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Extended statistical linearization approach for estimating non-stationary response statistics of systems comprising fractional derivative elements

Beatrice Pomaro [a,b,*], Pol D. Spanos [c]

[a] Department of Civil, Environmental and Architectural Engineering, University of Padova, Via F. Marzolo 9, 35131, Padova, Italy
[b] Department of Civil and Environmental Engineering, Rice University, 6100 Main, Houston, TX 77005-1827, USA
[c] L. B. Ryon Chair in Engineering, Rice University, 6100 Main, Houston, TX 77005-1827, USA

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ABSTRACT

An efficient technique to analyze non-stationary nonlinear random vibrations of dynamic systems endowed with a fractional derivative term is presented. The technique itself represents an extension of the concept of statistical linearization to this kind of systems, and it is applicable for both analytic and hysteretic system nonlinearities. The technique first resorts to harmonic balancing in deriving response-amplitude dependent equivalent damping and stiffness. This enables representation of the fractional derivative term as a linear combination of the system response displacement and velocity with amplitude dependent coefficients. Then, the expected values of these parameters are considered in proceeding to formulate a statistical linearization solution scheme. In this context, the solution procedure is completed by integrating in time the covariance Lyapunov equation associated with the derived equivalent linear system. The reliability of the proposed technique is tested by a series of germane Monte Carlo studies. This juxtaposition is also used to elucidate salient features of the technique, by varying the order of the fractional derivative term, and of the degree of the nonlinearity in the system. It also points out the versatility of the technique in determining the non-stationary values of auto-correlation and cross-correlations response parameters involving even the fractional derivative term.

1. Introduction

In estimating the response statistics of nonlinear dynamic systems, the analytical approximations by Statistical Linearization (SL) [1–9] have attracted considerable interest in the literature. Further, engineering analyses require commonly the study of systems under evolutionary in time excitations, that usually are stochastic in nature (e.g. earthquake, wind, waves). Thus, the study of the non-stationary response of the system is often desirable [10,11]. In this context, when dealing with random excitations, SL is a quite versatile tool for estimating the evolution in time of the relevant response statistics.

A special kind of problems pertains to nonlinear dynamics systems comprising an additional term with a fractional derivative operator. This occurs when the system response dictates modeling its frequency-dependent dissipative damping properties [12–14], or its viscoelastic rheological features [15–19]. This operator derives its memory-persistent feature from its integral-based representation [20–23]. As a consequence, any problem involving a fractional derivative is "memory persistent". For this reason, specific solution strategies exist in the literature to conduct deterministic dynamic response analyses of systems endowed with fractional derivative terms. Among the analytical methods used to treat fractional equations are: the Laplace or Fourier integral transformation method [24]; the perturbation method [25]; the Harmonic Balancing (HB) method [26–28]; the deterministic averaging method [29]; the semi-analytical decomposition method [30]; and the variational iteration method [31]. Numerical procedures include the differential transform method [32]; numerical quadrature, such as product-integration and collocation methods [33,34]; the Finite Difference Method (FDM) [35]; the Finite Element Method (FEM) [36]; the Boundary Element Method (BEM) [37]; and predictor–corrector schemes [38].

For stochastic excitations, the problem of the response determination of systems comprising fractional terms has been addressed in the literature both in the frequency domain [39], and in the time domain [40]. This has set useful bases for subsequent research on randomly excited, fractionally endowed systems. Research efforts to treat this kind of systems under stochastic actions have focused mainly on SL [41], Stochastic Averaging (SA) [11,42–44], perturbation methods [45], methods involving system dimension augmentation [46–48], and wavelet-based methods [49].

The majority of the existing studies of fractionally endowed systems subject to random loads are restricted to the investigation of stationary

* Corresponding author at: Department of Civil, Environmental and Architectural Engineering, University of Padova, Via F. Marzolo 9, 35131, Padova, Italy.
E-mail addresses: beatrice.pomaro@unipd.it (B. Pomaro), spanos@rice.edu (P.D. Spanos).

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response statistics [50–52]. Among the non-stationary studies, considerable attention has received the application of the SA technique in the evaluation of the system survival probability, or its counterpart (first-passage failure), for risk assessment purposes related to systems with fractional elements [43,44,53]. Further, the SA technique has been successfully applied in the literature in the assessment of the stability of the stochastic system [42,54,55]. Comparatively, the non-stationary behavior of response statistical moments of fractionally endowed systems has been less investigated [56].

In this paper, the determination of non-stationary statistics of the response state variables associated with the aforementioned problem is addressed. The work is motivated by the desire to make use of averaging in time schemes in the original nonlinear system, while adhering to the classical SL procedure. Simultaneously, the complete derivation of the system non-stationary statistics is pursued, inclusive of the auto-correlation and cross-correlations involving the fractional term. This is achieved without resorting to the solution of the system associated Fokker–Planck–Kolmogorov (FPK) equation. Numerical results for a representative Duffing oscillator with fractional damping are shown. Evidence of statistical correlations between the fractional term, and the response displacement and velocity, is reported and captured by the proposed technique. This is also confirmed by pertinent Monte Carlo (MC) results obtained by treating the system with a standard Newmark-method-based integration algorithm, where the fractional term is numerically treated by resorting to the Grünwald–Letnikov representation.

2. Mathematical formulation

2.1. Preliminary remarks

The fractional operator is a well-established [20–23] mathematical tool. Perhaps, a most intuitive definition is [22]

$$D^{\beta}_{t_0,t}x(t) = D^n J^{n-\beta}_{t_0,t}x(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\left[\int_{t_0}^{t}(t-\tau)^{n-\beta-1}x(\tau)\,d\tau\right], \beta\in\mathbb{R}^+ \tag{1}$$

where $J^{\beta}_{t_0,t}x(t)$ is the Riemann–Liouville (RL) fractional integral of order $\beta\in\mathbb{R}^+$, of a function $x(t)$, and origin at terminal t_0

$$J^{\beta}_{t_0,t}x(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-\tau)^{\beta-1}x(\tau)\,d\tau. \tag{2}$$

Thus, the RL definition of the β-order derivative in Eq. (1) is obtained by computing the nth order derivative of the integral of order $(n-\beta)$, n being the smallest integer greater than β, that is: $n-1\le\beta<n\in\mathbb{Z}^+$. It is clear from this definition that the β -order fractional derivative of a function $x(t)$ involves a convolution integral of the function itself. This points out the fading memory feature of this mathematical operator. This feature distinguishes the fractional derivative from the integer-order derivative, which is defined locally. For this reason, the fractional derivative is treated as a non-local or memory-persistent operator. As such, it is defined in terms of the present and of the "past" values (when time is involved), or of the values "in the surrounding space" (when space is involved).

Several representations of this operator exist in the literature [22,23]. They mainly differ in the sequence of differentiation or integration [22]. For the numerical simulations in this work the Grünwald–Letnikov (GL) representation is used, because it is particularly convenient to treat computationally. The latter can be seen as a special case of the RL definition. It can be obtained from (1) by repeatedly performing integration by parts and differentiation. The GL definition of the fractional operator yields [22]

$${}_{GL}D^{\beta}_{t_0,t}x(t) = \sum_{k=0}^{n-1}\frac{x^{(k)}(t_0)(t-t_0)^{-\beta+k}}{\Gamma(-\beta+k+1)} + \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-\tau)^{n-\beta-1}x^{(n)}(\tau)\,d\tau \tag{3}$$

where the derivatives $x^{(k)}(t)$ for $k=1,2,\ldots,n-1$ must be continuous in the closed interval $[t_0,t]$, and $n-1\le\beta<n\in\mathbb{Z}^+$. Alternatively to the RL definition, the Caputo representation [22,23], to be adopted in an ensuing section, can be used in dealing with fading memory problems.

2.2. Equation of motion

Consider a single-degree-of-freedom nonlinear oscillator

$$m\ddot{x}(t) + cD^{\beta}_{0,t}x(t) + f(t,x,\dot{x}) = w(t), \tag{4}$$

with $x(0)=\dot{x}(0)=0$, subject to a zero-mean Gaussian white process $w(t)$, of two-sided power spectral density $S_w(\omega)=S_0$. In Eq. (4) m is the mass coefficient; c can be construed as a damping coefficient (for $\beta=1$), or a stiffness coefficient (for $\beta=0$). In this sense the fractional operator embodies both effects simultaneously for intermediate values of β. The nonlinear restoring force is represented by the function $f(t,x,\dot{x})$.

For the purpose of conducting pertinent MC simulations, and determining the time history $x(t)$ of the system, the governing equation of motion is considered into N timesteps of magnitude Δt, and solved in time by a Newmark integration scheme. The fractional term is treated by the G1-Algorithm, as outlined in [20], and elucidated by [41], whereas, the nonlinearity term is treated by a Newton–Raphson scheme.

Expanding the series in Eq. (3), and setting $t_0=0$, yields [41]

$${}_{GL}D^{\beta}_{0,t}x(t) = \lim_{\Delta t\to 0}\Delta t^{-\beta}\sum_{k=0}^{N}GL_k x(t-k\Delta t), \tag{5}$$

with the GL_k coefficients represented by factorial terms

$$GL_k = (-1)^k\binom{\beta}{k}. \tag{6}$$

They can, more conveniently, be recast in a recursive form [20] by resorting to the properties of the Gamma function. Specifically,

$$GL_k = \frac{\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)} = \frac{k-\beta-1}{k}\frac{\Gamma(k-\beta-1)}{\Gamma(-\beta)\Gamma(k)} = \frac{k-\beta-1}{k}GL_{k-1}. \tag{7}$$

Note from Eq. (7), that $GL_{k=0}=1$.

Hence, the discretized equation of motion of the oscillator yields

$$m\Delta\ddot{x}_{i+1} + c\Delta t^{-\beta}\left[\sum_{k=0}^{i+1}GL_k x(t_{i+1}-k\Delta t) - \sum_{k=0}^{i}GL_k x(t_i-k\Delta t)\right] + \Delta f(t,x,\dot{x})_{i+1} = \Delta w_{i+1}, \tag{8}$$

with the notation

$$\Delta w_{i+1} = w(x_{i+1}) - w(x_i);\ \text{and}$$
$$\Delta f(t,x,\dot{x})_{i+1} = f(t_{i+1},x_{i+1},\dot{x}_{i+1}) - f(t_i,x_i,\dot{x}_i). \tag{9}$$

A more compact form of Eq. (8), particularly advantageous for computations, was suggested in [41]

$$m\Delta\ddot{x}_{i+1} + c\Delta t^{-\beta}\Delta x_{i+1} + \Delta f(t,x,\dot{x})_{i+1} = \Delta w_{i+1} - c\Delta t^{-\beta}P_1, \tag{10}$$

where $P_1=P_{k=1}$, with

$$P_k = \sum_{k=0}^{i}GL_k\Delta x_{i+1-k} + GL_{i+1}x_0, \tag{11}$$

and with the notation

$$\Delta x_{i+1} = x_{i+1} - x_i. \tag{12}$$

It can be argued that the G1-Algorithm treats the fractional derivative by truncating the infinite number of past terms required by the GL representation beyond the most significant ones in magnitude; that is, selecting a finite but sufficiently large N in Eq. (5). This captures the non-locality of the operator, while limiting the requisite computational effort.

2.3. Statistical linearization approach

Following the pioneering works by Booton [1], Kazakov [2], and Caughey [3,57], with an early review by Spanos [58], the nonlinear oscillator of Eq. (4) can be transformed into an optimal equivalent linear system of the form

$$m\ddot{x}(t) + cD^{\beta}_{0,t}x(t) + c_{eq,SL}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right)\dot{x}(t) + k_{eq,SL}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right)x(t) = w(t), \quad (13)$$

in such a manner that the response statistics of the linear system approximate satisfactorily those of the nonlinear oscillator. According to the classical SL technique, the optimal equivalent damping $c_{eq,SL}$ and stiffness $k_{eq,SL}$ can be computed by minimizing the expected value of the error $\epsilon^2(t)$

$$E\left\{\epsilon^2(t)\right\} = \langle\left[f(t,x,\dot{x}) - c_{eq,SL}\dot{x}(t) - k_{eq,SL}x(t)\right]^2\rangle, \quad (14)$$

where $\langle\rangle$ denotes mathematical expectation.

The minimization is pursued with respect to $c_{eq,SL}$ and $k_{eq,SL}$ by using the equations

$$\frac{\partial E\left\{\epsilon^2(t)\right\}}{\partial c_{eq,SL}} = 0; \qquad \frac{\partial E\left\{\epsilon^2(t)\right\}}{\partial k_{eq,SL}} = 0. \quad (15)$$

If the analysis adopts approximations of $x(t)$ and $\dot{x}(t)$ as Gaussian processes, these expressions simplify based on the properties of the expectation of the product of Gaussian variables [59]. Specifically,

$$c_{eq,SL}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right) = \langle\frac{\partial f(t,x,\dot{x})}{\partial \dot{x}}\rangle, \text{and} \qquad k_{eq,SL}(\sigma^2_x, \sigma^2_{\dot{x}}) = \langle\frac{\partial f(t,x,\dot{x})}{\partial x}\rangle. \quad (16)$$

Note that, in doing this, it is assumed that the input process is Gaussian, hence the response of the equivalent linear system is also Gaussian.

It is also pointed out that the equivalent quantities in Eq. (16) account for the linearization of the restoring force $f(t,x,\dot{x})$, only, while keeping the fractional term as it is. Clearly, they are time-dependent in the non-stationary case.

2.4. Harmonic balancing

In treating the fractional term in Eq. (4), a solution to the equivalent quantities for the fractional term in Eq. (13) involving temporal averaging [57,60] is next pursued. In doing so, it is assumed that the system exhibits pseudo-harmonic response. For the special case of Eq. (4) with $\beta = 1$, and $f(t,x,\dot{x}) = kx$, linear system, $\omega_0 = \sqrt{k/m}$ is the natural angular frequency, and c_0 is the damping coefficient of the corresponding linear oscillator; $\zeta = c_0/(2\omega_0 m) \ll 1$ is the associated ratio of critical damping. Under these assumptions the response $x(t)$ can be treated as a narrow-band process [61]. As such, its pseudo-harmonic behavior involves amplitude $a(t)$, and phase $\theta(t)$ characterized by slowly varying in time statistics behavior [62,63]. For such functions it makes sense to introduce individual cycles phase and amplitude. Specifically, set

$$x(t) = a(t)\cos[\omega(a)t + \theta(t)], \quad (17)$$

and

$$\dot{x}(t) = -\omega(a)a(t)\sin[\omega(a)t + \theta(t)]. \quad (18)$$

In Eqs. (17) and (18) $\omega(a)$ denotes the undamped natural angular frequency of the nonlinear oscillator of Eq. (1), to be determined as a function of the response amplitude $a(t)$.

Eqs. (17) and (18) can be regarded as two transformations between the original variables, $x(t)$ and $\dot{x}(t)$, and the new ones, $a(t)$ and $\theta(t)$. Indeed, combining the two equations, yields

$$a(t) = \sqrt{x^2(t) + \frac{\dot{x}^2(t)}{\omega^2(a)}}, \quad (19)$$

and

$$\theta(t) = -\omega(a)t - \arctan\left(\frac{\dot{x}(t)}{\omega(a)x(t)}\right). \quad (20)$$

Adopting the HB method [7,64], the fractional contribution over one cycle of oscillation to the effective damping and stiffness of the system, $c_{HB,\beta}$ and $k_{HB,\beta}$, respectively, is determined as function of the amplitude a. This is done by minimizing the mean square of the error between the original and the linearized equation. In doing so, a and θ are treated as constants in this interval. This leads to [52,53]

$$c_{HB,\beta}(a) = -\frac{1}{\pi a\omega(a)}\left[c\int_0^{2\pi} D^{\beta}_{0,t}(a\cos\psi)\sin\psi d\psi\right], \quad (21)$$

and

$$k_{HB,\beta}(a) = \frac{1}{\pi a}\left[c\int_0^{2\pi} D^{\beta}_{0,t}(a\cos\psi)\cos\psi d\psi\right], \quad (22)$$

where $\psi = [\omega(a)t + \theta(t)]$.

Introduce the compact expressions [52,53,56]

$$S_\beta(a) = \int_0^{2\pi} D^{\beta}_{0,t}(a\cos\psi)\sin\psi d\psi \quad (23)$$

and

$$F_\beta(a) = \int_0^{2\pi} D^{\beta}_{0,t}(a\cos\psi)\cos\psi d\psi. \quad (24)$$

Next, approximating the fractional derivatives according to [43,44], which is consistent with the Caputo representation of the fractional derivative [22,23], yields

$$c_{HB,\beta}(a) = -c\frac{S_\beta(a)}{\pi a\omega(a)} = \frac{c}{\omega^{1-\beta}(a)}\sin\left(\frac{\beta\pi}{2}\right), \quad (25)$$

and

$$k_{HB,\beta}(a) = c\frac{F_\beta(a)}{\pi a} = c\omega^\beta(a)\cos\left(\frac{\beta\pi}{2}\right). \quad (26)$$

It is clear form Eqs. (25) and (26) that the fractional term merely introduces additional amplitude-dependent, hence time-dependent, damping and stiffness. Thus, the fading memory of the operator with this approach is captured by the amplitude itself. The latter depends on the past history of the system oscillations.

Further, in the presence of the random excitation $w(t)$ it is reasonable to replace $c_{HB,\beta}(a)$ and $k_{HB,\beta}(a)$ by their expected values $c_{eq,\beta}$ and $k_{eq,\beta}$ [7]. That is,

$$c_{eq,\beta} = \langle -c\frac{S_\beta(a)}{\pi a\omega(a)}\rangle, \quad (27)$$

and

$$k_{eq,\beta} = \langle c\frac{F_\beta(a)}{\pi a}\rangle. \quad (28)$$

Next, that the probability density function of the response amplitude is approximated by a Rayleigh distribution with time-dependent parameters [7,65], which depend on the time-dependent variance of the oscillator response displacement. That is,

$$p(a) = \frac{a}{\sigma^2_x}e^{-\frac{a^2}{2\sigma^2_x}}; \text{ with } \qquad \sigma^2_x = \sigma^2_x(t). \quad (29)$$

Then, the corresponding equivalent quantities can be determined by the equations

$$c_{eq,\beta}\left(\sigma^2_x\right) = \int_0^\infty\left[-c\frac{S_\beta(a)}{\pi a\omega(a)}\right]p(a)\,da, \quad (30)$$

and

$$k_{eq,\beta}\left(\sigma^2_x\right) = \int_0^\infty\left[c\frac{F_\beta(a)}{\pi a}\right]p(a)\,da. \quad (31)$$

Thus, the initial system of Eq. (4) can be linearized as

$$m\ddot{x}(t) + c_{eq}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right)\dot{x}(t) + k_{eq}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right)x(t) = w(t) \quad (32)$$

with

$$c_{eq}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right) = c_{eq,SL}\left(\sigma^2_x, \sigma^2_{\dot{x}}\right) + c_{eq,\beta}\left(\sigma^2_x\right), \quad (33)$$

and

$$k_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)=k_{eq,SL}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)+k_{eq,\beta}\left(\sigma_x^2\right). \tag{34}$$

In computing the expressions of Eqs. (30) and (31), an approximation to $\omega(a)$ is used, involving the equivalent stiffness associated with system of Eq. (13), $k_{eq,SL}$, at the stationary regime. The latter can be obtained by the classical SL technique, as shown in Section 2.3, accounting, as well, for the contribution to the effective stiffness of the system by the fractional term. In this regard, first, the stationary response variance of the initial nonlinear oscillator σ^2_{xSL} is computed by resorting to the input–output relationship applicable for the considered oscillator, endowed with a fractional term, as proposed in [41]. Specifically, σ^2_{xSL} is derived iteratively by setting [7,41]

$$\sigma^2_{xSL}=\int_{-\infty}^{\infty}S_x(\omega)\,d\omega=\int_{-\infty}^{\infty}\frac{S_w(\omega)}{\left|-m\omega^2+k_{eq,SL}+c_{eq,SL}i\omega+(i\omega)^{\beta}c\right|^2}d\omega, \tag{35}$$

where, in the specific case: $S_w(\omega)=S_0$. It can be argued, once more, from Eq. (35) that the fractional derivative term induces in the system dynamics both additional damping and stiffness terms, frequency-dependent, through the term $(i\omega)^{\beta}$. Specifically, to account approximately for the fractional term, the equation

$$\omega(a)\cong\sqrt{\frac{\hat{k}_{eq,SL}}{m}}=\sqrt{\frac{k_{eq,SL}+\mathrm{Re}[(i\omega)^{\beta}c]}{m}} \tag{36}$$

is used for estimating the values of Eqs. (27) and (28), where $\hat{k}_{eq,SL}$ is the modified stationary equivalent stiffness, due to the fractional term.

2.5. Non-stationary response statistics

Next, by direct manipulation of the equation of motion, expressions for the non-stationary variance of the response displacement and velocity, and covariance between the two, are sought. This can be achieved by solving the associated oscillator response covariance Lyapunov equation [7]. Specifically, Eq. (32) is written in the state variables form

$$\dot{\mathbf{z}}=\mathbf{G}\mathbf{z}+\mathbf{f}, \tag{37}$$

where

$$\mathbf{z}=\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix},\mathbf{f}=\begin{bmatrix}0\\ \frac{w(t)}{m}\end{bmatrix},\text{ and }\quad \mathbf{G}=\begin{bmatrix}0 & 1\\ -\frac{k_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)}{m} & -\frac{c_{eq}(\sigma_x^2,\sigma_{\dot{x}}^2)}{m}\end{bmatrix}. \tag{38}$$

In Eq. (38) the terms $c_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)$ and $k_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)$ are the equivalent damping, and the equivalent stiffness defined in Eqs. (33) and (34), in the pursuit to linearize the equation of motion by time-averaging the system response. They depend on time through their dependence on the system response statistics. Then the evolution of the covariance matrix $\mathbf{V}$ of the oscillator, that is,

$$\mathbf{V}=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}\sigma_x^2 & \sigma_{x\dot{x}}\\ \sigma_{\dot{x}x} & \sigma_{\dot{x}}^2\end{bmatrix} \tag{39}$$

is governed by the differential covariance Lyapunov equation

$$\dot{\mathbf{V}}=\mathbf{G}\mathbf{V}^{\mathbf{T}}+\mathbf{V}\mathbf{G}^{\mathbf{T}}+\mathbf{D};\qquad \mathbf{D}=\begin{bmatrix}0 & 0\\ 0 & \frac{2\pi S_0}{m^2}\end{bmatrix}. \tag{40}$$

This is claimed with the provision that $w(t)$ is a zero-mean white noise of spectral level S_0, and correlation function $2\pi S_0\delta(\tau)$, with $\delta(\tau)$ the Dirac delta function. Note that for the random Gaussian excitation $w(t)/m$ the correlation function is $2\pi S_0\delta(\tau)/m^2$. Through Eq. (40) the non-stationary statistical moments of the system response are determined, by setting appropriate equivalent linear quantities for the damping and stiffness of the original nonlinear oscillator endowed with a fractional term.

Note that the linearizing terms $c_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)$ and $k_{eq}\left(\sigma_x^2,\sigma_{\dot{x}}^2\right)$ both depend on the variances of the displacement and of the velocity, in general. Thus, Eq. (40) constitutes a system of nonlinear coupled differential equations.

Further, note that when restricting the analysis to the stationary regime, and to zero nonlinearity, and null fractional term, the variances of the stationary response displacement and velocity to white noise excitation obtained by Eq. (40) are given by the diagonal components of $\mathbf{V}$. That is,

$$v_{11}=\sigma^2_{x0}=\frac{\pi S_0}{2\zeta\omega_0^3m^2};\text{and}\qquad v_{22}=\sigma^2_{\dot{x}0}=\frac{\pi S_0}{2\zeta\omega_0 m^2}. \tag{41}$$

It is understood that information on the auto-correlation of the fractional term, and on the cross-correlations between the integer and the fractional state variables, is not directly retrievable by using the Lyapunov equation. Further, it is worth noting that Eq. (40) can handle arbitrary forms of the excitations, even non-white, by making use of pre-filtering [7].

At this point, it is perhaps worthwhile to juxtapose the herein proposed technique versus the time-dependent formulation proposed in [56], as a generalization to the results reported in a former paper [11]. Specifically, the formulation in [56] applies HB both to the fractional, and to the non-fractional nonlinear term in the problem. Then, it utilizes the solution of the FPK partial differential equation associated with the probability density function of the amplitude response. Specifically, the surrogate system with time-dependent quantities

$$m\ddot{x}(t)+c_s(t)\dot{x}(t)+k_s(t)x(t)=w(t), \tag{42}$$

is used, with [56]

$$\begin{aligned}c_s(t)&=\int_0^{\infty}\left[\frac{S(a)}{a\omega(a)}-c\frac{S_{\beta}(a)}{\pi a\omega(a)}\right]p(a,t)\,da\\ &=\int_0^{\infty}\frac{S(a)}{a\omega(a)}p(a,t)\,da+c\sin\left(\frac{\beta\pi}{2}\right)\int_0^{\infty}\frac{1}{\omega^{1-\beta}(a)}p(a,t)\,dA,\end{aligned} \tag{43}$$

and

$$\begin{aligned}k_s(t)&=\int_0^{\infty}\left[\frac{F(a)}{a}+c\frac{F_{\beta}(a)}{\pi a}\right]p(a,t)\,da\\ &=\int_0^{\infty}\frac{F(a)}{a}p(a,t)\,da+c\cos\left(\frac{\beta\pi}{2}\right)\int_0^{\infty}\omega^{\beta}(a)\,p(a,t)\,dA,\end{aligned} \tag{44}$$

with $S_{\beta}(a)$ and $F_{\beta}(a)$ in Eqs. (43) and (44) taking the expressions of Eqs. (23) and (24), and $S(a)$, $F(a)$ as defined in [7].

It is pointed out that expressions in Eqs. (43) and (44) require the knowledge of the variance of the system at each time step via the probability $p(a,t)$. In this context, a first order stochastic differential equation for the response amplitude a is derived under Gaussian noise excitation [66]. The amplitude probability density is also represented by a Rayleigh distribution with time-dependent parameters

$$p(a,t)=\frac{a}{\sigma_x^2(t)}e^{-\frac{a^2}{2\sigma_x^2(t)}}, \tag{45}$$

where the response standard deviation satisfies the differential equation [56]

$$\dot{\sigma}_x^2(t)=-c_s\left(\sigma_x^2(t)\right)\sigma_x^2(t)+\frac{\pi S\left[k_s\left(\sigma_x^2(t)\right),t\right]}{k_s(\sigma_x^2(t))}. \tag{46}$$

Again, it is seen that Eqs. (43) and (44) contain $\omega(a)$, whose dependence on amplitude is captured by Eq. (44), since $\sqrt{\frac{k_s(a)}{m}}=\omega(a)$. Further, for the computations of the expressions in Eqs. (43) and (44), the expected value of $\omega(a)$ is used. Clearly, compared to Eq. (36) of the herein proposed technique, the procedure adopted in [56] introduces the additional complexity and computational cost in determining the effective natural frequency in time step when integrating Eq. (46).

2.6. Fractional term correlation statistics

Revisiting expressions in Eqs. (25) and (26), derived from an approximation of the Caputo derivative [22,23,67], and following the developments of [43,44], the fractional term can be conveniently expressed as

$$D_{0,t}^{\beta} x(t) = \omega^{\beta-1}(a)\left[\dot{x}(t)\sin\left(\frac{\beta\pi}{2}\right) + x(t)\,\omega(a)\cos\left(\frac{\beta\pi}{2}\right)\right] + \frac{\omega^{\beta-1}(a)}{\Gamma(1-\beta)}\frac{\dot{x}(t)\sin[\omega(a)t] - x(t)\,\omega(a)\cos[\omega(a)t]}{[\omega(a)t]^{\beta}} + o\,[\omega(a)t]^{-\beta-1}, \tag{47}$$

in this manner simplifying the calculations of expressions in Eqs. (23) and (24). Notably, the fractional derivative term can be construed as a combination of the system displacement and velocity. This corroborates the argument that, in effect, the contribution of the fractional derivative term to the dynamic system is the introduction of time-dependent additional damping, and stiffness. It does not itself reflect memory of the past events experienced by the system. This must, rather, be attributed implicitly to the system amplitude quantity.

By considering the analytical representation for the fractional term of Eq. (47), approximated to $o\,[\omega(a)t]^{-\beta-1}$, and after neglecting this last term, the fractional term can be recast as a combination of the system displacement and velocity. That is,

$$D_{0,t}^{\beta} x(t) = C(a,t)x(t) + D(a,t)\dot{x}(t), \tag{48}$$

where $C(a,t)$ and $D(a,t)$ are the time-dependent trigonometric expressions

$$C(a,t) = \omega^{\beta-1}(a)\left[\omega(a)\cos\left(\frac{\beta\pi}{2}\right)\right] - \frac{\omega^{\beta-1}(a)}{\Gamma(1-\beta)}\frac{\omega(a)\cos[\omega(a)t]}{[\omega(a)t]^{\beta}}, \tag{49}$$

and

$$D(a,t) = \omega^{\beta-1}(a)\sin\left(\frac{\beta\pi}{2}\right) + \frac{\omega^{\beta-1}(a)}{\Gamma(1-\beta)}\frac{\sin[\omega(a)t]}{[\omega(a)t]^{\beta}}. \tag{50}$$

These equations lead to

$$D_{0,t}^{\beta} x(t)^2 = C^2(a,t)x^2(t) + D^2(a,t)\dot{x}^2(t) + 2C(a,t)D(a,t)x(t)\dot{x}(t). \tag{51}$$

Thus, the auto-correlation of the fractional term $\sigma^2_{D_{0,t}^{\beta}x}$ is computed as a function of the variances of the displacement and the velocity, σ_x^2 and $\sigma_{\dot{x}}^2$, respectively, and of the cross-correlation between the two, $\sigma_{x\dot{x}}$, that are the components of $\mathbf{V}$ in Eq. (39). Specifically,

$$\sigma^2_{D_{0,t}^{\beta}x} = \langle D_{0,t}^{\beta}x(t)^2\rangle = \langle C^2(a,t)\rangle\sigma_x^2 + \langle D^2(a,t)\rangle\sigma_{\dot{x}}^2 + 2\langle C(a,t)D(a,t)\rangle\sigma_{x\dot{x}}. \tag{52}$$

In a similar manner, the two cross-correlation terms, $\sigma_{xD_{0,t}^{\beta}x}$ and $\sigma_{\dot{x}D_{0,t}^{\beta}x}$, can be computed as

$$\sigma_{xD_{0,t}^{\beta}x} = \langle D_{0,t}^{\beta}x(t)\,x(t)\rangle = \langle C(a,t)\rangle\sigma_x^2 + \langle C(a,t)D(a,t)\rangle\sigma_{x\dot{x}}, \tag{53}$$

and

$$\sigma_{\dot{x}D_{0,t}^{\beta}x} = \langle D_{0,t}^{\beta}x(t)\,\dot{x}(t)\rangle = \langle C(a,t)\,D(a,t)\rangle\sigma_{x\dot{x}} + \langle D(a,t)\rangle\sigma_{\dot{x}}^2. \tag{54}$$

Clearly, Eqs. (52)–(54) point out the statistical independence between the response amplitude and the displacement. It is noted that other schemes [48] that require the solution of an augmented transient problem of the kind of Eq. (40) due to the presence of the fractional derivative term can be used. Nevertheless, the cross-correlations between the fractional term and the displacement or the velocity, as well as the auto-correlation of the fractional derivative term, can only be determined indirectly, from the main statistics obtained by solving the same problem of Eq. (40).

3. Numerical results

Without lack of generality, for the purpose of assessing the reliability of the proposed linearization scheme, a Duffing oscillator with a fractional element is considered. The restoring function in this case is represented by the equation

$$f(t,x,\dot{x}) = kx(1+\phi x^2), \tag{55}$$

where the constant ϕ captures the strength of the nonlinearity. Further, the oscillator is described by the equation

$$m\ddot{x}(t) + c\,D_{0,t}^{\beta}x(t) + kx(1+\phi x^2) = w(t), \tag{56}$$

with $x(0) = \dot{x}(0) = 0$ as initial conditions.

Classical SL yields

$$c_{eq,SL}\left(\sigma_x^2, \sigma_{\dot{x}}^2\right) = 0; \quad \text{and} \quad k_{eq,SL}\left(\sigma_x^2, \sigma_{\dot{x}}^2\right) = k + 3\phi k\sigma_x^2. \tag{57}$$

Further, from Eqs. (33)–(35) one derives

$$c_{eq}\left(\sigma_x^2\right) = c_{eq,\beta}\left(\sigma_x^2\right), \tag{58}$$

$$k_{eq}\left(\sigma_x^2\right) = k + 3\phi k\sigma_x^2 + k_{eq,\beta}\left(\sigma_x^2\right), \tag{59}$$

and

$$\omega(a) \cong \sqrt{\frac{\hat{k}_{eq,SL}}{m}} = \sqrt{\frac{k + 3\phi k\sigma_x^2 + \mathrm{Re}[(i\omega)^{\beta}c]}{m}}. \tag{60}$$

These expressions are substituted into Eq. (38) to derive from Eq. (40) the non-stationary evolution of the variances of the system response using the proposed technique.

MC simulations (500 in number) corresponding to various values of the order of the fractional derivative have been conducted. Further, various values for the strength of the nonlinearity ϕ have been considered. The oscillator has $m = 1$, and $k = 4$. The c coefficient in Eq. (56) is set to a critical damping value of $\zeta = 5\%$, that is: $c = 2m\zeta\omega_0 = 0.2$. A white noise process of unit spectral level ($S_0 = 1$) has been considered. The time integration step has been selected equal to 0.01. The GL coefficients have been retained up to the order of 10^{-5}. The reliability of the Newmark algorithm in solving the equation of motion has been examined by juxtaposing results obtained via the algorithm proposed by Katsikadelis [68]. A comparison between two time-histories obtained via the two time-integration schemes, for $\beta = 0.7$, and $\phi = 10/\sigma^2_{x0}$ is shown in Fig. 1.

In Figs. 2 to 4 results for a system characterized by $\beta = 0.5$, and $\phi = 10/\sigma^2_{x0}$ are shown. Specifically, in Figs. 2 and 3 the non-stationary results are presented for the variance of the response displacement and velocity, respectively, normalized with respect to the corresponding theoretical variance of the stationary response of the linear part of the system, computed for $\beta = 1$. They are referred to as σ^2_{x0} and $\sigma^2_{\dot{x}0}$, respectively, henceforth. In Fig. 2 the solution obtained by the proposed technique is compared to the one obtained using the formulation provided in [56]. Both are found in a good agreement with the MC simulations. Nevertheless, a slight difference in the non-stationary behavior is observed by using the two approaches. Interestingly, the proposed technique yields at the stationary regime a solution which coincides with the solution derived by using the classical SL technique (Eq. (35)), as proposed in [41]. This is evident for all of the examined orders of β; only partial results are reported herein, for succinctness. It is observed from Fig. 3 that the proposed technique yields satisfactory results also in terms of the velocity variance, in juxtaposition with the corresponding MC solution.

In Fig. 4 the non-stationary statistics involving the fractional derivative term is shown. Specifically, the auto-correlation of the fractional derivative term, $\sigma^2_{D_{0,t}^{\beta}x}$, and the cross-correlations between the latter and displacement, or velocity, $\sigma_{xD_{0,t}^{\beta}x}$ or $\sigma_{\dot{x}D_{0,t}^{\beta}x}$, respectively, are shown. Results obtained as discussed in Section 2.6 are juxtaposed with

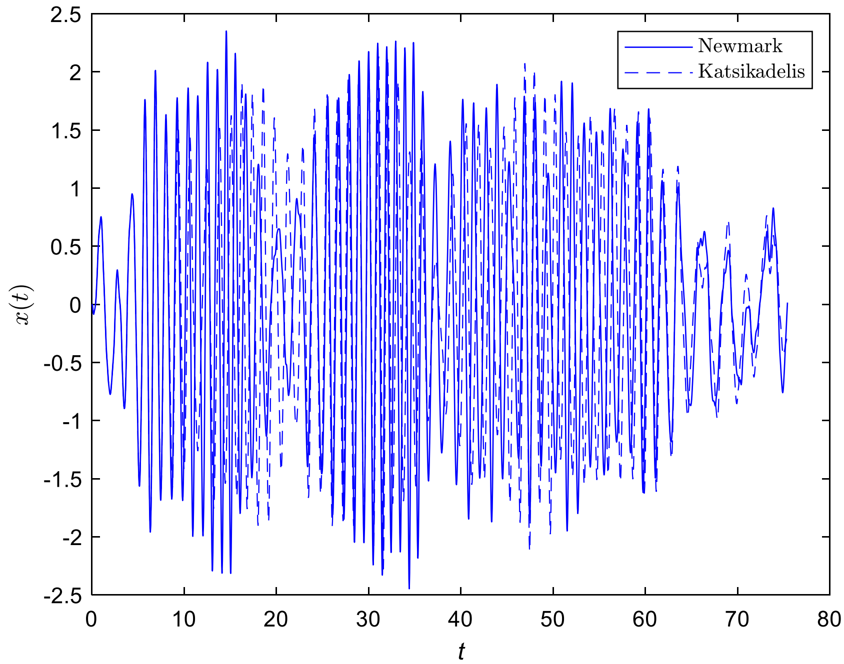

Fig. 1. Oscillator with Duffing nonlinearity $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$, and $\beta = 0.7$, excited by white noise of unit spectral density; comparison of time-histories obtained by time-integrating the equation of motion via Newmark algorithm, and by using the algorithm in Ref. [68].

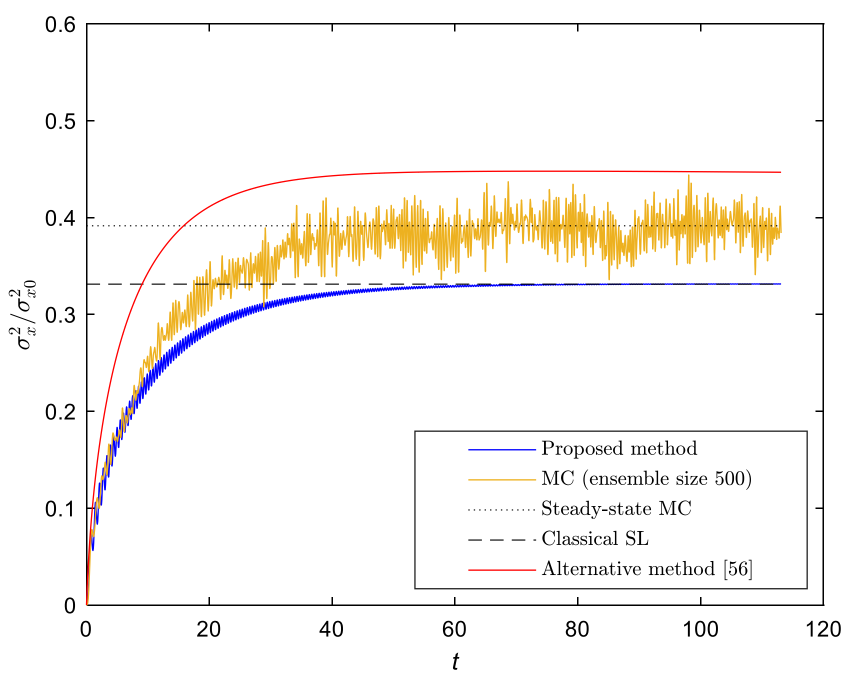

Fig. 2. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 0.5$ and white noise excitation of unit spectral density; comparison involving various techniques for estimating the non-stationary response displacement variance.

those obtained via MC analyses, showing a good agreement. Notably, these quantities capture a persistent non-zero correlation between the fractional derivative and the system displacement and velocity. Nevertheless, the response displacement and integer velocity become zero correlated, as expected, in the stationary regime, as captured by the black solid line in the same figure.

An estimate of the expected value of the response amplitude is shown in Fig. 5, and compared with MC results. Specifically, the expected value of the amplitude has been estimated as a function of the numerical values of $x(t)$ and $\dot{x}(t)$ by using Eq. (19). In doing this, two approaches are compared. They differ in terms of the specific estimate of $\omega(a)$: in one case $\omega(a)$ is approximated as proposed in

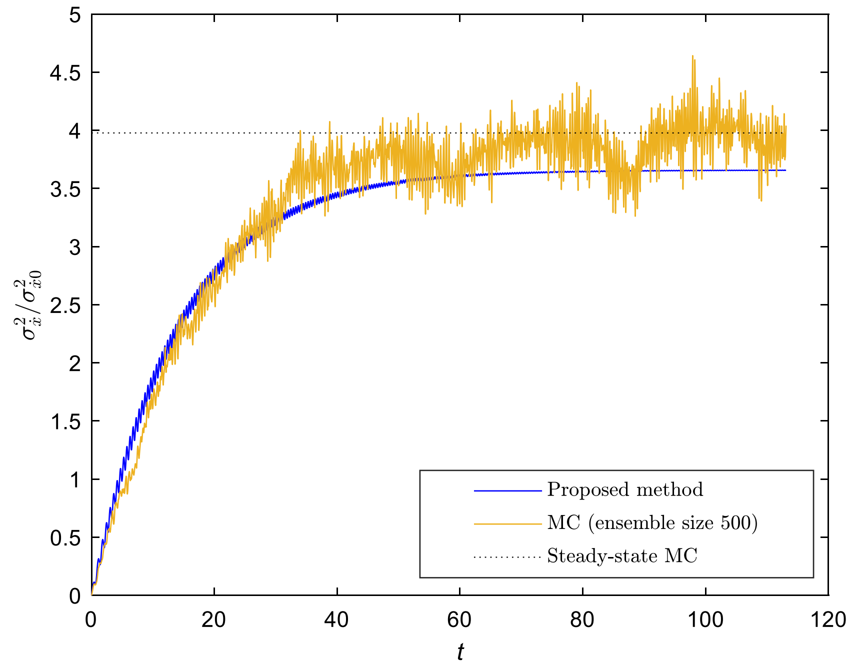

Fig. 3. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1$; $k = 4$; $c = 0.2$; $\beta = 0.5$ and white noise excitation of unit spectral density; non-stationary response velocity variance.

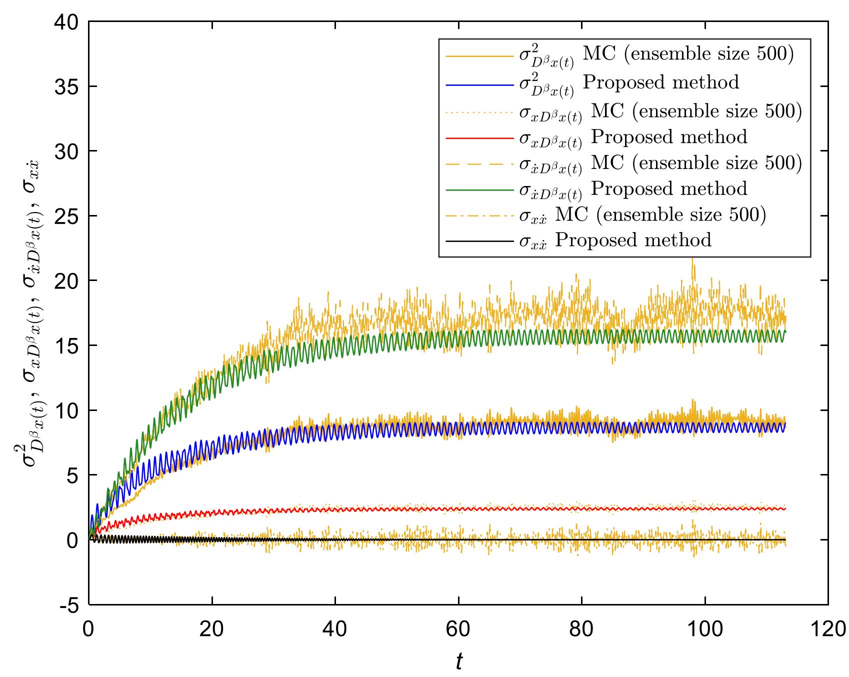

Fig. 4. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1$; $k = 4$; $c = 0.2$; $\beta = 0.5$ and white noise excitation of unit spectral density; comparison between MC results and results obtained by the proposed technique for estimating the non-stationary statistics involving the fractional derivative term, and the cross-correlation between response displacement and velocity.

Eq. (36), starting from the stationary regime effective stiffness derived from the classical SL technique; in the second case $\omega(a)$ is approximated by its expected value. The numerical results from MC simulations are compared with the analytic expression of the expected value of the response amplitude. That is, by means of the averaging expression

$$\langle a(t)\rangle = \int_0^\infty ap(a)\,da. \tag{61}$$

The probability density in Eq. (61) is taken as defined by Eq. (29) in the case of the proposed technique, or from the expression in Eq. (45) as proposed in [56], for comparison. If the expected value of the amplitude is computed analytically, the dependence in time of the amplitude is reflected by the time-dependence of σ^2_x. This quantity is obtained either by solving the system of Eqs. (40) in the case of the proposed technique, or by solving Eq. (46).

In Figs. 6 to 9 similar results are shown for $\beta = 1$. In particular, for this value of β a comparison of the non-stationary results obtained by the classical SL technique is done (see dashed black line in Figs. 6 and 7). Further, note that for unit β the variance of the fractional term, and

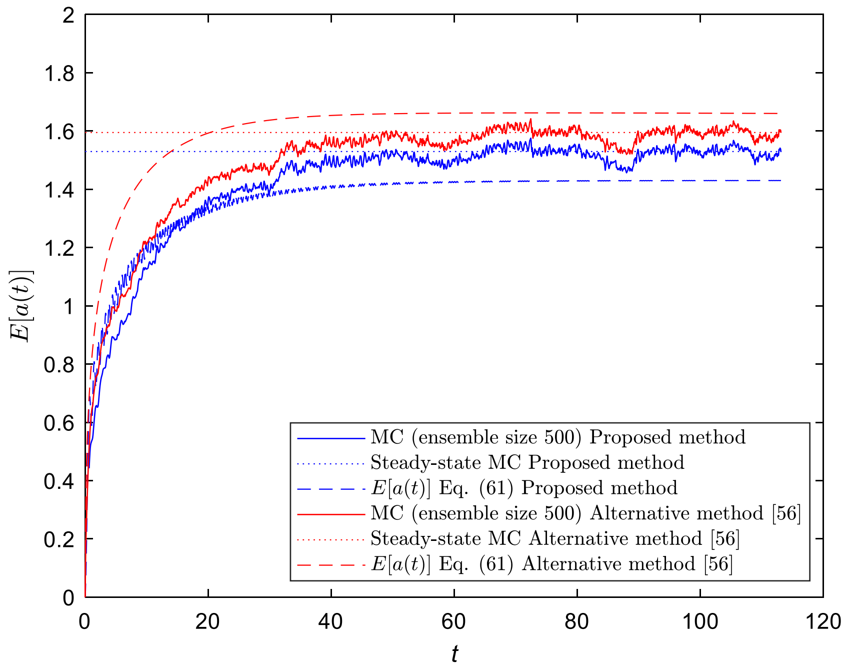

Fig. 5. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 0.5$ and white noise excitation of unit spectral density; non-stationary expected value of the response amplitude.

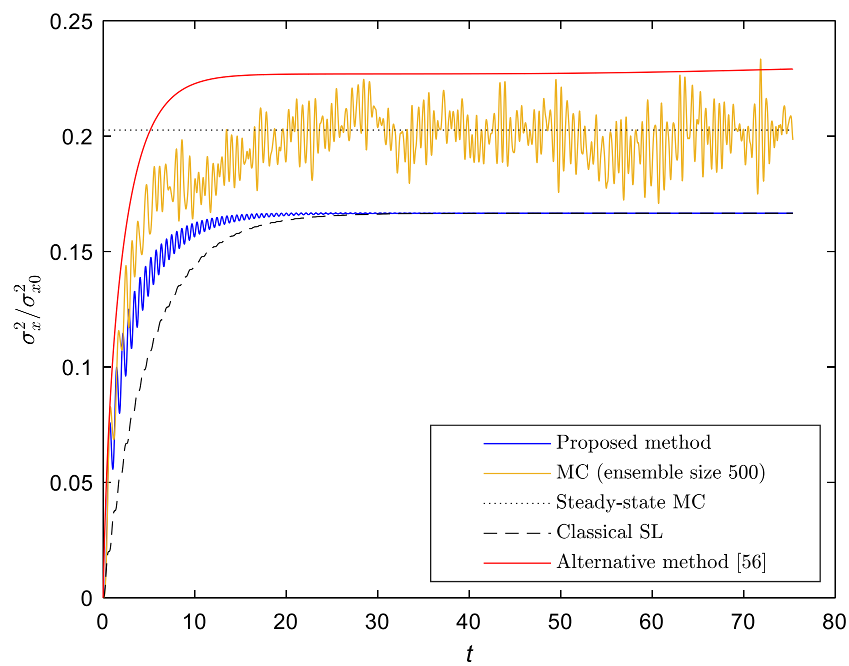

Fig. 6. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 1.0$ and white noise excitation of unit spectral density; comparison involving various techniques for estimating the non-stationary response displacement variance.

the cross-correlation between the velocity and the fractional term coincide. Similarly, the cross-correlation term between the displacement and the fractional term coincides with the cross-correlation between the displacement and the velocity, leading to null correlation between the two at the stationary regime, as theoretically expected; see Fig. 8.

Further, in Fig. 9 the expected value of the response amplitude is shown versus time for a system involving unit value of β. It is seen that the MC results yield identical solutions using either of the approaches. With regards to the analytic estimate, it is seen that the proposed technique tends to underestimate the MC results, while the averaging method proposed in [56] tends to slightly overestimate them.

For a value of the order of the fractional derivative term $\beta = 1.3$, and with caution on the range of validity of the approximation in Eq. (47), similar results are shown in Figs. 10 to 13.

Finally, in Fig. 14 the stationary value of the response displacement variance, normalized with respect to the theoretical variance of the stationary response of the linear part of the system corresponding to $\beta = 1$, σ_{x0}^2, is shown versus the degree of the nonlinearity for a

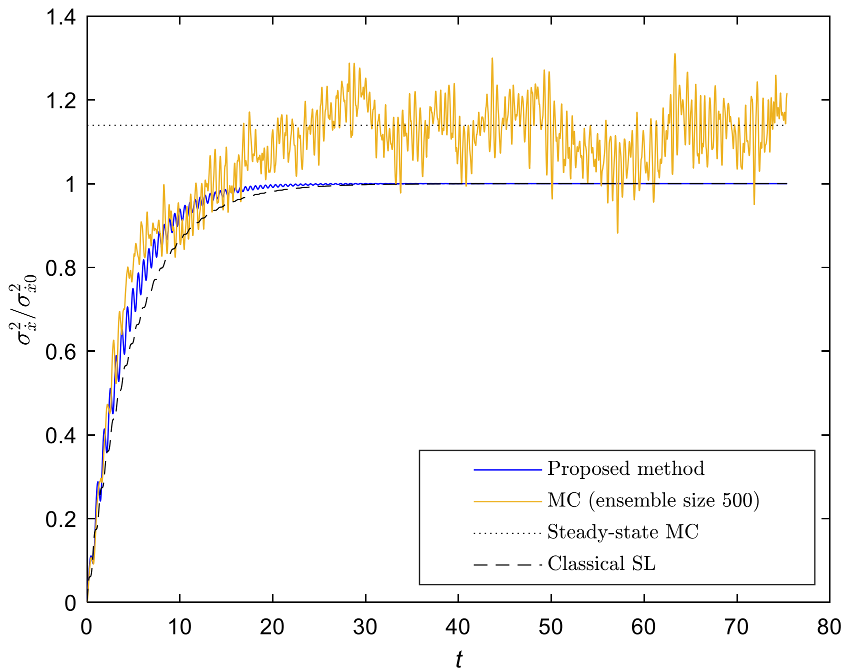

Fig. 7. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1; k = 4; c = 0.2$; $\beta = 1.0$ and white noise excitation of unit spectral density; non-stationary response velocity variance.

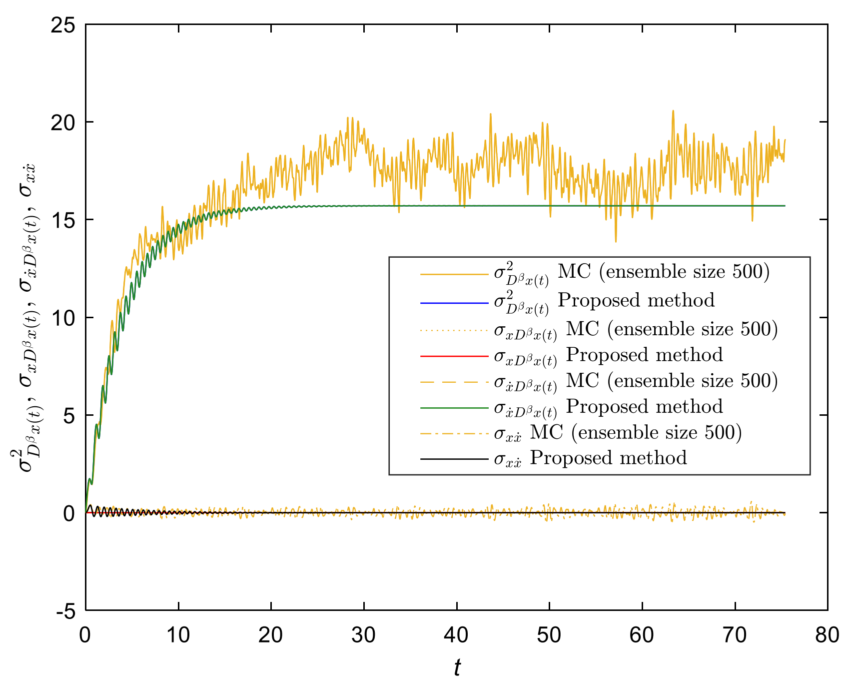

Fig. 8. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1; k = 4; c = 0.2$; $\beta = 1.0$ and white noise excitation of unit spectral density; comparison between MC results and results obtained by the proposed technique for estimating the non-stationary statistics involving the fractional derivative term.

fixed fractional order $\beta = 0.7$, just a representative value, where the nonlinearity factor $\rho = \phi\sigma^2_{x0}$ is introduced. It is noted that the degree of reliability in estimating the stationary regime response displacement variance by the proposed technique is almost invariant versus the degree of the nonlinearity, even for strong nonlinearity levels.

4. Concluding remarks

In this paper a generalization of the Statistical Linearization technique for determining the non-stationary response statistics of nonlinear oscillators endowed with fractional elements, subject to white

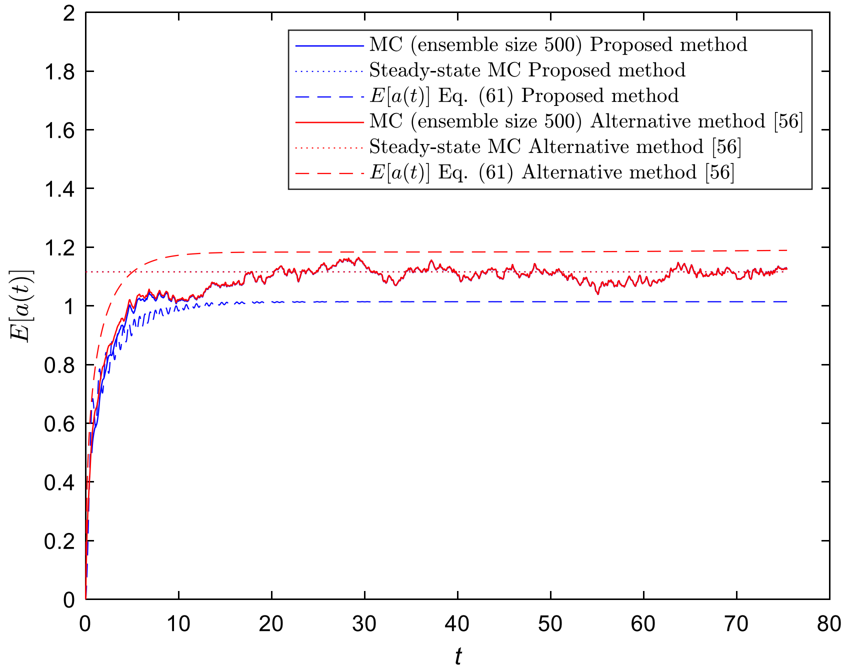

Fig. 9. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 1.0$ and white noise excitation of unit spectral density; non-stationary expected value of the response amplitude.

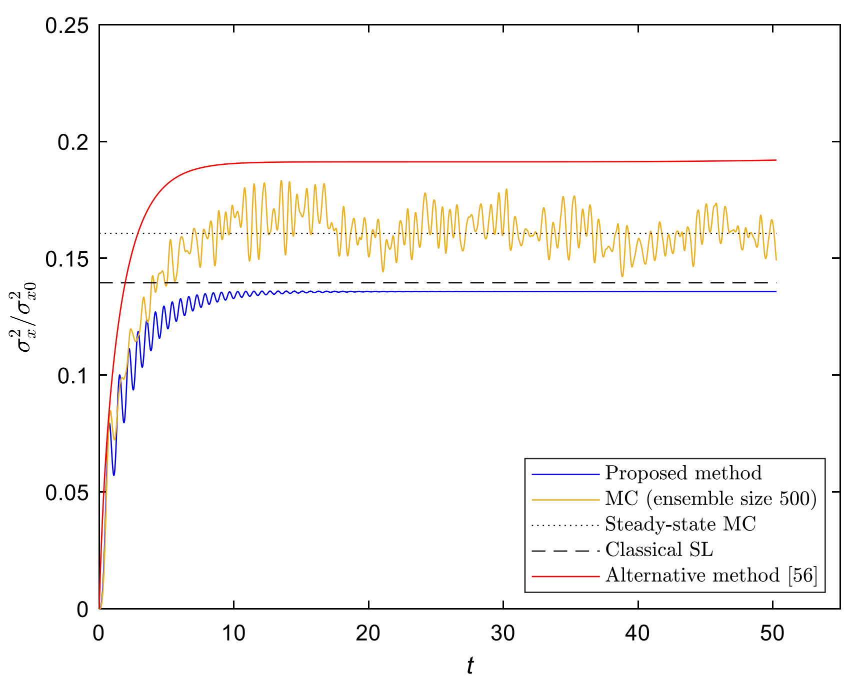

Fig. 10. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 1.3$ and white noise excitation of unit spectral density; comparison involving various techniques for estimating the non-stationary response displacement variance.

noise excitation, has been developed. The technique resorts to a pre-treatment of the system response by the Harmonic Balancing technique, limited to the fractional term, in deriving response-amplitude dependent equivalent damping and stiffness. Expectations of these values are next derived by averaging them through the probability density function of the response amplitude, for which a Rayleigh distribution with time-dependent parameters is used. In this manner, the expected values of the equivalent quantities are determined as functions of the variance of the system response itself. Further, the covariance Lyapunov equation associated with the derived equivalent system is solved

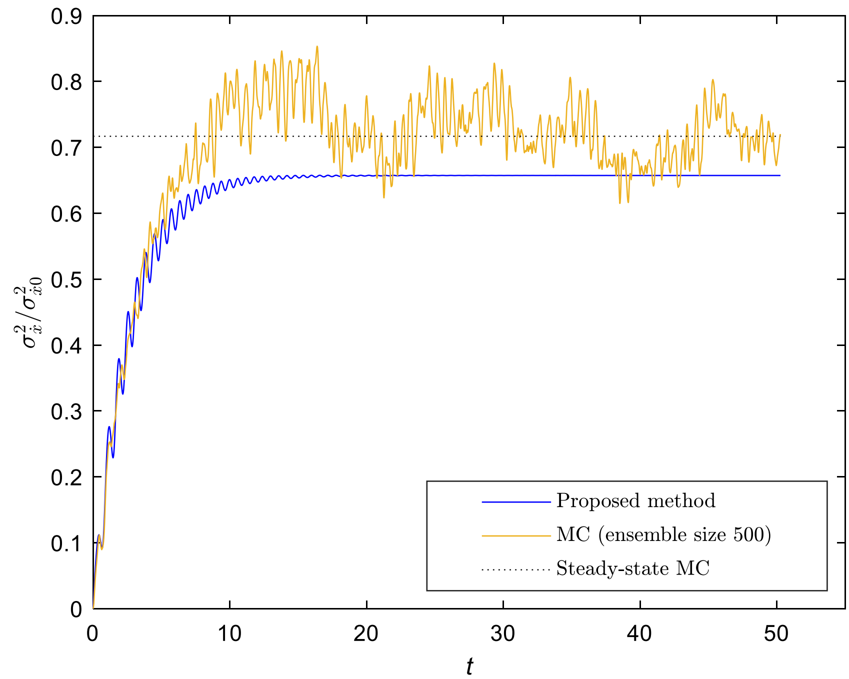

Fig. 11. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1; k = 4; c = 0.2$; $\beta = 1.3$ and white noise excitation of unit spectral density; non-stationary response velocity variance.

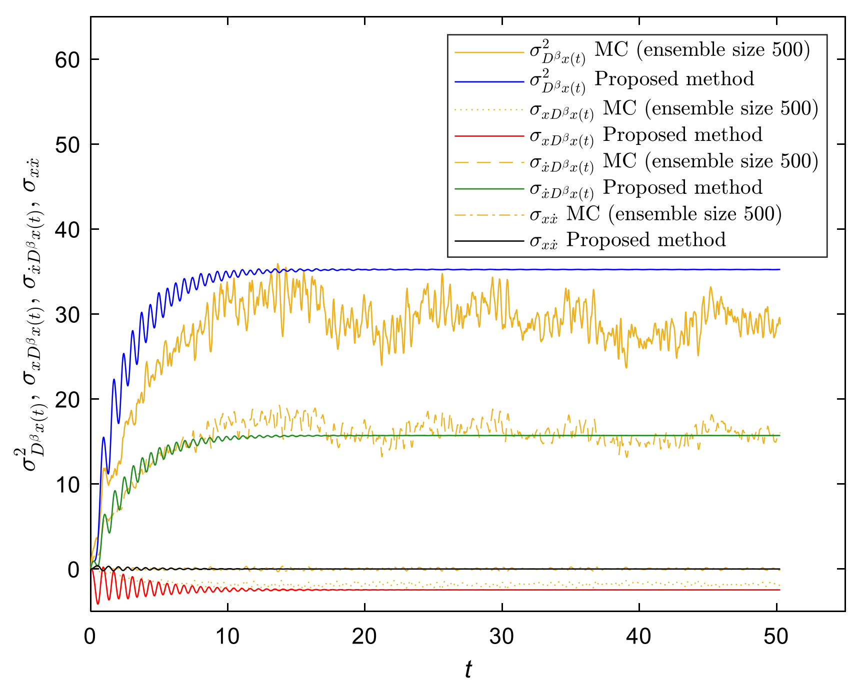

Fig. 12. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma^2_{x0}$; $m = 1; k = 4; c = 0.2$; $\beta = 1.3$ and white noise excitation of unit spectral density; comparison between MC results and results obtained by the proposed technique for estimating the non-stationary statistics involving the fractional derivative term.

numerically, to derive non-stationary estimates of the auto-correlations, and cross-correlation functions of the response displacement, and velocity. These are also used to determine a posteriori similar expressions regarding the fractional term itself and the integer-order state variables of the oscillator.

The attractiveness of the developed technique relates to its applicability to a broad class of nonlinear systems, and stochastic excitations, as well as to the possibility of deriving important oscillator response statistics at a low computational cost. It is pointed out that the versatility of the approach is contingent upon the assumption of

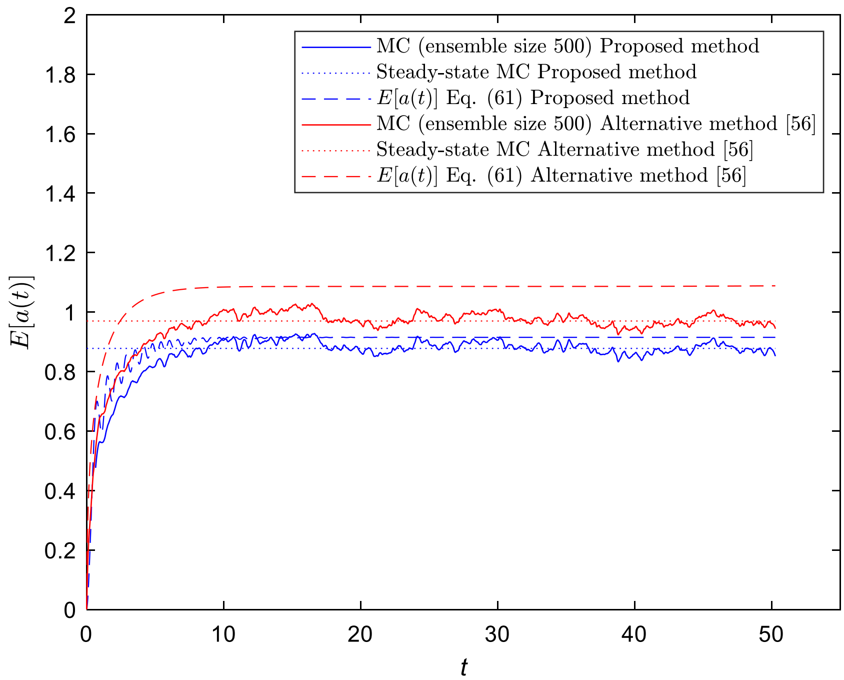

Fig. 13. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 1.3$ and white noise excitation of unit spectral density; non-stationary expected value of the response amplitude.

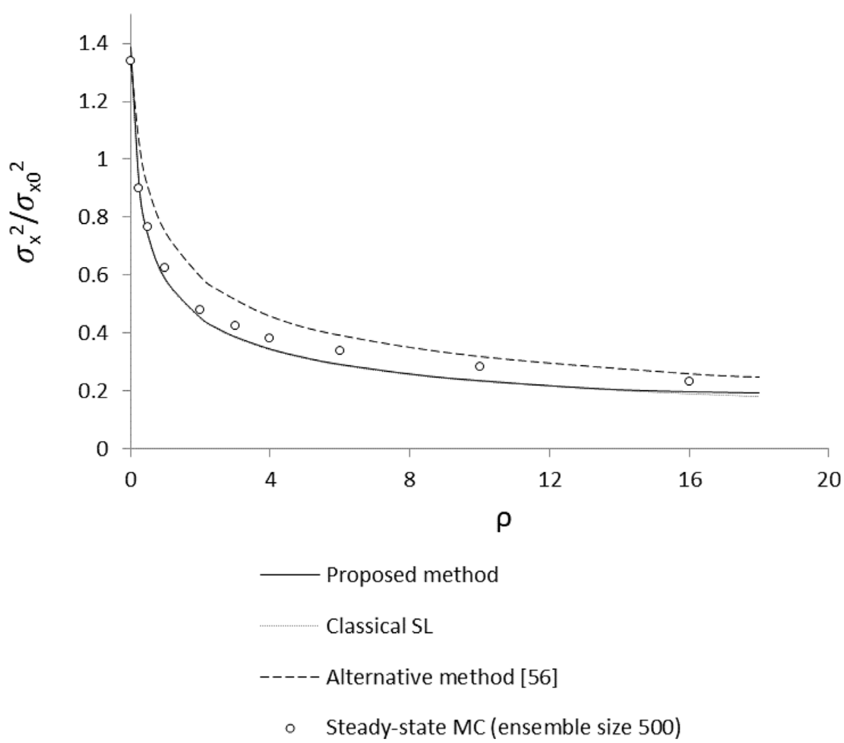

Fig. 14. Oscillator with Duffing nonlinearity: $\phi = 10/\sigma_{x0}^2$; $m = 1; k = 4; c = 0.2$; $\beta = 0.7$ and white noise excitation of unit spectral density; comparison involving various techniques for estimating the stationary response displacement variance versus the nonlinearity factor $\rho = \phi\sigma_{x0}^2$.

a pseudo-harmonic response process, and within the range of validity of the Harmonic Balancing approximation of the fractional term. The juxtaposition of results obtained by the developed technique and by pertinent Monte Carlo simulations has demonstrated its accuracy and reliability for a broad range of the order of the fractional derivative, and even for strongly nonlinear oscillators.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request

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