Composition operators in Grand Lebesgue spaces

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Abstract: Let Ω be an open subset of $\mathbb{R}^n$ of finite measure. Let f be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$. We prove necessary and sufficient conditions on f in order that the composite function $T_f[g] = f \circ g$ belongs to the Grand Lebesgue space $L_{p),\theta}(\Omega)$ whenever g belongs to $L_{p),\theta}(\Omega)$.

We also study continuity, uniform continuity, Hölder and Lipschitz continuity of the composition operator $T_f[g]$ in $L_{p),\theta}(\Omega)$.

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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$ of finite measure, where

$$n \in \mathbb{N} \setminus \{0\},$$

and where $\mathbb{N}$ denotes the set of natural numbers including 0. Let f be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$. The composition operator T_f in the space of measurable functions in Ω is defined as follows

$$T_f[g] \equiv f \circ g, \tag{1.1}$$

for all measurable functions g from Ω to $\mathbb{R}$. Here we study the operator (1.1) in the framework of Grand Lebesgue $L_{p),\theta}(\Omega)$ spaces. For the definition of $L_{p),\theta}(\Omega)$ we refer to Section 3. More precisely, we look for necessary and sufficient conditions on f in order that T_f acts in $L_{p),\theta}(\Omega)$, *i.e.*,

$$T_f[L_{p),\theta}(\Omega)] \subseteq L_{p),\theta}(\Omega),$$

(*cf.* Section 4) and for conditions on f in order that T_f be continuous in $L_{p),\theta}(\Omega)$ (*cf.* Section 5), uniformly continuous in $L_{p),\theta}(\Omega)$ (*cf.* Section 6), and Hölder and Lipschitz continuous in $L_{p),\theta}(\Omega)$ (*cf.* Section 7).

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The theory of Grand Lebesgue spaces and related spaces, such as Small Lebesgue spaces, was widely developed in connection with the applications to PDE's. Grand Lebesgue spaces $L_{p)}(\Omega)$ have been introduced in Iwaniec and Sbordone [14], and the generalized Grand Lebesgue spaces $L_{p),\theta}(\Omega)$ have been treated by Di Fratta and Fiorenza [9], Fiorenza and Karadzhov [12], Greco, Iwaniec and Sbordone [13]. Here and in what follows we will omit the word "generalized", as it is now customary in many recent papers to call $L_{p),\theta}(\Omega)$ simply a 'Grand Lebesgue space'.

To the best of our knowledge, the analysis of composition operators of the type (1.1) in the framework of the Grand Lebesgue spaces $L_{p),\theta}(\Omega)$ has not been carried out. However such analysis is natural in view of the applications of the theory of Grand Lebesgue spaces in analysis and PDE's, in which composition operators naturally arise. The study of composition operators (1.1) combines function space theory and operator theory, and is motivated by quite natural questions coming from applications. For a detailed study of such operators and for a more circumstantial explanation of the interest in the study of operators such as (1.1), we refer to the monographs of Appell and Zabrejko [1], Runst and Sickel [19], Dudley and Norvaiša [10].

The Grand Lebesgue space $L_{p),\theta}(\Omega)$ contains classical Lebesgue space $L_p(\Omega)$, and is contained in $L_{p-\delta}(\Omega)$ for all $\delta \in]0, p-1[$. However the known theory of composition operators in Lebesgue spaces $L_p(\Omega)$ cannot be directly applied to the case of the Grand Lebesgue spaces in view of a different nature of these scales of spaces.

For the action, uniform continuity, α-Hölder continuity, and Lipschitz continuity of T_f, we can prove necessary and sufficient conditions. Instead for the continuity of T_f we have only sufficient conditions and necessary conditions. Then we have necessary and sufficient conditions for the continuity of T_f only in the case of vanishing Grand Lebesgue spaces and with some extra restriction.

2 Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$. We retain the standard notation of $L_p(\Omega)$ for the space of all real-valued p-summable measurable (equivalence classed of) functions on $\Omega \subseteq \mathbb{R}^n$. We use the symbol $L_{1\,\mathrm{loc}}(\Omega)$ for the space of (equivalence classed of) locally integrable functions on Ω. By $\mathcal{D}(\Omega)$ we denote the set of compactly supported $C^\infty(\Omega)$ functions.

A normed space of functions E that is contained in the space $L_{1\,\mathrm{loc}}(\Omega)$ of (equivalence classes of) real valued locally integrable functions in Ω is called a $\mathcal{D}(\Omega)$-module provided that for each $\varphi \in \mathcal{D}(\Omega)$ we have $\varphi g \in E$ for all $g \in E$ and that the multiplication operator by φ is continuous in E.

We note that the action condition of T_f implies a property of local boundedness on bounded sets for T_f by exploiting the following Lemma of Bourdaud [2], which develops ideas of Katznelson [17, Ch. VIII, § 8.3]. For a proof due to Bourdaud, we refer to [15, §2].

Lemma 2.1 *Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let E_i, $i = 1, 2$, be subspaces of $L_{1\,\mathrm{loc}}(\Omega)$ endowed with a norm such that the canonical injection $E_i \subseteq L_{1\,\mathrm{loc}}(\Omega)$ is continuous. Suppose E_1 is complete and that E_2 is a $\mathcal{D}(\Omega)$-module. Let f be a Borel measurable function*