Journal of
Risk and Financial Management

MDPI

Article

How Fast Does the Clock of Finance Run?—A Time-Definition Enforcing Stationarity and Quantifying Overnight Duration

Michele Caraglio [1], Fulvio Baldovin [2] and Attilio L. Stella [2,*]

[1] Institut für Theoretische Physik, Universität Innsbruck, Technikerstraße 21A, A-6020 Innsbruck, Austria; michele.caraglio@uibk.ac.at
[2] Dipartimento di Fisica e Astronomia and Sezione INFN, Università di Padova, Via Marzolo 8, I-35131 Padova, Italy; fulvio.baldovin@pd.infn.it
* Correspondence: stella@pd.infn.it

Abstract: A definition of time based on the assumption of scale invariance may enhance and simplify the analysis of historical series with cyclically recurrent patterns and seasonalities. By enforcing simple-scaling and stationarity of the distributions of returns, we identify a successful protocol of time definition in finance, functional from tens of minutes to a few days. Within this time definition, the significant reduction of cyclostationary effects allows analyzing the structure of the stochastic process underlying the series on the basis of statistical sampling sliding along the whole time series. At the same time, the duration of periods in which markets remain inactive is properly quantified by the novel clock, and the corresponding returns (e.g., overnight or weekend) can be consistently taken into account for financial applications. The method is applied to the S&P500 index recorded at a 1 min frequency between September 1985 and June 2013.

Keywords: nonstationary time series; cyclostationary; scale invariance; financial time

Citation: Caraglio, Michele, Fulvio Baldovin, and Attilio L. Stella. 2021. How Fast Does the Clock of Finance Run?—A Time-Definition Enforcing Stationarity and Quantifying Overnight Duration. *Journal of Risk and Financial Management* 14: 384. https://doi.org/10.3390/jrfm14080384

Academic Editor: Tihana Skrinjaric

Received: 9 July 2021
Accepted: 12 August 2021
Published: 17 August 2021

1. Introduction

Nonstationary time series are very frequently encountered in fields such as meteorology Ivanova and Ausloos (1999), seismology Corral (2004, 2006), physiology Ashkenazy et al. (2001); Hausdorff et al. (1995); Peng et al. (1995), and finance Bassler et al. (2007); Galluccio et al. (1997); Liu et al. (1997). The nonstationary mark limits the applicability of statistical methods in the quest for the process generating the series Tsay (2002) and may convey spurious effects in the detection of correlations Kantelhardt et al. (2002); Peng et al. (1994). Depending on the suspected cause of nonstationarity, different strategies may be applied to suppress it and to simplify the signals. For instance, various detrending procedures have been proposed Kantelhardt et al. (2002); Peng et al. (1994), although their success is not always guaranteed Bryce and Sprague (2012); Chen et al. (2002). Other approaches Dacorogna et al. (1996, 2001); Corral (2004); Zumbach (1997) are instead based on a reassessment of the time stamp in terms of which series is recorded. In particular, in the case of seismology, it has been advanced Corral (2004, 2006) in that the distribution of inter-occurrence times between earthquakes becomes invariant under rescaling of a properly redefined time: this redefinition renders inter-occurrences stationary on the basis of Omori's aftershock law Omori (1894).

Finance is a field in which the very definition of time constitutes a long-standing problem, since a clock properly adjusted to financial activity is not easy to identify. Financial markets have to cope with daily human routines throughout the world and bursts and doldrums occur at various time scales. In addition, during nights, weekends, and festivities, transactions stop in most cases. These interruptions, at the end of which asset prices turn out to have changed anyhow, together with the recurrent patterns in the activity (seasonalities), immediately signal the inadequacy of "natural" physical time as the appropriate one in terms of which to describe the stochastic evolution of financial markets. Thus, various alternatives have been studied Clark (1973); Dacorogna et al. (2001); Ané and Geman (1999); Gillemot et al. (2006); Jensen et al. (2004); Mantegna and Stanley (2000), including trading time, volume time, and tick-by-tick time. A time definition (theta-time) has also been put forward with the specific intent of removing the seasonalities in financial time series Dacorogna et al. (1996, 2001); Zumbach (1997). Theta-time is designed to record the progress of market activity through the increase of the volatility as measured by the average absolute return. For the latter, a power-law behavior is assumed as a function of physical time. However, theta-time suffers some limitations. In particular, as in most theoretical studies, inactive market periods are cut from the analyzed data set, and the corresponding returns ignored Dacorogna et al. (2001), with the consequence that the correspondence between the sum of returns and the real asset price is destroyed. On the other hand, the alternative of assigning an arbitrary time interval to periods of market inactivity would preclude the possibility of establishing a meaningful link between returns and financial time. In addition, a discussion of the effectiveness of theta-time in enforcing the stationarity of the returns probability density function (PDF) has yet to be addressed.

In summary, the appropriate duration to be ascribed to overnight and similar returns remains an open issue, and in general, there is little focus on the requirement that increments over intervals of equal span should be identically distributed in order to make standard empirical sampling (e.g., on intervals sliding along the historical series of an asset) applicable. This, for instance, is a difficulty in studies of return predictability Potí (2018), which rely on regression analysis. More generally, one can say that establishing a consistent time scale for the financial series is a requisite for studies aimed at assessing the efficient market hypothesis or to model leverage effects in portfolio risk evaluation.

Within the context of time series affected by cyclic nonstationary behavior in finance (see, e.g., Gardner et al. (2006)), we show here that one can promote the approximate fulfillment of a statistical scaling symmetry as an operative criterion for the definition of a novel time scale. Our basic results are that: (i) the returns distribution becomes stationary if recorded over the novel time scale; (ii) we identify a proper time duration to be attributed to the overnight returns; (iii) we are able to properly include overnight returns in the time series sequence. In terms of this time scale, empirical sampling on sliding time intervals bridging intraday and interday durations become possible.

2. Scaling and Hurst Exponent

Simple scaling is an approximate scale invariance revealed when cumulative distribution functions of returns of certain financial indexes over different time intervals are compared. Within specific ranges, this symmetry is reasonably well satisfied by the distributions of logarithmic interday returns of various indexes and assets. If we indicate the return r over the time span Δt (here, Δt is an integer number of days) as the log-difference of the asset price value s, $r \equiv \ln s(t) - \ln s(t - \Delta t)$, this means that the empirical cumulative distribution functions (CDFs) of r over different Δt's can be approximately collapsed onto each other in force of the scaling law

$$P(r, \Delta t) \simeq G\left(\frac{r}{\Delta t^H}\right), \tag{1}$$

where H is called the Hurst exponent in view of his seminal work on the Nile floods Hurst (1951), and G, whose corresponding PDF is not Gaussian, is a scaling function. The estimation of H for various markets has been addressed, e.g., in Reference Di Matteo et al. (2005); Galluccio et al. (1997). The Hurst exponent is also known to be very close to $1/2$ for indexes of mature markets, a value which we will assume in our analysis. Collapse means that the plots of the CDF's of returns r_1 and r_2 for intervals Δt_1 and Δt_2, respectively, coincide if, e.g., the first is plotted as a function of r_1 and the latter as a function of $r_2(\Delta t_2/\Delta t_1)^H$. The deceptive simplicity of Equation (1) already suggests the strategy we are going to follow: Assuming the value $H = 1/2$, we will argue the appropriate financial duration of a given interval as that which best allows extending the validity of Equation (1).

This procedure will be followed to quantify especially the financial duration of periods of market inactivity or of intraday intervals, which are affected by cyclic nonstationarity.

Equation (1) is approximately valid with $H = 1/2$ and the G function characterizing the market, for intervals of integer numbers of days in physical time. It thus establishes a simple bi-univocal correspondence between return CDF's and interval durations. Stationarity of the CDF's for such intervals qualifies it as a proper clock for financial activity. Indeed, the collapse protocol outlined above allows one to argue the ratios of durations for such intervals. However, Equation (1) definitely does not hold at the intraday time scales if intervals are measured in physical time. In the intraday, nonstationarities affect the CDFs for returns defined over intervals with equal physical time duration Allez and Bouchaud (2011); Bassler et al. (2007); Galluccio et al. (1997), and H can be very different from $1/2$ in selected time windows Baldovin et al. (2015); Bassler et al. (2007). Due to nonstationarity, these intraday determinations of H are based on sampling the return CDF from corresponding intervals selected from an ensemble of daily histories and not by letting the intervals slide along the time series. Furthermore, even in the interday domain, multiscaling effects Borland et al. (2005); Calvet and Fisher (2002); Di Matteo (2007) and a slow crossover to Gaussianity of the return PDF Cont (2001) prevent from a fully satisfactory realization of the collapses implied by Equation (1), especially for Δt exceeding a few days. In spite of these limitations, we show below that the outlined program can be carried out to a satisfactory approximation on time scales spanning from a few minutes to a few days, thus allowing the definition of a financial scaling time (FST) within this range.

Some aspects of our method have interesting precursors in the history of financial markets analysis. Indeed, stimulated by the discovery of fat tails Cont (2001); Mandelbrot and Taylor (1967); Mantegna and Stanley (1995), previous studies Ané and Geman (2000); Clark (1973); Mandelbrot (1973) attempted to modify the returns PDFs through the introduction of a stochastic redefinition of time, in order to cast them into Gaussians. Here, no reference to probabilistic models is made, and our FST is not stochastic. However, similar to these earlier studies, our time redefinition is intended to alter some fundamental property of the returns PDFs: in our case, it enforces a simple scaling symmetry.

Our construction does not amount to the proposal of a specific parameter-dependent model of market dynamics. However, one should mention that the simple scaling expressed by Equation (1) has been recently assumed as a main modeling ingredient of the stochastic dynamics of financial indexes Baldovin and Stella (2007); Baldovin et al. (2015); Peirano and Challet (2012); Stella and Baldovin (2010); Zamparo et al. (2013). Combining scaling with ideas of fine graining inspired by the renormalization group approach of statistical mechanics Kadanoff (2005), these models reproduce many stylized facts exposed by interday time series analysis, including some multiscaling effects.

3. FST Definition and Construction

Our basic ansatz is that, once expressed in terms of FST, Equation (1) turns into a scaling law with $H = 1/2$:

$$P(r, \Delta\tau) = G\left(\frac{r}{\Delta\tau^{1/2}}\right), \tag{2}$$

where $\Delta\tau > 0$ represents the FST duration of an interval, and G is a suitable scaling function. For simplicity, we denote by the same symbols used in Equation (1) the CDF of returns and the scaling function expressed as functions of $\Delta\tau$. As it will be shown below, we are able to extend the validity of this scaling law in both intra-and interday regimes.

The CDF in Equation (2) may be used to express expectation values, which are empirically based on sampling determinations. Concerning the sampling, we will operate differently for interday and intraday intervals. For $\Delta\tau$'s corresponding to an integer number of days in physical time, we will sample from intervals of equal duration sliding along the historical series. In the intraday regime, the expectation values will be initially based on sampling over corresponding intervals in an ensemble of daily histories. Indeed, in spite of the manifest nonstationarities at the intraday level Allez and Bouchaud (2011); Bassler et al. (2007); Dacorogna et al. (2001); Galluccio et al. (1997), the process of return aggregation appears compatible with the assumption of one-day ciclostationarity Gardner et al. (2006). In other words, the evolution of the aggregate return from the opening during each market day can be regarded as a realization of the same stochastic process Baldovin et al. (2015); Bassler et al. (2007); Dacorogna et al. (2001), and empirical averages can be taken over correspondent time windows within different days.

Before describing our method, some remarks concerning the implications of Equation (2) and the choice of intervals are necessary: Treating r as a continuous variable, the q-th moment of the absolute return in an interval of duration $\Delta\tau$ is clearly given by

$$\mathbb{E}[|r|^q] = \int_{-\infty}^{\infty} dr \frac{g(r/\Delta\tau^{1/2})}{\Delta\tau^{1/2}} \, |r|^q = A_q \, \Delta\tau^{q/2} \, , \tag{3}$$

where A_q is an amplitude depending on q and g, the derivative of G or the PDF of rescaled returns.[1] In particular, for $q = 2$, we obtain $\mathbb{E}[|r|^2] = A_2\Delta\tau$. If we consider an interval of duration $\Delta\tau = \Delta\tau_1 + \Delta\tau_2$, the aggregated return $r = r_1 + r_2$ must satisfy

$$\Delta\tau = \frac{\mathbb{E}[|r_1 + r_2|^2]}{A_2} = \frac{\mathbb{E}[|r_1|^2] + \mathbb{E}[|r_2|^2] + 2\mathbb{E}[r_1 r_2]}{A_2} \, , \tag{4}$$

which implies

$$\mathbb{E}[r_1 r_2] = 0 \, , \tag{5}$$

since $\mathbb{E}[|r_i|^2] = A_2\Delta\tau_i$. Thus, the validity of Equation (2) and time additivity impose a linear uncorrelation between returns in successive intervals of time. This linear uncorrelation, which does not imply independence Cont (2001), corresponds to the martingale property for the stochastic process defining the price formation Bouchaud and Potters (2003) and is generally valid if the market is efficient and the interval duration is above a few minutes Bouchaud and Potters (2003); Cont (2001). Therefore, in the choice of intervals for which we will define FST by means of collapses, we will not consider durations so short to imply sensible violations of the condition in Equation (5). Another obvious limitation when considering markets with interruptions of activity is, for example, that the interval between closure and opening in the following day (overnight) can not be split into shorter intervals due to the absence of intermediate price determinations.

To best fit the collapses implied by Equation (2), we apply the Kolmogorov–Smirnov (KS) two-sample test DeGroot (2010). Assuming the CDF of the daily return as that corresponding to a unit of FST ($\Delta\tau = 1$ day $= 1$ fst), for any other chosen interval, the appropriate $\Delta\tau$ will be determined as the value realizing the optimal collapse of its CDF with the reference unit one as implied by (2). Indeed, the KS test allows to identify this value of $\Delta\tau$ as that which minimizes the distance between the reference CDF and the one under consideration plotted as a function of $r\Delta\tau^{1/2}$. This distance will be discussed in the next section.

The FST construction proceeds as follows: on the physical time axis, we first operate a partition into intervals within the intraday consistent with the one-day periodicity. To do so, the time between opening and closure in each trading day is split into a certain number of basic intervals with an identical physical time duration $\overline{\Delta t}$ long enough to guarantee approximate satisfaction of Equation (5) for returns in contiguous intervals. In such a way, the whole daily time span open-to-open can be viewed as the union of these consecutive basic intervals plus the last one coinciding with the full overnight (or weekend) closure. Other time intervals, either within or exceeding the duration of one day, are similarly constructed as the appropriate union of consecutive intraday basic intervals and overnights. To associate an adequate FST-duration to each interval with extremes belonging to the above partition, we proceed by application of the KS test to each elementary interval chosen in the intraday and to the overnight interval. The values of $\Delta\tau$ assigned to such intervals allow mapping the partition of the physical time axis onto the FST τ axis.

The above partition of the physical time axis is conveniently specified in terms of a set of time marks $\{t_{l,m}\}$, where $l = 0, 1, \ldots$ is trading day and $m = 0, 1, \ldots, m_{\max}$ is a multiple of $\overline{\Delta t}$ within the open market phase of each trading day. As shown in the next section, the application of the KS test identifies the duration in FST of the various $\overline{\Delta t}$ intervals occurring during the day, $\Delta\tau_m$, and overnight, $\Delta\tau_{\text{night}}$. Note that the CDF of returns in the intraday and overnight are sampled from the ensemble of daily histories. Specifically, the FST corresponding to this partition of the physical time axis is then defined in such a way that intervals on the τ axis have duration

$$\tau_{l',m'} - \tau_{l,m} \equiv \begin{cases} \sum_{n=m+1}^{m'} \Delta\tau_n & \text{if} \quad l' = l \\ \sum_{n=m+1}^{M} \Delta\tau_n + \Delta\tau_{\text{night}} + \sum_{n=1}^{m'} \Delta\tau_n & \text{if} \quad l' = l+1 \\ \sum_{n=m+1}^{M} \Delta\tau_n + \Delta\tau_{\text{night}} + (l'-l-1)\left(\sum_{n=1}^{M} \Delta\tau_n + \Delta\tau_{\text{night}}\right) + \sum_{n=1}^{m'} \Delta\tau_n & \text{if} \quad l' > l+1 \end{cases}, \quad (6)$$

where $\{\tau_{l,m}\}$ is the FST partition corresponding to $\{t_{l,m}\}$, and $\{t_{l',m'}\} > \{t_{l,m}\}$. Notice that in Equation (6), consistently with the following analysis, the FST duration of over-weekends and other periods of market closure is assumed to be equal to that of overnights.

The above partition of the τ axis is thus obtained by imposing an optimal scaling collapse between the CDF of each individual interval chosen on the t axis and that of the reference open-to-open interval. Such collapses are not full, since scaling is not obeyed exactly. At the same time, also in view of the fact that returns on contiguous intervals are not strictly uncorrelated, one can not expect additivity of τ durations to be strictly satisfied as if Equation (2) would be exactly obeyed. Therefore, for example, we can anticipate that adding the two $\Delta\tau$'s corresponding to a full daily opening and to overnight will not give exactly 1 as a result. Then, for a consistency assessment, the KS test is also applied to other intraday intervals belonging to the partition, as well as to multi-day intervals, i.e., intervals of two, three, and more days.[2] In the latter case, the τ scale differs much less from the t scale due to the fact that scaling is already reasonably well satisfied in physical time.

4. Methods and Results

Here, we describe in detail the implementation of the above ideas to the S&P500 index, recorded at 1-min frequency between 9:40 and 16:00 New York time from September 1985 to June 2013.[3] After excluding those days for which the records are not complete (e.g., holidays and market anticipated closures or delayed openings), the data set includes $L = 6852$ trading days. For each single day l ($l = 1, \ldots, L$), there is a total of 381 index values $s_{l,n}$ ($n = 0, \ldots, N$, with $N = 380$). Thus, a first partition of the physical-time axis is conveniently identified as $\{t_{l,n}\}_{l=0,1,\ldots;\, n=0,1,\ldots,N}$, and in terms of this partition, returns are defined as $r_{l,n}^{l',n'} \equiv \ln s_{l',n'} - \ln s_{l,n}$, with $t_{l',n'} > t_{l,n}$. In order to work with zero empirical averages, returns are detrended: $r_{l,n}^{l',n'} \mapsto r_{l,n}^{l',n'} - \langle r_{l,n}^{l',n'} \rangle$, where the average is calculated over all possible returns with the same n, n' and same $\Delta l = l' - l$, with $t_{l',n'} > t_{l,n}$. In this notation, overnight returns are indicated as $r_{l,N}^{l+1,0}$ and they include single night returns, returns over weekends, returns over holidays and other market closures. Closures due to specific market reasons should, in principle, be treated differently from weekends and normal holidays (e.g., several days following the black Monday 1987 are missing in the dataset because of anticipated market closure). Because we verified that they introduce only minor effects, we do not make such distinctions in our analysis.

In the identification of the FST duration of the various intervals, a key step consists in checking whether two random variables X and Y are identically distributed. A basic tool to establish this is the KS two-sample test DeGroot (2010). Given two samples $\{x_i\}_{i=1}^{n_x}$, $\{y_i\}_{i=1}^{n_y}$, with empirical cumulative distribution functions F_x, F_y, respectively, the test is based on the rescaled supremum of the distributions difference:

$$D_{x,y} := \left(\frac{n_x\, n_y}{n_x + n_y}\right)^{1/2} \sup_{z\in\mathbb{R}} \{F_x(z) - F_y(z)\}\,. \quad (7)$$

In cases in which the samples refer to independent variables, the test can rely on the KS theorem to assign a significance level to any $D_{x,y}$, under the null hypothesis that $\{x_i\}$, $\{y_i\}$ come from the same distribution. If, however, samples are extracted from the same time series and data display long-range dependence, such as in the case examined in the present study, the mapping between $D_{x,y}$ and the associated significance level is expected to be strongly affected Chicheportiche and Bouchaud (2011), and only model-dependent statements can be produced. Still, $D_{x,y}$ quantifies how close the distributions F_x and F_y are.

In Table 1, we report $D_{x,y}$ calculated for different Y samples relative to various intervals in physical time, once the one-day open-to-open return distribution is taken as the reference X sample. The table also displays the values of the FST duration, $\Delta\tau_y$, and of $D^{\mathrm{FST}}_{x,y}$ for the chosen Y-samples obtained as follows. Within a variational strategy, we rescale each Y-sample as $Y\Delta\tau^{1/2}$ according to Equation (2), and obtain the FST duration of the interval as the value $\Delta\tau_y$ of $\Delta\tau$ for which $D_{x,y}$ reaches its minimal value $D^{\mathrm{FST}}_{x,y}$. We notice that the $D(x,y)$ values are all rather large in comparison to the corresponding $D^{\mathrm{FST}}_{x,y}$. While entries for 10, 20, 38 min in Table 1 refer to averages over the whole day of intervals of the respective duration, as explained below, Figure 1 details the values of $\Delta\tau_y$ for all individual 20-min intervals from day opening to closure.

Table 1. Outcome of KS test and of the minimization procedure when the one-day open-to-open return distribution is taken as x-sample. For $y = 10,\ 20,\ 38$ min, the last three columns, in fact, correspond to the average of $D_{x,y}$, $\Delta\tau_y$ and $D^{\mathrm{FST}}_{x,y}$ over the (380 min$/y$) contiguous time intervals existing within a trading day.

y	$D_{x,y}$	$\Delta\tau_y$ (fst)	$D^{\mathrm{FST}}_{x,y}$
10-min	21.8	0.011	1.56
20-min	19.4	0.028	1.13
38-min	16.8	0.055	0.90
morning	6.7	0.337	0.79
afternoon	8.6	0.260	0.99
trading day	3.1	0.661	0.60
overnights	9.7	0.216	0.86
2 days	4.7	1.965	0.57
3 days	7.6	3.251	0.88
5 days	10.8	5.260	1.15
10 days	14.9	11.19	1.80

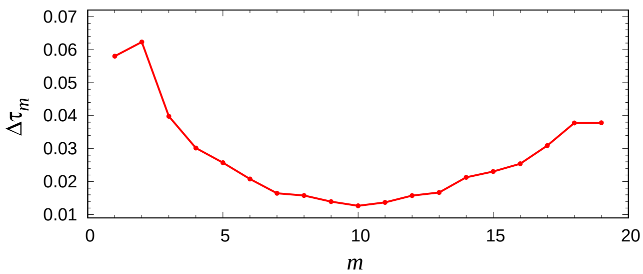

Figure 1. Values of 20-min $\Delta\tau$, $\Delta\tau_m$ ($m = 1, \ldots, 19$), from day opening to closure.

Indeed, the KS two-sample test enables us to determine the duration in FST of every physical time interval occurring during a day. For instance, if we split the trading day in 19 contiguous intervals with a physical time duration of 20-min and we call $\Delta\tau_m$ ($m = 1, \ldots, 19$) their duration in terms of FST, the first 20-min interval of the day corresponds to $\Delta\tau_1 \simeq 0.06$ fst, whereas twenty minutes at half and at the end of the day amount to $\Delta\tau_{11} \simeq 0.01$ fst and $\Delta\tau_{19} \simeq 0.04$ fst, respectively. We can also establish the duration of an overnight interval in the FST: $\Delta\tau_{\mathrm{night}} \simeq 0.21$ fst. As a side note, we observe that, in terms of FST, overnights are much shorter than trading days. Notice also that, assuming Equation (2), returns over different intervals with the same duration in terms of FST are by construction identically distributed.

A qualitative inspection of how the FST definition emphasizes the simple scaling properties of the empirical CDFs is offered in Figure 2. The comparison of Figure 2a with Figure 2b makes it evident that only in terms of the FST the data-collapse implied by Equation (2) can be assumed to hold to a reasonable approximation from 20 min up to a few days.

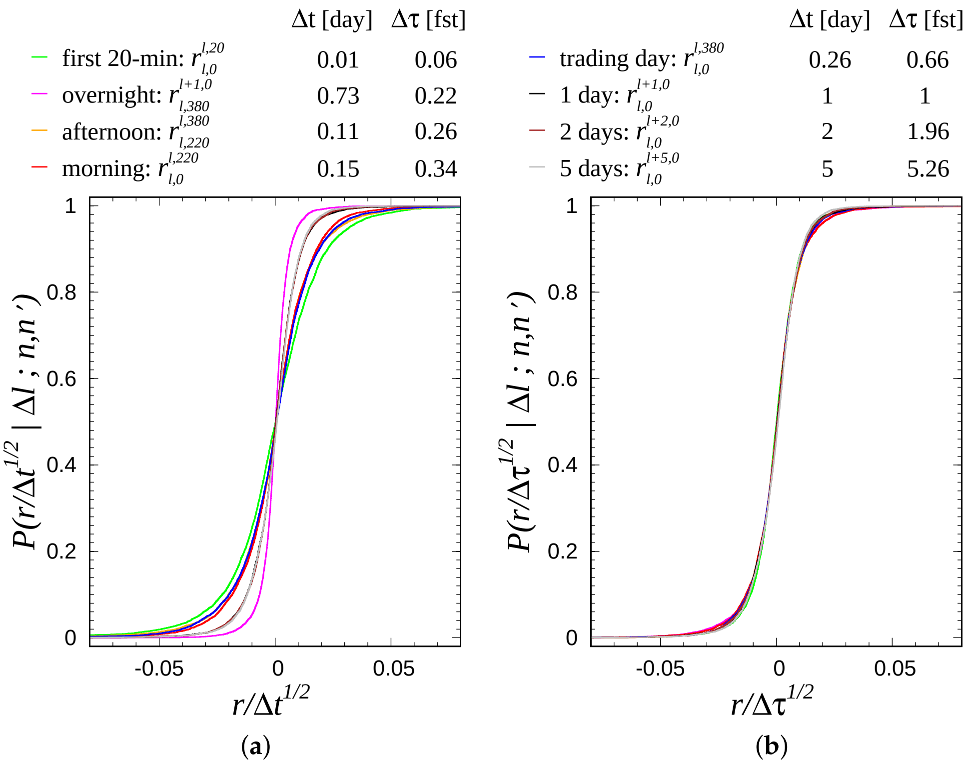

Figure 2. Empirical CDFs rescaled according to Equation (1) with $H = 1/2$ in (**a**), and Equation (2) in (**b**). In the key label, we make use of the notation $r_{l,n}^{l',n'} \equiv \ln s_{l',n'} - \ln s_{l,n}$ with $n = 0, 1, \ldots, N = 380$ and the CDFs $P(r/\Delta t^{1/2}|\Delta l; n, n')$ ($P(r/\Delta\tau^{1/2}|\Delta l; n, n')$) are evaluated on the ensemble of all possible $r_{l,n}^{l',n'}$ with n, n' and $\Delta l = l' - l$ fixed. While in the physical clock, the rescaled CDFs do not satisfy a simple scaling property since they do not collapse onto each other (**a**), when proper time is adopted (the FST), the scaling behavior nicely emerges as indicated by the satisfactory data-collapse (**b**).

Table 1 allows to further clarify our approach: Equation (2), if amounting to a fully satisfied scaling relation, would imply values of $D_{x,y}^{\mathrm{FST}}$ smaller than the threshold set by the significance level chosen in the KS test for every considered Y-sample. Such possibility is prevented by the fact that even in the interday range Equation (2) can not be satisfied completely due to both, multiscaling effects and a slow crossover to Gaussianity for the pdf, g, of the returns. The crossover to Gaussianity is due to the progressive loss of dependence of successive returns Cont (2001). However, Table 1 suggests that Equation (2) can be considered approximately valid in a time window ranging from about 20 min to a few days. To provide a comparison, if in place of the one-day distribution we take a zero-average Gaussian, the KS analysis would return $D_{x,y}^{\mathrm{FST}}$ values between 1.8 and 4.2, with the lower values corresponding to y's associated with longer time intervals, as one would expect in view of the slow crossover to Gaussianity.

Results (again, see Table 1) suggest that for the 10-min returns, the approximate scaling relation of Equation (2) is not well satisfied. Indeed, for this specific dataset (including entries of thirty years ago) a duration $\overline{\Delta t}$ over which contiguous returns can be considered linearly uncorrelated corresponds to 20 min (See Appendix B for details). However, note that, due to the impact of information technology on trading practice, the more recent the data are, the lower the threshold can be Bouchaud and Potters (2003). Thus, the 10 min interval is excluded from the FST window, and a more convenient partition of time can be indicated as $\{t_{l,m}\}$, where index l is run over the days ($l = 0, 1, \ldots, L$) and the index m over the 20-min intervals in which the trading day is split, ($m = 0, 1, \ldots, 19$).

In Figure 3a, we outline the FST pattern resulting from such physical time partition. $t_{l,m+1} - t_{l,m} = \overline{\Delta t} = 20$ min, $m_{\max} = 19$. The result is obtained by averaging over all possible days in the dataset. Figure 3a highlights the non-linear character of the FST vs. the physical one. If, reversely, one plots equally-spaced contiguous $\Delta\tau$-intervals as a function of the physical time (Figure 3b), it becomes evident that the FST runs faster at the beginning and at the end of the open market day and slower at noon, New York time.

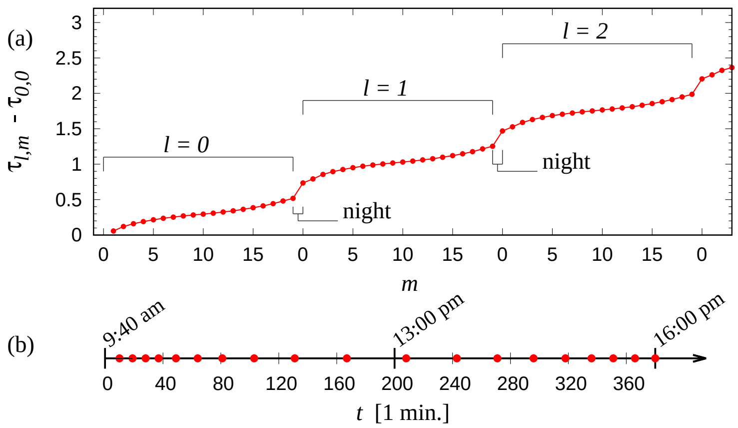

Figure 3. FST vs. physical time. In (**a**), the integer l labels the day after the chosen reference one and the integer m, the 20-min multiple after the opening time. In (**b**), instants occurring at multiples of $\Delta\tau = 0.028$ FST from the opening (red circles) are plotted on the physical time axis.

We already anticipated that, since Equations (2) and (5) are not exactly satisfied, one should not expect the FST scale, such as that reported in Figure 3, to be fully consistent as far as additivity properties are concerned. For our choice $\overline{\Delta t} = 20$ min, we verified that the difference between the $\Delta\tau$'s guaranteeing the best data collapse according to Equation (2) and those obtained by simply assuming additivity, does in general not exceed 25% of the value of the scaling factor itself. For instance, $\Delta\tau_{1\text{ day}}/(\Delta\tau_{\text{morning}} + \Delta\tau_{\text{afternoon}} + \Delta\tau_{\text{night}}) \simeq 1.22$. Furthermore, consistently with the positive value of linear correlations, this difference is typically positive, with exceptions for some intervals, including overnights, possibly due to a negative correlation between overnights and the first 20 min returns. Notice that it is because of this discrepancy that in Figure 3a the values for $m = 0$ are not exact integers.

5. Further Results and Applications

Volatility is a central quantity in financial practice, assessing the intensity of market fluctuations. In physical time it may be defined as $\sigma_{\Delta t}(t) := \mathbb{E}\left[(r_{\Delta t}(t))^2\right]^{1/2}$, and in FST as $\sigma_{\Delta\tau}(\tau) := \mathbb{E}\left[(r_{\Delta\tau}(\tau))^2\right]^{1/2}$, where $r_{\Delta t}(t)$ ($r_{\Delta\tau}(\tau)$) is the return in an interval of duration Δt ($\Delta\tau$) starting at t (τ). Clear evidence of the nonstationarity of returns in the physical time scale is given by the characteristic "U" shape assumed by the intraday volatility defined over the day-by-day ensemble of returns Admati and Pfleiderer (2015); Andersen and Bollerslev (1997); Baldovin et al. (2015); Bassler et al. (2007) (see Figure 4a). Once analyzed in terms of FST, volatility stationarity is instead sensibly recovered. Indeed, the plot in (Figure 4b) is obtained after performing a partition into 19 intervals of equal FST duration ($\Delta\tau = 0.028$) of the open market period. One clearly sees that the variation of this volatility is much less pronounced, as a sign of partial restoration of stationarity.

The availability of a time series with almost stationary increments, thus strongly reducing the seasonalities appearing in the physical time scale, conveys the methodological advantage that empirical analyses can be performed through ordinary, sliding-window techniques in the intraday regime, hence extending the sample size considerably in favor of higher statistical confidence. An example of how this extension can be performed is offered by the equal $\Delta\tau$ intervals identified for the plot in Figure 4b. For an interval of FST duration equal to 0.028, the sampling of CDF can be based on a data set 19 times larger than that available for intraday t-intervals.

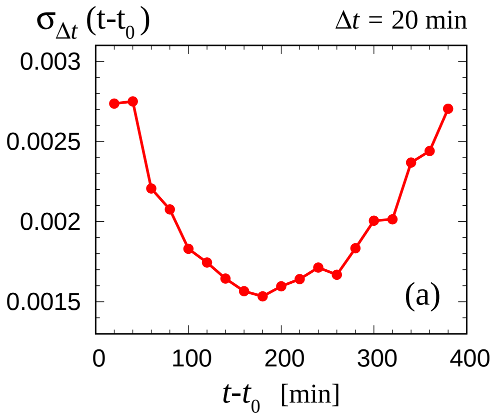

Figure 4. Intraday volatility. Each market day is assumed to be an independent realization of the same stochastic process Baldovin et al. (2015); Bassler et al. (2007). Volatility over the scale Δt (**a**) and $\Delta\tau$ (**b**) is plotted as a function of the time elapsing from the market opening (t_0 and τ_0, respectively). While in physical time, volatility is markedly nonstationary (**a**), apart from fluctuations, it becomes stationary in FST (**b**).

Let us now recall that Equation (2) straightforwardly descends

$$\mathbb{E}[|r_{\Delta\tau}|^q] = (\Delta\tau)^{q/2} \int \mathrm{d}x\, g(x)\, |x|^q\,. \tag{8}$$

In the presence of a general Hurst exponent $H \neq 1/2$, such as in Equation (1), one gets instead

$$\mathbb{E}[|r_{\Delta\tau}|^q] = (\Delta\tau)^{qH} \int \mathrm{d}x\, g(x)\, |x|^q. \tag{9}$$

A consequence of Equation (9) is that a log–log plot of the qth-moment $\mathbb{E}[|r_{\Delta\tau}|^q]$ vs. $\Delta\tau$ should be a straight line with slope qH. Therefore, the empirical analysis of the returns' moment as a function of the time interval duration is particularly revealing about the meaning of the FST. As a matter of fact, Figure 5 highlights that the recurrent patterns deviating from straight behavior in physical time (Figure 5a) are wiped out in the FST (Figure 5b).

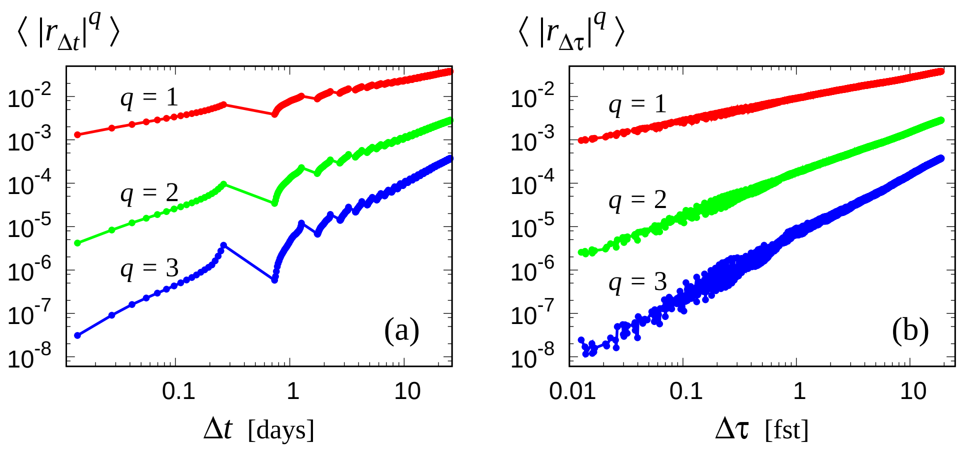

Figure 5. Empirical moment analysis. Δt (**a**) and $\Delta\tau$ (**b**) are the duration of the time intervals over which returns $r_{\Delta t}$ and $r_{\Delta\tau}$ are defined, respectively.

As already mentioned in the introduction, one of the factors preventing a full satisfaction of scaling, i.e., an equation such as Equation (2) either in physical time or in FST, is the presence of multiscaling. This means that the Hurst exponent is not strictly constant but might show a certain dependence on q, especially when analyzing the behavior as a function of t or of τ of relatively high order moments of the absolute return. Therefore, multiscaling may be easily detected by analyzing as a function of q the slopes of the log–log plots in Figure 5. Quite remarkably, multiscaling features that manifest in physical time (Figure 6a) are still present but become much less pronounced when the FST is adopted (Figure 6b). Since our definition of FST emphasizes simple scaling, there should be no surprise in realizing this attenuation of multiscaling effects.

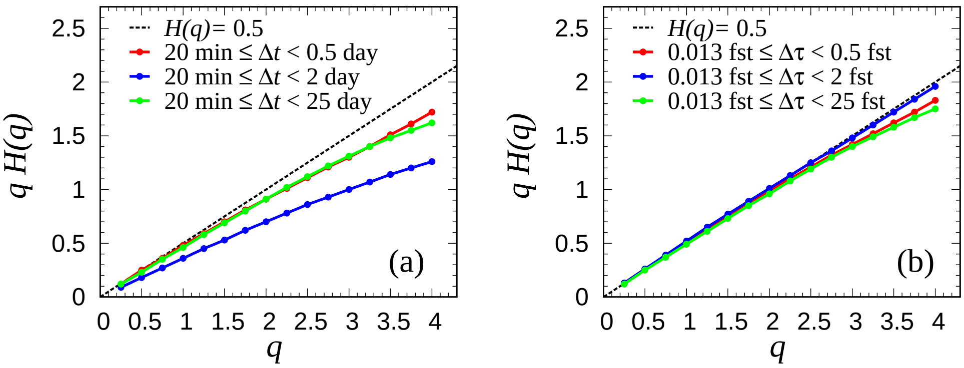

Figure 6. Multiscaling analysis. (**a**,**b**) Slopes of linear regression of the log–log plots in Figure 5a over different ranges of the time interval are respectively reported as a function of the moment-order q.

Thus far, results concern single-point statistics. In finance, a fundamental two-point indicator is the volatility autocorrelation function Cont (2001); Bouchaud and Potters (2003); Dacorogna et al. (2001); Mantegna and Stanley (2000), i.e., the correlation between absolute values of two returns at a given lag. At variance with the linear one, the volatility autocorrelation is known to decay very slowly with the time lag Cont (2001); Bouchaud and Potters (2003); Dacorogna et al. (2001); Mantegna and Stanley (2000). In Figure 7a, the $\Delta t = 20$ min volatility autocorrelation at lag t, $c_{\sigma,\Delta t}(t)$,[4] is plotted; while Figure 7b reports the analogous quantity in FST, $c_{\sigma,\Delta\tau}(\tau)$, for $\Delta\tau = 0.028$ fst. In both cases, the autocorrelation has been evaluated through a sliding-window procedure. The ciclostationary quality of $c_{\sigma,\Delta t}(t)$ in physical time is a direct consequence of the "U" shape in Figure 4a. It reveals that the lack of stationarity in the intraday prevents a successful application of sliding interval sampling. In contrast, in FST-lag, the periodic structure of $c_{\sigma,\Delta\tau}(\tau)$ is almost completely removed, pointing out the methodological advantage in such an empirical estimation. It is interesting to notice that if we try to fit the decay of the volatility autocorrelation in FST with a power law of the form $c_{\sigma,\Delta\tau}(\tau) \sim A/\tau^{\beta}$,2 we obtain a power law exponent $\beta \simeq 0.14$, in agreement with what is expected from the study of stylized facts in finance Cont (2001).

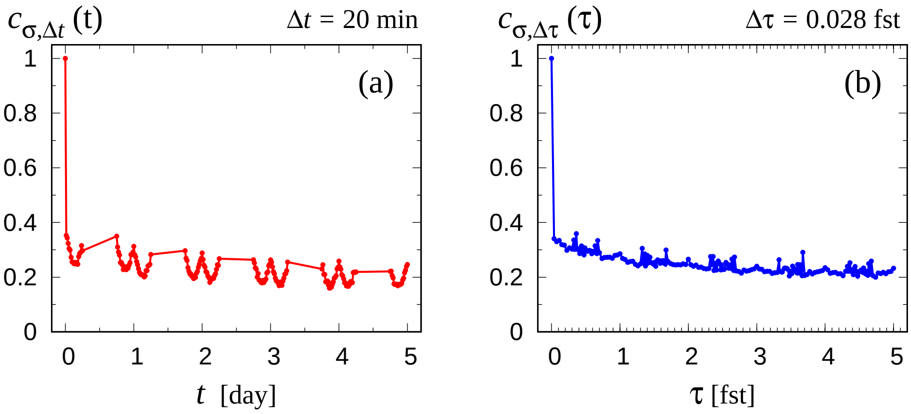

Figure 7. Volatility autocorrelation as a function of the physical (**a**) and FST (**b**) lag. Plots have been realized taking into account all possible pairs of lagged returns in the dataset, with t (τ) an an integer multiple of Δt ($\Delta\tau$). The ciclostationary character of the autocorrelation in physical time (**a**) is suppressed by the adoption of FST (**b**).

6. Discussion of Methods and Results

Our definition of FST is based on a minimization of the KS distance of the CDF of suitably scaled returns in any given time interval, with respect to a reference one. A question one could raise is if such a minimization could be avoided by simply enforcing, rather than Equation (2), one of its consequences, such as the one expressed by Equation (3). One could indeed simply obtain a $\Delta\tau$ by imposing, for a given q, a relation

$$\frac{E[|r|^q]}{E[|r|^q]_{\text{day}}} = \frac{\Delta\tau^{q/2}}{1} \tag{10}$$

where the numerator is the moment obtained with the CDF of an interval we want to quantify in terms of FST and the denominator is the same moment extracted from the CDF of the reference interval (one day). This definition is, in fact, similar to that adopted in the case of theta-time Dacorogna et al. (1996, 2001); Zumbach (1997). In Appendix A, we analyze this type of definition and show that it results to be less effective than our one in ensuring reasonable stationarity of the distributions.

While the choice $H = 1/2$ appears appropriate for mature markets, it is worth outlining what would change in applications to time series displaying $H \neq 1/2$. In the latter case, Equation (9) carries along the consequence that $\mathbb{E}[|r|^q] = \Delta\tau$ for $q = 1/H$. Following reasonings analogous to those leading to Equations (4) and (5), one would conclude that time additivity implies

$$\Delta\tau = \Delta\tau_1 + \Delta\tau_2 = \frac{\mathbb{E}[|r_1 + r_2|^{1/H}]}{A_{1/H}} = \frac{\mathbb{E}[|r_1|^{1/H}] + \mathbb{E}[|r_2|^{1/H}]}{A_{1/H}}\,. \tag{11}$$

Whenever $H \neq 1/2$, this establishes a more articulated constraint on the correlation structure of adjacent returns than Equation (5). The implications of such a constraint on the martingale property and thus on market efficiency require further analysis, which we postpone to future research.

7. Conclusions

Considering the paradigmatic case of the S&P500 index, we have shown that the assumption of simple-scaling invariance with Hurst exponent 1/2 for the return CDF allows the construction of a time scale, which cures the inner cyclostationarity of processes in finance on scales ranging from tens of minutes to a few days, by making them approximately stationary. Our construction opens interesting perspectives in the statistical analysis of financial time series with cyclostationarity that can be explored in future studies considering other financial indexes or assets. For instance, the recovered stationarity may sensibly enrich empirical estimates enlarging data sets by switching from sampling at cyclic positions along the history to sliding-window procedures. An example of this was offered by the above estimates of volatility and volatility correlations. Moreover, we have solved the problem of including overnight and over-weekend returns in financial analysis, bridging intra and interday regimes. This makes markets with discontinuous trading activity closer to those not experiencing transaction interruptions, thus opening novel possibilities for meaningful comparisons. Such a possibility is of major interest in problems such as return predictability, studies of market efficiency or leverage effects at high frequency. Indeed, the availability of a meaningful time definition suppressing cyclostationary effects and bridging overnight gaps amounts to a prerequisite for the application of regression analysis methods currently employed at low frequency.

Finally, our FST construction does not conflict with the possible presence of stylized facts related to the breaking of time reversal invariance, such as the leverage effect whereby a negative price change is on average followed by a volatility increase. Indeed, the scale-invariant distribution of returns in FST could, for instance, present skewness.

Author Contributions: Conceptualization, methodology, writing—review and editing, all authors; data curation, M.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

To better clarify the differences between the FST and a time scale constructed based on Equation (10), we report in Table A1 the outcome of different moment q choices.

Results show that a time definition based on the second moment may differ up to about 25% with respect to our FST, defined via the Kolmogorov–Smirnov test and that time scales defined on the other considered moments ($q = 1, 3$) do not perform better as far as KS distance is concerned. Furthermore, Table A1 shows that to define a time scale that enforces the scaling symmetry Equation (2), there is not a preferential moment since the best choice would depend on the considered time interval. For example, in the case of overnight returns, a time definition based on the third moment would perform much better in the attempt to satisfy Equation (2). This shows that if the goal is to promote in the best possible way the stationarity of the return PDF's (and this is our goal), the KS criterion on which we rely in our procedure is much more adequate than any moment-based procedure.

Table A1. Outcome of a time definition procedure when the one-day open-to-open return distribution is taken as x-sample. For $y = 10,\ 20,\ 38$ min, the last two columns correspond, in fact, to the average of $\Delta\tau_y$ and $D_{x,y}$ over the $(380\ \text{min}/y)$ contiguous time intervals existing within a trading day. Columns with the "fst" title report data already presented in Table 1 coming from the KS minimization procedure, while the other columns are obtained for a time definition based on a single moment q; see Equation (10).

	fst		$q = 1$		$q = 2$		$q = 3$	
y	$\Delta\tau_y$	$D_{x,y}$	$\Delta\tau_y$	$D_{x,y}$	$\Delta\tau_y$	$D_{x,y}$	$\Delta\tau_y$	$D_{x,y}$
10-min	0.011	1.56	0.014	2.37	0.015	3.03	0.015	3.14
20-min	0.028	1.13	0.029	1.73	0.032	2.06	0.032	2.27
38-min	0.055	0.90	0.059	1.45	0.063	1.64	0.066	1.89
morning	0.337	0.79	0.339	0.81	0.321	1.03	0.273	2.01
afternoon	0.260	0.99	0.298	1.66	0.349	2.62	0.406	3.51
trading day	0.661	0.60	0.693	0.91	0.723	1.18	0.811	1.91
overnights	0.216	0.86	0.244	1.34	0.259	1.70	0.235	1.19
2 days	1.965	0.57	1.955	0.64	1.936	0.61	1.886	0.73
3 days	3.251	0.88	2.956	1.25	2.791	1.45	2.533	1.83
5 days	5.260	1.15	4.785	1.74	4.440	2.04	3.924	2.55
10 days	11.19	1.80	8.860	2.16	8.320	2.41	7.269	2.94

Appendix B

Let us indicate with $[\Delta t] \equiv \Delta t/(1\ \text{min})$ the number of minutes characterizing Δt (for instance, $[10\ \text{min}] = 10$). The empirical linear correlation between two contiguous intervals of span Δt may be calculated as

$$c_{r,\Delta t}(\Delta t) \equiv \frac{1}{N - 2[\Delta t] + 1} \sum_{n=[\Delta t]}^{N-[\Delta t]} \cdot \frac{\frac{1}{L}\sum_{l=1}^{L} r_{l,n-[\Delta t]}^{l,n}\, r_{l,n}^{l,n+[\Delta t]}}{\sqrt{\frac{1}{L}\sum_{l=1}^{L}\left(r_{l,n-[\Delta t]}^{l,n}\right)^2}\sqrt{\frac{1}{L}\sum_{l=1}^{L}\left(r_{l,n}^{l,n+[\Delta t]}\right)^2}}, \tag{A1}$$

and it is plotted in Figure A1.

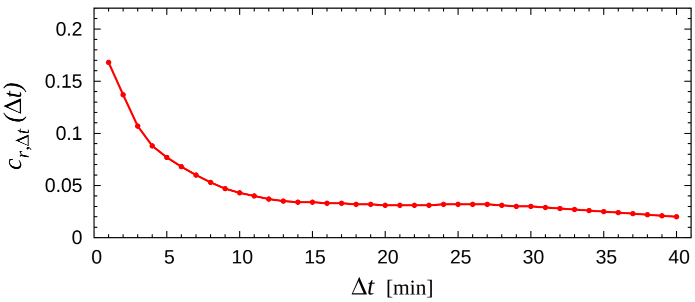

Figure A1. Empirical linear correlation function between two contiguous intervals of span Δt.

Notes

[1] We make use of the common notation, which associates capital letters to CDFs and lower case letters to the corresponding PDFs: $G(x) = \int_{-\infty}^{x} dx' g(x')$.

[2] Note that while the CDF of returns in the intra-day are sampled from the ensemble of daily histories, for intervals of an integer number of days, including the one-day reference, the sampling is based on sliding along the time series.

[3] The New York market opens at 9:30 a.m., whereas our dataset records the start at 9:40 a.m. instead. Due to the end of research funding sponsoring the project, we could not access S&P500 high-frequency data after 2013.

[4] The volatility autocorrelation is defined as

$$c_{\sigma,\Delta t}(t) := \frac{\langle |r_{\Delta t}(t'+t)\, r_{\Delta t}(t')| \rangle_{t'} - \langle |r_{\Delta t}(t'+t)| \rangle_{t'}\, \langle |r_{\Delta t}(t')| \rangle_{t'}}{\sqrt{\left(\langle [r_{\Delta t}(t'+t)]^2 \rangle_{t'} - \langle [r_{\Delta t}(t'+t)] \rangle_{t'}^2\right)\left(\langle [r_{\Delta t}(t')]^2 \rangle_{t'} - \langle [r_{\Delta t}(t')] \rangle_{t'}^2\right)}},$$

where $\langle\cdot\rangle_{t'}$ is a sliding-window empirical average with respect to t', performed over all available couples $(t', t'+t)$ in the dataset. An analogous definition follows for $c_{\sigma,\Delta\tau}(\tau)$ in FST.

References

Admati, Anat R., and Paul Pfleiderer. 2015. A Theory of Intraday Patterns: Volume and Price Variability. *The Review of Financial Studies* 1: 3. [CrossRef]

Allez, Romain, and Jean-Philippe Bouchaud. 2011. Individual and collective stock dynamics: Intra-day seasonalities. *New Journal of Physics* 13: 025010. [CrossRef]

Andersen, Torben G., and Tim Bollerslev. 1997. Intraday periodicity and volatility persistence in financial markets. *Journal of Empirical Finance* 4: 115. [CrossRef]

Ané, Thierry, and Hélyette Geman. 1999. Stochastic volatility and transaction time: An activity-based volatility estimator. *Journal of Risk* 1: 57. [CrossRef]

Ané, Thierry, and Hélyette Geman. 2000. Order Flow, Transaction Clock, and Normality of Asset Returns. *The Journal of Finance* 55: 2259. [CrossRef]

Ashkenazy, Yosef, Plamen Ch. Ivanov, Shlomo Havlin, Chung-K. Peng, Ary L. Goldberger, and H. Eugene Stanley. 2001. Magnitude and sign correlations in heartbeat fluctuations. *Physical Review Letters* 86: 1900. [CrossRef] [PubMed]

Baldovin, Fulvio, and Attilio L. Stella. 2007. Scaling and efficiency determine the irreversible evolution of a market. *Proceedings of the National Academy of Sciences* 104: 19741. [CrossRef]

Baldovin, Fulvio, Francesco Camana, Massimiliano Caporin, Michele Caraglio, and Attilio L. Stella. 2015. Ensemble properties of high-frequency data and intraday trading rules. *Quantitative Finance* 15: 231. [CrossRef]

Baldovin, Fulvio, Massimiliano Caporin, Michele Caraglio, Attilio L. Stella, and Marco Zamparo. 2015. Option pricing with non-Gaussian scaling and infinite-state switching volatility. *Journal of Econometrics* 187: 486. [CrossRef]

Bassler, Kevin E., Joseph L. McCauley, and Gemunu H. Gunaratne. 2007. Nonstationary increments, scaling distributions, and variable diffusion processes in financial markets. *Proceedings of the National Academy of Sciences* 104: 17287. [CrossRef] [PubMed]

Borland, Lisa, Jean-Philippe Bouchaud, Jean-Francois Muzy, and Gilles Zumbach. 2005. The dynamics of financial markets—Mandelbrot's multifractal cascades, and beyond. *Wilmott Magazine* , January 12.

Bouchaud, Jean-Philippe, and Marc Potters. 2003. *Theory of Financial Risk and Derivative Pricing*. Cambridge: Cambridge University Press.

Bryce, R. M., and K. B. Sprague. 2012. Revisiting detrended fluctuation analysis. *Scientific Reports* 2: 315. [CrossRef] [PubMed]

Calvet, Laurent, and Adlai Fisher. 2002. Multifractality in Asset Returns: Theory and Evidence. *The Review of Economics and Statistics* 84: 381. [CrossRef]

Chen, Zhi, Plamen Ch. Ivanov, Kun Hu, and H. Eugene Stanley. 2002. Effect of nonstationarities on detrended fluctuation analysis. *Physical Review E* 65: 041107. [CrossRef]

Chicheportiche, Rémy, and Jean-Philippe Bouchaud. 2011. Goodness-of-fit tests with dependent observations. *Journal of Statistical Mechanics: Theory and Experiment* 2011: P09003. [CrossRef]

Clark, Peter K. 1973. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41: 135–55. [CrossRef]

Cont, Rama. 2001. Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance* 1: 223. [CrossRef]

Corral, Álvaro. 2004. Long-Term Clustering, Scaling, and Universality in the Temporal Occurrence of Earthquakes. *Physical Review Letters* 92: 108501. [CrossRef]

Corral, Álvaro. 2006. Statistical features of earthquake temporal occurrence. *Lecture Notes in Physics* 705: 191. [CrossRef]

Dacorogna, Michel M., Cindy L. Gauvreau, Ulrich A. Müller, Richard B. Olsen, and Olivier V. Pictet. 1996. Changing time scale for short-term forecasting in financial markets. *Journal of Forecasting* 15: 203–27. [CrossRef]

Dacorogna, Michel M., Ramazan Gençay, Ulrich A. Müller, Richard B. Olsen, and Olivier V. Pictet. 2001. *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

DeGroot, Morris H. 2010. *Probability and Statistics*, 4th ed. Boston: Addison-Wesley.

Di Matteo, T. 2007. Multi-scaling in finance. *Quantitative Finance* 7: 21. [CrossRef]

Di Matteo, Tiziana, Tommaso Aste, and Michel M. Dacorogna. 2005. Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development. *Journal of Banking & Finance* 29: 827. [CrossRef]

Galluccio, Stefano, Guido Caldarelli, Matteo Marsili, and Yicheng Zhang. 1997. Scaling in currency exchange. *Physica A: Statistical Mechanics and its Applications* 245: 423. [CrossRef]

Gardner, William A., Antonio Napolitano, and Luigi Paura. 2006. Cyclostationarity: Half a century of research. *Signal Processing* 86: 639. [CrossRef]

Gillemot, László, J. Doyne Farmer, and Fabrizio Lillo. 2006. There's more to volatility than volume. *Quantitative Finance* 6: 371. [CrossRef]

Hausdorff, Jeffrey, Chung-Kang Peng, Z. Ladin, J. Y. Wei, and Ary L. Goldberger. 1995. Is walking a random walk? Evidence for long-range correlations in stride interval of human gait. *Journal of Applied Physiology* 78: 349. [CrossRef] [PubMed]

Hurst, Harold Edwin. 1951. Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers* 116: 770. [CrossRef]

Ivanova, Kristinka, and Marcel Ausloos. 1999. Application of the detrended fluctuation analysis (DFA) method for describing cloud breaking. *Physica A: Statistical Mechanics and its Applications* 274: 349. [CrossRef]

Jensen, Mogens H., Anders Johansen, Filippo Petroni, and Ingve Simonsen. 2004. Inverse statistics in the foreign exchange market. *Physica A: Statistical Mechanics and Its Applications* 340: 678. [CrossRef]

Kadanoff, Leo P. 2005. *Statistical Physics, Statics, Dynamics and Renormalization*. Singapore: World Scientific.

Kantelhardt, Jan W., Stephan A. Zschiegner, Eva Bunde, Shlomo Havlin, Armin Bunde, and H. Eugene Stanley. 2002. Multifractal detrended fluctuation analysis of nonstationary time series. *Physica A: Statistical Mechanics and Its Applications* 316: 87. [CrossRef]

Liu, Yanhui, Pierre Cizeau, Martin Meyer, Chung-K. Peng, and H. Eugene Stanley. 1997. Correlations in economic time series. *Physica A: Statistical Mechanics and Its Applications* 245: 437. [CrossRef]

Mandelbrot, Benoit B. 1973. Comments on: "A subordinated stochastic process model with finite variance for speculative prices," by Peter K. Clark. *Econometrica* 41: 157–59. [CrossRef]

Mandelbrot, Benoit B., and Howard M. Taylor. 1967. On the distribution of stock price differences. *Operations Research* 15: 1057–62. [CrossRef]

Mantegna, Rosario N., and H. Eugene Stanley. 1995. Scaling behaviour in the dynamics of an economic index. *Nature* 376: 46. [CrossRef]

Mantegna, Rosario N., and H. Eugene Stanley. 2000. *Introduction to Econophysics: Correlations and Complexity in Finance*. Cambridge: Cambridge University Press.

Omori, Fukuyoshi. 1894. On the after-shocks of earthquakes. *Journal of the College of Science, Imperial University of Tokyo, Japan* 7: 111.

Peirano, Pier Paolo, and Damien Challet. 2012. Baldovin-Stella stochastic volatility process and Wiener process mixtures. *The European Physical Journal B* 85: 276. [CrossRef]

Peng, Chung-K., Shlomo Havlin, H. Eugene Stanley, and Ary L. Goldberger. 1995. Quantification of scaling exponents and crossover phenomena in nonstationary heartbeat time series. *Chaos: An Interdisciplinary Journal of Nonlinear Science* 5: 82. [CrossRef] [PubMed]

Peng, Chung-K., S. V. Buldyrev, Shlomo Havlin, M. Simons, H. Eugene Stanley, and Ary L. Goldberger. 1994. Mosaic organization of DNA nucleotides. *Physical Review E* 49: 1685. [CrossRef]

Potí, Valerio. 2018. A new tight and general bound on return predictability. *Economics Letters* 162: 140–45. [CrossRef]

Stella, Attilio L., and Fulvio Baldovin. 2010. Anomalous scaling due to correlations: Limit theorems and self-similar processes. *Journal of Statistical Mechanics: Theory and Experiment* 2010: P02018. [CrossRef]

Tsay, Ruey S. 2002. *Analysis of Financial Time Series*. New York: John Wiley & Sons.

Zamparo, Marco, Fulvio Baldovin, Michele Caraglio, and Attilio L. Stella. 2013. Scaling symmetry, renormalization, and time series modeling: The case of financial assets dynamics. *Physical Review E* 88: 062808. [CrossRef]

Zumbach, Gilles. 1997. Considering time as the random variable: A new point of view for studying financial time series. Paper presented at 1st Workshop on Econophysics, Budapest, Hungary, July 21–27.