in t). Then we can define $P_0(\tau | \mathbf{x}(t))$ as the probability that no reaction will occur at least until $t + \tau$. In each

$d\tau$ we have therefore the probability $a_j(\mathbf{x})d\tau$ for the reaction j of occurring, and the corresponding one $1 - a_j(\mathbf{x})d\tau$ of not occurring. We can then estimate P_0 as following:

$$P_0(\tau|\mathbf{x}(t)) = \left[1 - d\tau \sum_{i=1}^M a_i(\mathbf{x})\right]^k = \left[1 - \frac{\tau}{k} \sum_{i=1}^M a_i(\mathbf{x})\right]^k \xrightarrow{k \rightarrow \infty} \exp\left(-\tau \sum_{i=1}^M a_i(\mathbf{x})\right), \quad (\text{A.10})$$

where we have used the fact that $[1 - b/k]^k \rightarrow e^{-b}$ as $k \rightarrow \infty$.

Moreover, we have:

$$P(\tau j|\mathbf{x}(t))d\tau = P_0(\tau|\mathbf{x}(t))a_j(\mathbf{x})d\tau \sim a_j(\mathbf{x})d\tau \exp\left\{\left(-\tau \sum_{i=1}^M a_i(\mathbf{x})\right)\right\} \quad (\text{A.11})$$

and by summing over all the possible reactions j we get:

$$P(\tau|\mathbf{x}(t)) = \sum_{j=1}^M P(\tau j|\mathbf{x}(t)) = \sum_{j=1}^M a_j(\mathbf{x})e^{-\tau \sum_{i=1}^M a_i(\mathbf{x})} \equiv \gamma e^{-\tau\gamma} \quad (\text{A.12})$$

where we have defined $\gamma = \sum_{j=1}^M a_j(\mathbf{x}) = a_0(\mathbf{x})$. $P(\tau|\mathbf{x}(t))$ is the probability of having at least one reaction in the interval $[t + \tau, t + \tau + d\tau]$. Then, the main purpose of Gillespie is to extract the time of the next reaction, τ , from the previous exponential distribution; practically, we generate a random number from an exponential distribution of parameter γ . Notice that, proceeding in this way, instead of taking fixed time steps, we are considering a distribution of steps of different lengths, assuming that an interaction occurs every time.

We can obtain the final $P(\tau j|\mathbf{x}(t))$ by imposing the normalization:

$$\frac{P(\tau j|\mathbf{x}(t))}{P(\tau|\mathbf{x}(t))} = \frac{P(\tau j|\mathbf{x}(t))}{\sum_{j=1}^M P(\tau j|\mathbf{x}(t))} = \frac{a_j(\mathbf{x})}{a_0(\mathbf{x})} = p_j \quad (\text{A.13})$$

Algorithmically, we can proceed with the following steps:

1. Initialize the system with the configuration of abundances $\mathbf{x}_0 = \mathbf{x}(t = 0)$
2. As per equation (A.6), we evaluate $a_j(\mathbf{x}) \forall j$ in such a way to determine $a_0(\mathbf{x}) = \sum_{j=1}^M a_j(\mathbf{x})$

3. Compute the time at which the next interaction will occur. As we saw before, $P(\tau|\mathbf{x}(t))$, that is the probability of having at least a reaction in $[t+\tau, t+\tau+d\tau]$, is exponentially distributed, $\gamma e^{-\tau\gamma}$ with $\gamma = a_0(\mathbf{x})$. We extract the time τ from an exponential random distribution, with parameter $a_0(\mathbf{x})$.
4. Compute which is the next reaction: extract a random number $u \in [0, 1]$ from a uniform distribution and multiply it by $a_0(\mathbf{x})$, which is the sum of all the $a_j(\mathbf{x})$, namely the total exit rate from state $\mathbf{x}$. The next reaction is the i^{th} one, where $i \in \{1 \dots n\}$ is the minimum integer number which satisfies the following condition:

$$\sum_{k=1}^i a_k(\mathbf{x}) > u a_0(\mathbf{x}) \quad (\text{A.14})$$

Notice that only one of the possible reactions has to happen.

5. Update $\mathbf{x}(t + \tau) = \mathbf{x}(t) + \mathbf{v}_i$.
6. Repeat the sequence starting from the step 3.

The resulting time series $\mathbf{x}(t)$ satisfies the Master Equation, since it is obtained from the underlying stochastic processes themselves.

Example

Let us consider an epidemic system with two possible states: *infected* (i) and *susceptible* (s) (SIS model). Then, let I be the number of infected and S the number of susceptible individuals. The state vector is $\mathbf{x}(t) = (N_I(t), N_S(t))$. We define two reactions: an infected individual recovers with rate μ and becomes susceptible again to the disease; a susceptible individual comes in contact with an infected one and becomes sick as well with a rate λ . With the previous notation, we have:



If $N_I + N_S = N$, namely the total number of individuals in the community N doesn't vary, the propensity rates are:

$$a_1(N_I) = \mu N_I \equiv W(N_I \longrightarrow N_I - 1) \equiv W^- \quad (\text{A.16})$$

$$a_2(N_S, N_I) = \lambda \frac{N_S N_I}{N} = \lambda \frac{(N - N_I) N_I}{N} \equiv W(N_I \longrightarrow N_I + 1) \equiv W^+ \quad (\text{A.17})$$

Notice that N is the analogous of the previous Ω , while μ and λ correspond to the probabilities per unit of time and per individual which we called C_1 and C_2 in our general notation. At this point we can rewrite the Master equation (A.9) as follows:

$$\begin{aligned} \frac{\partial P(N_I, t)}{\partial t} = & -\lambda \frac{N_I(N - N_I)}{N} P(N_I, t) + \lambda \frac{(N_I - 1)(N - (N_I - 1))}{N} P(N_I - 1, t) + \\ & -\mu N_I P(N_I, t) + \mu(N_I + 1) P(N_I + 1, t) \end{aligned} \quad (\text{A.18})$$

where the 1^{st} and the 3^{rd} terms are two outgoing fluxes from the configuration with N_I infected individuals, while the 2^{nd} and the 4^{th} terms are two incoming fluxes to the state N_I :

$$\begin{array}{ccccc} \bullet & \xrightarrow{\lambda} & \bullet & \xrightarrow{\lambda} & \bullet \\ N_{I-1} & & N_I & & N_{I+1} \end{array} \quad (\text{A.19})$$

$$\begin{array}{ccccc} \bullet & \xleftarrow{\mu} & \bullet & \xleftarrow{\mu} & \bullet \\ N_{I-1} & & N_I & & N_{I+1} \end{array} \quad (\text{A.20})$$

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*Lay down, black gives way to blue
Lay down, I'll remember you*

Alice in Chains — Black Gives Way to Blue