Research Article

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On Aharonov–Bohm Operators with Two Colliding Poles

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Abstract: We consider Aharonov–Bohm operators with two poles and prove sharp asymptotics for simple eigenvalues as the poles collapse at an interior point out of nodal lines of the limit eigenfunction.

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1 Introduction

The present paper is concerned with asymptotic estimates of the eigenvalue variation for magnetic Schrödinger operators with Aharonov–Bohm potentials. These special potentials generate localized magnetic fields, as they are produced by infinitely long thin solenoids intersecting perpendicularly the plane at fixed points (poles), as the radius of the solenoids goes to zero and the magnetic flux remains constant.

The aim of the present paper is the investigation of eigenvalues of these operators as functions of the poles on the domain. This study was initiated by the set of papers [1, 2, 4, 10, 19], where a single point moving in the domain was considered, providing sharp asymptotics as it goes to an interior point or to a boundary point. On the other hand, to the best of our knowledge, the only paper considering different poles is [18], providing a continuity result for the eigenvalues and an improved regularity for simple eigenvalues as the poles are distinct and far from the boundary.

Additional motivations for the study of eigenvalue functions of these operators appear in the theory of spectral minimal partitions. We refer the interested reader to [7, 9, 14, 20] and references therein.

For $a = (a_1, a_2) \in \mathbb{R}^2$, the Aharonov–Bohm magnetic potential with pole a and circulation $\frac{1}{2}$ is defined as

$$\mathbf{A}_a(x) = \frac{1}{2}\Big(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\Big), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\}.$$

In this paper we consider potentials which are the sum of two different Aharonov–Bohm potentials whose singularities are located at two different points in the domain moving towards each other. For $a > 0$ small, let $a^- = (-a, 0)$ and $a^+ = (a, 0)$ be the poles of the following Aharonov–Bohm potential:

$$\mathbf{A}_{a^-,a^+}(x) := -\mathbf{A}_{a^-}(x) + \mathbf{A}_{a^+}(x) = -\frac{1}{2}\frac{(-x_2, x_1 + a)}{(x_1 + a)^2 + x_2^2} + \frac{1}{2}\frac{(-x_2, x_1 - a)}{(x_1 - a)^2 + x_2^2}. \tag{1.1}$$

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Let Ω be an open, bounded and connected set in $\mathbb{R}^2$ such that $0 \in \Omega$. We consider the Schrödinger operator

$$H^{\Omega}_{a^-,a^+} = (i\nabla + \mathbf{A}_{a^-,a^+})^2 \tag{1.2}$$

with homogeneous Dirichlet boundary conditions (see Section 3.1 for the notion of magnetic Hamiltonians) and its eigenvalues $(\lambda_k^a)_{k\geq 1}$, counted with multiplicities. We denote by $(\lambda_k)_{k\geq 1}$ the eigenvalues of the Dirichlet Laplacian $-\Delta$ in Ω. As already mentioned, we know from [18] that for every $k \geq 1$,

$$\lim_{a\to 0} \lambda_k^a = \lambda_k. \tag{1.3}$$

The main result of the present paper is a sharp asymptotic for the eigenvalue variation $\lambda_k^a - \lambda_k$ as the two poles a^- and a^+ coalesce towards a point where the limit eigenfunction does not vanish.

A first result in this direction was given in [3], under a symmetry assumption on the domain.

Theorem 1.1 ([3, Theorem 1.13]). *Let $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(x_1, x_2) = (x_1, -x_2)$. Let Ω be an open, bounded and connected set in $\mathbb{R}^2$ satisfying $\sigma(\Omega) = \Omega$ and $0 \in \Omega$. Let λ_N be a simple eigenvalue of the Dirichlet Laplacian on Ω, and let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to λ_N. Let $k \in \mathbb{N} \cup \{0\}$ be the order of vanishing of u_N at 0, and let $\alpha \in [0, \pi)$ be such that the minimal slope of nodal lines of u_N is equal to $\frac{\alpha}{k}$, so that*

$$u_N(r(\cos t, \sin t)) \sim r^k \beta \sin(\alpha - kt) \quad \text{as } r \to 0^+ \text{ for all } t,$$

for some $\beta \in \mathbb{R} \setminus \{0\}$ (see, e.g., [12]). Let us assume that $\alpha \neq 0$ (by symmetry of Ω, this forces $\alpha = \frac{\pi}{2}$, i.e, the x_1-axis is the bisector of two nodal lines of u_N).

For $a > 0$ small, let $a^- = (-a, 0)$, $a^+ = (a, 0) \in \Omega$, and let λ_N^a be the N-th eigenvalue for $(i\nabla + \mathbf{A}_{a^-,a^+})^2$. Then

$$\lambda_N^a - \lambda_N = \begin{cases} \dfrac{2\pi}{|\log a|}|u_N(0)|^2(1 + o(1)) & \text{if } k = 0,\\ C_k \pi \beta^2 a^{2k}(1 + o(1)) & \text{if } k \geq 1 \end{cases}$$

as $a \to 0^+$, with $C_k > 0$ being a positive constant depending only on k.

In the present paper we are able to remove, in the case $k = 0$ (i.e., when the limit eigenfunction u_N does not vanish at the collision point), the assumption on the symmetry of the domain, proving the following result.

Theorem 1.2 ([3, Theorem 1.17]). *Let Ω be an open, bounded and connected set in $\mathbb{R}^2$ such that $0 \in \Omega$. Let us assume that there exists $N \geq 1$ such that the N-th eigenvalue λ_N of the Dirichlet Laplacian in Ω is simple. Let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to λ_N. If $u_N(0) \neq 0$, then*

$$\lambda_N^a - \lambda_N = \frac{2\pi u_N^2(0)}{|\log a|}(1 + o(1)) \quad \text{as } a \to 0^+.$$

It is worthwhile mentioning that in [18] simple magnetic eigenvalues are proved to be analytic functions of the configuration of the poles, provided the limit configuration is made of interior distinct poles. A consequence of our result is that the latter assumption is even necessary, and simple eigenvalues are not analytic in a neighborhood of configurations of poles collapsing outside nodal lines of the limit eigenfunction.

The proof of Theorem 1.2 relies essentially on the characterization of the magnetic eigenvalue as an eigenvalue of the Dirichlet Laplacian in Ω with a small set removed, in the flavor of [3] (see Section 3.2 below). In [3] only the case of symmetric domains was considered and the magnetic problem was shown to be spectrally equivalent to the eigenvalue problem for the Dirichlet Laplacian in the domain obtained by removing the segment joining the poles. In the general non-symmetric case, we can still derive a spectral equivalence with a Dirichlet problem in the domain obtained by removing from Ω the nodal lines of magnetic eigenfunctions close to the collision point. The general shape of this removed set (which is not necessarily a segment as in the symmetric case) creates some further difficulties. In particular, precise information about the diameter of such a set is needed in order to apply the following result from [3].

Theorem 1.3 ([3, Theorem 1.7]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set containing 0. Let λ_N be a simple eigenvalue of the Dirichlet Laplacian in Ω, and let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to λ_N such that $u_N(0) \neq 0$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact connected sets contained in Ω such that for every $r > 0$, there exists $\bar\varepsilon$ such that $K_\varepsilon \subseteq D_r$ for every $\varepsilon \in (0, \bar\varepsilon)$ (D_r denoting the disk of radius r centered at 0). Then*

$$\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N = u_N^2(0)\frac{2\pi}{|\log(\operatorname{diam} K_\varepsilon)|} + o\Big(\frac{1}{|\log(\operatorname{diam} K_\varepsilon)|}\Big) \quad \text{as } \varepsilon \to 0,$$

where $\lambda_N(\Omega \setminus K_\varepsilon)$ denotes the N-th eigenvalue of the Dirichlet Laplacian in $\Omega \setminus K_\varepsilon$.

In order to apply Theorem 1.3, a crucial intermediate step in the proof of Theorem 1.2 is the estimate of the diameter of nodal lines of magnetic eigenfunctions near the collision point. More precisely, we prove that when a is sufficiently small, locally near 0 suitable (magnetic-real) eigenfunctions have a nodal set consisting in a single regular curve connecting a^- and a^+. If d_a denotes the diameter of such a curve, we obtain that

$$\lim_{a\to 0^+} \frac{|\log a|}{|\log d_a|} = 1, \tag{1.4}$$

see Section 4.

The paper is organized as follows. In Section 2 we obtain some preliminary upper bounds for the eigenvalue variation $\lambda_N^a - \lambda_N$, testing the Rayleigh quotient for eigenvalues with proper test functions constructed by suitable manipulation of limit eigenfunctions. In Section 3 we prove that, as the two poles of the operator (1.2) move towards each other colliding at 0, λ_N^a is equal to the N-th eigenvalue of the Laplacian in Ω with a small piece of nodal line of the magnetic eigenfunction removed. Combining the upper estimates of Section 2 with Theorem 1.3, in Section 4 we succeed in estimating the diameter of the removed small set as in (1.4); we then conclude the proof of Theorem 1.2 by combining (1.4) and Theorem 1.3.

2 Estimates from Above

We denote by $\mathcal{H}_a$ the closure of $C_c^\infty(\Omega \setminus \{a^+, a^-\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{H}_a} = \Big(\int_\Omega |(i\nabla + \mathbf{A}_{a^-,a^+})u|^2\,dx\Big)^{1/2}.$$

We observe that, by Poincaré and diamagnetic inequalities together with the Hardy type inequality proved in [16], $\mathcal{H}_a \subset H_0^1(\Omega)$ with continuous inclusion. In order to estimate from above the eigenvalue λ_N^a, we recall the well-known Courant–Fisher *minimax characterization*:

$$\lambda_N^a = \min\Big\{\max_{u\in F\setminus\{0\}} \frac{\int_\Omega |(i\nabla + \mathbf{A}_{a^-,a^+})u|^2\,dx}{\int_\Omega |u|^2\,dx} : F \text{ is a subspace of } \mathcal{H}_a,\ \dim F = N\Big\}. \tag{2.1}$$

Lemma 2.1. *Let $\tau \in (0,1)$. For every $0 < \varepsilon < 1$, there exists a continuous radial cut-off function $\rho_{\varepsilon,\tau} : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho_{\varepsilon,\tau} \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, which also has the following properties:*
(i) $0 \le \rho_{\varepsilon,\tau}(x) \le 1$ *for all* $x \in \mathbb{R}^2$,
(ii) $\rho_{\varepsilon,\tau}(x) = 0$ *if* $|x| \le \varepsilon$, *and* $\rho_{\varepsilon,\tau}(x) = 1$ *if* $|x| \ge \varepsilon^\tau$,
(iii) $\int_{\mathbb{R}^2} |\nabla\rho_{\varepsilon,\tau}|^2\,dx = \frac{2\pi}{(\tau-1)\log\varepsilon}$,
(iv) $\int_{\mathbb{R}^2} (1-\rho_{\varepsilon,\tau}^2)\,dx = O(\varepsilon^{2\tau})$ *as* $\varepsilon \to 0^+$.

Proof. We set

$$\rho_{\varepsilon,\tau}(x) = \begin{cases} 0 & \text{if } |x| \le \varepsilon,\\ \dfrac{\log|x| - \log(\varepsilon)}{\log(\varepsilon^\tau) - \log(\varepsilon)} & \text{if } \varepsilon < |x| < \varepsilon^\tau,\\ 1 & \text{if } |x| \ge \varepsilon^\tau. \end{cases}$$

The function $\rho_{\varepsilon,\tau}$ is continuous and locally in H^1, with $0 \le \rho_{\varepsilon,\tau} \le 1$. The function $1-\rho^2_{\varepsilon,\tau}$ is supported in the disk of radius ε^τ centered at 0. We therefore have

$$\int_{\mathbb{R}^2} (1-\rho^2_{\varepsilon,\tau}(x))\,dx \le \pi\varepsilon^{2\tau},$$

which proves (iv). We have $\nabla\rho_{\varepsilon,\tau}(x)=0$ if $|x|<\varepsilon$ or $|x|>\varepsilon^\tau$, and

$$\nabla\rho_{\varepsilon,\tau}(x) = \frac{x}{(\tau-1)\log(\varepsilon)|x|^2}$$

if $\varepsilon < |x| < \varepsilon^\tau$. From this we directly obtain identity (iii). □

Lemma 2.2. *For all $a>0$, there exists a smooth function $\psi_a : \mathbb{R}^2 \setminus s_a \to \mathbb{R}$ satisfying*

$$\nabla\psi_a = \mathbf{A}_{a^-,a^+},$$

where s_a is the segment in $\mathbb{R}^2$ defined by $s_a := \{(t,0) : -a \le t \le a\}$. Furthermore, for every $x \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\lim_{a\to 0^+}\psi_a(x)=0$.

Proof. See [3, Lemma 3.1]. □

The first step in the proof of Theorem 1.2 is the following upper bound for the eigenvalue λ_N^a.

Proposition 2.3. *For every $\tau \in (0,1)$,*

$$\lambda_N^a \le \lambda_N + \frac{2\pi}{(1-\tau)|\log a|}(u_N^2(0)+o(1)) \quad \text{as } a \to 0^+.$$

The proof of Proposition 2.3 is based on estimates from above of the Rayleigh quotient for λ_N^a computed at some proper test functions constructed by suitable manipulation of limit eigenfunctions. To this end, let us consider, for each $j \in \{1,\dots,N\}$, a real eigenfunction u_j of $-\Delta$ with homogeneous Dirichlet boundary conditions associated with λ_j, with $\|u_j\|_{L^2(\Omega)}=1$. Furthermore, we choose these eigenfunctions so that

$$\int_\Omega u_j u_k\,dx = 0 \quad \text{for } j \ne k. \tag{2.2}$$

For $j \in \{1,\dots,N\}$ and $a>0$ small enough, we set

$$v^a_{j,\tau} := e^{i\psi_a}\rho_{2a,\tau}u_j. \tag{2.3}$$

We have that $v^a_{j,\tau} \in \mathcal{H}_a$. Lemma 2.1 and the Dominated Convergence Theorem imply that $v^a_{j,\tau}$ tends to u_j in $L^2(\Omega)$ when $a \to 0^+$. This implies, in particular, that the functions $v^a_{j,\tau}$ are linearly independent for a small enough.

Hence, for $a>0$ small enough, $E^a_{N,\tau} = \operatorname{span}\{v^a_{1,\tau},\dots,v^a_{N,\tau}\}$ is an N-dimensional subspace of $\mathcal{H}_a$, so that, in view of (2.1),

$$\lambda_N^a \le \max_{u \in E^a_{N,\tau}\setminus\{0\}} \frac{\int_\Omega |(i\nabla + \mathbf{A}_{a^-,a^+})u|^2\,dx}{\int_\Omega |u|^2\,dx} = \frac{\int_\Omega |(i\nabla + \mathbf{A}_{a^-,a^+})v^a_\tau|^2\,dx}{\int_\Omega |v^a_\tau|^2\,dx} \tag{2.4}$$

with

$$v^a_\tau = \sum_{j=1}^N \alpha^a_{j,\tau}v^a_{j,\tau} \quad \text{for some } \alpha^a_{1,\tau},\dots,\alpha^a_{N,\tau} \in \mathbb{C} \text{ such that } \sum_{j=1}^N |\alpha^a_{j,\tau}|^2 = 1. \tag{2.5}$$

Lemma 2.4. *For $a>0$ small, let v^a_τ be as in (2.4)–(2.5). Then*

$$\int_\Omega |v^a_\tau|^2\,dx = 1 + O(a^{2\tau}) \quad \text{as } a \to 0^+. \tag{2.6}$$

Proof. Taking into account (2.5), (2.3) and (2.2), we can write

$$\begin{aligned}\int_\Omega |v_\tau^a|^2\,dx &= \sum_{j,k=1}^N \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \int_\Omega \rho_{2a,\tau}^2 u_j u_k\,dx \\ &= 1 + \sum_{j=1}^N |\alpha_{j,\tau}^a|^2 \int_\Omega (\rho_{2a,\tau}^2 - 1) u_j^2\,dx + \sum_{j\neq k} \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \int_\Omega (\rho_{2a,\tau}^2 - 1) u_j u_k\,dx.\end{aligned}$$

Hence, the conclusion follows from Lemma 2.1 (iv). □

Lemma 2.5. *For $a > 0$ small, let v_τ^a be as in (2.4)–(2.5). Then*

$$\int_\Omega |(i\nabla + \mathbf{A}_{a^-,a^+}) v_\tau^a|^2\,dx = \sum_{j,k=1}^N \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \bigg(\frac{\lambda_j + \lambda_k}{2} \int_{\Omega\setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k\,dx + \int_{D_{(2a)^\tau}\setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2\,dx \bigg), \tag{2.7}$$

where, for all $r > 0$, $D_r = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r\}$ denotes the disk of center $(0, 0)$ and radius r.

Proof. Let us fix j and k in $\{1, \dots, N\}$ (possibly equal). In $\Omega \setminus D_{2a}$, we have that

$$\begin{aligned}(i\nabla + \mathbf{A}_{a^-,a^+}) v_{j,\tau}^a \cdot \overline{(i\nabla + \mathbf{A}_{a^-,a^+}) v_{k,\tau}^a} &= \nabla(\rho_{2a,\tau} u_j) \cdot \nabla(\rho_{2a,\tau} u_k) \\ &= \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k + u_j u_k |\nabla \rho_{2a,\tau}|^2 + (u_j \nabla u_k + u_k \nabla u_j) \cdot \rho_{2a,\tau} \nabla \rho_{2a,\tau}\end{aligned}$$

and, since $\rho_{2a,\tau} \nabla \rho_{2a,\tau} = \frac{1}{2} \nabla(\rho_{2a,\tau}^2)$,

$$\begin{aligned}&\int_\Omega (i\nabla + \mathbf{A}_{a^-,a^+}) v_{j,\tau}^a \cdot \overline{(i\nabla + \mathbf{A}_{a^-,a^+}) v_{k,\tau}^a}\,dx \\ &\quad= \int_{\Omega\setminus D_{2a}} \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k\,dx + \int_{D_{(2a)^\tau}\setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2\,dx + \frac{1}{2} \int_{\Omega\setminus D_{2a}} (u_j \nabla u_k + u_k \nabla u_j) \cdot \nabla(\rho_{2a,\tau}^2)\,dx.\end{aligned} \tag{2.8}$$

An integration by parts on the last term of (2.8) gives

$$\begin{aligned}&\int_\Omega (i\nabla + \mathbf{A}_{a^-,a^+}) v_{j,\tau}^a \cdot \overline{(i\nabla + \mathbf{A}_{a^-,a^+}) v_{k,\tau}^a}\,dx \\ &\quad= \int_{\Omega\setminus D_{2a}} \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k\,dx + \int_{D_{(2a)^\tau}\setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2\,dx - \frac{1}{2} \int_{\Omega\setminus D_{2a}} (u_j \Delta u_k + 2\nabla u_k \cdot \nabla u_j + \Delta u_j u_k) \rho_{2a,\tau}^2\,dx.\end{aligned}$$

After cancellations, we get

$$\int_\Omega (i\nabla + \mathbf{A}_{a^-,a^+}) v_{j,\tau}^a \cdot \overline{(i\nabla + \mathbf{A}_{a^-,a^+}) v_{k,\tau}^a}\,dx = \frac{\lambda_k + \lambda_j}{2} \int_{\Omega\setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k\,dx + \int_{D_{(2a)^\tau}\setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2\,dx. \tag{2.9}$$

From bilinearity, (2.5) and (2.9), we obtain (2.7). □

From (2.4) and (2.7), it follows that

$$\lambda_N^a - \lambda_N \le \frac{1}{\int_\Omega |v_\tau^a|^2\,dx} \bigg[\mathcal{Q}_a(\alpha_{1,\tau}^a, \alpha_{2,\tau}^a, \dots, \alpha_{N,\tau}^a) + \lambda_N \bigg(1 - \int_\Omega |v_\tau^a|^2\,dx \bigg) \bigg], \tag{2.10}$$

where $\mathcal{Q}_a : \mathbb{C}^N \to \mathbb{R}$ is the quadratic form defined as

$$\mathcal{Q}_a(z_1, z_2, \dots, z_N) = \sum_{j,k=1}^N M_{jk}^a z_j \overline{z_k}, \tag{2.11}$$

where

$$M_{jk}^a = \frac{\lambda_j + \lambda_k}{2} \int_{\Omega\setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k\,dx + \int_{D_{(2a)^\tau}\setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2\,dx - \lambda_N \delta_{jk}, \tag{2.12}$$

with δ_{jk} being the Kronecker delta.

To estimate the largest eigenvalue of the quadratic form $\mathcal{Q}_a$, we will use the following technical lemma.

Lemma 2.6. *For every $\varepsilon > 0$, let us consider the quadratic form*

$$Q_\varepsilon : \mathbb{C}^N \to \mathbb{R}, \quad Q_\varepsilon(z_1, z_2, \dots, z_N) = \sum_{j,k=1}^{N} m_{j,k}(\varepsilon) z_j \overline{z_k},$$

with $m_{j,k}(\varepsilon) \in \mathbb{C}$ such that $m_{j,k}(\varepsilon) = \overline{m_{k,j}(\varepsilon)}$. Assume that there exist real numbers $C > 0$ and $K_1, K_2, \dots, K_{N-1} < 0$ such that

$$\begin{aligned} m_{N,N}(\varepsilon) &= C\varepsilon(1 + o(1)) && \text{as } \varepsilon \to 0^+, \\ m_{j,j}(\varepsilon) &= K_j + o(1) && \text{as } \varepsilon \to 0^+ \text{ for all } j < N, \\ m_{j,k}(\varepsilon) &= \overline{m_{k,j}(\varepsilon)} = O(\varepsilon) && \text{as } \varepsilon \to 0^+ \text{ for all } j \neq k. \end{aligned}$$

Then

$$\max\Big\{ Q_\varepsilon(z_1, \dots, z_N) : (z_1, \dots, z_N) \in \mathbb{C}^N, \sum_{j=1}^{N} |z_j|^2 = 1 \Big\} = C\varepsilon(1 + o(1)) \quad \text{as } \varepsilon \to 0^+.$$

Proof. The result is contained in [1, Lemma 6.1], hence we omit the proof. □

Lemma 2.7. *For $a > 0$ small, let $\mathcal{Q}_a : \mathbb{C}^N \to \mathbb{R}$ be the quadratic form defined in (2.11)–(2.12). Then*

$$\max\Big\{ \mathcal{Q}_a(z_1, \dots, z_N) : (z_1, \dots, z_N) \in \mathbb{C}^N, \sum_{j=1}^{N} |z_j|^2 = 1 \Big\} = \frac{2\pi u_N^2(0)}{(1-\tau)|\log(a)|}(1 + o(1)) \quad \text{as } a \to 0^+.$$

Proof. Since $\int_\Omega u_N^2 = 1$, we can write

$$M_{NN}^a = \lambda_N \int_\Omega (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx + \int_{D_{(2a)^\tau} \setminus D_{2a}} u_N^2 |\nabla \rho_{2a,\tau}|^2 \, dx.$$

Since $u_N \in L^\infty_{\mathrm{loc}}(\Omega)$, from Lemma 2.1 (iv), it follows that

$$\int_\Omega (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx = \int_{D_{(2a)^\tau}} (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx = O(a^{2\tau}) \quad \text{as } a \to 0^+.$$

Since $u_N \in C^\infty_{\mathrm{loc}}(\Omega)$, we have that $u_N^2(x) - u_N^2(0) = O(|x|)$ as $|x| \to 0^+$. Then Lemma 2.1 (iii) implies that

$$\begin{aligned} \int_{D_{(2a)^\tau} \setminus D_{2a}} u_N^2 |\nabla \rho_{2a,\tau}|^2 \, dx &= u_N^2(0) \int_{D_{(2a)^\tau} \setminus D_{2a}} |\nabla \rho_{2a,\tau}|^2 \, dx + \int_{D_{(2a)^\tau} \setminus D_{2a}} (u_N^2(x) - u_N^2(0)) |\nabla \rho_{2a,\tau}(x)|^2 \, dx \\ &= (u_N^2(0) + O(a^\tau)) \int_{D_{(2a)^\tau} \setminus D_{2a}} |\nabla \rho_{2a,\tau}|^2 \, dx \\ &= \frac{2\pi}{(\tau - 1)\log(2a)} (u_N^2(0) + O(a^\tau)) \\ &= \frac{2\pi}{(\tau - 1)\log(a)} u_N^2(0)(1 + o(1)) \quad \text{as } a \to 0^+. \end{aligned}$$

Then

$$M_{NN}^a = \frac{2\pi}{(\tau - 1)\log(a)} u_N^2(0)(1 + o(1)) \quad \text{as } a \to 0^+. \tag{2.13}$$

For all $1 \le j < N$, we have that

$$\begin{aligned} M_{jj}^a &= \lambda_j \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 u_j^2 \, dx + \int_{D_{(2a)^\tau} \setminus D_{2a}} u_j^2 |\nabla \rho_{2a,\tau}|^2 \, dx - \lambda_N \\ &= (\lambda_j - \lambda_N) + \lambda_j \int_\Omega (\rho_{2a,\tau}^2 - 1) u_j^2 \, dx + \int_\Omega u_j^2 |\nabla \rho_{2a,\tau}|^2 \, dx, \end{aligned}$$

and hence, since $u_j \in C^\infty_{\mathrm{loc}}(\Omega)$ and in view of Lemma 2.1,

$$M^a_{jj} = (\lambda_j - \lambda_N) + O\Big(\frac{1}{|\log a|}\Big) = (\lambda_j - \lambda_N) + o(1) \quad \text{as } a \to 0^+. \tag{2.14}$$

Moreover, for all $j, k = 1, \dots, N$ with $j \neq k$, in view of (2.2) and Lemma 2.1, we have that

$$M^a_{jk} = \frac{\lambda_j + \lambda_k}{2} \int\limits_{\Omega \setminus D_{2a}} (\rho^2_{2a,\tau} - 1) u_j u_k \, dx + \int\limits_{D_{(2a)^\tau} \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx = O\Big(\frac{1}{|\log a|}\Big) \quad \text{as } a \to 0^+. \tag{2.15}$$

In view of estimates (2.13), (2.14) and (2.15), we have that $\mathcal{Q}_a$ satisfies the assumption of Lemma 2.6 (with $\varepsilon = \frac{1}{|\log a|}$), hence the conclusion follows from Lemma 2.6. □

Proof of Proposition 2.3. Combining (2.10), Lemma 2.7 and estimate (2.6), we obtain that

$$\begin{aligned} \lambda^a_N - \lambda_N &\leq \frac{1}{1 + O(a^{2\tau})} \Big[\frac{2\pi u_N^2(0)}{(1-\tau)|\log(a)|} (1 + o(1)) + O(a^{2\tau}) \Big] \\ &= \frac{2\pi u_N^2(0)}{(1-\tau)|\log(a)|} (1 + o(1)) \quad \text{as } a \to 0^+, \end{aligned}$$

thus completing the proof. □

3 Gauge Invariance, Nodal Sets and Reduction to the Dirichlet–Laplacian

In the following, by a *path* γ we mean a piecewise C^1 map $\gamma : I \mapsto \mathbb{R}^2$, with $I = [a, b] \subset \mathbb{R}$ being a closed interval. It follows from the definition of $\mathbf{A}_{a^-,a^+}$ (see (1.1)) that for any closed path γ (i.e., $\gamma(a) = \gamma(b)$),

$$\frac{1}{2\pi} \oint\limits_\gamma \mathbf{A}_{a^-,a^+} \cdot d\mathbf{s} = \frac{1}{2} \mathrm{ind}_\gamma(a^+) - \frac{1}{2} \mathrm{ind}_\gamma(a^-), \tag{3.1}$$

where $\mathrm{ind}_\gamma(a^+)$ (resp. $\mathrm{ind}_\gamma(a^-)$) is the winding number of γ around a^+ (resp. a^-).

3.1 Gauge Invariance

Let us give some results concerning the gauge invariance of our operators. In view of applying them to several different situations, we give statements valid for a magnetic Hamiltonian in an open and connected domain D, without restricting ourselves to the Aharonov–Bohm case.

In the following, the term *vector potential* (in an open connected domain D) stands for a smooth real vector field $\mathbf{A} : D \to \mathbb{R}^2$. In order to define the quantum mechanical Hamiltonian for a particle in D, under the action of the magnetic field derived from the vector potential $\mathbf{A}$, we first consider the differential operator

$$P = (i\nabla + \mathbf{A})^2$$

acting on smooth functions compactly supported in D. Using integration by parts (Green's formula), one can easily see that P is symmetric and positive. This is formally the desired Hamiltonian, but to obtain a self-adjoint Schrödinger operator, we have to specify the boundary conditions on ∂D, which we choose to be Dirichlet boundary conditions everywhere. More specifically, our Hamiltonian is the Friedrichs extension of the differential operator P. We denote it by $H^D_{\mathbf{A}}$, and we call it the magnetic Hamiltonian on D associated with $\mathbf{A}$.

We observe that the Aharonov–Bohm operator $H^\Omega_{a^-,a^+}$, introduced in (1.2) with the poles $a^- = (-a, 0)$ and $a^+ = (a, 0)$ in Ω, can be defined as the magnetic Hamiltonian $H^{\dot\Omega}_{\mathbf{A}_{a^-,a^+}}$ on $\dot\Omega$, where $\dot\Omega = \Omega \setminus \{a^-, a^+\}$, and that the spectrum of $H^\Omega_{a^-,a^+}$ consists of the eigenvalues defined by (2.1). The space $\mathcal{H}_a$ is the form domain of $H^\Omega_{a_-,a^+}$.

Definition 3.1. We call *gauge function* a smooth complex valued function $\psi\colon D \to \mathbb{C}$ such that $|\psi| \equiv 1$. To any gauge function ψ, we associate a *gauge transformation* acting as $(\mathbf{A}, u) \mapsto (\mathbf{A}^*, u^*)$, with

$$\mathbf{A}^* = \mathbf{A} - i\frac{\nabla\psi}{\psi}, \quad u^* = \psi u,$$

where $\nabla\psi = \nabla(\operatorname{Re}\psi) + i\nabla(\operatorname{Im}\psi)$. We notice that, since $|\psi| = 1$, $i\frac{\nabla\psi}{\psi}$ is a real vector field. Two magnetic potentials are said to be gauge equivalent if one can be obtained from the other by a gauge transformation (this is an equivalence relation).

The following result is a consequence of [17, Theorem 1.2].

Proposition 3.2. *If $\mathbf{A}$ and $\mathbf{A}^*$ are two gauge equivalent vector potentials, then the operators $H^D_{\mathbf{A}}$ and $H^D_{\mathbf{A}^*}$ are unitarily equivalent.*

The equivalence between two vector potentials (which is equivalent to the fact that their difference is gauge-equivalent to 0) can be determined using the following criterion.

Lemma 3.3. *Let $\mathbf{A}$ be a vector potential in D. This is gauge equivalent to 0 if and only if*

$$\frac{1}{2\pi}\oint_\gamma \mathbf{A}(s)\cdot d\mathbf{s} \in \mathbb{Z} \tag{3.2}$$

for every closed path γ contained in D.

Remark 3.4. The reverse implication in Lemma 3.3 is contained in [13, Theorem 1.1] for the Neumann boundary condition.

Proof. Let us first prove the direct implication. We assume that $\mathbf{A}$ is gauge equivalent to 0, that is to say that there exists a gauge function ψ such that

$$\mathbf{A} \equiv i\frac{\nabla\psi}{\psi}.$$

Fix a closed path $\gamma\colon I = [a, b] \to D$ and consider the mapping $z = \psi\circ\gamma$ from I to $\mathbb{U}$, where $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. By the lifting property, there exists a piecewise C^1 function $\theta\colon I \to \mathbb{R}$ such that $z(t) = \exp(i\theta(t))$ for all $t \in I$. This implies that

$$\nabla\psi(\gamma(t))\cdot\gamma'(t) = (\psi\circ\gamma)'(t) = z'(t) = i\theta'(t)\exp(i\theta(t)),$$

and therefore

$$i\frac{\nabla\psi}{\psi}(\gamma(t))\cdot\gamma'(t) = -\theta'(t).$$

This implies that

$$\oint_\gamma \mathbf{A}(s)\cdot d\mathbf{s} = \int_a^b i\frac{\nabla\psi}{\psi}(\gamma(t))\cdot\gamma'(t)\,dt = -\int_a^b \theta'(t)\,dt = \theta(a) - \theta(b).$$

Since γ is a closed path, $\exp(i\theta(a)) = \exp(i\theta(b))$, and therefore

$$\frac{\theta(a) - \theta(b)}{2\pi} \in \mathbb{Z}.$$

Let us now consider the reverse implication. We define a gauge function ψ in the following way. We fix an (arbitrary) point $X_0 = (x_0, y_0) \in D$. Let us show that for $X = (x, y) \in D$, the quantity

$$\exp\Big(-i\int_\gamma \mathbf{A}(s)\,d\mathbf{s}\Big)$$

does not depend on the choice of the path γ from X_0 to X. Indeed, let γ_1 and γ_2 be two such paths, and let γ_3 be the closed path obtained by going from X_0 to X along γ_1 and then from X to X_0 along γ_2. On the one hand, we have

$$\oint_{\gamma_3} \mathbf{A}(s)\,d\mathbf{s} = \int_{\gamma_1} \mathbf{A}(s)\,d\mathbf{s} - \int_{\gamma_2} \mathbf{A}(s)\,d\mathbf{s}.$$

On the other hand, if (3.2) holds, we have

$$\oint_{\gamma_3} \mathbf{A}(s)\,d\mathbf{s} \in 2\pi\mathbb{Z}.$$

This implies that

$$\exp\Big(-i\int_{\gamma_1} \mathbf{A}(s)\,d\mathbf{s}\Big) = \exp\Big(-i\int_{\gamma_2} \mathbf{A}(s)\,d\mathbf{s}\Big).$$

By the connectedness of D, there exists a path from X_0 to X for any $X \in \Omega$ (we can even choose it piecewise linear). We can therefore define, without ambiguity, a function $\psi\colon \Omega \to \mathbb{C}$ by

$$\psi(X) = \exp\Big(-i\int_{\gamma} \mathbf{A}(s)\,d\mathbf{s}\Big).$$

It is immediate from the definition that $|\psi| \equiv 1$ and that ψ is smooth, with

$$\nabla\psi(X) = -i\psi(X)\mathbf{A}(X).$$

It is therefore a gauge function sending $\mathbf{A}$ to 0. □

Lemma 3.3 can be used to define a set of eigenfunctions for $H^{\Omega}_{a^-,a^+}$ having special nice properties, as was done in [13, Section 3] for the Neumann boundary condition. It is analogous to the set of real eigenfunctions for the usual Dirichlet–Laplacian. To define it, we will construct a conjugation, that is, an antilinear antiunitary operator, which commutes with $H^{\Omega}_{a^-,a^+}$. To simplify the notation, we denote $\mathbf{A}_{a^-,a^+}$ by $\mathbf{A}$ and $H^{\Omega}_{a^-,a^+}$ by H in the rest of this section.

According to (3.1), the vector potential $2\mathbf{A}$ satisfies condition (3.2) of Lemma 3.3 on $\dot{\Omega}$, and therefore is gauge equivalent to 0. Therefore, there exists a gauge function ψ in $\dot{\Omega}$ such that

$$2\mathbf{A} = -i\frac{\nabla\psi}{\psi} \quad \text{in } \dot{\Omega}.$$

We now define the antilinear antiunitary operator K by

$$Ku = \psi\bar{u}.$$

For all $u \in C_0^\infty(\dot{\Omega},\mathbb{C})$,

$$(i\nabla + \mathbf{A})(\psi\bar{u}) = \psi\Big(i\nabla + i\frac{\nabla\psi}{\psi} + \mathbf{A}\Big)\bar{u} = \psi(i\nabla - \mathbf{A})\bar{u} = -\psi\overline{(i\nabla + \mathbf{A})u}.$$

The above formula and the fact that K is antilinear and antiunitary, imply that for all u and v in $C_0^\infty(\dot{\Omega},\mathbb{C})$,

$$\langle K^{-1}HKu, v\rangle = \langle Kv, HKu\rangle = \int_{\Omega} (i\nabla + \mathbf{A})(\psi\bar{v})\cdot\overline{(i\nabla + \mathbf{A})(\psi\bar{u})}\,dx = \int_{\Omega} \overline{(i\nabla + \mathbf{A})v}\cdot(i\nabla + \mathbf{A})u\,dx = \langle Hu, v\rangle,$$

where $\langle f, g\rangle = \int_\Omega f\bar{g}\,dx$ denotes the standard scalar product on the complex Hilbert space $L^2(\Omega,\mathbb{C})$. By density, we conclude that

$$K^{-1}HK = H.$$

Definition 3.5. We say that a function $u \in L^2(\Omega,\mathbb{C})$ is *magnetic-real* when $Ku = u$.

Let us denote by $\mathcal{R}$ the set of magnetic-real functions in $L^2(\Omega,\mathbb{C})$. The restriction of the scalar product to $\mathcal{R}$ gives it the structure of a real Hilbert space. The commutation relation $HK = KH$ implies that $\mathcal{R}$ is stable under the action of H. We denote by H^R the restriction of H to $\mathcal{R}$. There exists an orthonormal basis of $\mathcal{R}$ formed by eigenfunctions of H^R. Such a basis can be seen as a basis of magnetic-real eigenfunctions of the operator H in the complex Hilbert space $L^2(\Omega,\mathbb{C})$.

Let us now fix an eigenfunction u of H^R (or, equivalently, a magnetic-real eigenfunction of H). We define its *nodal set* $\mathcal{N}(u)$ as the closure in $\overline{\Omega}$ of the zero-set $u^{-1}(\{0\})$. Let us describe the local structure of $\mathcal{N}(u)$. In the sequel, by a *regular curve* or *regular arc* we mean a curve admitting a $C^{1,\alpha}$ parametrization for some $\alpha \in (0,1)$.

Theorem 3.6. *The set $\mathcal{N}(u)$ has the following properties:*
(i) *$\mathcal{N}(u)$ is, locally in $\dot{\Omega}$, a regular curve, except possibly at a finite number of singular points $\{X_j\}_{j\in\{1,\dots,n\}}$.*
(ii) *For $j \in \{1,\dots,n\}$, in the neighborhood of X_j, $\mathcal{N}(u)$ consists of an* even *number of regular half-curves meeting at X_j with equal angles (so that X_j can be seen as a cross-point).*
(iii) *In the neighborhood of a^+ (resp. a^-), $\mathcal{N}(u)$ consists of an* odd *number of regular half-curves meeting at a^+ (resp. a^-) with equal angles (in particular this means that a^+ and a^- are always contained in $\mathcal{N}(u)$).*

Proof. The proof is essentially contained in [20, Theorem 1.5] (see also [5]); for the sake of completeness we present a sketch of it. Let the eigenfunction u be associated with the eigenvalue λ, so that $Hu = \lambda u$. Let x_0 be a point in $\dot{\Omega}$. For $\varepsilon > 0$, we denote by $D(x_0,\varepsilon)$ the open disk $\{x : |x - x_0| < \varepsilon\}$. Let us show that we can find $\varepsilon > 0$ small enough and a local gauge transformation $\varphi\colon D(x_0,\varepsilon) \to \mathbb{C}$ such that $\mathbf{A}^* = \mathbf{A} - i\frac{\nabla\varphi}{\varphi} = 0$ and $u^* = \varphi u$ is a real-valued function in $D(x_0,\varepsilon)$. Indeed, let us define, as before, a gauge function ψ such that $2\mathbf{A} = -i\frac{\nabla\psi}{\psi}$. For $\varepsilon > 0$ small enough, we can define a smooth function $\varphi\colon D(x_0,\varepsilon) \to \mathbb{C}$ such that $\overline{\psi}(x) = (\varphi(x))^2$ for all $x \in D(x_0,\varepsilon)$, by taking

$$\varphi(x) = \exp\Big(-\frac{i}{2}\arg(\psi(x))\Big),$$

with arg a determination of the argument in $\psi(D(x_0,\varepsilon))$. A direct computation shows that for $x \in D(x_0,\varepsilon)$,

$$i\frac{\nabla\varphi(x)}{\varphi(x)} = \frac{i}{2}\frac{\nabla\overline{\psi}(x)}{\overline{\psi}(x)} = \mathbf{A}(x).$$

The gauge transformation on $D(x_0,\varepsilon)$ associated with φ therefore sends $\mathbf{A}$ to 0. Furthermore, since u is K-real, we have $\psi\bar{u} = \overline{\varphi^2 u} = u$ in $D(x_0,\varepsilon)$, and therefore $\overline{\varphi u} = \varphi u$. The real-valued function $v = \varphi u$ satisfies $-\Delta v = \lambda v$, and, since $|\varphi| \equiv 1$ on $D(x_0,\varepsilon)$, we have that $\mathcal{N}(v) \cap D(x_0,\varepsilon) = \mathcal{N}(u) \cap D(x_0,\varepsilon)$. Parts (i) and (ii) of Theorem 3.6 then follow from classical results on the nodal set of Laplacian eigenfunctions (see, for instance, [15, Theorem 2.1] and [20, Theorem 4.2]).

To prove part (iii) of Theorem 3.6, we use the regularity result of [20] for the Dirichlet problem associated with a one-pole Aharonov–Bohm operator. Indeed, let $\varepsilon > 0$ be small enough so that $D = D(a^+,\varepsilon) \subset \Omega$ and $a^- \notin D$. By this choice of ε, $\mathbf{A}_{a^-} = \nabla f$ on D, with f a smooth function, so that the domain D and the magnetic potential $\mathbf{A}$, restricted to D, satisfy the hypotheses of [20, Theorem 1.5]. The function u is a solution of the Dirichlet problem

$$\begin{cases} (i\nabla + \mathbf{A})^2 u - \lambda u = 0 & \text{in } D,\\ u = \gamma & \text{on } \partial D,\end{cases}$$

with $\gamma = u|_{\partial D} \in W^{1,\infty}(\partial D)$. A direct application of [20, Theorem 1.5] gives property (iii) around a^+. We can obtain property (iii) around a^- by exchanging the role of a^+ and a^-. □

3.2 Reduction to the Dirichlet–Laplacian

Our aim in this subsection is to show that, as the two poles of the operator (1.2) coalesce into a point at which u_N does not vanish, λ_N^a is equal to the N-th eigenvalue of the Laplacian in Ω with a small subset concentrating at 0 removed.

Theorem 3.7. *Let us assume that there exists $N \geq 1$ such that the N-th eigenvalue λ_N of the Dirichlet Laplacian in Ω is simple. Let u_N be a $L^2(\Omega)$-normalized eigenfunction associated λ_N and assume that $u_N(0) \neq 0$. Then, for all $a > 0$ sufficiently small, there exists a compact connected set $K_a \subset \Omega$ such that*

$$\lambda_N^a = \lambda_N(\Omega \setminus K_a),$$

and K_a concentrates around 0 as $a \to 0^+$, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a < \delta$, then $K_a \subset D_\varepsilon$.

We will divide the proof into two lemmas.

Lemma 3.8. *Let $R > 0$ be such that $\overline{D_R} \subset \Omega$ and $u_N(x) \neq 0$ for all $x \in \overline{D_R}$. Let $r \in (0, R)$. We denote by $C_{r,R}$ the closed ring*

$$C_{r,R} = \{x \in \mathbb{R}^2 : r \leq |x| \leq R\}.$$

There exists $\delta > 0$ such that if $0 < a < \delta$ and u is a magnetic-real eigenfunction associated with λ_N^a, then u does not vanish in $C_{r,R}$.

Proof. Let us assume, by contradiction, that there exists a sequence $a_n \to 0^+$ such that for all $n \geq 1$, $\lambda_N^{a_n}$ admits an eigenfunction φ_n which vanishes somewhere in $C_{r,R}$. Let us denotes by X_n a zero of φ_n in $C_{r,R}$.

According to [18, Section III], we can assume, up to extraction and a suitable normalization of φ_n, that $\varphi_n \to u_N$ in $L^2(\Omega)$. Since H is a uniformly regular elliptic operator in a neighborhood of $C_{r,R}$, φ_n converges to u_N uniformly on $C_{r,R}$. Furthermore, up to one additional extraction, we can assume that $X_n \to X_\infty \in C_{r,R}$. This implies that $u_N(X_\infty) = 0$, contradicting the fact that $u_N(x) \neq 0$ for all $x \in \overline{D_R}$. □

Lemma 3.9. *For all $R > 0$ such that $\overline{D_R} \subset \Omega$ and $u_N(x) \neq 0$ for all $x \in \overline{D_R}$, there exists $\delta > 0$ such that if $0 < a < \delta$ and u_N^a is a magnetic-real eigenfunction associated with λ_N^a, then $\mathcal{N}(u_N^a) \cap D_R$ consists in a single regular curve connecting a^- and a^+.*

Proof. By the continuity of $(a^-, a^+) \mapsto \lambda_N^a$ (see [18]), we have that

$$\Lambda = \max_{a \in [0,R]} \lambda_N^a \in (0, +\infty). \tag{3.3}$$

Let us choose $r \in (0, R)$ such that

$$r < \sqrt{\frac{\lambda_1(D_1)}{\Lambda}}, \tag{3.4}$$

where $\lambda_1(D_1)$ is the 1-st eigenvalue of the Laplacian in the unit disk D_1. According to Lemma 3.8, there exists $\delta(r) > 0$ such that if $a < \delta(r)$, then any eigenfunction associated to λ_N^a does not vanish in the closed ring $C_{r,R}$.

Let us assume that $0 < a < \delta(r)$ and $a < r$, and let u_N^a be an eigenfunction associated with λ_N^a. The proof relies on a topological analysis of $\mathcal{N}' := \mathcal{N}(u_N^a) \cap D_R$, inspired by previous work on nodal sets and minimal partitions (see [8, Section 6] and references therein). Lemma 3.8 implies that $\mathcal{N}'$ is compactly included in D_r. Theorem 3.6 implies that $\mathcal{N}'$ consists of a finite number of regular arcs connecting a finite number of singular points. In other words, $\mathcal{N}'$ is a regular planar graph. Let us denote by V the set of vertices of $\mathcal{N}'$, by b_1 the number of its connected components and by μ the number of its faces. By face, we mean a connected component of $\mathbb{R}^2 \setminus \mathcal{N}'$. There is always one unbounded face, so $\mu \geq 1$. Furthermore, for all $w \in V$, we denote by $\nu(w)$ the degree of the vertex w, that is to say the number of half-curves ending at w. Let us note that, according to Theorem 3.6, both a^- and a^+ belong to V and have an odd degree, and any other vertex can only have an even degree. These quantities are related through Euler's formula for planar graphs:

$$\mu = b_1 + \sum_{w \in V} \Big(\frac{\nu(w)}{2} - 1\Big) + 1. \tag{3.5}$$

For this classical formula, see, for instance, [6, Theorems 1.1 and 9.5]. Note that this reference treats the case of a connected graph. The generalization used here is easily obtained by linking the b_1 connected components of the graph with $b_1 - 1$ edges, in order to go back to the connected case.

Let us show by contradiction that $\mu = 1$. If $\mu \geq 2$, there exists a bounded face of the graph $\mathcal{N}'$, which is a nodal domain of u_N^a entirely contained in D_r. Let us call it ω. We denote by $\lambda_k(\omega, a^-, a^+)$ the k-th eigenvalue of the operator $(i\nabla + \mathbf{A}_{a^-,a^+})^2$ in ω, with homogeneous Dirichlet boundary condition on $\partial\omega$. Since ω is a nodal domain, for some $k(a) \in \mathbb{N} \setminus \{0\}$ depending on a, we have that

$$\lambda_N^a = \lambda_{k(a)}(\omega, a^-, a^+) \geq \lambda_1(\omega, a^-, a^+).$$

By the diamagnetic inequality,

$$\lambda_1(\omega, a^-, a^+) \geq \lambda_1(\omega),$$

where $\lambda_1(\omega)$ is the 1-st eigenvalue of the Dirichlet Laplacian in ω. By domain monotonicity,

$$\lambda_1(\omega) \geq \lambda_1(D_r) = \frac{\lambda_1(D_1)}{r^2}.$$

Hence, we obtain

$$r \geq \sqrt{\frac{\lambda_1(D_1)}{\lambda_N^a}},$$

thus contradicting (3.4). We conclude that $\mu = 1$.

Going back to Euler's formula (3.5), we obtain

$$\sum_{w\in V}\Big(\frac{\nu(w)}{2} - 1\Big) = -b_1 \leq -1. \tag{3.6}$$

According to Theorem 3.6, we have $\frac{\nu(w)}{2} - 1 \geq -\frac{1}{2}$ if $w \in \{a^-, a^+\}$, and $\frac{\nu(w)}{2} - 1 \geq 1$ if $w \in V \setminus \{a^-, a^+\}$. Inequality (3.6) can therefore be satisfied only if $V = \{a^-, a^+\}$ and $\nu(a^-) = \nu(a^+) = 1$, that is to say if $\mathcal{N}'$ is a regular arc connecting a^- and a^+. □

We are now in position to prove Theorem 3.7.

Proof of Theorem 3.7. From Lemma 3.9, it follows that for a sufficiently small, there exists a curve K_a in $\mathcal{N}(u_N^a)$ connecting a^- and a^+ and (in view of Lemma 3.8) concentrating at 0, where u_N^a is a magnetic-real eigenfunction associated with λ_N^a.

Let us write $\Omega_a' = \Omega \setminus K_a$. Since K_a is contained in $\mathcal{N}(u_N^a)$, we have that there exists $k(a) \in \mathbb{N} \setminus \{0\}$ (depending on a) such that

$$\lambda_N^a = \lambda_{k(a)}(\Omega_a', a^-, a^+), \tag{3.7}$$

where $\lambda_{k(a)}(\Omega_a', a^-, a^+)$ denotes the $k(a)$-th eigenvalue of $H^{\Omega_a'}_{a^-,a^+}$.

Let us consider a closed path γ in Ω_a'. By the definition of Ω_a', γ does not meet K_a, which means that K_a is contained in a connected component of $\mathbb{R}^2 \setminus \gamma$. Since the function $X \mapsto \mathrm{Ind}_\gamma(X)$ is constant on all connected components of $\mathbb{R}^2 \setminus \gamma$, we have that $\mathrm{Ind}_\gamma(a^-) = \mathrm{Ind}_\gamma(a^+)$. According to (3.1), this implies that

$$\frac{1}{2\pi}\oint_\gamma \mathbf{A}_{a^-,a^+} \cdot d\mathbf{s} = 0.$$

In view of Lemma 3.3, we conclude that $\mathbf{A}_{a^-,a^+}$ is gauge equivalent to 0 in Ω_a', and hence Proposition 3.2 ensures that

$$\lambda_{k(a)}(\Omega_a', a^-, a^+) = \lambda_{k(a)}(\Omega_a'). \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\lambda_N^a = \lambda_{k(a)}(\Omega_a'). \tag{3.9}$$

We observe that $a \mapsto k(a)$ stays bounded as $a \to 0^+$. Indeed if, by contradiction, $k(a_n) \to +\infty$ along some sequence $a_n \to 0^+$, by (3.9), we should have

$$\lambda_N^{a_n} = \lambda_{k(a_n)}(\Omega_{a_n}') \geq \lambda_{k(a_n)}(\Omega) \to +\infty,$$

thus contradicting (3.3).

Then, for any sequence $a_n \to 0^+$, there exists a subsequence a_{n_j} such that $k(a_{n_j}) \to k$ for some k. Since $k(a)$ is integer-valued we have that necessarily $k(a_{n_j}) = k \in \mathbb{N} \setminus \{0\}$ for j sufficiently large. Hence, (3.9) yields $\lambda_N^{a_{n_j}} = \lambda_k(\Omega \setminus K_{a_{n_j}})$. It is well known (see, e.g., [11, Theorem 1.2]) that $\lambda_k(\Omega \setminus K_{a_{n_j}}) \to \lambda_k(\Omega)$ as $j \to +\infty$; hence, taking into account (1.3), we conclude that $k = N$. Moreover, since the limit of $k(a_{n_j})$ does not depend on the subsequence and $a \mapsto k(a)$ is integer-valued, we conclude that $k(a) = N$ for all a sufficiently small, so that (3.9) becomes

$$\lambda_N^a = \lambda_N(\Omega_a'),$$

and the proof is complete. □

4 Proof of Theorem 1.2

We are in position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. For $a > 0$ small, let $K_a \subset \Omega$ be as in Theorem 3.7. We denote as

$$d_a := \operatorname{diam} K_a$$

the diameter of K_a. From Theorem 1.3, it follows that

$$\lambda_N(\Omega \setminus K_a) - \lambda_N = u_N^2(0)\frac{2\pi}{|\log d_a|} + o\Big(\frac{1}{|\log d_a|}\Big) \quad \text{as } a \to 0^+.$$

Hence, in view of Theorem 3.7,

$$\lambda_N^a - \lambda_N = u_N^2(0)\frac{2\pi}{|\log d_a|} + o\Big(\frac{1}{|\log d_a|}\Big) \quad \text{as } a \to 0^+. \tag{4.1}$$

From (4.1) and Proposition 2.3, it follows that for every $\tau \in (0, 1)$,

$$\frac{1}{|\log d_a|}(1 + o(1)) \le \frac{1}{(1-\tau)|\log a|}(1 + o(1)),$$

and then

$$\frac{|\log a|}{|\log d_a|} \le \frac{1}{1-\tau}(1 + o(1)) \quad \text{as } a \to 0^+. \tag{4.2}$$

On the other hand, since $a^-, a^+ \in K_a$, we have that $d_a \ge 2a$, so that $|\log a| \ge |\log d_a| + \log 2$ and

$$\frac{|\log a|}{|\log d_a|} \ge 1 + O\Big(\frac{1}{|\log d_a|}\Big) = 1 + o(1), \quad \text{as } a \to 0^+. \tag{4.3}$$

Combining (4.2) and (4.3), we conclude that

$$1 \le \liminf_{a\to 0^+} \frac{|\log a|}{|\log d_a|} \le \limsup_{a\to 0^+} \frac{|\log a|}{|\log d_a|} \le \frac{1}{1-\tau}$$

for every $\tau \in (0, 1)$, and then, letting $\tau \to 0^+$, we obtain that

$$\lim_{a\to 0^+} \frac{|\log a|}{|\log d_a|} = 1. \tag{4.4}$$

The conclusion then follows from (4.1) and (4.4). □

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