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Periodic sequences, binomials modulo a prime power, and a math/music application ☆

Luisa Fiorot [a,b,*], Riccardo Gilblas [a,*], Alberto Tonolo [a,b,*]

[a] *Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Padova, Italy*
[b] *Centro interdipartimentale di ricerca "I–APPROVE – International Auditory Processing Project in Venice" c/o Dipartimento di Neuroscienze - DNS, Università di Padova, Padova, Italy*

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We study, through new recurrence relations for certain binomial coefficients modulo a power of a prime, the evolution of the iterated anti-differences of periodic sequences modulo m. We prove that one can reduce to study iterated anti-differences of constant sequences. Finally we apply our results to describe the dynamics of the iterated applications of the *Vieru operator* to the sequence considered by the Romanian composer Vieru in his *Book of Modes* [20].

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* Corresponding authors.
E-mail addresses: luisa.fiorot@unipd.it (L. Fiorot), riccardo.gilblas@phd.unipd.it (R. Gilblas), alberto.tonolo@unipd.it (A. Tonolo).

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In this paper, we investigate periodic sequences modulo a positive integer m, i.e., $f = [a_0, a_1, \dots, a_{\pi-1}] \in (\mathbb{Z}/m\mathbb{Z})^\pi$ extended by periodicity in $(\mathbb{Z}/m\mathbb{Z})^\mathbb{N}$ and their transforms when the difference operator Δ and the anti-difference operator Σ are applied to them. The operator Δ (i.e. $\Delta(f) = [a_1 - a_0, a_2 - a_1, \dots, a_0 - a_{\pi-1}]$) allows us to examine the differences between consecutive terms in a sequence, while the operator Σ (i.e. $\Sigma(f) = [0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots]$) serves as a complementary tool, enabling us to study the cumulative sums of sequences and understand their overall behavior. The study of the period of sequences modulo a positive integer has been and continues to be of interest in combinatorics and its applications to computer science and cryptography [19,23].

It is well known that any periodic sequence can be uniquely decomposed into the sum of an idempotent sequence and a nilpotent sequence. A sequence is considered idempotent when applying the difference operator Δ multiple times results in the same sequence. Conversely, nilpotent sequences vanish after a certain number of applications of the difference operator Δ. By studying the idempotent and nilpotent parts separately, we gain a deeper understanding of the dynamics and behavior of the original sequence.

In the context of periodic sequences taking values in $\mathbb{Z}/m\mathbb{Z}$ with $m \in \mathbb{N}$, we focus on studying the evolution of their periods and the p-adic valuation of their elements when the sum operator Σ is applied. The first fundamental step has been the reduction of this study from periodic sequences to constant ones. Our first result is the proof that the study of the period of the anti-differences of any nilpotent or idempotent sequence, and hence of any periodic sequence, reduces to that of the anti-differences of a constant sequence (refer to Theorem 3.7, Theorem 3.11).

By Remark 3.1 the n-th entry of the s-anti-difference of a constant c is equal to c times the binomial coefficient $\binom{n}{s}$ in $\mathbb{Z}/m\mathbb{Z}$. Consequently, the study of binomial coefficients modulo a positive integer m becomes significant. Thanks to the Chinese Remainder Theorem we reduce to $m = p^\ell$ where p is a prime number and $\ell \geq 1$. Many mathematicians of the nineteenth century considered problems involving binomial coefficients modulo a prime power (for instance Kummer and Lucas). Several attempts of generalizing these classical results can be found in [1,4–6,8–10,12,18]. We try to give a new contribution in this research area.

One of our main result consists in providing new recurrence relations for certain binomial coefficients, enabling efficient computation of their p-adic valuation. This occurs precisely when the lower index in the binomial coefficient exhibits patterns of the following types: $\underbrace{p-1, \dots, p-1}_{\ell}$, $\underbrace{0, \dots, 0}_{\ell}$, and $\underbrace{p-1, 0, \dots, 0}_{\ell}$ (refer to Lemmas 4.7, 4.9 and 4.12).

As an application, we provide a comprehensive answer to three questions posed by the Romanian composer Anatol Vieru (1926-1998) [21, 3.1, 3.2, 3.3]. In the context of 1960s musical serialism, Vieru in his *Book of Modes* [20] explores a composition technique based on periodic sequences with values in $\mathbb{Z}/12\mathbb{Z}$. If a sequence f in $\mathbb{Z}/12\mathbb{Z}$ represents the pitch classes of an initial musical theme, Vieru decodes a musical aspect (such as rhythm, harmony, tone color, or dynamics) from the anti-differences of f using a suitable dictionary. Employing this technique, Vieru composes several pieces, including Symphony No. 2 and "Zone d'oublie". Manipulating Messiaen's second mode of limited transposition, he got the sequence $V := (2, 1, 2, 4, 8, 1, 8, 4)$. He applied to this sequence the *Vieru operator* $\mathscr{V}$ which is equal to the anti-difference operator plus the constant sequence [8]. The questions were to provide a formula for the period of $\mathscr{V}^s V$ [21, 3.1], to explain why never the numbers 3, 6, 9 appear in $\mathscr{V}^s V$ [21, 3.2], and, additionally, to explain why the values 4 and 8 proliferate among the coefficients of $\mathscr{V}^s V$ [21, 3.3]. These questions have been formalized in precise mathematical terms in the papers [2,3]. In [14,15] the authors studied the operators Δ and Σ from the point of view of automata proposing some applications to Vieru periodic sequences.

Our complete solutions to these questions have been announced without proofs in the paper [7] and are given in Vieru's question I 5.1, Vieru's question II 5.2, and Vieru's question III 5.3.

We believe that the techniques and ideas we have developed in our application to address Vieru's questions can be applicable in various other scenarios involving the study of anti-differences of periodic sequences.

In Section 1, we define periodic sequences with values on integers modulo $m > 0$. We introduce the difference operator Δ and the anti-difference operator Σ.

In Section 2, we present some results about decomposing the $\mathbb{Z}/m\mathbb{Z}$-module P_m of periodic sequences: the decomposition in nilpotent and idempotent part, and the decomposition in primes through the Chinese Remainder Theorem.

In Section 3, we present our first new fundamental result, reducing the study of anti-differences of generic sequences to anti-differences of constant ones (see Theorem 3.7 and Theorem 3.11).

In Section 4, we provide our main tool giving new recursive formulas for binomial coefficients modulo the power of a prime integer p (refer to Lemmas 4.7, 4.9 and 4.12). These formulas allow to reduce the complexity of the computation of the p-adic valuation of sequences of binomial coefficients.

In Section 5, as an example of effectiveness of our results, we apply the previous results to the peculiar periodic sequence $V = [2, 1, 2, 4, 8, 1, 8, 4]$ with coefficients in $\mathbb{Z}/12\mathbb{Z}$ that arises from the mathematical-musical problem posed by Vieru in [20].

1. Periodic sequences in $\mathbb{Z}_m$ and the operators Δ and Σ

In this section we introduce the periodic sequences over $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 2$, the difference operator Δ and the sum operator Σ.

Periodic sequences in $\mathbb{Z}_m$

Let $m \geq 2$ be a natural number. We denote by $S_m := \mathbb{Z}_m^{\mathbb{N}}$ the $\mathbb{Z}_m$-module of all sequences with values in $\mathbb{Z}_m$.

The *shifting* operator θ is the endomorphism of S_m acting on $f \in S_m$ as:

$$\theta(f)(n) := f(n+1) \quad \forall n \in \mathbb{N}.$$

A sequence $f \in S_m$ is said *periodic* if there exists $j \geq 1$ such that $\theta^j(f) = f$, i.e. $f \in \ker(\theta^j - \mathrm{id})$. We denote by P_m the $\mathbb{Z}_m$-submodule of periodic sequences in S_m:

$$P_m := \bigcup_{j \geq 1} \ker(\theta^j - \mathrm{id}).$$

Given a periodic sequence $f \in P_m$, we say that it has *period* $\pi(f)$ if $\pi(f)$ is the minimum positive integer such that $\theta^{\pi(f)} f = f$. Furthermore $\theta^k f = f$ if and only if $\pi(f) \mid k$. Since $\pi(f) = \pi(\theta(f))$, θ restricts to an endomorphism of P_m.

Let f be a sequence of period π. Since it is determined by its values $f(0), \ldots, f(\pi - 1)$, we will write

$$f = (f(0), f(1), \ldots, f(\pi), f(\pi+1), \ldots) =: [f(0), f(1), \ldots, f(\pi - 1)].$$

In particular, for any $c \in \mathbb{Z}_m$, $[c]$ denotes a constant sequence.

We define the trace of f to be:

$$\mathrm{tr} f := \sum_{i=0}^{\pi-1} f(i).$$

Definition 1.1 *(The operator Δ).* We define on S_m the difference operator:

$$\Delta := \theta - \mathrm{id}.$$

It restricts to an operator of P_m since the period of Δf divides the period of f.

We say that a periodic sequence $0 \neq f \in P_m$ is *nilpotent* (resp. *idempotent*) if there exists $\eta \geq 1$ such that $\Delta^\eta f = 0$ (resp. $\Delta^\eta f = f$). The minimal η satisfying this condition is said to be the *nilpotency* (resp. *idempotency*) *index* of f. We denote by I_m^Δ the submodule of P_m of idempotent sequences and by N_m^Δ the submodule of nilpotent sequences.

Example 1.2. The sequence $f = [0, 1, 2, 3] \in P_4$ is nilpotent of index 2, while the sequence $g = [2, 1] \in P_3$ is idempotent of index 1.

Definition 1.3 *(The operator Σ).* We define on S_m the anti-difference operator Σ as follows: for every sequence $f \in S_m$,

$$(\Sigma f)(n) := \begin{cases} 0 \text{ if } n = 0 \\ f(n-1) + (\Sigma f)(n-1) \text{ if } n \geq 1. \end{cases}$$

The operator Σ acts as right inverse for Δ, i.e. $\Delta \circ \Sigma = \mathrm{id}$. More, for every $c \in \mathbb{Z}_m$ and $f \in S_m$, one has $\Delta(\Sigma f + [c]) = f$. Also it is a matter of explicit computation to find that

$$(\Sigma \circ \Delta)(f) + [f(0)] = f.$$

Notice that Σ defines an endomorphism both of S_m and of P_m. If $f \in P_m$ has period π, then $\theta^{\pi m}(\Sigma f) = \Sigma f$. Precisely we have

Lemma 1.4. *For any $f \in P_m$, if h is the additive order of* $\mathrm{tr} f$ *in $\mathbb{Z}_m$, then*

$$\pi(\Sigma f) = h \cdot \pi(f).$$

Proof. We already observed that $\pi(\Delta g) \mid \pi(g)$, so for $g = \Sigma f$ we have

$$\pi(f) = \pi(\Delta(\Sigma f)) \mid \pi(\Sigma f).$$

Since by the Fundamental Theorem of finite calculus [17, Th. 6.27]

$$(\Sigma f)(\pi m + i) - (\Sigma f)(i) = \sum_{j=i}^{\pi m+i-1} f(j) = m \sum_{j=i}^{\pi+i-1} f(j) = m \cdot \mathrm{tr} f,$$

we deduce that $\pi(\Sigma f) = h \cdot \pi(f)$. $\square$

Example 1.5. Given $c \in \mathbb{Z}_m$, for the constant sequence $[c]$, one has:

$$(\Sigma[c])(0) = 0,\ (\Sigma[c])(1) = c,\ (\Sigma[c])(2) = 2c,\ \ldots,\ (\Sigma[c])(n) = nc.$$

Hence the constant sequence $[c]$ has period equal to the additive order of c in $\mathbb{Z}_m$.

2. Decomposition of P_m

In this section we introduce in cascade two decompositions for the $\mathbb{Z}_m$-module P_m. The first one decomposes P_m as the direct sum of I_m^Δ and N_m^Δ, the submodules of idempotent and nilpotent sequences respectively. The second one is the standard decomposition into primes, using the factorization of m.

Decomposition in idempotent and nilpotent part

Let $f \in P_m$ be a sequence of period π and consider the set $A = \{\Delta^i f \mid i \in \mathbb{N}\}$. A is a subset of the set of sequences having the period dividing π, hence A is finite. So take the minimal $M \in \mathbb{N}$ such that there exists $u < M$ satisfying

$$\Delta^M f = \Delta^u f$$

If $t := M - u$ then $\Delta^{t+u} f = \Delta^u f$ and $\Delta^u f, \Delta^{u+1} f, \ldots, \Delta^{M-1} f$ are distinct sequences. Define $\bar{k}$ to be the minimal $k \in \mathbb{N}$ such that $kt \geq u$. It is $u \leq \bar{k}t < M$. Denote:

$$f_I := \Delta^{\bar{k}t} f \qquad f_N := f - f_I.$$

Lemma 2.1. *With the above notation, $f = f_I + f_N$ is the unique decomposition of f as a sum of an idempotent and a nilpotent sequence. The sequence f_N (resp. f_I) has nilpotency (resp. idempotency) index $\bar{k}t$ (resp. t). Moreover $\pi(f) = \operatorname{lcm}\{\pi(f_I), \pi(f_N)\}$.*

Proof. The sequence f_I is idempotent since

$$\Delta^t f_I = \Delta^t(\Delta^{\bar{k}t} f) = \Delta^{\bar{k}t-u}(\Delta^{u+t} f) = \Delta^{\bar{k}t-u}(\Delta^u f) = \Delta^{\bar{k}t} f = f_I.$$

The minimality of t comes from the fact that $\{\Delta^{\bar{k}t+i} f \mid 0 \leq i \leq t-1\}$ has cardinality t. The sequence f_N is nilpotent since

$$\Delta^{\bar{k}t} f_N = \Delta^{\bar{k}t}(f - f_I) = \Delta^{\bar{k}t} f - \Delta^{\bar{k}t} f_I = f_I - f_I = 0.$$

The minimality of $\bar{k}t$ follows from the minimality of $\bar{k}$.

This decomposition is unique: by contradiction take $f = f'_I + f'_N$. One has that $f_I - f'_I = f'_N - f_N$ is both nilpotent and idempotent thus it is equal to 0.

Furthermore, one clearly has $\pi(f) \mid \operatorname{lcm}\{\pi(f_I), \pi(f_N)\}$. Since Δ and θ commute, $\theta^{\pi(f)}(f_N)$ (resp. $\theta^{\pi(f)}(f_I)$) is nilpotent (resp. idempotent). From

$$f = \theta^{\pi(f)}(f) = \theta^{\pi(f)}(f_N) + \theta^{\pi(f)}(f_I)$$

and the uniqueness of the decomposition, one gets $\theta^{\pi(f)}(f_N) = f_N$ and $\theta^{\pi(f)}(f_I) = f_I$, thus $\pi(f_N), \pi(f_I) \mid \pi(f)$. Hence $\pi(f) = \operatorname{lcm}\{\pi(f_I), \pi(f_N)\}$. □

Decomposition with primes

Given the prime factorization $m = \prod_{i=1}^t p_i^{\ell_i}$, the group isomorphism $\mathbb{Z}_m \to \bigoplus_{i=1}^t \mathbb{Z}_{p_i^{\ell_i}}$ gives rise to an isomorphism of $\mathbb{Z}_m$-modules

$$\begin{aligned} P_m &\longrightarrow \bigoplus_{i=1}^t P_{p_i^{\ell_i}} \\ f &\longmapsto (f_{p_i})_{1 \leq i \leq t} \end{aligned}$$

where $f_{p_i}(n) \equiv f(n) \mod p_i^{\ell_i}$. The sequence f_{p_i} is the p_i-part of f. The inverse of this morphism is given by the Chinese Remainder Theorem.

As a consequence, one can easily prove the following lemma.

Lemma 2.2. *[2, Prop. 13 and Prop. 16] A sequence $f \in P_m$ is nilpotent (resp. idempotent) if and only if the p_i-part f_{p_i} is nilpotent (resp. idempotent) for every i. The nilpotency (resp. idempotency) index η coincides with the maximum (resp. least common multiple) of the nilpotency (resp. idempotency) indices η_i of f_{p_i} for $i = 1, \dots, t$. Moreover, the period of f satisfies:*

$$\pi(f) = \operatorname{lcm}\{\pi(f_{p_i})\}_{1 \le i \le t}.$$

The primes decomposition, Lemmas 2.1 and 2.2 imply the following isomorphisms:

$$P_m = I_m^\Delta \oplus N_m^\Delta \qquad I_m^\Delta = \bigoplus_{i=1}^{t} I_{p_i^{\ell_i}}^\Delta \qquad N_m^\Delta = \bigoplus_{i=1}^{t} N_{p_i^{\ell_i}}^\Delta.$$

Thus we can always reduce to study sequences on $\mathbb{Z}_{p^\ell}$.

Theorem 2.3. *[2, Th. 7] Let $f \in P_{p^\ell}$ be a periodic sequence. Then $f \in N_{p^\ell}^\Delta$ if and only if $\pi(f) = p^t$ for $t \in \mathbb{N}$.*

Remark 2.4. The period of an idempotent sequence $f \in I_{p^\ell}^\Delta$ may or may not be divisible by p. Using a generic computer algebra system one can easily check that the sequence $[1, 1, 1, 0, 0, 2, 0, 0, 0, 2, 2, 2, 0, 0, 1, 0, 0, 0] \in P_3$ is idempotent (of index 9) and it has period 18, and the sequence $[0, 2, 0, 0, 1] \in P_3$ is idempotent (of index 80) and it has period 5.

Definition 2.5. Consider $f \in P_{p^\ell}$ of period $\pi = qp^t$ with $p \nmid q$. The p^t*-periodised sequence* of f is the sequence:

$$\sum_{j=1}^{q} \theta^{jp^t} f = \theta^{p^t} f + \theta^{2p^t} f + \dots + \theta^{\pi - p^t} f + f.$$

It is easy to verify that it has period dividing p^t and hence it is nilpotent.

Proposition 2.6. *[2, Th. 17] Given $f \in P_{p^\ell}$ of period $\pi = qp^t$ with $p \nmid q$, the nilpotent part f_N of f coincides with the p^t-periodised sequence of f multiplied by $q^{-1} \mod p^\ell$.*

Corollary 2.7. *Let $f \in P_{p^\ell}$ be a periodic sequence.*

1. *If f is idempotent, then $\operatorname{tr} f = 0$.*
2. *If $\operatorname{tr} f = 0$ and $p \nmid \pi(f)$ then f is idempotent.*

Proof. 1. If f is idempotent with idepotency index η, from $\Delta^\eta f = f$ one gets $\Delta^{\eta-1} f - [(\Delta^{\eta-1} f)(0)] = \Sigma f$. By the idempotency of f and Lemma 1.4 one has

$$\pi(f) = \pi\left(\Delta^{\eta-1} f\right) = \pi\left(\Delta^{\eta-1} f - [(\Delta^{\eta-1} f)(0)]\right) = \pi\left(\Sigma f\right) = h \cdot \pi(f)$$

where h is the additive order of $\mathrm{tr}f$. Then $h = 1$ and hence $\mathrm{tr}f = 0$.
2. If $p \nmid \pi$, the p^t-periodised of f coincides with the constant sequence $[\mathrm{tr}f] = [0]$. By Proposition 2.6, f is idempotent. □

3. The period of the iterated anti-differences of nilpotent and idempotent periodic sequences

In this section we provide some new results reducing the study of anti-differences of any periodic sequence to that of anti-differences of constant sequences.

The first remark we are presenting in this section displays the connection between the iterated anti-differences of constant sequences and the binomial coefficients.

Remark 3.1. Since in $\mathbb{Z}^{\mathbb{N}}$ (and hence in S_m and P_m for each $m \geq 2$)

$$(\Sigma^0[1])(n) = [1](n) = 1, \ (\Sigma^s[1])(0) = 0 \text{ and}$$
$$(\Sigma^s[1])(n+1) = (\Sigma^{s-1}[1])(n) + (\Sigma^s[1])(n) \quad \forall n \geq 0, \ s \geq 1,$$

by the Stifel recursive formula [16, Prop. 2.22] we have

$$(\Sigma^s[c])(n) = c(\Sigma^s[1])(n) = c\binom{n}{s} \quad \forall n, s \geq 0.$$

Given a prime p and a natural number m, we denote by $\nu_p(m)$ (or simply by $\nu(m)$ when the prime p is clear in the context) the p-adic valuation of m, i.e., the highest power of p dividing m. In particular the p-adic valuation of 0 is infinite. The elements in any non zero coset in $\mathbb{Z}_{p^\ell}$ have the same p-adic valuation: therefore setting $\nu_p(0 + p^\ell\mathbb{Z}) = \infty$ the p-adic valuation can be defined also on $\mathbb{Z}_{p^\ell}$. For each real number r, we denote by $\lfloor r \rfloor$ the greatest integer number less or equal than r.

In 1956 Śviatomir Ząbek [22, Th. 3] proved the following result:

Theorem 3.2 *(Ząbek). For each natural number $2 \leq m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ the sequence $\Sigma^s[1]$ in P_m has period $mp_1^{\lfloor \log_{p_1} s \rfloor} \cdots p_r^{\lfloor \log_{p_r} s \rfloor}$. In particular for $m = p^\ell$ we have*

$$\pi(\Sigma^s[c]) = p^{\ell + \lfloor \log_p s \rfloor - \nu_p(c)}.$$

Definition 3.3. Let $f \in S_{p^\ell}$ be a sequence with values in $\mathbb{Z}_{p^\ell}$. Let us set

$$e_{f,i} := (\Delta^i f)(0) \quad \forall i \geq 0.$$

We write simply e_i when the sequence we refer to is clear from the context.

The numbers $e_{f,i}$, $i \geq 0$, determine uniquely the sequence f:

Proposition 3.4. *For any $f \in S_{p^\ell}$ and $0 \le j$ it is*

$$f(j) = \sum_{i=0}^{j} \binom{j}{i} e_{f,i}.$$

Proof. For any sequence f in S_{p^ℓ} we have $f(0) = (\Delta^0 f)(0) = e_{f,0} = \binom{0}{0} e_{f,0}$. Assume that the result is true for $j \ge 0$ for any sequence in S_{p^ℓ}. Then

$$\begin{aligned} f(j+1) = f(j) + (\Delta f)(j) &= \sum_{i=0}^{j} \binom{j}{i} e_{f,i} + \sum_{i=0}^{j} \binom{j}{i} e_{\Delta f,i} \\ &= \sum_{i=0}^{j} \binom{j}{i} e_{f,i} + \sum_{i=0}^{j} \binom{j}{i} e_{f,i+1} \\ &= \binom{j}{0} e_{f,0} + \sum_{i=1}^{j} (\binom{j}{i} + \binom{j}{i-1}) e_{f,i} + \binom{j}{j} e_{f,j+1} \\ &= \sum_{i=0}^{j+1} \binom{j+1}{i} e_{f,i} \quad \square \end{aligned}$$

Definition 3.5. Let $f \in I^{\Delta}_{p^\ell} \cup N^{\Delta}_{p^\ell}$ be a either nilpotent or idempotent periodic sequence with nilpotency or idempotency index η. We call *generating vector* of f the ordered η-tuple

$$\mathrm{vec}(f) = (e_{f,0}, e_{f,1}, \ldots, e_{f,\eta-1}) \in \mathbb{Z}^{\eta}_{p^\ell} \quad e_{f,i} = (\Delta^i f)(0),\ 0 \le i < \eta.$$

The last entry of $\mathrm{vec}(f)$ with minimal p-adic valuation is called the *leading component* of f.

Example 3.6. Consider the sequence $V = [2, 1, 2, 4, 8, 1, 8, 4] \in P_{12}$. The 2- and the 3-parts of V are

$$V_2 = [2, 1, 2, 0, 0, 1, 0, 0] \in P_4 \qquad V_3 = [2, 1] \in P_3.$$

The sequence V_2 has period 8 and hence by Theorem 2.3 it is nilpotent. Since $\Delta^5 V_2 = 0$ while $\Delta^4 V_2 = [2]$, the sequence V_2 has nilpotency index 5; then $\mathrm{vec}(V_2) = (2, 3, 2, 3, 2)$. The sequence V_3 has period 2 and $\mathrm{tr}(V_3) = 0$: hence it is idempotent by Corollary 2.7. Clearly it has idempotency index 1; then $\mathrm{vec}(V_3) = (2)$. By the Chinese Remainder Theorem, the sequences V_2 and V_3 correspond respectively to the following sequences in P_{12}:

$$\tilde{V}_2 = [6, 9, 6, 0, 0, 9, 0, 0] \qquad \tilde{V}_3 = [8, 4]$$

and $V = \tilde{V}_2 + \tilde{V}_3$.

By Proposition 3.4, any nilpotent (resp. idempotent) sequence $f \in I^{\Delta}_{p^\ell} \cup N^{\Delta}_{p^\ell}$ is determined uniquely by its generating vector $\mathrm{vec}(f)$.

The next results provide a rationale for the name *leading component* of the generating vectors of nilpotent and idempotent periodic sequences introduced in Definition 3.5. Indeed, given $f \in I^{\Delta}_{p^\ell} \cup N^{\Delta}_{p^\ell}$, we will prove that for s sufficiently large (shortly $s \gg 0$) the period of $\Sigma^s f$ will be driven by that of the iterated anti-difference of the leading component of the generating vector $\mathrm{vec}(f)$.

Theorem 3.7. *Let $f \in N^{\Delta}_{p^\ell}$ be a nilpotent sequence of nilpotency index η with generating vector $\mathrm{vec}(f) = (e_0, \dots, e_{\eta-1})$. Then*

$$\Sigma^s f = \sum_{i=0}^{\eta-1} \Sigma^{i+s}[e_i] \quad \forall s \geq 0.$$

If e_γ is the leading component of the generating vector $\mathrm{vec}(f)$, then

$$\pi(\Sigma^s f) = \pi(\Sigma^{s+\gamma}[e_\gamma]) = p^{\ell + \lfloor \log_p(s+\gamma) \rfloor - \nu_p(e_\gamma)} \quad \textit{for } s \gg 0.$$

Proof. Let us prove the first statement. By the linearity of Σ it is sufficient to prove the result for $s = 0$. We proceed by induction on η:

- $\eta = 1$ means $\Delta f = 0$, so $f = [c] = [f(0)]$.
- Suppose that the statement holds for $\eta = t$. If f has nilpotency index $\eta = t+1$, then Δf has nilpotency index t and by inductive hypothesis:

$$\Delta f = \sum_{i=0}^{\eta-1} \Sigma^i[e_i^{\Delta f}] = \sum_{i=0}^{\eta-1} \Sigma^i[(\Delta^i(\Delta f))(0)] = \sum_{i=0}^{\eta-1} \Sigma^i[\Delta^{i+1} f(0)] = \sum_{i=0}^{\eta-1} \Sigma^i[e_{i+1}].$$

Since $f = [f(0)] + \Sigma\Delta f$ we obtain that

$$f = [f(0)] + \Sigma\left(\sum_{i=0}^{\eta-1} \Sigma^i[e_{i+1}]\right) = [e_0] + \sum_{i=1}^{\eta} \Sigma^i[e_i] = \sum_{i=0}^{\eta} \Sigma^i[e_i].$$

If e_γ is the leading component, then $\nu_p(e_\gamma) \leq \nu_p(e_i)$, $0 \leq i < \gamma$, and $\nu_p(e_\gamma) < \nu_p(e_i)$, $\gamma < i < \eta$. Let us prove that $\pi(\Sigma^s f) = \pi(\Sigma^{s+\gamma}[e_\gamma])$ for $s \gg 0$. Let μ be the minimal natural number such that $p^\mu - \gamma \geq 0$ and $\eta - \gamma - 1 < p^\mu(p-1)$. Notice that for any $k \geq \mu$ both p^k and $p^k + \eta - \gamma - 1$ are strictly less than p^{k+1} and hence $k = \log_p p^k = \lfloor \log_p(p^k + \eta - \gamma - 1) \rfloor$. In order to conclude the proof, we show that for any $k \geq \mu$ one has:

$$\pi(\Sigma^s f) = \pi(\Sigma^{s+\gamma}[e_\gamma]) \qquad \forall\, p^k - \gamma \leq s < p^{k+1} - \gamma,$$

hence the statement holds for any $s \geq p^\mu - \gamma$.

For $s = p^k - \gamma$ we have:

$$\Sigma^{p^k-\gamma} f = \Sigma^{p^k-\gamma}[e_0] + \Sigma^{1+p^k-\gamma}[e_1] + \cdots + \Sigma^{\gamma+p^k-\gamma}[e_\gamma] + \cdots + \Sigma^{\eta-1+p^k-\gamma}[e_{\eta-1}].$$

By Theorem 3.2, $\Sigma^{p^k}[e_\gamma]$ has period $p^{\ell+k-\nu_p(e_\gamma)}$. The other summands have period strictly dividing $p^{\ell+k-\nu_p(e_\gamma)}$:

- For every $\gamma < i < \eta$, $p^k + i - \gamma < p^{k+1}$ by construction and so $\lfloor \log_p(p^k + i - \gamma) \rfloor = k$; hence the period of $\Sigma^{p^k+i-\gamma}[e_i]$ is $p^{\ell+k-\nu_p(e_i)} \mid p^{\ell+k-\nu_p(e_\gamma)-1}$ (since $\nu_p(e_i) > \nu_p(e_\gamma)$).
- For every $0 \leq i < \gamma$, $\nu_p(e_\gamma) \leq \nu_p(e_i)$ and $p^k + i - \gamma < p^k$ and so $\lfloor \log_p(p^k + i - \gamma) \rfloor \leq k - 1$. Hence the period of $\Sigma^{p^k+i-\gamma}[e_i]$ is a divisor of $p^{\ell+k-1-\nu_p(e_i)}$ and so it divides $p^{\ell+k-1-\nu_p(e_\gamma)}$.

Thus the period $\pi(\Sigma^{p^k-\gamma} f)$ is equal to $p^{\ell+k-\nu_p(e_\gamma)}$.

For $p^k - \gamma < s < p^{k+1} - \gamma$, the period of $\Sigma^{s+\gamma}[e_\gamma]$ is $p^{\ell+k-\nu_p(e_\gamma)}$, and by Lemma 1.4 $\pi(\Sigma^s f) \mid \pi(\Sigma^{p^k-\gamma} f) = p^{\ell+k-\nu_p(e_\gamma)}$. Furthermore, since $p^{k+1} + \eta - \gamma - 1 < p^{k+2}$ we have

$$p^k - \gamma < s \leq s + \eta - 1 < p^{k+1} - \gamma + \eta - 1 < p^{k+2}.$$

Then $\pi(\Sigma^{s+i}[e_i]) \mid p^{\ell+k+1-(\nu_p(e_\gamma)+1)}$ for $\gamma < i \leq \eta - 1$, and $\pi(\Sigma^{s+i}[e_i]) \mid p^{\ell+k-\nu_p(e_\gamma)}$ for $0 \leq i \leq \gamma - 1$. Thus $\pi(\Sigma^s f) \mid p^{\ell+k-\nu_p(e_\gamma)}$, and hence $\pi(\Sigma^s f) = p^{\ell+k-\nu_p(e_\gamma)}$. □

Corollary 3.8. *Denoted by e_γ the leading component of the generating vector of $f \in N^\Delta_{p^\ell}$, one has that for $t \gg 0$*

$$\pi\Big(\sum_{s=0}^{t} \Sigma^s f\Big) = \pi\Big(\Sigma^{t+\gamma}[e_\gamma]\Big) = p^{\ell+\lfloor \log_p(t+\gamma)\rfloor-\nu_p(e_\gamma)}.$$

Proof. Observe that if $\mathrm{vec}(f) = (e_0, \ldots, e_{\eta-1})$, then

$$\sum_{s=0}^{t} \Sigma^s f = \sum_{s=0}^{t} (\sum_{i=0}^{\eta-1} \Sigma^{i+s}[e_i]).$$

Hence, repeating the same reasoning of the proof of Theorem 3.7, we get that the period of $\sum_{s=0}^{t} \Sigma^s f$ is equal to $\pi(\Sigma^{t+\gamma}[e_\gamma])$. □

Remark 3.9. With the notation of Theorem 3.7, for each $s \geq 0$

$$\pi(\Sigma^s f) \mid \mathrm{lcm}\{\pi(\Sigma^{i+s}[e_{f,i}]) \,:\, 0 \leq i < \eta)\} = \max\{\pi(\Sigma^{i+s}[e_{f,i}]) \,:\, 0 \leq i < \eta\}.$$

In general we have not the equality. For example $f = [0, 2] \in N^\Delta_8$ is a nilpotent sequence of nilpotency index $\eta = 3$. It has period 2 and $\mathrm{vec}(f) = (0, 2, 4)$. By Theorem 3.7 one has $f = \Sigma[2] + \Sigma^2[4]$, and both $\Sigma[2]$ and $\Sigma^2[4]$ have period 4. Hence $\pi(f) = 2$ divides properly $\mathrm{lcm}\{\pi(\Sigma[2]), \pi(\Sigma^2[4])\} = 4$.

Remark 3.10. Observe that Theorem 3.7 generalizes [11, Th. 8] to the case of sequences in $\mathbb{Z}_{p^\ell}$, and hence, by the Chines Remainder Theorem, to that of sequences in $\mathbb{Z}_m$ for each $m \in \mathbb{N}$. Indeed for $s = 0$ we obtain that the nilpotent sequences in $\mathbb{Z}_{p^\ell}$, i.e. those with period a power of p, are sums of anti-differences of constant sequences, i.e., linear combinations of binomials.

Theorem 3.11. *Consider $f \in I^{\Delta}_{p^\ell}$ with idempotency index η and generating vector* $\mathrm{vec}(f) = (e_0, \dots, e_{\eta-1})$. *For every $s \geq 1$, one has:*

$$\Sigma^s f = \Delta^{\overline{-s}} f - \sum_{j=0}^{s-1} \Sigma^j [e_{\overline{j-s}}]$$

where $\overline{j-s}$ is the remainder in the division of $j - s$ by η for $j = 0, 1, \dots, s-1$. This provides the explicit decomposition in idempotent and nilpotent part of $\Sigma^s f$. Moreover if e_γ is the leading component of $\mathrm{vec}(f)$, *one has*

$$\pi(\Sigma^s f) = \mathrm{lcm}\left(\pi(f), \pi(\Sigma^{s-\eta+\gamma}[e_\gamma])\right) = \mathrm{lcm}\left(\pi(f), p^{\ell + \lfloor \log_p(s-\eta+\gamma) \rfloor - \nu_p(e_\gamma)}\right) \quad \forall s \gg 0.$$

Proof. We proceed by induction on s.

- For $s = 1$ one has $\Sigma f = \Sigma(\Delta^\eta f) = \Delta^{\eta-1} f - [e_{\eta-1}] = \Delta^{\overline{-1}} f - [e_{\overline{-1}}]$, and hence the thesis.
- Suppose that the statement is true for $1 \leq s = t\eta + \bar{s}$, $t \geq 0$, $0 \leq \bar{s} < \eta$; let us prove it for $s+1 = t'\eta + \overline{s+1}$. Notice that $(t', \overline{s+1}) = (t+1, 0)$ if $\bar{s} = \eta - 1$ and $(t', \overline{s+1}) = (t, \bar{s}+1)$ otherwise. By inductive hypothesis we have:

$$\begin{aligned}
\Sigma^{s+1} f = \Sigma(\Sigma^s f) =& \Sigma(\Delta^{\overline{-s}} f - \sum_{j=0}^{s-1} \Sigma^j [e_{\overline{j-s}}]) \\
=& \Delta^{\overline{-s-1}} f - [e_{\overline{-s-1}}] - \sum_{j=1}^{s} \Sigma^j [e_{\overline{j-1-s}}] \\
=& \Delta^{\overline{-(s+1)}} f - \sum_{j=0}^{s} \Sigma^j [e_{\overline{j-(s+1)}}].
\end{aligned}$$

By the uniqueness of the decomposition of a periodic sequence in its idempotent and nilpotent parts, $\Delta^{\overline{-s}} f$ and $-\sum_{j=0}^{s-1} \Sigma^j [e_{\overline{j-s}}]$ are the idempotent and nilpotent parts of $\Sigma^s f$, respectively. Let us now compute the period of $\Sigma^s f$ for s sufficiently large. Firstly we have $\pi\left(\Delta^j f\right) = \pi(f)$ for any $j \in \mathbb{N}$ (since $f \in I^{\Delta}_{p^\ell}$). Now let us denote by g the nilpotent sequence $\sum_{j=0}^{\eta-1} \Sigma^j[e_j]$. The nilpotency index of g is $\eta_g := \max\{j : e_j \neq 0\} + 1$; clearly $\eta_g \leq \eta$, but the leading component of $\mathrm{vec}(g)$ and $\mathrm{vec}(f)$ is the same: $[e_\gamma]$. Notice that for any $s \geq \eta$:

$$\sum_{j=0}^{s-1} \Sigma^j[e_{\overline{j-s}}] = \sum_{0\leq j<\bar{s}} \Sigma^j[e_{\overline{j-s}}] + \sum_{j=\bar{s}}^{s-1} \Sigma^j[e_{\overline{j-s}}] = \sum_{0\leq j<\bar{s}} \Sigma^j[e_{\overline{j-s}}] + \sum_{i=0}^{t-1} \Sigma^{i\eta+\bar{s}} g.$$

By Corollary 3.8, for $s \gg 0$ one has:

$$\pi\left(\sum_{j=0}^{s-1} \Sigma^j[e_{\overline{j-s}}]\right) = \pi\left(\sum_{i=0}^{t-1} \Sigma^{i\eta+\bar{s}} g\right) = \pi\left(\Sigma^{(t-1)\eta+\bar{s}+\gamma}[e_\gamma]\right) = \pi\left(\Sigma^{s-\eta+\gamma}[e_\gamma]\right).$$

We conclude by Theorem 3.2. □

Remark 3.12. With the notation of the previous theorem, for each $s \geq 0$

$$\pi(\Sigma^s f) \mid \mathrm{lcm}\{\pi(\Delta^{\overline{-s}} f), \pi(\Sigma^j[e_{f,\overline{j-s}}]) \,:\, 0 \leq i < \eta\}.$$

In general we have not the equality. For example $f = [6, 0, 2] \in I_8^\Delta$ is an idempotent sequence of idempotency index $\eta = 6$. It has period 3 and $\mathrm{vec}(f) = (6, 2, 0, 2, 2, 4)$. By Theorem 3.7 one has $\Sigma^3 f = \Delta^3 f - \big([2] + \Sigma[2] + \Sigma^2[4]\big)$. Now $\Sigma^3 f$ has period 6, $\Delta^3 f$ has period 3, [2] has period 1, $\Sigma[2]$ has period 4 and $\Sigma^2[4]$ has period 4. Hence $\pi(\Sigma^3 f) = 6$ divides properly $\mathrm{lcm}\{\pi(\Delta^3 f), \pi([2]), \pi(\Sigma[2]), \pi(\Sigma^2[4])\} = 12$.

Remark 3.13. In the proofs of Theorems 3.7, Corollary 3.8 and 3.11, we computed explicitly how big s has to be for the statements to hold. It is easy to check that $s \geq \eta - \gamma - 1$ and $s \geq \eta$ are sufficient in the nilpotent and in the idempotent cases respectively to satisfy the required conditions.

Example 3.14. Consider the sequences

$$V_2 = [2, 1, 2, 0, 0, 1, 0, 0] \in P_4, \quad V_3 = [2, 1] \in P_3.$$

As we observed in Example 3.6, V_2 is nilpotent of nilpotency index 5 and V_3 is idempotent of idempotency index 1. Their generating vectors are $\mathrm{vec}(V_2) = (2, 3, 2, 3, 2)$, and $\mathrm{vec}(V_3) = (2)$. The leading component of $\mathrm{vec}(V_2)$ is $e_3 = 3$, that of $\mathrm{vec}(V_3)$ is $e_0 = 2$. By Theorem 3.7 we have

$$\Sigma^s V_2 = \Sigma^s[2] + \Sigma^{s+1}[3] + \Sigma^{s+2}[2] + \Sigma^{s+3}[3] + \Sigma^{s+4}[2].$$

By Remark 3.13 and Theorem 3.2, for $s \geq 1$ we have

$$\pi(\Sigma^s V_2) = \pi(\Sigma^{s+3}[3]) = 2^{2+\lfloor \log_2(s+3) \rfloor}.$$

By Theorem 3.11 we have

$$\Sigma^s V_3 = f - \sum_{j=0}^{s-1} \Sigma^j [2].$$

By Remark 3.13, for $s \geq 1$ we have

$$\pi(\Sigma^s V_3) = \operatorname{lcm}\big(2, \pi(\Sigma^{s-1}[2])\big) = 2 \times 3^{1+\lfloor \log_3(s-1) \rfloor}.$$

4. Recursive formula for binomials coefficients in $\mathbb{Z}_{p^\ell}$

In this section we will prove some new recursive formulas describing the p-adic valuation of the coefficients of the periodic sequence of binomial coefficients $\binom{n}{s}_{n\geq 0} \mod p^\ell$, $p^k \leq s < p^{k+1}$, in terms the p-adic valuation of the coefficients of the periodic sequence of binomial coefficients $\binom{n}{s'}_{n\geq 0} \mod p^\ell$, $p^{k-1} \leq s' < p^k$.

One of the main tools to study binomial coefficients modulo p^ℓ is *Kummer's Theorem* [13]. This result says that, given a prime p, for given integers $n \geq m \geq 0$ with p-adic representation

$$n = \langle a_s a_{s-1} \dots a_1 a_0 \rangle_p, \quad s = \langle b_s b_{s-1} \dots b_1 b_0 \rangle_p,$$

the p-adic valuation $\nu_p\left(\binom{n}{s}\right)$ of the binomial coefficient n over s is equal to the number of borrows in the subtraction $\langle a_s a_{s-1} \dots a_1 a_0 \rangle_p - \langle b_s b_{s-1} \dots b_1 b_0 \rangle_p$.

Example 4.1. Consider the numbers

$$798 = \langle 1002120 \rangle_3, \quad 454 = \langle 121211 \rangle_3.$$

Let us compute the 3-adic valuation of $\binom{798}{454}$:

$$\begin{array}{cccccccc} \check{1} & \check{0} & 0 & \check{2} & 1 & \check{2} & 0 & - \\ & 1 & 2 & 1 & 2 & 1 & 1 & = \\ \hline \\ & 1 & 1 & 0 & 2 & 0 & 2 & \end{array} \quad \Longrightarrow \quad \nu_3\left(\binom{798}{454}\right) = 4.$$

In this section, we focus on the s-th binomial function:

$$\begin{aligned} \mathbf{b}_s : \mathbb{N} &\longrightarrow \mathbb{Z}_{p^\ell} \\ n &\longmapsto \binom{n}{s}. \end{aligned}$$

As observed in Remark 3.1, this function coincides with the s-th anti-difference $\Sigma^s[1]$ of the constant sequence $[1] \in P_{p^\ell}$. If the expression of s in base p is one of the following:

$$\langle b_k \cdots b_{k-m} \underbrace{(p-1)\cdots(p-1)}_{\ell} b_{k-m-\ell-1}\cdots b_0\rangle_p$$

$$\langle b_k \cdots b_{k-m} \underbrace{0 \quad \cdots \quad 0}_{\ell} b_{k-m-\ell-1}\cdots b_0\rangle_p$$

$$\langle b_k \cdots b_{k-m} \underbrace{(p-1)\, 0 \ \cdots \ 0}_{\ell} b_{k-m-\ell-1}\cdots b_0\rangle_p$$

where $k > \ell$ and $0 \le m \le k-\ell-1$, we prove that it is possible to link the s-th binomial function $\mathbf{b}_s$ to $\mathbf{b}_{s'}$ where s' is obtained from s by removing one of the explicit coefficients in its p-base expression. Of course, such patterns do not always occur in the expression of s in base p, a part the case $p = 2$ and $\ell = 2$.

Firstly we need some definitions.

Definition 4.2. Given a sequence $f \in S_m := \mathbb{Z}_m^{\mathbb{N}}$, a prime q and an integer $t \ge 1$, we call *j-th q^t-subsequence of f* the element $h_j \in \mathbb{Z}_m^{q^t}$ defined as

$$h_j = (f(jq^t), f(jq^t+1), \dots, f((j+1)q^t-1)) \qquad j \in \mathbb{N}.$$

We denote by $\mathrm{R}(f, q^t) \in S_m$ the sequence obtained repeating q times the j-th q^t-subsequences of f for $j = 0, 1, 2, \dots$:

$$\mathrm{R}(f, q^t) = (\underbrace{h_0, \dots, h_0}_{q}, \underbrace{h_1, \dots, h_1}_{q}, \dots).$$

We denote by $\mathrm{A}(f, q^t) \in S_m$ the sequence obtained alternating $(q-1)q^t$ zeros and the j-th q^t-subsequences of f for $j = 0, 1, 2, \dots$:

$$\mathrm{A}(f, q^t) = (\underbrace{0, \dots, 0}_{(q-1)q^t}, h_0, \underbrace{0, \dots, 0}_{(q-1)q^t}, h_1, \dots).$$

Proposition 4.3. *For any $f \in S_m$, $t \ge 1$, and $n' = \langle a_r \dots a_t a_{t-1} \dots a_0\rangle_q$ one has*

$$\mathrm{R}(f, q^t)(n) = f(n') \quad \text{if } n = \langle a_r \dots a_t\, \alpha\, a_{t-1} \dots a_0\rangle_q,\ \forall\, 0 \le \alpha < q.$$

$$\mathrm{A}(f, q^t)(n) = \begin{cases} f(n') & \text{if } n = \langle a_r \dots a_t (q-1) a_{t-1} \dots a_0\rangle_q \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Definition 4.2, given $\xi \in \mathbb{N}$, $0 \le \alpha < q$, $0 \le i < q^t$, one has

$$\mathrm{R}(f, q^t)(\xi q^{t+1} + \alpha q^t + i) = f(\xi q^t + i)$$

$$\mathrm{A}(f, q^t)(\xi q^{t+1} + \alpha q^t + i) = \begin{cases} f(\xi q^t + i) & \text{if } \alpha = q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Translating in the q-adic representation, we get the claim. □

Example 4.4.

- The set of 2-subsequences of $f = [0, 1, 2, 3, 4, 5] \in P_7$ is

$$\{[0, 1], [2, 3], [4, 5]\}.$$

- If $h = [1, 2, 3, 4, 5, 6, 7, 8] \in P_{11}$, then:

$$\begin{aligned} \mathrm{R}(h, 2^2) =& [1, 2, 3, 4, 1, 2, 3, 4, 5, 6, 7, 8, 5, 6, 7, 8] \\ \mathrm{A}(h, 2^2) =& [0, 0, 0, 0, 1, 2, 3, 4, 0, 0, 0, 0, 5, 6, 7, 8]. \end{aligned}$$

  Moreover

$$\begin{aligned} \mathrm{R}(h, 2^2)(2^3 + 2^2 + 3) = 8 = h(2^2 + 3) \qquad & \mathrm{R}(h, 2^2)(2^3 + 2) = 7 = h(2^2 + 2) \\ \mathrm{A}(h, 2^2)(2^3 + 2^2 + 3) = 8 = h(2^2 + 3) \qquad & \mathrm{A}(h, 2^2)(2^3 + 2) = 0. \end{aligned}$$

Remark 4.5. Observe the following facts:

- For any q^t both R and A are linear operators: for any $c_1, c_2 \in \mathbb{Z}_m$ and $f_1, f_2 \in S_m$, it is

$$\begin{aligned} \mathrm{R}(c_1 f_1 + c_2 f_2, q^t) =& c_1 \mathrm{R}(f_1, q^t) + c_2 \mathrm{R}(f_2, q^t) \\ \mathrm{A}(c_1 f_1 + c_2 f_2, q^t) =& c_1 \mathrm{A}(f_1, q^t) + c_2 \mathrm{A}(f_2, q^t). \end{aligned}$$

- If $f \in P_m$ has period π and $q^t \mid \pi$, then both $\mathrm{R}(f, q^t)$ and $\mathrm{A}(f, q^t)$ have period $q\pi$.

Definition 4.6. If $f, g \in P_{p^\ell}$, we write:

- $f \equiv_\nu g$ if for any $n \geq 0$, $f(n) = 0$ if and only if $g(n) = 0$, and otherwise $\nu_p(f(n)) = \nu_p(g(n)) \in \{0, \cdots, \ell - 1\}$.
- $\Pi_i(f) := \#\{f(x) \mid 0 \leq x < \pi(f),\ \nu_p(f(x)) = i\}$ the number of coefficients with p-adic valuation i, for every $0 \leq i < \ell$.
- $Z(f) := \#\{f(x) \mid 0 \leq x < \pi(f),\ f(x) = 0\}$ the number of zeros.

Let us consider now the s-th anti-difference $\Sigma^s[1] = \mathbf{b}_s$ of the constant sequence [1] in P_{p^ℓ}. Suppose that $p^k \leq s < p^{k+1}$. The next results allow to link in certain cases the quantities $\Pi_i(\mathbf{b}_s), Z(\mathbf{b}_s)$ to the quantities $\Pi_i(\mathbf{b}_{s'}), Z(\mathbf{b}_{s'})$ for some s' with $p^{k-1} \leq s' < p^k$.

Lemma 4.7. *With the notation above, suppose that $k > \ell$, $-1 \leq m \leq k - \ell - 1$, and that the expression of s in base p is:*

$$s = \langle b_k \cdots b_{k-m} \underbrace{(p-1)\cdots(p-1)}_{\ell} b_{k-m-\ell-1} \cdots b_0 \rangle_p.$$

Denote by

$$\begin{aligned} s' &:= s - \big(b_k p^k + (b_{k-1} - b_k)p^{k-1} + \cdots + (p-1-b_{k-m})p^{k-m-1}\big) \\ &= \langle b_k \cdots b_{k-m} \underbrace{(p-1)\cdots(p-1)}_{\ell-1} b_{k-m-\ell-1} \cdots b_0 \rangle_p. \end{aligned}$$

Then $\mathbf{b}_s \equiv_\nu \mathrm{A}(\mathbf{b}_{s'}, p^{k-m-\ell})$. *In particular* $\Pi_i(\mathbf{b}_s) = \Pi_i(\mathbf{b}_{s'})$ *and* $Z(\mathbf{b}_s) = Z(\mathbf{b}_{s'}) + (p-1)p^{k+\ell-1}$.

Proof. The case $m = -1$ corresponds to $s = \langle \underbrace{(p-1)\cdots(p-1)}_{\ell} b_{k-\ell} \cdots b_0 \rangle_p$. The sequence $\mathbf{b}_s$ has period $p^{\ell+k}$ by Theorem 3.2. For any $0 \le n < p^{\ell+k}$, let $n = \langle a_{k+\ell-1} \dots a_0 \rangle_p$ be its expression in base p. The n-th coefficient of $\mathbf{b}_s$ is:

$$\begin{pmatrix} a_{k+\ell-1} & \cdots & a_{k+1} & a_k & \cdots & a_{k-m} & a_{k-m-1} \cdots a_{k-m-\ell} & a_{k-m-\ell-1} & \cdots & a_0 \\ & & & b_k & \cdots & b_{k-m} & \underbrace{(p-1)\cdots(p-1)}_{\ell} & b_{k-m-\ell-1} & \cdots & b_0 \end{pmatrix}.$$

Let n' be obtained from n by removing the coefficient $a_{k-m-\ell}$. The n'-th coefficient of $\mathbf{b}_{s'}$ is:

$$\begin{pmatrix} a_{k+\ell-1} & \cdots & a_{k+1} & a_k & \cdots & a_{k-m} & a_{k-m-1} \cdots a_{k-m-\ell+1} & a_{k-m-\ell-1} & \cdots & a_0 \\ & & & b_k & \cdots & b_{k-m} & \underbrace{(p-1)\cdots(p-1)}_{\ell-1} & b_{k-m-\ell-1} & \cdots & b_0 \end{pmatrix}.$$

By Proposition 4.3, to conclude that $\mathbf{b}_s \equiv_\nu \mathrm{A}(\mathbf{b}_{s'}, p^{k-m-\ell})$, it is enough to show that $\nu_p(\mathbf{b}_s(n)) = \nu_p(\mathbf{b}_{s'}(n'))$ if $a_{k-m-\ell} = p-1$ and $\mathbf{b}_s(n) = 0$ otherwise. To prove this, we use Kummer's Theorem studying the number of borrows in the subtractions $n - s$ and $n' - s'$ in base p:

- If $a_{k-m-\ell} = p-1$:
  - If $a_{k-m-\ell}$ lends, the number of borrows in $n - s$ is at least $\ell + 1$ and one more than the number of borrows in $n' - s'$. However in both binomials there are at least ℓ borrows (given by the remaining $(\ell-1)$ coefficients equal to $p-1$), hence both binomials are zero modulo p^ℓ.
  - If $a_{k-m-\ell}$ does not lend, the number of borrows is the same for $n - s$ and $n' - s'$.
- If $a_{k-m-\ell} < p-1$: the binomial $\mathbf{b}_s(n) = 0$ since again there are at least ℓ borrows.

From the considerations above, we conclude that $\mathbf{b}_s \equiv_\nu \mathrm{A}(\mathbf{b}_{s'}, p^{k-m-\ell})$. Then immediately follows

$$\Pi_i(\mathbf{b}_s) = \Pi_i(\mathbf{b}_{s'}),$$
$$Z(\mathbf{b}_s) = Z(\mathbf{b}_{s'}) + (p-1)\pi(\mathbf{b}_{s'}) = Z(\mathbf{b}_{s'}) + (p-1)p^{k+\ell-1}. \quad \square$$

Example 4.8. Let $p = 3$, $\ell = 2$ and $s = 51 = \langle 1220 \rangle$. Then $s' = \langle 120 \rangle = 15$. The sequence $\mathbf{b}_{51}$ has period $3^{2+3} = 243$, while the sequence $\mathbf{b}_{15}$ has period $3^{2+2} = 81$. We have

$$\mathbf{b}_{15} = [0_{15}1716663332520_6 3330_6 4146663335850_6 6660_6 747666333828],$$

where 0_i denotes a sequence of i zeros. By Lemma 4.7 the elements of the sequence $\mathbf{b}_{51}$ have the same 3-adic valuation of the elements of $\mathrm{A}(\mathbf{b}_{15}, 3)$, which is obtained alternating 6 zeros to the 3-subsequences of $\mathbf{b}_{15}$:

$$\mathbf{b}_{51} \equiv_\nu [\underbrace{0_6 0_3 \dots 0_6 0_3}_{45} \underbrace{0_6\, 171\, 0_6\, 666 \dots 0_6\, 828}_{198}.]$$

In particular $\Pi_0(\mathbf{b}_{51}) = \Pi_0(\mathbf{b}_{15}) = 18$, $\Pi_1(\mathbf{b}_{51}) = \Pi_1(\mathbf{b}_{15}) = 24$, $Z(\mathbf{b}_{51}) = Z(\mathbf{b}_{15}) + (3 - 1)\pi(\mathbf{b}_{15}) = 39 + 2 \times 3^{2+3-1} = 201$.

Lemma 4.9. *With the notation above, suppose that $k > \ell$, $0 \le m \le k - \ell - 1$ and that the expression of s in base p is:*

$$s = \langle b_k \cdots b_{k-m} \underbrace{0 \cdots 0}_{\ell} b_{k-m-\ell-1} \cdots b_0 \rangle_p.$$

Denote by

$$\begin{aligned} s' &:= s - \left(b_k p^k + (b_{k-1} - b_k)p^{k-1} + \cdots + (b_{k-m} - b_{k-m+1})p^{k-m} - b_{k-m}p^{k-m-1}\right) \\ &= \langle b_k \cdots b_{k-m} \underbrace{0 \cdots 0}_{\ell-1} b_{k-m-\ell-1} \cdots b_0 \rangle_p. \end{aligned}$$

Then $\mathbf{b}_s \equiv_\nu \mathrm{R}(\mathbf{b}_{s'}, p^{k-m-1})$. In particular, $\Pi_i(\mathbf{b}_s) = p \cdot \Pi_i(\mathbf{b}_{s'})$ and $Z(\mathbf{b}_s) = p \cdot Z(\mathbf{b}_{s'})$.

Proof. The sequence $\mathbf{b}_s$ has period $p^{\ell+k}$ by Theorem 3.2. Similarly to the previous lemma, for $0 \le n < p^{\ell+k}$ with $n = \langle a_{k+\ell-1} \dots a_0 \rangle_p$, the coefficient $\mathbf{b}_s(n) = \binom{n}{s}$ is:

$$\begin{pmatrix} a_{k+\ell-1} & \cdots & a_{k+1} & a_k & \cdots & a_{k-m} & a_{k-m-1} \cdots a_{k-m-\ell} & a_{k-m-\ell-1} & \cdots & a_0 \\ & & & b_k & \cdots & b_{k-m} & \underbrace{0 \quad \cdots \quad 0}_{\ell} & b_{k-m-\ell-1} & \cdots & b_0 \end{pmatrix}.$$

Let n' be obtained from n by removing the coefficient a_{k-m-1}, hence the n'-th coefficient of $\mathbf{b}_{s'}$ is:

$$\begin{pmatrix} a_{k+\ell-1} & \cdots & a_{k+1} & a_k & \cdots & a_{k-m} & a_{k-m-2} \cdots a_{k-m-\ell} & a_{k-m-\ell-1} & \cdots & a_0 \\ & & & b_k & \cdots & b_{k-m} & \underbrace{0 \quad \cdots \quad 0}_{\ell-1} & b_{k-m-\ell-1} & \cdots & b_0 \end{pmatrix}.$$

By Proposition 4.3, to conclude that $\mathbf{b}_s \equiv_\nu \mathrm{R}(\mathbf{b}_{s'}, p^{k-m-1})$, it is enough to show that, for any value of a_{k-m-1}, $\mathbf{b}_{s'}(n') = 0$ whenever $\mathbf{b}_s(n) = 0$, otherwise $\nu_p(\mathbf{b}_s(n)) = \nu_p(\mathbf{b}_{s'}(n'))$. To prove this, we use Kummer's Theorem studying the number of borrows in the subtractions $n - s$ and $n' - s'$ in base p:

- If a_{k-m-1} lends, then $a_{k-m-2} = \cdots = a_{k-m-\ell} = 0$ and they all lend. So in this case in both s and s' there are at least ℓ borrows (notice that a_{k-m} lends in s'); so the binomials are both equal to zero.
- If a_{k-m-1} does not lend, then the number of borrows remains the same in both the binomials.

Henceforth we can conclude that $\mathbf{b}_s \equiv_\nu \mathrm{R}(\mathbf{b}_{s'}, p^{k-m-1})$, thus:

$$\Pi_i(\mathbf{b}_s) = p \cdot \Pi_i(\mathbf{b}_{s'}) \qquad Z(\mathbf{b}_s) = p \cdot Z(\mathbf{b}_{s'}). \quad \square$$

Example 4.10. Let $p = 3$, $\ell = 2$ and $s = 55 = \langle 2001 \rangle$. Then $s' = \langle 201 \rangle = 19$. The sequence $\mathbf{b}_{55}$ has period $3^{2+3} = 243$, while the sequence $\mathbf{b}_{19}$ has period $3^{2+2} = 81$. We have

$$\mathbf{b}_{19} = [0_{18}012318678036036036063063063012318678063063063036036036012318678];$$

by Lemma 4.9 the elements of the sequence $\mathbf{b}_{55}$ have the same 3-adic valuation of the elements of $\mathrm{R}(\mathbf{b}_{19}, 3^2)$, which is obtained repeating 3 times the 9-subsequences of $\mathbf{b}_{19}$:

$$\begin{aligned} \mathbf{b}_{55} \equiv_\nu [&0_9 0_9 0_9\, 0_9 0_9 0_9\, 012318678\, 012318678\, 012318678\, 036036036 \\ &036036036\, 036036036 \ldots 012318678\, 012318678\, 012318678.] \end{aligned}$$

In particular $\Pi_0(\mathbf{b}_{55}) = 3 \times \Pi_0(\mathbf{b}_{19}) = 54$, $\Pi_1(\mathbf{b}_{55}) = 3 \times \Pi_1(\mathbf{b}_{19}) = 90$, $Z(\mathbf{b}_{55}) = 3 \times Z(\mathbf{b}_{19}) = 99$.

In order to present the last result of this section, we need some preliminary definitions.

Definition 4.11. Given $s = \langle b_k \cdots b_{k-m} \underbrace{(p-1)\, 0 \cdots 0}_{\ell} b_{k-m-\ell-1} \cdots b_0 \rangle_p \in \mathbb{N}$ with $k > \ell$ and $-1 \le m \le k - \ell - 1$, we denote by E_s the following subset of $\{0, \ldots, p^{k+\ell} - 1\}$:

$$\begin{aligned} E_s := \Big\{ n \in \mathbb{N} : 0 \le n < p^{k+\ell},\ n = \langle a_{k+\ell-1} \ldots a_0 \rangle_p \text{ such that:} \\ a_{k-m-1} = p - 1 \qquad a_{k-m-2} \neq 0 \\ a_{k-m-i} = 0 \quad \forall\, 3 \le i \le \ell \qquad a_{k-m-\ell-1} < b_{k-m-\ell-1} \\ a_j \ge b_j \quad \forall\, 0 \le j \le k - m - \ell - 2 \text{ and } k - m \le j \le k \Big\}. \end{aligned}$$

The case $m = -1$ corresponds to $s = \langle\underbrace{(p-1)\,0\cdots 0}_{\ell}\,b_{k-\ell}\cdots b_0\rangle_p$. We denote by $\chi_{E_s} \in P_{p^\ell}$ the sequence:

$$\chi_{E_s} = [e_0, \ldots, e_{p^{k+\ell}-1}] \quad \text{where } e_i = \begin{cases} 1 \text{ if } i \in E_s \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to check that

$$|E_s| = p^{\ell-1}\left(\prod_{j=k-m}^{k}(p-b_j)\right)(p-1)\,b_{k-m-\ell-1}\left(\prod_{i=0}^{k-\ell-m-2}(p-b_i)\right)$$

and hence $E_s = \emptyset$ if $b_{k-m-\ell-1} = 0$.

Lemma 4.12. *With the notation above, suppose that $k > \ell, -1 \le m \le k-\ell-1$ and that the expression of s in base p is:*

$$s = \langle b_k\cdots b_{k-m}\,\underbrace{(p-1)\;0\cdots 0}_{\ell}\;b_{k-m-\ell-1}\cdots b_0\rangle_p.$$

Denote by

$$\begin{aligned} s' &:= s - \big(b_k p^k + (b_{k-1}-b_k)p^{k-1} + \cdots + (p-1-b_{k-m})p^{k-m-1} - (p-1)p^{k-m-2}\big) \\ &= \langle b_k\cdots b_{k-m}\,\underbrace{(p-1)\;0\cdots 0}_{\ell-1}\;b_{k-m-\ell-1}\cdots b_0\rangle_p. \end{aligned}$$

Then $\mathbf{b}_s \equiv_\nu \mathrm{R}(\mathbf{b}_{s'}, p^{k-m-2}) + p^{\ell-1}\chi_{E_s}$ and thus

$$\begin{aligned} \Pi_i(\mathbf{b}_s) &= p\cdot\Pi_i(\mathbf{b}_{s'}) \qquad 0 \le i \le \ell-2 \\ \Pi_{\ell-1}(\mathbf{b}_s) &= p\cdot\Pi_{\ell-1}(\mathbf{b}_{s'}) + |E_s| \\ Z(\mathbf{b}_s) &= p\cdot Z(\mathbf{b}_{s'}) - |E_s|. \end{aligned}$$

Proof. The case $m = -1$ corresponds to $s = \langle\underbrace{(p-1)\cdots(p-1)}_{\ell}\,b_{k-\ell}\cdots b_0\rangle_p$. The sequence $\mathbf{b}_s$ has period $p^{\ell+k}$ by Theorem 3.2. Similarly to the previous lemmas, for $0 \le n < p^{\ell+k}$ with $n = \langle a_{k+\ell-1}\ldots a_0\rangle_p$, the coefficient $\mathbf{b}_s(n) = \binom{n}{s}$ is:

$$\begin{pmatrix} a_{k+\ell-1}\cdots a_{k+1} & a_k\cdots a_{k-m} & a_{k-m-1} & a_{k-m-2}\cdots a_{k-m-\ell} & a_{k-m-\ell-1}\cdots a_0 \\ & b_k\cdots b_{k-m} & p-1 & \underbrace{0\quad\cdots\quad 0}_{\ell-1} & b_{k-m-\ell-1}\cdots b_0 \end{pmatrix}.$$

Let n' be obtained from n by removing the coefficient a_{k-m-2}, hence the n'-th coefficient of $\mathbf{b}_{s'}$ is:

$$\begin{pmatrix} a_{k+\ell-1}\cdots a_{k+1} & a_k\cdots a_{k-m} & a_{k-m-1} & a_{k-m-3}\cdots a_{k-m-\ell} & a_{k-m-\ell-1}\cdots a_0 \\ & b_k\cdots b_{k-m} & p-1 & \underbrace{0\ \ \cdots\ \ 0}_{\ell-2} & b_{k-m-\ell-1}\cdots b_0 \end{pmatrix}.$$

Let us use Kummer's Theorem to study the number of borrows in the subtractions $n-s$ and $n'-s'$ in base p:

- if $a_{k-m-\ell-1}$ does not lend, the two binomials have the same number of borrows.
- if $a_{k-m-\ell-1}$ lends, we have the following cases:
  - if $a_{k-m-2}=a_{k-m-3}=\cdots=a_{k-m-\ell}=0$, then both binomials have at least ℓ borrows and hence they are zero.
  - If $a_{k-m-3}=\cdots=a_{k-m-\ell}=0$ but $a_{k-m-2}\neq 0$, there are at least ℓ borrows in s'. In this situation there are at least $\ell-1$ borrows in s and they are precisely $\ell-1$ when $n\in E_s$.
  - In the remaining cases, there exists an index $k-m-\ell\leq i\leq k-m-3$ such that $a_i\neq 0$, thus a_{k-m-2} does not lend, so the borrows in s and s' are the same.

This proves the statement. □

Remark 4.13. Observe that Lemma 4.12 with $m\geq 0$ generalizes Lemma 4.9 if $p=2$: indeed the hypotheses of Lemma 4.9 imply $b_{k-m-\ell-1}=0$ in Lemma 4.12 and hence $E_s=\emptyset$.

Remark 4.14. Let $s=\langle b_k\cdots b_0\rangle$. The construction of s' in Lemmas 4.7, 4.9 and 4.12 does not depend on the $(k-m-\ell)$-tail $b_{k-m-\ell-1}\ldots b_0$. Therefore if s and $s+i$ differ only on their $(k-m-\ell)$-tails, then $(s+i)'=s'+i$.

Example 4.15. Let $p=3$, $\ell=2$ and $s=47=\langle 1202\rangle$. Then $s'=\langle 122\rangle=17$. The sequence $\mathbf{b}_{47}$ has period $3^{2+3}=243$, while the sequence $\mathbf{b}_{17}$ has period $3^{2+2}=81$. We have

$$\mathbf{b}_{17}=[0_{17}10060030020_8 30_8 40060030050_8 60_8 7006003008];$$

by Lemma 4.12 the elements of the sequence $\mathbf{b}_{47}$ have the same 3-adic valuation of the elements of $\mathrm{R}(\mathbf{b}_{17},3)+3\chi_{E_{47}}$, which is obtained adding 3 times the sequence $\chi_{E_{47}}$ to that obtained repeating 3 times the 3-subsequences of $\mathbf{b}_{17}$: the effect is the substitution of the zeros appearing in $\mathrm{R}(\mathbf{b}_{17},3)$ in the positions belonging to E_{47} with 3, which has 3-adic valuation equal to 1. It is

$$E_{47}=\{48,49,51,52,75,76,78,79,129,130,132,133,156,157,159,160,\\ 210,211,213,214,237,238,240,241\}$$

and hence $|E_{47}|=24$. Therefore $\Pi_0(\mathbf{b}_{47})=3\times\Pi_0(\mathbf{b}_{17})=18$, $\Pi_1(\mathbf{b}_{47})=3\times\Pi_1(\mathbf{b}_{17})+|E_{47}|=48$, $Z(\mathbf{b}_{47})=3\times Z(\mathbf{b}_{17})-|E_{47}|=177$.

5. The case of $\mathbb{Z}_4$ and Vieru's sequence

In this section we apply the previous results to answer two questions posed by Anatol Vieru [21, 3.1, 3.2, 3.3]. The questions regarded the period of, and the proliferation of 4 and 8 in, the sequences obtained iterating the *Vieru operator* $\mathscr{V}$ on the sequence $V = [2, 1, 2, 4, 8, 1, 8, 4]$ in P_{12} where

$$\mathscr{V}V := \Sigma V + [8].$$

As observed in Example 3.6

$$V = \tilde{V}_2 + \tilde{V}_3 = [6, 9, 6, 0, 0, 9, 0, 0] + [8, 4];$$

hence

$$\mathscr{V}V = \Sigma V + [8] = \Sigma[6, 9, 6, 0, 0, 9, 0, 0] + \Sigma[8, 4] + [8] = \Sigma[6, 9, 6, 0, 0, 9, 0, 0] + [8, 4].$$

Therefore iterating s times the Vieru operator we get

$$\mathscr{V}^s V = \Sigma^s[6, 9, 6, 0, 0, 9, 0, 0] + [8, 4].$$

The period of $\mathscr{V}^s V$ is equal to the period of $\Sigma^s[6, 9, 6, 0, 0, 9, 0, 0]$. Reducing modulo 4, the latter is equal to the period of the iterated anti-differences of $V_2 = [2, 1, 2, 0, 0, 1, 0, 0]$ in P_4.

Vieru's question I 5.1. *The first Vieru's question [21, 3.1] was about calculating the period of the sequences $\mathscr{V}^s V$. Using what was observed in Example 3.14 for $s \geq 1$ we have that the period of $\mathscr{V}^s V$ is*

$$\pi(\mathscr{V}^s V) = \pi(\Sigma^s V) = \pi(\Sigma^s V_2) = 2^{2+\lfloor \log_2(s+3) \rfloor}.$$

Vieru's question II 5.2. *The second Vieru's question [21, 3.2] was about the possible values appearing in $\mathscr{V}^s V$, $s \geq 0$. Since in the iterated anti-differences of $[6, 9, 6, 0, 0, 9, 0, 0]$ appear only the numbers 0, 3, 6, 9, in $\mathscr{V}^s V$ appear only the numbers $8 + \{0, 3, 6, 9\} = \{8, 11, 2, 5\}$ and $4 + \{0, 3, 6, 9\} = \{4, 7, 10, 1\}$.*

Vieru's question III 5.3. *The third Vieru's question [21, 3.3] was about the proliferation of 8 and 4 in $\mathscr{V}^s V$. This corresponds to the proliferation of zeros in $\Sigma^s \tilde{V}_2$. The study of the iterated anti-differences $\Sigma^s \tilde{V}_2$ in P_{12} is equivalent to the study of the iterated anti-differences $\Sigma^s V_2 = \Sigma^s[2, 1, 2, 0, 0, 1, 0, 0]$ in P_4. Let us therefore focus on $\mathbb{Z}_4$: with the notation of the previous section, we are considering $p = 2$ and $\ell = 2$. Notice that in this case Lemmas 4.7 and 4.12 allow to reduce each binomial coefficient to a smaller* *one, permitting to link the s-th anti-difference* $\Sigma^s[1]$ *with* $2^k \le s < 2^{k+1}$ *to a* s'*-th anti-difference* $\Sigma^{s'}[1]$ *with* $2^{k-1} \le s' < 2^k$*.*

We provide now a recursive formula for the zeros $Z(s) := Z(\Sigma^s V_2)$ *of the s-th anti-difference of the sequence*

$$V_2 = [2, 1, 2, 0, 0, 1, 0, 0] \in P_4,$$

when $2^k \le s < 2^{k+1}$ *for* $k \ge 5$*. The sequence* $Z(s)$ *is clearly a sequence of natural numbers.*

To state our formula, we need some technical results. Firstly observe that since $2 \cdot 2 = 0$ *in* $\mathbb{Z}_4$*, if* $2^k \le s, t < 2^{k+1}$*, then*

$$2\chi_{E_s \triangle E_t} := 2(\chi_{E_s} + \chi_{E_t})(n) = \begin{cases} 2 & \text{if } n \in E_s \triangle E_t, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore if $s = \langle 10b_{k-2} \dots b_0 \rangle_2$*, the quantity* $|E_s|$ *is linked with the number* $\mathfrak{z}(s)$ *of 0's in the binary expansion of s in the following way:*

$$\begin{aligned} |E_s| &= 2 \cdot b_{k-2} \cdot 2^{\mathfrak{z}(\langle b_{k-3} \cdots b_0 \rangle_2)} \\ &= b_{k-2} \cdot 2^{\mathfrak{z}(s)} = \begin{cases} 2^{\mathfrak{z}(s)} & \text{if } b_{k-2} = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The quantities $\Pi_0(f)$*,* $\Pi_1(f)$*,* $Z(f)$*,* $f \in P_4$*, introduced in Definition 4.6 represent the number of 1 or 3, the number of 2, and the number of 0 in* f*, respectively.*

If $s = \langle 1b_{k-1} \cdots b_0 \rangle_2$ *and* $t = \langle 1b'_{k-1} \cdots b'_0 \rangle_2$*, denote by* $(s \mid t)$ *the bitwise OR of s and t, i.e., the number whose 2-adic representation has 1 in each bit position for which the corresponding bit of either s or t is 1.*

Lemma 5.4. *Let* $k \ge 5$ *and* $2^k + 2^{k-2} \le s < 2^k + 2^{k-1} - 4$*. Set* $\mathfrak{d}_k$ *equal to the* $(2^{k-2} - 4)$*-sequence* $\mathfrak{d}_k(s) := \Pi_1(2(\chi_{E_{s+1} \triangle E_{s+3}}))$*. Then*

$$\mathfrak{d}_k(s) = 2^{\mathfrak{z}(s+1)} + 2^{\mathfrak{z}(s+3)} - 2 \times 2^{\mathfrak{z}((s+1|s+3))}$$

and

$$\mathfrak{d}_5 = (4, 8, 4, 4) \quad \text{and} \quad \mathfrak{d}_{k+1} = (2 \times \mathfrak{d}_k, 4, 2^{k-1}, 2^{k-2}, 2^{k-2}, \mathfrak{d}_k) \ \forall k \ge 5.$$

Proof. Observe that, by Remark 4.13

$$\mathfrak{d}_k(s) = |E_{s+1}| + |E_{s+3}| - 2 \times |E_{s+1} \cap E_{s+3}| = 2^{\mathfrak{z}(s+1)} + 2^{\mathfrak{z}(s+3)} - 2 \times 2^{\mathfrak{z}((s+1|s+3))}$$

since $|E_{s+1} \cap E_{s+3}| = 2^{\mathfrak{z}((s+1|s+3))}$ (see Definition 4.11 with $\ell = 2$ and $m = -1$).

If $k = 5$, then $s \in \{40, 41, 42, 43\}$. It is easy to verify that

$$\mathfrak{d}_5 = (2^{\mathfrak{z}(41)} + 2^{\mathfrak{z}(43)} - 2^{\mathfrak{z}(41|43)+1}, \ldots, 2^{\mathfrak{z}(44)} + 2^{\mathfrak{z}(46)} - 2^{\mathfrak{z}(44|46)+1}) = (4, 8, 4, 4).$$

Fixed k, the binary representation of the numbers s between $2^k + 2^{k-2}$ and $2^k + 2^{k-1} - 4$ are of the following three types

- I_k: $2^k + 2^{k-2} \leq s < 2^k + 2^{k-2} + 2^{k-3} - 4$,
- II_k: $2^k + 2^{k-2} + 2^{k-3} - 4 \leq s < 2^k + 2^{k-2} + 2^{k-3}$,
- III_k: $2^k + 2^{k-2} + 2^{k-3} \leq s < 2^k + 2^{k-1} - 4$.

Given $s' \in III_{k+1}$, hence $s' = 2^{k+1} + 2^{k-1} + 2^{k-2} + t$ with $0 \leq t < 2^{k-2} - 4$. Set $s = 2^k + 2^{k-2} + t$, we get

$$(\mathfrak{z}(s'+1),\ \mathfrak{z}(s'+3),\ \mathfrak{z}(s'+1 \mid s'+3)) = (\mathfrak{z}(s+1),\ \mathfrak{z}(s+3),\ \mathfrak{z}(s+1 \mid s+3)).$$

Given $s' \in I_{k+1}$, hence $s' = 2^{k+1} + 2^{k-1} + t$ with $0 \leq t < 2^{k-2} - 4$. Set $s = 2^k + 2^{k-2} + t$, we get

$$(\mathfrak{z}(s'+1),\ \mathfrak{z}(s'+3),\ \mathfrak{z}(s'+1 \mid s'+3)) = (1 + \mathfrak{z}(s+1),\ 1 + \mathfrak{z}(s+3),\ 1 + \mathfrak{z}(s+1 \mid s+3)).$$

Finally the binary representation of s' in the group II_{k+1} is the following:

$$s' : \quad \langle 10101_{k+1-5}00\rangle_2,\ \langle 10101_{k+1-5}01\rangle_2,\ \langle 10101_{k+1-5}10\rangle_2,\ \langle 10101_{k+1-5}11\rangle_2.$$

Therefore $(\mathfrak{z}(s'+1), \mathfrak{z}(s'+3), \mathfrak{z}(s'+1 \mid s'+3))$ are

1. $(3, 2, 2)$ for $s' = \langle 10101_{k+1-5}00\rangle_2$,
2. $(3, k-1, 2)$ for $s' = \langle 10101_{k+1-5}01\rangle_2$,
3. $(2, k-2, 1)$ for $s' = \langle 10101_{k+1-5}10\rangle_2$,
4. $(k-1, k-2, k-2)$ for $s' = \langle 10101_{k+1-5}11\rangle_2$.

Thus we get the wanted claim. $\square$

Remark 5.5. We can link the sequence $\mathfrak{d}_k$ to two well known integer sequences. Fixed $k \geq 5$ the sequence $\mathfrak{d}_k$ coincides with

$$(2^{k-a(4)}, 2^{k-a(5)} \ldots, 2^{k-a(2^{k-2}-1)})$$

where $a(2^t) = t + 1$ and $a(2^t + i) = 1 + a(i)$ for $t \geq 0$ and $0 < i < 2^t$ (see A063787 in the OEIS, the online encyclopedia of integer sequences). Noticed that $a(2^{t_1} + \cdots + 2^{t_h}) = h + t_h$ for $t_1 > \cdots > t_h \geq 0$, one can directly prove that $\mathfrak{d}_k(2^k + 2^{k-2} + 2^{t_1} + \cdots 2^{t_h} - 4) = 2^{k-h-t_h}$.

Denoted by $wt(n)$ the Hamming weight of n, i.e., the number of 1's in the binary expansion of n, we have, for $2^k + 2^{k-2} \leq s < 2^k + 2^{k-1} - 4$

$$\mathfrak{d}_k(s) = 2^{wt(2^k+2^{k-1}-4-s)+1}.$$

The recurrence relation for $\mathfrak{d}_k(s)$ permits to compute a recurrence relation for the Hamming weight. Denoted by w_h the Hamming weight of the numbers $\langle 1\rangle_2, \ldots, \langle 2^{h+1} - 4\rangle_2$, we have

$$w_2 = (1, 1, 2, 1), \quad w_{h+1} = (w_h, h, h, h+1, 1, w_h + 1) \quad \forall h \geq 2$$

where $w_h + 1$ is the sequence obtained by w_h increasing by one each entrance.

Main Recursive Formula 5.6. *For $k \geq 5$ and $2^k \leq s < 2^{k+1}$, denote:*

$$\begin{aligned}(c_1, c_2, c_3, c_4) &:= (48, 32, 40, 44)\\ (c'_1, c'_2, c'_3, c'_4) &:= (48, 40, 44, 48)\\ (c''_1, c''_2, c''_3, c''_4) &:= (32, 32, 48, 64)\\ \mathcal{Z}_k &:= \big(Z(s)\big)_{2^k \leq s < 2^{k+1}}.\end{aligned}$$

The initial condition is

$$\begin{aligned}\mathcal{Z}_5 = (&88, 64, 80, 88, 92, 64, 80, 88, 104, 92, 104, 108, 94, 78, 88, 96,\\ &108, 96, 104, 108, 110, 102, 108, 112, 118, 114, 118, 120, 64, 64, 96, 128).\end{aligned}$$

For $k \geq 6$, the 2^k-tuple $\mathcal{Z}_k$ coincides with

$$Z(s) = \begin{cases} 2Z(s - 2^{k-1}) & \text{if } 2^k \leq s \leq 2^k + 2^{k-2} - 5 \quad (\mathbf{A})\\ Z(s - 2^{k-1} - 2^{k-3}) + 2^{k-5}c_i & \text{if } s = 2^k + 2^{k-2} - 5 + i,\ i = 1, 2, 3, 4 \quad (\mathbf{B})\\ 2Z(s - 2^{k-1}) - \mathfrak{d}_k(s) & \text{if } 2^k + 2^{k-2} \leq s \leq 2^k + 2^{k-1} - 5 \quad (\mathbf{C})\\ Z(s - 2^{k-1} - 2^{k-2}) + 2^{k-5}c'_i & \text{if } s = 2^k + 2^{k-1} - 5 + i,\ i = 1, 2, 3, 4 \quad (\mathbf{D})\\ Z(s - 2^k) + 2^{k+1} & \text{if } 2^k + 2^{k-1} \leq s \leq 2^{k+1} - 5 \quad (\mathbf{E})\\ Z(s - 2^k) + 2^{k-5}c''_i & \text{if } s = 2^{k+1} - 5 + i,\ i = 1, 2, 3, 4 \quad (\mathbf{F}). \end{cases}$$

Proof. We give year a sketch of the proof. W invite the interested reader to see the detailed proof in the following Appendix. The s-th anti-difference of the sequence $V_2 = [2, 1, 2, 0, 0, 1, 0, 0]$ in P_4 is equal to

$$\Sigma^s V_2 = 2\,\mathbf{b}_{s+4} + 3\,\mathbf{b}_{s+3} + 2\,\mathbf{b}_{s+2} + 3\,\mathbf{b}_{s+1} + 2\,\mathbf{b}_s \quad \forall s \geq 0. \tag{1}$$

In base 2 we have

$$2^k = \langle 10_k \rangle_2 := \langle 1\underbrace{0\cdots0}_{k\text{ times}}\rangle_2,$$

therefore $\langle 10_k \rangle_2 \leq [s]_2 \leq \langle 11_k \rangle_2$. Set $h = k - 5$, we will consider in order the following cases:

$$\begin{aligned}
&\mathbf{A}: && \langle 1000_h000\rangle_2 \leq s \leq \langle 1001_h011\rangle_2;\\
&\mathbf{C}: && \langle 1010_h000\rangle_2 \leq s \leq \langle 1011_h011\rangle_2;\\
&\mathbf{E}: && \langle 1100_h000\rangle_2 \leq s \leq \langle 1111_h011\rangle_2;\\
&\mathbf{B}: && \langle 1001_h100\rangle_2 \leq s \leq \langle 1001_h111\rangle_2;\\
&\mathbf{D}: && \langle 1011_h100\rangle_2 \leq s \leq \langle 1011_h111\rangle_2;\\
&\mathbf{F}: && \langle 1111_h100\rangle_2 \leq s \leq \langle 1111_h111\rangle_2.
\end{aligned}$$

In the cases **A**, **C** and **E**, the 2-adic notation of $s+4$, $s+3$, $s+2$, $s+1$, and s (see the summands in Equation (1)) have the same prefix: 10 in the first two cases, and 11 in the last. This allows to apply in parallel the recursive lemmas of Section 4. The remaining twelve cases require *ad hoc* analysis. □

The Main Recursive Formula 5.6 describes precisely how the number of zeros vary in $\Sigma^s V_2$ *passing from* $s \in [2^{k-2}, 2^k[$*, to* $s \in [2^k, 2^{k+1}[$*. As we observed before, the number of zeros in* $\Sigma^s V_2$ *corresponds to the number of 4 and 8 in* $\mathscr{V}^s V$*. In such a way we are able to measure the proliferation of 4 and 8 perceived by Vieru: more, we can compute recursively the exact number of 4 and 8 in the iterated applications* $\mathscr{V}^s V$*,* $s \geq 0$*, of the Vieru operator to the sequence* V*. E.g., if* s *satisfy* $2^{k+1} - 16 \leq s \leq 2^{k+1} - 5$*, looking at the recursive formula (**E**), one has*

$$Z(s) = Z(s - 2^k) + 2^{k+1} \quad \forall k \geq 6;$$

by induction we get

$$Z(s) = Z(s-2^k) + 2^{k+1} = \sum_{i=7}^{k+1} 2^i + Z(s - \sum_{j=6}^{k} 2^j) = 2^{k+2} - 128 + Z(s - 2^{k+1} + 64)),$$

where $48 \leq s - 2^{k+1} + 64 \leq 59$*. Since* $\pi(\Sigma^s V_2) = 2^{k+2}$*, the percentage of zeros inside the period for* $k \to \infty$ *tends to* 100%*.*

Appendix A. Proof Main Recursive Formula 5.6

Proof of Main Recursive Formula 5.6. Using a generic computer algebra system one can easily compute the sequence $\mathcal{Z}_5$, the initial condition for the recursive formula.

Cases **A** and **C**. In both the cases s, $s+1$, $s+2$, $s+3$, and $s+4$ have a binary representation $\langle 10b_{k-2}\dots b_0\rangle$ with the two most representative figures equal to 10. If $f \in P_4$, we denote shortly

$$\mathrm{R}\, f^s := \mathrm{R}(\Sigma^s f, 2^{k-1}), \quad \mathrm{A}\, f^s := \mathrm{A}(\Sigma^s f, 2^{k-1}).$$

Using Lemma 4.12, we lead back the study of $\mathbf{b}_s, \dots, \mathbf{b}_{s+4}$ to the study of $\mathbf{b}_{s'}, \dots,$ $\mathbf{b}_{s'+4}$ where $s' = s - (2^k - 2^{k-1}) = s - 2^{k-1}$. It is

$$\begin{aligned}\Sigma^s V_2 &= 2\,\mathbf{b}_{s+4} + 3\,\mathbf{b}_{s+3} + 2\,\mathbf{b}_{s+2} + 3\,\mathbf{b}_{s+1} + 2\,\mathbf{b}_s \\ &\equiv_\nu 2\left(\mathrm{R}\,\mathbf{b}_{s'+4} + 2\chi_{E_{s+4}}\right) + 3\left(\mathrm{R}\,\mathbf{b}_{s'+3} + 2\chi_{E_{s+3}}\right) + 2\left(\mathrm{R}\,\mathbf{b}_{s'+2} + 2\chi_{E_{s+2}}\right) + \\ &+ 3\left(\mathrm{R}\,\mathbf{b}_{s'+1} + 2\chi_{E_{s+1}}\right) + 2\left(\mathrm{R}\,\mathbf{b}_{s'} + 2\chi_{E_s}\right) \\ &\equiv_\nu \mathrm{R}\, V_2^{s'} + 3\cdot 2\chi_{E_{s+1}} + 3\cdot 2\chi_{E_{s+3}} \\ &\equiv_\nu \mathrm{R}\, V_2^{s'} + 2\chi_{E_{s+1}\triangle E_{s+3}}.\end{aligned}$$

In case **A** it is $E_{s+1} = \emptyset = E_{s+3}$: indeed the condition $a_{k+1-2-1} < b_{k+1-2-1} = 0$ is impossible (see Definition 4.11 with $\ell = 2$ and $m = -1$). Hence $\Sigma^s V_2 \equiv_\nu \mathrm{R}\, V_2^{s'}$. Therefore

$$Z(\Sigma^s V_2) = Z\left(\mathrm{R}\, V_2^{s'}\right) = 2 \times Z(V_2^{s'}).$$

In case **C**, if $n \in E_{s+1}\triangle E_{s+3}$, then $\mathrm{R}\, V_2^{s'}(n)$ is equal to zero. Indeed it is easy to check that $n = \langle a_{k+1}\dots a_0\rangle_2 \in E_{s+1}\triangle E_{s+3}$ implies $a_k = 1$, $a_{k-1} = 1$ and $a_{k-2} = 0$. Since the binary representation of $t \in \{s, s+1, s+2, s+3, s+4\}$ is $\langle 101b_{k-3}\dots b_0\rangle_2$, using Kummer's Theorem one has for $t' = t - 2^{k-1}$:

$$\mathrm{R}\,\mathbf{b}_{t'}(n) = \mathbf{b}_{t'}(n') = \binom{\langle a_{k+1}10a_{k-3}\dots a_0\rangle_2}{\langle 11b_{k-3}\dots b_0\rangle_2} = 0,$$

hence $\mathrm{R}\, V_2^{s'}(n) = 0$. Therefore, we have

$$Z(\Sigma^s V_2) = Z\left(\mathrm{R}\, V_2^{s'}\right) - \Pi_1(2\chi_{E_{s+1}\triangle E_{s+3}}) = 2 \times Z(\Sigma^{s'} V_2) - \mathfrak{d}_k(s).$$

Case **E**. The numbers s, $s+1$, $s+2$, $s+3$, and $s+4$ have a binary representation $\langle 11b_{k-2}\dots b_0\rangle$ with the two most representative figures equal to 11. If $f \in P_4$, we denote shortly $\mathrm{A}\, f^s := \mathrm{A}(\Sigma^s f, 2^{k-1})$. Using Lemma 4.7, we lead back the study of $\mathbf{b}_s, \dots, \mathbf{b}_{s+4}$ to the study of $\mathbf{b}_{s'}, \dots, \mathbf{b}_{s'+4}$ where $s' = s - 2^k$. Thanks to the linearity of A we have:

$$\begin{aligned}\Sigma^s V_2 &= 2\,\mathbf{b}_{s+4} + 3\,\mathbf{b}_{s+3} + 2\,\mathbf{b}_{s+2} + 3\,\mathbf{b}_{s+1} + 2\,\mathbf{b}_s \\ &\equiv_\nu 2\,\mathrm{A}\,\mathbf{b}_{s'+4} + 3\,\mathrm{A}\,\mathbf{b}_{s'+3} + 2\,\mathrm{A}\,\mathbf{b}_{s'+2} + 3\,\mathrm{A}\,\mathbf{b}_{s'+1} + 2\,\mathrm{A}\,\mathbf{b}_{s'}\end{aligned}$$

$$\equiv_\nu \mathrm{A}\,(2\,\mathbf{b}_{s'+4} + 3\,\mathbf{b}_{s'+3} + 2\,\mathbf{b}_{s'+2} + 3\,\mathbf{b}_{s'+1} + 2\,\mathbf{b}_{s'})$$
$$\equiv_\nu \mathrm{A}\,V_2^{s'}.$$

Therefore

$$Z(\Sigma^s V_2) = Z\left(\mathrm{A}\,V_2^{s'}\right) = Z(\Sigma^{s'} V_2) + 2^{k+1}.$$

Case **B** and **D**. The number s has a binary representation $\langle 10b_{k-2}1_h 1b_1b_0\rangle$ with $b_0, b_1, b_{k-2} \in \{0,1\}$. If $f \in P_4$, we denote shortly

$$\mathrm{R}\,f^s := \mathrm{R}(\Sigma^s f, 2^{k-4}), \quad \mathrm{A}\,f^s := \mathrm{A}(\Sigma^s f, 2^{k-4}).$$

B. Using Lemma 4.7 with $m = 2$ and Lemma 4.9 with $m = 3$, we lead back the study of $\mathbf{b}_s, \ldots, \mathbf{b}_{s+4}$ to the study of $\mathbf{b}_{s'}, \ldots, \mathbf{b}_{s'+4}$ where $s' = s - 2^{k-1} - 2^{k-3}$ in case **B**, and $s' = s - 2^{k-1} - 2^{k-2}$ in case **D**.

- If $(b_1b_0) = (00)$, then we have

$$s+1 = \langle 10b_{k-2}1_h101\rangle_2,\ s+2 = \langle 10b_{k-2}1_h110\rangle_2,\ s+3 = \langle 10b_{k-2}1_h111\rangle_2,$$

and $s+4 = \langle 1b'_{k-1}b'_{k-2}0_h000\rangle_2$ with $b'_{k-1}b'_{k-2} = 01$ in case **B** and $b'_{k-1}b'_{k-2} = 10$ in case **D**. By Lemma 4.7 with $m = 2$ for $s+i$, $i = 0,1,2,3$ and Lemma 4.9 with $m = 3$ for $s+4$ we have

$$\mathbf{b}_{s+i} \equiv_\nu \mathrm{A}\,\mathbf{b}_{s'+i},\ i = 0,1,2,3, \text{ and } \mathbf{b}_{s+4} \equiv_\nu \mathrm{R}\,\mathbf{b}_{s'+4}.$$

Then

$$\Sigma^s V_2 \equiv_\nu 2\,\mathrm{R}\,\mathbf{b}_{s'+4} + 3\,\mathrm{A}\,\mathbf{b}_{s'+3} + 2\,\mathrm{A}\,\mathbf{b}_{s'+2} + 3\,\mathrm{A}\,\mathbf{b}_{s'+1} + 2\,\mathrm{A}\,\mathbf{b}_{s'}.$$

Analysing the previous equation in blocks of length 2^{k-4}, one obtains:

$$Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + Z\,(2\,\mathbf{b}_{s'+4}).$$

Since $s'+4 = \langle 1b'_{k-1}b'_{k-2}0_{h-1}000\rangle_2$, applying h-times Lemma 4.9, we get

$$\begin{aligned} Z\,(2\,\mathbf{b}_{s'+4}) &= Z\,(\mathbf{b}_{s'+4}) + \Pi_1\,(\mathbf{b}_{s'+4}) \\ &= \begin{cases} 2^h\Big(Z\,(\mathbf{b}_{20}) + \Pi_1\,(\mathbf{b}_{20})\Big) = 48 \cdot 2^{k-5} & \text{in case } \mathbf{B}, \\ 2^{h-1}\Big(Z\,(\mathbf{b}_{24}) + \Pi_1\,(\mathbf{b}_{24})\Big) = 48 \cdot 2^{k-5} & \text{in case } \mathbf{D} \end{cases} \end{aligned}$$

Therefore in both the cases **B** and **D** we have $Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + 2^{k-5} \times 48$.

- If $(b_1b_0) = (01)$, then we have $Z(v^s) = \begin{cases} Z(v^{s'}) + 2^{k-5} \times 32 & \text{in case } \mathbf{B}, \\ Z(v^{s'}) + 2^{k-5} \times 40 & \text{in case } \mathbf{D}. \end{cases}$

- If $(b_1b_0) = (10)$, then we have $Z(\Sigma^s V_2 s) = \begin{cases} Z(\Sigma^{s'} V_2) + 2^{k-5} \times 40 & \text{in case } \mathbf{B}, \\ Z(\Sigma^{s'} V_2) + 2^{k-5} \times 44 & \text{in case } \mathbf{D}. \end{cases}$

- If $(b_1b_0) = (11)$, then we have $Z(\Sigma^s V_2) = \begin{cases} Z(\Sigma^{s'} V_2) + 2^{k-5} \times 44 & \text{in case } \mathbf{B}, \\ Z(\Sigma^{s'} V_2) + 2^{k-5} \times 48 & \text{in case } \mathbf{D}. \end{cases}$

Case **F**. The number s has a binary representation $\langle 1111_h 1 b_1 b_0 \rangle$ with $b_0, b_1 \in \{0, 1\}$. If $f \in P_4$, and $2^k \leq t < 2^{k+1}$ we denote shortly

$$\mathrm{R}\, f^t := \mathrm{R}(\Sigma^t f, 2^{k-2}), \quad \mathrm{A}\, f^t := \mathrm{A}(\Sigma^t f, 2^{k-1}).$$

We lead back the study of $\mathbf{b}_s, \ldots, \mathbf{b}_{s+4}$ to the study of $\mathbf{b}_{s'}, \ldots, \mathbf{b}_{s'+4}$ where $s' = s - 2^k$

- If $(b_1b_0) = (00)$, then we have

$$s+1 = \langle 1111_h 101 \rangle_2, \; s+2 = \langle 1111_h 110 \rangle_2, \; s+3 = \langle 1111_h 111 \rangle_2, \; s+4 = \langle 10000_h 000 \rangle_2.$$

For $0 \leq i \leq 3$ the sequence $\mathbf{b}_{s+i}$ has period 2^{k+2}, while $\mathbf{b}_{s+4}$ has period 2^{k+3}. Nevertheless, the period of

$$\Sigma^s V_2 = 2\, \mathbf{b}_{s+4} + 3\, \mathbf{b}_{s+3} + 2\, \mathbf{b}_{s+2} + 3\, \mathbf{b}_{s+1} + 2\, \mathbf{b}_s$$

is 2^{k+2}: indeed the sequence $2\, \mathbf{b}_{s+4}$ has period 2^{k+2} by Theorem 3.2. By Lemma 4.7 with $m = -1$, Lemma 4.9 with $m = 1$, and Remark 4.14 we have

$$\Sigma^s V_2 = 2\, \mathrm{R}\, \mathbf{b}_{s'+4} + 3\, \mathrm{A}\, \mathbf{b}_{s'+3} + 2\, \mathrm{A}\, \mathbf{b}_{s'+2} + 3\, \mathrm{A}\, \mathbf{b}_{s'+1} + 2\, \mathrm{A}\, \mathbf{b}_{s'}$$

where $s' = s - 2^k$. Then one gets

$$Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + Z(2\, \mathbf{b}_{s'+4}).$$

Notice that $Z(2\, \mathbf{b}_{s'+4}) = \frac{1}{2}\big(Z\left(\mathbf{b}_{s'+4}\right) + \Pi_1\left(\mathbf{b}_{s'+4}\right)\big)$. Indeed $2\, \mathbf{b}_{s'+4}$ has period equal to one half of the period of $\mathbf{b}_{s'+4}$ and the zeros of $2\, \mathbf{b}_{s'+4}$ correspond to the 0's and 2's of $\mathbf{b}_{s'+4}$. Applying h-times Lemma 4.9 with $m = 1$, we get

$$Z\left(\mathbf{b}_{s'+4}\right) + \Pi_1\left(\mathbf{b}_{s'+4}\right) = 2^h \big(Z(\mathbf{b}_{32}) + \Pi_1(\mathbf{b}_{32})\big) = 2^{k-5} \cdot 64.$$

Hence $Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + 2^{k-5} \cdot 32$.

- If $(b_1 b_0) = (01)$, we have

$$s + 1 = \langle 1111_h 110\rangle_2,\ \ s + 2\langle 1111_h 111\rangle_2,\ \ s + 3 = \langle 10000_h 000\rangle_2,\ \ \ s + 4 = \langle 10000_h 001\rangle_2.$$

By Lemma 4.7 with $m = -1$, Lemma 4.9 with $m = 1$, and Remark 4.14 we have

$$\Sigma^s V_2 = 2\,\mathrm{R}\,\mathbf{b}_{s'+4} + 3\,\mathrm{R}\,\mathbf{b}_{s'+3} + 2\,\mathrm{A}\,\mathbf{b}_{s'+2} + 3\,\mathrm{A}\,\mathbf{b}_{s'+1} + 2\,\mathrm{A}\,\mathbf{b}_{s'}\,.$$

Observe that $3\,\mathrm{R}\,\mathbf{b}_{s'+3}$ has period 2^{k+3}, while $2\,\mathrm{R}\,\mathbf{b}_{s'+4}$, $\mathrm{A}\,\mathbf{b}_{s'+i}$, $i = 0, 1, 2$, have period 2^{k+2}. We have that

$$Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + Z(2\,\mathbf{b}_{s'+4} + 3\,\mathbf{b}_{s'+3}).$$

Applying h times Lemma 4.9 with $m = 1$, we get

$$Z(2\,\mathbf{b}_{s'+4} + 3\,\mathbf{b}_{s'+3}) = Z(2R^h\,\mathbf{b}_{33} + 3R^h\,\mathbf{b}_{32}) = 2^h Z(2\,\mathbf{b}_{33} + 3\,\mathbf{b}_{32}) = 2^{k-5}\cdot 32.$$

Therefore $Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + 2^{k-5}\cdot 32$.

- If $(b_1 b_0) = (10)$, then we have $Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + 2^{k-5}\cdot 48$.
- If $(b_1 b_0) = (11)$, then we have $Z(\Sigma^s V_2) = Z(\Sigma^{s'} V_2) + 2^{k-5}\cdot 64$. □

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