Chapter III

Regularity: formal theory

Introduction

With the example of Gauss hypergeometric equation at hand, we first recall the classical Fuchs-Frobenius theory of meromorphic linear differential systems: Fuchs' numerical criterion of regularity and Frobenius' convergence of the asymptotic expansions of the solutions.

We then turn to the algebraic viewpoint which will be prevalent in the book, and present the classical theory of regularity of formal differential modules in one variable (after Yu. Manin, N. Katz, ...). We abstract the canonical decomposition according to exponents into a general formalism of Jordan decompositions which will be also used later in the theory of irregularity.

We then tackle the theory of regular formal integrable connections in several variables, and present a detailed account of the (little known) theory of Gérard-Levelt lattices.

6 Hypergeometric equations and local monodromy

6.1 Singular points of hypergeometric equations

Let us consider again, from the complex-analytic point of view, the hypergeometric differential operator $L_{a,b,c}$ of (1.1.1), with $a, b, c \in \mathbb{C}$, namely

(6.1.1) $$L_{a,b,c} = x(1-x)\,\partial_x^2 + \big(c-(a+b+1)x\big)\,\partial_x - ab,$$

or, equivalently, using the operator $\vartheta_x = x\partial_x$,

(6.1.2) $$xL_{a,b,c} = (1-x)\,\vartheta_x^2 + \big(c-1-(a+b)x\big)\,\vartheta_x - abx.$$

We already pointed out that, if $c \notin \mathbb{Z}$, a full set of solutions of $L_{a,b,c}$ at 0 is given by the two functions ${}_2F_1(a,b,c;x)$ and $x^{1-c}{}_2F_1(a+1-c, b+1-c, 2-c; x)$, where

x^{1-c} is the multivalued function obtained by analytic continuation from one fixed determination at a fixed point $x_0 \neq 0$, for example

$$x^{1-c} = \Big(1 + \frac{x - x_0}{x_0}\Big)^{1-c} = \sum_n \binom{1-c}{n} \Big(\frac{x - x_0}{x_0}\Big)^n.$$

The power series part of the previous multivalued solutions has radius of convergence 1, as one can easily check from d'Alembert's ratio test, while any determination of x^{1-c} at any $x = x_0 \neq 0$, converges in the maximal open disk centered at x_0 and contained in $D^* = D \smallsetminus \{0\}$. One could hardly expect a more complete description of the analytic solutions of the differential equation $L_{a,b,c}y = 0$ in the open unit disk D centered at $0 \in \mathbb{C}$. The point $x = 0$ is a *singular point* of the previous equation in D, while every other point $x_0 \in D^*$ is *non-singular*[1] for (6.1.1), in the sense that there exist two linearly independent converging power series solutions in the variable $x - x_0$. At 0 instead, the vector space of power series solutions is one-dimensional and spanned by ${}_2F_1(a, b, c; x)$. This is the general principle:

Theorem 6.1.3 (Cauchy)**.** *Let $U \subseteq \mathbb{C}$ be an open connected domain and let*

$$L = \partial_x^{\mu} + a_{\mu-1}\,\partial_x^{\mu-1} + \cdots + a_1\,\partial_x + a_0, \tag{6.1.4}$$

be a differential operator with coefficients $a_i \in \mathcal{O}(U)$ that are holomorphic on U, $i = 0, \ldots, \mu - 1$. For any $x_0 \in U$, let $D_{x_0} \subseteq U$ be the maximal open disk in U centered at x_0. Then, for any $x_0 \in U$, the solutions of L in the differential ring $(\mathbb{C}[[x - x_0]], \partial_x)$ form a vector space of dimension μ. Moreover, they are Taylor expansions at x_0 of functions holomorphic on the full disk D_{x_0}.

In simple words, *the solutions of a differential equation at a non-singular point converge up to the nearest singularity.* An enhanced version of Cauchy's theorem states that *the power series part of solutions of a differential equation at a* regular *singular point converge up to the next nearest singularity.* A typical example of the latter situation is precisely (6.1.1). We refer to [87, Chap. V Sect. 17] for the proof, or the second appendix of [38] for more details. This chapter is dedicated to the explanation of the formal meaning of the term *regular* singular point appearing in the previous statement, and to its higher-dimensional generalizations.

We remark that Cauchy's theorem generalizes in the complex-analytic context in this form: every integrable connection is locally trivial (see 29.1.1).

Cauchy's theorem leads to the classical *theory of monodromy*, as we now explain.

[1] The classical name for *non-singular point* is *ordinary point*. However, we prefer to avoid this term, because the term *ordinary differential equation* is also used in the classical literature as an antonym to *partial differential equation*.

6.2 Local monodromy

6.2.1. Let D_ϵ^* be the punctured disk of radius ϵ centered at $0 \in \mathbb{C}$, and let $\mathbb{C}\{\{x\}\}$ be $\bigcup_\epsilon \mathcal{O}(D_\epsilon^*)$ viewed as a differential field. The field $\mathbb{C}(\{x\})$ of germs of meromorphic functions at 0 is a differential subfield of $\mathbb{C}\{\{x\}\}$. We consider a system of linear differential equations of the form

$$\frac{d}{dx}\,\vec{y} = G(x)\,\vec{y}, \tag{6.2.2}$$

where $G(x) \in M_\mu(\mathbb{C}\{\{x\}\})$. Here $\vec{y}$ is a column vector of unknown functions of size μ, but it also makes sense to replace $\vec{y}$ by a $\mu \times \mu$ matrix Y. We may assume that 0 is the only singularity of G in some disk D around 0.

6.2.3. Let $z_0 \in D^* := D \smallsetminus \{0\}$, and let $\mathcal{O}_{z_0}$ be the ring of germs of holomorphic functions at z_0. By Cauchy's theorem, the system has a solution matrix Y at z_0 in $\mathrm{GL}_\mu(\mathcal{O}_{z_0})$. This is called a *fundamental solution matrix at* z_0. Any other solution matrix Z of equation (6.2.2) in $M_\mu(\mathcal{O}_{z_0})$, is necessarily of the form $Z = YC$, for $C \in M_\mu(\mathbb{C})$, because $C := Y^{-1}Z \in M_\mu(\mathcal{O}_{z_0})$, and by a trivial manipulation, $\frac{dC}{dx} = 0$. There is a unique fundamental solution matrix at z_0 such that $Y(z_0) = I_\mu$. We shall denote it by $Y_{z_0}(x)$.

6.2.4. If $\gamma_0 : [0,1] \to D^*$, continuous with $\gamma_0(0) = \gamma_0(1) = z_0$, denotes a closed circuit in D^*, starting at z_0 and turning once counterclockwise around 0, analytic continuation along γ_0 then defines an automorphism of $\mathbb{C}$-algebras

$$T_{\gamma_0} : \mathcal{O}_{z_0} \longrightarrow \mathcal{O}_{z_0}, \tag{6.2.5}$$

which is non-trivial in general. For example, for any determination of $\log x$ (resp. x^a) at z_0, $T_{\gamma_0} \log x = 2\pi i + \log x$ (resp. $T_{\gamma_0} x^a = e^{2\pi i a} x^a$).

6.2.6. Analytic continuation commutes with differentiation, and therefore transforms the solution matrix Y_{z_0} of equation (6.2.2) at z_0 into another fundamental solution matrix at z_0, $T_{\gamma_0} Y_{z_0} = Y_{z_0} C(\gamma_0)$, with $C(\gamma_0) \in \mathrm{GL}_\mu(\mathbb{C})$. Analytic continuation along a homotopic path $\gamma_0' \sim \gamma_0$, leads to the same result. The fundamental group of D^* with base point z_0 is the group of homotopy classes of continuous paths $\gamma : [0,1] \to D^*$, with $\gamma(0) = \gamma(1) = z_0$, where the product $[\gamma_0][\gamma_1]$ is represented by the path

$$\gamma(s) = \begin{cases} \gamma_0(2s), & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \gamma_1(2s-1), & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

It follows that, if we let analytic continuation of $Y_{z_0} C(\gamma_0)$ along γ_1 follow analytic continuation of Y_{z_0} along γ_0, we obtain the matrix $Y_{z_0} C(\gamma_1) C(\gamma_0)$, which then represents analytic continuation of Y_{z_0} along the product path $\gamma_0\gamma_1$. The *monodromy representation* associated to (6.2.2) is the group anti-homomorphism

$$\begin{array}{rccc} \rho : & \pi_1(D^*, z_0) & \longrightarrow & \mathrm{GL}_\mu(\mathbb{C}) \\ & \gamma & \longmapsto & C(\gamma). \end{array} \tag{6.2.7}$$

For any $A \in M_\mu(\mathbb{C})$, let us define x^A as

$$x^A = \exp(A \log x) = \sum_{s=0}^{\infty} \frac{(A \log x)^s}{s!},$$

for some fixed determination of $\log x$ at z_0. Then

$$T_{\gamma_0}(x^A) = \exp(A \log x + 2\pi i A) = x^A \exp(2\pi i A).$$

So, if we take A such that $\exp(2\pi i A) = C(\gamma_0)$, then

$$T_{\gamma_0}(Y_{z_0} x^{-A}) = Y_{z_0} C(\gamma_0) C(\gamma_0)^{-1} x^{-A} = Y_{z_0} x^{-A},$$

so that

(6.2.8) $$W = Y_{z_0} x^{-A}$$

is *uniform*, that is, analytic in D^*. This is the *complex monodromy theorem*.

6.2.9. Let us now assume that the coefficients of the system (6.2.2) lie in the differential subfield $\mathbb{C}(\{x\})$ of $\mathbb{C}\{\{x\}\}$. It then makes sense to distinguish between two cases:

(1) W has coefficients in $\mathbb{C}(\{x\})$, i.e., has at worst a pole at 0, or

(2) W has an essential singularity at 0.

In the first case the singularity 0 is called (meromorphically) *regular*; in the second *irregular*. This dichotomy depends only on the differential module M over $\mathbb{C}(\{x\})$ attached to the system (6.2.2). It turns out that it can be detected on the *formal* differential module $\widehat{M}$ over $\mathbb{C}((x))$ (the formal completion of M). As we will see, the notion of regularity makes sense for differential modules over $\mathbb{C}((x))$.

6.2.10. The convergence of formal solutions at a regular singularity will be at the root of the comparison theorem for algebraic and complex-analytic De dham cohomology (cf. 31.3.3).

6.3 Fuchs-Frobenius theory

6.3.1. By the lemma of the cyclic vector (3.3.1), the system (6.2.2) is equivalent to a scalar equation $Ly = 0$, where

(6.3.2) $$L = \partial_x^\mu + a_{\mu-1} \partial_x^{\mu-1} + \cdots + a_1 \partial_x + a_0$$

is a linear differential operator of order μ, with the coefficients a_i holomorphic in a possibly smaller punctured disk contained in D^*, and with at most a pole at 0.

In the next definitions we relax the assumptions on L, as in (6.3.2), and only assume that the coefficients $a_i \in K((x))$, for a field K of characteristic 0; ord_0 denotes again the x-adic valuation.

Definition 6.3.3 (**Fuchs number**). *We define the* Fuchs number (*or irregularity, following* [78, Def. 1.5]) *at* 0 *of a differential operator*

$$L = a_\mu \partial_x^\mu + a_{\mu-1} \partial_x^{\mu-1} + \cdots + a_1 \partial_x + a_0$$

with $a_i \in K((x))$ *by*

$$i_{\mathrm{F}}(L) = \max\{(i - \mathrm{ord}_0 a_i) - (\mu - \mathrm{ord}_0 a_\mu) : i = 0, \ldots, \mu\}.$$

It is clear from the definition that $i_{\mathrm{F}}(L) \geqslant 0$. *In the case of the monic operator* L *as in* (6.3.2), *we have*

$$i_{\mathrm{F}}(L) = \max\{0, \max\{(i - \mathrm{ord}_0 a_i) - \mu : i = 0, \ldots, \mu - 1\}\}.$$

We say that L *satisfies the* Fuchs condition *at* 0, *or that* 0 *is a* regular singularity *for* L, *if* $i_{\mathrm{F}}(L) = 0$, *namely if*

$$\mathrm{ord}_0(a_i) \geqslant i - \mu, \quad \text{for } i = 0, \ldots, \mu - 1.$$

A classical theorem due to Fuchs and Frobenius (see [42], [41]) holds true.

Theorem 6.3.4 (**Frobenius-Fuchs**). *Let* $F = \mathbb{C}(\{x\})$ *be the field of germs of meromorphic functions on* $\mathbb{C}$ *at* $x = 0$. *The differential system* (6.2.2), *with coefficients in* F, *is regular at* 0 *if and only if one* (*and therefore any*) *scalar differential equation obtained from* (6.2.2) *by application of* (3.3.2) *over the differential field* (F, ∂_x) *satisfies the Fuchs condition at* 0.

Most of this chapter is dedicated to the purely formal theory of regular singularities in one and several variables, which greatly generalizes the formal questions raised by the theorem of Frobenius and Fuchs.

6.3.5. Thanks to this theorem, if the singularity is regular, the calculation of the monodromy matrix $C(\gamma_0)$ is easy. The logarithms of its eigenvalues are the zeros of the indicial polynomial of L at 0 (see Definition 7.3.1).

A similar discussion can be formally carried out in the irregular case, but does not lead to a calculation of $C(\gamma_0)$, because of the appearance of divergent series which only represent asymptotic expansions at 0 of holomorphic solutions of L in circular sectors of D^*.

7 The classical formal theory of regular singular points

The classical definition of regularity at 0 which we gave in the previous section has a meaning in pure differential algebra, as we now explain. The purpose of this section is to expose the formal aspects of the classical one-dimensional theory of regular singular points, on which the rest of the theory is ultimately based.

7.1 The exponential formalism x^A

7.1.1. Let K be a field of characteristic 0, and $\overline{K}$ be a fixed algebraic closure of K. We work with the differential field $F = K((x))$ of formal Laurent series equipped with derivation

$$\partial = \partial_x = \frac{d}{dx} \quad \text{or} \quad \vartheta_x = x\frac{d}{dx}.$$

First notice that there exists a differential field extension $(\mathcal{F}, \vartheta_x)$ of $(K((x)), \vartheta_x)$ such that:

(1) $\mathcal{F}$ contains a solution $\log x$ of the differential equation $\vartheta_x y = 1$;

(2) the natural homomorphism $(\mathbb{Z}, +) \to (\mathcal{F}^\times, \cdot)$, $m \mapsto x^m$, extends to an injective homomorphism $(\mathbb{Q}, +) \to (\mathcal{F}^\times, \cdot)$, $\frac{m}{n} \mapsto x^{\frac{m}{n}}$;

(3) the homomorphism $(\mathbb{Q}, +) \to (\mathcal{F}^\times, \cdot)$ extends to an injective map $(\overline{K}, +) \to (\mathcal{F}^\times, \cdot)$, $\alpha \mapsto x^\alpha$, such that

  (a) $x^{\alpha+m} = x^\alpha x^m$, for $m \in \mathbb{Z}$;

  (b) $x^{-\alpha} = (x^\alpha)^{-1}$;

  (c) x^α is a solution of the differential equation $\vartheta_x y = \alpha y$;

(4) $\mathcal{F}^{\vartheta_x} = \overline{K}$.

The map $\alpha \mapsto x^\alpha$ is not, in general, a group homomorphism, but for any $\alpha, \beta \in \overline{K}$, we have $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta}$, for some constant $c_{\alpha,\beta} \in \overline{K}$ satisfying suitable compatibility conditions (more precisely: $c_{\alpha,-\alpha} = 1 = c_{\alpha,m}$, $c_{\alpha,\beta} = c_{\beta,\alpha}$ and $c_{\alpha,\beta} c_{\alpha+\beta,\gamma} = c_{\alpha,\beta+\gamma} c_{\beta,\gamma}$ for any $\alpha, \beta, \gamma \in \overline{K}$ and $m \in \mathbb{Z}$).

If $\overline{K}$ embeds into $\mathbb{C}$, then upon choosing a branch of the logarithm we may define x^α to be $\exp(\alpha \log(x))$; with this choice we have $c_{\alpha,\beta} = 1$ for all $\alpha, \beta \in \overline{K}$.[2]

Lemma 7.1.2. *Let $q(t) \in \overline{K}[t]$, be a non-zero polynomial with constant coefficients, and let $\alpha \in \overline{K}$. Then $x^\alpha q(\log x) \in K((x))$ if and only if $\alpha \in \mathbb{Z}$, $\deg q = 0$, and $q \in K$.*

Proof. Let d be the degree of q. Let $w = x^\alpha q(\log x) = \sum_n b_n x^n \in K((x))$. We have $\vartheta_x q(\log x) = (\partial_t q)(\log x)$, hence $\vartheta_x^{d+1}(q(\log x)) = (\partial_t^{d+1} q)(\log x) = 0$. On the other hand, $\vartheta_x^{d+1}(q(\log x)) = \vartheta_x^{d+1}(x^{-\alpha} w) = x^{-\alpha}(\vartheta_x - \alpha)^{d+1} w$. Therefore $\sum_n b_n (n-\alpha)^{d+1} x^n = 0$. We deduce that either $w = 0$, or $\alpha \in \mathbb{Z}$ and $w = b_\alpha x^\alpha$. In either case $x^{-\alpha} w \in K$, so $q(\log x) \in K$. Since $\log x$ is not in $\overline{K}$, we conclude that $d = 0$ and $q \in K$. □

[2] An abstract construction of $\mathcal{F}$ is provided by Picard-Vessiot theory: let $A \sqcup 0$ be a set of representatives in $\overline{K}$ of elements of $\overline{K}$ modulo translation by integers and modulo sign. Let us endow the Laurent polynomial algebra $\overline{K}((x))[x_\alpha, x_\alpha^{-1}]_{\alpha \in A}$ with an action of ϑ_x given by $\vartheta_x(x_\alpha) = \alpha x_\alpha$. Its quotient by any maximal differential ideal is a simple differential algebra, hence a domain, and its fraction field $\mathcal{F}$, together with the images x^α of x_α, have the required properties (see e.g., [88, ch. 1])).

7.1.3. Let $A \in M_\mu(K)$, and let consider the differential system (7.2.1), with $G = A$ (that is, with constant coefficients). Then we may specify a solution matrix $x^A \in \mathrm{GL}_\mu(\mathcal{F})$ as follows:

(1) if $A = \Delta = \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_\mu \end{pmatrix}$ is a diagonal matrix, then
$$x^\Delta = \begin{pmatrix} x^{\Delta_1} & 0 & \cdots & 0 \\ 0 & x^{\Delta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{\Delta_\mu} \end{pmatrix};$$

(2) if $A = N$ is nilpotent, then $x^N = \exp(N \log x) = \sum_{j=0}^{\infty} \frac{N^j}{j!} (\log x)^j$;

(3) if $A = P(\Delta + N)P^{-1}$ gives the Jordan canonical form (with $P \in \mathrm{GL}_\mu(\overline{K})$), then $x^A = P x^\Delta x^N P^{-1}$.

Definition 7.1.4 (non-resonance). *We say that a matrix $A \in M_\mu(K)$ is* non-resonant *if the difference between any eigenvalues A in $\overline{K}$ is not a non-zero integer.*

This is a widespread, but not universal terminology: [38] says that A has *prepared eigenvalues*.

7.2 Non-resonance

We consider a differential system

$$\vartheta_x Y = GY, \quad \text{with } G \in M_\mu(K((x))). \tag{7.2.1}$$

Lemma 7.2.2. *Assume that the matrix G satisfies*

(1) $G \in M_\mu(K[[x]])$;

(2) $G(0)$ *is non-resonant.*

Then, if K' denotes any extension of K containing all eigenvalues of $G(0)$, the system has a fundamental solution matrix of the form $Y = W x^{G(0)}$, with $W \in \mathrm{GL}_\mu(K'[[x]])$ and $W(0) = I_\mu$.

Proof. Let $G(x) = \sum_{i=0}^{\infty} G_i x^i$ (with $G_i \in M_\mu(K)$; in particular, $G_0 = G(0)$ is non-resonant). We look for a matrix $W = \sum_{i=0}^{\infty} W_i x^i$ with $W \in M_\mu(K')$ and $W_0 = I_\mu$ such that $W x^{G_0}$ is a solution of (7.2.1). The condition gives

$$\vartheta_x W + W G_0 = GW,$$

that is,

$$\vartheta_x \big(\sum_{i=0}^{\infty} W_i x^i\big) + \big(\sum_{i=0}^{\infty} W_i x^i\big) G_0 = \big(\sum_{i=0}^{\infty} W_i x^i\big)\big(\sum_{i=0}^{\infty} G_i x^i\big)$$

and finally

$$\sum_{i=0}^{\infty}\big(iW_i + W_iG_0\big)x^i = \sum_{i=0}^{\infty}\big(\sum_{j=0}^{i} G_jW_{i-j}\big)x^i.$$

Equating the coefficient of x^i for any $i = 0, 1, \ldots$ we have an infinite system of equations

$$iW_i + W_iG_0 - G_0W_i = \sum_{j=1}^{i} G_jW_{i-j},$$

which allows us to use recursion on i. Starting with $W_0 = I_\mu$, the i-th equation may be solved for W_i (the right-hand side depends only on $W_0, \ldots, W_{i-1}$) if the linear transformation $\psi_{G_0} : M_\mu(K') \to M_\mu(K')$ given by $X \mapsto G_0X - XG_0$, has no non-zero integer eigenvalues. Since the eigenvalues of ψ_{G_0} are precisely the differences of eigenvalues of G_0, that condition is satisfied, because G_0 is non-resonant. □

The method of *shearing transformations* allows to pass from a system (7.2.1) with $G \in M_\mu(K[[x]])$, to an equivalent system

$$\vartheta_x Y = G_P Y, \tag{7.2.3}$$

where $G_P \in M_\mu(K'[[x]])$ and $G_P(0)$ is non-resonant, where K' is the splitting field over K of the characteristic polynomial of $G(0)$ and $P \in \mathrm{GL}_\mu(K'[x, x^{-1}])$. Here is the precise statement.

Lemma 7.2.4 (**Shearing transformations).** *Let $G \in M_\mu(K[[x]])$, and let $\alpha \in K'$ be an eigenvalue of $G(0)$. There exists $P \in \mathrm{GL}_\mu(K'[x, x^{-1}])$ such that $G_P \in M_\mu(K'[[x]])$ and the eigenvalues of $G_P(0)$ and of $G(0)$ are the same and have the same multiplicities, except for α which is replaced by $\alpha + 1$. Similarly for $\alpha - 1$.*

Proof. We may assume that $G(0) = \begin{pmatrix} J & B \\ 0 & D \end{pmatrix}$ with $J \in M_{\mu-s}(K')$, $D \in M_s(K')$ upper-triangular and $J = \begin{pmatrix} \alpha & * & \cdots & * \\ 0 & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \alpha \end{pmatrix}$, $D = \begin{pmatrix} \alpha_1 & * & \cdots & * \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \alpha_s \end{pmatrix}$, $\alpha \neq \alpha_i$, for $i = 1, \ldots, s$. We then take $P = \begin{pmatrix} xI_{\mu-s} & 0 \\ 0 & I_s \end{pmatrix}$ and check directly that $G_P(0) = \begin{pmatrix} J + I_{\mu-s} & 0 \\ * & D \end{pmatrix}$. To get the eigenvalue $\alpha - 1$, one works in a similar way with lower triangular matrices. □

7.3 Indicial polynomials

Definition 7.3.1 (indicial polynomial). *Let*

$$\Lambda = a_{\mu-1}\partial_x^\mu + a_{\mu-1}\partial_x^{\mu-1} + \cdots + a_j\partial_x^j + \cdots + a_0 \in K((x))\langle\partial_x\rangle$$

be a differential polynomial. We define the indicial polynomial of Λ at $x = 0$ *as the polynomial* $\mathrm{ind}_{\Lambda,0}(t) \in K[t]$ *determined by the condition*

(7.3.2) $$\Lambda(x^t) = (\mathrm{ind}_{\Lambda,0}(t) + o(x))x^{t+r},$$

where $o(x)$ *denotes an element of* $xK[t][[x]]$. *The indicial polynomial* $\mathrm{ind}_{\Lambda,0}(t)$ *will be denoted also* $\mathrm{ind}_0(t)$ *if the there is no ambiguity on* Λ.

7.3.3. The dominant coefficient of $\mathrm{ind}_0(t)$ will be indicated by γ_0 or γ, and the roots by $\alpha_{0,i}$ or α_i, so that

$$\mathrm{ind}_0(t) = \gamma_0 \prod_i (t - \alpha_{0,i}).$$

7.3.4. If $\Lambda \in K(x)\langle\partial_x\rangle$, then for any $x_0 \in K \cup \{\infty\}$ we define the indicial polynomial $\mathrm{ind}_{\Lambda,x_0}(t)$ of Λ at $x = x_0$ in the same way (by the change of variable $x' = x - x_0$ if $x_0 \in K$, or $x' = 1/x$ for $x_0 = \infty$).

Example 7.3.5. Let $L_{n,m} = x^n\partial_x^m + 1$. Then we have

$$\mathrm{ind}_{L_{n,m},0}(t) = \begin{cases} t(t-1)\cdots(t-m+1), & \text{if } n < m, \\ t(t-1)\cdots(t-m+1)+1, & \text{if } n = m, \\ 1, & \text{if } n > m. \end{cases}$$

Remark 7.3.6. Let write the differential polynomial Λ of 7.3.1 in the form

$$\Lambda = b_\mu\vartheta_x^\mu + b_{\mu-1}\vartheta_x^{\mu-1} + \cdots + b_j\vartheta_x^j + \cdots + b_0 \in K((x))\langle\vartheta_x\rangle.$$

Set

$$r = \min_i \mathrm{ord}_0 b_i \quad \text{and} \quad \nu = \max\{i : \mathrm{ord}_0 b_i = r\}.$$

Then $\mathrm{ind}_{\Lambda,0}(t) = \sum_{i=0}^{\nu}(x^{-r}b_i)_{x=0}t^i$. In particular, the degree of the indicial polynomial $\mathrm{ind}_0(t)$ is always between 0 and μ. It is exactly μ if and only if $b_i \in \mathbb{C}[[x]]$, for all i.

Remark 7.3.7 (Indicial polynomial as a characteristic polynomial). Let us consider the system $\vartheta_x Y = GY$ constructed from a differential operator

$$\Lambda = \vartheta_x^\mu + b_{\mu-1}\vartheta_x^{\mu-1} + \cdots + b_j\vartheta_x^j + \cdots + b_0 \in K[[x]]\langle\vartheta_x\rangle.$$

Then the shape (2.3.5) of that system shows that the characteristic polynomial of $G(0)$ is

(7.3.8) $$\det(I_\mu t - G(0)) = t^\mu + b_{\mu-1}(0)t^{\mu-1} + \cdots + b_j(0)t^j + \cdots + b_0(0),$$

and therefore coincides with the indicial polynomial $\mathrm{ind}_{\Lambda,0}(t)$ of Λ at 0.

7.4 Regularity of differential systems

The first half of the algebraic version of the theorem of Fuchs and Frobenius (6.3.4) is the following result about the system (7.2.1).

Proposition 7.4.1. *The following conditions are equivalent:*

(1) *there exists a finite Galois extension K' of K such that the system admits a solution of the form $Y = Wx^A$, with $A \in M_\mu(K')$ and $W \in \mathrm{GL}_\mu(K'((x)))$;*

(2) *there exists P in $\mathrm{GL}_\mu(K(x))$ (or, equivalently, in $\mathrm{GL}_\mu(K((x)))$), such that $G_P \in M_\mu(K[[x]])$;*

(3) *there exists a finite Galois extension K' of K and $P \in \mathrm{GL}_\mu(K'((x)))$ such that $G_P \in M_\mu(K')$ is in standard Jordan canonical form*

$$G_P = \begin{pmatrix} \alpha_1 + N_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 + N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_r + N_r \end{pmatrix}$$

where $\alpha_i - \alpha_j \notin \mathbb{Z}$, if $i \neq j$, and

$$N_i = \begin{pmatrix} 0 & \varepsilon_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_{n_1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a standard upper-triangular nilpotent matrix with $\varepsilon_i \in \{0, 1\}$ for all i.

Proof. (1) $\Rightarrow$ (2) By remark (2.6.10), it suffices to show that (1) implies (2) with K replaced by K'. This is clear if $W \in \mathrm{GL}_\mu(K'[[x]])$, since, by differentiation, we get $G = \vartheta_x(W)W^{-1} + WAW^{-1} \in M_\mu(K'[[x]])$. In general, $W \in \mathrm{GL}_\mu(K'((x)))$ and one can always find $H \in \mathrm{GL}_\mu(K'(x))$ such that $HW \in \mathrm{GL}_\mu(K'[[x]])$. But HWx^A is then a solution of the system $(\vartheta_x - G_H)Y = 0$, so that it follows that $G_H \in M_\mu(K'[[x]])$.

The implication (2) $\Rightarrow$ (1) follows directly from lemmas (7.2.2) and (7.2.4).

The equivalence (1) $\Leftrightarrow$ (3) is clear (for the "only if" part, use $P = W^{-1}$ to obtain $G_P = A$). □

Definition 7.4.2 (Regularity for differential systems). *One says that the differential system (7.2.1)*

$$\vartheta_x Y = GY \quad (\text{or the equivalent system } \partial_x Y = \tfrac{1}{x} GY)$$

is regular *– or that* 0 is a regular singularity *– if the equivalent conditions of the previous proposition are satisfied.*

Remark 7.4.3. The second point of 7.4.1 suggests that regularity may be defined "formally", i.e., requiring the existence of a $K[[x]]$-lattice of the differential module over $K((x))$ stable under the action of ϑ_x, as we will do in the next section.

7.5 Regularity criterion for differential equations

We say that a differential operator L is equivalent to the differential system (7.2.1) if the associated differential system is equivalent in the sense of definition 2.3.9.

Proposition 7.5.1 (Fuchs regularity criterion). *The following conditions are equivalent:*

(1) *the system* (7.2.1) *is regular;*

(2) *for* some (*resp.* any) *scalar differential equation* $Ly = 0$ *equivalent to* (7.2.1), *where*

$$L = \partial_x^{\mu} + a_{\mu-1}\,\partial_x^{\mu-1} + \cdots + a_1\,\partial_x + a_0, \quad a_i \in K((x)),$$

we have $\mathrm{ord}_0(a_i) \geqslant i - \mu$, *for* $i = 0, \ldots, \mu - 1$;

(3) *for* some (*resp.* any) *scalar differential equation* $\Lambda y = 0$ *equivalent to* (7.2.1), *where*

$$\Lambda = \vartheta_x^{\mu} + b_{\mu-1}\,\vartheta_x^{\mu-1} + \cdots + b_1\,\vartheta_x + b_0, \quad b_i \in K((x)),$$

we have $b_i \in K[[x]]$, *for* $i = 0, \ldots, \mu - 1$.

Proof. The equivalence of (2) and (3) follows from 2.2.6.

The implication (3) $\Rightarrow$ (1) is also clear by the construction of the system (2.3.4) associated to the scalar differential equation (2.3.1).

We are only left to prove (1) $\Rightarrow$ (3); we will apply Proposition 7.4.1. Thus, the system (7.2.1) admits a fundamental solution matrix of the form Wx^A, with $W \in \mathrm{GL}_{\mu}(K((x)))$ and $A \in M_{\mu}(K)$. Any scalar differential operator of the form of Λ, that is equivalent to our system may be obtained using $Y = x^A$: we choose a row $(y_1, \ldots, y_{\mu}) = (u_1, \ldots, u_{\mu})x^A$, with $u_i \in K((x))$, such that the wronskian

$$w(y_1, \ldots, y_{\mu}) := \det \begin{pmatrix} y_1 & \cdots & y_{\mu} \\ \vartheta_x y_1 & \cdots & \vartheta_x y_{\mu} \\ \vdots & \ddots & \vdots \\ \vartheta_x^{\mu-1} y_1 & \cdots & \vartheta_x^{\mu-1} y_{\mu} \end{pmatrix} \tag{7.5.2}$$

is non-zero, and construct the monic linear differential operator $\Lambda \in \mathcal{F}\langle\vartheta_x\rangle$ of minimal order that has the solutions $(y_1, \ldots, y_{\mu})$. By the condition on the wronskian, $y_1, \ldots, y_{\mu}$ are linearly independent over the field of constants $\overline{K}$, and the order of Λ is μ. We have

$$\vartheta_x(y_1, \ldots, y_{\mu}) = (\vartheta_x(u_1, \ldots, u_{\mu}) + (u_1, \ldots, u_{\mu})A)x^A,$$

and $(\vartheta_x(u_1,\dots,u_\mu)+(u_1,\dots,u_\mu)A)$ is a row of elements of $K((x))$. Iterating, we get

$$(7.5.3)\qquad \begin{pmatrix} y_1 & \cdots & y_\mu \\ \vartheta_x y_1 & \cdots & \vartheta_x y_\mu \\ \vdots & \ddots & \vdots \\ \vartheta_x^\mu y_1 & \cdots & \vartheta_x^\mu y_\mu \end{pmatrix} = \begin{pmatrix} v_{01} & \cdots & v_{0\mu} \\ v_{11} & \cdots & v_{1\mu} \\ \vdots & \ddots & \vdots \\ v_{\mu 1} & \cdots & v_{\mu\mu} \end{pmatrix} x^A = Vx^A,$$

with $v_{ij}\in K((x))$. Let V_i be the matrix obtained by removing from V the row of index i, e.g.,

$$(7.5.4)\qquad V_\mu = \begin{pmatrix} v_{01} & \cdots & v_{0\mu} \\ v_{11} & \cdots & v_{1\mu} \\ \vdots & \ddots & \vdots \\ v_{\mu-1,1} & \cdots & v_{\mu-1,\mu} \end{pmatrix}.$$

The wronskian matrix of $(y_1,\dots,y_\mu)$ is

$$(7.5.5)\qquad W(y_1,\dots,y_\mu) = V_\mu x^A,$$

so that $w(y_1,\dots,y_\mu)\neq 0$ is equivalent to $\det V_\mu \neq 0$. Explicitly,

$$(7.5.6)\qquad \Lambda(y) = (-1)^\mu (\det V_\mu x^A)^{-1} \det \begin{pmatrix} y & y_1 & \cdots & y_\mu \\ \vartheta_x y & \vartheta_x y_1 & \cdots & \vartheta_x y_\mu \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_x^{\mu-1} y & \vartheta_x^{\mu-1} y_1 & \cdots & \vartheta_x^{\mu-1} y_\mu \\ \vartheta_x^{\mu} y & \vartheta_x^{\mu} y_1 & \cdots & \vartheta_x^{\mu} y_\mu \end{pmatrix}$$

We get $\Lambda = \vartheta_x^\mu + b_{\mu-1}\vartheta_x^{\mu-1} + \cdots + b_j\vartheta_x^j + \cdots + b_0$, where

$$(7.5.7)\qquad b_j = (-1)^{\mu-j}\frac{\det(V_j x^A)}{\det(V_\mu x^A)} = (-1)^{\mu-j}\frac{\det V_j}{\det V_\mu}$$

showing that $\Lambda\in F\langle\vartheta_x\rangle$. Next, we have to prove that $b_j\in K[[x]]$, for all j. For this we may use the elementary proof of [38, III, Theorem 8.9], that we repeat for the reader convenience. By base change with a matrix $C\in \mathrm{GL}_\mu(K')$ we may assume that the matrix A is of the form $\left(\begin{smallmatrix}\alpha I+N & 0\\ 0 & A'\end{smallmatrix}\right)$ where N is a standard nilpotent matrix (all the entries are 0, outside the upper diagonal with entries all equal to 1). Changing Λ to $x^{-\alpha}\Lambda x^\alpha$ (which is in $K[[x]]\langle\vartheta_x\rangle$ if and only if Λ is) we may assume that $A=\left(\begin{smallmatrix}N & 0\\ 0 & A'\end{smallmatrix}\right)$. In particular, $y_1=u_1\neq 0$ and changing Λ to $u_1^{-1}\Lambda u_1$ (which is in $K[[x]]\langle\vartheta_x\rangle$ if and only if Λ is) we may assume that $y_1=u_1=1$ and the equality

$$(1,y_2,\dots,y_\mu) = (1,u_2,\dots,u_\mu)x^{\left(\begin{smallmatrix}N & 0\\ 0 & A'\end{smallmatrix}\right)}$$

holds, with $w(1, y_2, \ldots, y_\mu) = w(y_2, \ldots, y_\mu) \neq 0$. The differential system is then

$$\begin{aligned}\vartheta_x(1, y_2, \ldots, y_\mu) &= \vartheta_x\Big((1, u_2, \ldots, u_\mu)x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)}\Big) \\ &= ((0, \vartheta_x u_2, \ldots, \vartheta_x u_\mu) + (1, u_2, \ldots, u_\mu)\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right))x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)} \\ &= (0, v_2, \ldots, v_\mu)x^{\left(\begin{smallmatrix} N & 0 \\ 0 & A' \end{smallmatrix}\right)},\end{aligned}$$

and erasing the first terms we have a system

$$\vartheta_x(y_2, \ldots, y_\mu) = (v_2, \ldots, v_\mu)x^{\left(\begin{smallmatrix} N' & 0 \\ 0 & A' \end{smallmatrix}\right)},$$

where N' is the standard nilpotent matrix obtained by removing first row and column of N. We can now conclude by induction on μ that the differential operator Λ' associated to this system has coefficients in $K[[x]]$. Therefore also $\Lambda = \Lambda'\vartheta_x$ has the same property. □

Remark 7.5.8. We shall give another proof of these criteria in 15.2.

7.6 Exponents

The remaining part of the classical theory of regular singular points deals with exponents.

Proposition-Definition 7.6.1 (Exponents). *Let (7.2.1) be a regular system. Let G_P be an equivalent system with entries in $K[[x]]$. Then the classes in $\overline{K}/\mathbb{Z}$ of the eigenvalues of $G_P(0)$ (counted with multiplicity), do not depend on G_P. They are called the* exponents *of the differential system (7.2.1).*

Here is a more precise form of this result.

Lemma 7.6.2. *Let $G, H \in M_\mu(K((x))$, and $P \in \mathrm{GL}_\mu(K((x)))$ be such that $H = G_P$. Assume that there exist matrix solutions of $(\vartheta_x - G)Y = 0$ (resp. of $(\vartheta_x - H)Y = 0$) of the form $Y_G x^A$ (resp. $Y_H x^B$) with $A, B \in M_\mu(K)$, $Y_G, Y_H \in \mathrm{GL}_\mu(K((x)))$. Let*

$$A = C_2^{-1}(\Delta + N)C_2, \quad B = C_1(\Delta' + N')C_1^{-1}, \quad C_1, C_2 \in \mathrm{GL}_\mu(K)$$

with Δ, Δ' diagonal, N, N' nilpotent, $\Delta N = N\Delta$, $\Delta' N' = N'\Delta'$. Then there exists $\overline{C} \in \mathrm{GL}_\mu(K)$ such that

$$x^{\Delta'}\overline{C}x^{-\Delta} \in \mathrm{GL}_\mu(K[x, x^{-1}]).$$

Proof. Let us write $\Delta = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_\mu \end{pmatrix}$ and $\Delta' = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_\mu \end{pmatrix}$. We have $PY_G x^A = Y_H x^B \widetilde{C}$, for some $\widetilde{C} \in \mathrm{GL}_\mu(\overline{K})$. Let $\overline{P} = Y_H^{-1} P Y_G \in \mathrm{GL}_\mu(K((x)))$, so

$\overline{P}x^A = x^B\widetilde{C}$. But this is

$$\overline{P}C_2^{-1}x^{\Delta+N}C_2 = C_1x^{\Delta'+N'}C_1^{-1}\widetilde{C},$$

so that

$$x^{\Delta'}x^{N'}C_1^{-1}\widetilde{C}C_2^{-1}x^{-N}x^{-\Delta} = C_1^{-1}\overline{P}C_2^{-1} \in \mathrm{GL}_\mu(K((x))).$$

Set $\overline{C} := C_1^{-1}\widetilde{C}C_2^{-1} = (c_{ij})_{ij} \in \mathrm{GL}_\mu(\overline{K})$; then

$$x^{\Delta'}x^{N'}\overline{C}x^{-N}x^{-\Delta} \in \mathrm{GL}_\mu(K((x))).$$

Now $x^{N'}\overline{C}x^{-N}$ is of the form $(P_{ij}(\log x))_{ij}$, for some $P_{ij}(T) \in \overline{K}[T]$ so that

$$x^{\Delta'}x^{N'}\overline{C}x^{-N}x^{-\Delta} = (x^{\beta_i-\alpha_j}P_{ij}(\log x))_{ij} \in \mathrm{GL}_\mu(K((x))).$$

By Lemma 7.1.2, we conclude that each $P_{ij}(T)$ is an element of K, so that $x^{N'}\overline{C}x^{-N} = \overline{C}$ and

(7.6.3) $$x^{\Delta'}\overline{C}x^{-\Delta} = (c_{ij}x^{\beta_i-\alpha_j})_{ij} \in \mathrm{GL}_\mu(K((x))).$$ □

Definition 7.6.4 (Multiplicities). *Formula (7.6.3) shows that* $(\alpha_1+\mathbb{Z},\ldots,\alpha_\mu+\mathbb{Z}) \in (\overline{K}/\mathbb{Z})^\mu$ *is a permutation of* $(\beta_1+\mathbb{Z},\ldots,\beta_\mu+\mathbb{Z})$*. The number of times a given element* $\overline{\alpha}$ *of* $\overline{K}/\mathbb{Z}$ *appears in those rows is called the* multiplicity of the exponent $\overline{\alpha}$ *of the system* $(\vartheta_x - G)Y = 0$*.*

8 Jordan decomposition of differential modules

In this section we introduce some general tools which we will use in this chapter for the theory of regular differential modules, and in Chapter V for the formal theory of irregular differential modules. The main tool is a sort of generalization of the classical Jordan decomposition of linear operators to the case of differential modules, and its stability under the action of commuting derivations. The main results of this chapter are the existence of the Jordan decomposition in the regular case, and a variant with parameters.

8.1 Jordan theory for differential modules

8.1.1. Let (F,∂) be a differential field and let $K = F^\partial$ be its field of constants (of characteristic 0, as always). For any differential module (M,∇_∂) over (F,∂), for any $\phi \in F$, and any $\nu = 0,1,\ldots$, we set (identifying ϕ with a homothety of M)

$$K_\phi^{(\nu)} = K_\phi^{(\nu)}(M) = \mathrm{Ker}_M(\nabla_\partial - \phi)^\nu,$$

and

$$M_\phi^{(\nu)} = \mathrm{Im}(F \otimes_K K_\phi^{(\nu)}(M) \longrightarrow M).$$

The $K_\phi^{(\nu)}$'s are K-vector subspaces of M (endowed with a nilpotent endomorphism $\nabla_\partial - \phi$), and the $M_\phi^{(\nu)}$'s are differential submodules of M. We have

$$(0) = K_\phi^{(0)} \subseteq K_\phi^{(1)} \subseteq \cdots$$

and

$$(0) = M_\phi^{(0)} \subseteq M_\phi^{(1)} \subseteq \cdots$$

For all $\phi, \psi \in F$ and for any $\lambda, \nu = 0, 1, \ldots$, we have

$$K_\phi^{(\lambda)}(M_\psi^{(\nu)}) = K_\phi^{(\lambda)}(M) \cap M_\psi^{(\nu)}$$

while, *a priori*, only

$$(M_\psi^{(\nu)})_\phi^{(\lambda)} \subseteq M_\phi^{(\lambda)} \cap M_\psi^{(\nu)}.$$

Lemma 8.1.2. *If* $K_\phi^{(\nu)} = K_\phi^{(\nu+1)}$*, then* $K_\phi^{(\nu+1)} = K_\phi^{(\nu+2)} = \cdots$.

Proof. Let $m \in K_\phi^{(\nu+2)}$, so that $(\nabla_\partial - \phi)m \in K_\phi^{(\nu+1)} = K_\phi^{(\nu)}$. Then $(\nabla_\partial - \phi)^{\nu+1}m = 0$, and $m \in K_\phi^{(\nu+1)}$. □

8.1.3 (Condition $(*)$). The following conditions on (F, ∂) are equivalent:

(1) $\mathrm{Ker}_F \partial^n \subseteq \mathrm{Ker}_F \partial$ $(= K)$ for all $n \geqslant 1$ (so the equalities hold);

(2) $\mathrm{Ker}_F \partial^2 \subseteq \mathrm{Ker}_F \partial$ $(= K)$ (so the equality holds);

(3) $\mathrm{Im}_F(\partial) \cap \mathrm{Ker}_F(\partial) = 0$, that is, for all $a \in F$ we have: $\partial(a) \in K$ (if and) only if $\partial(a) = 0$.

(4) for all $a \in F$ we have: $\partial^n(a) \in K$ for some $n \geqslant 1$ (if and) only if $\partial(a) = 0$.

The non-trivial implication "(2) implies (1)" is proved by induction; the equivalence of (2) and (3) follows from the trivial remark that $\partial^2(a) = 0$ if and only if $\partial(a) \in K (= \mathrm{Ker}_F \partial)$; the equivalence of (4) and (3) (or (4) and (1)) is straightforward.

Remark 8.1.4. The above conditions are satisfied by $F = K((x))$ and $\partial = x^r \partial_x$ if and only if $r = 1$. More generally, a derivation (of $F = K((x))$) of the form $g(x)x\partial_x$ satisfies the conditions if and only if $g(x)$ is a unit of $K[[x]]$.

Proposition 8.1.5. *Let* (M, ∇_∂) *be a differential module over a differential field* (F, ∂) *satisfying conditions* $(*)$ *of 8.1.3. Then, for any* $\phi \in F$ *and* $\nu = 1, 2, \ldots$*, the natural map*

$$F \otimes_K K_\phi^{(\nu)}(M) \longrightarrow M$$

is an injection.

Proof. We proceed by induction on ν. Let $m_1, \ldots, m_s$ be elements of $K_\phi^{(1)} = \mathrm{Ker}_M(\nabla_\partial - \phi)$ with the following properties:

(1) $m_1, \dots, m_s$ are linearly independent over K;

(2) there exists a relation $m_s = \sum_{i=1}^{s-1} a_i m_i$, $a_i \in F$;

(3) s is minimal for the properties (1) and (2).

We apply $\nabla_\partial - \phi$ to m_s and obtain a relation $\displaystyle\sum_{i=1}^{s-1} (\partial a_i) m_i = 0$. By minimality, $\partial a_i = 0$, so that $a_i \in K$, for all $i = 1, \dots, s-1$, which contradicts property (1). This proves the case $\nu = 1$, for any (M, ∇_∂) and $\phi \in F$. Let us now assume that $F \otimes_K K_\phi^{(\nu)}(M)$ injects into M. Let N be the quotient differential module $M/M_\phi^{(1)}$. The kernel of the natural projection

$$\pi : K_\phi^{(\nu+1)}(M) \longrightarrow K_\phi^{(\nu)}(N)$$

is $\mathrm{Ker}(\pi) = K_\phi^{(\nu+1)}(M) \cap M_\phi^{(1)} = K_\phi^{(\nu+1)}(M) \cap (F \otimes_K K_\phi^{(1)}(M))$, by the case $\nu = 1$. Let $\displaystyle\sum_{i=1}^{s} a_i \otimes m_i \in \mathrm{Ker}\, \pi$, with $(m_1, \dots, m_s)$ a K-basis of $K_\phi^{(1)}$ and $a_i \in F$. We have

$$\begin{aligned} 0 = (\nabla_\partial - \phi)^{\nu+1} \sum_{i=1}^{s} a_i m_i &= (\nabla_\partial - \phi)^{\nu} \Big(\sum_{i=1}^{s} (\partial a_i) m_i + \sum_{i=1}^{s} a_i \underbrace{(\nabla_\partial - \phi) m_i}_{=0} \Big) \\ &= (\nabla_\partial - \phi)^{\nu} \sum_{i=1}^{s} (\partial a_i) m_i = \dots = \sum_{i=1}^{s} (\partial^{\nu+1} a_i) m_i . \end{aligned}$$

Therefore, $\partial^{\nu+1} a_i = 0$, hence (by the condition $(*)$ of 8.1.3) $a_i \in K$, $\forall\, i = 1, \dots, s$, so that $\mathrm{Ker}\, \pi = K_\phi^{(1)}$. We then have an isomorphism

$$K_\phi^{(\nu+1)}(M) / K_\phi^{(1)}(M) \xrightarrow{\sim} K_\phi^{(\nu)}(N).$$

By the induction assumption (for the differential module $N = M/M_\phi^{(1)}$)

$$\frac{F \otimes_K K_\phi^{(\nu+1)}(M)}{F \otimes_K K_\phi^{(1)}(M)} = F \otimes_K \frac{K_\phi^{(\nu+1)}(M)}{K_\phi^{(1)}(M)} \cong F \otimes_K K_\phi^{(\nu)}(N) \hookrightarrow N = \frac{M}{F \otimes_K K_\phi^{(1)}(M)} .$$

Therefore, $F \otimes_K K_\phi^{(\nu+1)}(M)$ injects into M. □

Corollary 8.1.6. *For any $\lambda, \nu = 0, 1, \dots,$ and any $\phi \in F$, we have*

(8.1.7) $$(M/M_\phi^{(\lambda)})_\phi^{(\nu)} = M_\phi^{(\lambda+\nu)} / M_\phi^{(\lambda)} .$$

Proof. We already showed, in the proof of the lemma, that the natural projection $\pi : M \to N = M/M_\phi^{(1)}$ identifies $M_\phi^{(\nu+1)}/M_\phi^{(1)}$ to $N_\phi^{(\nu)}$. This is the present statement for $\lambda = 1$, which we call the *basic case*. So, we now assume the statement to hold for some λ, for any M and any $\nu = 0, 1, \ldots$, and show that it holds for $\lambda + 1$ and any M and ν. We write

$$\begin{aligned}
&(M/M_\phi^{(\lambda+1)})_\phi^{(\nu)} \\
&= ((M/M_\phi^{(1)})/(M_\phi^{(\lambda+1)}/M_\phi^{(1)}))_\phi^{(\nu)} \quad \text{(by the basic case)} \\
&= ((M/M_\phi^{(1)})/(M/M_\phi^{(1)})_\phi^{(\lambda)})_\phi^{(\nu)} \quad \text{(by the inductive assumption)} \\
&= (M/M_\phi^{(1)})_\phi^{(\lambda+\nu)}/(M/M_\phi^{(1)})_\phi^{(\lambda)} \quad \text{(by two instances of the basic case)} \\
&= (M_\phi^{(\lambda+\nu+1)}/M_\phi^{(1)})/(M_\phi^{(\lambda+1)}/M_\phi^{(1)}) = M_\phi^{(\lambda+\nu+1)}/M_\phi^{(\lambda+1)}.
\end{aligned}$$

□

Corollary 8.1.8. *For any $\nu = 0, 1, \ldots,$ and $\phi \in F$, the two conditions are equivalent:*

(1) $K_\phi^{(\nu)} = K_\phi^{(\nu+1)}$;

(2) $M_\phi^{(\nu)} = M_\phi^{(\nu+1)}$.

In particular, $\dim_K K_\phi^{(\nu)} \leqslant \mu := \dim_F M$, *and the sequence* $K_\phi^{(\nu)}$ *(resp.* $M_\phi^{(\nu)}$*) is stationary from* $\nu = \mu$ *on.*

Proposition 8.1.9. *Let (M, ∇_∂) be a differential module over a differential field (F, ∂) satisfying assumption $(*)$ of 8.1.3, and let $\phi, \psi \in F$. If* $\mathrm{Ker}_F(\partial - \phi + \psi) = (0)$, *then* $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$ *for any (λ, ν). Otherwise,* $M_\phi^{(\nu)} = M_\psi^{(\nu)}$ *for any ν.*

Proof. Let us assume that $\mathrm{Ker}_F(\partial - \phi + \psi) = (0)$. We first prove that $M_\phi^{(1)} \cap M_\psi^{(1)} = (0)$. Let $m = \sum_{i=1}^s a_i m_i = \sum_{j=1}^r b_j n_j \in M_\phi^{(1)} \cap M_\psi^{(1)}$, with $m_1, \ldots, m_s$ linearly independent in $K_\phi^{(1)}$ (resp. $n_1, \ldots, n_r$ linearly independent in $K_\psi^{(1)}$), and with s *minimal*. Then

$$\nabla_\partial m = \sum_{i=1}^s (\partial a_i) m_i + \phi m = \sum_{j=1}^r (\partial b_j) n_j + \psi m.$$

Combining the two equations

$$\sum_{i=1}^s (\partial a_i + \phi a_i) m_i = \sum_{j=1}^r (\partial b_j + \psi b_j) n_j \quad \text{and} \quad \sum_{i=1}^s a_i m_i = \sum_{j=1}^r b_j n_j$$

we get

$$\sum_{i=1}^{s-1} ((\partial a_s + \phi a_s) a_i - a_s(\partial a_i + \phi a_i)) m_i = \sum_{j=1}^r ((\partial a_s + \phi a_s) b_j - a_s(\partial b_j + \psi b_j)) n_j\,,$$

so that all coefficients of the m_i's and n_j's figuring here must vanish (by the minimality of s). In particular, $(\partial a_s + \phi a_s)b_j - a_s(\partial b_j + \psi b_j)) = 0\ \forall\, j$, may be read, if $b_j \neq 0$, as $\partial(a_s/b_j) = (\psi - \phi)a_s/b_j$, which implies (by hypothesis) $a_s = 0$, a contradiction to the minimality of s. So, $b_j = 0\ \forall\, j$, hence $m = 0$.

We now prove by induction on λ that $M_\phi^{(\lambda)} \cap M_\psi^{(1)} = (0)$, for all $\lambda = 1, 2, \dots$. We set again $N = M/M_\phi^{(1)}$, and denote by $\pi : M \to N$ the natural projection. Then π identifies $M_\phi^{(\lambda+1)}/M_\phi^{(1)}$ to $N_\phi^{(\lambda)}$, by Corollary (8.1.6), and $M_\psi^{(1)}$ to $N_\psi^{(1)}$, by the case $\lambda = 1$ treated above. By the induction assumption, $N_\phi^{(\lambda)} \cap N_\psi^{(1)} = (0)$. We get

$$M_\phi^{(\lambda+1)} \cap (M_\phi^{(1)} + M_\psi^{(1)}) = M_\phi^{(1)},$$

whence

$$M_\phi^{(\lambda+1)} \cap M_\psi^{(1)} = M_\phi^{(1)} \cap M_\psi^{(1)} = (0).$$

We now prove by induction on ν that $M_\phi^{(\lambda)} \cap M_\psi^{(\nu)} = (0)$, for all $\nu = 1, 2, \dots$, the case $\nu = 1$ being already discussed. We set here $N = M/M_\psi^{(1)}$, and denote by $\pi : M \to N$ the natural projection. As before, π identifies $M_\psi^{(\nu+1)}/M_\psi^{(1)}$ to $N_\psi^{(\nu)}$, by Corollary (8.1.6), and $M_\phi^{(\lambda)}$ to $N_\phi^{(\lambda)}$, by the previous case. Induction on ν shows that $N_\phi^{(\lambda)} \cap N_\psi^{(\nu)} = (0)$, so that $(M_\phi^{(\lambda)} + M_\psi^{(1)}) \cap M_\psi^{(\nu+1)} = M_\psi^{(1)}$, and finally $M_\phi^{(\lambda)} \cap M_\psi^{(\nu+1)} = (0)$. Let us now prove the second assertion. We first remark that if $\partial f = cf$, for $c \in F$ and $f \in F^\times$, then $K_{\phi+c}^{(\nu)} = fK_\phi^{(\nu)}$ for every $\phi \in F$ and every ν. Therefore, if $\phi, \psi \in F$ are such that $\mathrm{Ker}_F(\partial - \phi + \psi) \neq (0)$, then $M_\phi^{(\nu)} = M_\psi^{(\nu)}$, for every ν. □

8.1.10 (Logarithmic derivatives). We denote by $\partial \log F^\times$ the additive subgroup of F consisting of *logarithmic derivatives*, i.e., elements of the form $f^{-1}\partial f$ for $f \in F^\times$.

An element u of F is in $\partial \log F^\times$ if and only if there exists $f \in F^\times$ such that $\partial f = uf$, that is, if and only if $\mathrm{Ker}(\partial - u) \neq \{0\}$.

As an example, take $F = K((x))$ and $\partial = \vartheta_x = x\partial_x$. Then $\partial \log F^\times$ is $\mathbb{Z} \oplus xK[[x]]$ (in particular, we have $\partial \log F^\times \cap K\big[\frac{1}{x}\big] = \mathbb{Z}$ and $\partial \log F^\times + K\big[\frac{1}{x}\big] = F$) and the quotient $F/\partial \log F^\times$ is isomorphic to $K\big[\frac{1}{x}\big]/\mathbb{Z} \cong K/\mathbb{Z} \oplus \frac{1}{x}K\big[\frac{1}{x}\big]$.

8.1.11. Notice that for each ϕ_i and for any logarithmic derivative $\psi = \partial \log(u)$ we have a canonical isomorphism of K-vector spaces $K_\phi^{(\nu)} \cong K_{\phi+\psi}^{(\nu)}$ (sending f to uf), and the equality $M_\phi^{(\nu)} = M_{\phi+\psi}^{(\nu)}$. In particular one may add a logarithmic derivative without changing the differential submodule $M_{\phi_i}^{(\mu)}$.

Definition 8.1.12 (Jordan differential modules, characters). *Let (F, ∂) be a differential field satisfying $(*)$ of 8.1.3. We say that the differential module (M, ∇_∂) of rank μ over (F, ∂)* admits a Jordan decomposition*, or* is a Jordan module*, if there exist $\phi_1, \ldots, \phi_r \in F$, such that*

$$M = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)},$$

where the ϕ_i's are pairwise distinct modulo $\partial \log F^\times$. Their classes $\overline{\phi}_i$ in $F/\partial \log F^\times$ are called the characters[3] *of the Jordan module. We define also the multiplicities of the character ϕ to be $\dim_K(K_\phi^{(\mu)}) = \dim_F(M_\phi^{(\mu)})$.*

Remark 8.1.13. The characters $\overline{\phi}_1, \ldots, \overline{\phi}_r \in F/\partial \log F^\times$ of (M, ∇_∂) and their multiplicities are obviously uniquely defined by our construction. In case $F = K((x))$ and $\partial' = u(x)\partial$ for a unit $u(x) \in K[[x]]^\times$, the F-vector space M which supports the differential module (M, ∇_∂) over (F, ∂) also supports the differential module $(M, \nabla_{\partial'} = u(x)\nabla_\partial)$ over (F, ∂'). The characters $\overline{\phi_1}, \ldots, \overline{\phi_r}$ of the former get respectively changed into $\overline{u(x)\phi_1}, \ldots, \overline{u(x)\phi_r}$, the modules $(M_{\phi_1}^{(\mu)}, \ldots, M_{\phi_r}^{(\mu)})$ into $(M_{u(x)\phi_1}^{(\mu)}, \ldots, M_{u(x)\phi_r}^{(\mu)})$, and the multiplicity of $\overline{u(x)\phi_i}$ for $(M, \nabla_{\partial'})$ over (F, ∂') coincides with the one of $\overline{\phi}_i$ for (M, ∇_∂) over (F, ∂).

8.1.14. The set of characters of M will be denoted by $\mathrm{Chr}(M)$. It is the set of (classes of) $\phi \in F$ for which

$$K_\phi^{(\mu)}(M) = \mathrm{Ker}_M(\nabla_\partial - \phi)^\mu \neq 0.$$

For example, consider again $F = K((x))$ and $\partial = \vartheta_x = x\partial_x$. Then the trivial differential module $M = F$ has $\mathrm{Chr}(M) = \{0\}$, while $M = F.x^\alpha$ for $\alpha \in K$ has $\mathrm{Chr}(M) = \{\alpha\}$. The differential module $M = F.\exp(\phi)$ for $\phi \in F$ has $\mathrm{Chr}(M) = \{\partial(\phi)\}$.

In general, differential modules of rank 1 are Jordan modules, and they are parametrized by characters as elements of $F/\partial \log F^\times$. By a recursive argument, any differential module is a direct sum of a Jordan module and a differential module without rank 1 submodules or quotients.

Proposition 8.1.15 (Structure of Jordan modules). *Every Jordan module over a differential field (F, ∂), satisfying $(*)$ of 8.1.3, is an iterated extension of differential modules of rank 1. Conversely, a differential module (M, ∇_∂) over a differential field (F, ∂) which is an iterated extension of differential modules of rank 1, is a Jordan module. Its characters are the characters of its components of rank 1 and*

[3] This terminology is not standard but will be convenient. It is justified by the tannakian viewpoint: the characters of a Jordan module M are the characters of the tannakian group attached to the tannakian category of differential modules generated by M.

the multiplicity of a character coincides with the multiplicity of the corresponding component in the sense of Jordan-Hölder.

Proof. This is achieved by writing $K_{\phi_i}^{(\mu)}$ as a successive extension of $K[\nabla_\partial - \phi_i]$-modules of K-rank 1 (note that $\nabla_\partial - \phi_i$ is a nilpotent endomorphism of $K_{\phi_i}^{(\mu)}$), tensoring by F and summing up over i. □

8.1.16. The structure of Jordan modules up to isomorphism is completely determined by the characters ϕ_i and for any character the sequence of integer numbers

$$d_{\phi_i}^{(\nu)} = \dim_K(K_{\phi_i}^{(\nu)}) = \dim_F(M_{\phi_i}^{(\nu)})$$

for $\nu = 1, \ldots, \mu$. The following immediate consequence of Corollary 8.1.6 will be used in the next subsection.

Lemma 8.1.17. *Any element m of a Jordan module M such that $(\nabla_\partial - \phi)(m) \in M_\phi^{(\nu)}$ belongs to $M_\phi^{(\nu+1)}$.*

Lemma 8.1.18. *Subquotients, extensions, tensor products and duals of Jordan modules are Jordan modules.*

If M is a Jordan module over F, then $M_{F'}$ is a Jordan module over any differential extension F' of F with the same characters, taken modulo $\partial \log F'^\times$.

For characters, one has the formulas (comparing subsets of the abelian group $F/\partial \log F^\times$):

$$\mathrm{Chr}(M) = \mathrm{Chr}(M_1) \cup \mathrm{Chr}(M_2)$$

if M is an extension of M_1 by M_2, and

$$\mathrm{Chr}(M \otimes M') = \mathrm{Chr}(M) + \mathrm{Chr}(M'),$$
$$\mathrm{Chr}(M^\vee) = -\mathrm{Chr}(M),$$
$$\mathrm{Chr}(M_{F'}) = \mathrm{Chr}(M).$$

The proof is straightforward. □

Proposition 8.1.19. *Let (M, ∇_∂) be a differential module over a differential field (F, ∂) satisfying assumption $(*)$ of 8.1.3. Suppose that there exists a differential extension (G, ∂) such that the module M_G admits a Jordan decomposition (over G). Then there exists a minimal extension F' of F under which $M_{F'}$ admits a Jordan decomposition. Moreover, F' is a Galois extension of F, it is generated over F by the characters of M (in G), and the degree is a divisor of $\mu!$.*

Proof. We may suppose that G is a Galois extension of F of Galois group Γ. Let $\phi_1, \ldots, \phi_r \in G$ the elements appearing in the Jordan decomposition of M_G (pairwise distinct modulo $\partial \log G^\times$). Then the elements of Γ permute the classes $\overline{\phi}_i$. Therefore, using the normal subgroup Γ' of Γ acting trivially on the characters of M_G, the projectors which define the Jordan decomposition of M_G descend to define a Jordan decomposition of $M_{F'}$ where F' is the Galois extension of F corresponding to Γ'. The degree of F' over F is $[\Gamma : \Gamma']$, which divides $\mu!$. □

8.2 Action of commuting derivations

We keep the notation of the previous subsection: namely, (F, ∂) is a differential field of characteristic 0, and $K = F^{\partial}$ is its field of constants. Let Δ be a set of derivations δ of F which commute with ∂. One thus has $\Delta(K) \subseteq K$. Let $k = K^{\Delta} = F^{\partial,\Delta}$ be the field of simultaneous constants of ∂ and the δ's. Notice that k is algebraically closed in K.

Let us assume (M, ∇_∂) is endowed with a map

$$\Delta \longrightarrow \mathrm{End}_{k\langle\partial\rangle} M, \quad \delta \longmapsto \nabla_\delta$$

such that every ∇_δ satisfies the Leibniz rule (hence makes (M, ∇_δ) into a differential module over (F, δ)).

Proposition 8.2.1 (**Stability of Jordan decomposition**). *If (M, ∇_∂) has a Jordan decomposition*

$$M = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)},$$

this decomposition is stable under ∇_δ for any $\delta \in \Delta$.

Proof. We prove, by induction on ν, that $\nabla_\delta(M_\phi^{(\nu)}) \subseteq M_\phi^{(\nu+1)}$. This is trivial for $\nu = 0$, so let us deduce the case ν from the case $\nu - 1$. It is enough to show that for any $m \in K_\phi^{(\nu)}$, $\nabla_\delta(m)$ belongs to $M_\phi^{(\nu+1)}$. We know that $(\nabla_\partial - \phi)(m)$ belongs to $K_\phi^{(\nu-1)}$. By the induction hypothesis, $\nabla_\delta(\nabla_\partial - \phi)(m) \in M_\phi^{(\nu)}$. It follows that

$$(\nabla_\partial - \phi)\nabla_\delta(m) = \nabla_\delta(\nabla_\partial - \phi)(m) + \partial(\phi)m$$

belongs to $M_\phi^{(\nu)}$. Then, according to Lemma 8.1.17, $\nabla_\delta(m)$ belongs to $M_\phi^{(\nu+1)}$. □

When the ϕ_i's belong to K, there is a more straightforward way to get this result. In fact, one has the following lemma, which holds without assuming that M is a Jordan module.

Lemma 8.2.2. *For any $\alpha \in K$, we have that $\nabla_\delta(K_\alpha^{(\nu)}) \subseteq K_\alpha^{(\nu+1)}$. A fortiori, $(K_\alpha^{(\mu)}, \nabla_\delta)$ is a differential module over (K, δ).*

Proof. The statement is trivial for $\nu = 0$. We proceed by induction: assume that the statement holds for $\nu < \nu_0$ ($\nu_0 \geqslant 1$) and let $m \in K_\alpha^{(\nu_0)}$. So, $(\nabla_\partial - \alpha)^{\nu_0} m = 0$ and $(\nabla_\partial - \alpha)m \in K_\alpha^{(\nu_0-1)}$. Then

$$(\nabla_\partial - \alpha)^{\nu_0+1}\nabla_\delta m = (\nabla_\partial - \alpha)^{\nu_0}\nabla_\delta(\nabla_\partial - \alpha)m + (\nabla_\partial - \alpha)^{\nu_0}(\delta\alpha)m = 0,$$

since $\delta\alpha \in K$ and by the induction assumption. □

Theorem 8.2.3. *Assume that* (M, ∇_∂) *has a Jordan decomposition*

$$M = \bigoplus_{i=1}^{r} M_{\alpha_i}^{(\mu)}$$

with $\alpha_i \in K$. *Then:*

(1) *there exists a basis* $\mathbf{m} = (m_1, \ldots, m_\mu)$ *of* M *over* F *such that*

$$\nabla_\partial \, \mathbf{m} = \mathbf{m} \, H_\partial, \quad \nabla_\delta \, \mathbf{m} = \mathbf{m} \, H_\delta \ (\forall \, \delta \in \Delta),$$

with both H_∂ *and* H_δ *in* $M_\mu(K)$.

(2) *For any basis* $\mathbf{m}$ *of* M *as in* (1)*, the eigenvalues of* H_∂ *are in* $k = K^\Delta$.

Proof. (1): by Lemma 8.2.2, the $K_\alpha^{(\mu)}((M, \nabla_\partial))$ are stable under ∇_δ. This shows (1).

(2): from $\nabla_\partial \nabla_\delta = \nabla_\delta \nabla_\partial$ we get (cf. (4.4.4)) $[H_\partial, H_\delta] = \delta(H_\partial) - \partial(H_\delta) = \delta(H_\partial)$, and by induction, $[H_\partial^m, H_\delta] = \delta(H_\partial^m)$. To show that the case m implies the case $m+1$, we compute

$$\begin{aligned} \delta(H_\partial^{m+1}) &= \delta(H_\partial^m) H_\partial + H_\partial^m \delta(H_\partial) \\ &= (H_\partial^m H_\delta - H_\delta H_\partial^m) H_\partial + H_\partial^m (H_\partial H_\delta - H_\delta H_\partial) \\ &= -H_\delta H_\partial^{m+1} + H_\partial^{m+1} H_\delta \ . \end{aligned}$$

So, $\delta(\mathrm{Tr}(H_\partial^m)) = \mathrm{Tr}(\delta(H_\partial^m)) = 0$. Therefore, the coefficients of the characteristic polynomial of H_∂ are δ-constants, for any $\delta \in \Delta$, hence are in k. A fortiori, the eigenvalues of H_∂ are algebraic over k. Since they coincide with the $\alpha_i \in K$ modulo logarithmic derivatives, hence are in K, and since k is algebraically closed in K, we conclude that the eigenvalues of H_∂ are in k. □

8.3 The regular case

8.3.1. We come back to the case $(F, \partial) = (K((x)), \vartheta_x = x\partial_x)$. Condition $(*)$ of 8.1.3 is then satisfied, and the only logarithmic derivatives which belong to K are the integers $(n = \vartheta_x(x^n)/x^n)$:

$$K/\mathbb{Z} \subseteq F/\partial \log F^\times .$$

For any $u \in K((x))^\times$, passing from $(M, \nabla_{u\partial_x})$ to $(M, \nabla_{\vartheta_x} = x\nabla_{\partial_x})$ identifies differential modules over $(K((x)), u\partial_x)$ and differential modules over $(K((x)), \vartheta_x)$. We shall just say that M is a differential module over $K((x))$.

The following definition formalizes the notion of regular differential module over $(F, \partial) = (K((x)), \vartheta_x = x\partial_x)$.

Definition 8.3.2 (**Regularity for differential modules).** *A differential module M over $K((x))$ is* regular *if it contains a $K[[x]]$-lattice (i.e., a free $K[[x]]$-module which spans M) stable under $x\nabla_{\partial_x} = \nabla_{\vartheta_x}$.*

The discussion of Section 7 on regular differential systems may be summarized in the following statements in the language of differential modules.

Theorem 8.3.3 (**Regularity criteria for differential modules).** *The following conditions on a differential module M of rank μ over $K((x))$ are equivalent:*

(1) *M is regular (that is, M admits a $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice);*

(2) *for every $m \in M$, the smallest $K[[x]]$-submodule of M containing m and stable under $x\nabla_{\partial_x}$ is finitely generated;*

(3) *the monic differential operator $\Lambda \in K((x))\langle x\partial_x\rangle$ attached to some (resp. any) cyclic vector m of M has coefficients in $K[[x]]$.*

Proof. The equivalence of the first two items is clear. The equivalence with the third item follows from the Fuchs criterion 7.5.1. □

Theorem 8.3.4 (**Structure of regular differential modules).** *Let M be a regular differential module M of rank μ over $K((x))$. Then there exists a finite extension $K' = K[\alpha_1, \dots, \alpha_r]$ of K such that $M_{K'((x))} = K' \otimes_K M$ admits a Jordan decomposition (with respect to $\vartheta_x = x\partial_x$)*

$$M_{K'((x))} = \bigoplus_{i=1}^{r} M_{\alpha_i}^{(\mu)}, \tag{8.3.5}$$

where $M_{\alpha_i}^{(\mu)} = K'((x)) \otimes_{K'} \mathrm{Ker}_{M_{K'((x))}}(x\nabla_{\partial_x} - \alpha_i)^{\mu}$ and the $\alpha_i \in K'$ are pairwise distinct modulo $\mathbb{Z}$. This decomposition is independent of the choice of the α_i's modulo $\mathbb{Z}$ and is canonical.

Proof. The theorem follows from 7.4.1 and the definition of a regular module. The sense in which uniqueness and canonicity hold was described in general in Remark 8.1.13. □

In fact the theorem asserts an equivalence, since the converse is obvious: a differential module over $K((x))$ is regular if and only if it admits a Jordan decomposition with characters in a finite extension of K.

Theorem 8.3.6 (**Exponents).** *Let M be a regular differential module M of rank μ over $K((x))$ as in the previous theorem. The characters $\overline{\alpha}_i = \alpha_i + \mathbb{Z} \in \overline{K}/\mathbb{Z}$ are called the* exponents of M, *and the set of exponents is denoted by* $\mathrm{Exp}\,(M)$, *instead of* $\mathrm{Chr}(M)$. *The following conditions on an element of $\overline{K}/\mathbb{Z}$ are equivalent:*

(1) *$\overline{\alpha} \in \mathrm{Exp}\,(M)$;*

(2) *M has a non-zero solution in $x^{\overline{\alpha}}\,\overline{K}((x))$, in the sense that for any lift α of $\overline{\alpha}$ in $\overline{K}$, there is a horizontal morphism from M to the $\overline{K}((x))$-differential module generated by an element x^α such that $\vartheta_x(x^\alpha) = \alpha x^\alpha$;*

(3) *$\overline{\alpha}$ is the class modulo $\mathbb{Z}$ of an eigenvalue of the value at $x = 0$ of the matrix of $x\nabla_{\partial_x}$ in some basis of some (or any) $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice of M (notice that the exponents of M are the exponents of any associated differential system (cf. 2.6), because the relation $G(0) = {}^tH(0)$ preserves eigenvalues);*

(4) *$\overline{\alpha}$ is the class modulo $\mathbb{Z}$ of a root of the indicial polynomial of Λ (more precisely, the dimension of $\operatorname{Ker}_{M_{K'((x))}}(x\nabla_{\partial_x} - \alpha_i)^\mu$ is the sum of the multiplicities of the roots of the indicial polynomial of Λ which are congruent to α_i modulo $\mathbb{Z}$).*

Proof. This follows from the definition 8.1.12 of characters and 7.6. □

Any one of the previous conditions allows one to associate to each exponent a well-defined multiplicity (coherenty with 7.6.4).

Theorem 8.3.7 (Descent of the decomposition). *Let $\widetilde{M}$ be a $x\nabla_{\partial_x}$-stable $K[[x]]$-lattice such that the value at $x = 0$ of the matrix of $x\nabla_{\partial_x}$ in some basis of $\widetilde{M}$ is non-resonant (in the sequel, we simply say that $\widetilde{M}$ is* non-resonant*). Then, taking α_i to be the eigenvalues of that matrix, the decomposition (8.3.5) descends to a decomposition*

$$\widetilde{M}_{K'[[x]]} = \bigoplus_{i=1}^{r} K'[[x]] \otimes_{K'} \operatorname{Ker}_{\widetilde{M}_{K'[[x]]}}(x\nabla_{\partial_x} - \alpha_i)^\mu. \tag{8.3.8}$$

In the class $\overline{\alpha}_i$, viewed as a ordered set, α_i is characterized as the supremum among the lifts α for which $\widetilde{M}$ has a solution in $x^\alpha K'[[x]]$.

Proof. This is a translation of item (3) of Proposition 7.4.1 in terms of differential modules, using the correspondence established in 2.6. □

The proof of the following stability properties with respect to the usual operations (2.7, 2.8) is straightforward and left to the reader (see also [38, III,8]).

Proposition 8.3.9. *Regularity of differential modules over $F = K((x))$ is stable under taking subquotients, extensions, tensor products, and duals.*

If M is a regular differential module over F, then $M_{F'}$ is regular over any differential extension of the form $F' = K'((x^{1/e}))$ (equipped with the K'-linear extension of the derivation ∂_x, K' being some extension of K). If K'/K is finite, and if M' is a regular differential module over F', then ${}_FM'$ is regular over F; moreover, if M is a differential module over F and $M_{F'}$ is regular as a differential module over F', then M is a regular differential module over F.

For exponents, one has the formulas (comparing subsets of the divisible abelian group $\overline{K}/\mathbb{Z}$):

$$\operatorname{Exp}(M) = \operatorname{Exp}(M_1) \cup \operatorname{Exp}(M_2)$$

if M is an extension of M_1 by M_2, and

$$\begin{aligned}\operatorname{Exp}(M\otimes M') &= \operatorname{Exp}(M)+\operatorname{Exp}(M'),\\ \operatorname{Exp}(M^\vee) &= -\operatorname{Exp}(M),\\ \operatorname{Exp}(M_{F'}) = e\cdot\operatorname{Exp}(M),\ \ & \operatorname{Exp}({}_F M') = \frac{1}{e}\cdot\operatorname{Exp}(M').\end{aligned}$$

In the last two formulas, it is understood that F' is endowed with the derivation $x'\frac{d}{dx'} = \frac{x}{e}\frac{d}{dx}$. □

Finally, as in Subsection 8.2, we consider a set Δ of derivations of F commuting with $x\partial_x$ (for instance, derivations of K, which one extends to F by setting $\delta(x) = 0$), and $k = K^\Delta$.

Theorem 8.3.10. *Assume M is a regular differential module over $K((x))$, endowed, for every $\delta \in \Delta$, with an action ∇_δ commuting with ∇_{∂_x} and making (M, ∇_δ) into a differential module over (F, δ). Assume that the exponents of (M, ∇_{∂_x}) are in $K/\mathbb{Z}$ (rather than just in $\overline{K}/\mathbb{Z}$). Then:*

(1) *there exists a basis $\mathbf{m} = (m_1, \ldots, m_\mu)$ of M over $K((x))$ such that*

$$\nabla_{\partial_x}\mathbf{m} = \mathbf{m}\,H_{\partial_x},\quad \nabla_\delta\,\mathbf{m} = \mathbf{m}\,H_\delta\ \ (\forall\,\delta\in\Delta),$$

with both H_{∂_x} and H_δ in $M_\mu(K)$.

(2) *For any basis $\mathbf{m}$ of M as in (1), the eigenvalues of H_{∂_x} are in k.*

Proof. This follows from point (3) of theorem 8.3.4, and theorem 8.2.3. □

8.4 Variant with parameters

8.4.1. We address now the situation where K is the fraction field of a noetherian, integrally closed k-algebra R. We consider a differential module (M, ∇_{∂_x}) over the differential ring $(R((x)), \partial_x)$, where $R((x)) = R[[x]][\frac{1}{x}]$, which is "generically regular", i.e., such that $M_{K((x))}$ is regular.

The theme of this section is: to which extent does the decomposition (8.3.5) descend to $R'((x))$, where R' denotes the integral closure of R in K'?

Theorem 8.4.2. *In the setting 8.4.1:*

(1) *The exponents $\overline{\alpha}_i$ belong to $R'/\mathbb{Z}$.*

(2) *If the differences between exponents are constant, i.e., belong to the algebraic closure k' of k in R' (modulo $\mathbb{Z}$), the decomposition (8.3.5) descends to a decomposition of $M_{R'((x))}$:*

$$M_{R'((x))} = \bigoplus_{i=1}^{r} R'((x))\otimes_{R'}\operatorname{Ker}_{M_{R'((x))}}(x\nabla_{\partial_x}-\alpha_i)^\mu. \tag{8.4.3}$$

Remark 8.4.4. Some condition on the exponents is required to descend the decomposition: for instance, the differential module M attached to the differential operator $(x\partial_x)^2 - y^2$ with parameter y (for which one can take $\overline{\alpha}_1 = y + \mathbb{Z}$, $\overline{\alpha}_2 = -y + \mathbb{Z}$) decomposes over $K[y, \frac{1}{y}]((x))$, but not over $K[y]((x))$, since $M_{|y=0}$ is not semisimple.

Of course, such a phenomenon cannot occur in the integrable case, where the α's are indeed constant.

Proof. For simplicity, let us replace M by $M_{R'((x))}$ and drop all $'$ from the notation. Since R is normal, it suffices to show that for any divisorial valuation (that is, a valuation associated to prime ideal of height 1) v of R, the α's are v-integral. We may thus replace R by its v-adic completion $\widehat{R}$ (which is a complete discrete valuation ring, with fraction field $\widehat{K}$), and M by $\widehat{M} = M_{\widehat{R}((x))}$. We fix an isomorphism $\widehat{R} \cong \kappa[[y]]$, κ being the residue field (which we regard as a subfield of $\widehat{R}$). Weierstrass' division theorem implies that the ring $\widehat{R}((x))$ is principal. Therefore, the projective $\widehat{R}((x))$-module $\widehat{M}$ is free (of rank μ).

Up to replacing $\widehat{R}$ by a finite extension, we first construct a ϑ_x-stable $\widehat{R}[[x]]$-lattice in $\widehat{M}$. For this purpose, we use a cyclic vector, at the cost of introducing apparent singularities.

More precisely, we first choose a cyclic vector $\overline{m}$ of the $\kappa((x))$-differential module $\widehat{M}_{\kappa((x))} = \widehat{M}/y\widehat{M}$. Any lift $m \in \widehat{M}$ of $\overline{m}$ is a cyclic vector of the differential module $\widehat{M}_{\mathrm{Frac}(\widehat{R}((x)))}$. Indeed, it suffices to see that it becomes a cyclic vector over some extension of $\mathrm{Frac}(\widehat{R}((x))) = \mathrm{Frac}(\kappa[[x, y]])$, and the fraction field of the v-adic completion of $\widehat{R}((x)) = \kappa[[y]]((x))$ (which is nothing but $\kappa((x))((y))$) is such an extension: the smallest ϑ_x-stable $\kappa((x))[[y]]$-submodule of $\widehat{M}_{\kappa((x))[[y]]}$ containing m coincides with $\widehat{M}_{\kappa((x))[[y]]}$ (by Nakayama's lemma, since this holds modulo y by assumption).

Since the connection is generically regular, the matrix of ϑ_x in the cyclic basis

$$\mathbf{m} = (m, \nabla_{\vartheta_x} m, \dots, \nabla_{\vartheta_x}^{\mu-1} m)$$

of $\widehat{M}_{\mathrm{Frac}(\widehat{R}((x)))}$ has no pole in the variable x. This means that

$$\nabla_{\vartheta_x}(\mathbf{m}) = \mathbf{m}\widetilde{H},$$

with $\widetilde{H} \in M_\mu((\widehat{R}[[x]])_{(x)}) \subseteq M_\mu(\widehat{K}[[x]])$.

In order to clear out the denominators, we use a classical technique (cf. e.g., [38, V.5.1]). We fix a basis $\mathbf{n}$ of $\widehat{M}$ over $\widehat{R}((x))$, and consider the matrix from $\mathbf{n}$ to $\mathbf{m}$:

$$\mathbf{m} = \mathbf{n}Q.$$

Since the elements of $\mathbf{m}$ are in $\widehat{M}$, we have

$$Q \in \mathrm{GL}_\mu(\mathrm{Frac}(\widehat{R}((x)))) \cap M_\mu(\widehat{R}((x))).$$

On the other hand, since $\mathbf{m}$ modulo y is a cyclic basis of $\widehat{M} \otimes \kappa((x))$, Q modulo y lies in $\mathrm{GL}_\mu(\kappa((x)))$. We use now the following assertion.

8.4.5. Claim. *Replacing $\widehat{R} \cong \kappa[[y]]$ by a finite extension if necessary, one can write Q as a product $Q'(Q'')^{-1}$, where*

$$Q' \in \mathrm{GL}_\mu(\widehat{R}((x))), \quad Q'' \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)}).$$

Proof of the claim. Since $\det Q$ modulo y is non-zero, $\det Q$ can be written, according to Weierstrass' preparation theorem ([25, VII, 3.8, prop. 6]), as the product of a monic polynomial $q \in \widehat{R}[x]$ and a unit $u \in \widehat{R}((x))^\times$. Replacing $\widehat{R}$ by a finite extension, we may and shall assume that the zeroes of q belong to $y\widehat{R}$.

Let ξ be one of these zeroes. By induction, it suffices to find a matrix

$$Q''_1 \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)})$$

such that $QQ''_1 \in M_\mu(\widehat{R}((x)))$, $\det(QQ''_1)$ divides $\det(Q)$, and $\mathrm{ord}_\xi \det(QQ''_1) < \mathrm{ord}_\xi \det(Q)$. Let $\lambda_1, \ldots, \lambda_\mu \in \widehat{R}$ be the coefficients of a non-trivial dependence relation between the *columns* of $Q_{|x=\xi}$. We assume, as we may, that one of these numbers, say $\lambda_i = 1$. Then

$$Q''_1 = \left(\begin{array}{c|c|c} & \lambda_1/(x-\xi) & \\ \mathbf{I}_{i-1} & \vdots & \mathbf{O}_{i-1,\mu-i} \\ & \lambda_{i-1}/(x-\xi) & \\ \hline 0\cdots 0 & 1/(x-\xi) & 0\cdots 0 \\ \hline & \lambda_{i+1}/(x-\xi) & \\ \mathbf{O}_{\mu-i,i-1} & \vdots & \mathbf{I}_{\mu-i} \\ & \lambda_\mu/(x-\xi) & \end{array}\right),$$

whose inverse is

$$\left(\begin{array}{c|c|c} & -\lambda_1 & \\ \mathbf{I}_{i-1} & \vdots & \mathbf{O}_{i-1,\mu-i} \\ & -\lambda_{i-1} & \\ \hline 0\cdots 0 & x-\xi & 0\cdots 0 \\ \hline & -\lambda_{i+1} & \\ \mathbf{O}_{\mu-i,i-1} & \vdots & \mathbf{I}_{\mu-i} \\ & -\lambda_\mu & \end{array}\right),$$

is easily seen to satisfy the required property. □

We continue the proof of the theorem. The basis

$$\mathbf{n}' := \mathbf{n}Q' = \mathbf{m}Q''$$

generates a ϑ_x-stable $\widehat{R}[[x]]$-lattice in $\widehat{M}$. On the one hand, $\mathbf{n}'$ is a basis of $\widehat{M}$ over $\widehat{R}((x))$, and therefore the matrix $H = \widetilde{H}_{[Q'']}$ such that

$$\nabla_{\vartheta_x}\mathbf{n}' = \mathbf{n}'H$$

belongs to $M_\mu(\widehat{R}((x)))$. On the other hand,

$$\mathbf{n}' = \mathbf{m}Q'',$$

with $Q'' \in \mathrm{GL}_\mu(\widehat{K}[x]_{(x)}) \subseteq \mathrm{GL}_\mu(\widehat{K}[[x]])$, guarantees that $H \in M_\mu(\widehat{K}[[x]])$. But $\widehat{R}((x)) \cap \widehat{K}[[x]] = \widehat{R}[[x]]$, so that $H \in M_\mu(\widehat{R}[[x]])$. The eigenvalues of $H_{|x=0}$ then coincide, modulo $\mathbb{Z}$, with the α's. In particular, the α's are v-integral. This proves item (1) of the theorem.

Let us now prove item (2). We first notice that if M is projective of finite type over a ring R, K' an extension of $K = \mathrm{Frac}(R)$ and R' a subring of K' s.t. $R' \cap K = R$, then a decomposition of M_K descends to a decomposition of M if and only if the corresponding decomposition of $M_{K'}$ descends to a decomposition of $M_{R'}$. In fact, the decomposition is just defined by its projectors $n \in \mathrm{End}(M)$. Now $\mathrm{End}(M) = M^\vee \otimes M$ is projective of finite type, therefore there exists a set of generators $v_1, \dots, v_n$ of $\mathrm{End}(M)$ and a set $v_1^\vee, \dots, v_n^\vee$ of $\mathrm{End}(M)^\vee$ (R-dual of M) such that $n = \sum_\ell v_\ell^\vee(n) v_\ell$ (see for example [23, ch.II, par. 2, prop. 12]). In particular, the projectors have coordinates in R' if and only if they have coordinates in R.

Therefore, it is sufficient to show that the decomposition extended to $\widehat{K}((x))$ descends to $\widehat{R}((x))$ (for one completion). From the hypothesis we deduce that the exponents are constant, and by applying suitable shearing transformations in $\mathrm{GL}_\mu(\widehat{R}[x, x^{-1}])$ (cf. Lemma 7.2.4), we may assume that the eigenvalues of $H_{|x=0}$ are prepared.

It suffices now to find a basis $\mathbf{n}''$ of $\widehat{M}$ in which ϑ_x has matrix $H_0 := H_{|x=0}$. We know that there is a (unique) matrix $Y \in \mathrm{GL}_\mu(\widehat{K}[[x]])$ with $Y_{|x=0} = I$ and $\vartheta_x(Yx^{-H_0}) = -HYx^{-H_0}$. It is thus enough to show that, under the condition that the α's are constant and distinct modulo $\mathbb{Z}$, the entries of Y lie in $\widehat{R}[[x]]$. The condition on the exponents implies that, for every $n \in \mathbb{Z}_{>0}$, the endomorphism

$$\overline{Y} \longmapsto n\overline{Y} + H_0\overline{Y} - \overline{Y}H_0$$

of $M_\mu(\widehat{R}/y\widehat{R}) = M_\mu(\kappa)$ is injective, hence bijective by reason of dimension. Let H_n and Y_n denote the n^{th} coefficients of H and Y, respectively. The Y_n's satisfy the recursion ([38, III.8.5])

$$nY_n + H_0Y_n - Y_nH_0 = -(H_1Y_{n-1} + H_2Y_{n-2} + \cdots + H_n).$$

It order to conclude, it suffices to notice that the endomorphism $Y_n \mapsto nY_n + H_0Y_n - Y_nH_0$ of $M_\mu(\widehat{R})$ is surjective, since it is so modulo y. □

9 Formal theory of integrable connections (several variables)

In this section we introduce the theory of formal connections in several variables, following in the exposition the fundamental paper [44] of Gérard and Levelt, with some complements. The main results are the decomposition of Gérard-Levelt structures (a generalization to several variables of the Jordan decomposition for regular modules), and the characterization of regularity for differential modules in several variables, in terms of regularity for each variable separately.

9.1 Outline of Gérard-Levelt theory

The basic object in the paper [44] is the following:

Definition 9.1.1 (Gérard-Levelt structure). *A d-dimensional Gérard-Levelt structure (a "GL-structure") over k is a set of data $(\Lambda, (\nabla_1, \ldots, \nabla_d))$ where Λ is a torsion free $k[[x_1, \ldots, x_d]]$-module of finite type, endowed with k-linear endomorphisms $\nabla_1, \ldots, \nabla_d$ which commute and satisfy the Leibniz rule*

(9.1.2) $$\nabla_j(fm) = x_j \frac{\partial f}{\partial x_j} m + f\nabla_j(m),$$

for any $f \in k[[x_1, \ldots, x_d]]$ and $m \in \Lambda$.

9.1.3 (Generic and special ranks). We denote by $k((\mathbf{x}))$ the fraction field of $k[[\mathbf{x}]] = k[[x_1, \ldots, x_d]]$ and regard Λ as a "lattice" in the $k((\mathbf{x}))$-vector space $V = \Lambda \otimes k((\mathbf{x}))$, i.e., as a finitely generated sub-$k[[\mathbf{x}]]$-module of V which spans V over $k((\mathbf{x}))$. The action of $\nabla_1, \ldots, \nabla_d$ may be uniquely extended to V by imposing the rule (9.1.2) for any $f \in k((\mathbf{x}))$.

Let $p = \dim_{k((\mathbf{x}))} V$ (resp. $q = \dim_k \Lambda/(x_1, \ldots, x_d)\Lambda$) denote the *generic* (resp. *special*) rank of Λ . Then $q \geqslant p$ and Λ is a free $k[[\mathbf{x}]]$-module if and only if $p = q$.

9.1.4. For any d-dimensional GL-structure $(\Lambda, (\nabla_1, \ldots, \nabla_d))$ over k, the operators $\nabla_1, \ldots, \nabla_d$ induce commuting k-linear operators $\delta_1, \ldots, \delta_d$ on $\Lambda/(x_1, \ldots, x_d)\Lambda$. A straightforward Jordan theory[4] attaches to the k-vector space $\Lambda/(x_1, \ldots, x_d)\Lambda$ equipped with the commuting operators $(\delta_1, \ldots, \delta_d)$ a notion of *multiplicity* e_α *of a vector* $\alpha = (\alpha_1, \ldots, \alpha_d) \in k^d$ *as a multiexponent of* $(\delta_1, \ldots, \delta_d)$, namely

$$e_\alpha = \dim_k \bigcap_{i=1}^{d} \mathrm{Ker}(\delta_i - \alpha_i)^M,$$

for any $M \gg 0$. Obviously, $e_\alpha = 0$ for almost all α ("α is not a multiexponent of $(\delta_1, \ldots, \delta_d)$" in that case) and $\sum_\alpha e_\alpha = q$.

[4] developed in a more general context in Section 3 of [15].

Definition 9.1.5 (*GL*-**exponents, multiplicities).** *With the previous notation, we will say that* e_α *is the multiplicity of the GL-exponent* $\alpha \in k^d$ *of* $(\Lambda, (\nabla_1, \dots, \nabla_d))$.

Theorem 9.1.6 ([44, Thm. 3.4]). *Let* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *be a* d*-dimensional GL-structure over* k*, of generic rank* p *and special rank* q*. Let* $\alpha_k = (\alpha_{1,k}, \dots, \alpha_{d,k})$*, for* $k = 1, \dots, q$ *be the GL-exponents of* $(\Lambda, (\nabla_1, \dots, \nabla_d))$*, repeated according to their multiplicities.*

(i) *If*

$$\alpha_{i,j} - \alpha_{i,h} \notin \mathbb{Z} \setminus \{0\} \quad \text{for all } i, j, h, \tag{9.1.7}$$

then $p = q$ *(hence* Λ *is a free module), and there exists a* $k[[\mathbf{x}]]$*-basis* $\mathbf{e} = (e_1, \dots, e_p)$ *of* Λ *such that, for all* $i = 1, \dots, d$*,*

$$\nabla_i \, \mathbf{e} = \mathbf{e} \, C_i, \tag{9.1.8}$$

for a $p \times p$ *matrix* C_i *with coefficients in* k*.*

(ii) *There exists a free lattice* Λ' *in* V*, stable under the action of* $\nabla_1, \dots, \nabla_d$*, satisfying assumption* (9.1.7)*.*

Proof. (i) The main ingredient is the Jordan decomposition of a k-linear operator ϕ on a finite-dimensional k-vector space as the sum of its semisimple part ${}^{\mathrm{s}}\phi$ and of its nilpotent part ${}^{\mathrm{n}}\phi$, $\phi = {}^{\mathrm{s}}\phi + {}^{\mathrm{n}}\phi$, with $[{}^{\mathrm{s}}\phi, {}^{\mathrm{n}}\phi] = 0$. This elementary fact is applied to the filtered k-vector space $(\Lambda, \{(x_1, \dots, x_d)^M \Lambda\}_M)$, and to its endomorphisms $\nabla_1, \dots, \nabla_d$. One thus gets a Jordan decomposition $\nabla_i = {}^{\mathrm{s}}\nabla_i + {}^{\mathrm{n}}\nabla_i$, for $i = 1, \dots, d$, with the property that, for all $i = 1, \dots, d$ and $M = 1, 2, \dots$, the k-linear endomorphism induced by ${}^{\mathrm{s}}\nabla_i$ (resp. by ${}^{\mathrm{n}}\nabla_i$) on the finite-dimensional k-vector space $\Lambda/(x_1, \dots, x_d)^M \Lambda$ is semisimple (resp. nilpotent). The k-linear endomorphisms $\nabla_1, \dots, \nabla_d, {}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d, {}^{\mathrm{n}}\nabla_1, \dots, {}^{\mathrm{n}}\nabla_d$, commute. Moreover, ${}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d$ satisfy

$${}^{\mathrm{s}}\nabla_j(fm) = x_j \frac{\partial f}{\partial x_j} m + f \, {}^{\mathrm{s}}\nabla_j(m) \tag{9.1.9}$$

for any $f \in k[[\mathbf{x}]]$ and $m \in \Lambda$, while ${}^{\mathrm{n}}\nabla_1, \dots, {}^{\mathrm{n}}\nabla_d$ are $k[[\mathbf{x}]]$-linear and nilpotent. By semisimplicity, there exists a k-linear subspace W of Λ that complements $(x_1, \dots, x_d)\Lambda$, and is stable under ${}^{\mathrm{s}}\nabla_1, \dots, {}^{\mathrm{s}}\nabla_d$. The projection

$$W \xrightarrow{\sim} \Lambda/(x_1, \dots, x_d)\Lambda$$

identifies the restriction $({}^{\mathrm{s}}\nabla_i)_{|W}$ with ${}^{\mathrm{s}}\delta_i$, for every i. Hence, there exists a k-basis $(e_1, \dots, e_q)$ of W, such that

$${}^{\mathrm{s}}\nabla_i(e_j) = \alpha_{i,j} e_j \;, \tag{9.1.10}$$

for all i, j. We note that in any case $(e_1, \dots, e_q)$ generate Λ over $k[[\mathbf{x}]]$. So, in order to show that Λ is free, it suffices to prove that $(e_1, \dots, e_q)$ are linearly independent over $k((\mathbf{x}))$. Let us recall the argument of [44]. If $p < q$, we can write $e_{r+1} = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with $\lambda_i \in k((\mathbf{x}))$ (not all in k) and r minimal. Applying ${}^s\nabla_j$, we get $\alpha_{j,r+1}e_{r+1} = \sum_{\ell \leqslant r}(x_j \frac{\partial}{\partial x_j}\lambda_\ell + \alpha_{j,\ell}\lambda_\ell)e_\ell$, and on combining these equalities,

$$x_j \frac{\partial}{\partial x_j}\lambda_1 + (\alpha_{j,1} - \alpha_{j,r+1})\lambda_1 = \cdots = x_j \frac{\partial}{\partial x_j}\lambda_r + (\alpha_{j,r} - \alpha_{j,r+1})\lambda_r = 0.$$

Let us remark that for any $\lambda \in k((\mathbf{x}))$, $\lambda \neq 0$, $\beta_j \in k$, the simultaneous equations $x_j \frac{\partial}{\partial x_j}\lambda = \beta_j \lambda$ imply that all β_j are integers. This shows that if some $\lambda_\ell \neq 0$, then $\alpha_{j,\ell} = \alpha_{j,r+1}$ for all j, so that $\lambda_\ell \in k$. This shows $\lambda_\ell \in k$ for *all* ℓ, a contradiction. One also needs to show that ${}^{\mathrm{n}}\nabla_i(W) \subseteq W$, for all i. This is a general fact, under the assumption (9.1.7): for any $k((\mathbf{x}))$-linear endomorphism T of V such that $[{}^{\mathrm{s}}\nabla_i, T] = 0$, for all i, a simple argument shows that $T(W) \subseteq W$. So, the isomorphism $W \xrightarrow{\sim} \Lambda/(x_1, \ldots, x_d)\Lambda$ also identifies the restriction $({}^{\mathrm{n}}\nabla_i)_{|W}$ with ${}^{\mathrm{n}}\delta_i$, hence $(\nabla_i)_{|W}$ with δ_i, for every i.

(ii) One uses the method of *shearing transformations* 7.2.4 Let us pick $(e_1, \ldots, e_q)$ as in (9.1.10). They generate Λ over $k[[\mathbf{x}]]$. If we multiply each e_ℓ by a suitable Laurent monomial

$$m_\ell = \prod_{i=1}^{d} x_i^{\gamma_{i,\ell}}$$

in the $(x_1, \ldots, x_d)$, with $\gamma_{i,\ell} \in \mathbb{Z}$, we can arrange that condition (9.1.7) of the theorem is satisfied for the lattice Λ' spanned in V by $(e'_1, \ldots, e'_q)$, where $e'_\ell = m_\ell e_\ell$, for each ℓ. Of course, $(e'_1, \ldots, e'_q)$ need not be linearly independent over k, but they satisfy

$$({9.1.11}) \qquad {}^{\mathrm{s}}\nabla_i(e'_j) = \alpha'_{i,j}e'_j,$$

for all i, j, with $\alpha'_{i,j} = \alpha_{i,j} + \gamma_{i,j}$. The k-vector space W' generated by $(e'_1, \ldots, e'_q)$ in Λ' is a complement of $(x_1, \ldots, x_d)\Lambda'$, and is stable under ${}^{\mathrm{s}}\nabla_1, \ldots, {}^{\mathrm{s}}\nabla_d$. So, we may apply the discussion of part (i) to any maximal k-linearly independent subset of $(e'_1, \ldots, e'_q)$. Such a subset will be a free system of generators of Λ'. □

Remark 9.1.12. Notice that if in the previous discussion we replace $(e_1, \ldots, e_q)$ by a vector $(v_1, \ldots, v_q)$ of elements of Λ, with $v_j - e_j \in (x_1, \ldots, x_d)^M\Lambda$, for $M \gg 0$, the lattice spanned by $(v'_1, \ldots, v'_q)$, where $v'_\ell = m_\ell v_\ell$, for each ℓ, coincides with Λ'. This remark will be used in the sequel, when we have to replace the constructions of Gérard-Levelt in the formal setting by algebraic approximations.

Corollary 9.1.13. *Let $(\Lambda, (\nabla_1, \ldots, \nabla_d))$ be as in Theorem 9.1.6 and let $\tau = (\tau_1, \ldots, \tau_d)$ be any vector of set-theoretic sections τ_i of the projection $k \to k/\mathbb{Z}$.*

(i) *There is a unique free lattice Λ_τ in V, stable under the action of $\nabla_1, \ldots, \nabla_d$, and such that for every i the eigenvalues of the k-linear endomorphism induced by ∇_i on $\Lambda_\tau/(x_1, \ldots, x_d)\Lambda_\tau$ are in the image of τ_i.*

(ii) *There exists a free system of generators* $(e_1,\dots,e_p)$ *of* Λ_τ *over* $k[[x_1,\dots,x_d]]$*, such that* (9.1.10) *holds, with* $\alpha_{i,j}\in\operatorname{Im}\tau_i$*.*

(iii) *If* $\nabla_i\Lambda\subseteq x_i\Lambda$ *and* $\tau_i(\mathbb{Z})=0$ *for for* $i=s+1,\dots,d$*, the lattice* Λ_τ *satisfies* $\nabla_i\Lambda_\tau\subseteq x_i\Lambda_\tau$ *for those* i*'s.*

Proof. It is clear that in the discussion of (9.1.11) we can arrange that the $\alpha'_{i,j}$'s are in the image of τ_i, for all i,j. It is also clear from the previous discussion that the lattice Λ' generated by $(e'_1,\dots,e'_q)$ is free of rank p, and admits a basis $(e_1,\dots,e_p)$ as in the statement. Suppose now that we have two bases $\mathbf{e}=(e_1,\dots,e_p)$ and $\mathbf{e}'=(e'_1,\dots,e'_p)$ of V such that

$$^s\nabla_i(e_j)=\alpha_{i,j}e_j,\quad {}^s\nabla_i(e'_j)=\alpha'_{i,j}e'_j$$

with $\alpha_{i,j},\alpha'_{i,j}\in\operatorname{Im}\tau_i$ for all i,j. Suppose one of the $\{e'_1,\dots,e'_p\}$, say e'_1, were k-linearly independent of $(e_1,\dots,e_p)$. For all i,j, $\alpha_{i,j}-\alpha'_{i,1}\notin\mathbb{Z}\smallsetminus\{0\}$. So, the argument in the proof of 9.1.6 (i), shows that $(e'_1,e_1,\dots,e_p)$ is a set of $k((\mathbf{x}))$-linearly independent vectors in V. This is a contradiction; therefore $\mathbf{e}$ and $\mathbf{e}'$ span the same k-linear subspace of V[5]. □

Remark 9.1.14. The following special case of the previous theorem deserves attention. Assume the d-dimensional GL-structure $(\Lambda,(\nabla_1,\dots,\nabla_d))$ has the property that, for $i=s+1,\dots,d$, $\nabla_i\Lambda\subseteq x_i\Lambda$. Then $\delta_i=0$ for those i's. Let us pick a k-supplement W of $(x_1,\dots,x_d)\Lambda$, stable under all the ${}^s\nabla_i$'s . Then $({}^s\nabla_i)_{|W}=0$ for $i=s+1,\dots,d$. Assume that (9.1.7) is verified. Then Λ is free and W is also stable under the ${}^n\nabla_i$'s and the ∇_i's. Since $({}^n\nabla_i)_{|W}$ and $(\nabla_i)_{|W}$ identify with ${}^n\delta_i$ and δ_i respectively, they are both 0 and we deduce that ${}^n\nabla_i=0$ for $i=s+1,\dots,d$.

Notice however that, if we apply part (ii) to modify Λ so that (9.1.7) will be satisfied, but insist to preserve the property $\nabla_i\Lambda\subseteq x_i\Lambda$ for $i=s+1,\dots,d$, then we must take $\gamma_{i,j}=0$, for $i=s+1,\dots,d$ and all j's. In fact, $\alpha_{i,j}=0$ already for those i,j.

This is the reason why one assumes that the sections $\tau_1,\dots,\tau_N$ satisfy $\tau_i(\mathbb{Z})=0$, for all i. In down-to-earth terms, one wants to avoid introducing unnecessary "apparent singularities".

Theorem 9.1.15 (**Decomposition of** GL**-structures).** *Let* $(\Lambda,(\nabla_1,\dots,\nabla_d))$ *be a* d*-dimensional* GL*-structure over* k*. Then there exist a finite extension* k' *of* k *and a canonical decomposition*

$$\Lambda_{k'((\mathbf{x}))}=\bigoplus_{\alpha\in k'^d}k'[[\mathbf{x}]]\Big[\frac{1}{x_1\cdots x_d}\Big]\otimes_k K_\alpha,$$

where the $\alpha\in k'^d$ *are pairwise distinct modulo* $\mathbb{Z}^d$ *and*

$$K_\alpha=\bigcap_{i=1}^{d}\operatorname{Ker}_{\Lambda_{k'[[\mathbf{x}]][(x_1\cdots x_d)^{-1}]}}(\nabla_{\vartheta_i}-\alpha_i)^\mu$$

[5] for more information see Section 3.4 of [15].

(for $\mu \gg 0$) is different from 0 if and only if $\alpha \in k'^d$ is (up to elements of $\mathbb{Z}^d$) a GL-exponent of the Gérard-Levelt structure. Moreover, if the GL-structure satisfies the condition (9.1.7), then the decomposition descends to $\Lambda \otimes_k k'$.

Proof. We proceed by induction on d, the case $d = 1$ being given by Theorem 8.3.4. The inductive step follows from Lemma 8.2.2, taking into account that in the proof of 9.1.6 part (ii) we need only for Laurent monomials (and not the whole fraction field $k'((\mathbf{x}))$). □

9.1.16. Let $(\Lambda, (\nabla_1, \ldots, \nabla_d))$ be a *GL*-structure of dimension d over k. For every i, let $\kappa_i := k((x_1, \ldots, \widehat{x_i}, \ldots, x_d))$ (where the single variable x_i is missing). The x_i-adic completion of the discrete valuation ring $k[[\mathbf{x}]]_{(x_i)}$ can be identified with $\kappa_i[[x_i]]$.

Similarly, the x_i-adic completion of the quotient field $k((\mathbf{x}))$ of $k[[\mathbf{x}]]_{(x_i)}$, can be identified with $\kappa_i((x_i))$. The standard κ_i-linear, x_i-adically continuous derivation $x_i \frac{\partial}{\partial x_i}$ of $\kappa_i((x_i))$ coincides with the extension by continuity of the derivation $x_i \frac{\partial}{\partial x_i}$ of $k((\mathbf{x}))$ considered in our previous discussion of Theorem 9.1.6. We consider the x_i-adic completion of Λ (resp. V) and call it $\Lambda_i = \kappa_i[[x_i]] \otimes \Lambda$ (resp. $V_i = \kappa_i((x_i)) \otimes V$). Since $\kappa_i[[x_i]]$ is a discrete valuation ring, the lattice Λ_i of V_i is a free $\kappa_i[[x_i]]$-module of rank p.

The extension $\nabla_i^{(1)}$ of the endomorphism ∇_i of Λ (resp. V) by x_i-adic continuity to Λ_i (resp. V_i) produces a 1-dimensional *GL*-structure over the field κ_i. The characteristic polynomial of the κ_i-linear endomorphism $\delta_i^{(1)}$ induced by $\nabla_i^{(1)}$ on $\Lambda_i / x_i \Lambda_i$ has coefficients in k and will be denoted by $P_i(t)$.

Corollary 9.1.17. *With the notation of 9.1.16, $P_i(t)$ divides the characteristic polynomial of the k-linear endomorphism δ_i of $\Lambda/(x_1, \ldots, x_d)\Lambda$. If Λ is a free lattice, the two polynomials coincide.*

Proof. The natural map $\kappa_i \otimes_k \Lambda/(x_1, \ldots, x_d)\Lambda \longrightarrow \Lambda_i / x_i \Lambda_i$ is a surjection and $\delta_i^{(1)}$ is induced by $1 \otimes \delta_i$. Therefore, ${}^s\delta_i^{(1)}$ is induced by $1 \otimes {}^s\delta_i$. If Λ is a free lattice, the last map is an isomorphism. □

9.2 Regularity and logarithmic extensions

We keep the notation of 9.1.16.

Remark 9.2.1. Let (V, ∇) a differential module over $k((x_1, \ldots, x_d))$. Notice that if we consider $V_i = V \otimes \kappa_i((x_i))$ as a differential module over $\kappa_i((x_i))$ with derivation $\vartheta_{x_i} = x_i \frac{\partial}{\partial x_i}$, then the regularity condition (along x_i) is equivalent to the existence of a free 1-dimensional *GL*-structure (Λ_i, ∇_i) over κ_i inducing (V_i, ∇_i) by base change.

Theorem 9.2.2. *Let (M, ∇) be a finite locally free differential module over the ring $R_{d,s} = k[[x_1, \ldots, x_d]]\left[\frac{1}{x_1 \cdots x_s}\right]$. Then the following conditions are equivalent:*

(1) *for all* $i = 1, \dots, s$, $M_i = M \otimes \kappa_i((x_i))$ *is regular* (*i.e., admits a free* 1-*dimensional GL-structure* (Λ_i, ∇_i) *over* κ_i);

(2) (M, ∇) *admits a d-dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over k, i.e., there exists a d-dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over k inducing* (M, ∇) *by base change;*

(3) (M, ∇) *admits a free d-dimensional GL-structure* $(\Lambda, (\nabla_1, \dots, \nabla_d))$ *over k and a basis of* Λ *such that the matrices of* ∇_i *are non-resonant with coefficients in k* (*and zero if* $i > s$).

Remark 9.2.3. Item (2) may be considered as a strong regularity condition for (M, ∇). The equivalence of (1) and (2) says that regularity in the strong sense is equivalent to regularities "for all variables" separately.

On the other hand, one can relax the assumption of local freeness by using the notion of stable freeness, cf. [79, 1.3].

Proof. The equivalence of (2) and (3) follows from Theorem 9.1.6 of the Gérard-Levelt theory, and Remark 9.1.14. Clearly (2) implies (1).

The crucial point is to show that (1) implies (2). By hypothesis, for all i we have free $\kappa_i[[x_i]]$-lattices Λ_i in M_i, stable under ∇_i. Let $c_i : M \to M_i$ be the canonical morphisms. We define

$$\Lambda = \bigcap_i c_i^{-1}(\Lambda_i) = \{v \in M \ : \ c_i(v) \in \Lambda_i \quad \forall i\},$$

which is clearly a $k[[x]]$-submodule of M (ttence torsion-free), stable under $\nabla_i = \nabla_{\vartheta_{x_i}}$ and such that $\Lambda \otimes_{k[[x]]} R_{d,s} = M$. We have to show that Λ is finite as $k[[x]]$-module. Since M is finite projective, we may consider a set of generators $(v_1, \dots, v_n)$ of M and the set $(v_1^\vee, \dots, v_n^\vee)$ of $M^\vee$ ($R_{d,s}$-dual of M) with the property that for any $x \in M$ we have $x = \sum_\ell v_\ell^\vee(x) v_\ell$ (see, for example, [23, Ch.II, par. 2, Prop. 12]). Let us consider now the $\kappa_i[[x_i]]$-lattice $\Lambda_i^\vee$ inside $M_i^\vee \cong M^\vee \otimes_{R_{d,s}} \kappa_i((x_i))$. Then there exists an integer d_i such that $x_i^{d_i} v_\ell^\vee \in \Lambda_i^\vee$ for any $\ell = 1, \dots, n$. Therefore, $\big(\prod_i x_i^{d_i}\big)\Lambda$ is contained in the $k_i[[\mathbf{x}]]$-lattice Λ' generated by $(v_1, \dots, v_n)$. We deduce that $\Lambda \subseteq \big(\prod_i x_i^{-d_i}\big)\Lambda'$, which is of finite type over $k[[\mathbf{x}]]$, and so is also Λ (since $k[[\mathbf{x}]]$ is noetherian). □

Definition 9.2.4 (**logarithmic extensions**). *The data of a free GL-structure as in item* (3) *is called a logarithmic extension of* (M, ∇). *It is unique if we ask that the GL-exponents are in the images of sections* τ_i *of the projection* $k \to k/\mathbb{Z}$ (*see* 9.1.13).

Remark 9.2.5. Theorem 9.2.2 does not hold for differential modules over the field $F = k((x_1, \dots, x_d))$, as the example $M = F.\exp\big(\frac{1}{x_1 + x_2}\big)$ over $F = k((x_1, x_2))$ shows.