

# On the Herdability of Linear Time-Invariant Systems with Special Topological Structures

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## Abstract

In this paper we investigate the herdability property, namely the capability of a system to be driven towards the (interior of the) positive orthant, for linear time-invariant state space models. Herdability of certain matrix pairs  $(A, B)$ , where  $A$  is the adjacency matrix of a multi-agent network, and  $B$  a selection matrix that singles out a subset of the agents (the “network leaders”), is explored. The cases when the graph associated with  $A$ ,  $\mathcal{G}(A)$ , is directed and clustering balanced (in particular, structurally balanced), or it has a tree topology and there is a single leader, are investigated.

*Key words:* Herdability, multi-agent systems, social networks, linear systems, signed graphs, clustering balance.

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## 1 Introduction

Networked multi-agent systems have been the subject of an impressive number of contributions in the last two decades, due to their wide range of application [4,5,21,34]. As a result, the controllability of this class of systems, namely the property of the system state to be driven towards any point of the state space, has attracted a lot of interest, mainly aimed at deriving conditions that rely on the communication graph structure, rather than on the specific weights attributed to the graph edges [22,23,26,30]. However, there are many research fields, such as biology [18], chemistry [6], sociology [27], neuroscience [15], social networks [17,19] etc. for which, due to the nature of the describing variables, investigating if the system state can be brought towards any point of the state space is not of practical interest, and may lead to overly restrictive conditions on the model into play. Consider, for example, the model of a chemical reactor and assume that the state vector represents some reactant concentrations. In this context, it is pointless to impose that the state entries may assume any real value, including the negative ones, while it makes sense to impose that the concentrations of all the elements in the chemical reactor can be brought over a minimum level. In the context of marketing advertise-

ment, it is of interest to devise strategies targeting some individuals to bring the consumption level/usage of a certain good/service for a group of consumers over a certain threshold. In many electoral systems there is an election threshold that represents the minimum share of votes which a candidate or political party has to achieve to become entitled to any representation in a legislature. It is in contexts like these, in which (positive) thresholds come into play, that the investigation of a weaker concept with respect to controllability, known in the literature as herdability [24,25], becomes of interest. Herdability refers to the possibility of driving the state variable towards the interior of the positive orthant. More precisely, a system is herdable if, for every choice of the initial conditions, there exists a control input that drives all the state variables over a positive threshold. Clearly, controllable systems are also herdable, but the converse is not true.

While there is an extensive literature on controllability and structural controllability of networked systems [22,23,33,26], the research on herdability is still at an early stage. In [20] the herdability of leader/follower networks is studied. The leaders are assumed to be equipped with an external control and the relationships among the agents in the network can be either cooperative or competitive. The leader group selection problem is investigated in this context, with a special focus on structurally balanced network configurations. In [28] the herdability of leader/follower signed networks is investigated starting from the network topology and some sufficient conditions for herdability based on 1-walks and 2-walks in

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the graph are provided. The results are then extended to the case of acyclic graphs with walks of arbitrary length. In [25] a connection between system herdability and sign distribution over some specific graph topologies is established. The notion of *sign herdability* is introduced in association with classes of systems whose herdability is deduced from their sign patterns. In [29], a characterization of the controllable subspace is given based on (generalised equitable) graph partitions, and some sufficient conditions for the system herdability are provided. The concept of quotient graph is also exploited in the study of herdability of the original graph. Finally, in [24] how the underlying graph structure affects the system herdability is investigated for signed and directed graphs, and the herdability of a subset of nodes in a graph is studied, also focusing on the herdability of directed out-branching rooted graphs with a single input.

In this paper we study the herdability of linear time invariant (LTI) systems described by a matrix pair  $(A, B)$ , assuming that the matrix  $A$  (and the associated graph  $\mathcal{G}(A)$ ) represents the interaction network of a multi-agent system and in particular of a social network, while  $B$  is a selection matrix that singles out the agents that are subject to a direct control action (the “leaders”). We focus on special topologies of the graph  $\mathcal{G}(A)$ , such as the tree topology, or the case when the graph is directed and structurally/clustering balanced, configurations that are of interest in social networks. Indeed, advertisement/political systems, that lend themselves to the study of herdability, are special cases of social networks/opinion dynamics models. Our focus on LTI systems is motivated by the fact that there are many social context in which LTI models turn out to be accurate enough to capture certain aspects of opinion formation, e.g. consensus, polarization [3,10,13]. Additional meaningful examples are represented by models that account for the influence of prejudice in opinion dynamics [14] or models that describe the individual attitude of a group of agents to innovation [32]. In particular, in [14] a case study is presented in which a linear time invariant social network model, the well known Friedkin-Johnsen model, is shown to fit real data. Generally, in social contexts, LTI models are suitable to capture situations in which the change in agents’ opinions is supposed to occur at a much smaller time scale compared with the changes in the network topology, as it happens for some real life social networks, where the network can be considered as static. All these LTI models can be tailored to the case when a subset of individuals is selected as “leaders”, subject to an external influence (possibly a marketing campaign), with the goal of impacting on the opinions of all the others (the “followers”). When so, herdability naturally arises as the target, for instance, of propaganda and marketing strategies.

In detail, in Section 2 some preliminaries, together with the definition of herdability of a matrix pair  $(A, B)$  and the main characterisation available in the literature, are

given. Section 3 proposes structural conditions for herdability of networked systems with leader/follower topologies and a directed communication graph that is structurally or clustering balanced. Such conditions deeply rely on the splitting of the graph into (two or more) clusters and on the signs of the minimal paths from leaders to followers. Indeed, each transition from a cluster to another changes the sign of the overall path, and herdability is ensured if minimal paths of a fixed length from a certain leader have the same sign. On the other hand, Section 4 focuses on the case of (undirected) tree topologies and a single leader. For (undirected) trees we provide a general sufficient condition (Proposition 6) and two necessary and sufficient conditions (Propositions 10 and 11) for trees with 2 or 3 levels. In all cases the conditions are structural, namely they depend on the signs but not on the specific weights of the edges. The first two results easily extend to the case of directed tree with a single root node, while the third result becomes quite simple in the directed case. Section 5 brings some conclusions, while the Appendix provides all the technical results required to prove the results of Sections 3 and 4.

This paper extends the preliminary results included in our recent paper [9]. Specifically, Propositions 9 and 13 in section 4 can also be found in [9], while Proposition 14 has appeared in [9] without a proof. Lemma 18 extends Lemma 3 in [9], while Proposition 20 provides a generalisation of Proposition 7 in [9]. All the remaining results are new.

## 2 Preliminaries and definition of herdability of a pair $(A, B)$

We start the paper by providing some basic definitions and notation that will be used in the following.

Given  $k, n \in \mathbb{Z}$ , with  $k < n$ , the symbol  $[k, n]$  denotes the integer set  $\{k, k + 1, \dots, n\}$ . The  $(i, j)$ -th entry of a matrix  $A$  is denoted by  $[A]_{ij}$ , while the  $i$ -th entry of a vector  $\mathbf{v}$  by  $[\mathbf{v}]_i$ . The notation  $M = \text{diag}\{m_1, m_2, \dots, m_n\}$  indicates a diagonal matrix with diagonal entries  $m_1, m_2, \dots, m_n$ . We let  $\mathbf{e}_i$  denote the  $i$ -th vector of the canonical basis of  $\mathbb{R}^n$ , where the dimension  $n$  will be clear from the context. Accordingly,  $M\mathbf{e}_j$  denotes the  $j$ -th column of  $M$ , and  $\mathbf{e}_i^\top M$  the  $i$ -th row of  $M$ . Every nonzero multiple of a canonical vector is called *monomial vector*. The vectors  $\mathbf{1}_n$  and  $\mathbf{0}_n$  denote the  $n$ -dimensional vectors whose entries are all 1 or 0, respectively. Similarly, the symbol  $\mathbf{0}_{p \times m}$  denotes the  $p \times m$  matrix with all zero entries. Given a vector  $\mathbf{v} \in \mathbb{R}^n$ , the set  $\overline{ZP}(\mathbf{v}) = \{i \in [1, n] : [\mathbf{v}]_i \neq 0\}$  denotes the *non-zero pattern* of  $\mathbf{v}$  [31]. A nonzero vector  $\mathbf{v}$  is said to be *unsigned* [25] if all its nonzero entries have the same sign. Given a matrix  $A \in \mathbb{R}^{n \times m}$ , the notation  $\text{Im}(A)$  denotes the image of the matrix  $A$ . A matrix (in particular, a vector)  $A$  is *nonnegative* ( $A \geq 0$ ) [12] if all its entries are nonnegative.  $A$  is *strictly positive* ( $A \gg 0$ ) if all its entries are positive. To any matrix

$A \in \mathbb{R}^{n \times n}$ , we associate the *signed and weighted directed graph*  $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}, A)$ , where  $\mathcal{V} = [1, n]$  is the set of nodes. The set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the set of arcs (edges) connecting the nodes, while the matrix  $A \in \mathbb{R}^{n \times n}$  is the *adjacency matrix* of the graph. There is an arc  $(j, i) \in \mathcal{E}$  from  $j$  to  $i$  if and only if  $[A]_{ij} \neq 0$ . When so,  $[A]_{ij}$  is the *weight* of the arc. A sequence of  $k$  consecutive arcs  $(j, j_2), (j_2, j_3), \dots, (j_k, i) \in \mathcal{E}$  is a *walk* of length  $k$  from  $j$  to  $i$ . A walk from  $j$  to  $i$  is said to be *positive (negative)* if the product of the weights of the edges that compose the walk is positive (negative). A *minimum walk* from  $j$  to  $i$  is a walk of minimum length connecting the two nodes. We define the *distance*  $d(j, i)$  from the node  $j$  to the node  $i$  as the length of the minimum walk from  $j$  to  $i$ . If there is no walk from  $j$  to  $i$  then  $d(j, i) = +\infty$ . If  $A$  is a symmetric matrix, namely  $A = A^\top$ , the graph  $\mathcal{G}(A)$  is (signed, weighted and) *undirected*, and the concepts of walk and distance become symmetric. An undirected graph  $\mathcal{G}(A)$  is *connected* if for every pair of vertices there is a walk connecting them. A graph  $\mathcal{G}(A)$  is said to be *clustering balanced* (with  $k \geq 2$  clusters) [7] if all its nodes can be partitioned into  $k$  disjoint subsets  $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k$  in such a way that  $\forall i, j \in \mathcal{V}_p, p \in [1, k]$ , we have  $[A]_{ij} \geq 0$ , and  $\forall i \in \mathcal{V}_p$  and  $\forall j \in \mathcal{V}_q, p, q \in [1, k], p \neq q$ , we have  $[A]_{ij} \leq 0$ . Note that clustering balance for  $k = 2$  clusters is generally known in the literature as *structural balance* [7].

The concept of herdability of linear and time-invariant state space models described by a matrix pair  $(A, B)$ , with  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ , has been defined in various ways [24,25,28]. In this paper we are interested in the behavior of all state variables, rather than in the behavior of a subset of them. Consequently, we assume the following definition (which is equivalent to Definition 3 in [25]).

**Definition 1** *Given a (continuous-time or discrete-time) (linear and time-invariant) state space model of dimension  $n$  with  $m$  inputs, described by a pair  $(A, B)$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ , the system (the pair) is said to be herdable if for every  $\mathbf{x}(0)$  and every  $h > 0$ , there exists a time  $t_f > 0$  and an input  $\mathbf{u}(t), t \in [0, t_f)$ , that drives the state of the system from  $\mathbf{x}(0)$  to  $\mathbf{x}(t_f) \geq h\mathbf{1}_n$ <sup>1</sup>.*

Both in the continuous-time case and in the discrete-time case, herdability reduces to a condition on the controllability matrix associated with the pair  $(A, B)$ .

**Proposition 2 (Corollary 1, [25])** *A pair  $(A, B)$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ , is herdable if and only if*

<sup>1</sup> It is worth noticing that, when dealing with linear systems, requiring that each of the agents' states can be brought above a common positive threshold  $h$  is equivalent to requiring that each state variable  $x_i$  can be brought above a specific positive threshold  $h_i$  or that at some time instant the state vector becomes strictly positive.

$\text{Im}(\mathcal{R}(A, B))$  includes a strictly positive vector, where

$$\mathcal{R}(A, B) := \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \quad (1)$$

is the controllability matrix of the pair  $(A, B)$ .

### 3 Herdability of pairs $(A, B)$ corresponding to a directed graph $\mathcal{G}(A)$ with $m$ leaders

In this section we investigate the herdability of the pairs  $(A, B)$ , where  $A \in \mathbb{R}^{n \times n}$  is any real matrix and  $B \in \mathbb{R}^{n \times m}, n > m$ , is a *selection matrix*, namely an  $n \times m$  submatrix of the identity matrix  $I_n$ . This set-up, previously considered in [20,28,29], can be interpreted as the description of a network of mutually interacting agents, each of them associated with a scalar describing variable. In the network, a subset of  $m$  agents (the indices of the nonzero rows of  $B$ ) is selected as “leaders”, by this meaning that such agents are the target of a direct external action, aiming to influence the states of the remaining agents, the “followers”<sup>2</sup>. This set-up is also very similar to the one adopted in [33,26], where controllability properties of continuous systems, known in the literature as “networks of diffusively coupled agents” and described by a pair  $(-\mathcal{L}, B)$ , where  $\mathcal{L}$  is a Laplacian matrix and  $B$  a selection matrix, have been investigated. Both in references [33,26] and in [20,28,29], the key idea is to exploit the structure of the signed and weighted directed graph  $\mathcal{G}(A)$  and the specific selection of the leaders to deduce some (in general, sufficient) conditions for controllability/herdability to hold. Indeed, herdability of a system described by such a pair  $(A, B)$  represents the capability of the group of  $m$  leaders to simultaneously bring their own states and those of the followers over a certain threshold. The results we will derive in this setting can be used not only to analyse an existing network, but also for design purposes. Indeed, one can look for all possible leaders' selections that allow this property to hold and, in particular, for the smallest sets of leaders that ensure herdability of the resulting network. The interest in this kind of problems is quite immediate if we think, for instance, of the marketing or the electoral system examples, where it is quite relevant to understand which choices of the leaders ensure the success of the marketing/propaganda policies.

Finally, based on Lemma 17 in the Appendix, we can always reduce ourselves to the case  $(A, B)$ , with  $B$  a selection matrix, every time all the nonzero rows of  $B$  are linearly independent.

We first consider the case when  $\mathcal{G}(A)$ , the communication graph associated with  $A$ , is structurally balanced

<sup>2</sup> Note that it is the choice of where to apply the external inputs, and not the communication network, that determines the leaders.

or, more generally, clustering balanced [8]. These configurations are of strong interest in sociological contexts since they describe the case when individuals split in factions: individuals within the same faction have friendly/cooperative behaviors, while individuals belonging to different factions behave in a competitive/antagonistic way. A structurally balanced network represents an intrinsically stable social configuration [10,16].

It must be remarked that bringing all the agents' states over a certain positive threshold in the presence of competitive interactions is nontrivial, especially when the antagonistic relationships between individuals of different factions tend to stimulate somewhat opposite reactions that lead to opposite signs of the variables involved in the system dynamics.

Proposition 3, below, addresses the case when the directed graph  $\mathcal{G}(A)$  is clustering balanced, the set of leaders coincides with one of the clusters (without loss of generality the first one) and for each follower in the other clusters there is a leader whose distance from that follower is smaller than the distance from any other follower belonging to a different cluster.

**Proposition 3** *Assume that  $\mathcal{G}(A)$  is a clustering balanced directed graph with  $k$  clusters,  $\mathcal{V}_1, \dots, \mathcal{V}_k$ , and that the set of leaders coincides with one of the clusters, e.g.,  $\mathcal{L} = \mathcal{V}_1$ . If for every  $p \in [2, k]$  and every  $i \in \mathcal{V}_p$  there exists  $\ell_i \in \mathcal{L} = \mathcal{V}_1$  such that  $d(\ell_i, i) < d(\ell_i, j), \forall j \in \cup_{h \notin \{1, p\}} \mathcal{V}_h$ , then the pair  $(A, B)$  is herdable.*

*Proof.* It entails no loss of generality assuming that  $\mathcal{L} = \mathcal{V}_1 = [1, m]$ . Therefore  $B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$  and the reachability matrix takes the form

$$\mathcal{R} = \left[ B \mid AB \ A^2B \ \dots \ A^{n-1}B \right] = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}.$$

Under the statement assumptions, for every  $p \in [2, k]$  and  $\forall i \in \mathcal{V}_p$  there exists  $\ell_i \in \mathcal{L} = \mathcal{V}_1 = [1, m]$  and  $k_i > 0$  (the distance from  $\ell_i$  to  $i$ ) such that  $[A^{k_i}B]_{i, \ell_i} \neq 0$  and if  $[A^{k_i}B]_{j, \ell_i} \neq 0$  for some  $j \neq i$  then either  $j \in \mathcal{L}$  or  $j \in \mathcal{V}_p$ . Clearly, the vector  $A^{k_i}B\mathbf{e}_{\ell_i}$  represents the  $(mk_i + \ell_i)$ -th column of  $\mathcal{R}$ , and its restriction to the last  $n - m$  entries is the  $(mk_i - m + \ell_i)$ -th column of  $\Phi_{22}$ . We want to prove that such a restriction is a unisigned vector. To this end, we first observe that assumption  $d(\ell_i, i) < d(\ell_i, j), \forall j \in \cup_{h \notin \{1, p\}} \mathcal{V}_h$ , implies that the walk of length  $k_i$  from  $\ell_i$  to  $i$  cannot pass through any cluster  $\mathcal{V}_h, h \notin \{1, p\}$ , therefore the nodes belonging to the walk either belong to  $\mathcal{V}_1$  or to  $\mathcal{V}_p$ . This immediately implies that if  $[A^{k_i}B]_{j, \ell_i} \neq 0$  for some  $j \in \mathcal{L} = \mathcal{V}_1$  then  $[A^{k_i}B]_{j, \ell_i} > 0$ , while if  $[A^{k_i}B]_{j, \ell_i} \neq 0$  for some  $j \in \mathcal{V}_p$

then  $[A^{k_i}B]_{j, \ell_i} < 0$ . This implies that the  $(mk_i - m + \ell_i)$ -th column of  $\Phi_{22}$  is a unisigned vector. On the other hand, since  $\cup_{i \in [2, k]} \mathcal{ZP}(A^{k_i}B\mathbf{e}_{\ell_i}) \supseteq [m+1, n]$ , this means that  $\text{Im}(\Phi_{22})$  includes a strictly positive vector. Therefore, by Lemma 18 in the Appendix (see also Remark 19),  $\text{Im}(\mathcal{R})$  includes a strictly positive vector, and hence the pair  $(A, B)$  is herdable.  $\square$

Proposition 4 considers the case when the directed graph  $\mathcal{G}(A)$  is structurally balanced and there are leaders in both classes.

**Proposition 4** *Assume that the directed graph  $\mathcal{G}(A)$  is structurally balanced, with nodes split into clusters  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , and that the set of leaders  $\mathcal{L}$  intersects both  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . If*

- a)  $\forall i \in \mathcal{V}_1 \setminus \mathcal{L}$  there exists  $\ell \in \mathcal{L} \cap \mathcal{V}_1$  such that  $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_2 \setminus \mathcal{L}$ ;
- b)  $\forall i \in \mathcal{V}_2 \setminus \mathcal{L}$  there exists  $\ell \in \mathcal{L} \cap \mathcal{V}_2$  such that  $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_1 \setminus \mathcal{L}$ ;

*then the pair  $(A, B)$  is herdable.*

*Proof.* First of all, we observe that under the structural balance assumption if two nodes (leader or follower) belong to the same class, every walk (and hence, in particular, every minimum walk) that connects them has a positive weight. As a result, if  $i, j \in \mathcal{V}_p$  for some  $p \in [1, 2]$  and  $[A^k B]_{ij} \neq 0$  for some  $k > 0$ , then  $[A^k B]_{ij} > 0$ . Condition a) ensures that for every  $i \in \mathcal{V}_1 \setminus \mathcal{L}$  there exists  $\ell \in \mathcal{L} \cap \mathcal{V}_1$  and  $k_i > 0$  such that  $[A^{k_i} B]_{i\ell} \neq 0$  and hence  $[A^{k_i} B]_{i\ell} > 0$ . On the other hand,  $[A^{k_i} B]_{j\ell} = 0$  for every  $j \in \mathcal{V}_2 \setminus \mathcal{L}$ . Therefore if  $[A^{k_i} B]_{j\ell} \neq 0$  and  $j \notin \mathcal{L}$  then  $[A^{k_i} B]_{j\ell} > 0$ . Consequently, for every  $i \in \mathcal{V}_1 \setminus \mathcal{L}$  there exists  $\ell \in \mathcal{V}_1 \setminus \mathcal{L}$  and  $k_i > 0$  such that the vector  $A^{k_i}B\mathbf{e}_{\ell}$  has the  $i$ -th entry which is nonzero and its restriction to the entries that correspond to the followers (i.e., with indices in  $[1, n] \setminus \mathcal{L}$ ) is a unisigned vector. By exploiting b), we can claim the same result for all indices  $i \in \mathcal{V}_2 \setminus \mathcal{L}$ . So, keeping in mind the structure of  $B$ , we can claim that there exists a permutation matrix  $P$  and a selection matrix  $S$  such that

$$P\mathcal{R}(A, B)S = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix},$$

where all columns of  $\Phi_{22}$  are unisigned (in fact, non-negative) and  $\Phi_{22}$  has no zero rows. By Lemma 18, in the Appendix, we can claim the herdability of the pair  $(A, B)$ .  $\square$

**Remark 5** *The results in this section hold also for undirected graphs. It is worth noticing that conditions a) and b) in Proposition 4 amount to requiring that for each follower there is a leader in the same cluster that is closer to that follower than to any other follower belonging to the other cluster, something reasonable to assume when dealing with social networks.*

**Remark 6** The proofs of Propositions 3 and 4 show some similarities. Indeed, they are both based on structural conditions (based on the graph partitioning) that ensure that for each node  $i$  one can find a leader  $l_i$  that reaches for the first time  $i$  in a certain number of steps, and all the other nodes that are reached from  $l_i$  for the first time in that same number of steps are reached via a path of the same sign. Clearly, the sign depends on the number of transitions from one cluster to another.

On the other hand, neither of the results follows from the other, since they deal with a graph partition into different numbers of clusters and the leaders are differently distributed among the clusters in the two cases. The following two examples illustrate the two results.

**Example 7** (Example for Proposition 3) Figure 1 shows an example of clustering balanced network that is herdable. Solid lines represent positive ties while dashed lines represent negative ties. Cluster  $\mathcal{V}_1$  corresponds to the set of leaders. For both clusters  $\mathcal{V}_2, \mathcal{V}_3$  the case occurs that for each node, all nodes in the other cluster can be reached from a specific leader (leader 1 for cluster  $\mathcal{V}_2$  and leader 2 for  $\mathcal{V}_3$ ) in a larger number of steps. The herdability of the network can be deduced according to Proposition 3.

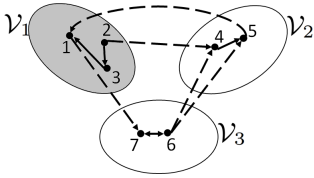


Fig. 1. Clustering balanced herdable network.

**Example 8** (Example for Proposition 4) Figure 2 shows an example of structurally balanced network that is herdable. Nodes in the shaded area represent the leaders. Solid lines represent positive ties while dashed lines represent negative ties. One can notice that node 6 can be reached from leader 1 in one step and  $\forall j \in \mathcal{V}_2 \setminus \mathcal{L}, d(1, j) \geq 3$ . Similarly, node 7 can be reached from leader 2 in one step and  $\forall j \in \mathcal{V}_2 \setminus \mathcal{L}, d(2, j) \geq 2$ . Analogously, for each node that is not a leader one can find a leader in the same cluster at a shorter distance with respect to the one of any other node in the other cluster. From Proposition 4 it follows that the network in Figure 2 is herdable.

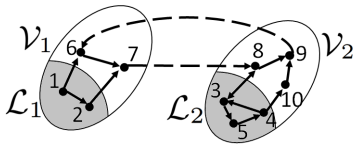


Fig. 2. Structurally balanced herdable network.

#### 4 Herdability of pairs $(A, B)$ with $\mathcal{G}(A)$ an undirected tree with a single leader

We now consider the case when  $B$  is a canonical vector and the matrix  $A$  is a symmetric real matrix whose associated undirected graph  $\mathcal{G}(A)$  is a tree [11]. Undirected graphs are reasonably common when describing social networks, due to the fact that in the long term the friendly/antagonistic attitude that an individual has toward another one tends to be reciprocated. Also, it has been shown that moderate size scale social graphs exhibit non-trivial tree-like structures and tree-like decomposition properties [1,2]. In fact, trees are typically used to represent social networks that exhibit a multi-layer organisation (for instance, employees in a company, members of a sport association...).

When an undirected tree represents a social network topology, it makes sense to assume, as in the previous section, that the leader of the network is the individual who is subject to a direct external influence. All the other  $n - 1$  members of the network will be referred to as followers. This case has been investigated in [28], where a sufficient condition for the herdability of such pairs  $(A, B)$  has been provided. In this section we provide a sufficient condition for herdability that is less restrictive, and in the case of trees whose followers have distance at most 2 from the leader we provide necessary and sufficient conditions.

To investigate the problem we adopt the following non restrictive

**Assumption:** The graph  $\mathcal{G}(A)$  is a signed, weighted, connected and acyclic undirected graph, namely a tree [11]. Let us assume  $B = \mathbf{e}_1$ , and hence the leader is  $\mathcal{L} = \{1\}$ , while the followers split into classes, based on their distance from the leader. The followers at distance 1 from the leader are  $\mathcal{F}_1 = [2, m_1 + 1]$ , the followers at distance 2 from the leader are  $\mathcal{F}_2 = [m_1 + 2, m_1 + m_2 + 1]$ , and so on till the last class  $\mathcal{F}_k = [m_1 + \dots + m_{k-1} + 2, n]$ , where  $k$  is the maximum distance between the leader and one of its followers.

**Proposition 9** Consider a pair  $(A, B)$ , with  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^n$  satisfying the previous Assumption. If, for every  $d \in [0, k - 1]$ , all the edges from the vertices in  $\mathcal{F}_d$  to the vertices in  $\mathcal{F}_{d+1}$  have the same sign, then the pair  $(A, B)$  is herdable.

*Proof.* Under the previous assumption, it is easy to see that every vertex in  $\mathcal{F}_d$  is reached for the first time by the leader in  $d$  steps,  $d \in [0, k]$ , and subsequently it is reached after  $d + 2h$  steps for every  $h \in \{1, 2, 3, \dots\}$  (since each undirected edge of the graph can be crossed back and forth, and hence condition  $[A^d B]_i \neq 0$  implies  $[A^{d+2} B]_i \neq 0$ ). Therefore the controllability matrix of

the pair  $(A, B)$  takes the form

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & * & 0 & * & \dots \\ 0 & \mathbf{v}_1 & 0 & * & 0 & \dots \\ 0 & 0 & \mathbf{v}_2 & 0 & * & \dots \\ 0 & 0 & 0 & \mathbf{v}_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \mathbf{v}_k & \dots \end{bmatrix}, \quad (2)$$

where  $\mathbf{v}_d \in \mathbb{R}^{m_d}, d \in [1, k]$ , are, by assumption, unisigned, while  $*$  denotes (nonzero) vectors/entries whose values are not relevant. So, by Remark 19 in the Appendix, we immediately deduce that there exists a strictly positive vector in the image of  $\mathcal{R}$ , and hence  $(A, B)$  is herdable.  $\square$

**Remark 10** *Theorem 3 in [28] follows as a corollary of the previous proposition, since it imposes that all paths from the leader to the followers in  $\mathcal{V}_o := \cup_{h \in \mathbb{Z}_+} \mathcal{F}_{1+2h}$  have the same sign and, at the same time, all paths from the leader to the followers in  $\mathcal{V}_e := \cup_{h \in \mathbb{Z}_+} \mathcal{F}_{2+2h}$  have the same sign. This means that not only all the edges from vertices in  $\mathcal{F}_d$  to vertices in  $\mathcal{F}_{d+1}, d \in [0, k-1]$ , (where  $\mathcal{F}_0 := \mathcal{L}$ ) have the same signs, but such signs are uniquely determined for  $d \geq 1$  once we choose the signs of the edges from  $\mathcal{F}_1$  to  $\mathcal{F}_2$ .*

**Example 11** *Consider a pair  $(A, B)$ , with  $A = A^\top \in \mathbb{R}^{9 \times 9}$  and  $B = \mathbf{e}_1$ , and assume that the undirected graph  $\mathcal{G}(A)$  associated with the matrix  $A$  is a tree whose structure and edge signs are described in Figure 3. The nodes*

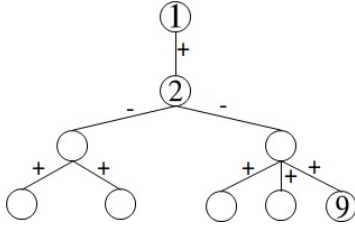


Fig. 3. Tree structure of the herdable system of Example 11.

$i = 2$  and  $j = 9$  both belong to  $\mathcal{V}_o$ , since both of them are reached from the leader (node 1 in Fig. 3) in an odd number of steps ( $1 + 2h$  and  $3 + 2h, h \in \{0, 1, 2, \dots\}$ , respectively). The node  $i$  is reached by the leader with positive walks, while  $j$  with negative ones, so the hypotheses of Theorem 3 in [28] are violated. However, the controllability matrix of the pair takes the structure in (2) for  $k = 3$ , with unisigned vectors  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$ , the first one with a positive entry, while the other two with negative entries, thus the pair is herdable by Proposition 9.

**Remark 12** *As previously remarked, the choice of the leader is not intrinsic to the structure of the tree, but is just the specific node to which we apply the input. In general, herdability is achieved only when selecting as*

*leaders certain nodes, rather than others, as it is related to the path signs that connect the leader to the other nodes.*

Propositions 13 and 14, below, provide complete characterizations of herdability for trees in which followers have all distance 1 from the leader or distance at most 2 from the leader, respectively.

**Proposition 13** *Consider a pair  $(A, B)$ , with  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^n$  satisfying the aforementioned Assumption, and suppose that all the followers have distance one from the leader.*

*Then the pair  $(A, B)$  is herdable if and only if all the edges have the same sign.*

*Proof.* If all the followers have distance 1 from the leader, namely  $k = 1$ , then

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & \mathbf{0}_{(n-1) \times (n-1)} \end{bmatrix},$$

where  $A_{21} = A_{12}^\top \in \mathbb{R}^{n-1}$  is devoid of zero entries. By Proposition 20,  $(A, B)$  is herdable if and only if the pair  $(\mathbf{0}_{(n-1) \times (n-1)}, A_{21})$  is herdable, and this is the case if and only if  $A_{21}$  is unisigned.  $\square$

**Proposition 14** *Consider a pair  $(A, B)$ , with  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^n$  satisfying the aforementioned Assumption, and suppose that all the followers have distance at most 2 from the leader, and hence*

$$A = \begin{bmatrix} 0 & A_{12} & \mathbf{0}_{1 \times m_2} \\ A_{21} & \mathbf{0}_{m_1 \times m_1} & A_{23} \\ \mathbf{0}_{m_2 \times 1} & A_{32} & \mathbf{0}_{m_2 \times m_2} \end{bmatrix},$$

where  $A_{21} = A_{12}^\top \in \mathbb{R}^{m_1}$  and  $A_{32} = A_{23}^\top \in \mathbb{R}^{m_2 \times m_1}$ . Then the pair  $(A, B)$  is herdable if and only if for every  $i, j \in \mathcal{F}_1 = [2, m_1 + 1]$  (including  $i = j$ )<sup>3</sup> such that

$$[A_{23}A_{32}]_{ii} = [A_{23}A_{32}]_{jj}, \quad (3)$$

we have:

- i)  $[A_{21}]_i \cdot [A_{21}]_j > 0$  (namely the two edges from the leader  $\mathcal{L}$  to  $i$  and  $j$  have the same sign);
- ii)  $A_{32}(\mathbf{e}_i + \mathbf{e}_j)$  is either zero or unisigned (namely all edges from  $i$  and  $j$  to their followers in  $\mathcal{F}_2$  have the same sign).

*Proof.* First of all, we highlight that, by Assumption,  $\Gamma := A_{21}$  is devoid of zero entries, and for every  $i \in [1, m_2]$  the row vector  $\mathbf{e}_i^\top A_{32}$  is a monomial vector (namely it has a single nonzero entry). Consequently,

<sup>3</sup> Note that for  $i = j$  condition i) becomes trivial, while condition ii) becomes “ $A_{32}\mathbf{e}_i$  is either zero or unisigned”.

$\Lambda := A_{23}A_{32}$  is a diagonal matrix (with nonnegative diagonal entries).

By Proposition 20 in the Appendix,  $(A, B)$  is herdable if and only if the pair

$$\left( \begin{bmatrix} \mathbf{0}_{m_1 \times m_1} & A_{23} \\ A_{32} & \mathbf{0}_{m_2 \times m_2} \end{bmatrix}, \begin{bmatrix} A_{21} \\ \mathbf{0}_{m_2} \end{bmatrix} \right)$$

is herdable, and this is the case if and only if the image of the controllability matrix  $\hat{\mathcal{R}}$  of the previous pair, see (4), includes a strictly positive vector. This is the case if and only if the following conditions simultaneously hold:

- a) the image of the controllability matrix  $\mathcal{R}_1 := \begin{bmatrix} A_{21} & (A_{23}A_{32})A_{21} & (A_{23}A_{32})^2A_{21} & \dots \end{bmatrix}$  includes a strictly positive vector, namely the pair  $(A_{23}A_{32}, A_{21})$  is herdable;
- b) the image of the matrix  $A_{32}\mathcal{R}_1$  includes a strictly positive vector.

As the matrix  $\Lambda = A_{23}A_{32}$  is diagonal, while the column vector  $\Gamma = A_{21}$  has no zero entries, by Lemma 22, the pair  $(\Lambda, \Gamma) = (A_{23}A_{32}, A_{21})$  is herdable if and only if condition (3) implies  $[A_{21}]_i \cdot [A_{21}]_j > 0$ . This means that a) is equivalent to condition i).

So, we are now remained with proving that if i) (equivalently, a)) holds, then b) and ii) are equivalent. If i) holds, by referring to the proof of Lemma 22 in the Appendix, we can assume without loss of generality that  $\Lambda$  and  $\Gamma$  take the form given in (8) and claim that

$$\text{Im}(A_{32}\mathcal{R}_1) = \text{Im}(A_{32} \cdot \text{diag}\{\gamma_1, \dots, \gamma_p, \gamma_{p+1}, \dots, \gamma_s\}),$$

where  $\gamma_i \in \mathbb{R}^{n_i}$  is strictly positive if  $i \in [1, p]$  and strictly negative if  $i \in [p+1, s]$ .

Set  $W = \begin{bmatrix} \mathbf{w}_1 & | & \dots & | & \mathbf{w}_p & | & \mathbf{w}_{p+1} & | & \dots & | & \mathbf{w}_s \end{bmatrix} := A_{32} \cdot \text{diag}\{\gamma_1, \dots, \gamma_p, \gamma_{p+1}, \dots, \gamma_s\}$ , where each vector  $\mathbf{w}_i$  is obtained by combining with the coefficients of the vector  $\gamma_i$  (having all the same sign) the columns of  $A_{32}$  of indices  $[h_i + 1, h_i + n_i]$ , where by definition  $h_1 := 0$ , while  $h_i := n_1 + n_2 + \dots + n_{i-1}$  for  $i \in [2, s]$ .

We observe that all columns of  $A_{32}$  are either zero (if a vertex in  $\mathcal{F}_1 = [2, m_1 + 1]$  has no followers) or have disjoint nonzero patterns, meaning that for every  $\ell, m \in [h_i + 1, h_i + n_i], \ell \neq m$ ,  $\overline{\text{ZP}}(A_{32}\mathbf{e}_\ell) \cap \overline{\text{ZP}}(A_{32}\mathbf{e}_m) = \emptyset$ . As a result also the columns  $\mathbf{w}_i$  of  $W$  are either zero or have disjoint nonzero patterns.

We can now conclude that condition b) holds if and only if  $\text{Im}(A_{32}\mathcal{R}_1) = \text{Im}(W)$  contains a strictly positive vector, but this is possible if and only if all vectors  $\mathbf{w}_h$  are unsigned. By the way the vectors  $\mathbf{w}_h$  have been obtained, this is possible if and only if condition ii) holds.  $\square$

Proposition 14 states that if  $\mathcal{G}(A)$  is a tree and the distance from the unique leader is at most 2, then herdability is possible if and only if every time the sum of the squares of all arc weights from a node  $i \in \mathcal{F}_1$  to all the nodes in  $\mathcal{F}_2$  coincides with the sum of the squares of all arc weights from some node  $j \in \mathcal{F}_1$  to all the nodes in  $\mathcal{F}_2$ , then i) the edges from the leader to  $i$  and  $j$  must have the same sign; ii) all edges from  $i$  and  $j$  to the nodes in  $\mathcal{F}_2$  must have the same sign.

**Example 15** Consider the pair  $(A, B)$ , with

$$A = \begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix} = \begin{array}{c|ccc|cc} \hline 0 & 1 & a & 2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & b & c \\ \hline 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ \hline \end{array}, \quad B = \mathbf{e}_1,$$

whose graph is given in Fig. 4, where  $a, b$  and  $c$  are nonzero real values. Note that  $\mathcal{L} = \{1\}$ ,  $\mathcal{F}_1 = [2, 4]$  and  $\mathcal{F}_2 = [5, 6]$ , so that  $m_1 = 3$  and  $m_2 = 2$ . We first check for all indices  $i, j \in [1, 3], i \neq j$ , whether condition (3) holds. It is easily seen that  $[A_{23}A_{32}]_{11} = [A_{23}A_{32}]_{33} = 0$ . In fact, for the pair of indices  $(1, 3)$  both condition i) and condition ii) of Proposition 14 are satisfied, since  $[A_{21}]_1 \cdot [A_{21}]_3 = 2 > 0$  and both column 1 and column 3 of  $A_{32}$  are zero. On the other hand, if we assume  $i = j \in [1, 3]$ , for which condition (3) trivially holds, condition i) is straightforward, while condition ii) holds provided that  $bc > 0$  (namely  $b$  and  $c$  have the same sign), because all columns of  $A_{32}$  are zero or unsigned if and only if  $bc > 0$ . Therefore the pair  $(A, B)$  is herdable for every  $a \neq 0$  and for  $bc > 0$ .

**Remark 16** It is worth underlying that Propositions 9 and 13 easily extend to the case of directed trees, namely directed acyclic graphs, with a single root node having access to all the other nodes. On the other hand, in the case of Proposition 14, when moving from the undirected case to the directed ones the analysis significantly simplifies, since a directed tree with root node 1 and followers at distance at most 2 is described by the pair:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ A_{21} & 0 & 0 \\ 0 & A_{32} & 0 \end{bmatrix}, \quad B = \mathbf{e}_1.$$

Its reachability matrix is trivially

$$\mathcal{R}(A, B) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A_{21} & 0 & 0 \\ 0 & 0 & A_{32} & 0 \end{bmatrix},$$

and hence the pair  $(A, B)$  is herdable if and only if  $A_{21}$

$$\hat{\mathcal{R}} := \begin{bmatrix} A_{21} & 0 & (A_{23}A_{32})A_{21} & 0 & (A_{23}A_{32})^2A_{21} & 0 & \dots \\ 0 & A_{32}A_{21} & 0 & A_{32}(A_{23}A_{32})A_{21} & 0 & A_{32}(A_{23}A_{32})^2A_{21} & \dots \end{bmatrix} \quad (4)$$

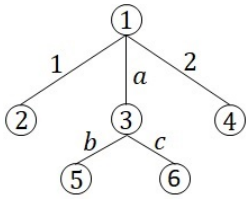


Fig. 4. Graph structure related to the pair  $(A, B)$  in Example 15.

and  $A_{32}$  are unsigned, namely all edges from the leader to nodes at distance 1 have the same sign, and all edges from nodes at distance 1 to nodes at distance 2 have the same sign.

## 5 Conclusions

In this paper herdability of linear time-invariant systems has been investigated. First, some results for pairs  $(A, B)$  corresponding to leader-follower networks  $\mathcal{G}(A)$  satisfying clustering balance have been derived. Subsequently, pairs  $(A, B)$  whose network  $\mathcal{G}(A)$  has a tree topology and a single leader have been addressed. The study of herdability based on the graph  $\mathcal{G}(A)$  seems particularly promising, especially because it may lead, for special graph topologies, to determine conditions for “structural herdability”, which is independent of the specific values of the matrix entries, and only relies on their signs. Also, herdability property for time varying linear systems will be the subject of future investigation.

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## Appendix

In this Appendix we provide some technical results used within the paper.

**Lemma 17** Consider a pair  $(A, B)$ , where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ . For every nonsingular matrix  $T \in \mathbb{R}^{m \times m}$ , the pair  $(A, B)$  is herdable if and only if the pair  $(A, BT)$  is herdable.

*Proof.* Follows from  $\text{Im}(\mathcal{R}(A, B)) = \text{Im}(\mathcal{R}(A, BT))$ .  $\square$

The following result is elementary and its proof, that can be obtained by recursion on  $k$ , the number of blocks on the diagonal, is omitted.

**Lemma 18** Given a matrix  $\Phi \in \mathbb{R}^{n \times d}$ , assume that there exist two permutation matrices  $P_1 \in \mathbb{R}^{n \times n}$  and  $P_2 \in \mathbb{R}^{d \times d}$  such that  $P_1 \Phi P_2$  is block-partitioned as

$$P_1 \Phi P_2 = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1k} \\ 0 & \Phi_{22} & \dots & \Phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Phi_{kk} \end{bmatrix}, \quad (5)$$

where the diagonal blocks  $\Phi_{ii}$  are not necessarily square matrices. If the image of each diagonal block  $\Phi_{ii}$ ,  $i \in [1, k]$ , includes a strictly positive vector, then there exists  $\mathbf{u} \in \mathbb{R}^k$  such that  $\Phi \mathbf{u} \gg 0$ .

**Remark 19** As a special case, if for every  $i \in [1, k]$  we can select a subset of the columns of  $\Phi_{ii}$  in (5) such that: 1) each of them is unisigned; 2) for each row index  $j$ , at least one of these unisigned columns has the  $j$ -th entry

which is nonzero, then each vector subspace  $\text{Im}(\Phi_{ii})$ ,  $i \in [1, k]$ , includes a strictly positive vector and hence Lemma 18 ensures herdability.

Proposition 20, below, provides a method for the dimensionality reduction of the herdability problem for matrix pairs  $(A, B)$ , with  $A$  and  $B$  block-partitioned.

**Proposition 20** Consider a pair  $(A, B)$ , where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are described as in as in

$$A = \begin{bmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad (6)$$

where  $B_1 \in \mathbb{R}^{r \times m}$  is of full row rank and  $A_{11} \in \mathbb{R}^{r \times r}$ . The pair  $(A, B)$  is herdable if and only if the pair  $(A_{22}, A_{21})$  is herdable.

*Proof.* Let  $\mathcal{R}(A, B)$  be the controllability matrix of  $(A, B)$  and  $\mathcal{R}(A_{22}, A_{21})$  the controllability matrix of  $(A_{22}, A_{21})$ . We preliminarily note that, since  $B_1$  is of full row rank, there exists a nonsingular matrix  $T \in \mathbb{R}^{m \times m}$  such that  $B_1 T = [I_r \ 0]$ . Since the zero columns of  $BT$  are irrelevant, in the following by making use of Lemma 17 we assume  $r = m$  and  $B_1 = I_m$ . Since

$$\mathcal{R}(A, B) = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}$$

where

$$\begin{aligned} \Phi_{22} &:= \begin{bmatrix} A_{21} & A_{21}A_{11} + A_{22}A_{21} \\ A_{21}(A_{11}^2 + A_{12}A_{21}) + A_{22}(A_{21}A_{11} + A_{22}A_{21}) & \dots \end{bmatrix} \\ &= \begin{bmatrix} 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix}, \end{aligned}$$

it is immediate to see that for every  $\mathbf{v}_1 \in \mathbb{R}^m$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \text{Im}(\mathcal{R}(A, B)) \quad \Leftrightarrow \quad \mathbf{v}_2 \in \text{Im}(\Phi_{22}).$$

We now prove that  $\text{Im}(\Phi_{22}) =$

$$\text{Im} \left( \begin{bmatrix} 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \right) = \text{Im}(\mathcal{R}(A_{22}, A_{21})).$$

To prove this result we show that for every  $k \in [1, n-1]$

$$\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} A^k B = \begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{k-1}A_{21} \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \\ I_m \end{bmatrix} \quad (7)$$

where  $*$  denotes a real matrix (whose value is not relevant). We proceed by induction on  $k$ . If  $k = 1$  the result



trix on the right of (9) is of full row rank. This ensures that

$$\text{Im}(\mathcal{R}(\Lambda, \Gamma)) = \text{Im} \left( \left[ \begin{array}{c|c} \gamma_1 & \\ \hline & \gamma_p \\ \hline & \gamma_{p+1} \\ & & \ddots \\ & & & \gamma_s \end{array} \right] \right)$$

and since all columns of this latter matrix are unsigned, it is immediate to see that there exists a strictly positive vector in its image, and hence the pair  $(\Lambda, \Gamma)$  is heritable.  $\square$