Forum of Mathematics, Sigma (2022), **Vol. 10**:e74 1–36
doi:10.1017/fms.2022.59

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Kernels of localities

Valentina Grazian[1] and Ellen Henke[2]

[1]Department of Mathematics and Applications, University of Milano - Bicocca, Via Roberto Cozzi 55, 20125 Milano, Italy;
E-mail: valentina.grazian@unimib.it
[2]Institut für Algebra, Fakultät Mathematik, Technische Universität Dresden, 01062 Dresden, Germany;
E-mail: ellen.henke@tu-dresden.de

Received: 2 November 2021; **Revised:** 10 June 2022; **Accepted:** 27 June 2022

2020 Mathematics Subject Classification: *Primary* – 20D20; *Secondary* – 20E25, 55R35

Abstract
We state a sufficient condition for a fusion system to be saturated. This is then used to investigate localities with kernels: that is, localities that are (in a particular way) extensions of groups by localities. As an application of these results, we define and study certain products in fusion systems and localities, thus giving a new method to construct saturated subsystems of fusion systems.

Contents

1. Introduction

The problem of showing that a given fusion system is saturated arises in different contexts. For example, proving that certain subsystems of fusion systems are saturated is one of the major difficulties in developing a theory of saturated fusion systems in analogy to the theory of finite groups. When studying

© The Author(s), 2022. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

extensions of fusion systems, it is also crucial to understand under which conditions such extensions are saturated. In this paper, we seek to take up both themes simultaneously in a systematic way. To formulate and study extension problems, it is common to work with linking systems or, more generally, with transporter systems associated to fusion systems (see, e.g., [6, 25, 23]). The equivalent framework of localities (introduced by Chermak [7]) was recently used by Chermak and the second named author of this paper to construct saturated subsystems of fusion systems, thereby enriching the theory of fusion systems by some new concepts (compare [10, Theorem C] and [20]).

In the present paper (apart from the appendix), we work with localities rather than transporter systems. Our first main result is, however, formulated purely in terms of fusion systems. It gives a sufficient condition for a fusion system to be saturated. The proof generalises an argument used by Oliver [23] to show that the fusion systems associated to certain extensions of groups by linking systems are saturated.

Our saturation criterion serves us as an important tool for studying *kernels* of localities as introduced in Definition 4 below. A kernel of a locality $\mathcal{L}$ is basically a partial normal subgroup $\mathcal{N}$ such that the factor locality $\mathcal{L}/\mathcal{N}$ is a group and $\mathcal{N}$ itself supports the structure of a locality. We show in an appendix that kernels of localities correspond to 'normal pairs of transporter systems' (compare Definition A.3).

In Section 6, we prove some results demonstrating that the theory of kernels can be used to construct saturated subsystems of fusion systems. More precisely, we study certain products in localities that give rise to 'sublocalities' whose fusion systems are saturated. As a special case, if $\mathcal{F}$ is a saturated fusion system over S, one can define a notion of a product of a normal subsystem with a subgroup of a model for $N_{\mathcal{F}}(S)$ (or equivalently with a saturated subsystem of $N_{\mathcal{F}}(S)$). In particular, our work generalises the notion of a product of a normal subsystem with a subgroup of S, which was introduced by Aschbacher [3]. Our results on products are also used by the second named author of this article [19] to construct normalisers and centralisers of subnormal subsystems of saturated fusion systems.

For the remainder of this introduction, let $\mathcal{F}$ be a fusion system over a finite p-group S.

We will adapt the terminology and notation regarding fusion systems from [2, Chapter 1], except that we will write homomorphisms on the right-hand side of the argument (similarly as in [2, Chapter 2]) and that we will define centric radical subgroups of $\mathcal{F}$ differently, namely as follows.

Definition 1. Define a subgroup $P \leq S$ to be *centric radical* in $\mathcal{F}$ if

- P is centric: that is, $C_S(Q) \leq Q$ for every $\mathcal{F}$-conjugate Q of P; and
- $O_p(N_{\mathcal{F}}(Q)) = Q$ for every fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate Q of P.

Write $\mathcal{F}^{cr}$ for the set of subgroups of S that are centric radical in $\mathcal{F}$.

If $\mathcal{F}$ is saturated, then our notion of centric radical subgroups of $\mathcal{F}$ coincides with the usual notion (compare Lemma 2.6). However, defining centric radical subgroups as above is crucial if we want to conclude that the fusion systems of certain localities are saturated.

1.1. A saturation criterion

The following definition will be used to formulate the previously mentioned sufficient condition for a fusion system to be saturated.

Definition 2. Let Δ be a set of subgroups of S.

- The set Δ is called $\mathcal{F}$*-closed* if Δ is closed under $\mathcal{F}$-conjugacy and overgroup-closed in S.
- $\mathcal{F}$ is called Δ-generated if every morphism in $\mathcal{F}$ can be written as a product of restrictions of $\mathcal{F}$-morphisms between subgroups in Δ.
- $\mathcal{F}$ is called Δ-saturated if each $\mathcal{F}$-conjugacy class in Δ contains a subgroup that is fully automised and receptive in $\mathcal{F}$ (as defined in [2, Definition I.2.2]).

Generalising arguments used by Oliver [23], we prove the following theorem.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Theorem A. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant saturated subsystem of $\mathcal{F}$. Suppose $\mathcal{F}$ is Δ-generated and Δ-saturated for some $\mathcal{F}$-closed set Δ of subgroups of S with $\mathcal{E}^{cr} \subseteq \Delta$. Then $\mathcal{F}$ is saturated.*

1.2. Kernels of localities

The reader is referred to Section 3 for an introduction to partial groups and localities. We say that a locality $(\mathcal{L}, \Delta, S)$ is a locality *over* $\mathcal{F}$ to indicate that $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. The set Δ is called the *object set* of $(\mathcal{L}, \Delta, S)$. It follows from the definition of a locality that the normaliser $N_{\mathcal{L}}(P)$ of any object $P \in \Delta$ is a subgroup of $\mathcal{L}$ and thus a finite group. We will use the following definition.

Definition 3.

- A locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ is called *cr-complete* if $\mathcal{F}^{cr} \subseteq \Delta$.
- A finite group G is said to be *of characteristic p* if $C_G(O_p(G)) \leq O_p(G)$.
- A locality $(\mathcal{L}, \Delta, S)$ is called a *linking locality* if it is cr-complete and $N_{\mathcal{L}}(P)$ is of characteristic p for every $P \in \Delta$.

The slightly nonstandard notion of centric radical subgroups introduced in Definition 1 ensures that the fusion system $\mathcal{F}_S(\mathcal{L})$ of a cr-complete locality $(\mathcal{L}, \Delta, S)$ is saturated (compare Proposition 3.18(c)). If $(\mathcal{L}, \Delta, S)$ is a cr-complete locality over $\mathcal{F}$, then it gives rise to a transporter system $\mathcal{T}$ associated to $\mathcal{F}$ whose object set is Δ and thus contains the set $\mathcal{F}^{cr}$. It follows from [25, Proposition 4.6] that the p-completed nerve of such a transporter system $\mathcal{T}$ is homotopy equivalent to the p-completed nerve of a linking system associated to $\mathcal{F}$. The results we present next are centred around the following concept.

Definition 4. A *kernel* of a locality $(\mathcal{L}, \Delta, S)$ is a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $P \cap \mathcal{N} \in \Delta$ for every $P \in \Delta$.

We show in Appendix A that kernels of localities correspond to 'normal pairs of transporter systems'. In particular, the results presented below can be translated to results on transporter systems. The reader is referred to Definition A.3, Proposition A.4, Theorem A.7 and Remark A.8 for details.

If $\mathcal{N}$ is a kernel of a locality $(\mathcal{L}, \Delta, S)$, then, setting

$$T := \mathcal{N} \cap S \text{ and } \Gamma := \{P \cap \mathcal{N} \colon P \in \Delta\},$$

it is easy to see that $(\mathcal{N}, \Gamma, T)$ is a locality (compare Lemma 5.2). We also say in this situation that $(\mathcal{N}, \Gamma, T)$ is a kernel of $(\mathcal{L}, \Delta, S)$.

Suppose now that $(\mathcal{N}, \Gamma, T)$ is a kernel of $(\mathcal{L}, \Delta, S)$. Observe that T is an element of $\Gamma \subseteq \Delta$, so $N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$. It follows therefore from [8, Theorem 4.3(b), Corollary 4.5] that $\mathcal{L}/\mathcal{N} \cong N_{\mathcal{L}}(T)/N_{\mathcal{N}}(T)$ is a group. Thus, $\mathcal{L}$ can be seen as an extension of the group $\mathcal{L}/\mathcal{N}$ by the locality $(\mathcal{N}, \Gamma, T)$.

If the kernel $(\mathcal{N}, \Gamma, T)$ is cr-complete, then the following theorem implies that $\mathcal{F}_S(\mathcal{L})$ is saturated. Its proof uses Theorem A.

Theorem B. *Let $(\mathcal{N}, \Gamma, T)$ be a kernel of a locality $(\mathcal{L}, \Delta, S)$. Then $(\mathcal{L}, \Delta, S)$ is cr-complete if and only if $(\mathcal{N}, \Gamma, T)$ is cr-complete. If so, then $\mathcal{F}_T(\mathcal{N})$ is a normal subsystem of $\mathcal{F}_S(\mathcal{L})$.*

Theorem C. *Let $(\mathcal{L}, \Delta, S)$ be a locality with a kernel $(\mathcal{N}, \Gamma, T)$. Then the following conditions are equivalent:*

(i) *$(\mathcal{L}, \Delta, S)$ is a linking locality;*
(ii) *$(\mathcal{N}, \Gamma, T)$ is a linking locality and $N_{\mathcal{L}}(T)$ is of characteristic p;*
(iii) *$(\mathcal{N}, \Gamma, T)$ is a linking locality and $C_{\mathcal{L}}(T)$ is of characteristic p.*

We now want to consider special kinds of linking localities. The object set of any linking locality over $\mathcal{F}$ is always contained in the set $\mathcal{F}^s$ of $\mathcal{F}$-subcentric subgroups (defined in Definition 3.19). If $\mathcal{F}$ is saturated, then the existence and uniqueness of centric linking systems imply conversely that for every

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

$\mathcal{F}$-closed set Δ of subgroups of $\mathcal{F}^s$, there is an essentially unique linking locality over $\mathcal{F}$ with object set Δ. Chermak introduced an $\mathcal{F}$-closed set $\delta(\mathcal{F}) \subseteq \mathcal{F}^s$, which by [10, Lemma 7.21] can be described as the set of all subgroups of S containing an element of $F^*(\mathcal{F})^s$ (where $F^*(\mathcal{F})$ is the generalised Fitting subsystem of $\mathcal{F}$ introduced by Aschbacher [3]). Notice that there always exists an essentially unique linking locality over $\mathcal{F}$ whose object set is the set $\delta(\mathcal{F})$. Such a linking locality is called a *regular locality*.

For an arbitrary locality $(\mathcal{L}, \Delta, S)$, there is a largest subgroup R of S with $\mathcal{L} = N_{\mathcal{L}}(R)$. This subgroup is denoted by $O_p(\mathcal{L})$. Setting $\tilde{\Delta} := \{P \leq S \colon PO_p(\mathcal{L}) \in \Delta\}$, the triple $(\mathcal{L}, \tilde{\Delta}, S)$ is also a locality (but with a possibly larger object set). It turns out that $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $(\mathcal{L}, \tilde{\Delta}, S)$ is a linking locality (compare [10, Lemma 3.28]). This flexibility in the choice of object sets makes it possible to formulate a result similar to Theorem C for regular localities.

Theorem D. *Let $(\mathcal{L}, \Delta, S)$ be a locality with a kernel $(\mathcal{N}, \Gamma, T)$. Set*

$$\tilde{\Delta} := \{P \leq S \colon PO_p(\mathcal{L}) \in \Delta\} \text{ and } \tilde{\Gamma} := \{Q \leq T \colon QO_p(\mathcal{N}) \in \Gamma\}.$$

Then the following conditions are equivalent:

(i) $(\mathcal{L}, \tilde{\Delta}, S)$ *is a regular locality;*
(ii) $(\mathcal{N}, \tilde{\Gamma}, T)$ *is a regular locality and $N_{\mathcal{L}}(T)$ is of characteristic p;*
(iii) $(\mathcal{N}, \tilde{\Gamma}, T)$ *is a regular locality and $C_{\mathcal{L}}(T)$ is of characteristic p.*

Moreover, if these conditions hold, then $E(\mathcal{L}) = E(\mathcal{N})$.

In an unpublished preprint, Chermak defined a locality $(\mathcal{L}, \Delta, S)$ to be *semiregular* if (in our language) it has a kernel $(\mathcal{N}, \Gamma, T)$ that is a regular locality. He observed furthermore that a locality is semiregular if and only if it is an image of a regular locality under a projection of localities. As a consequence, images of semiregular localities under projections are semiregular. Moreover, since partial normal subgroups of regular localities form regular localities, it follows that partial normal subgroups of semiregular localities form semiregular localities. Thus, the category of semiregular localities and projections might provide a good framework to study extensions. This is one of our motivations to study kernels of localities more generally.

Remark. Extensions of partial groups and localities have already been studied by Gonzalez [12]. He starts by giving important insights into the existence of extensions of partial groups. Basically, Gonzalez considers partial groups as simplicial sets and uses the concept of a simplicial fibre bundle. Gonzalez then states some results about extensions of localities in Section 7 of his paper. He calls a locality 'saturated' if it is cr-complete in our sense. Under certain conditions, it is shown that extensions of localities lead to (saturated) localities. To summarise, Gonzalez starts by defining *isotypical extensions* (compare [12, Definition 7.1]) and shows that an isotypical extension of a locality $(\mathtt{L}'', \Delta'', S'')$ by a locality $(\mathtt{L}', \Delta', S')$ leads to a locality $(\mathtt{T}, \Delta, S)$ (compare [12, Proposition 7.6]). Slightly more precisely, we have $\mathtt{T} \subseteq \mathtt{L}$ for an extension $\mathtt{L}$ of the partial group $\mathtt{L}''$ by the partial group $\mathtt{L}'$.

The situation Gonzalez studies is principally different from ours. However, in [12, Example 7.9, Corollary 7.10], he considers a setup where $\mathtt{L} = \mathtt{T}$ (with $\mathtt{L}$ and $\mathtt{T}$ as above). In this situation, one can observe easily that $(\mathtt{L}', \Delta', S')$ is a kernel of $(\mathtt{L}, \Delta, S)$. Indeed, our Theorem B shows that the assumption in [12, Corollary 7.10] that Δ' contains all $\mathcal{F}'$-centric subgroups is redundant. It would be the subject of further research to see how far our results have other interesting applications in the context of Gonzalez's work.

1.3. Products in regular localities and fusion systems

We now demonstrate that the theory of kernels can be used to study certain products in regular localities and thereby construct saturated subsystems of saturated fusion systems.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

We study regular localities rather than arbitrary (linking) localities, mainly because every partial normal subgroup of a regular locality can be given the structure of a regular locality. To be exact, if $(\mathcal{L}, \Delta, S)$ is a regular locality and $\mathcal{N} \trianglelefteq \mathcal{L}$, then $\mathcal{E} := \mathcal{F}_{S\cap\mathcal{N}}(\mathcal{N})$ is saturated and $(\mathcal{N}, \delta(\mathcal{E}), S \cap \mathcal{N})$ is a regular locality.

In localities or, more generally, in partial groups, there is a natural notion of products of subsets. More precisely, if $\mathcal{L}$ is a partial group with product $\Pi\colon \mathbf{D} \to \mathcal{L}$, then for $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{L}$, we set

$$\mathcal{X}\mathcal{Y} := \{\Pi(x, y)\colon x \in \mathcal{X},\ y \in \mathcal{Y},\ (x, y) \in \mathbf{D}\}.$$

The product of a partial normal subgroup with another partial subgroup is only in special cases known to be a partial subgroup. For example, the product of two partial normal subgroups of a locality $(\mathcal{L}, \Delta, S)$ is a partial normal subgroup (and thus forms a regular locality if $(\mathcal{L}, \Delta, S)$ is regular). Our next theorem gives a further example of a product that is a partial subgroup and can be given the structure of a regular locality.

For the theorem below to be comprehensible, a few preliminary remarks may be useful. For any linking locality $(\mathcal{L}, \Delta, S)$, one can define the generalised Fitting subgroup $F^*(\mathcal{L})$ as a certain partial normal subgroup of $\mathcal{L}$ (see [18, Definition 3]). If $(\mathcal{L}, \Delta, S)$ is a regular locality, then $F^*(\mathcal{L})$ is a kernel of $(\mathcal{L}, \Delta, S)$, and thus $T^* := S \cap F^*(\mathcal{L})$ is an element of Δ. In particular, $N_{\mathcal{L}}(T^*)$ forms a group of characteristic p. If $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup of $\mathcal{L}$, then $N_{\mathcal{N}}(T^*)$ is a normal subgroup of $N_{\mathcal{L}}(T^*)$. Hence, for any subgroup H of $N_{\mathcal{L}}(T^*)$, the product $N_{\mathcal{N}}(T^*)H$ is a subgroup of $N_{\mathcal{L}}(T^*)$.

Theorem E. *Let $(\mathcal{L}, \Delta, S)$ be a regular locality. Moreover, fix*

$$\mathcal{N} \trianglelefteq \mathcal{L},\ T := \mathcal{N} \cap S,\ \mathcal{E} := \mathcal{F}_T(\mathcal{N}),\ T^* := F^*(\mathcal{L}) \cap S \text{ and } H \le N_{\mathcal{L}}(T^*).$$

Then $\mathcal{N}H$ is a partial subgroup of $\mathcal{L}$. Moreover, for every Sylow p-subgroup S_0 of $N_{\mathcal{N}}(T^*)H$ with $T \le S_0$, the following hold:

(a) *There exists a unique set Δ_0 of subgroups of S_0 such that $(\mathcal{N}H, \Delta_0, S_0)$ is a cr-complete locality with kernel $(\mathcal{N}, \delta(\mathcal{E}), T)$.*
(b) *Let Δ_0 be as in (a), and set $\tilde{\Delta}_0 := \{P \le S\colon PO_p(\mathcal{N}H) \in \Delta_0\}$. Then $(\mathcal{N}H, \tilde{\Delta}_0, S_0)$ is a regular locality if and only if $N_{\mathcal{N}}(T^*)H$ is a group of characteristic p.*

If the hypothesis of Theorem E holds and $S \cap H$ is a Sylow p-subgroup of H, then

$$S_0 := T(S \cap H)$$

is a Sylow p-subgroup of $N_{\mathcal{N}}(T^*)H$ that is contained in S (compare Lemma 6.2). Thus, the cr-complete locality $(\mathcal{N}H, \Delta_0, S_0)$ from Theorem E(a) gives rise to a saturated subsystem $\mathcal{F}_{S_0}(\mathcal{N}H) = \mathcal{F}_{T(H\cap S)}(\mathcal{N}H)$ of $\mathcal{F}$. This leads us to a statement that can be formulated purely in terms of fusion systems. We use here that, for every regular locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$, there is by [10, Theorem A] a bijection from the set of partial normal subgroups of $\mathcal{L}$ to the set of normal subsystems of $\mathcal{F}$ given by $\mathcal{N} \mapsto \mathcal{F}_{S\cap\mathcal{N}}(\mathcal{N})$. By [10, Theorem E(d)], this bijection takes $F^*(\mathcal{L})$ to $F^*(\mathcal{F})$. In particular, $F^*(\mathcal{F})$ is a fusion system over $F^*(\mathcal{L}) \cap S$.

To formulate the result we obtain, we rely on the fact that every constrained fusion system is realised by a model: that is, by a finite group of characteristic p. Furthermore, if $\mathcal{F}$ is constrained and G is a model for $\mathcal{F}$, then every normal subsystem of $\mathcal{F}$ is realised by a normal subgroup of G. We also use that, for every saturated fusion system $\mathcal{F}$ over S, every normal subsystem $\mathcal{E}$ of $\mathcal{F}$ and every subgroup R of S, there is a product subsystem $\mathcal{E}R$ defined (compare [3, Chapter 8] or [13]).

Corollary F. *Let $\mathcal{F}$ be a saturated fusion system over S and $\mathcal{E} \trianglelefteq \mathcal{F}$ over $T \le S$. Let $T^*, T_0 \le S$ such that*

$$F^*(\mathcal{F}) \text{ is a fusion system over } T^* \text{ and } E(\mathcal{E}) \text{ is a fusion system over } T_0.$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Then $N_{\mathcal{F}}(T^*)$ is constrained and $N_{\mathcal{E}}(T_0) \trianglelefteq N_{\mathcal{F}}(T^*)$. Thus we may choose a model G for $N_{\mathcal{F}}(T^*)$ and $N \trianglelefteq G$ with $\mathcal{F}_{S\cap N}(N) = N_{\mathcal{E}}(T_0)$. Let $H \leq G$ with $S \cap H \in \mathrm{Syl}_p(H)$. Set

$$S_0 := T(S \cap H) \text{ and } \mathcal{E}H := \langle E(\mathcal{E})S_0, \mathcal{F}_{S_0}(NH)\rangle.$$

Then the following hold:

(a) *$\mathcal{E}H$ is a saturated fusion system over S_0 with $\mathcal{E} \trianglelefteq \mathcal{E}H$.*
(b) *If $\mathcal{D}$ is a saturated subsystem of $\mathcal{F}$ with $E(\mathcal{E}) \trianglelefteq \mathcal{D}$ and $\mathcal{F}_{S_0}(NH) \subseteq \mathcal{D}$, then $\mathcal{E}H \subseteq \mathcal{D}$.*

For a saturated fusion system $\mathcal{F}$ with $\mathcal{E} \trianglelefteq \mathcal{F}$, it is shown, for example, in [10, Lemma 7.13(a)] that $E(\mathcal{E}) \trianglelefteq \mathcal{F}$. Hence, it makes sense in the situation above to form the product subsystem $E(\mathcal{E})S_0$. Moreover, part (b) of Corollary F implies the following statement: If $\mathcal{D}$ is a saturated subsystem of $\mathcal{F}$ with $\mathcal{E} \trianglelefteq \mathcal{D}$ and $\mathcal{F}_{S_0}(NH) \subseteq \mathcal{D}$, then $\mathcal{E}H \subseteq \mathcal{D}$.

Organisation of the paper

We start by proving our saturation criterion (Theorem A) in Section 2. An introduction to partial groups and localities is given in Section 3. After proving some preliminary results, we study kernels of localities in Section 5. This is used in Section 6 to prove Theorem E and Corollary F as well as some more detailed results on products.

2. Proving saturation

Throughout this section, let $\mathcal{F}$ be a fusion system over S.

In this section, we prove Theorem A. The reader is referred to [2, Chapter I] for an introduction to fusion systems. We will adopt the notation and terminology from there with the following two caveats: firstly, we write homomorphisms on the right-hand side of the arguments similarly as in [2, Chapter II]. Secondly, we define the set $\mathcal{F}^r$ of $\mathcal{F}$-radical subgroups differently, namely as in Definition 2.5 below.

Definition 2.1. A subgroup $P \leq S$ is said to respect $\mathcal{F}$-saturation if there exists an element of $P^{\mathcal{F}}$ that is fully automised and receptive in $\mathcal{F}$ (as defined in [2, Definition I.2.2]).

If Δ is a set of subgroups of S that is closed under $\mathcal{F}$-conjugacy, then notice that $\mathcal{F}$ is Δ-saturated (as defined in the introduction) if and only if every element of Δ respects $\mathcal{F}$-saturation. On the other hand, P respects $\mathcal{F}$-saturation if $\mathcal{F}$ is $P^{\mathcal{F}}$-saturated. Observe also that a fusion system is saturated if every subgroup of S respects $\mathcal{F}$-saturation, or equivalently if $\mathcal{F}$ is Δ-saturated, where Δ is the set of all subgroups of S. Roberts and Shpectorov [26] proved the following lemma, which we will use from now on, most of the time without reference.

Lemma 2.2. *Let $\mathcal{C}$ be an $\mathcal{F}$-conjugacy class of $\mathcal{F}$. Then $\mathcal{F}$ is $\mathcal{C}$-saturated if and only if the following two conditions hold:*

(I) *(Sylow axiom) Each subgroup $P \in \mathcal{C}$ that is fully $\mathcal{F}$-normalised is also fully $\mathcal{F}$-centralised and fully automised in $\mathcal{F}$.*
(II) *(Extension axiom) Each subgroup $P \in \mathcal{C}$ that is fully $\mathcal{F}$-centralised is also receptive in $\mathcal{F}$.*

Furthermore, if $\mathcal{F}$ is $\mathcal{C}$-saturated, then for every fully $\mathcal{F}$-normalised $P \in \mathcal{C}$ and every $Q \in P^{\mathcal{F}}$, there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $Q\alpha = P$.

Proof. If (I) and (II) hold, then every fully $\mathcal{F}$-normalised subgroup $P \in \mathcal{C}$ is fully automised and receptive, and thus $\mathcal{F}$ is $\mathcal{C}$-saturated. On the other hand, if $\mathcal{F}$ is $\mathcal{C}$-saturated, it follows from [2, Lemma I.2.6(c)] that (I) and (II) and the statement of the lemma hold. □

Corollary 2.3. *If $\mathcal{F}$ is saturated and $P \leq S$ is fully $\mathcal{F}$-normalised, then for every $Q \in P^{\mathcal{F}}$, there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $Q\alpha = P$.*

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Recall that a subgroup $P \leq S$ is called $\mathcal{F}$*-centric* if $C_S(Q) \leq Q$ for every $Q \in P^{\mathcal{F}}$. Write $\mathcal{F}^c$ for the set of $\mathcal{F}$-centric subgroups of S. We will need the following lemma.

Lemma 2.4. *Let $\mathcal{F}$ be a fusion system over S, and let $P \leq S$ be fully $\mathcal{F}$-centralised. Then P is $\mathcal{F}$-centric if and only if $C_S(P) \leq P$.*

Proof. If P is $\mathcal{F}$-centric, then clearly $C_S(P) \leq P$. Suppose now that $C_S(P) \leq P$. Then for any $Q \in P^{\mathcal{F}}$, we have $C_S(Q) \geq Z(Q) \cong Z(P) = C_S(P)$. Hence, as P is fully $\mathcal{F}$-centralised, we have $C_S(Q) = Z(Q) \leq Q$ for all $Q \in P^{\mathcal{F}}$: that is, P is $\mathcal{F}$-centric. □

We recall that $O_p(\mathcal{F})$ denotes the largest subgroup of S that is normal in $\mathcal{F}$ ([2, Definition I.4.3]). Moreover, if $\mathcal{F}$ is saturated and $O_p(\mathcal{F}) \in \mathcal{F}^c$, then $\mathcal{F}$ is called *constrained* ([2, Definition I.4.8]). The Model Theorem for constrained fusion systems [2, Theorem III.5.10] guarantees that every constrained fusion system $\mathcal{F}$ over S has a model: that is, there is a finite group G such that $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G) = \mathcal{F}$ and $C_G(O_p(G)) \leq O_p(G)$.

Definition 2.5.

- A subgroup $P \leq S$ is called $\mathcal{F}$*-radical* if there exists a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate Q of P such that $O_p(N_{\mathcal{F}}(Q)) = Q$. We denote by $\mathcal{F}^r$ the set of $\mathcal{F}$-radical subgroups of S.
- Set $\mathcal{F}^{cr} = \mathcal{F}^c \cap \mathcal{F}^r$, and call the elements of $\mathcal{F}^{cr}$ the $\mathcal{F}$-centric radical subgroups of S.
- A subgroup $P \leq S$ is called $\mathcal{F}$*-critical* if P is $\mathcal{F}$-centric and, for every $\mathcal{F}$-conjugate Q of P, we have

$$\mathrm{Out}_S(Q) \cap O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1.$$

As remarked before, our definition of radical subgroups differs from the usual one given, for example, in [2, Definition I.3.1]. We show in part (b) of our next lemma that, for a saturated fusion system $\mathcal{F}$, the set $\mathcal{F}^{cr}$ equals the set of $\mathcal{F}$-centric radical subgroups in the usual definition.

Lemma 2.6.

(a) *For every $R \leq S$, the following implications hold:*

$$R \in \mathcal{F}^c \text{ and } O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R) \Longrightarrow R \text{ is } \mathcal{F}\text{-critical} \Longrightarrow R \in \mathcal{F}^{cr}.$$

(b) *If $\mathcal{F}$ is saturated, then we have*

$$\mathcal{F}^{cr} = \{R \in \mathcal{F}^c : O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)\} = \{R \leq S : R \text{ is } \mathcal{F}\text{-critical}\}.$$

Proof. If $\mathrm{Inn}(R) = O_p(\mathrm{Aut}_{\mathcal{F}}(R))$, then $\mathrm{Inn}(Q) = O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_S(Q) \cap O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ for every $\mathcal{F}$-conjugate Q of R, so R is $\mathcal{F}$-critical if in addition $R \in \mathcal{F}^c$. This shows the first implication in (a).

Now let $R \in \mathcal{F}^c$ such that $R \notin \mathcal{F}^r$. If we pick a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate Q of R, we have $Q < Q^* := O_p(N_{\mathcal{F}}(Q))$. So $\mathrm{Inn}(Q) < \mathrm{Aut}_{Q^*}(Q)$ as $Q \in \mathcal{F}^c$. Moreover, $\mathrm{Aut}_{Q^*}(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$, as by definition of Q^* every element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(Q^*)$. Hence, $\mathrm{Inn}(Q) < \mathrm{Aut}_{Q^*}(Q) \leq \mathrm{Aut}_S(Q) \cap O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. This shows that R is not $\mathcal{F}$-critical, so (a) holds. In particular,

$$\mathcal{F}^{cr} \supseteq \{R \leq S : R \text{ is } \mathcal{F}\text{-critical}\} \supseteq \{R \in \mathcal{F}^c : O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)\}.$$

For the proof of (b), it is thus sufficient to show that $O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)$ for every $R \in \mathcal{F}^{cr}$. Now fix $R \in \mathcal{F}^{cr}$. Since the property $O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)$ is preserved if R is replaced by an $\mathcal{F}$-conjugate, we may assume without loss of generality that R is fully $\mathcal{F}$-normalised and $R = O_p(N_{\mathcal{F}}(R))$. Note that $N_{\mathcal{F}}(R)$ is saturated. So as $R \in \mathcal{F}^c$, the subsystem $N_{\mathcal{F}}(R)$ is constrained. Thus, we may choose a model G for $N_{\mathcal{F}}(R)$. Then $O_p(G) = R = O_p(N_{\mathcal{F}}(R))$ and

$$\mathrm{Aut}_{\mathcal{F}}(R) \cong G/C_G(R) \cong G/Z(R).$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Hence, $O_p(\mathrm{Aut}_{\mathcal{F}}(R)) \cong O_p(G/Z(R)) = R/Z(R) \cong \mathrm{Inn}(R)$. Thus $O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)$. This proves that $\mathcal{F}^{cr} \subseteq \{R \in \mathcal{F}^c : O_p(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Inn}(R)\}$, so (b) holds. □

The $\mathcal{F}$-critical subgroups play a crucial role in showing that a fusion system is saturated. This is made precise in the following theorem, which we restate for the reader's convenience.

Theorem 2.7. *Suppose Δ is a set of subgroups of S that is closed under $\mathcal{F}$-conjugacy and contains every $\mathcal{F}$-critical subgroup. If $\mathcal{F}$ is Δ-generated and Δ-saturated, then $\mathcal{F}$ is saturated.*

Proof. This is a reformulation of [5, Theorem 2.2]. The reader might also want to note that the theorem follows from Lemma 2.9 below. □

Later, we will need to prove saturation in a situation where it appears impossible to apply Theorem 2.7 directly. We will therefore have a closer look at the arguments used in the proof of that theorem.

Define a partial order $\leq$ on the set of $\mathcal{F}$-conjugacy classes by writing $\mathcal{P} \leq \mathcal{Q}$ if some (and thus every) element of $\mathcal{Q}$ contains an element of $\mathcal{P}$.

Lemma 2.8. *Let $\mathcal{H}$ be a set of subgroups of S closed under $\mathcal{F}$-conjugacy such that $\mathcal{F}$ is $\mathcal{H}$-generated and $\mathcal{H}$-saturated. Let $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class that is maximal with respect to $\leq$ among those $\mathcal{F}$-conjugacy classes that are not contained in $\mathcal{H}$.*

Write $\mathcal{S}_{\geq P} \supseteq \mathcal{S}_{>P}$ for the sets of subgroups of $N_S(P)$ that contain, or properly contain, an element $P \in \mathcal{P}$. Then the following hold for every $P \in \mathcal{P}$ that is fully $\mathcal{F}$-normalised:

(a) *The subsystem $N_{\mathcal{F}}(P)$ is $\mathcal{S}_{>P}$-generated and $\mathcal{S}_{>P}$-saturated.*
(b) *If $N_{\mathcal{F}}(P)$ is $\mathcal{S}_{\geq P}$-saturated, then $\mathcal{F}$ is $\mathcal{H} \cup \mathcal{P}$-saturated.*
(c) *P is fully $\mathcal{F}$-centralised. Moreover, for every $Q \in \mathcal{P}$, there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $Q\alpha = P$.*
(d) *If $C_S(P) \leq P$, then P is $\mathcal{F}$-centric, and if $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, then $\mathrm{Out}_S(Q) \cap O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$ for all $Q \in \mathcal{P}$. In particular, if $C_S(P) \leq P$ and $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, then P is $\mathcal{F}$-critical.*

Proof. Basically, this follows from [5, Lemma 2.4] and its proof. We take the opportunity to point out the following small error in the proof of part (b) of that lemma: in l.9 on p.334 of [5] it says 'replacing each φ_i by $\chi_i \circ \varphi_i \circ \chi_i^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\chi_i(Q_i), S)$'. However, it should be 'replacing each φ_i by $\chi_{i+1} \circ \varphi_i \circ \chi_i^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\chi_i(Q_i), S)$'.

Let us now explain in detail how the assertion follows. By [5, Lemma 2.4(a), (c)], $N_{\mathcal{F}}(P)$ is $\mathcal{S}_{>P}$-saturated and (b) holds. To prove (a), one needs to argue that $N_{\mathcal{F}}(P)$ is also $\mathcal{S}_{>P}$-generated. If $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(P)}(A, B)$, then φ extends to a morphism $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(AP, BP)$ that normalises P. If $A \not\leq P$, then $AP \in \mathcal{S}_{>P}$, and thus φ is the restriction of a morphism between subgroups in $\mathcal{S}_{>P}$. If $A \leq P$, then $\hat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(P)$, and by [5, Lemma 2.4(b)], $\hat{\varphi}$ (and thus φ) is a composite of restrictions of morphisms in $N_{\mathcal{F}}(P)$ between subgroups in $\mathcal{S}_{>P}$. Hence, $N_{\mathcal{F}}(P)$ is $\mathcal{S}_{>P}$ generated, and (a) holds.

The following property is shown in the proof of Lemma 2.4 in [5] (see p.333, property (3)).

(*) There is a subgroup $\hat{P} \in \mathcal{P}$ fully $\mathcal{F}$-centralised such that, for all $Q \in \mathcal{P}$, there exists a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(\hat{P}))$ with $Q\varphi = \hat{P}$.

In particular, there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(\hat{P}))$ such that $P\psi = \hat{P}$. Since P is fully $\mathcal{F}$-normalised, the map ψ is an isomorphism. In particular, P is fully $\mathcal{F}$-centralised as $\hat{P}$ is fully $\mathcal{F}$-centralised. Moreover, if $Q \in \mathcal{P}$, then by (*), there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(\hat{P}))$ with $Q\varphi = \hat{P}$, and we have $\alpha := \varphi\psi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$, with $Q\alpha = \hat{P}\psi^{-1} = P$. Hence, (c) holds.

It follows from Lemma 2.4 and the first part of (c) that P is $\mathcal{F}$-centric if $C_S(P) \leq P$. If $Q \in \mathcal{P}$, then the second part of (c) allows us to choose $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ with $Q\alpha = P$. Notice that the map $\alpha^*\colon \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(P), \gamma \mapsto \alpha^{-1}\gamma\alpha$ is a group isomorphism with $\mathrm{Inn}(Q)\alpha^* = \mathrm{Inn}(P)$ and $\mathrm{Aut}_S(Q)\alpha^* \leq \mathrm{Aut}_S(P)$. So α^* induces an isomorphism from $\mathrm{Out}_{\mathcal{F}}(Q)$ to $\mathrm{Out}_{\mathcal{F}}(P)$ that maps $\mathrm{Out}_S(Q) \cap O_p(\mathrm{Out}_{\mathcal{F}}(Q))$ into $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P))$. This implies (d). □

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma 2.9. *Suppose that $\mathcal{H}$ is one of the following two sets:*

(i) *the set of all subgroups of S respecting $\mathcal{F}$-saturation; or*
(ii) *the set of all $P \leq S$ such that every subgroup of S containing an $\mathcal{F}$-conjugate of P respects $\mathcal{F}$-saturation.*

Assume that $\mathcal{F}$ is $\mathcal{H}$-generated. Let $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class, which is maximal with respect to $\leq$ among the $\mathcal{F}$-conjugacy classes not contained in $\mathcal{H}$. Then the elements of $\mathcal{P}$ are $\mathcal{F}$-critical.

Proof. Observe first that $\mathcal{H}$ is in either case closed under $\mathcal{F}$-conjugacy and $\mathcal{F}$ is $\mathcal{H}$-saturated. Moreover, note that $\mathcal{F}$ is not $\mathcal{H} \cup \mathcal{P}$-saturated; this is clear if $\mathcal{H}$ is as in (i); if $\mathcal{H}$ is as in (ii), then note that because of the maximal choice of $\mathcal{P}$, $\mathcal{F}$ being $\mathcal{H} \cup \mathcal{P}$-saturated would imply that every subgroup containing an element of $\mathcal{P}$ would be either in $\mathcal{P}$ or in $\mathcal{H}$ and thus respect $\mathcal{F}$-saturation.

Now fix $P \in \mathcal{P}$ fully $\mathcal{F}$-normalised. By Lemma 2.8(a),(b) (using the notation introduced in that lemma), $N_{\mathcal{F}}(P)$ is $\mathcal{S}_{>P}$ generated and $\mathcal{S}_{>P}$-saturated, but not $\mathcal{S}_{\geq P}$-saturated as $\mathcal{F}$ is not $\mathcal{H} \cup \mathcal{P}$-saturated. Thus, by [5, Lemma 2.5] applied with $N_{\mathcal{F}}(P)$ in place of $\mathcal{F}$, we have $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ and $P \in N_{\mathcal{F}}(P)^c$. It now follows from Lemma 2.8(d) that P is $\mathcal{F}$-critical. □

Lemma 2.10. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$ over $T \leq S$. Let $P \leq S$, and set $P_0 := P \cap T$. Then the following hold:*

(a) *If $C_S(P) \leq P$ and $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, then $C_T(P_0) \leq P_0$ and $\mathrm{Out}_T(P_0) \cap O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) = 1$.*
(b) *If $\mathcal{E}$ is saturated, P_0 is fully $\mathcal{E}$-normalised and P is $\mathcal{F}$-critical, then $P_0 \in \mathcal{E}^{cr}$.*

Proof. Assume that $C_T(P_0) \not\leq P_0$ or $\mathrm{Out}_T(P_0) \cap O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) \neq 1$. Let Q be the preimage of $\mathrm{Out}_T(P_0) \cap O_p(\mathrm{Out}_{\mathcal{E}}(P_0))$ in $N_T(P_0)$. Note that $C_T(P_0) \leq Q$, so our assumption implies in any case that $P_0 < Q$ and thus $Q \not\leq P$. As P normalises P_0, P also normalises Q. Hence, PQ is a p-group with $P < PQ$. This yields $P < N_{PQ}(P) = PN_Q(P)$, and thus $N_Q(P) \not\leq P$. Note that $[P, N_Q(P)] \leq P \cap T = P_0$. So $X := \langle \mathrm{Aut}_Q(P)^{\mathrm{Aut}_{\mathcal{F}}(P)} \rangle$ acts trivially on P/P_0. At the same time, by definition of Q, the elements of $\mathrm{Aut}_Q(P)$, and thus the elements of X, restrict to automorphisms of P_0 that lie in $O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P_0)$. Hence, a p'-element of X acts trivially on P/P_0 and on P_0, and therefore it is trivial by properties of coprime action (compare [21, 8.2.2(b)] or [2, Lemma A.2]). This shows that X is a p-group, so $\mathrm{Aut}_Q(P) \leq \mathrm{Aut}_S(P) \cap O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. As $N_Q(P) \not\leq P$, this implies either $C_S(P) \not\leq P$ or $1 \neq \mathrm{Out}_Q(P) \leq \mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P))$. This proves (a).

Assume now that $\mathcal{E}$ is saturated, P_0 is fully $\mathcal{E}$-normalised and P is $\mathcal{F}$-critical. The latter condition implies $C_S(P) \leq P$ and $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. So by (a), $C_T(P_0) \leq P_0$ and $\mathrm{Out}_T(P_0) \cap O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) = 1$. As $\mathcal{E}$ is saturated and P_0 is fully $\mathcal{E}$-normalised, P_0 is fully $\mathcal{E}$-centralised and fully $\mathcal{E}$-automised. Hence, P_0 is $\mathcal{E}$-centric by Lemma 2.4 and $O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) \leq \mathrm{Out}_T(P_0)$. The latter condition implies $O_p(\mathrm{Out}_{\mathcal{E}}(P_0)) = 1$. Using Lemma 2.6(a), we conclude that $P_0 \in \mathcal{E}^{cr}$. □

The basic idea in the proof of the following theorem is taken from Step 5 of the proof of [23, Theorem 9]. As we argue afterwards, it easily implies Theorem A. If $P, Q \leq S$, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $A \leq \mathrm{Aut}_{\mathcal{F}}(P)$, note that $A^{\varphi} := \varphi^{-1} A \varphi \leq \mathrm{Aut}_{\mathcal{F}}(P\varphi)$.

Theorem 2.11. *Let $\mathcal{H}$ be the set of all subgroups $P \leq S$ such that every subgroup of S containing an $\mathcal{F}$-conjugate of P respects $\mathcal{F}$-saturation. Assume that $\mathcal{F}$ is $\mathcal{H}$-generated. Suppose furthermore that there exists an $\mathcal{F}$-invariant saturated subsystem $\mathcal{E}$ of $\mathcal{F}$ with $\mathcal{E}^{cr} \subseteq \mathcal{H}$. Then $\mathcal{F}$ is saturated.*

Proof. Let $T \leq S$ be such that $\mathcal{E}$ is a subsystem of $\mathcal{F}$ over T. By $\mathcal{H}_0$, denote the set of subgroups of T that are elements of $\mathcal{H}$. Write $\mathcal{H}^{\perp}$ for the set of subgroups of S not in $\mathcal{H}$ and $\mathcal{H}_0^{\perp}$ for the set of elements of $\mathcal{H}^{\perp}$ that are subgroups of T. If $P \leq S$, write P_0 for $P \cap T$, and similarly for subgroups of S with different names.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

For the proof of the assertion, we will proceed in three steps. In the first two steps, we will argue that property (2.1) below implies the assertion, and in the third step, we will prove that (2.1) holds.

$$\text{Let } P \leq S \text{ such that } P_0 \text{ is an element of } \mathcal{H}_0^{\perp} \text{ of maximal order. If } P_0 \text{ is not fully } \mathcal{E}\text{-normalised, then there exists } P'' \in P^{\mathcal{F}} \text{ such that } |N_T(P_0)| < |N_T(P_0'')|. \tag{2.1}$$

Step 1: We argue that (2.1) implies the following property:

$$\text{If } P \leq S \text{ such that } P_0 \text{ is an element of } \mathcal{H}_0^{\perp} \text{ of maximal order, then there exists an } \mathcal{F}\text{-conjugate } Q \text{ of } P \text{ such that } Q_0 \text{ is fully } \mathcal{E}\text{-normalised.} \tag{2.2}$$

This can be seen as follows. If P is as in (2.2) and P_0 is fully $\mathcal{E}$-normalised, then we are done. If not, then by (2.1), we can pass on to an $\mathcal{F}$-conjugate P'' of P such that $|N_T(P_0)| < |N_T(P_0'')|$. Again, if P_0'' is fully $\mathcal{E}$-normalised, then we are done; otherwise, we can repeat the process. Since T is finite, we will eventually end up with an $\mathcal{F}$-conjugate Q of P such that Q_0 is fully $\mathcal{E}$-normalised. So (2.1) implies (2.2).

Step 2: We show that property (2.2) implies the assertion. Assume that (2.2) holds and $\mathcal{F}$ is not saturated. Then $\mathcal{H}^{\perp} \neq \emptyset$. By construction, $\mathcal{H} \supseteq \mathcal{H}_0$ is $\mathcal{F}$-closed. So, for every subgroup $P \leq S$, $P_0 \in \mathcal{H}_0$ implies $P \in \mathcal{H}$. In particular, as $\mathcal{H}^{\perp} \neq \emptyset$, we can conclude $\mathcal{H}_0^{\perp} \neq \emptyset$. Set

$$m := \max\{|Q| : Q \in \mathcal{H}_0^{\perp}\},$$
$$\mathcal{M} := \{P \in \mathcal{H}^{\perp} : |P_0| = m\}.$$

For any $Q \in \mathcal{H}_0^{\perp}$ with $|Q| = m$, we have $Q = Q_0$, and thus $Q \in \mathcal{M}$. Hence, $\mathcal{M} \neq \emptyset$. As T is strongly closed, $\mathcal{M}$ is closed under $\mathcal{F}$-conjugacy. Thus we can choose an $\mathcal{F}$-conjugacy class $\mathcal{P} \subseteq \mathcal{M}$ such that the elements of $\mathcal{P}$ are of maximal order among the elements of $\mathcal{M}$.

We argue first that $\mathcal{P}$ is maximal with respect to $\leq$ among the $\mathcal{F}$-conjugacy classes contained in $\mathcal{H}^{\perp}$. For that, let $P \in \mathcal{P}$ and $P \leq R \leq S$. We need to show that $R = P$ or $R \in \mathcal{H}$. Notice that $m = |P_0| \leq |R_0|$. If $m < |R_0|$, then by definition of m, we have $R_0 \in \mathcal{H}_0 \subseteq \mathcal{H}$, and thus $R \in \mathcal{H}$. If $m = |R_0|$, then $R \in \mathcal{M}$ or $R \in \mathcal{H}$. So the maximality of $|P|$ yields $R = P$ or $R \in \mathcal{H}$. Hence, $\mathcal{P}$ is maximal with respect to $\leq$ among the $\mathcal{F}$-conjugacy classes contained in $\mathcal{H}^{\perp}$. Thus Lemma 2.9 implies that the elements of $\mathcal{P}$ are $\mathcal{F}$-critical. By (2.2), there exists $P \in \mathcal{P}$ such that P_0 is fully $\mathcal{E}$-normalised. Then by Lemma 2.10(b), we have $P_0 \in \mathcal{E}^{cr}$. Since by assumption $\mathcal{E}^{cr} \subseteq \mathcal{H}$, it follows that $P_0 \in \mathcal{H}$, so $P \in \mathcal{H}$. This contradicts the choice of $\mathcal{P} \subseteq \mathcal{M} \subseteq \mathcal{H}^{\perp}$. Hence, (2.2) implies that $\mathcal{F}$ is saturated.

Step 3: We complete the proof by proving (2.1). Whenever we have subgroups $Q_1 \leq Q \leq S$, we set $\mathrm{Aut}_{\mathcal{F}}(Q : Q_1) := N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(Q_1)$, $\mathrm{Aut}_S(Q : Q_1) := N_{\mathrm{Aut}_S(Q)}(Q_1)$, $N_S(Q : Q_1) := N_S(Q) \cap N_S(Q_1)$, and so on. Now let $P \leq S$ be such that P_0 is an element of $\mathcal{H}_0^{\perp}$ of maximal order that is not fully $\mathcal{E}$-normalised. Note that $T \in \mathcal{E}^{cr} \subseteq \mathcal{H}$ and thus $P_0 \neq T$. Hence, $P_0 < N_T(P_0)$. So the maximality of $|P_0|$ yields $R := N_T(P_0) \in \mathcal{H}_0$. In particular, every $\mathcal{F}$-conjugate of R respects $\mathcal{F}$-saturation. Since $\mathcal{E}$ is saturated, there exists $\rho \in \mathrm{Hom}_{\mathcal{E}}(R, T)$ such that $P_0' := P_0\rho$ is fully $\mathcal{E}$-normalised. As $R' := R\rho$ respects $\mathcal{F}$-saturation, there exists $\sigma \in \mathrm{Hom}_{\mathcal{F}}(R', S)$ such that $R'' := R'\sigma$ is fully automised and receptive. Set $P_0'' := P_0'\sigma = P_0\rho\sigma$. As R'' is fully automised, by Sylow's Theorem, there exists $\xi \in \mathrm{Aut}_{\mathcal{F}}(R'')$ such that

$$\mathrm{Aut}_{\mathcal{F}}(R'' : P_0'')^{\xi} \cap \mathrm{Aut}_S(R'') = \mathrm{Aut}_S(R'' : P_0''\xi)$$

is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R'' : P_0'')^{\xi} = \mathrm{Aut}_{\mathcal{F}}(R'' : P_0''\xi)$. So replacing σ by $\sigma\xi$, we may assume

$$\mathrm{Aut}_S(R'' : P_0'') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R'' : P_0'')).$$

Notice that $\mathrm{Aut}_S(R : P_0)^{\rho\sigma}$ and $\mathrm{Aut}_S(R' : P_0')^{\sigma}$ are p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(R'' : P_0'')$. Hence, again by Sylow's Theorem, there exist $\gamma, \delta \in \mathrm{Aut}_{\mathcal{F}}(R'' : P_0'')$ such that $\mathrm{Aut}_S(R : P_0)^{\rho\sigma\gamma}$ and $\mathrm{Aut}_S(R' : P_0')^{\sigma\delta}$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

are contained in $\mathrm{Aut}_S(R'' : P_0'')$. This implies $N_S(R : P_0) \leq N_{\rho\sigma\gamma}$ and $N_S(R' : P_0') \leq N_{\sigma\delta}$. Hence, as R'' is receptive, the map $\rho\sigma\gamma \in \mathrm{Hom}_{\mathcal{F}}(R, R'')$ extends to $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(R : P_0), S)$, and similarly $\sigma\delta \in \mathrm{Hom}_{\mathcal{F}}(R', R'')$ extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(R' : P_0'), S)$. Notice that

$$P_0\varphi = P_0\rho\sigma\gamma = P_0''\gamma = P_0'' \text{ and } P_0'\psi = P_0'\sigma\delta = P_0''\delta = P_0''.$$

Moreover, since $P_0 = P \cap T$, $R = N_T(P_0)$ and T is strongly closed, we have $P \leq N_S(P_0) = N_S(R : P_0)$. Hence

$$P'' := P\varphi$$

is a well-defined $\mathcal{F}$-conjugate of P, for which we have indeed that $P'' \cap T = P\varphi \cap T = (P \cap T)\varphi = P_0\varphi = P_0''$ as our notation suggests. So it only remains to show that $|N_T(P_0)| < |N_T(P_0'')|$. We will show this by arguing that

$$|N_T(P_0)| = |R| < |N_T(R' : P_0')| \leq |N_T(P_0'')|. \tag{2.3}$$

Recall that P_0 is by assumption not fully $\mathcal{E}$-normalised, whereas $P_0' = P_0\rho$ is fully $\mathcal{E}$-normalised. Therefore, we have $R' = R\rho = N_T(P_0)\rho < N_T(P_0')$. Thus $R' < N_{N_T(P_0')}(R') = N_T(R' : P_0')$. As $|R| = |R'|$, this shows the first inequality in (2.3). As $P_0'\psi = P_0''$ and T is strongly closed, we have $N_T(R' : P_0')\psi \leq N_T(P_0'')$. Since ψ is injective, this shows the second inequality in (2.3). So the proof of (2.1) is complete. As argued in Steps 1 and 2, this proves the assertion. □

Proof of Theorem A. Write $\mathcal{H}$ for the set of all subgroups $P \leq S$ such that every subgroup of S containing an $\mathcal{F}$-conjugate of P respects $\mathcal{F}$-saturation. Since Δ is $\mathcal{F}$-closed and $\mathcal{F}$ is Δ-saturated, it follows that $\Delta \subseteq \mathcal{H}$. Hence, as $\mathcal{F}$ is Δ-generated, $\mathcal{F}$ is also $\mathcal{H}$-generated. Moreover, $\mathcal{E}^{cr} \subseteq \Delta \subseteq \mathcal{H}$. Thus the assertion follows from Theorem 2.11 above. □

3. Partial groups and localities

In this section, we introduce some basic definitions and notations that will be used in the remainder of the paper. We refer the reader to [8] for a more comprehensive introduction to partial groups and localities.

3.1. *Partial groups*

Following the notation introduced in [8], we will write $\mathbf{W}(\mathcal{L})$ for the set of words in a set $\mathcal{L}$. The elements of $\mathcal{L}$ will be identified with the words of length one, and $\emptyset$ denotes the empty word. The concatenation of words $u_1, u_2, \ldots, u_k \in \mathbf{W}(\mathcal{L})$ will be denoted $u_1 \circ u_2 \circ \cdots \circ u_k$.

Definition 3.1 [8, Definition 1.1]. Suppose $\mathcal{L}$ is a nonempty set and $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$. Let $\Pi\colon \mathbf{D} \longrightarrow \mathcal{L}$ be a map, and let $(-)^{-1}\colon \mathcal{L} \longrightarrow \mathcal{L}$ be an involutory bijection, which we extend to a map

$$(-)^{-1}\colon \mathbf{W}(\mathcal{L}) \longrightarrow \mathbf{W}(\mathcal{L}), w = (g_1, \ldots, g_k) \mapsto w^{-1} = (g_k^{-1}, \ldots, g_1^{-1}).$$

Then $\mathcal{L}$ is called a *partial group* with product Π and inversion $(-)^{-1}$ if the following hold for all words $u, v, w \in \mathbf{W}(\mathcal{L})$:

- $\mathcal{L} \subseteq \mathbf{D}$ and

$$u \circ v \in \mathbf{D} \Longrightarrow u, v \in \mathbf{D}.$$

  (So in particular, $\emptyset \in \mathbf{D}$.)
- Π restricts to the identity map on $\mathcal{L}$.
- $u \circ v \circ w \in \mathbf{D} \Longrightarrow u \circ (\Pi(v)) \circ w \in \mathbf{D}$, and $\Pi(u \circ v \circ w) = \Pi(u \circ (\Pi(v)) \circ w)$.
- $w \in \mathbf{D} \Longrightarrow w^{-1} \circ w \in \mathbf{D}$ and $\Pi(w^{-1} \circ w) = \mathbf{1}$, where $\mathbf{1} := \Pi(\emptyset)$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

For the remainder of this section, let $\mathcal{L}$ always be a partial group with product $\Pi\colon \mathbf{D} \to \mathcal{L}$. As above, set $\mathbf{1} := \Pi(\emptyset)$.

If $w = (f_1, \ldots, f_n) \in \mathbf{D}$, then we sometimes write $f_1 f_2 \cdots f_n$ for $\Pi(f_1, \ldots, f_n)$. In particular, if $(x, y) \in \mathbf{D}$, then xy denotes the product $\Pi(x, y)$.

Notation 3.2.

- For every $f \in \mathcal{L}$, write $\mathbf{D}(f) := \{x \in \mathcal{L}\colon (f^{-1}, x, f) \in \mathbf{D}\}$ for the set of all x such that the conjugate $x^f := \Pi(f^{-1}, x, f)$ is defined.
- By c_f, denote the conjugation map $c_f : \mathbf{D}(f) \to \mathcal{L}, x \mapsto x^f$.
- For $f \in \mathcal{L}$ and $\mathcal{X} \subseteq \mathbf{D}(f)$, set $\mathcal{X}^f := \{x^f : x \in \mathcal{X}\}$.
- Given $\mathcal{X} \subseteq \mathcal{L}$, set

$$N_{\mathcal{L}}(\mathcal{X}) := \{f \in \mathcal{L}\colon \mathcal{X} \subseteq \mathbf{D}(f) \text{ and } \mathcal{X}^f = \mathcal{X}\}$$

  and

$$C_{\mathcal{L}}(\mathcal{X}) := \{f \in \mathcal{L}\colon x \in \mathbf{D}(f) \text{ and } x^f = x \text{ for all } x \in \mathcal{X}\}.$$

- For $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{L}$, define $N_{\mathcal{Y}}(\mathcal{X}) = \mathcal{Y} \cap N_{\mathcal{L}}(\mathcal{X})$ and $C_{\mathcal{Y}}(\mathcal{X}) = \mathcal{Y} \cap C_{\mathcal{L}}(\mathcal{X})$.

We call $N_{\mathcal{Y}}(\mathcal{X})$ the *normaliser* of $\mathcal{X}$ in $\mathcal{Y}$ and $C_{\mathcal{Y}}(\mathcal{X})$ the *centraliser* of $\mathcal{X}$ in $\mathcal{Y}$.

Definition 3.3.

- A subset $\mathcal{H} \subseteq \mathcal{L}$ is called a *partial subgroup* of $\mathcal{L}$ if $h^{-1} \in \mathcal{H}$ for all $h \in \mathcal{H}$, and moreover $\Pi(w) \in \mathcal{H}$ for all $w \in \mathbf{D} \cap \mathbf{W}(\mathcal{H})$.
- A partial subgroup $\mathcal{H}$ of $\mathcal{L}$ is a called a *subgroup* of $\mathcal{L}$ if $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}(\mathcal{L})$.
- By a *p-subgroup* of $\mathcal{L}$, we mean a subgroup S of $\mathcal{L}$ such that $|S|$ is a power of p.
- Let $\mathcal{N}$ be a partial subgroup of $\mathcal{L}$. Then we call $\mathcal{N}$ a *partial normal subgroup* of $\mathcal{L}$ (and write $\mathcal{N} \trianglelefteq \mathcal{L}$) if $n^f \in \mathcal{N}$ for all $f \in \mathcal{L}$ and all $n \in \mathcal{N} \cap \mathbf{D}(f)$.
- A partial subgroup $\mathcal{H}$ of $\mathcal{L}$ is called a *partial subnormal subgroup* of $\mathcal{L}$ if there exists a series $\mathcal{H} = \mathcal{H}_0 \trianglelefteq \mathcal{H}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{H}_k = \mathcal{L}$ of partial subgroups of $\mathcal{L}$.

If $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$, notice that $\mathcal{H}$ is itself a partial group with product $\Pi|_{\mathbf{W}(\mathcal{H})\cap\mathbf{D}}$. If $\mathcal{H}$ is a subgroup of $\mathcal{L}$, then $\mathcal{H}$ forms a group with binary product defined by $x \cdot y := \Pi(x, y)$ for all $x, y \in \mathcal{H}$. In particular, every p-subgroup of $\mathcal{L}$ forms a p-group. The following definition will be crucial in the definition of a locality.

Definition 3.4. Let $\mathcal{L}$ be a partial group, and let Δ be a collection of subgroups of $\mathcal{L}$. Define $\mathbf{D}_\Delta$ to be the set of words $w = (g_1, \ldots, g_k) \in \mathbf{W}(\mathcal{L})$ such that there exist $P_0, \ldots, P_k \in \Delta$ with $P_{i-1} \subseteq \mathbf{D}(g_i)$ and $P_{i-1}^{g_i} = P_i$ for all $1 \le i \le k$. If such w and $P_0, \ldots, P_k$ are given, then we also say that $w \in \mathbf{D}_\Delta$ via $P_0, P_1, \ldots, P_k$, or just that $w \in \mathbf{D}_\Delta$ via P_0.

Lemma 3.5. *Let $\mathcal{L}$ be a partial group and $\mathcal{N} \trianglelefteq \mathcal{L}$.*

(a) *If $f \in \mathcal{L}$ and $P \subseteq \mathbf{D}(f)$, then $(P \cap \mathcal{N})^f = P^f \cap \mathcal{N}$.*
(b) *Let S be a subgroup of $\mathcal{L}$ and Γ a set of subgroups of $\mathcal{N} \cap S$. Set*

$$\Delta := \{P \le S\colon P \cap \mathcal{N} \in \Gamma\}.$$

Then $\mathbf{D}_\Gamma = \mathbf{D}_\Delta$.

Proof. **(a)** Notice that $(P \cap \mathcal{N})^f \subseteq P^f \cap \mathcal{N}$ as $\mathcal{N} \trianglelefteq \mathcal{L}$. Since $(c_f)^{-1} = c_{f^{-1}}$ by [8, Lemma 1.6(c)], we have $(P^f)^{f^{-1}} = P$. So, similarly, $(P^f \cap \mathcal{N})^{f^{-1}} \subseteq P \cap \mathcal{N}$, and thus $P^f \cap \mathcal{N} \subseteq (P \cap \mathcal{N})^f$. This shows $(P \cap \mathcal{N})^f = P^f \cap \mathcal{N}$ as required.

(b) As $\Gamma \subseteq \Delta$, we have $\mathbf{D}_\Gamma \subseteq \mathbf{D}_\Delta$. Conversely, if $w = (f_1, \ldots, f_n) \in \mathbf{D}_\Delta$ via $P_0, P_1, \ldots, P_n \in \Delta$, then (a) yields that $w \in \mathbf{D}_\Gamma$ via $P_0 \cap \mathcal{N}, P_1 \cap \mathcal{N}, \ldots, P_n \cap \mathcal{N}$. Thus $\mathbf{D}_\Delta \subseteq \mathbf{D}_\Gamma$, implying $\mathbf{D}_\Gamma = \mathbf{D}_\Delta$. □

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

3.2. Localities

Definition 3.6. Let $\mathcal{L}$ be a partial group, and let S be a p-subgroup of $\mathcal{L}$. For $f \in \mathcal{L}$, set

$$S_f := \{x \in S : x \in \mathbf{D}(f) \text{ and } x^f \in S\}.$$

More generally, if $w = (f_1, \dots, f_k) \in \mathbf{W}(\mathcal{L})$, then write S_w for the set of $s \in S$ such that there exists a sequence $s = s_0, \dots, s_k$ of elements of S with $s_{i-1} \in \mathbf{D}(f_i)$ and $s_{i-1}^{f_i} = s_i$ for $i = 1, \dots, k$.

Definition 3.7. Let $\mathcal{L}$ be a finite partial group with product $\Pi\colon \mathbf{D} \to \mathcal{L}$, let S be a p-subgroup of $\mathcal{L}$, and let Δ be a nonempty set of subgroups of S. We say that $(\mathcal{L}, \Delta, S)$ is a *locality* if the following hold:

1. S is maximal with respect to inclusion among the p-subgroups of $\mathcal{L}$;
2. $\mathbf{D} = \mathbf{D}_\Delta$;
3. Δ is closed under passing to $\mathcal{L}$-conjugates and overgroups in S; that is, Δ is overgroup-closed in S and $P^f \in \Delta$ for all $P \in \Delta$ and $f \in \mathcal{L}$ with $P \subseteq S_f$.

If $(\mathcal{L}, \Delta, S)$ is a locality, then Δ is also called the set of *objects* of $(\mathcal{L}, \Delta, S)$.

As argued in [16, Remark 5.2], Definition 3.7 is equivalent to the definition of a locality given by Chermak [8, Definition 2.7]. We will use this fact throughout.

For the remainder of this subsection, let $(\mathcal{L}, \Delta, S)$ be a locality.

Lemma 3.8 (Important properties of localities). *The following hold:*

(a) *$N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$ for each $P \in \Delta$.*
(b) *Let $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq S_g$. Then $Q := P^g \in \Delta$, $N_{\mathcal{L}}(P) \subseteq \mathbf{D}(g)$, and*

$$c_g\colon N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q)$$

is an isomorphism of groups.
(c) *Let $w = (g_1, \dots, g_n) \in \mathbf{D}$ via $(X_0, \dots, X_n)$. Then*

$$c_{g_1} \circ \cdots \circ c_{g_n} = c_{\Pi(w)}$$

is a group isomorphism $N_{\mathcal{L}}(X_0) \to N_{\mathcal{L}}(X_n)$.
(d) *$S_g \in \Delta$ for every $g \in \mathcal{L}$. In particular, S_g is a subgroup of S. Moreover, $S_g^g = S_{g^{-1}}$, and $c_g\colon S_g \to S$ is an injective group homomorphism.*
(e) *For every $w \in \mathbf{W}(\mathcal{L})$, S_w is a subgroup of S. Moreover, $S_w \in \Delta$ if and only if $w \in \mathbf{D}$.*
(f) *If $w \in \mathbf{D}$, then $S_w \leq S_{\Pi(w)}$.*

Proof. Properties (a),(b) and (c) correspond to the statements (a),(b) and (c) in [8, Lemma 2.3] except for the fact stated in (b) that $Q \in \Delta$, which is however clearly true if one uses the definition of a locality above. Property (d) holds by [8, Proposition 2.5(a), (b)], and property (e) is stated in [8, Corollary 2.6]. Property (f) follows from (e) and (c). □

We use from now on without further reference that $c_g\colon S_g \to S$ is an injective group homomorphism for all $g \in \mathcal{L}$. By $\mathcal{F}_S(\mathcal{L})$, we denote the fusion system over S generated by all maps of this form. More generally, we have the following definition.

Definition 3.9. Let $\mathcal{H}$ be a partial subgroup of $\mathcal{L}$. Write $\mathcal{F}_{S\cap\mathcal{H}}(\mathcal{H})$ for the fusion system over $S \cap \mathcal{H}$ generated by the injective group homomorphisms of the form $c_h\colon S_h \cap \mathcal{H} \to S \cap \mathcal{H}$ with $h \in \mathcal{H}$.

Lemma 3.10. *Let $\mathcal{H}$ be a partial subgroup of $\mathcal{L}$ and $P \in \Delta$ with $P \leq S \cap \mathcal{H}$. Set $\mathcal{D} := \mathcal{F}_{S\cap\mathcal{H}}(\mathcal{H})$. Then the following hold:*

(a) *For every $\varphi \in \mathrm{Hom}_{\mathcal{D}}(P, S \cap \mathcal{H})$, there exists $h \in \mathcal{H}$ with $P \leq S_h$ and $\varphi = c_h$.*
(b) $N_{\mathcal{D}}(P) = \mathcal{F}_{N_{S\cap\mathcal{H}}(P)}(N_{\mathcal{H}}(P))$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Proof. For part (a), see [18, Lemma 3.11(b)]. Set $T := S \cap \mathcal{H}$. Clearly, $\mathcal{F}_{N_T(P)}(N_{\mathcal{H}}(P)) \subseteq N_{\mathcal{D}}(P)$ and $N_{\mathcal{D}}(P)$ is a fusion system over $N_T(P)$. If $\varphi \in \mathrm{Hom}_{N_{\mathcal{D}}(P)}(X,Y)$, where $X,Y \leq N_T(P)$ is a morphism in $N_{\mathcal{D}}(P)$, then we may assume that $P \leq X \cap Y$ and $P\varphi = P$. As $P \in \Delta$, we have $X,Y \in \Delta$. Hence, by (a), there exists $h \in \mathcal{H}$ such that $X \leq S_h$ and $\varphi = c_h|_X$. As $P\varphi = P$, it follows $h \in N_{\mathcal{H}}(P)$, and thus φ is a morphism in $\mathcal{F}_{N_T(P)}(N_{\mathcal{H}}(P))$, as required. □

Definition 3.11. If $(\mathcal{L}, \Delta, S)$ is a locality, then set

$$O_p(\mathcal{L}) := \bigcap\{S_w : w \in \mathbf{W}(\mathcal{L})\}.$$

Lemma 3.12. *If $(\mathcal{L}, \Delta, S)$ is a locality, then $O_p(\mathcal{L})$ is the unique largest p-subgroup of $\mathcal{L}$ that is a partial normal subgroup of $\mathcal{L}$. Moreover, a subgroup $P \leq S$ is a partial normal subgroup of $\mathcal{L}$ if and only if $N_{\mathcal{L}}(P) = \mathcal{L}$.*

Proof. This is proved as [18, Lemma 3.13] based on [8, Lemma 2.13]. □

We will use the characterisation of $O_p(\mathcal{L})$ given in Lemma 3.12 most of the time from now on without further reference.

Lemma 3.13. *Let $(\mathcal{L}, \Delta, S)$ be a locality, $\mathcal{N} \trianglelefteq \mathcal{L}$ and $T := \mathcal{N} \cap S$. Then for every $g \in \mathcal{L}$, there exist $n \in \mathcal{N}$ and $f \in N_{\mathcal{L}}(T)$ such that $(n, f) \in \mathbf{D}$, $g = nf$ and $S_g = S_{(n,f)}$.*

Proof. By the Frattini Lemma [8, Corollary 3.11], there exist $n \in \mathcal{N}$ and $f \in \mathcal{L}$ such that $(n, f) \in \mathbf{D}$, $g = nf$ and f is $\uparrow$-maximal (in the sense of [8, Definition 3.6]). By [8, Proposition 3.9] and then [8, Lemma 3.1(a)], every $\uparrow$-maximal element is in $N_{\mathcal{L}}(T)$. Moreover, the Splitting Lemma [8, Lemma 3.12] gives $S_g = S_{(n,f)}$. □

Definition 3.14. Let $(\mathcal{L}^+, \Delta^+, S)$ be a locality with a partial product $\Pi^+ \colon \mathbf{D}^+ \longrightarrow \mathcal{L}^+$. Suppose that $\emptyset \neq \Delta \subseteq \Delta^+$ such that Δ is $\mathcal{F}_S(\mathcal{L}^+)$-closed. Set

$$\mathcal{L}^+|_{\Delta} := \{f \in \mathcal{L}^+ : S_f \in \Delta\}.$$

Note that $\mathbf{D} := \mathbf{D}_{\Delta} \subseteq \mathbf{D}^+ \cap \mathbf{W}(\mathcal{L}^+|_{\Delta})$ and, by Lemma 3.8(c), $\Pi^+(w) \in \mathcal{L}|_{\Delta}$ for all $w \in \mathbf{D}$. We call $\mathcal{L} := \mathcal{L}^+|_{\Delta}$ together with the partial product $\Pi^+|_{\mathbf{D}} \colon \mathbf{D} \longrightarrow \mathcal{L}$ and the restriction of the inversion map on $\mathcal{L}^+$ to $\mathcal{L}$ the *restriction* of $\mathcal{L}^+$ to Δ.

With the hypothesis and notation as in the preceding definition, it turns out that the restriction of $\mathcal{L}^+$ to Δ forms a partial group and the triple $(\mathcal{L}^+|_{\Delta}, \Delta, S)$ is a locality (see [7, Lemma 2.21(a)] and [17, Lemma 2.23(a), (c)] for details). It might be worth pointing out that in the definition of the restriction, S_f and $\mathbf{D}_{\Delta}$ are a priori formed inside of $\mathcal{L}^+$, but as argued in [17, Lemma 2.23(b)], it does not matter whether one forms S_f and $\mathbf{D}_{\Delta}$ inside of $\mathcal{L}^+$ or inside of the partial group $\mathcal{L}^+|_{\Delta}$.

3.3. Homomorphisms of partial groups and projections

Definition 3.15. Let $\mathcal{L}$ and $\mathcal{L}'$ be partial groups with products $\Pi \colon \mathbf{D} \to \mathcal{L}$ and $\Pi' \colon \mathbf{D}' \to \mathcal{L}'$, respectively. Let $\alpha \colon \mathcal{L} \to \mathcal{L}'$, $f \mapsto f\alpha$ be a map.

- Write α^* for the induced map:

$$\alpha^* \colon \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}'), (f_1, \ldots, f_n) \mapsto (f_1\alpha, \ldots, f_n\alpha).$$

- We call α a *homomorphism of partial groups* if $\mathbf{D}\alpha^* \subseteq \mathbf{D}'$ and $\Pi(w)\alpha = \Pi'(w\alpha^*)$.
- The map α is called a *projection of partial groups* if α is a homomorphism of partial groups and $\mathbf{D}\alpha^* = \mathbf{D}'$.
- A bijective projection of partial groups is called an *isomorphism* of partial groups and an isomorphism from $\mathcal{L}$ to itself is called an *automorphism* of $\mathcal{L}$. Write $\mathrm{Aut}(\mathcal{L})$ for the set of automorphisms of $\mathcal{L}$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

- If $\alpha\colon \mathcal{L} \longrightarrow \mathcal{L}'$ is a homomorphism of partial groups and $\mathbf{1}'$ denotes the identity in $\mathcal{L}'$, then $\ker(\alpha) := \{f \in \mathcal{L}\colon f\alpha = \mathbf{1}'\}$ is called the kernel of α.

If $\mathcal{L}'$ is a partial group, then $\mathcal{L}' \subseteq \mathbf{D}'$. Hence, a projection of partial groups $\mathcal{L} \longrightarrow \mathcal{L}'$ is always surjective.

If $\alpha\colon \mathcal{L} \longrightarrow \mathcal{L}'$ is a homomorphism of partial groups, then $\ker(\alpha) \trianglelefteq \mathcal{L}$ by [8, Lemma 1.14]. If $(\mathcal{L}, \Delta, S)$ is a locality and $\mathcal{N} \trianglelefteq \mathcal{L}$, then conversely, one can construct a partial group $\mathcal{L}'$ and a projection of partial groups $\mathcal{L} \longrightarrow \mathcal{L}'$ with kernel $\mathcal{N}$. Namely, define a *coset* of $\mathcal{N}$ in $\mathcal{L}$ to be a subset of the form

$$\mathcal{N}f := \{\Pi(n, f)\colon n \in \mathcal{N},\ f \in \mathcal{L} \text{ such that } (n, f) \in \mathbf{D}\}.$$

Call a coset *maximal* if it is maximal with respect to inclusion among all cosets of $\mathcal{N}$ in $\mathcal{L}$. Write $\mathcal{L}/\mathcal{N}$ for the set of maximal cosets of $\mathcal{N}$ in $\mathcal{L}$. By [8, Proposition 3.14(d)], $\mathcal{L}/\mathcal{N}$ forms a partition of $\mathcal{L}$. Thus there is a natural map

$$\alpha\colon \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{N}$$

that sends every element $f \in \mathcal{L}$ to the unique maximal coset containing f. It turns out that the set $\mathcal{L}/\mathcal{N}$ can be (in a unique way) given the structure of a partial group such that the map α above is a projection of partial groups (compare [8, Lemma 3.16]). We therefore call the map α the *natural projection* from $\mathcal{L}$ to $\mathcal{L}/\mathcal{N}$. The identity element of this partial group is the maximal coset $\mathcal{N} = \mathcal{N}\mathbf{1}$. In particular, $\mathcal{N}$ is the kernel of the natural projection.

Given a locality $(\mathcal{L}, \Delta, S)$ and $\mathcal{N} \trianglelefteq \mathcal{L}$, we will sometimes write $\overline{\mathcal{L}}$ for the set $\mathcal{L}/\mathcal{N}$. We mean then implicitly that we use a kind of 'bar notation' similar to the way it is commonly used for groups. Namely, if X is an element or subset of $\mathcal{L}$, then $\overline{X}$ denotes the image of X under the natural projection $\alpha\colon \mathcal{L} \longrightarrow \overline{\mathcal{L}}$. Later, we will consider cases where $\mathcal{N} \cap S = \mathbf{1}$. Since $\mathcal{N}$ equals the kernel of α, the restriction $\alpha|_S\colon S \longrightarrow \overline{S}$ is then a bijection. Thus we will be able to identify S with $\overline{S}$.

Definition 3.16. Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities, and let $\alpha\colon \mathcal{L} \longrightarrow \mathcal{L}'$ be a projection of partial groups. We say that α is a *projection of localities* from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ if the set $\Delta\alpha := \{P\alpha \mid P \in \Delta\}$ equals Δ'.

If α is a projection of localities from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$, then notice that α maps S to S', as S and S' are the unique maximal elements of Δ and Δ' respectively.

3.4. Localities with special properties

The following definition was partly introduced in the introduction. Recall that a finite group G is of characteristic p if $C_G(O_p(G)) \leq O_p(G)$.

Definition 3.17. Let $(\mathcal{L}, \Delta, S)$ be a locality.

- We call $(\mathcal{L}, \Delta, S)$ *cr-complete* if $\mathcal{F}_S(\mathcal{L})^{cr} \subseteq \Delta$.
- A locality $(\mathcal{L}, \Delta, S)$ is said to be of *objective characteristic* p if $N_{\mathcal{L}}(P)$ is of characteristic p for every $P \in \Delta$.
- A locality is called a *linking locality* if it is cr-complete and of objective characteristic p.

Proposition 3.18. *Let $(\mathcal{L}, \Delta, S)$ be a locality.*

(a) *$\mathcal{F}_S(\mathcal{L})$ is Δ-generated and Δ-saturated.*
(b) *If every $\mathcal{F}_S(\mathcal{L})$-critical subgroup is an element of Δ, then $\mathcal{F}_S(\mathcal{L})$ is saturated and $(\mathcal{L}, \Delta, S)$ is cr-complete.*
(c) *If $(\mathcal{L}, \Delta, S)$ is cr-complete, then $\mathcal{F}_S(\mathcal{L})$ is saturated. Hence, the fusion system of a linking locality is saturated.*

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Proof. The fusion system $\mathcal{F}_S(\mathcal{L})$ is Δ-generated by definition and Δ-saturated by [8, Lemma 2.9] and [18, Lemma 3.27]. Thus Theorem 2.7 implies that $\mathcal{F}_S(\mathcal{L})$ is saturated if Δ contains every $\mathcal{F}_S(\mathcal{L})$-critical subgroup. Part (b) follows now from Lemma 2.6(b) and part (c) from Lemma 2.6(a). Alternatively, (c) is proved in [18, Theorem 3.26]. □

It is of crucial importance that cr-complete localities (and thus linking localities) lead to saturated fusion systems. This fact was first observed by Chermak. Conversely, it follows from the existence and uniqueness of centric linking systems (see [7, 24, 11]) that there is a linking locality attached to every saturated fusion system. In this context, the following definition plays a crucial role.

Definition 3.19 [16, Definition 1, Lemma 3.1]. Let $\mathcal{F}$ be a saturated fusion system. A subgroup $P \leq S$ is called $\mathcal{F}$*-subcentric* if $N_{\mathcal{F}}(Q)$ is constrained for every fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate Q of P. We denote by $\mathcal{F}^s$ the set of $\mathcal{F}$-subcentric subgroups of S.

It can be shown that for every $\mathcal{F}$-closed set Δ of subgroups of S with $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^s$, there exists an (essentially unique) linking locality over $\mathcal{F}$ with object set Δ. Moreover, $\mathcal{F}^s$ is $\mathcal{F}$-closed, and thus there is an (essentially unique) linking locality over $\mathcal{F}$ with object set $\mathcal{F}^s$, which can be seen as the 'largest' linking locality over $\mathcal{F}$. The reader is referred to [16, Theorem A] for a precise statement of these results.

3.5. Regular localities

Regular localities were introduced by Chermak [9], but we will refer to the treatment of the subject in [18]. If $(\mathcal{L}, \Delta, S)$ is a linking locality over a saturated fusion system $\mathcal{F}$, then there is the *generalised Fitting subgroup* $F^*(\mathcal{L})$ of $\mathcal{L}$ defined as a certain partial normal subgroup of $\mathcal{L}$ (compare [18, Definition 3]). It turns out that the set $\{P \leq S\colon P \cap F^*(\mathcal{L}) \in \mathcal{F}^s\}$ depends only on $\mathcal{F}$ and not on the linking locality $(\mathcal{L}, \Delta, S)$ (see [18, Lemma 10.2]). Thus the following definition makes sense.

Definition 3.20. If $\mathcal{F}$ is a saturated fusion system and $(\mathcal{L}, \Delta, S)$ is a linking locality over $\mathcal{F}$, then set

$$\delta(\mathcal{F}) := \{P \leq S\colon P \cap F^*(\mathcal{L}) \in \mathcal{F}^s\}.$$

A linking locality $(\mathcal{L}, \Delta, S)$ is called a *regular locality* if $\Delta = \delta(\mathcal{F})$.

For every saturated fusion system $\mathcal{F}$, the set $\delta(\mathcal{F})$ is $\mathcal{F}$-closed, and thus there exists a regular locality over $\mathcal{F}$ (compare [18, Lemma 10.4]).

It turns out that regular localities have particularly nice properties. To describe these, suppose that $(\mathcal{L}, \Delta, S)$ is a regular locality. If $\mathcal{H}$ is a partial normal subgroup of $\mathcal{L}$ or, more generally, a partial subnormal subgroup of $\mathcal{L}$, then by [18, Theorem 3, Corollary 10.19], $\mathcal{E} := \mathcal{F}_{S\cap\mathcal{H}}(\mathcal{H})$ is saturated and $(\mathcal{H}, \delta(\mathcal{E}), S \cap \mathcal{H})$ is a regular locality over $\mathcal{E}$. This leads to a natural notion of components of $\mathcal{L}$ (see [18, Definition 11.1]) and to a theory of components of regular localities that mirrors results from finite group theory (see [18, Chapter 11]). In particular, the product $E(\mathcal{L})$ of components of $\mathcal{L}$ forms a partial normal subgroup with $F^*(\mathcal{L}) = E(\mathcal{L})O_p(\mathcal{L})$.

If $(\mathcal{L}, \Delta, S)$ is a regular locality over $\mathcal{F}$ (or, more generally, a linking locality over $\mathcal{F}$ with $\delta(\mathcal{F}) \subseteq \Delta$), then it is shown in [10, Theorem A] that the assignment $\mathcal{N} \mapsto \mathcal{F}_{S\cap\mathcal{N}}(\mathcal{N})$ defines a bijection Φ from the set of partial normal subgroups of $\mathcal{L}$ to the set of normal subsystems of $\mathcal{F}$. This bijection sends $F^*(\mathcal{L})$ to $F^*(\mathcal{F})$ and $E(\mathcal{L})$ to $E(\mathcal{F})$. Moreover, in the case that $(\mathcal{L}, \Delta, S)$ is regular, we have $C_{\mathcal{L}}(\mathcal{N}) \trianglelefteq \mathcal{L}$ and $\Phi(C_{\mathcal{L}}(\mathcal{N})) = C_{\mathcal{F}}(\Psi(\mathcal{N}))$ (see [18, Theorem 3] and [10, Proposition 6.7]). We refer the reader here to [3, p.3] or [10, Definition 7.2] for the definitions of $F^*(\mathcal{F})$ and $E(\mathcal{F})$, and to [3, Chapter 6] or [15] for a definition of the centraliser $C_{\mathcal{F}}(\mathcal{E})$ of a normal subsystem $\mathcal{E}$.

The results we just summarised are used to prove our next lemma, which collects most of the information we need in Section 6. It will be convenient to use the following notation.

Notation 3.21. If $\mathcal{F}$ is a fusion system over S and $\mathcal{E}$ is a subsystem of $\mathcal{F}$ over $T \leq S$, then we set $\mathcal{E} \cap P := T \cap P$ for every subgroup $P \leq S$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma 3.22. *Let $(\mathcal{L}, \Delta, S)$ be a regular locality over $\mathcal{F}$ and $\mathcal{N} \trianglelefteq \mathcal{L}$. Set $T := \mathcal{N} \cap S$, $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$, $T^* := F^*(\mathcal{L}) \cap S$, $T_0 := E(\mathcal{N}) \cap S$ and $R := E(C_{\mathcal{L}}(\mathcal{N})) \cap S$. Then the following hold:*

(a) *$N_{\mathcal{L}}(T^*) = N_{\mathcal{L}}(T_0) \cap N_{\mathcal{L}}(R)$ and $N_{\mathcal{N}}(T^*) = N_{\mathcal{N}}(T_0)$.*
(b) *We have $T_0 = E(\mathcal{E}) \cap S$ and $R = E(C_{\mathcal{F}}(\mathcal{E})) \cap S$.*
(c) *$N_{\mathcal{E}}(T_0) = \mathcal{F}_T(N_{\mathcal{N}}(T_0))$.*
(d) *$\delta(\mathcal{E})$ is closed under passing to $\mathcal{L}$-conjugates in S.*

Proof. By [18, Theorem 3], we have $C_{\mathcal{L}}(\mathcal{N}) \trianglelefteq \mathcal{L}$; moreover, $(\mathcal{N}, \delta(\mathcal{E}), T)$ is a regular locality over $\mathcal{E}$, and $C_{\mathcal{L}}(\mathcal{N})$ can also be given the structure of a regular locality. In particular, $E(\mathcal{N})$ and $E(C_{\mathcal{L}}(\mathcal{N}))$ are well-defined.

(a) By [18, Lemma 11.13], $E(\mathcal{N}) \trianglelefteq \mathcal{L}$ and similarly $E(C_{\mathcal{L}}(\mathcal{N})) \trianglelefteq \mathcal{L}$. Moreover, by [18, Lemma 11.16], $E(\mathcal{L}) = E(\mathcal{N})E(C_{\mathcal{L}}(\mathcal{N}))$, which by [14, Theorem 1] implies $T_0R = E(\mathcal{L}) \cap S \leq T^*$. In particular, $N_{\mathcal{L}}(T^*)$ normalises $T_0 = T^* \cap E(\mathcal{N})$ and $R = T^* \cap E(C_{\mathcal{L}}(\mathcal{N}))$. On the other hand, by [18, Lemma 11.9], we have $T^* = (E(\mathcal{L}) \cap S)O_p(\mathcal{L})$, so $N_{\mathcal{L}}(T_0) \cap N_{\mathcal{L}}(R) \leq N_{\mathcal{L}}(T^*)$. Thus, $N_{\mathcal{L}}(T^*) = N_{\mathcal{L}}(T_0) \cap N_{\mathcal{L}}(R)$. As $R \subseteq C_{\mathcal{L}}(\mathcal{N})$, it follows from [18, Lemma 3.5] that $\mathcal{N} \subseteq C_{\mathcal{L}}(R) \subseteq N_{\mathcal{L}}(R)$. Hence, we can conclude that $N_{\mathcal{N}}(T_0) = \mathcal{N} \cap N_{\mathcal{L}}(T_0) \cap N_{\mathcal{L}}(R) = \mathcal{N} \cap N_{\mathcal{L}}(T^*) = N_{\mathcal{N}}(T^*)$. This proves (a).

(b) By [10, Proposition 6.7], we have $\mathcal{F}_{C_S(\mathcal{N})}(C_{\mathcal{L}}(\mathcal{N})) = C_{\mathcal{F}}(\mathcal{E})$. This means $(C_{\mathcal{L}}(\mathcal{N}), \delta(C_{\mathcal{F}}(\mathcal{E})), C_S(\mathcal{N}))$ is a regular locality over $C_{\mathcal{F}}(\mathcal{E})$. Recall also that $(\mathcal{N}, \delta(\mathcal{E}), T)$ is a regular locality over $\mathcal{E}$. Hence, [10, Theorem E(d)] yields (b).

(c) By [18, Corollary 11.10], $T_0 \in \delta(\mathcal{E})$. Hence, (c) follows from Lemma 3.10(b) applied with $(\mathcal{N}, \delta(\mathcal{E}), T)$ in place of $(\mathcal{L}, \Delta, S)$.

(d) The set $\delta(\mathcal{E})$ is $\mathcal{E}$-closed by [18, Lemma 10.4]. Moreover, it follows from [18, Theorem 10.16(f)] that $\delta(\mathcal{E})$ is closed under passing to $N_{\mathcal{L}}(T)$-conjugates. By Lemma 3.13, for every $g \in \mathcal{L}$, there exist $n \in \mathcal{N}$ and $f \in N_{\mathcal{L}}(T)$ such that $(n, f) \in \mathbf{D}$, $g = nf$ and $S_g = S_{(n,f)}$. Now (d) follows using Lemma 3.8(c),(e). □

4. Some additional lemmas

4.1. Lemmas on groups

Lemma 4.1. *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and $N \trianglelefteq G$ a normal subgroup of G. If $H \leq G$ is such that $S \cap H \in \mathrm{Syl}_p(H)$, then*

$$(S \cap N)(S \cap H) \in \mathrm{Syl}_p(NH).$$

Proof. Notice that $P := (S \cap N)(S \cap H) \leq S \cap NH$ and $|P| = \frac{|S\cap N||S\cap H|}{|S\cap N\cap H|}$. By assumption, $S \cap H \in \mathrm{Syl}_p(H)$. As $N \trianglelefteq G$ and $N \cap H \trianglelefteq H$, it follows that $S \cap N \in \mathrm{Syl}_p(N)$ and $S \cap N \cap H \in \mathrm{Syl}_p(N \cap H)$. The assertion follows now from $|NH| = \frac{|N|\,|H|}{|N\cap H|}$. □

Recall that a finite group G is said to be of characteristic p if $C_G(O_p(G)) \leq O_p(G)$.

Lemma 4.2. *Let G be a finite group of characteristic p. Then the following hold:*

(a) *$N_G(P)$ and $C_G(P)$ are of characteristic p for all nontrivial p-subgroups $P \leq G$.*
(b) *Every subnormal subgroup of G is of characteristic p.*
(c) *If N is a normal subgroup of G of p-power index, then G is of characteristic p if and only if N is of characteristic p.*

Proof. Let P be a p-subgroup of G. By [22, Lemma 1.2(a), (c)], every subnormal subgroup of G is of characteristic p, and $N_G(P)$ is of characteristic p. In particular, $C_G(P) \trianglelefteq N_G(P)$ is of characteristic p.

Now let N be as in (c). By [22, Lemma 1.3], G is of characteristic p if and only if $O^p(G)$ is of characteristic p. Similarly, N is of characteristic p if and only if $O^p(N)$ is of characteristic p. As $O^p(N) = O^p(G)$, this implies (c). □

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma 4.3. *Let G be a finite group, and let N be a normal subgroup of G with Sylow p-subgroup T. Then the following are equivalent:*

(i) *G is of characteristic p.*
(ii) *N and $N_G(T)$ are of characteristic p.*
(iii) *N and $C_G(N)$ are of characteristic p.*

Proof. It follows from Lemma 4.2 that (i) implies (ii). As N is normal in G, $C_G(N)$ is also normal in G. Notice that $C_G(N) \leq C_G(T) \leq N_G(T)$. Thus $C_G(N)$ is a normal subgroup of $N_G(T)$. So by Lemma 4.2(b), $C_G(N)$ has characteristic p if $N_G(T)$ has characteristic p. Hence, (ii) implies (iii). Assume now that (iii) holds. It follows from [21, 6.5.2] that every component of G is a component either of N or of $C_G(N)$. As N and $C_G(N)$ do not have any components, it follows that $E(G) = 1$. Moreover, $[N, O_{p'}(G)] \leq N \cap O_{p'}(G) = O_{p'}(N) = 1$. Thus $O_{p'}(G) \leq C_G(N)$, so $O_{p'}(G) = O_{p'}(C_G(N)) = 1$. Hence, (i) holds, and the proof is complete. □

4.2. Some properties of fusion systems

It will be convenient to use the following definition.

Definition 4.4. Suppose $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are fusion systems over S and $\tilde{S}$, respectively.

- If $\alpha\colon S \longrightarrow \tilde{S}$ is a group isomorphism, then α is said to *induce an isomorphism from $\mathcal{F}$ to $\tilde{\mathcal{F}}$* if $\{\alpha^{-1}\varphi\alpha\colon \varphi \in \mathrm{Mor}_{\mathcal{F}}(P,Q)\} = \mathrm{Mor}_{\tilde{\mathcal{F}}}(P\alpha, Q\alpha)$ for all $P, Q \leq S$.
- We say that $\alpha \in \mathrm{Aut}(S)$ induces an automorphism of $\mathcal{F}$ if α induces an isomorphism from $\mathcal{F}$ to $\mathcal{F}$. Write $\mathrm{Aut}(\mathcal{F})$ for the set of all $\alpha \in \mathrm{Aut}(S)$ that induce an automorphism of $\mathcal{F}$.

Lemma 4.5. *Let $\mathcal{F}$ be a fusion system over S, and let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$. Then $\mathcal{E}^c$, $\mathcal{E}^r$ and $\mathcal{E}^{cr}$ are closed under $\mathcal{F}$-conjugacy. Similarly, the set of $\mathcal{E}$-critical subgroups is closed under $\mathcal{F}$-conjugacy.*

Proof. Let $T \leq S$ be such that $\mathcal{E}$ is a subsystem over T. Let Γ be one of the sets $\mathcal{E}^c$, $\mathcal{E}^r$, $\mathcal{E}^{cr}$ or the set of $\mathcal{E}$-critical subgroups. It follows directly from the definition of these sets that Γ is closed under $\mathcal{E}$-conjugacy. By the Frattini condition for $\mathcal{F}$-invariant subsystems [2, Definition I.6.1], it is thus sufficient to argue that Γ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. The definition of $\mathcal{F}$-invariant subsystems implies, moreover, that every element of $\mathrm{Aut}_{\mathcal{F}}(T)$ induces an automorphism of $\mathcal{E}$. Therefore, $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ maps $\mathcal{E}$-conjugacy classes to $\mathcal{E}$-conjugacy classes and fully $\mathcal{E}$-normalised subgroups to fully $\mathcal{E}$-normalised subgroups. Moreover, if $P \leq T$ with $C_T(P) \leq P$, then $C_T(P\alpha) \leq P\alpha$. So $\mathcal{E}^c$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. Notice also that $\alpha|_{N_T(Q)}$ induces an isomorphism from $N_{\mathcal{E}}(Q)$ to $N_{\mathcal{E}}(Q\alpha)$ for every $Q \leq T$. Hence, α maps $\mathcal{E}$-radical subgroups to $\mathcal{E}$-radical subgroups. So $\mathcal{E}^r$ and $\mathcal{E}^{cr} = \mathcal{E}^c \cap \mathcal{E}^r$ are $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. For $Q \leq T$, the map $\mathrm{Aut}_{\mathcal{E}}(Q) \to \mathrm{Aut}_{\mathcal{E}}(Q\alpha), \varphi \mapsto \alpha^{-1}\varphi\alpha$ is an isomorphism that takes $\mathrm{Inn}(Q)$ to $\mathrm{Inn}(Q\alpha)$ and $\mathrm{Aut}_T(Q)$ to $\mathrm{Aut}_T(Q\alpha)$. Hence, the set of $\mathcal{E}$-critical subgroups is $\mathrm{Aut}_{\mathcal{F}}(T)$ invariant. This proves the assertion. □

For the remainder of this subsection, let $\mathcal{F}$ be a saturated fusion system over a p-group S.

In the proofs of the following two lemmas, we cite [1]. It should be pointed out that normal subsystems in the sense of [1, Definition 1.18] correspond to weakly normal subsystems in the language of [2] (i.e., in the language that we are using in this paper).

Lemma 4.6. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ over T. If $R \in \mathcal{F}^{cr}$, then $R \cap T \in \mathcal{E}^{cr}$.*

Proof. This is [1, Lemma 1.20(d)] but also follows easily from the results stated before. Namely, by Corollary 2.3, there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(R \cap T), S)$ such that $(R \cap T)^\alpha$ is fully normalised. Since $R \leq N_S(R \cap T)$, the subgroup R^α is well-defined. As T is strongly closed, $R^\alpha \cap T = (R \cap T)^\alpha$. By Lemma 4.5, $\mathcal{E}^{cr}$ and $\mathcal{F}^{cr}$ are closed under $\mathcal{F}$-conjugacy. So, replacing R by R^α, we may assume that $R \cap T$ is fully $\mathcal{F}$-normalised. If Q is an $\mathcal{E}$-conjugate of $R \cap T$, then by Corollary 2.3, there is an $\mathcal{F}$-morphism mapping $N_S(Q)$ into $N_S(R \cap T)$ and thus $N_T(Q)$ into $N_T(R \cap T)$. Hence, $R \cap T$ is fully $\mathcal{E}$-normalised. Now the claim follows from Lemma 2.6(b) and Lemma 2.10. □

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma 4.7. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant (not necessarily saturated) subsystem of $\mathcal{F}$ over $T \le S$. Let P be an $\mathcal{E}$-critical subgroup of T such that P is fully $\mathcal{F}$-normalised. Then there exists $R \in \mathcal{F}^{cr}$ with $R \cap T = P$.*

Proof. This is a special case of [1, Lemma 1.19]. □

We will now look at some properties of the set $\delta(\mathcal{F})$ introduced in Subsection 3.5. As shown in [10, Lemma 7.21], $\delta(\mathcal{F})$ can be characterised as the set of subgroups of S containing an element of $F^*(\mathcal{F})^s$. The reader is referred to [3] or [10, Definition 7.2] for the definition of the *generalised Fitting subsystem* $F^*(\mathcal{F})$ and the *layer* $E(\mathcal{F})$ of $\mathcal{F}$. Recall Notation 3.21.

Lemma 4.8. *Let $P \le S$ with $O_p(\mathcal{F}) \le P$. Then $P \in \delta(\mathcal{F})$ if and only if $E(\mathcal{F}) \cap P \in E(\mathcal{F})^s$.*

Proof. We use that $F^*(\mathcal{F}) = E(\mathcal{F}) * \mathcal{F}_{O_p(\mathcal{F})}(O_p(\mathcal{F}))$ by [10, Theorem 7.10(e)]. In particular, $F^*(\mathcal{F}) \cap S = (E(\mathcal{F}) \cap S)O_p(\mathcal{F})$. As $O_p(\mathcal{F}) \le P$, a Dedekind Argument yields thus $F^*(\mathcal{F}) \cap P = (E(\mathcal{F}) \cap P)O_p(\mathcal{F})$. It is shown in [10, Lemma 7.21] that $P \in \delta(\mathcal{F})$ if and only if $F^*(\mathcal{F}) \cap P \in F^*(\mathcal{F})^s$. The assertion follows now from [18, Lemma 2.14(g)]. □

Lemma 4.9. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ and $O_p(\mathcal{F}) \le P \le S$. Then the following hold:*

(a) *If $E(\mathcal{F}) = E(\mathcal{E})$, then $P \in \delta(\mathcal{F})$ if and only if $\mathcal{E} \cap P \in \delta(\mathcal{E})$.*
(b) *Suppose $E(C_{\mathcal{F}}(\mathcal{E})) \cap S \le P$. Then*

$$P \in \delta(\mathcal{F}) \Longleftrightarrow E(\mathcal{E}) \cap P \in E(\mathcal{E})^s \Longleftrightarrow \mathcal{E} \cap P \in \delta(\mathcal{E}).$$

Proof. By Lemma 4.8, we have $P \in \delta(\mathcal{F})$ if and only if $E(\mathcal{F}) \cap P \in E(\mathcal{F})^s$. As $O_p(\mathcal{E}) \le O_p(\mathcal{F}) \le P$ by [16, Lemma 2.12(b)], it follows similarly that $\mathcal{E} \cap P \in \delta(\mathcal{E})$ if and only if $E(\mathcal{E}) \cap P \in E(\mathcal{E})^s$. This implies (a). By [10, Lemma 7.13(c)], $E(\mathcal{F}) = E(\mathcal{E}) * E(C_{\mathcal{F}}(\mathcal{E}))$. As $R := E(C_{\mathcal{F}}(\mathcal{E})) \cap S \le P$, we have $E(\mathcal{F}) \cap P = (E(\mathcal{E}) \cap P)R$. Notice that $R \in E(C_{\mathcal{F}}(\mathcal{E}))^s$. Hence, by [18, Lemma 2.14(g)], $E(\mathcal{F}) \cap P \in E(\mathcal{F})^s$ if and only if $E(\mathcal{E}) \cap P \in E(\mathcal{E})^s$. This implies (b). □

Remark 4.10. If $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$ defined over $T \le S$ and R is a subgroup of S, then there is a unique saturated subsystem $\mathcal{D}$ of $\mathcal{F}$ over TR with $O^p(\mathcal{D}) = O^p(\mathcal{E})$. It is denoted by $(\mathcal{E}R)_{\mathcal{F}}$ (or sometimes simply by $\mathcal{E}R$). This was first shown by Aschbacher [3, Chapter 8]. The result was revisited in [13], where a concrete description is given.

We conclude this section with the following lemma, which was first proved by Aschbacher [4, 1.3.2]. We give a new proof using localities.

Lemma 4.11. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \le S$. Then $\mathcal{F} = \langle \mathcal{E}S, N_{\mathcal{F}}(T) \rangle$.*

Proof. By [16, Theorem A], there exists a linking locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ with $\Delta = \mathcal{F}^s$. Moreover, by [10, Theorem A], there exists a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ with $T = S \cap \mathcal{N}$ and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. Now, by [10, Corollary 4.10], we have $\mathcal{E}S = \mathcal{F}_S(\mathcal{N}S)$. As $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ is generated by maps of the form $c_g : S_g \to S$ with $g \in \mathcal{L}$, it is sufficient to prove that such maps are in $\langle \mathcal{E}S, N_{\mathcal{F}}(T) \rangle$. Let $g \in \mathcal{L}$. By Lemma 3.13, there exist $n \in \mathcal{N}$ and $f \in N_{\mathcal{L}}(T)$ such that $(n, f) \in \mathbf{D}$, $g = nf$ and $P := S_g = S_{(n,f)}$. Hence, $c_g = (c_n|_P)(c_f|_{P^n})$. Notice that $c_n|_P \in \mathcal{F}_S(\mathcal{N}S) = \mathcal{E}S$ and that $c_f|_{P^n}$ is a morphism in $N_{\mathcal{F}}(T)$. This shows the assertion. □

5. Kernels of localities

In this section, we investigate the properties of kernels of localities. In particular, we prove Theorems B, C and D. Recall the following definition.

Definition 5.1. Let $(\mathcal{L}, \Delta, S)$ be a locality. A *kernel* of $\mathcal{L}$ is a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $P \cap \mathcal{N} \in \Delta$ for every $P \in \Delta$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma 5.2. *Let $\mathcal{N}$ be a kernel of a locality $(\mathcal{L}, \Delta, S)$. Set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, $T := S \cap \mathcal{N}$, $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$ and $\Gamma := \{P \in \Delta : P \leq T\} = \{P \cap T : P \in \Delta\}$. Then the following hold:*

(a) *$(\mathcal{N}, \Gamma, T)$ is a locality over $\mathcal{E}$, and Γ is closed under $\mathcal{F}$-conjugacy.*
(b) *For every $f \in N_{\mathcal{L}}(T)$, $\mathcal{L} = \mathbf{D}(f)$, $c_f \in \mathrm{Aut}(\mathcal{L})$ and $c_f|_{\mathcal{N}} \in \mathrm{Aut}(\mathcal{N})$.*
(c) *$O_p(\mathcal{N}) = O_p(\mathcal{L}) \cap \mathcal{N} \trianglelefteq \mathcal{L}$.*
(d) *The subsystem $\mathcal{E}$ is $\mathcal{F}$-invariant.*

Proof. **(a,b)** It follows from [18, Lemma 3.29(b), (c)] and the definition of $\mathcal{E}$ that $(\mathcal{N}, \Gamma, T)$ is a locality over $\mathcal{E}$ and that (b) holds. Since Δ is closed under passing to $\mathcal{L}$-conjugates in S, $\mathcal{N} \trianglelefteq \mathcal{L}$ and $\Gamma = \{P \in \Delta : P \subseteq \mathcal{N}\}$, it follows that Γ is closed under passing to $\mathcal{L}$-conjugates in S. So Γ is closed under $\mathcal{F}$-conjugacy, and (a) holds.

(c) Notice that $O_p(\mathcal{N})$ is invariant under conjugation by elements of $N_{\mathcal{L}}(T)$ as the elements of $N_{\mathcal{L}}(T)$ induce automorphisms of $\mathcal{N}$. Since $O_p(\mathcal{N}) \trianglelefteq \mathcal{N}$, it follows from [8, Corollary 3.13] that $O_p(\mathcal{N}) \trianglelefteq \mathcal{L}$ and thus $O_p(\mathcal{N}) \leq O_p(\mathcal{L})$. Clearly, $O_p(\mathcal{L}) \cap \mathcal{N} \leq O_p(\mathcal{N})$, so (c) holds.

(d) As $\mathrm{Aut}_{\mathcal{F}}(T)$ is generated by maps of the form $c_f|_T$ with $f \in N_{\mathcal{L}}(T)$ and these maps induce automorphisms of $\mathcal{E}$ by [18, Lemma 3.29(d)], it follows that the elements of $\mathrm{Aut}_{\mathcal{F}}(T)$ induce automorphisms of $\mathcal{E}$. Now (d) is a consequence of [18, Lemma 3.28]. □

Definition 5.3. We say that $(\mathcal{N}, \Gamma, T)$ is a kernel of $(\mathcal{L}, \Delta, S)$ to indicate that $\mathcal{N}$ is a kernel of $(\mathcal{L}, \Delta, S)$, $T = S \cap \mathcal{N}$ and $\Gamma = \{P \in \Delta : P \leq T\} = \{P \cap T : P \in \Delta\}$.

The following lemma can be seen as a converse to Lemma 5.2(a).

Lemma 5.4. *Suppose we are given a finite partial group $\mathcal{L}$ with product $\Pi : \mathbf{D} \to \mathcal{L}$. Let S be a maximal p-subgroup of $\mathcal{L}$ and $\mathcal{N} \trianglelefteq \mathcal{L}$. Set $T := \mathcal{N} \cap S$, let Γ be a set of subgroups of T, and set $\Delta := \{P \leq S : P \cap T \in \Gamma\}$. Suppose the following hold:*

- *Γ is closed under passing to overgroups and $\mathcal{L}$-conjugates that lie in T.*
- *$\mathbf{D} = \mathbf{D}_\Gamma$.*

Then $(\mathcal{L}, \Delta, S)$ is a locality with kernel $(\mathcal{N}, \Gamma, T)$.

Proof. If $(\mathcal{L}, \Delta, S)$ is a locality, then clearly $(\mathcal{N}, \Gamma, T)$ is a kernel of $(\mathcal{L}, \Delta, S)$. By Lemma 3.5(b), we have $\mathbf{D}_\Delta = \mathbf{D}_\Gamma = \mathbf{D}$. Recall that S is a maximal p-subgroup of $\mathcal{L}$ by assumption. As Γ is overgroup-closed in T, it follows that Δ is overgroup-closed in S. Hence, it remains to show that Δ is closed under passing to $\mathcal{L}$-conjugates in S.

Let $P \in \Delta$ and $f \in \mathcal{L}$ with $P \subseteq \mathbf{D}(f)$ and $P^f \subseteq S$. Observe that $Q := P \cap \mathcal{N} \in \Gamma$ is normal in P and $Q^f \in \Gamma$ by assumption. Hence, if $x, y \in P$, then $u := (f^{-1}, x, f, f^{-1}, y, f) \in \mathbf{D} = \mathbf{D}_\Gamma$ via Q^f. Thus, by the axioms of a partial group, $x^f y^f = \Pi(u) = \Pi(f^{-1}, x, y, f) = (xy)^f$, so $c_f : P \to S, x \mapsto x^f$ is a group homomorphism. Therefore, P^f is a subgroup of S. By Lemma 3.5(a), $P^f \cap T = P^f \cap \mathcal{N} = Q^f \in \Gamma$, so $P^f \in \Delta$. □

We will use from now on without further reference that, by Lemma 5.2(a), $(\mathcal{N}, \Gamma, T)$ is a locality if $(\mathcal{N}, \Gamma, T)$ is a kernel of a locality $(\mathcal{L}, \Delta, S)$. In particular, it makes sense to say that a kernel $(\mathcal{N}, \Gamma, T)$ is cr-complete.

Proposition 5.5. *Let $(\mathcal{L}, \Delta, S)$ be a locality with kernel $(\mathcal{N}, \Gamma, T)$. Then $(\mathcal{L}, \Delta, S)$ is cr-complete if and only if $(\mathcal{N}, \Gamma, T)$ is cr-complete.*

Proof. Set $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$ and $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$. By Lemma 5.2(a),(d), $(\mathcal{N}, \Gamma, T)$ is a locality and $\mathcal{E}$ is $\mathcal{F}$-invariant.

Assume first that $(\mathcal{L}, \Delta, S)$ is cr-complete. Then in particular, $\mathcal{F}$ is saturated by Proposition 3.18(c). We show:

$$\text{If } Q \leq T \text{ is } \mathcal{E}\text{-critical, then } Q \in \Gamma. \tag{5.1}$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

For the proof, let Q be an $\mathcal{E}$-critical subgroup of T. Then there exists a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate P of Q, which by Lemma 4.5 is also $\mathcal{E}$-critical. Hence, Lemma 4.7 implies the existence of an element $R \in \mathcal{F}^{cr}$ with $R \cap T = P$. As $\mathcal{F}^{cr} \subseteq \Delta$ and $(\mathcal{N}, \Gamma, T)$ is a kernel of $(\mathcal{L}, \Delta, S)$, it follows that $P \in \Gamma$. Hence, Lemma 5.2(a) yields $Q \in \Gamma$, and thus equation (5.1) holds. Proposition 3.18(b) implies now that $\mathcal{E}$ is saturated and $(\mathcal{N}, \Gamma, T)$ is cr-complete.

To prove the other implication, assume that $(\mathcal{N}, \Gamma, T)$ is cr-complete. Then by Proposition 3.18(c), $\mathcal{E}$ is saturated. Observe also that $\mathcal{E}^{cr} \subseteq \Gamma \subseteq \Delta$. As $\mathcal{F}$ is Δ-saturated and Δ-generated by Proposition 3.18(a), it follows now from Theorem A that $\mathcal{F}$ is saturated. In particular, $\mathcal{E}$ is weakly normal in $\mathcal{F}$. So Lemma 4.6 gives $R \cap T \in \mathcal{E}^{cr} \subseteq \Gamma \subseteq \Delta$ for every $R \in \mathcal{F}^{cr}$. As Δ is overgroup-closed, this implies that $(\mathcal{L}, \Delta, S)$ is cr-complete. □

Lemma 5.6. *Let $(\mathcal{N}, \Gamma, T)$ be a kernel of a locality $(\mathcal{L}, \Delta, S)$. Set $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$ and $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$. Suppose $C_{\mathcal{N}}(T) \leq T$. Then the following hold:*

(a) $N_{\mathcal{N}}(T) \subseteq N_{\mathcal{N}}(TC_S(T))$.
(b) *If $\mathcal{E}$ and $\mathcal{F}$ are saturated, then $\mathcal{E}$ is normal in $\mathcal{F}$.*

Proof. **(a)** Notice that $T \in \Gamma \subseteq \Delta$, and thus $N_{\mathcal{L}}(T)$ is a group. The assumption $C_{\mathcal{N}}(T) \leq T$ yields that $N_{\mathcal{N}}(T)$ is of characteristic p.

Let $g \in N_{\mathcal{N}}(T)$, and set $P := S_g$. Then $T \leq P$, so $P^g = P$ by [8, Lemma 3.1(b)]. Hence, $g \in K := N_{\mathcal{N}}(P)$. Note that $N_{\mathcal{L}}(P) \leq N_{\mathcal{L}}(T)$, so $K = N_{N_{\mathcal{N}}(T)}(P) \leq N_{\mathcal{L}}(T)$. Calculating inside the group $N_{\mathcal{L}}(T)$, we have

$$[K, N_{C_S(T)}(P)] \leq \mathcal{N} \cap C_{\mathcal{L}}(T) = C_{\mathcal{N}}(T) \leq T,$$

where the last inclusion holds by assumption. Thus $[g, N_{C_S(T)}(P)] \leq T$, so $N_{C_S(T)}(P) \leq S_g = P$. Therefore, we have $N_{PC_S(T)}(P) = PN_{C_S(T)}(P) = P$. As $PC_S(T)$ is a p-group, this implies $C_S(T) \leq P$ and thus g acts on T and $C_S(T)$. Hence, (a) holds.

(b) Notice that part (a) implies $[TC_S(T), N_{\mathcal{N}}(T)] \leq (TC_S(T)) \cap \mathcal{N} = T$. Since $\mathrm{Aut}_{\mathcal{F}}(T)$ is generated by maps of the form $c_f|_T$, with $f \in N_{\mathcal{L}}(T)$, the extension condition as stated in [2, Definition I.6.1] follows. Now (b) follows from Lemma 5.2(b). □

Lemma 5.7. *A projection from a locality $(\mathcal{L}^*, \Delta^*, S^*)$ to a locality $(\mathcal{L}, \Delta, S)$ sends every kernel of $(\mathcal{L}^*, \Delta^*, S^*)$ to a kernel of $(\mathcal{L}, \Delta, S)$.*

Proof. Let $\varphi\colon \mathcal{L}^* \to \mathcal{L}$ be a projection of localities, and let $\mathcal{N}^*$ be a kernel of $\mathcal{L}^*$. Then $\mathcal{N} := (\mathcal{N}^*)\varphi$ is a partial normal subgroup of $\mathcal{L}$ by [20, Lemma 2.5]. Let $P \in \Delta$. Then $P = (P^*)\varphi$ for some $P^* \in \Delta^*$ since $\Delta^*\varphi = \Delta$. Since $\mathcal{N}^*$ is a kernel of $\mathcal{L}^*$, we have $\mathcal{N}^* \cap P^* \in \Delta^*$. Hence

$$\mathcal{N} \cap P = (\mathcal{N}^*)\varphi \cap P^*\varphi \geq (\mathcal{N}^* \cap P^*)\varphi \in \Delta^*\varphi = \Delta.$$

As Δ is overgroup-closed, it follows that $\mathcal{N} \cap P \in \Delta$, and thus $\mathcal{N}$ is a kernel of $\mathcal{L}$. □

For the next lemma, the reader might want to recall Definition 3.14. By $\mathcal{E}^q$, we denote the quasicentric subgroups of a saturated fusion system $\mathcal{E}$; for the definition, see [2, Definition III.4.5].

Lemma 5.8. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$ with cr-complete kernel $(\mathcal{N}, \Gamma, T)$. Set $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$. Let $\mathcal{E}^{cr} \subseteq \Gamma_0 \subseteq \Gamma \cap \mathcal{E}^q$ such that Γ_0 is $\mathcal{E}$-closed and $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant, and let Δ_0 be the set of overgroups in S of the elements of Γ_0. Set*

$$\mathcal{L}_0 := \mathcal{L}|_{\Delta_0},\ \mathcal{N}_0 := \mathcal{N} \cap \mathcal{L}_0 \text{ and } \Theta = \bigcup_{P \in \Gamma_0} O_{p'}(N_{\mathcal{N}}(P)).$$

Then the following hold:

(a) *$(\mathcal{L}_0, \Delta_0, S)$ is a cr-complete locality over $\mathcal{F}$ with cr-complete kernel $(\mathcal{N}_0, \Gamma_0, T)$. Moreover, $\mathcal{N}_0 = \mathcal{N}|_{\Gamma_0}$ and $\mathcal{F}_T(\mathcal{N}_0) = \mathcal{E}$.*

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

(b) *Θ is a partial normal subgroup of $\mathcal{L}_0$ with $\Theta \cap S = 1$. In particular, setting $\overline{\mathcal{L}_0} = \mathcal{L}_0/\Theta$ and using the 'bar notation' as usual, S and T are naturally isomorphic to $\overline{S}$ and $\overline{T}$ via the restriction of the natural projection map $\mathcal{L}_0 \to \overline{\mathcal{L}_0}$.*
(c) *Identify $\overline{S}$ and $\overline{T}$ with S and T. Then the following conditions hold:*
  - *$(\overline{\mathcal{L}_0}, \Delta_0, S)$ is a locality over $\mathcal{F}$.*
  - *$(\overline{\mathcal{N}_0}, \Gamma_0, T)$ is a kernel of $(\overline{\mathcal{L}_0}, \Delta_0, S)$, which is a linking locality over $\mathcal{E}$.*

Proof. **(a)** Recall from Lemma 5.2(d) that $\mathcal{E}$ is $\mathcal{F}$-invariant. As Γ_0 is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant and closed under $\mathcal{E}$-conjugacy, it follows from the Frattini condition for $\mathcal{F}$-invariant subsystems (compare [2, Definition I.6.1]) that Γ_0 is closed under $\mathcal{F}$-conjugacy. Thus, Δ_0 is $\mathcal{F}$-closed, $\mathcal{L}_0$ is well-defined, and $(\mathcal{L}_0, \Delta_0, S)$ is a locality. It is easy to observe that $\mathcal{N}_0$ is a partial normal subgroup of $\mathcal{L}_0$. It follows from the definition of Δ_0 that $(\mathcal{N}_0, \Gamma_0, T)$ is a kernel of $(\mathcal{L}_0, \Delta_0, S)$ and $\mathcal{N}_0 = \mathcal{N}|_{\Gamma_0}$ (see also Lemma 3.5). As $\mathcal{E}^{cr} \subseteq \Gamma_0$, we can conclude from Lemma 4.6 that $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta$. In particular, $\mathcal{F}$ and $\mathcal{E}$ are saturated, by Proposition 3.18(c). Note furthermore that for all $P \in \Delta$, $N_{\mathcal{L}}(P) = N_{\mathcal{L}_0}(P)$, and thus, by Lemma 3.10(a), $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}_S(\mathcal{L}_0)}(P)$. Similarly, $\mathrm{Aut}_{\mathcal{E}}(Q) = \mathrm{Aut}_{\mathcal{F}_T(\mathcal{N}_0)}(Q)$ for all $Q \in \Gamma$. Hence, it follows from Alperin's Fusion Theorem [2, Theorem I.3.6] (combined with Lemma 2.6(b)) that $\mathcal{F} = \mathcal{F}_S(\mathcal{L}_0)$ and $\mathcal{E} = \mathcal{F}_T(\mathcal{N}_0)$. This proves (a).

(b,c) We argue first that

$$N_{\mathcal{N}}(P)/O_{p'}(N_{\mathcal{N}}(P)) \text{ is of characteristic } p \text{ for all } P \in \Gamma_0. \tag{5.2}$$

By Lemma 3.8(b), we may reduce to the case that P is fully $\mathcal{E}$-normalised. So let $P \in \Gamma_0$ be fully $\mathcal{E}$-normalised. As $\Gamma_0 \subseteq \mathcal{E}^q$, [16, Proposition 1(c)] gives that $C_{\mathcal{N}}(P)/O_{p'}(C_{\mathcal{N}}(P))$ is a p-group. In particular, $C_{\mathcal{N}}(P)/O_{p'}(C_{\mathcal{N}}(P))$ is of characteristic p, and thus $C_{\mathcal{N}}(P) = C_{N_{\mathcal{N}}(P)}(P)$ is 'almost of characteristic p' in the sense of [16, Definition 2.6]. It follows therefore from [16, Lemma 2.9] (applied with $N_{\mathcal{N}}(P)$ in place of G) that $N_{\mathcal{N}}(P)$ is 'almost of characteristic p', which means $N_{\mathcal{N}}(P)/O_{p'}(N_{\mathcal{N}}(P))$ is of characteristic p. This proves equation (5.2).

As $N_{\mathcal{N}_0}(P) = N_{\mathcal{N}}(P)$ for all $P \in \Gamma_0$, it follows from equation (5.2) and from [16, Proposition 6.4] applied with $(\mathcal{N}_0, \Gamma_0, T)$ in place of $(\mathcal{L}, \Delta, S)$ that Θ is a partial normal subgroup of $\mathcal{N}_0$, that $T \cap \Theta = 1$, that the canonical projection $\mathcal{N}_0 \to \mathcal{N}_0/\Theta$ is injective on T and that $(\mathcal{N}_0/\Theta, \Gamma_0, T)$ is a linking locality over $\mathcal{E}$ if we identify T with its image in $\mathcal{N}_0/\Theta$. The elements of $N_{\mathcal{L}}(T) = N_{\mathcal{L}_0}(T)$ induce $\mathcal{F}$-automorphisms of T and act thus by assumption on Γ_0 via conjugation. Hence, it follows from Lemma 3.8(b) that Θ is invariant under conjugation by elements of $N_{\mathcal{L}_0}(T)$. Now [8, Corollary 3.13] gives that Θ is a partial normal subgroups of $\mathcal{L}_0$. Moreover, $S \cap \Theta = S \cap \mathcal{N} \cap \Theta = T \cap \Theta = 1$. This means the kernel of the natural projection $\rho\colon \mathcal{L}_0 \to \overline{\mathcal{L}_0} := \mathcal{L}_0/\Theta$ intersects trivially with S and restricts thus to isomorphisms $S \to \overline{S}$ and $T \to \overline{T}$. Hence, (b) holds.

We now use the notation introduced in (b) and (c). By [8, Theorem 4.3], $(\overline{\mathcal{L}_0}, \Delta_0, S_0)$ is a locality. It is a special case of [17, Lemma 2.21(b)] that $\mathcal{F}_S(\overline{\mathcal{L}_0}) = \mathcal{F}_S(\mathcal{L}_0) = \mathcal{F}$. Using [8, Lemma 3.15], one can observe that $\rho|_{\mathcal{N}_0}$ coincides with the canonical projection $\mathcal{N}_0 \to \mathcal{N}_0/\Theta$. Thus, $(\overline{\mathcal{N}_0}, \Gamma_0, T) = (\mathcal{N}_0/\Theta, \Gamma_0, T)$ is a linking locality over $\mathcal{E}$. Now (c) follows from Lemma 5.7. □

Proposition 5.9. *Let $\mathcal{F}$ be a fusion system over S with a subsystem $\mathcal{E}$ over T. Then the following conditions are equivalent:*

(i) *There exists a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ with a cr-complete kernel $(\mathcal{N}, \Gamma, T)$ over $\mathcal{E}$.*
(ii) *There exists a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ with a kernel $(\mathcal{N}, \Gamma, T)$, which is a linking locality over $\mathcal{E}$.*

If either of these two conditions holds, then $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$.

Proof. Clearly, (ii) implies (i) as every linking locality is cr-complete. Assume now that (i) holds. Notice that $\mathcal{E}^{cr} \subseteq \Gamma_0 := \Gamma \cap \mathcal{E}^c \subseteq \Gamma \cap \mathcal{E}^q$. Moreover, Γ_0 is $\mathcal{E}$-closed and $\mathrm{Aut}_{\mathcal{F}}(T)$ invariant as the same holds for Γ and $\mathcal{E}^c$ (compare Lemma 4.5). Hence, (ii) follows from Lemma 5.8(c).

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Assume now that (ii) holds. If $(\mathcal{N}, \Gamma, T)$ is a kernel of a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ such that $(\mathcal{N}, \Gamma, T)$ is a linking locality over $\mathcal{E}$, then $(\mathcal{N}, \Gamma, T)$ is in particular cr-complete. Thus, $(\mathcal{L}, \Delta, S)$ is cr-complete, by Proposition 5.5. Hence, $\mathcal{E}$ and $\mathcal{F}$ are saturated, by Proposition 3.18(c). So $\mathcal{E} \trianglelefteq \mathcal{F}$, by Lemma 5.6(b). This proves the assertion. □

Proof of Theorem B. The statement follows from Proposition 5.5 and Proposition 5.9. □

Proposition 5.10. *Let $(\mathcal{L}, \Delta, S)$ be a locality with a kernel $(\mathcal{N}, \Gamma, T)$. Then the following conditions are equivalent:*

(i) *$(\mathcal{L}, \Delta, S)$ is of objective characteristic p.*
(ii) *$N_{\mathcal{L}}(T)$ is of characteristic p, and $(\mathcal{N}, \Gamma, T)$ is of objective characteristic p.*
(iii) *$C_{\mathcal{L}}(T)$ is of characteristic p, and $(\mathcal{N}, \Gamma, T)$ is of objective characteristic p.*
(iv) *$N_{\mathcal{L}}(P)$ is of characteristic p for every $P \in \Gamma$.*

Proof. By Lemma 5.2(a), $(\mathcal{N}, \Gamma, T)$ is a locality. Notice that $T = S \cap \mathcal{N} \in \Gamma \subseteq \Delta$, and thus $N_{\mathcal{L}}(T)$ is a finite group with $C_{N_{\mathcal{L}}(T)}(T) = C_{\mathcal{L}}(T)$. So properties (ii) and (iii) are equivalent by [16, Lemma 2.9]. Hence, it is sufficient to show that properties (i),(ii) and (iv) are equivalent.

If (i) holds, then $N_{\mathcal{L}}(P)$ is of characteristic p for every $P \in \Gamma$. In particular, $N_{\mathcal{L}}(T)$ is of characteristic p. Moreover, by Lemma 4.2(b), $N_{\mathcal{N}}(P) \trianglelefteq N_{\mathcal{L}}(P)$ is of characteristic p for every $P \in \Gamma$, showing that $(\mathcal{N}, \Gamma, T)$ is of objective characteristic p. So (i) implies (ii).

To show that (ii) implies (iv), assume now that (ii) holds. Suppose furthermore that (iv) is false: that is, there exists $P \in \Gamma$ such that $G := N_{\mathcal{L}}(P)$ is not of characteristic p. Choose such P of maximal order. By Lemma 3.8(b) and [8, Lemma 2.9], we may replace P by a suitable $\mathcal{L}$-conjugate of P and assume that $N_S(P)$ is a Sylow p-subgroup of G. As (ii) holds, we have $P \neq T$ and thus $P < N_T(P)$. Hence, by the maximality of $|P|$, the group $N_{\mathcal{L}}(N_T(P))$ has characteristic p. Hence, by Lemma 4.2(a), $N_G(N_T(P)) = N_{N_{\mathcal{L}}(N_T(P))}(P)$ is of characteristic p. As $(\mathcal{N}, \Gamma, T)$ is of objective characteristic p, we also know that $N := N_{\mathcal{N}}(P) \trianglelefteq G$ is of characteristic p. Notice that $N_T(P) = N_S(P) \cap N$ is a Sylow p-subgroup of N, as $N_S(P)$ is a Sylow p-subgroup of G. So it follows from Lemma 4.3 that G is of characteristic p, contradicting our assumption. Hence, (ii) implies (iv).

Assume now that (iv) holds. If $P \in \Delta$ is arbitrary, then $P \cap \mathcal{N} \in \Gamma$, so $N_{\mathcal{L}}(P \cap \mathcal{N})$ is of characteristic p. Observe that $N_{\mathcal{L}}(P) \subseteq N_{\mathcal{L}}(P \cap \mathcal{N})$, and thus $N_{\mathcal{L}}(P) = N_{N_{\mathcal{L}}(P \cap \mathcal{N})}(P)$ is of characteristic p by Lemma 4.2(a). Hence, (iv) implies (i). □

Proof of Theorem C. The statement follows from Proposition 5.5 and Proposition 5.10. □

Proof of Theorem D. Set $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$ and $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$. By [10, Lemma 3.28(c)], $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $(\mathcal{L}, \tilde{\Delta}, S)$ is a linking locality, and similarly, $(\mathcal{N}, \Gamma, T)$ is a linking locality if and only if $(\mathcal{N}, \tilde{\Gamma}, T)$ is a linking locality.[1] Thus, supposing from now on that $(\mathcal{L}, \Delta, S)$ and $(\mathcal{N}, \Gamma, T)$ are linking localities, it is by Theorem C enough to show that $\tilde{\Delta} = \delta(\mathcal{F})$ if and only if $\tilde{\Gamma} = \delta(\mathcal{E})$ and that $E(\mathcal{L}) = E(\mathcal{N})$.

By Proposition 5.9, $\mathcal{E} \trianglelefteq \mathcal{F}$. Moreover, our assumption yields that $N_{\mathcal{L}}(T)$ is of characteristic p. Hence, by Lemma 3.10(b), $N_{\mathcal{F}}(T) = \mathcal{F}_S(N_{\mathcal{L}}(T))$ is constrained. It follows then from [10, Corollary 7.18, Lemma 7.19] that $E(C_{\mathcal{F}}(\mathcal{E})) = E(N_{\mathcal{F}}(T))$ is the trivial fusion system. By [10, Lemma 7.13(c)], this implies $E(\mathcal{F}) = E(\mathcal{E})$.

By [18, Lemma 11.13], $E(\mathcal{N}) \trianglelefteq \mathcal{L}$. Moreover, by [10, Theorem E(d)], $\mathcal{F}_{S \cap E(\mathcal{N})}(E(\mathcal{N})) = E(\mathcal{E}) = E(\mathcal{F})$, and $E(\mathcal{L})$ is the unique partial normal subgroup of $\mathcal{L}$ with $\mathcal{F}_{S \cap E(\mathcal{L})}(E(\mathcal{L})) = E(\mathcal{F})$. Hence, $E(\mathcal{N}) = E(\mathcal{L})$.

By Lemma 3.12 and [16, Proposition 5], $O_p(\mathcal{F}) = O_p(\mathcal{L})$ and $O_p(\mathcal{E}) = O_p(\mathcal{N})$. Moreover, $O_p(\mathcal{L}) \cap T = O_p(\mathcal{L}) \cap \mathcal{N} = O_p(\mathcal{N})$ by Lemma 5.2(c).

[1] It should be noted here that the term ‘proper locality’ used in [10] means exactly the same as the term ‘linking locality’.

Assume first that $\tilde{\Gamma} = \delta(\mathcal{E})$. Let $P \leq S$ with $O_p(\mathcal{L}) = O_p(\mathcal{F}) \leq P$. Then $O_p(\mathcal{N}) \leq P \cap T$ and thus $P \cap T \in \Gamma$ if and only if $P \cap T \in \tilde{\Gamma}$. Hence,

$$\begin{aligned} P \in \delta(\mathcal{F}) &\Longleftrightarrow P \cap T \in \delta(\mathcal{E}) \text{ (by Lemma 4.9(a))} \\ &\Longleftrightarrow P \cap T \in \tilde{\Gamma} (\text{ as } \tilde{\Gamma} = \delta(\mathcal{E})) \\ &\Longleftrightarrow P \cap T \in \Gamma \text{ (as noted above)} \\ &\Longleftrightarrow P \in \Delta \text{ (since } (\mathcal{N}, \Gamma, T) \text{is a kernel).} \end{aligned}$$

Since the above equivalences hold for every $P \leq S$ with $O_p(\mathcal{L}) \leq P$, it follows now from [18, Lemma 10.6] that $\delta(\mathcal{F}) = \{P \leq S \colon PO_p(\mathcal{L}) \in \delta(\mathcal{F})\} = \{P \leq S \colon PO_p(\mathcal{L}) \in \Delta\} = \tilde{\Delta}$. This proves one direction.

Assume now the other way around that $\delta(\mathcal{F}) = \tilde{\Delta}$. Let $Q \leq T$ with $O_p(\mathcal{N}) \leq Q$. Then $QO_p(\mathcal{L}) \cap T = Q(O_p(\mathcal{L}) \cap T) = QO_p(\mathcal{N}) = Q$. Hence,

$$\begin{aligned} Q \in \delta(\mathcal{E}) &\Longleftrightarrow QO_p(\mathcal{L}) \cap T \in \delta(\mathcal{E}) \text{ (as } Q = QO_p(\mathcal{L}) \cap T) \\ &\Longleftrightarrow QO_p(\mathcal{L}) \in \delta(\mathcal{F}) \text{ (by Lemma 4.9(a) and since } O_p(\mathcal{L}) = O_p(\mathcal{F})) \\ &\Longleftrightarrow QO_p(\mathcal{L}) \in \tilde{\Delta} \text{ (as } \delta(\mathcal{F}) = \tilde{\Delta}) \\ &\Longleftrightarrow QO_p(\mathcal{L}) \in \Delta \text{ (by definition of} \tilde{\Delta}) \\ &\Longleftrightarrow QO_p(\mathcal{L}) \cap T \in \Gamma \text{ (as } (\mathcal{N}, \Gamma, T) \text{ is a kernel)} \\ &\Longleftrightarrow Q \in \Gamma \text{ (as } Q = QO_p(\mathcal{L}) \cap T). \end{aligned}$$

The above equivalences hold for all $Q \leq T$ with $O_p(\mathcal{N}) \leq Q$. Hence, it follows from [18, Lemma 10.6] that $\delta(\mathcal{E}) = \{Q \leq T \colon QO_p(\mathcal{N}) \in \delta(\mathcal{E})\} = \{Q \leq T \colon QO_p(\mathcal{N}) \in \Gamma\} = \tilde{\Gamma}$. This proves the assertion. □

Lemma 5.11. *Let* $(\mathcal{L}, \Delta, S)$ *be a locality with kernel* $(\mathcal{N}, \Gamma, T)$ *such that* $(\mathcal{N}, \Gamma, T)$ *is a regular locality. Set* $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$, $T_0 := E(\mathcal{N}) \cap S = E(\mathcal{E}) \cap S$ *and* $\tilde{\Delta} := \{P \leq S \colon PO_p(\mathcal{L}) \in \Delta\}$. *Then the following are equivalent:*

(i) $(\mathcal{L}, \Delta, S)$ *is a linking locality.*
(ii) $(\mathcal{L}, \tilde{\Delta}, S)$ *is a regular locality.*
(iii) $N_{\mathcal{L}}(T)$ *is a group of characteristic* p.
(iv) $N_{\mathcal{L}}(T_0)$ *is a group of characteristic* p.

Proof. Set $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$. By [10, Theorem E(d)], $E(\mathcal{E}) = \mathcal{F}_{E(\mathcal{N}) \cap S}(E(\mathcal{N}))$, and in particular $T_0 := E(\mathcal{N}) \cap S = E(\mathcal{E}) \cap S$. As $(\mathcal{N}, \Gamma, T)$ is regular (and in particular a linking locality), it follows from Theorem C that (i) and (iii) are equivalent. Moreover, by [18, Lemma 10.6], $\Gamma = \delta(\mathcal{E}) = \{Q \leq T \colon QO_p(\mathcal{N}) \in \Gamma\}$. Therefore, properties (ii) and (iii) are equivalent by Theorem D. Hence, it remains to prove that (iii) and (iv) are equivalent.

As $T_0 \in E(\mathcal{E})^s \subseteq \delta(\mathcal{E}) = \Gamma \subseteq \Delta$ by [10, Lemma 7.22], it follows that $G := N_{\mathcal{L}}(T_0)$ is a group with $N := N_{\mathcal{N}}(T_0) \trianglelefteq G$. Indeed, since $(\mathcal{N}, \Gamma, T)$ is a regular locality, N is of characteristic p. Hence, by Lemma 4.3, G is of characteristic p if and only if $N_G(T)$ is of characteristic p. By [18, Lemma 11.12], every automorphism of $\mathcal{N}$ leaves $E(\mathcal{N})$ invariant, so $N_{\mathcal{L}}(T)$ acts by Lemma 5.2(b) on $E(\mathcal{N})$ via conjugation. In particular, $N_{\mathcal{L}}(T) \subseteq G$, and thus $N_{\mathcal{L}}(T) = N_G(T)$. This shows the equivalence of (iii) and (iv) as required. □

6. Products

In this section, we prove Theorem E and Corollary F. Indeed, we state and prove here some more detailed results. The following theorem implies Theorem E. The reader might want to recall the notation introduced in Remark 4.10.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Theorem 6.1. *Let $(\mathcal{L}, \Delta, S)$ be a regular locality, and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of $\mathcal{L}$. Set*

$$T^* := F^*(\mathcal{L}) \cap S,\ T_0 := E(\mathcal{N}) \cap S,\ T := S \cap \mathcal{N} \text{ and } \mathcal{E} := \mathcal{F}_T(\mathcal{N}).$$

Let $H \leq N_{\mathcal{L}}(T^*)$ and set $\tilde{H} := N_{\mathcal{N}}(T^*)H$. Fix $S_0 \in \mathrm{Syl}_p(\tilde{H})$ with $T \leq S_0$. Set

$$\Delta_0 := \{P \leq S_0 \mid P \cap \mathcal{N} \in \delta(\mathcal{E})\}.$$

Then the following hold:

(a) *$\mathcal{N}H = H\mathcal{N}$ is a partial subgroup of $\mathcal{L}$.*
(b) *$N_{\mathcal{N}}(T_0) = N_{\mathcal{N}}(T^*)$, and $\tilde{H} = N_{\mathcal{N}H}(T^*) = N_{\mathcal{N}H}(T_0) = N_{\mathcal{N}}(T_0)H$.*
(c) *$(\mathcal{N}H, \Delta_0, S_0)$ is a cr-complete locality with kernel $(\mathcal{N}, \delta(\mathcal{E}), T)$.*
(d) *Set $\tilde{\Delta}_0 := \{P \leq S_0 \colon PO_p(\mathcal{N}H) \in \Delta_0\}$. Then the following are equivalent:*
   - *$\tilde{H}$ is of characteristic p;*
   - *$(\mathcal{N}H, \Delta_0, S_0)$ is a linking locality;*
   - *$(\mathcal{N}H, \tilde{\Delta}_0, S_0)$ is a regular locality.*

(e) *$\mathcal{F}_0 := \mathcal{F}_{S_0}(\mathcal{N}H)$ is saturated. Moreover, $\mathcal{E}$ and $E(\mathcal{E})$ are normal subsystems of $\mathcal{F}_0$, $N_{\mathcal{F}_0}(T_0) = \mathcal{F}_{S_0}(\tilde{H})$, and*

$$\mathcal{F}_0 = \langle (E(\mathcal{E})S_0)_{\mathcal{F}_0}, \mathcal{F}_{S_0}(\tilde{H})\rangle.$$

Proof. Set $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$. As usual, we write $\Pi\colon \mathbf{D} \to \mathcal{L}$ for the product on $\mathcal{L}$.

(a) By definition of a regular locality, for every $P \leq S$, we have $P \in \Delta$ if and only if $P \cap T^* \in \Delta$. Thus, Lemma 3.8(e) yields that

$$\text{for all } u \in \mathbf{W}(\mathcal{L}), \text{ we have } u \in \mathbf{D} \text{ if and only if } S_u \cap T^* \in \Delta. \tag{6.1}$$

In particular, $S_f \cap T^* \in \Delta$ for all $f \in \mathcal{L}$. If $n \in \mathcal{N}$ and $g \in \mathcal{N}_{\mathcal{L}}(T^*)$, then we can conclude that $(g^{-1}, g, n) \in \mathbf{D}$ via $S_n \cap T^*$. Hence, by the axioms of a partial group, gn is defined and $g^{-1}(gn) = \Pi(g^{-1}, g, n) = n$. Note also that $S_{gn} \cap T^* \leq S_{(g,g^{-1},gn)}$. Using Lemma 3.8(f), we conclude that

$$S_{(g,n)} \cap T^* \leq S_{gn} \cap T^* \leq S_{(g,g^{-1},gn)} \cap T^* \leq S_{(g,g^{-1}(gn))} \cap T^* \leq S_{(g,n)} \cap T^*$$

and so

$$S_{(g,n)} \cap T^* = S_{gn} \cap T^* \text{ for all } g \in N_{\mathcal{L}}(T^*) \text{ and all } n \in \mathcal{N}. \tag{6.2}$$

Let $n \in \mathcal{N}$ and $g \in H$. We argue first that $\mathcal{N}H = H\mathcal{N}$. Observe that $u_1 := (g, g^{-1}, n, g) \in \mathbf{D}$ via $S_n \cap T^*$ and $u_2 := (g, n, g^{-1}, g) \in \mathbf{D}$ via $(S_n \cap T^*)^{g^{-1}}$. Hence, $ng = \Pi(u_1) = g(n^g) \in H\mathcal{N}$ and $gn = \Pi(u_2) = (n^{g^{-1}})g \in \mathcal{N}H$. This shows that $\mathcal{N}H = H\mathcal{N}$. In particular, if $f = ng \in \mathcal{N}H$, then [8, Lemma 1.4(f)] yields $(g^{-1}, n^{-1}) \in \mathbf{D}$ and $f^{-1} = g^{-1}n^{-1} \in H\mathcal{N} = \mathcal{N}H$.

It remains thus to show that $\mathcal{N}H = H\mathcal{N}$ is closed under the partial product $\Pi\colon \mathbf{D} \to \mathcal{L}$ on $\mathcal{L}$. Let $w = (f_1, \ldots, f_k) \in \mathbf{D} \cap \mathbf{W}(H\mathcal{N})$. Then for every $1 \leq i \leq k$, we have $f_i = g_i n_i$ for some $g_i \in H$ and $n_i \in \mathcal{N}$. Set $u = (g_1, n_1, \ldots, g_k, n_k)$. Then by (6.2), we get $S_u \cap T^* = S_w \cap T^*$. Now equation (6.1) yields first $S_u \cap T^* = S_w \cap T^* \in \Delta$ and then $u \in \mathbf{D}$. By [8, Lemma 3.4], $\Pi(u) = \Pi(g_1, \ldots, g_k, n)$ for some $n \in \mathcal{N}$. Hence, by the axioms of a partial group,

$$\Pi(w) = \Pi(u) = \Pi(g_1, \ldots, g_k, n) = \Pi(g_1, \ldots, g_k)n \in H\mathcal{N} = \mathcal{N}H.$$

Thus $\mathcal{N}H$ is closed under the partial product and thus a partial subgroup. Hence, (a) holds.

(b) Lemma 3.22(a) gives $N_{\mathcal{L}}(T_0) \leq N_{\mathcal{L}}(T^*)$ and $N_{\mathcal{N}}(T_0) = N_{\mathcal{N}}(T^*)$. It follows from the Dedekind Lemma [8, Lemma 1.10] that $N_{\mathcal{N}H}(T^*) = \mathcal{N}H \cap N_{\mathcal{N}H}(T^*) = N_{\mathcal{N}}(T^*)H = \tilde{H}$. As $H \leq N_{\mathcal{L}}(T^*) \subseteq$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

$N_{\mathcal{L}}(T_0)$ and $N_{\mathcal{L}}(T_0)$ is by [8, Lemma 2.12(a)] a partial subgroup of $\mathcal{L}$, the Dedekind Lemma yields similarly that $N_{\mathcal{N}H}(T_0) = \mathcal{N}H \cap N_{\mathcal{N}H}(T_0) = N_{\mathcal{N}}(T_0)H$. This implies (b).

(c,d) Set $\Gamma := \delta(\mathcal{E})$ and $R := E(C_{\mathcal{L}}(\mathcal{N})) \cap S$. We show first that

$$\mathbf{D} \cap \mathbf{W}(\mathcal{N}H) = \mathbf{D}_\Gamma \cap \mathbf{W}(\mathcal{N}H). \tag{6.3}$$

As $R \subseteq C_{\mathcal{L}}(\mathcal{N})$, [18, Lemma 3.5] yields $\mathcal{N} \subseteq C_{\mathcal{L}}(R) \subseteq N_{\mathcal{L}}(R)$. Moreover, by Lemma 3.22(a), $H \subseteq N_{\mathcal{L}}(T^*) \subseteq N_{\mathcal{L}}(R)$. As $N_{\mathcal{L}}(R)$ is by [8, Lemma 2.12(a)] a partial subgroup of $\mathcal{L}$, it follows that $\mathcal{N}H \subseteq N_{\mathcal{L}}(R)$. Moreover, we use that $O_p(\mathcal{L}) = O_p(\mathcal{F})$, by [16, Proposition 5] and Lemma 3.12. So fixing $w = (f_1, \ldots, f_n) \in \mathbf{W}(\mathcal{N}H)$, we have $RO_p(\mathcal{F}) = RO_p(\mathcal{L}) \leq S_w$. Hence, Lemma 4.9(b) gives

$$S_w \in \delta(\mathcal{F}) = \Delta \Longleftrightarrow S_w \cap T \in \Gamma.$$

Notice that $S_w \in \mathbf{D} = \mathbf{D}_\Delta$ if and only if $S_w \in \Delta$. By [18, Lemma 10.4], Γ is overgroup-closed in T. Hence, $w \in \mathbf{D}_\Gamma$ implies $S_w \cap T \in \Gamma$. Moreover, by Lemma 3.22(d), Γ is closed under passing to $\mathcal{L}$-conjugates in S. Therefore, $S_w \cap T \in \Gamma$ implies $w \in \mathbf{D}_\Gamma$. So $S_w \cap T \in \Gamma$ if and only if $w \in \mathbf{D}_\Gamma$. This proves equation (6.3). We show next

$$S_0 \text{ is maximal with respect to inclusion among the } p\text{-subgroups of } \mathcal{N}H. \tag{6.4}$$

Let S_1 be a p-subgroup of $\mathcal{N}H$ such that $S_0 \leq S_1$. Notice that $T \leq S_0 \leq S_1$, so $T_0 = E(\mathcal{N}) \cap S \leq E(\mathcal{N}) \cap S_1$. By [18, Lemma 11.13], $E(\mathcal{N}) \trianglelefteq \mathcal{L}$. In particular, [8, Lemma 3.1(c)] yields that T_0 is a maximal p-subgroup of $E(\mathcal{N})$. Hence, $T_0 = E(\mathcal{N}) \cap S_1 \trianglelefteq S_1$. Now (b) yields $S_1 \leq N_{\mathcal{N}H}(T_0) = \tilde{H}$. As $S_0 \in \mathrm{Syl}_p(\tilde{H})$, it follows that $S_1 = S_0$. This proves equation (6.4).

Recall that by Lemma 3.22(d), Γ is closed under passing to $\mathcal{L}$-conjugates in S. As T is by [8, Lemma 3.1(c)] a maximal p-subgroup of $\mathcal{N}$, we have $T = S_0 \cap \mathcal{N}$. Since the elements of Γ are contained in T and $\mathcal{N} \trianglelefteq \mathcal{N}H$, it follows that Γ is closed under passing to $\mathcal{N}H$-conjugates in S_0. So by equations (6.3) and (6.4), the hypothesis of Lemma 5.4 is fulfilled with $(\mathcal{N}H, S_0)$ in place of $(\mathcal{L}, S)$. Hence, it follows from this lemma that $(\mathcal{N}H, \Delta_0, S_0)$ is a locality with kernel $(\mathcal{N}, \Gamma, T)$. By [18, Theorem 10.16(a)], $(\mathcal{N}, \Gamma, T)$ is a regular locality, and in particular it is cr-complete. Hence, $(\mathcal{N}H, \Delta_0, S_0)$ is cr-complete by Theorem B: that is, (c) holds. Part (d) follows from (b) and Lemma 5.11.

(e) By [18, Corollary 11.10], we have $T_0 \in \delta(\mathcal{E}) = \Gamma \subseteq \Delta_0$. Hence, parts (b) and (c) together with Lemma 3.10(b) yield that $N_{\mathcal{F}_0}(T_0) := \mathcal{F}_{S_0}(N_{\mathcal{N}H}(T_0)) = \mathcal{F}_{S_0}(\tilde{H})$. Moreover, by Theorem B and Proposition 3.18(c), part (c) implies that $\mathcal{F}_0$ is saturated and $\mathcal{E} \trianglelefteq \mathcal{F}_0$. Hence, $E(\mathcal{E}) \trianglelefteq \mathcal{F}_0$ by [10, Lemma 7.13(a)]. By Lemma 3.22(b), we have $T_0 = E(\mathcal{E}) \cap S$. Hence, (e) follows from Lemma 4.11. □

If the subgroup H in Theorem 6.1 has the property that $S \cap H \in \mathrm{Syl}_p(H)$, then the following lemma says that we can choose the p-subgroup S_0 as a subgroup of S.

Lemma 6.2. *Let $(\mathcal{L}, \Delta, S)$ be a regular locality, and set $T^* := F^*(\mathcal{L}) \cap S$. Let $\mathcal{N} \trianglelefteq \mathcal{L}$, $T := S \cap \mathcal{N}$ and $H \leq N_{\mathcal{L}}(T^*)$ such that $S \cap H \in \mathrm{Syl}_p(H)$. Then*

$$S_0 := T(S \cap H) = S \cap (\mathcal{N}H) \in \mathrm{Syl}_p(N_{\mathcal{N}}(T^*)H).$$

Proof. Note that $N_{\mathcal{L}}(T^*)$ is a group, S is a Sylow p-subgroup of $N_{\mathcal{L}}(T^*)$, $N_{\mathcal{N}}(T^*) \trianglelefteq N_{\mathcal{L}}(T^*)$, and $H \leq N_{\mathcal{L}}(T^*)$ by assumption. As $S \cap H \in \mathrm{Syl}_p(H)$, Lemma 4.1 yields that

$$S_0 := (S \cap N_{\mathcal{N}}(T^*))(S \cap H) = T(S \cap H)$$

is a Sylow p-subgroup of $N_{\mathcal{N}}(T^*)H$. In particular, it is a consequence of Theorem 6.1(a),(c) that S_0 is a maximal p-subgroup of the partial group $\mathcal{N}H$ and thus equals $S \cap (\mathcal{N}H)$. □

Assuming the hypothesis of Corollary 6.3 below, we have $\mathcal{E} \trianglelefteq \mathcal{F}$ by [10, Theorem A]. Hence, it is a consequence of [10, Lemma 7.13(a)] that $E(\mathcal{F}) \trianglelefteq \mathcal{F}$, and thus $(E(\mathcal{E})S_0)_{\mathcal{F}}$ is well-defined. This is implicitly used in the statement of part (b).

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Corollary 6.3. *Let $(\mathcal{L}, \Delta, S)$ be a regular locality, $T^* := F^*(\mathcal{L}) \cap S$, $\mathcal{N} \trianglelefteq \mathcal{L}$, and $H \le N_{\mathcal{L}}(T^*)$ with $S \cap H \in \mathrm{Syl}_p(H)$. Set $T := S \cap T$, $\mathcal{E} := \mathcal{F}_T(\mathcal{N})$, $S_0 := T(S \cap H)$, $\tilde{H} := N_{\mathcal{N}}(T^*)H$ and*

$$\mathcal{E}H := \mathcal{F}_{S_0}(\mathcal{N}H).$$

Then the following hold:

(a) *$\mathcal{E}H$ is a saturated subsystem of $\mathcal{F}$, $\mathcal{E} \trianglelefteq \mathcal{E}H$ and $E(\mathcal{E}) \trianglelefteq \mathcal{E}H$.*
(b) *$S_0 \in \mathrm{Syl}_p(\tilde{H})$ and $\mathcal{E}H = \langle E(\mathcal{E})S_0, \mathcal{F}_{S_0}(\tilde{H})\rangle$, where $E(\mathcal{E})S_0 := (E(\mathcal{E})S_0)_{\mathcal{F}} = (E(\mathcal{E})S_0)_{\mathcal{E}H}$.*
(c) *If $\mathcal{D}$ is a saturated subsystem of $\mathcal{F}$ such that $E(\mathcal{E}) \trianglelefteq \mathcal{D}$ and $\mathcal{F}_{S_0}(\tilde{H}) \subseteq \mathcal{D}$, then $\mathcal{E}H \subseteq \mathcal{D}$.*

Proof. By Lemma 6.2, $S_0 \in \mathrm{Syl}_p(\tilde{H})$. Therefore, setting $\Delta_0 := \{P \le S_0 \colon P \cap T \in \delta(\mathcal{E})\}$, the hypothesis and thus the conclusion of Theorem 6.1 hold. Theorem 6.1(e) implies now that (a) holds and $\mathcal{E}H = \langle (E(\mathcal{E})S_0)_{\mathcal{E}H}, \mathcal{F}_{S_0}(\tilde{H})\rangle$. By [10, Remark 2.27], $(E(\mathcal{E})S_0)_{\mathcal{E}H} = (E(\mathcal{E})S_0)_{\mathcal{F}}$. Hence, (b) holds as well.

Now let $\mathcal{D}$ be as in (c), and suppose $\mathcal{D}$ is a subsystem over $R \le S$. As $\mathcal{F}_{S_0}(\tilde{H}) \subseteq \mathcal{D}$, we have in particular that $S_0 \le R$. Using [10, Remark 2.27] again, we observe that $(E(\mathcal{E})S_0)_{\mathcal{F}} = (E(\mathcal{E})S_0)_{\mathcal{D}} \subseteq \mathcal{D}$. As $\mathcal{F}_{S_0}(\tilde{H}) \subseteq \mathcal{D}$ by assumption, it follows now from (b) that $\mathcal{E}H \subseteq \mathcal{D}$, so (c) holds. □

Proof of Corollary F. By [18, Lemma 10.4], there exists a regular locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$. Moreover, by [10, Theorem A], there exists $\mathcal{N} \trianglelefteq \mathcal{L}$ with $T = S \cap \mathcal{N}$ and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. By [10, Theorem E(d)], we have $T^* := F^*(\mathcal{F}) \cap S = F^*(\mathcal{L}) \cap S$. Moreover, Lemma 3.22(b) gives $T_0 := E(\mathcal{E}) \cap S = E(\mathcal{N}) \cap S$. In particular, by [18, Lemma 10.4, Remark 10.12], we have $T^* \in \delta(\mathcal{F}) = \Delta \subseteq \mathcal{F}^s$. So $N_{\mathcal{F}}(T^*)$ is constrained, and $N_{\mathcal{L}}(T^*)$ is by Lemma 3.10(b) a model for $N_{\mathcal{F}}(T^*)$. Using Lemma 3.22(a), one observes that $N_{\mathcal{N}}(T_0) = N_{\mathcal{N}}(T^*) \trianglelefteq N_{\mathcal{L}}(T^*)$. Moreover, Lemma 3.22(c) gives $N_{\mathcal{E}}(T_0) = \mathcal{F}_T(N_{\mathcal{N}}(T^*))$. Hence, it follows from [2, Proposition I.6.2] that $N_{\mathcal{E}}(T_0) \trianglelefteq N_{\mathcal{F}}(T^*)$.

Now let G be an arbitrary model for $N_{\mathcal{F}}(T^*)$. As $N_{\mathcal{E}}(T_0) \trianglelefteq N_{\mathcal{F}}(T^*)$, it follows from [2, Theorem II.7.5] that there exists a unique normal subgroup N of G with $S \cap N = T$ and $\mathcal{F}_T(N) = N_{\mathcal{E}}(T_0)$. Now let $H \le G$ with $S \cap H \in \mathrm{Syl}_p(H)$. Set $S_0 := T(S \cap H)$. By [2, Theorem III.5.10], there exists an isomorphism $\alpha \colon G \to N_{\mathcal{L}}(T^*)$ with $\alpha|_S = \mathrm{id}_S$. Notice that $N\alpha \trianglelefteq N_{\mathcal{L}}(T^*)$ and $H\alpha \le N_{\mathcal{L}}(T^*)$ with

$$T = S \cap (N\alpha), \ \mathcal{F}_T(N\alpha) = \mathcal{F}_T(N) = N_{\mathcal{E}}(T^*),$$

$$S \cap H = S \cap (H\alpha) \in \mathrm{Syl}_p(H\alpha) \text{ and } \mathcal{F}_{S_0}(NH) = \mathcal{F}_{S_0}((NH)\alpha).$$

By [2, Theorem II.7.5], $N_{\mathcal{N}}(T^*)$ is the unique normal subgroup of $N_{\mathcal{L}}(T^*)$ realising $N_{\mathcal{E}}(T^*)$. Hence, $N\alpha = N_{\mathcal{N}}(T^*)$ and $\mathcal{F}_{S_0}(NH) = \mathcal{F}_{S_0}(N_{\mathcal{N}}(T^*)(H\alpha))$. Therefore the assertion follows from Corollary 6.3 applied with $H\alpha$ in place of H. □

A. Normal pairs of transporter systems

It is not within the scope of this paper to construct extensions of localities, but Theorem A and our theorems on kernels provide some tools for examining the properties of existing extensions. The results in this appendix may help to compare our theorems on kernels to theorems on extensions of linking systems in the literature. Such extensions have been studied in various places, starting with [6]. A more transparent algebraic framework is used in [23, 1], where the definition of a normal pair of linking systems is crucial (see also [2, Definition III.4.12]). This definition naturally generalises to a definition of a 'normal pair of transporter systems', which we state below. Transporter systems were defined by Oliver and Ventura [25], generalising the concept of a linking system. The goal of this appendix is to show that localities with kernels correspond to normal pairs of transporter systems (compare Proposition A.4 and Theorem A.7). We use here a correspondence between localities and transporter systems that was observed by Chermak [7, Appendix A].

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

For any functor $\alpha\colon \mathcal{C} \to \mathcal{D}$ between categories and any objects $P, Q \in \mathcal{C}$, we let

$$\alpha_{P,Q}\colon \mathrm{Mor}_{\mathcal{C}}(P,Q) \to \mathrm{Mor}_{\mathcal{D}}(\alpha(P),\alpha(Q))$$

denote the induced map between morphisms sets. Moreover, we set $\alpha_P := \alpha_{P,P}$.

The literature on linking systems and transporter systems is mostly written in 'left-hand notation'. Therefore, in this appendix (unlike in the rest of the paper), we will write maps on the left-hand side of the argument and conjugate from the left. In particular, if G is a finite group and Δ is a set of subgroups of G, then $\mathcal{T}_\Delta(G)$ denotes the category whose object set is Δ and such that the morphism set $\mathrm{Mor}_{\mathcal{T}_\Delta(G)}(P,Q)$ between any two objects $P, Q \in \Delta$ is the set of all $g \in G$ with

$${}^gP := P^{g^{-1}} = gPg^{-1} \leq Q.$$

A *transporter system* associated to a fusion system $\mathcal{F}$ over S is a category $\mathcal{T}$ whose object set is an $\mathcal{F}$-closed collection of subgroups of S, together with functors

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{T})}(S) \xrightarrow{\delta} \mathcal{T} \xrightarrow{\pi} \mathcal{F}$$

subject to certain axioms. In particular, δ is the identity on objects and π is the inclusion on objects, δ is injective on morphism sets, π is surjective on morphism sets, and $\pi \circ \delta$ sends an element $g \in \mathrm{Mor}_{\mathcal{T}_{\mathrm{Ob}(\mathcal{T})}(S)}(P,Q)$ to the corresponding conjugation map $c_{g^{-1}}\colon P \to Q, x \mapsto {}^gx$. The reader is referred to [25, Definition 3.1] for the precise definition. If $P, Q \in \mathrm{Ob}(\mathcal{T})$ and $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$, we will usually write $\pi(\varphi)$ for $\pi_{P,Q}(\varphi)$. The following definition is nonstandard.

Definition A.1. If $(\mathcal{T},\delta,\pi)$ is a transporter system associated to a fusion system $\mathcal{F}$, then we will say that $(\mathcal{T},\delta,\pi)$ is a transporter system *over* $\mathcal{F}$ if $\mathcal{F}$ is $\mathrm{Ob}(\mathcal{T})$-generated: that is,

$$\mathcal{F} = \langle \pi(\varphi)\colon P,Q \in \mathrm{Ob}(\mathcal{T}),\ \varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)\rangle.$$

If $(\mathcal{T},\delta,\pi)$ is a linking system associated to a saturated fusion system $\mathcal{F}$, then it follows from Alperin's Fusion Theorem (compare [2, Theorem I.3.6]) that $(\mathcal{T},\delta,\pi)$ is a transporter system *over* $\mathcal{F}$. However, in general, there can be transporter systems associated to a fusion system $\mathcal{F}$ that are not transporter systems over $\mathcal{F}$ (but just transporter systems over a subsystem of $\mathcal{F}$).

For a locality $(\mathcal{L},\Delta,S)$, there is a transporter system $\mathcal{T}_\Delta(\mathcal{L})$ over $\mathcal{F}_S(\mathcal{L})$ defined. The object set of the category $\mathcal{T}_\Delta(\mathcal{L})$ is Δ, and for $P, Q \in \Delta$, the set $\mathrm{Mor}_{\mathcal{T}_\Delta(\mathcal{L})}(P,Q)$ consists of all $f \in \mathcal{L}$ with

$$P \leq S_{f^{-1}} \text{ and } {}^fP := P^{f^{-1}} \leq Q.$$

It turns out (see [7, Proposition A.3]) that $(\mathcal{T}_\Delta(\mathcal{L}),\delta,\pi)$ is a transporter system, where the functor $\delta\colon \mathcal{T}_\Delta(S) \to \mathcal{T}_\Delta(\mathcal{L})$ is the identity on objects and the inclusion on morphism sets, and the functor $\pi\colon \mathcal{T}_\Delta(\mathcal{L}) \to \mathcal{F}_S(\mathcal{L})$ is the inclusion on objects and sends a morphism $f \in \mathrm{Mor}_{\mathcal{T}_\Delta(\mathcal{L})}(P,Q)$ to $c_f|_P \in \mathrm{Hom}_{\mathcal{F}_S(\mathcal{L})}(P,Q)$.

To review some notation and basic results, let $(\mathcal{T},\delta,\pi)$ be a transporter system, and fix $P, Q \in \mathrm{Ob}(\mathcal{T})$. If $P \leq Q$, then the morphism

$$\iota_{P,Q} := \delta_{P,Q}(1)$$

is regarded as an 'inclusion map'. If $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ and $P' \leq P$, then set

$$\varphi(P') := \pi(\varphi)(P').$$

By [25, Lemma A.6], a morphism $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ is an isomorphism in the categorical sense if and only if $\pi(\varphi)$ is an isomorphism, which is the case if and only if $\varphi(P) = Q$. It is shown in [25, Lemma 3.2(c)]

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

that for all $P', Q' \in \Delta$ with $P' \leq P$, $Q' \leq Q$ and all $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ with $\varphi(P') \leq Q'$, there exists a unique morphism $\varphi|_{P',Q'} \in \mathrm{Mor}_{\mathcal{T}}(P', Q')$ with

$$\varphi \circ \iota_{P',P} = \iota_{Q',Q} \circ \varphi|_{P',Q'}.$$

The morphism $\varphi|_{P',Q'}$ is called the *restriction* of φ to a morphism from P' to Q'.

Lemma A.2. *Let $(\mathcal{T}, \delta, \pi)$ be a transporter system associated to some fusion system over a p-group S. Then the following hold:*

(a) *Let $P'' \leq P' \leq P$ and $Q'' \leq Q' \leq Q$ be objects in $\mathcal{T}$. Let $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ with $\varphi(P') \leq Q'$ and $\varphi(P'') \leq Q''$. Then*

$$\varphi|_{P'',Q''} = (\varphi|_{P',Q'})|_{P'',Q''}.$$

(b) *Let $P, Q, R, P', R' \in \mathrm{Ob}(\mathcal{T})$, $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{T}}(Q, R)$ with $P' \leq P$ and $(\psi \circ \varphi)(P') \leq R' \leq R$. Then*

$$(\psi \circ \varphi)|_{P',R'} = \psi|_{\varphi(P'),R'} \circ \varphi|_{P',\varphi(P')}.$$

(c) *Let $x \in S$ and $\alpha = \delta_S(x)$. For every $P \in \Delta$, we have $\alpha(P) = {}^xP \in \Delta$ and*

$$\alpha|_{P,\alpha(P)} = \delta_{P,\alpha(P)}(x).$$

(d) *Let $P, Q, P' \in \mathrm{Ob}(\mathcal{T})$ and $\psi, \varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ such that $P' \leq P$, $Q' := \psi(P') = \varphi(P')$ and $\varphi|_{P',Q'} = \psi|_{P',Q'}$. Then $\varphi = \psi$.*

Proof. **(a)** Setting $\varphi' = \varphi|_{P',Q'}$, it follows from the definition of the restriction that

$$\iota_{Q',Q} \circ \varphi' = \varphi \circ \iota_{P',P} \text{ and } \iota_{Q'',Q'} \circ \varphi'|_{P'',Q''} = \varphi' \circ \iota_{P'',P'}.$$

Hence,

$$\begin{aligned}\iota_{Q'',Q} \circ \varphi'|_{P'',Q''} &= \iota_{Q',Q} \circ \iota_{Q'',Q'} \circ \varphi'|_{P'',Q''}\\ &= \iota_{Q',Q} \circ \varphi' \circ \iota_{P'',P'}\\ &= \varphi \circ \iota_{P',P} \circ \iota_{P'',P'}\\ &= \varphi \circ \iota_{P'',P}.\end{aligned}$$

This implies (a).

(b) See [7, Lemma A.7(b)].

(c) By axiom (B) of a transporter system as stated in [25, Definition 3.1], we have $\pi(\alpha) = c_{x^{-1}}$, and thus ${}^xP = P^{x^{-1}} = \alpha(P)$. As δ is a functor, it follows that

$$\iota_{\alpha(P),S} \circ \delta_{P,\alpha(P)}(x) = \delta_{P,S}(1 \cdot x) = \delta_{P,S}(x \cdot 1) = \delta_S(x) \circ \delta_{P,S}(1) = \alpha \circ \iota_{P,S}.$$

This shows (c).

(d) Assume the hypothesis of (d) and set $\varphi_0 := \varphi|_{P',Q'} = \psi|_{P',Q'}$. It follows from the definition restrictions that

$$\varphi \circ \iota_{P',P} = \iota_{Q',Q} \circ \varphi_0 = \psi \circ \iota_{P',P}.$$

As every morphism in $\mathcal{T}$ is by [25, Lemma 3.2(d)] an epimorphism, it follows that $\varphi = \psi$, as required. □

Central to the considerations in this appendix is the following definition that generalises the definition of a normal pair of linking systems (compare [2, Definition III.4.12]).

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Definition A.3. Fix a pair of fusion systems $\mathcal{F}_0 \subseteq \mathcal{F}$ over p-groups $S_0 \leq S$ such that $\mathcal{F}_0$ is $\mathcal{F}$-invariant. Let $\mathcal{T}_0 \subseteq \mathcal{T}$ be transporter systems over $\mathcal{F}_0 \subseteq \mathcal{F}$, respectively. Then $\mathcal{T}_0$ is called *normal* in $\mathcal{T}$ (written as $\mathcal{T}_0 \trianglelefteq \mathcal{T}$) if the following hold:

(i) $\mathrm{Ob}(\mathcal{T}) = \{P \leq S \colon P \cap S_0 \in \mathrm{Ob}(\mathcal{T}_0)\}$;
(ii) for all $P \in \mathrm{Ob}(\mathcal{T}_0)$ and $\psi \in \mathrm{Mor}_{\mathcal{T}}(P, S_0)$, there are morphisms $\gamma \in \mathrm{Aut}_{\mathcal{T}}(S_0)$ and $\psi_0 \in \mathrm{Mor}_{\mathcal{T}_0}(P, S_0)$ such that $\psi = \gamma \circ \psi_0$; and
(iii) for all $\gamma \in \mathrm{Aut}_{\mathcal{T}}(S_0)$, $P, Q \in \mathrm{Ob}(\mathcal{T}_0)$, and $\psi \in \mathrm{Mor}_{\mathcal{T}_0}(P, Q)$,

$$\gamma|_{Q,\gamma(Q)} \circ \psi \circ (\gamma|_{P,\gamma(P)})^{-1} \in \mathrm{Mor}_{\mathcal{T}_0}(\gamma(P), \gamma(Q)).$$

We also say then that $\mathcal{T}_0 \trianglelefteq \mathcal{T}$ is a *normal pair of transporter systems* over $\mathcal{F}_0 \subseteq \mathcal{F}$.

Saying that $\mathcal{T}_0 \subseteq \mathcal{T}$ are transporter systems over $\mathcal{F}_0 \subseteq \mathcal{F}$, respectively, means here more precisely that $\mathcal{T}_0$ is a subcategory of $\mathcal{T}$, that $(\mathcal{T}, \delta, \pi)$ is a transporter system over $\mathcal{F}$ (for appropriate functors δ and π) and that $(\mathcal{T}_0, \delta|_{\mathcal{T}_{\mathrm{Ob}(\mathcal{T}_0)}(S_0)}, \pi|_{\mathcal{T}_0})$ is a transporter system over $\mathcal{F}_0$.

We now prove that kernels of localities lead to normal pairs of transporter systems.

Proposition A.4. *Let $(\mathcal{N}, \Gamma, T)$ be a kernel of a locality $(\mathcal{L}, \Delta, S)$. Then $\mathcal{T}_\Gamma(\mathcal{N}) \trianglelefteq \mathcal{T}_\Delta(\mathcal{L})$ is a normal pair of transporter systems over $\mathcal{F}_T(\mathcal{N}) \subseteq \mathcal{F}_S(\mathcal{L})$.*

Proof. It follows from the discussion above that $\mathcal{T}_\Gamma(\mathcal{N}) \subseteq \mathcal{T}_\Delta(\mathcal{L})$ are transporter systems over $\mathcal{F}_T(\mathcal{N}) \subseteq \mathcal{F}_S(\mathcal{L})$. By Lemma 5.2(d), $\mathcal{F}_T(\mathcal{N})$ is $\mathcal{F}_S(\mathcal{L})$-invariant. Referring to the properties (i),(ii),(iii) in Definition A.3, it follows from the definition of a kernel that (i) holds. Property (ii) is a consequence of Lemma 3.13. Property (iii) holds since $\mathcal{N} \trianglelefteq \mathcal{L}$. □

We outline now how a locality can be constructed from a transporter system and use this afterwards to show that normal pairs of transporter systems lead to localities with kernels.

Let $(\mathcal{T}, \delta, \pi)$ be a transporter system over a fusion system $\mathcal{F}$ on S. Write $\mathrm{Iso}(\mathcal{T})$ for the set of all isomorphisms in $\mathcal{T}$. Define a relation $\uparrow_{\mathcal{T}}$ on $\mathrm{Iso}(\mathcal{T})$ by writing

$$\varphi \uparrow_{\mathcal{T}} \psi$$

if ψ restricts to φ. More precisely, this means $\varphi \uparrow_{\mathcal{T}} \psi$, if $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P', Q')$ and $\psi \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$ for some $P, P', Q, Q' \in \mathrm{Ob}(\mathcal{T})$ with $P' \leq P$, $Q' \leq Q$, $\psi(P') = Q'$ and $\varphi = \psi|_{P',Q'}$. Let then $\equiv_{\mathcal{T}}$ be the smallest equivalence relation on $\mathrm{Iso}(\mathcal{T})$ containing $\uparrow_{\mathcal{T}}$. Write $[\varphi]$ for the $\equiv_{\mathcal{T}}$-equivalence class of $\varphi \in \mathrm{Iso}(\mathcal{T})$ and $\mathcal{L}(\mathcal{T})$ for the set of all equivalence classes of $\mathrm{Iso}(\mathcal{T})$.

Remark A.5. We have $\varphi \equiv_{\mathcal{T}} \psi$ if and only if there exists a sequence $\varphi_1, \varphi_2, \cdots, \varphi_k \in \mathrm{Iso}(\mathcal{T})$ such that, for all $i = 1, 2, \ldots, k-1$,

$$\varphi_i \uparrow_{\mathcal{T}} \varphi_{i+1} \text{ or } \varphi_{i+1} \uparrow_{\mathcal{T}} \varphi_i.$$

By $\mathbf{D}$ denote the set of tuples $w = (f_1, f_2, \ldots, f_k) \in \mathbf{W}(\mathcal{L}(\mathcal{T}))$ for which there exist $\varphi_i \in f_i$ for $i = 1, \ldots, k$ such that the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k$ is defined in the category $\mathcal{T}$. Moreover, given such w and φ_i, set

$$\Pi(w) := [\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k].$$

The map $\Pi \colon \mathbf{D} \longrightarrow \mathcal{L}(\mathcal{T})$ is well-defined. Together with Π and the map

$$\mathcal{L}(\mathcal{T}) \longrightarrow \mathcal{L}(\mathcal{T}), [\varphi] \mapsto [\varphi^{-1}],$$

which is also well-defined, the set $\mathcal{L}(\mathcal{T})$ forms a partial group by [7, Proposition A.9]. Moreover, the map

$$S \longrightarrow \mathcal{L}(\mathcal{T}),\ x \mapsto [\delta_S(x)]$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

is an injective homomorphism of partial groups, and its image is a subgroup of $\mathcal{L}(\mathcal{T})$. We will usually identify $x \in S$ with $[\delta_S(x)] \in \mathcal{L}(\mathcal{T})$. With this identification, S is a subgroup of $\mathcal{L}(\mathcal{T})$. Setting $\Delta := \mathrm{Ob}(\mathcal{T})$, it is shown in [7, Proposition A.13] that $(\mathcal{L}(\mathcal{T}), \Delta, S)$ is a locality. As we assume that $(\mathcal{T}, \delta, \pi)$ is a transporter system over $\mathcal{F}$, it follows from [17, Lemma 4.4(a)] that $(\mathcal{L}(\mathcal{T}), \Delta, S)$ is a locality over $\mathcal{F}$. We will use these properties throughout without further reference.

Remark A.6. The partial group structure on $\mathcal{L}(\mathcal{T})$ constructed above is not exactly the same as the one constructed by Chermak [7, Appendix A]. The reason is that Chermak consistently uses the 'right-hand notation' for maps and also for the category $\mathcal{T}$. So if $\Pi\colon \mathbf{D} \to \mathcal{L}(\mathcal{T})$ is the partial product defined above and $\Pi'\colon \mathbf{D}' \to \mathcal{L}(\mathcal{T})$ is the product constructed by Chermak, then

$$\mathbf{D}' = \{(f_n, f_{n-1}, \ldots, f_1)\colon (f_1, f_2, \ldots, f_n) \in \mathbf{D}\}$$

and $\Pi'(f_n, f_{n-1}, \ldots, f_1) = \Pi(f_1, f_2, \ldots, f_n)$ for all $(f_1, f_2, \ldots, f_n) \in \mathbf{D}$. In particular, conjugation by $f \in \mathcal{L}(\mathcal{T})$ with respect to the partial group with product Π' corresponds to conjugation by f^{-1} in the partial group constructed above.

The main goal of this appendix is to prove the following theorem.

Theorem A.7. *Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be fusion systems over $S_0 \leq S$ such that $\mathcal{F}_0$ is $\mathcal{F}$-invariant. Let $\mathcal{T}_0 \trianglelefteq \mathcal{T}$ be a normal pair of transporter systems over $\mathcal{F}_0 \subseteq \mathcal{F}$. Set $\Delta := \mathrm{Ob}(\mathcal{T})$, $\Gamma := \mathrm{Ob}(\mathcal{T}_0)$, $\mathcal{L} := \mathcal{L}(\mathcal{T})$ and*

$$\mathcal{N} := \{f \in \mathcal{L}\colon f \cap \mathrm{Iso}(\mathcal{T}_0) \neq \emptyset\}.$$

Then $(\mathcal{N}, \Gamma, S_0)$ is a kernel of $(\mathcal{L}, \Delta, S)$ with $\mathcal{F}_{S_0}(\mathcal{N}) = \mathcal{F}_0$. Moreover, there is an isomorphism $\mathcal{N} \to \mathcal{L}(\mathcal{T}_0)$, which is the identity on S_0, and there exists an invertible functor $\mathcal{T}_\Gamma(\mathcal{N}) \to \mathcal{T}_0$ that is the identity on Γ.

Remark A.8. Note that Theorem A.7 allows us to conclude results about normal pairs of transporter systems from our results about kernels of localities. Let us point out one example: Call a transporter system $(\mathcal{T}, \delta, \pi)$ over $\mathcal{F}$ cr-complete if $\mathcal{F}^{cr} \subseteq \mathrm{Ob}(\mathcal{T})$. Consider a normal pair of transporter systems $\mathcal{T}_0 \trianglelefteq \mathcal{T}$ over $\mathcal{F}_0 \subseteq \mathcal{F}$. Then it follows from Theorem B that $\mathcal{T}_0$ is cr-complete if and only if $\mathcal{T}$ is cr-complete. Moreover, if so, then $\mathcal{F}_0$ is normal in $\mathcal{F}$. As a particular consequence, if $\mathcal{T}_0 \trianglelefteq \mathcal{T}$ is a normal pair of linking systems over $\mathcal{F}_0 \subseteq \mathcal{F}$ (as defined in [1] and [23]), then $\mathcal{F}_0$ is always normal in $\mathcal{F}$.

To prove Theorem A.7, we assume the following hypothesis:

From now on, let $\mathcal{F}_0 \subseteq \mathcal{F}$ be fusion systems over $S_0 \leq S$, respectively, such that $\mathcal{F}_0$ is $\mathcal{F}$-invariant, and let $\mathcal{T}_0 \trianglelefteq \mathcal{T}$ be a normal pair of transporter systems over $\mathcal{F}_0 \subseteq \mathcal{F}$, respectively.

More precisely, let $(\mathcal{T}, \delta, \pi)$ be a transporter system over $\mathcal{F}$ and $(\mathcal{T}, \delta|_{\mathcal{T}_{\mathrm{Ob}(\mathcal{T}_0)}(S)}, \pi|_{\mathcal{T}_0})$ be a transporter system over $\mathcal{F}_0$.

Lemma A.9. *Let $P, Q \in \mathrm{Ob}(\mathcal{T}_0)$ and $\psi \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$. Then there exist $\gamma \in \mathrm{Aut}_{\mathcal{T}}(S_0)$, $R \in \mathrm{Ob}(\mathcal{T}_0)$ and $\psi_0 \in \mathrm{Iso}_{\mathcal{T}_0}(P, R)$ such that $\gamma(R) = Q$ and $\psi = \gamma|_{R,Q} \circ \psi_0$.*

Proof. Notice that $\hat{\psi} := \iota_{Q,S_0} \circ \psi \in \mathrm{Mor}_{\mathcal{T}_0}(P, S_0)$ with $\hat{\psi}|_{P,Q} = \psi$. Hence, by axiom (ii) of Definition A.3, there exist $\gamma \in \mathrm{Aut}_{\mathcal{T}}(S_0)$ and $\hat{\psi}_0 \in \mathrm{Mor}_{\mathcal{T}_0}(P, S_0)$ such that

$$\hat{\psi} = \gamma \circ \hat{\psi}_0.$$

Setting $R := \hat{\psi}_0(P)$ and $\psi_0 := \hat{\psi}_0|_{P,R} \in \mathrm{Iso}_{\mathcal{T}_0}(P, R)$, we have $\gamma(R) = Q$ and it follows from Lemma A.2(b) that $\psi = \hat{\psi}|_{P,Q} = \gamma|_{R,Q} \circ \psi_0$. □

Lemma A.10. *Let $P, \overline{P}, Q, \overline{Q} \in \mathrm{Ob}(\mathcal{T}_0)$ with $P \leq \overline{P}$ and $Q \leq \overline{Q}$. Let $\psi \in \mathrm{Iso}_{\mathcal{T}}(\overline{P}, \overline{Q})$ with $\psi(P) \leq Q$ and $\psi|_{P,Q} \in \mathrm{Mor}_{\mathcal{T}_0}(P, Q)$. Then ψ is a morphism in $\mathcal{T}_0$.*

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Proof. As $\overline{P}$ is a p-group, P is subnormal in $\overline{P}$. As $\mathrm{Ob}(\mathcal{T}_0)$ is overgroup-closed in S_0, induction on $|\overline{P}:P|$ allows us to reduce to the case that $P \trianglelefteq \overline{P}$. Moreover, replacing Q by $\psi(P)$, we may assume that $\psi(P) = Q$, $\varphi := \psi|_{P,Q} \in \mathrm{Iso}_{\mathcal{T}_0}(P,Q)$ and $Q = \psi(P) \trianglelefteq \psi(\overline{P}) = \overline{Q}$. It follows from [25, Lemma 3.3] that $\varphi \circ \delta_P(x) = \delta_Q(\pi(\psi)(x)) \circ \varphi$ for all $x \in \overline{P}$. As $\pi(\psi)(\overline{P}) \leq \overline{Q}$, this implies that $\varphi \circ \delta_P(\overline{P}) \circ \varphi^{-1} \leq \delta_Q(\overline{Q})$. Hence, by Axiom II of a transporter systems (as stated in [25, Definition 3.1]) applied to the transporter system $\mathcal{T}_0$, there exists $\overline{\varphi} \in \mathrm{Mor}_{\mathcal{T}_0}(\overline{P},\overline{Q})$ with $\overline{\varphi}|_{P,Q} = \varphi = \psi|_{P,Q}$. It follows now from Lemma A.2(d) that $\psi = \overline{\varphi}$ is a morphism in $\mathcal{T}_0$. □

Set now $\Delta := \mathrm{Ob}(\mathcal{T})$, $\Gamma := \mathrm{Ob}(\mathcal{T}_0)$, $\mathcal{L} := \mathcal{L}(\mathcal{T})$ and

$$\mathcal{N} := \{f \in \mathcal{L} \colon f \cap \mathrm{Iso}(\mathcal{T}_0) \neq \emptyset\}.$$

As before, we write $\Pi\colon \mathbf{D} \longrightarrow \mathcal{L}$ for the product on $\mathcal{L}$.

Notation A.11. For $P \leq S$, set $P_0 := P \cap S_0$. Similarly, define Q_0, P'_0, Q'_0 for subgroups $Q, P', Q' \leq S$. For $P, Q \in \Delta$ and $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$, set moreover

$$\varphi^0 := \varphi|_{P_0,Q_0}.$$

As S_0 is strongly closed, we have in the situation above that $\varphi(P_0) = \pi(\varphi)(P_0) = (\pi(\varphi)(P)) \cap S_0 = Q_0$. So φ^0 is well-defined and an element of $\mathrm{Iso}_{\mathcal{T}}(P_0,Q_0)$. Observe also that $\varphi^0 \uparrow_{\mathcal{T}} \varphi$ and thus $[\varphi] = [\varphi^0]$ for all $\varphi \in \mathrm{Iso}(\mathcal{T})$.

Lemma A.12. *If $\varphi, \psi \in \mathrm{Iso}(\mathcal{T})$ with $\psi \uparrow_{\mathcal{T}} \varphi$, then $\psi^0 \uparrow_{\mathcal{T}} \varphi^0$.*

Proof. Let $P' \leq P$ and $Q' \leq Q$ be objects in $\mathcal{T}$ such that $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$, $\psi \in \mathrm{Mor}_{\mathcal{T}}(P',Q')$, $\varphi(P') \leq Q'$ and $\varphi|_{P',Q'} = \psi$. Applying Lemma A.2(a) twice gives then

$$\varphi^0|_{P'_0,Q'_0} = \varphi|_{P'_0,Q'_0} = \psi|_{P'_0,Q'_0} = \psi^0.$$

Hence, $\psi^0 \uparrow_{\mathcal{T}} \varphi^0$. □

Lemma A.13. *The following hold:*

(a) *If $\varphi, \psi \in \mathrm{Iso}(\mathcal{T}_0)$, then $\varphi \equiv_{\mathcal{T}} \psi$ if and only if $\varphi \equiv_{\mathcal{T}_0} \psi$.*
(b) *Let $\varphi \in \mathrm{Iso}(\mathcal{T}_0)$ and $\psi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$ for some $P, Q \in \mathrm{Ob}(\mathcal{T}_0)$. If $\varphi \equiv_{\mathcal{T}} \psi$, then $\psi \in \mathrm{Iso}(\mathcal{T}_0)$.*
(c) *If $\varphi \in \mathrm{Iso}(\mathcal{T})$ with $[\varphi] \in \mathcal{N}$, then $\varphi^0 \in \mathrm{Iso}(\mathcal{T}_0)$.*

Proof. **(a,b)** Clearly, the relation $\uparrow_{\mathcal{T}_0}$ is contained in the relation $\uparrow_{\mathcal{T}}$, so, if $\varphi, \psi \in \mathrm{Iso}(\mathcal{T}_0)$ with $\varphi \equiv_{\mathcal{T}_0} \psi$, then $\varphi \equiv_{\mathcal{T}} \psi$.

Suppose now that $\varphi \in \mathrm{Iso}(\mathcal{T}_0)$ and $\psi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$ for some $P, Q \in \mathrm{Ob}(\mathcal{T}_0)$. Assume $\varphi \equiv_{\mathcal{T}} \psi$. To show (a) and (b), it remains to show that $\psi \in \mathrm{Iso}(\mathcal{T}_0)$ and $\varphi \equiv_{\mathcal{T}_0} \psi$. By Remark A.5, there exists a series $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_n = \psi \in \mathrm{Iso}(\mathcal{T})$ such that, for all $i = 1, 2, \ldots, n-1$, we have $\varphi_i \uparrow_{\mathcal{T}} \varphi_{i+1}$ or $\varphi_{i+1} \uparrow_{\mathcal{T}} \varphi_i$. Notice that $\varphi^0 = \varphi$ and $\psi^0 = \psi$ as $P, Q \in \mathrm{Ob}(\mathcal{T}_0)$. Thus, Lemma A.12 allows us to replace $\varphi_1, \varphi_2, \ldots, \varphi_n$ by $\varphi_1^0, \varphi_2^0, \ldots, \varphi_n^0$. Thus, we may assume that $\varphi_1, \varphi_2, \ldots, \varphi_n$ are isomorphisms in $\mathcal{T}$ between objects of $\mathcal{T}_0$. Then Lemma A.10 implies that for all $i = 1, 2, \ldots, n-1$, $\varphi_i \in \mathrm{Iso}(\mathcal{T}_0)$ if and only if $\varphi_{i+1} \in \mathrm{Iso}(\mathcal{T}_0)$. As $\varphi_1 = \varphi \in \mathrm{Iso}(\mathcal{T}_0)$ by assumption, it follows therefore inductively that $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathrm{Iso}(\mathcal{T}_0)$. In particular, $\psi = \varphi_n \in \mathrm{Iso}(\mathcal{T}_0)$. For $\alpha, \beta \in \mathrm{Iso}(\mathcal{T}_0)$ it is easy to observe that $\alpha \uparrow_{\mathcal{T}} \beta$ if and only if $\alpha \uparrow_{\mathcal{T}_0} \beta$. Hence, we also have $\varphi_i \uparrow_{\mathcal{T}_0} \varphi_{i+1}$ or $\varphi_{i+1} \uparrow_{\mathcal{T}_0} \varphi_i$ for $i = 1, 2, \ldots, n-1$. This shows $\varphi \equiv_{\mathcal{T}_0} \psi$. So (a) and (b) hold.

(c) If $\varphi \in \mathrm{Iso}(\mathcal{T})$ with $[\varphi] \in \mathcal{N}$, then by definition of $\mathcal{N}$, there exists $\psi \in \mathrm{Iso}(\mathcal{T}_0)$ with $\psi \in [\varphi]$. Then $\varphi^0 \equiv_{\mathcal{T}} \varphi \equiv_{\mathcal{T}} \psi$. Hence, it follows from part (b) that $\varphi^0 \in \mathrm{Iso}(\mathcal{T}_0)$. □

For the following lemma, recall that we identify $x \in S$ with $[\delta_S(x)] \in \mathcal{L}$.

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Lemma A.14.

(a) *$\mathcal{N}$ is a partial subgroup of $\mathcal{L}$ with $\mathcal{N} \cap S = S_0$.*
(b) *Set $\mathcal{L}_0 := \mathcal{L}(\mathcal{T}_0)$, write $[\varphi]_0$ for the $\equiv_{\mathcal{T}_0}$-equivalence class of $\varphi \in \mathrm{Iso}(\mathcal{T}_0)$, and identify $x \in S_0$ with $[\delta_{S_0}(x)]_0$. Then the map*

$$\theta\colon \mathcal{N} \to \mathcal{L}_0, f \mapsto f \cap \mathrm{Iso}(\mathcal{T}_0)$$

is an isomorphism of localities that restricts to the identity on S_0.

Proof. We prove first

$$\mathcal{N} \text{ is closed under inversion.} \tag{A.1}$$

For the proof, note that every $n \in \mathcal{N}$ can be written as $n = [\varphi]$ for some $\varphi \in \mathrm{Iso}(\mathcal{T}_0)$. Then $\varphi^{-1} \in \mathrm{Iso}(\mathcal{T}_0)$, and it follows from the definition of the inversion on $\mathcal{L}$ that $n^{-1} = [\varphi^{-1}] \in \mathcal{N}$. This shows (A.1). We argue next that

$$\text{the map } \theta \text{ is well-defined and a bijection.} \tag{A.2}$$

Indeed, it follows from Lemma A.13(a) that $[\varphi] \cap \mathrm{Iso}(\mathcal{T}_0) = [\varphi]_0$ for all $\varphi \in \mathrm{Iso}(\mathcal{T}_0)$. Hence, θ is well-defined and surjective. Note also that $f \cap g \neq \emptyset$ for all $f, g \in \mathcal{N}$ with $\theta(f) = \theta(g)$. As the equivalence classes of $\equiv_{\mathcal{T}}$ form a partition of $\mathrm{Iso}(\mathcal{T})$, we can therefore conclude that the map θ is injective. So (A.2) holds. We show next

$$S_0 \subseteq \mathcal{N} \cap S \text{ and } \theta|_{S_0} = \mathrm{id}_{S_0}\,. \tag{A.3}$$

For the proof, let $x \in S_0$. Lemma A.2(c) implies $\delta_S(x)|_{S_0} = \delta_{S_0}(x) \in \mathrm{Aut}_{\mathcal{T}_0}(S_0) \subseteq \mathrm{Iso}(\mathcal{T}_0)$, so $x = [\delta_S(x)] = [\delta_{S_0}(x)] \in \mathcal{N}$. Moreover, $x = [\delta_S(x)]$ gets mapped $[\delta_{S_0}(x)]_0$, which we also identify with x (as stated in part (b)). This proves (A.3). We show next

$$\mathcal{N} \text{ is a partial subgroup, and } \theta \text{ is a homomorphism of partial groups.} \tag{A.4}$$

For the proof, let $w := (n_1, n_2, \ldots, n_k) \in \mathbf{W}(\mathcal{N}) \cap \mathbf{D}$. As $w \in \mathbf{D}$, there exist $\varphi_i \in n_i$ for $i = 1, 2, \ldots, k$ such that the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k$ is defined in $\mathcal{T}$. Note that then the composition $\varphi_1^0 \circ \varphi_2^\circ \cdots \circ \varphi_k^0$ is also defined. Moreover, as $\varphi_i^0 \uparrow_{\mathcal{T}} \varphi_i$, we have $\varphi_i^0 \in n_i$ for $i = 1, 2, \ldots, k$. By Lemma A.13(c), $\varphi_i^0 \in \mathrm{Iso}(\mathcal{T}_0)$. So, replacing $\varphi_1, \varphi_2, \ldots, \varphi_k$ by $\varphi_1^0, \varphi_2^0, \ldots, \varphi_k^0$, we may assume $\varphi_i \in \mathrm{Iso}(\mathcal{T}_0)$. Then $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k \in \mathrm{Iso}(\mathcal{T}_0)$, and hence

$$\Pi(w) = [\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k] \in \mathcal{N}.$$

Together with (A.1), this shows that $\mathcal{N}$ is a partial subgroup.

Note also that $\theta(n_i) = n_i \cap \mathrm{Iso}(\mathcal{T}_0) = [\varphi_i]_0$ for $i = 1, 2, \ldots, k$. Moreover, the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k$ is defined in $\mathcal{T}_0$. Hence, writing $\Pi_0\colon \mathbf{D}_0 \to \mathcal{L}_0$ for the product on $\mathcal{L}_0$ (defined in the usual way), it follows that $\theta^*(w) = (\theta(n_1), \theta(n_2), \ldots, \theta(n_k)) \in \mathbf{D}_0$ and

$$\Pi_0(\theta^*(w)) = [\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n]_0 = [\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n] \cap \mathrm{Iso}(\mathcal{T}_0) = \Pi(w) \cap \mathrm{Iso}(\mathcal{T}_0) = \theta(\Pi(w)).$$

This proves that θ is a homomorphism of partial groups, and (A.4) holds.

For the proofs of the next two properties, recall Definition 3.15 and the notation introduced there. We show next that

$$\theta \text{ is an isomorphism of partial groups.} \tag{A.5}$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

Writing again $\Pi_0\colon \mathbf{D}_0 \to \mathcal{L}_0$ for the product on $\mathcal{L}_0$, it is by (A.2) and (A.4) sufficient to prove that $\mathbf{D}_0 \subseteq \theta^*(\mathbf{D}\cap\mathbf{W}(\mathcal{N}))$. For the proof of this property, let $u = (g_1, g_2, \dots, g_n) \in \mathbf{D}_0$, and fix $\varphi_i \in g_i$ for $i = 1, 2, \dots, n$ such that the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$ is defined in $\mathcal{T}_0$. Note that such φ_i exist by definition of $\mathbf{D}_0$. Then $\varphi_i \in \mathrm{Iso}(\mathcal{T}_0)$, and hence $f_i := [\varphi_i] \in \mathcal{N}$ with $\theta(f_i) = g_i$. As the composition $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$ is defined in $\mathcal{T}_0$, it is also defined in $\mathcal{T}$. Therefore, $w := (f_1, \dots, f_n) \in \mathbf{D}\cap\mathbf{W}(\mathcal{N})$ with $\theta^*(w) = u$. This shows $\mathbf{D}_0 \subseteq \theta^*(\mathbf{D}\cap\mathbf{W}(\mathcal{N}))$, and thus (A.5).

Note that (b) holds by (A.3) and (A.5). Moreover, $S_0 \subseteq S\cap\mathcal{N}$. Also recall that $\mathcal{N}$ is a partial subgroup of $\mathcal{L}$ by (A.4) and that $(\mathcal{L}_0, S_0, \Gamma)$ is by [7, Proposition A.13] a locality. In particular, S_0 is a maximal p-subgroup of $\mathcal{L}_0$ and thus by (b) also a maximal p-subgroup of $\mathcal{N}$. This implies $S\cap\mathcal{N} = S_0$. □

Lemma A.15. *Let $P, Q, R, P', Q', R' \in \mathrm{Ob}(\mathcal{T}_0)$, $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$, $\psi \in \mathrm{Iso}_{\mathcal{T}}(P', Q')$, $\varphi_0 \in \mathrm{Iso}_{\mathcal{T}_0}(P, R)$, $\psi_0 \in \mathrm{Iso}_{\mathcal{T}_0}(P', R')$, $\alpha, \gamma \in \mathrm{Aut}_{\mathcal{T}}(S_0)$ such that $\alpha(R) = Q$, $\gamma(R') = Q'$,*

$$\varphi = \alpha|_{R,Q} \circ \varphi_0 \text{ and } \psi = \gamma|_{R',Q'} \circ \psi_0.$$

If $\varphi \equiv_{\mathcal{T}} \psi$, then $\gamma^{-1} \circ \alpha \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$.

Proof. By Remark A.5, there exists a series $\varphi = \beta_1, \beta_2, \dots, \beta_n = \psi \in \mathrm{Iso}(\mathcal{T})$ such that $\beta_i \uparrow \beta_{i+1}$ or $\beta_{i+1} \uparrow \beta_i$ for $i = 1, \dots, n-1$. By Lemma A.9, every β_i can be factored as a restriction of an automorphism $\alpha_i \in \mathrm{Aut}_{\mathcal{T}}(S_0)$ with an isomorphism in $\mathcal{T}_0$. We may assume that $\alpha_1 = \alpha$ and $\alpha_n = \gamma$. If $\alpha_{i+1}^{-1} \circ \alpha_i \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$ for $i = 1, \dots, n-1$, then

$$\gamma^{-1} \circ \alpha = \alpha_n^{-1} \circ \alpha_1 = (\alpha_n^{-1} \circ \alpha_{n-1}) \circ (\alpha_{n-1}^{-1} \circ \alpha_{n-2}) \circ \cdots \circ (\alpha_2^{-1} \circ \alpha_1) \in \mathrm{Aut}_{\mathcal{T}_0}(S_0).$$

Thus, we may reduce to the case $\varphi \uparrow_{\mathcal{T}} \psi$ or $\psi \uparrow_{\mathcal{T}} \varphi$. As $\gamma^{-1} \circ \alpha \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$ if and only if $\alpha^{-1} \circ \gamma = (\gamma^{-1} \circ \alpha)^{-1} \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$, we may indeed assume that

$$\varphi \uparrow_{\mathcal{T}} \psi.$$

This means $P \le P'$, $\psi(P) = Q \le Q'$ and $\varphi = \psi|_{P,Q}$. Setting $X := \psi_0(P)$, Lemma A.2(b) gives

$$\alpha|_{R,Q} \circ \varphi_0 = \varphi = \psi|_{P,Q} = \gamma|_{X,Q} \circ \psi_0|_{P,X}.$$

As φ_0 and ψ_0 are morphisms in $\mathcal{T}_0$, this implies, together with Lemma A.2(b), that

$$(\gamma^{-1} \circ \alpha)|_{R,X} = (\gamma|_{X,Q})^{-1} \circ \alpha|_{R,Q} = \psi_0|_{P,X} \circ \varphi_0^{-1} \in \mathrm{Iso}_{\mathcal{T}_0}(R, X).$$

Hence, Lemma A.10 gives $\gamma^{-1} \circ \alpha \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$. □

Lemma A.16. *We have $\mathcal{N} \trianglelefteq \mathcal{L}$.*

Proof. Recall that $\mathcal{N}$ is a partial subgroup of $\mathcal{L}$ by Lemma A.14(a). Let $f \in \mathcal{L}$ and $n \in \mathcal{N}$ such that $(f^{-1}, n, f) \in \mathbf{D}$. Then there exist

$$\varphi^{-1} \in f^{-1},\ \rho \in n,\ \psi \in f$$

such that the composition $\varphi^{-1} \circ \rho \circ \psi$ is defined. Replacing φ, ρ, ψ by $\varphi^0, \rho^0, \psi^0$, we may assume that for some $P, Q, P', Q' \in \mathrm{Ob}(\mathcal{T}_0)$, we have

$$\psi \in \mathrm{Iso}_{\mathcal{T}}(Q', P'),\ \rho \in \mathrm{Iso}_{\mathcal{T}_0}(P', P),\ \varphi^{-1} \in \mathrm{Iso}_{\mathcal{T}}(P, Q).$$

By Lemma A.9, there exist then $R, R' \in \mathrm{Ob}(\mathcal{T}_0)$, $\varphi_0 \in \mathrm{Mor}_{\mathcal{T}_0}(P, R)$, $\psi_0 \in \mathrm{Mor}_{\mathcal{T}_0}(P', R')$ and $\alpha, \gamma \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$ such that $\alpha(R) = Q$, $\gamma(R') = Q'$,

$$\varphi^{-1} = \alpha|_{R,Q} \circ \varphi_0 \text{ and } \psi^{-1} = \gamma|_{R',Q'} \circ \psi_0.$$

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press

As $f = [\psi]$, we have $[\psi^{-1}] = f^{-1} = [\varphi^{-1}]$, so $\psi^{-1} \equiv_{\mathcal{T}} \varphi^{-1}$. Thus, Lemma A.15 (applied with $(\psi^{-1}, \varphi^{-1})$ in place of (φ, ψ)) implies that $\gamma^{-1} \circ \alpha \in \mathrm{Aut}_{\mathcal{T}_0}(S_0)$. By definition of the product Π on $\mathcal{L}$, we have

$$n^f = \Pi(f^{-1}, n, f) = [\varphi^{-1} \circ \rho \circ \psi].$$

Using Lemma A.2(b), observe that

$$\begin{aligned}\varphi^{-1} \circ \rho \circ \psi &= \alpha|_{R,Q} \circ \varphi_0 \circ \rho \circ \psi_0^{-1} \circ (\gamma|_{R',Q'})^{-1} \\ &= \alpha|_{R,Q} \circ \varphi_0 \circ \rho \circ \psi_0^{-1} \circ (\gamma^{-1} \circ \alpha)|_{\alpha^{-1}(Q'),R'} \circ (\alpha|_{\alpha^{-1}(Q'),Q'})^{-1}.\end{aligned}$$

As $\varphi_0 \circ \rho \circ \psi_0^{-1} \circ (\gamma^{-1} \circ \alpha)|_{\alpha^{-1}(Q'),R'} \in \mathrm{Iso}(\mathcal{T}_0)$, it follows from axiom (iii) in Definition A.3 that $\varphi^{-1} \circ \rho \circ \psi \in \mathrm{Iso}(\mathcal{T}_0)$. Hence, $n^f = [\varphi^{-1} \circ \rho \circ \psi] \in \mathcal{N}$. This shows the assertion. □

Proof of Theorem A.7. By Lemmas A.14(a) and A.16, $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$ with $S \cap \mathcal{N} = S_0$. Using that $\mathrm{Ob}(\mathcal{T}_0) \subseteq \mathrm{Ob}(\mathcal{T})$ and axiom (i) in Definition A.3 holds, we can see that $\Gamma = \{P \cap S_0 \colon P \in \Delta\} \subseteq \Delta$. Hence, $(\mathcal{N}, \Gamma, S_0)$ is a kernel of $(\mathcal{L}, \Delta, S)$.

By Lemma A.14(b), there is an isomorphism $\theta \colon \mathcal{N} \to \mathcal{L}_0 := \mathcal{L}(\mathcal{T}_0)$ that (with appropriate identifications) restricts to the identity on S_0. In particular, $\mathcal{F}_{S_0}(\mathcal{N}) = \mathcal{F}_{S_0}(\mathcal{L}_0)$, and there exists an invertible functor $\mathcal{T}_\Gamma(\mathcal{N}) \to \mathcal{T}_\Gamma(\mathcal{L}_0)$ that is the identity on Γ. We have seen in our general discussion above that $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. So we have similarly that $\mathcal{F}_0 = \mathcal{F}_{S_0}(\mathcal{L}_0) = \mathcal{F}_S(\mathcal{N})$. It is moreover shown in [7, Lemma A.15] that there is an invertible functor $\mathcal{T}_\Gamma(\mathcal{L}_0) \to \mathcal{T}_0$ that restricts to the identity on Γ. This implies the assertion. □

Conflicts of Interest. None.

Financial Support. The authors were partially supported by EPSRC grant no. EP/R010048/1. Moreover, they would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Groups, representations and applications: new perspectives*, where some of the work on this paper was undertaken and supported by EPSRC grant no EP/R014604/1. Open Access Funding by the Publication Fund of the TU Dresden.

References

[1] K. K. S. Andersen, B. Oliver and J. Ventura, Reduced, tame and exotic fusion systems. *Proc. Lond. Math. Soc. (3)*, **105**(2012), no. 1, 87–152. http://doi.org/10.1112/plms/pdr065
[2] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*. London Mathematical Society Lecture Note Series, **391**, Cambridge University Press, Cambridge, 2011.
[3] M. Aschbacher, The generalized Fitting subsystem of a fusion system. *Mem. Amer. Math. Soc.*, **209**(2011), no. 986, vi+110.
[4] M. Aschbacher, On fusion systems of component type. *Mem. Amer. Math. Soc.*, **257**(2018), no. 1236, vi+182.
[5] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver, Subgroup families controlling p-local finite groups. *Proc. London Math. Soc. (3)* **91**(2005), no. 2, 325–354. http://doi.org/10.1112/S0024611505015327
[6] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver, Extensions of p-local finite groups. *Trans. Amer. Math. Soc.* **359**(2007), no. 8, 3791–3858. http://doi.org/10.1090/S0002-9947-07-04225-0
[7] A. Chermak, Fusion systems and localities. *Acta Math.* **211**(2013), no. 1, 47-139. http://doi.org/10.1007/s11511-013-0099-5
[8] A. Chermak, Finite Localities I. *Forum Math. Sigma* **10**(2022), Paper No. e43, 31. http://doi.org/10.1017/fms.2022.31
[9] A. Chermak, *Finite localities III*. 2016. arxiv:1610.06161v2
[10] A. Chermak and E. Henke, *Fusion systems and localities – a dictionary*. *Advances in Mathematics* (in press) arxiv:1706.05343v2
[11] G. Glauberman and J. Lynd, Control of fixed points and existence and uniqueness of centric linking systems. *Invent. Math.* **206**(2016), no. 2, 441–484. http://doi.org/10.1007/s00222-016-0657-5
[12] A. Gonzalez, *An extension theory for partial groups and localities*. 2015. arxiv:1507.04392v2
[13] E. Henke, Products in Fusion systems. *J. Algebra* **376**(2013), 300–319. http://doi.org/10.1016/j.jalgebra.2012.11.037
[14] E. Henke, Products of partial normal subgroups. *Pacific J. Math.* **279**(2015), no. 1-2, 255–268. http://doi.org/10.2140/pjm.2015.279.255
[15] E. Henke, Centralizers of normal subsystems revisited. *J. Algebra* **511**(2018), 364–387. http://doi.org/10.1016/j.jalgebra.2018.05.034
[16] E. Henke, Subcentric linking systems. *Trans. Amer. Math. Soc.* **371**(2019), no. 5, 3325–3373. http://doi.org/10.1090/tran/7388

[17] E. Henke, Extensions of homomorphisms between localities. *Forum Math. Sigma* **9**(2021), Paper No. e63, 31. http://doi.org/10.1017/fms.2021.57

[18] E. Henke, *Commuting partial normal subgroups and regular localities*. 2021. arxiv:2103.00955v3

[19] E. Henke, *Normalizers and centralizers of subnormal subsystems of fusion systems*. 2021. arxiv:2109.14038

[20] E. Henke, Some products in fusion systems and localities. Preprint 2021, accepted to *Proc. Amer. Math. Soc.*, arxiv:2107.00301v2

[21] H. Kurzweil and B. Stellmacher, *The theory of finite groups – an introduction*. Universitext. Springer-Verlag, New York, 2004, xii+387. http://doi.org/10.1007/b97433

[22] U. Meierfrankenfeld and B. Stellmacher, Applications of the FF-module theorem and related results. *J. Algebra* **351**(2012), 64–106. http://doi.org/10.1016/j.jalgebra.2011.10.028

[23] B. Oliver, Extensions of linking systems and fusion systems. *Trans. Amer. Math. Soc.* **362**(2010), no. 10, 5483–5500. http://doi.org/10.1090/S0002-9947-2010-05022-6

[24] B. Oliver, Existence and uniqueness of linking systems: Chermak's proof via obstruction theory. *Acta Math.* **211**(2013), no. 1, 141–175. http://doi.org/10.1007/s11511-013-0100-3

[25] B. Oliver and J. Ventura, Extensions of linking systems with p-group kernel. *Math. Ann.* **338**(2007), no. 4, 983–1043. http://doi.org/10.1007/s00208-007-0104-4

[26] K. Roberts and S. Shpectorov, On the definition of saturated fusion systems. *J. Group Theory* **12**(2009), no. 5, 679–687. http://doi.org/10.1515/JGT.2009.003

https://doi.org/10.1017/fms.2022.59 Published online by Cambridge University Press