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Correction

A note on Insider information and its relation with the arbitrage condition and the utility maximization problem

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Abstract: We prove that Theorem 4.16 in [1] is false by constructing a strategy that generates $(FLVR)_{\mathcal{H}(\mathbb{G})}$. However, we success to prove that the no arbitrage property still holds when the agent only plays with strategies belonging to the admissible set called *buy-and-hold*.

Keywords: optimal portfolio; enlargement of filtration; arbitrage; no free lunch vanishing risk

In this note we show that the result of Theorem 4.16 of [1] is false by constructing a sequence of simple predictable strategies achieving *Free-Lunch-with-Vanishing-Risk* ($FLVR$) whose existence contradicts the conclusions of the theorem. The fault in the proof in [1] comes from the improper use of a bound on the compensator α^G. Indeed the bound holds only $\boldsymbol{P}$-almost surely, that is not strong enough to assure the required Novikov condition.

Using the notation introduced in [1], we consider the initial enlargement $\mathbb{G} \supset \mathbb{F}$ obtained by extending the natural filtration by the random variable

$$G = \mathbb{1}\left\{B_T \in \cup_{k=-\infty}^{+\infty}[2k-1, 2k]\right\} . \tag{1}$$

Assuming a constant proportional volatility $\xi > 0$, it follows that

$$S_T = \tilde{s}_0 \exp(\xi B_T) \ , \quad \tilde{s}_0 := s_0 \exp\left(\int_0^T \left(\eta_t - \xi^2/2\right) dt\right) ,$$

and the random variable G can be rewritten as $G = \mathbb{1}\left\{S_T \in \cup_{k=-\infty}^{+\infty}[c_{2k-1}, c_{2k}]\right\}$, where $c_k := \tilde{s}_0 e^{\xi k}$. The length of each interval is $\lambda_k := c_{2k} - c_{2k-1} = \tilde{s}_0 e^{\xi 2k}\left(1 - e^{-\xi}\right) > 0$. To simplify the computations, we assume that the interest rate $r = 0$.

Proposition 1. *Let G be as in* (1)*, the condition* $(\mathrm{FLVR})_{\mathcal{H}(\mathbb{G})}$ *is satisfied.*

Proof. Without loss of generality, we assume that, for some $t_0 < T$ there exists $k_0 \in \mathbb{Z}$ such that $c_{2k_0} - \lambda_{k_0}/4 \geq S_{t_0} \geq c_{2k_0-1} + \lambda_{k_0}/4$. We reason for the case $G = 0$, that in particular implies that $S_T \notin (c_{2k_0-1}, c_{2k_0})$, the case $G = 1$ is equivalent by symmetry. We define the following finite sets

$$A_n^\delta := \left\{c_{2k_0} - \frac{\lambda_{k_0}}{4} - \frac{\delta(k_0)}{2^n},\ c_{2k_0-1} + \frac{\lambda_{k_0}}{4} + \frac{\delta(k_0)}{2^n}\right\}, \quad n \geq 0\,,$$

together with the following sequence of stopping times, $\tau_0 = t_0$ and for $n \geq 1$

$$\begin{aligned}\tau_{2n-1} &:= \inf\{\tau_{2n-2} \leq t < T : S_{\tau_{2n-2}} \notin \{c_{2k_0},\ c_{2k_0-1}\},\ S_t \in A_\infty^\delta\}\,,\\ \tau_{2n} &:= \inf\{\tau_{2n-1} \leq t < T : S_t \in \{c_{2k_0},\ c_{2k_0-1}\} \cup A_n^\delta\}\,.\end{aligned}$$

where we define $\inf \emptyset = T$. With some abuse of notation, we construct a sequence of strategies $\{\Theta_n\}_n$ with $\Theta_0 = 0$ and $\Theta_n := \Theta_{n-1} + C_n \mathbb{1}_{]]\tau_{2n-1},\tau_{2n}]]}$ for $n \geq 1$, being C_n the following $\mathcal{F}_{\tau_{2n-1}}$-measurable random variable

$$C_n := \begin{cases} +1 & \text{if } S_{\tau_{2n-1}} = c_{2k_0} - \lambda_{k_0}/4 \\ -1 & \text{if } S_{\tau_{2n-1}} = c_{2k_0-1} + \lambda_{k_0}/4\,. \end{cases}$$

We prove that the sequence of strategies $\{\Theta_n\}_n$ achieves a gain greater than $\frac{\lambda_{k_0}}{4} - \delta(k_0)$, and by appropriately choosing $\delta(k_0)$ we can get $(\text{FLVR})_{\mathcal{H}(\mathbb{G})}$. To short the notation, we introduce the family $H_m := \mathbb{1}\{S_{\tau_{2n}} \notin \{c_{2k_0}, c_{2k_0-1}\},\ \forall n < m\}$

$$\begin{aligned}X_T^{\Theta_m} &= X_0 + \sum_{n=1}^{m} H_n C_n \left(S_{\tau_{2n}} - S_{\tau_{2n-1}}\right) = X_0 - \sum_{n=1}^{m} H_n \frac{\delta(k_0)}{2^n} + \frac{\lambda_{k_0}}{4}(1 - H_m)\\ &\geq X_0 - \delta(k_0)\left(1 - \frac{1}{2^m}\right) + \frac{\lambda_{k_0}}{4}(1 - H_m) \geq X_0 - \delta(k_0) + \frac{\lambda_{k_0}}{4}(1 - H_m)\,.\end{aligned}$$

We need to verify that $\lim_{m\to\infty} H_m = 0$, $\boldsymbol{P}(\cdot\,|\,G = 0)$-a.s. By definition of convergence a.s., it is equivalent to

$$\lim_{m\to\infty} \boldsymbol{P}(H_m < \varepsilon\,|\,G = 0) = 1\,, \quad \forall \varepsilon > 0\,.$$

The sequence of indicator functions is strictly decreasing by construction, so we need to check that

$$\begin{aligned}1 &= \lim_{m\to\infty} \boldsymbol{P}(H_m = 0\,|\,G = 0) = \lim_{m\to\infty} \boldsymbol{P}(S_{\tau_{2n}} \in \{c_{2k_0}, c_{2k_0-1}\} \text{ for some } n < m\,|\,G = 0)\\ &= \lim_{m\to\infty} \boldsymbol{P}(S_{\tau_{2n}} \in \{c_{2k_0}, c_{2k_0-1}\} \text{ for some } n < m\,|\,S_T \notin (c_{2k_0-1}, c_{2k_0}))\,,\end{aligned}$$

where the last condition is satisfied. □

Remark. By using an analogous technique, it can be proved that any random variable $G = \mathbb{1}_{\{B_T \in B\}}$ generates *(FLVR)* when B is a subset of positive probability less than one.

Since the result of Theorem 4.16 in [1] is false, we prove here a weaker result by showing that the strategies of type *buy-and-hold* do not generate arbitrage, *(NA)*, as it is shown in the following proposition.

Proposition 2. *Let G be as in* (1)*, the condition* $(\text{NA})_{\mathcal{H}(\mathbb{G})}$ *is satisfied with strategies of the type* $\Theta = C\mathbb{1}_{]]\sigma,T]]}$*, being σ any $\mathbb{G}$-stopping time and C a $\mathcal{G}_\sigma$-measurable random variable not identically zero.*

Proof. We claim that there exists some $\Theta = C\mathbb{1}_{]]\sigma,T]]}$ achieving arbitrage and we look for a contradiction. We start by computing the following conditional probabilities

$$\begin{aligned} \boldsymbol{P}(S_T < S_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T) &= \boldsymbol{P}(B_T < B_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T) > 0\,, \\ \boldsymbol{P}(S_T > S_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T) &= \boldsymbol{P}(B_T > B_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T) > 0\,. \end{aligned} \tag{2}$$

We introduce the event $A := \{C(S_T - S_\sigma) < 0\}$, by the definition of arbitrage we have $\boldsymbol{P}(A) = 0$ and jointly with the law of total probability we find the following contradiction,

$$\begin{aligned} 0 = \boldsymbol{P}(A) &= \boldsymbol{P}(C = 0)\boldsymbol{P}(A \,|\, C = 0) + \boldsymbol{P}(C < 0)\boldsymbol{P}(A \,|\, C < 0) + \boldsymbol{P}(C > 0)\boldsymbol{P}(A \,|\, C > 0) \\ &= \boldsymbol{P}(C < 0)\boldsymbol{P}(S_T - S_\sigma > 0) + \boldsymbol{P}(C > 0)\boldsymbol{P}(S_T - S_\sigma < 0) \\ &= \boldsymbol{P}(C < 0)\boldsymbol{E}[\boldsymbol{P}(S_T > S_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T)] + \boldsymbol{P}(C > 0)\boldsymbol{E}[\boldsymbol{P}(S_T < S_\sigma \,|\, \mathcal{G}_\sigma, \sigma < T)] > 0\,, \end{aligned}$$

which is positive because $\boldsymbol{P}(C \neq 0) > 0$ and the conditional probabilities given by (2). □

Conflict of interest

The authors declare there is no conflict of interest.

References

1. B. D'Auria, J. A. Salmerón, Insider information and its relation with the arbitrage condition and the utility maximization problem, *Math. Biosci. Eng.*, **17** (2020), 998–1019. 10.3934/mbe.2020053