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Research paper

A geometric framework for distributed frequency models

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ABSTRACT

Geometric control theory, developed by Basile and Marro, and independently, by Wonham and Morse in the 1970s revolves around characterizing the properties of finite dimensional, linear and time-invariant systems using geometry. Some examples of these properties are invariance, controllability and observability. The task addressed in this paper is to develop the geometric tools for fractional systems using the diffusive representation (also known as the distributed frequency) model. The mathematics involved in this approach is different from the classical case as fractional systems are inherently infinite dimensional. Unlike the integer order derivative, the fractional derivative is a non-local operator, and so the evolution of the so-called pseudo-state depends on not just its current value but also its past history. Thus, the notion of an initial condition for a fractional system can come in different forms. This leads to different kinds of fractional derivative operators. With some of these operators, the semigroup property is lost. With the distributed frequency model, the initial condition comes in the form of an initialization function. This takes care of the infinite dimensional nature of fractional systems. Furthermore, the distributed frequency model retains the semigroup property. This is useful in developing invariance and controlled invariance for fractional systems. Moreover, these properties of fractional systems are verified numerically using a higher order finite-dimensional approximation, which retains all the geometric properties of the distributed frequency model.

1. Introduction

Fractional systems are dynamical systems characterized by a non-integer order of differentiation of the variables involved. Fractional systems have been successfully used in a wealth of applications [1]. These include signal processing [2], control [3–6] electronics [7] and multi-agent systems [8], to name a few. In general, fractional models arise naturally in the mathematical description of systems characterized by memory behavior such as heat diffusion, and fractal structures such as repeated electric components and patterned systems [9,10].

Similarly to the integer case, a linear fractional system can be represented in the time domain or in the frequency domain (i.e., fractional transfer functions). The same input–output behavior, captured by a fractional transfer function, can correspond to different time-domain descriptions of the fractional system.

A sound approach that takes into account the infinite-dimensional nature of fractional systems in its entirety relies on the so-called frequency-distributed (or diffusive) representation of fractional systems, see [11–19]. This method hinges on a description of the

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fractional system that uses a partial differential equation in two variables: one is a time coordinate, and the other – usually denoted by ω – is used to characterize the infinite dimensional state of the system at any point in time. Importantly, this approach does not rely on any notion of fractional derivative, and it leads to the same input–output behavior of fractional systems defined using fractional derivatives. Interestingly, using this method the initial condition of the system becomes a function of ω, which entirely characterizes the natural evolution of the system from any point in time onward, thus guaranteeing the semigroup property.

Over the past two decades, the theory of fractional diffusive systems has been developed leading to important results such as stability, controllability, and techniques for initialization. However, there are important aspects of the theory of fractional diffusive systems which are yet to be addressed and understood. In particular, a geometric theory for this class of fractional systems has never been developed. Geometric control is instrumental in the understanding of structural system-theoretic properties of dynamical systems. In geometric control, the focus is on the subspaces where the trajectories of the pseudo-state of the system evolve, and their characterization in terms of invariance properties [20–22]. Moreover, the geometric approach to control theory provides the natural set of tools for the solution of important control and estimation problems including disturbance decoupling, model matching, unknown-input observation and monotonic tracking to name a few; see [23,24] and the references therein. The theory of geometric control for fractional systems has only been recently developed in [25] for fractional systems involving the Caputo derivative, which, however, suffers from the problem deriving from forcing a finite-dimensional structure onto an inherently infinite-dimensional system. In fact, fractional systems are inherently infinite dimensional, meaning that the corresponding state space is an infinite dimensional vector space. As such, the fractional derivative is not a local operator, unlike integer order derivatives. This means that in order to compute a fractional derivative of a function f at a point, say t, it is not enough to know the behavior of f on an arbitrarily small neighborhood around t. As a results, fractional models relying on any definition of fractional derivative do not enjoy the semigroup property. In other words, knowing the state at a specific point in time is not sufficient to uniquely determine the future evolution of the system. In addition, there exists many non-equivalent definitions of fractional derivatives [26,27], and any definition of fractional system that relies on the notion of a fractional derivative suffers from initialization issues [11]. Although these issues can be addressed by defining the initial condition of the system as a function of time (referred to as initialization function), see [28,29], the time invariance cannot be recovered, and the system defined using the notion of fractional derivative does not enjoy the semigroup property.

In this paper, we develop a geometric theory for fractional diffusive systems, and we use it to address the disturbance decoupling problem by pseudo-state feedback and dynamic output feedback. In particular, we first introduce a notion of invariance. Importantly, we extend the concept of analytic continuation from a segment to prove that this class of infinite dimensional systems enjoys the property that if a pseudo-state trajectory is entirely contained in an invariant subspace, then the initialization function is also contained in the same subspace almost everywhere. This implication normally ceases to be true for infinite dimensional systems, but it holds true for diffusive fractional systems, and it allows to retain some of the properties that finite LTI systems enjoy.

We then introduce and develop a theory of controlled invariance and its dual (i.e., conditioned invariance), and we show that if the initialization function lies on a controlled invariant subspace, then it is always possible to force the pseudo-state trajectory of the system to evolve on that subspace. We also conjecture the opposite implication, and we show that this is actually the case for some classes of control input functions. This suggests that it is impossible to force the trajectory of a fractional diffusive system on a controlled invariant subspace if the initialization function does not belong to the same controlled invariant subspace. While this is obvious if we restrict ourselves to a pseudo-state feedback, there are instances where other types of fractional systems can be forced to evolve on a controlled invariant subspace using a suitable feedforward control function, even if the past pseudo-state trajectory was not contained on such controlled invariant subspace [25]. This does not seem to be the case with diffusive fractional systems irrespective of the type of control utilized. This is a major difference which highlights the fact that using a control other than a pseudo-state feedback does not offer more flexibility, at least for what concerns maintaining trajectories on subspaces of the pseudo state-space.

As aforementioned, the last part of the paper deals with the most important application of geometric control theory: the disturbance decoupling problem. Surprisingly, when we consider fractional diffusive systems, the conditions under which the disturbance decoupling problem is solvable remain the same of systems based on Caputo Derivative. However, the formal derivation of the result is radically different, and, as such, we believe that it is valuable *per se* as it adds insight in this important class of systems.

We conclude by illustrating via numerical examples how the developed theory can be applied to the design of the controller, which is, at least in the latter case, another diffusive fractional system.

2. Distributed frequency model

In this paper we are concerned with models in the form

$$\frac{\partial z}{\partial t}(\omega, t) = -\omega z(\omega, t) + v(t), \qquad z(\omega, t_0) = z_0(\omega), \tag{1}$$

$$x(t) = \int_0^\infty \mu_\alpha(\omega) z(\omega, t)\mathrm{d}\omega, \tag{2}$$

where v is the input, z is referred to as a frequency state variable, and z_0 is the initialization function, and $\mu_\alpha(\omega) = \frac{\sin(\pi\alpha)}{\pi}\omega^{-\alpha}$, see [17].

We begin showing that, if we take the Laplace transform of (1)–(2), we obtain the input–output relation

$$X(s) = \frac{1}{s^\alpha} V(s), \tag{3}$$

where X and V are the Laplace transforms of x and v, respectively. It is well known that the frequency distributed model (1)–(2) and the fractional integrator $\frac{1}{s^\alpha}$ are equivalent. The standard proof of this property involves the integral identity

$$\int_0^\infty \frac{\sin(\pi\alpha)}{\pi\omega^\alpha}\frac{1}{s+\omega}\mathrm{d}\omega = \frac{1}{s^\alpha}, \tag{4}$$

which can be evaluated using a keyhole contour integral; see [17]. In this paper we propose an alternative derivation of (4) using the Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, where $\Gamma(z)$ is the Euler gamma function.[1]

Theorem 2.1. *The system described by (1)–(2) has the same transfer function as the system described by (3).*

Proof. Taking the Laplace transform of the partial differential equation in (1) under zero initial conditions yields

$$sZ(\omega,s) = -\omega Z(\omega,s) + V(s),$$

whence

$$Z(\omega,s) = \frac{1}{s+\omega}V(s). \tag{5}$$

Taking the Laplace transform of both sides of (2) and substituting for $Z(\omega,s)$ from (5) yields the following:

$$\begin{aligned} X(s) &= \int_0^\infty \mu_\alpha(\omega)Z(\omega,s)\mathrm{d}\omega \\ &= \int_0^\infty \mu_\alpha(\omega)\frac{1}{s+\omega}V(s)\mathrm{d}\omega \\ &= \left(\int_0^\infty \mu_\alpha(\omega)\frac{1}{s+\omega}\mathrm{d}\omega\right)V(s). \end{aligned}$$

It suffices to show the identity in (4). Recall the Euler reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{6}$$

where

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x.$$

Consider

$$\begin{aligned} \Gamma(\alpha)\Gamma(1-\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x \int_0^\infty y^{-\alpha}e^{-y}\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{-\alpha}e^{-(x+y)}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

For some $s \in \mathbb{C}$, let t and ω be a change of variables from x and y with $x = st$ and $y = \omega t$. Then the corresponding Jacobian determinant is

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \omega} \end{vmatrix} = \begin{vmatrix} s & 0 \\ \omega & t \end{vmatrix} = st.$$

The boundaries of integration with these new variables remain unchanged. Thus,

$$\begin{aligned} \Gamma(\alpha)\Gamma(1-\alpha) &= \int_0^\infty\int_0^\infty (st)^{\alpha-1}(\omega t)^{-\alpha}e^{-(st+\omega t)}st\mathrm{d}t\mathrm{d}\omega \\ &= \int_0^\infty\int_0^\infty s^{\alpha-1+1}t^{\alpha-1-\alpha+1}\omega^{-\alpha}e^{-(st+\omega t)}\mathrm{d}t\mathrm{d}\omega \\ &= \int_0^\infty\int_0^\infty s^\alpha\omega^{-\alpha}e^{-(st+\omega t)}\mathrm{d}t\mathrm{d}\omega \\ &= s^\alpha\int_0^\infty \omega^{-\alpha}\int_0^\infty e^{-st}e^{-\omega t}\mathrm{d}t\mathrm{d}\omega \\ &= s^\alpha\int_0^\infty \omega^{-\alpha}\mathcal{L}\{e^{-\omega t}\}\mathrm{d}\omega \\ &= s^\alpha\int_0^\infty \omega^{-\alpha}\frac{1}{s+\omega}\mathrm{d}\omega \end{aligned}$$

[1] Notice that another alternative proof, which makes use of the beta function can be found in [30].

By (6),

$$\frac{\pi}{\sin(\pi\alpha)} = s^{\alpha} \int_0^{\infty} \frac{\omega^{-\alpha}}{s+\omega} \mathrm{d}\omega,$$

whence,

$$\int_0^{\infty} \frac{\mu_\alpha(\omega)}{s+\omega} \mathrm{d}\omega = \int_0^{\infty} \frac{\sin(\pi\alpha)}{\pi} \frac{\omega^{-\alpha}}{s+\omega} \mathrm{d}\omega = \frac{1}{s^{\alpha}}.$$

Therefore,

$$X(s) = \frac{1}{s^{\alpha}} V(s),$$

as required. ■

It is common to represent a system such as (3) in the time domain as a fractional differential equation of the form $\mathcal{D}^{\alpha} x(t) = v(t)$, where the operator $\mathcal{D}^{\alpha}$ is a differential operator of fractional order α. We note that there are multiple definitions of fractional derivatives (the most well known include Riemann–Liouville, Caputo and Grünwald–Letnikov derivatives). However, irrespective of the chosen definition of fractional derivative, the input–output behavior of the above equation is the same, thereby leading to (3). The problem with any approach relying on fractional derivatives – irrespective of the particular definition – is that the resulting fractional system does not enjoy the semigroup property, as outlined in the Introduction. In other words, the use of a fractional derivative hides the infinite-dimensional nature of the system, which, however, cannot be ignored when solving an initial value problem.

To overcome this limitation, in this paper we use the diffusive representation of fractional systems that does not involve any fractional derivative.

Importantly, the distributed frequency model can be used to model fractional systems given by ratios of fractional polynomials. In fact, taking $V(s) = A\,X(s) + B\,U(s)$ we find that $X(s) = (s^{\alpha} I - A)^{-1} B\,U(s)$, and adding the output equation $Y(s) = C\,X(s) + D\,U(s)$ we obtain $Y(s) = G(s)\,U(s)$, where $G(s) = C\,(s^{\alpha} I - A)^{-1} B + D$ as required, see [17] for details of the derivation. It is now clear why the variable x is commonly referred to as pseudo-state: the information that x carries at any point in time is not sufficient to compute the future evolution of the system. In fact, in this context, the knowledge of the function $z(\omega,\cdot)$ at a given instant is required to integrate the differential equation. However, x has the same structure of the state of a classical integer system, in that it obeys $X(s) = (s^{\alpha} I - A)^{-1} B\,U(s)$.

3. Geometric foundations

A crucial building block for solving the disturbance decoupling problem is the notion of controlled invariance. It turns out to be more convenient and natural to consider an even more fundamental notion, namely, the concept of invariance.

3.1. Invariance

The classical notion of A-invariance for fractional systems using the distributed frequency model is considered in this section. We consider a model in the form (1)–(2). Solving the partial differential equation in (1) using the Laplace transform yields the following solution for $z(\omega,t)$:

$$z(\omega,t) = e^{-\omega(t-t_0)} z_0(\omega) + \int_{t_0}^{t} e^{-\omega(t-\tau)} v(\tau)\mathrm{d}\tau. \tag{7}$$

To characterize invariance for fractional systems, consider the homogeneous system

$$\frac{\partial z}{\partial t}(\omega,t) = -\,\omega z(\omega,t) + Ax(t), \qquad z(\omega,t_0) = z_0(\omega), \tag{8}$$

$$x(t) = \int_0^{\infty} \mu_\alpha(\omega) z(\omega,t)\mathrm{d}\omega, \tag{9}$$

Definition 3.1. A subspace $\mathcal{J}$ of the pseudo-state space $\mathbb{R}^n$ is said to be an invariant of the homogeneous fractional system in (8) if for every initialization function $z_0(\omega)$ that is in $\mathcal{J}$ almost everywhere, the corresponding trajectory $x(t)$ is in $\mathcal{J}$ for all $t \geqslant t_0$.

Recalling that a subspace $\mathcal{J}$ is A-invariant if $A\mathcal{J} \subseteq \mathcal{J}$, the following theorem characterizes A-invariant subspaces by conditions on trajectories of the pseudo-state.

Theorem 3.2. *Consider the system in (8) and let $\mathcal{J}$ be a subspace of $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) The subspace $\mathcal{J}$ is invariant;
(ii) The subspace $\mathcal{J}$ is A-invariant, i.e., $A\mathcal{J} \subseteq \mathcal{J}$;
(iii) For every trajectory $x(t)$ that is in $\mathcal{J}$ for all $t \geqslant t_0$, the corresponding initialization function $z_0(\omega)$ is in $\mathcal{J}$ almost everywhere.

The following lemmata are required to prove this theorem.

Lemma 3.3 (*Analytic Continuation From a Segment*)**.** *Let f be an analytic function on an open and connected set $\aleph$ in the complex plane containing the line segment (a,b) where $a,b \in \mathbb{R}$ with $a<b$. If $f(z)=0$ for all $z \in (a,b)$, then $f(z)=0$ for all $z \in \aleph$.*

Proof. Consider an arbitrary point $c \in (a,b)$. Because f is analytic, it can be locally expanded as a power series about c, that is, there exists a $\rho_0 > 0$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k$$

for all $z = c + \rho \exp(i\theta)$ with $\rho < \rho_0$ and $\theta \in [0, 2\pi)$. The claim is that $a_k = 0$ for every $k \in \mathbb{N}$. This can be argued by contradiction. Let k_0 be the smallest k such that $a_{k_0} \neq 0$. Locally around c, it follows that

$$f(z) = (z-c)^{k_0} \sum_{m=0}^{\infty} b_m (z-c)^m,$$

where $b_m \overset{\text{def}}{=} a_{m+k_0}$ for every $m \in \mathbb{N}$. Thus, $b_0 = a_{k_0} \neq 0$. Hence,

$$\varphi(z) \overset{\text{def}}{=} \sum_{m=0}^{\infty} b_m (z-c)^m$$

is an analytic function in a neighborhood of c and it is non-zero at c. By continuity of φ, it follows that $\varphi(z)$ is also non-zero in a ball of radius ε_1 with center c. Let $0 < \varepsilon < \min\{\varepsilon_1, \rho_0\}$ such that $z_0 \overset{\text{def}}{=} c + \varepsilon < b$. Note that $z_0 \in (a,b)$. Consequently, $f(z_0) = \varepsilon^{k_0} \varphi(z_0) \neq 0$, which is a contradiction since $z \in (a,b)$ and $f(z) = 0$ for all $z \in (a,b)$. It follows that in an open neighborhood of c in the complex plane, $f(z) = 0$. Therefore, f is zero in $\aleph$ by the principle of analytic continuation. ■

Remark 1. Note that this lemma is stronger than the usual principle of analytic continuation found in many complex analysis texts. The standard principle of analytic continuation requires the behavior of the function to be known in neighborhood $\aleph$. The line segment (a,b) is not a neighborhood in the complex plane.

Lemma 3.4 (*Analyticity of the Laplace Transform*)**.** *The Laplace transform of a function is analytic in the region of convergence.*

This result follows from Fubini's theorem and Morera's theorem and it can be found in [31]. This last lemma shows that the Leibniz rule holds for Eq. (2).

Lemma 3.5. *Consider a solution of (1) such that $\mu_\alpha(\omega) z(\omega,t)$ is integrable over $[0,\infty)$ with respect to ω for each $t \in [0,\infty)$. Then, for every $t \in [0,\infty)$*

$$\dot{x}(t) = \int_0^\infty \mu_\alpha(\omega) \frac{\partial z(\omega,t)}{\partial t} \mathrm{d}\omega.$$

Proof. We need to show the following two points:

L1. $\mu_\alpha(\omega)\frac{\partial z}{\partial t}$ exists for almost all ω and t in $[0,\infty)$; and
L2. $\mu_\alpha(\omega)\frac{\partial z}{\partial t}$ is integrable over $[0,\infty)$ with respect to ω for each $t \in [0,\infty)$.

The first point follows since (1) holds for almost all $\omega, t \geqslant 0$. Now we check the last point. Consider the following integral

$$\begin{aligned}
\int_0^\infty \mu_\alpha \frac{\partial z}{\partial t} \mathrm{d}\omega &= \int_0^\infty \mu_\alpha \big(-\omega z(\omega,t) + v(t)\big) \mathrm{d}\omega \\
&= \int_0^\infty \mu_\alpha \left(-\omega z_0(\omega) e^{-\omega t} - \omega \int_0^t e^{\omega(t-\tau)} v(\tau) \mathrm{d}\tau + v(t) \right) \mathrm{d}\omega \\
&= \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty -\omega^{1-\alpha} z_0(\omega) e^{-\omega t} - \frac{1}{\omega^\alpha} \left(\omega \int_0^t e^{\omega(t-\tau)} v(\tau) \mathrm{d}\tau - v(t) \right) \mathrm{d}\omega.
\end{aligned}$$

Since

$$\int_0^\infty \omega^{1-\alpha} z_0(\omega) e^{-\omega t} \mathrm{d}\omega$$

is finite, we only need to show that

$$\int_0^\infty \frac{1}{\omega^\alpha} \left(\omega \int_0^t e^{\omega(t-\tau)} v(\tau) \mathrm{d}\tau - v(t) \right) \mathrm{d}\omega$$

is finite. Indeed, we proceed as follows:

$$\int_0^\infty \frac{1}{\omega^\alpha} \left(\omega \int_0^t e^{-\omega(t-\tau)} v(\tau) \mathrm{d}\tau - v(t) \right) \mathrm{d}\omega$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{\omega^\alpha}\left(\omega\left(\frac{v(t)-e^{-\omega t}v(0)}{\omega}-\int_0^t \frac{e^{-\omega(t-\tau)}v'(\tau)}{\omega}\mathrm{d}\tau\right)-v(t)\right)\mathrm{d}\omega\\
&= \int_0^\infty \frac{1}{\omega^\alpha}\left(\left(v(t)-e^{-\omega t}v(0)-\int_0^t e^{-\omega(t-\tau)}v'(\tau)\mathrm{d}\tau\right)-v(t)\right)\mathrm{d}\omega\\
&= \int_0^\infty \frac{1}{\omega^\alpha}\left(-e^{-\omega t}v(0)-\int_0^t e^{-\omega(t-\tau)}v'(\tau)\mathrm{d}\tau\right)\mathrm{d}\omega.
\end{aligned}
$$

It is easy to check that

$$\int_0^\infty \frac{1}{\omega^\alpha}\left(-e^{-\omega t}v(0)\right)\mathrm{d}\omega$$

is finite. Moreover,

$$
\begin{aligned}
\int_0^\infty \frac{1}{\omega^\alpha}\int_0^t e^{-\omega(t-\tau)}v'(\tau)\mathrm{d}\tau\mathrm{d}\omega &\leqslant \int_0^\infty \frac{1}{\omega^\alpha}M\int_0^t e^{-\omega(t-\tau)}\mathrm{d}\tau\mathrm{d}\omega\\
&\leqslant M\int_0^\infty \frac{e^{-\omega t}-1}{\omega^{\alpha+1}}\mathrm{d}\omega < \infty,
\end{aligned}
$$

where $M \overset{\text{def}}{=} \sup\{v'(\xi)\mid \xi\in[0,t]\}$. ■

The proof of Theorem 3.2 is presented below.

Proof. First, consider the statement (i) implies (ii). Let $T=[T_1\ \ T_2]$ be a change of coordinates matrix where T_1 is a basis for $\mathcal{J}$ and T_2 is a basis for $\mathcal{J}^\perp$. Then letting $\widetilde{z}(\omega,t)$, $\widetilde{x}(t)$ and $\widetilde{A}$ represent the coordinates of $z(\omega,t)$, $x(t)$ and A, respectively in this new basis yields the following:

$$\widetilde{z}(\omega,t)=\begin{bmatrix} z'(\omega,t)\\ z''(\omega,t)\end{bmatrix},\quad \widetilde{x}(t)=\begin{bmatrix} x'(t)\\ x''(t)\end{bmatrix}\quad \text{and}\quad \widetilde{A}=\begin{bmatrix}\widetilde{A}_{11} & \widetilde{A}_{12}\\ \widetilde{A}_{21} & \widetilde{A}_{22}\end{bmatrix}. \tag{10}$$

To show that $A\mathcal{J}\subseteq\mathcal{J}$, it suffices to prove that $\widetilde{A}_{21}=0$. Notice that since $x(t)\in\mathcal{J}$ for all $t\geqslant t_0$, it follows that $\dot{x}''(t)=0$ for all $t\geqslant t_0$. Thus,

$$
\begin{aligned}
\dot{x}''(t_0) &= \int_0^\infty \mu_\alpha(\omega)\frac{\partial z''}{\partial t}(\omega,t_0)\mathrm{d}\omega\\
&= \int_0^\infty \mu_\alpha(\omega)(-\omega z_0''(\omega)+\widetilde{A}_{21}x'(t_0))\mathrm{d}\omega\\
&= \int_0^\infty \mu_\alpha(\omega)\widetilde{A}_{21}x'(t_0)\mathrm{d}\omega,\quad \text{since } z_0(\omega)\in\mathcal{J} \text{ for almost all } \omega\in[0,\infty).\\
&= \widetilde{A}_{21}x'(t_0)\int_0^\infty \mu_\alpha(\omega)\mathrm{d}\omega.
\end{aligned}\tag{11}
$$

It is easy to check that the integral in (11) diverges but $\dot{x}''(t_0)=0$. Therefore, $\widetilde{A}_{21}x'(t_0)=0$, which implies that $x'(t_0)\in\ker\widetilde{A}_{21}$. But recalling that $x'(t_0)$ can be made arbitrary by selecting an appropriate $z_0(\omega)$, it follows that $\widetilde{A}_{21}=0$. Thus, $A\mathcal{J}\subseteq\mathcal{J}$.

Next consider the statement (ii) implies (i). Suppose that a subspace $\mathcal{J}$ is A-invariant. That is, $A\mathcal{J}\subseteq\mathcal{J}$. Assume that $z_0(\omega)\in\mathcal{J}$ almost everywhere. Using the principle of induction over the continuum[2] (also known as real induction), to show that $x(t)\in\mathcal{J}$ for all $t\geqslant t_0$, it suffices to show the following three statements:

(a) $x(t_0)\in\mathcal{J}$;
(b) $x(t_0)\in\mathcal{J}$ implies that there exists a $t_1>t_0$ such that $x(t)\in\mathcal{J}$ for all $t\in[t_0,t_1)$;
(c) $x(t)\in\mathcal{J}$ for all $t\in[t_0,t_1)$ implies that there exists a $t_2>t_1$ such that $x(t)\in\mathcal{J}$ for all $t\in[t_0,t_2)$.

Clearly, condition (a) follows from the assumption $z_0(\omega)\in\mathcal{J}$ for almost all $\omega\in[0,\infty)$. Now assuming that $x(t_0)\in\mathcal{J}$, there are two cases[3] to consider: either there exists a $\bar{t}$ such that for all $t\in(t_0,\bar{t})$, $z(\omega,t)\notin\mathcal{J}$ for a set of ω with nonzero Lebesgue measure or there exists a $\bar{t}$ such that for some $t>\bar{t}>t_0$, $z(\omega,t)\notin\mathcal{J}$ on a set of ω with positive Lebesgue measure. In the former case, the following must be true:

$$\frac{\partial z}{\partial t}(\omega,t_0)\notin\mathcal{J}$$

[2] The principle of induction over the continuum, equivalent to the least upper bound property of the real numbers is the following. Let $a\in\mathbb{R}$ and $S\subseteq[a,\infty)$ with the following three properties:
(i) $a\in S$;
(ii) If $a\in S$ then there exists a real number $x>a$ such that $[a,x)\subseteq S$;
(iii) For all $x\in\mathbb{R}$, if $[a,x)\subseteq S$, then there exists a real number $y>x$ such that $[a,y)\subseteq S$.
Then $S=[a,\infty)$.

[3] The case where $z(\omega,t)\in\mathcal{J}$ for almost all $\omega\in[0,\infty)$ and all $t\geqslant t_0$ need not be considered in the following argument since that would imply $x(t)\in\mathcal{J}$ for all $t\geqslant t_0$.

for a set of ω with positive Lebesgue measure. But observe that

$$\frac{\partial z}{\partial t}(\omega, t_0) = -\omega z_0(\omega) + Ax(t_0) \in \mathcal{J}$$

for almost all $\omega \in [0,\infty)$ since $z_0(\omega) \in \mathcal{J}$ for almost all $\omega \in [0,\infty)$ and $Ax(t_0) \in \mathcal{J}$. This is a contradiction. Therefore, the former case cannot occur. In the latter case, choosing $t_1 = (t_0 + \bar{t})/2$ yields $z(\omega, t) \in \mathcal{J}$ for all $t \in [t_0, t_1)$, whence by (2), $x(t) \in \mathcal{J}$ for all $t \in [t_0, t_1)$. Hence, condition (b) is satisfied. Finally, to show (c), suppose that $x(t) \in \mathcal{J}$ for all $t \in [t_0, t_1)$. Similar to (b), there are two cases[4] for x: either there exists a $\bar{t} > t_1$ such that for all $t \in (t_1, \bar{t})$, $x(t) \notin \mathcal{J}$ or there exists a $\bar{t} > t_1$ for which $x(t) \in \mathcal{J}$ for some $t \in [t_1, \bar{t})$. Note that (c) follows immediately from the latter case. The former case cannot occur and it can be shown by contradiction. Notice that $x(t) \notin \mathcal{J}$ for all $t \in (t_1, \bar{t})$ only if $\dot{x}(t_1) \notin \mathcal{J}$. Moreover, $x(t) \in \mathcal{J}$ for all $t \in [t_0, t_1)$ implies that $z(\omega, t_1) \in \mathcal{J}$ for all $t \in [t_0, t_1)$ by (7). So (2) implies that $x(t_1) \in \mathcal{J}$. Thus,

$$\frac{\partial z}{\partial t}(\omega, t_1) = -\omega z(\omega, t_1) + Ax(t_1) \in \mathcal{J}.$$

Hence,

$$\dot{x}(t_1) = \int_0^\infty \mu_\alpha(\omega) \frac{\partial z}{\partial t}(\omega, t_1) \mathrm{d}\omega \in \mathcal{J},$$

which is a contradiction. Thus, condition (c) also holds. Therefore, by the principle of real induction, $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$.

Now, to show that (ii) implies (iii), suppose $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$. Note that making a change of coordinates with the matrix T yields the expression for $\tilde{z}$ in the first equation of (10). It suffices to show that $z_0''(\omega) = 0$ for almost all $\omega \in [0,\infty)$. The solution to (1) can then be written as:

$$z'(\omega, t) = e^{-\omega(t-t_0)} z_0'(\omega) + \int_{t_0}^t e^{-\omega(\tau - t_0)} Ax(\tau) \mathrm{d}\tau \tag{12}$$

$$z''(\omega, t) = e^{-\omega(t-t_0)} z_0''(\omega), \tag{13}$$

since $\mathcal{J}$ is A-invariant and $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$. Because $x(t) \in \mathcal{J}$, it must be that

$$\int_0^\infty \mu_\alpha(\omega) z''(\omega, t) \mathrm{d}\omega = 0.$$

Substituting the solution from (13) yields

$$\frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \omega^{-\alpha} e^{-\omega(t-t_0)} z_0''(\omega) \mathrm{d}\omega = 0.$$

for all $t \geqslant t_0$. This is equivalent to

$$\int_0^\infty \omega^{-\alpha} e^{-\omega\tau} z_0''(\omega) \mathrm{d}\omega = 0 \tag{14}$$

for all $\tau \geqslant 0$. Observe that exchanging the variables ω and τ with t and s, respectively, (14) can be rewritten as:

$$F(s) \stackrel{\text{def}}{=} \mathcal{L}\left\{ t^{-\alpha} z_0''(t) \right\} = 0,$$

for all real $s \geqslant 0$. Notice that this Laplace transform converges on the right half plane. Thus, it is analytic by Lemma 3.4. It follows from Lemma 3.3 that $F(s) = 0$ for all s in the right half plane. Hence, by the injectivity of the Laplace transform, it immediately follows that $t^{-\alpha} z_0''(t) = 0$ for almost all $t \in [0,\infty)$ which implies that $z_0''(\omega) = 0$ for almost all $\omega \in [0,\infty)$.

Next consider the statement (iii) implies (ii). It is required to show that if a subspace has the property that $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$ only if $z_0(\omega) \in \mathcal{J}$ for almost all ω, then that subspace must be A-invariant. By the semigroup property of this model, it follows that $x(t) \in \mathcal{J}$ for all $t \geqslant \bar{t} \geqslant t_0$ if and only if $z(\omega, \bar{t}) \in \mathcal{J}$ for almost all $\omega \in [0,\infty)$. Since $\bar{t}$ is arbitrary, it must be that $z(\omega, t) \in \mathcal{J}$ for all $t \geqslant t_0$ and for almost all $\omega \in [0,\infty)$. Therefore,

$$\frac{\partial z}{\partial t}(\omega, t_0) = \lim_{t \to t_0} \frac{z(\omega, t) - z(\omega, t_0)}{t - t_0} \in \mathcal{J}.$$

Moreover, by (1),

$$Ax(t_0) = \frac{\partial z}{\partial t}(\omega, t_0) + \omega z(\omega, t_0) \in \mathcal{J}.$$

Also, $x(t_0)$ can take any arbitrary value in $\mathcal{J}$ since there exists a $z(\omega, t_0)$ for which

$$x(t_0) = \int_0^\infty \mu_\alpha(\omega) z(\omega, t_0) \mathrm{d}\omega.$$

As a result, $A\mathcal{J} \subseteq \mathcal{J}$, as desired. ■

[4] Again, the case where $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$ need not be considered.

This shows the invariance property for fractional systems. In particular, note that condition (iii), the converse of condition (i), holds for fractional systems of this kind. This is rather obvious in the classical case, since if $x(t) \in \mathcal{J}$ for all $t \geqslant t_0$, then clearly $x_0 = x(t_0) \in \mathcal{J}$. However, this property is in general not guaranteed in infinite dimensional systems. The geometric theory with the distributed frequency model forms a nice parallel with the classical theory. The analogue of this for controlled invariance is still a subject of investigation.

3.2. Controlled invariance

In similar spirit, the concept of controlled invariant subspaces is stated below. In this case, the system considered is of the form

$$\frac{\partial z}{\partial t}(\omega,t) = -\omega z(\omega,t) + Ax(t) + Bu(t), \qquad z(\omega,t_0) = z_0(\omega), \tag{15}$$

$$x(t) = \int_0^\infty \mu_\alpha(\omega) z(\omega,t)\mathrm{d}\omega, \tag{16}$$

This system is also said to be inhomogeneous.

Definition 3.6. A subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be controlled invariant if for each initialization function $z_0(\omega)$ in $\mathcal{V}$, there exists a control u such that $x(t) \in \mathcal{V}$ for all $t \geqslant t_0$.

Controlled invariant subspaces are characterized geometrically in the following theorem. While the result matches the characterization of controlled invariance for classical systems, this characterization is not guaranteed to hold in general for infinite-dimensional systems.

Theorem 3.7. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. Then for a system of the form in (15), the following statements are equivalent:*

(i) $\mathcal{V}$ is controlled invariant;
(ii) $A\mathcal{V} \subseteq \mathcal{V} + \mathrm{im}\, B$;
(iii) There exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$,

where $\mathrm{im}\, B$ *denotes the image of matrix B.*

Proof. First, to show (i) implies (ii), assume $\mathcal{V}$ is controlled invariant. So for all $z_0(\omega) \in \mathcal{V}$, there exists a control u such that $x(t) \in \mathcal{V}$ for all $t \geqslant t_0$. For the sake of deriving a contradiction, suppose that $A\mathcal{V} \nsubseteq \mathcal{V} + \mathrm{im}\, B$. Take $z_0(\omega) \in \mathcal{V}$ such that $Ax(t_0) \notin \mathcal{V} + \mathrm{im}\, B$. Then

$$\begin{aligned}\dot{x}(t) &= \int_0^\infty \mu_\alpha(\omega)\frac{\partial z}{\partial t}(\omega,t)\mathrm{d}\omega \\ &= \int_0^\infty \mu_\alpha(\omega)(-\omega z(\omega,t) + Ax(t) + Bu(t))\mathrm{d}\omega,\end{aligned}$$

by (1). Now consider orthogonally decomposing this system into two components: one in $\mathcal{V}$ and one in $\mathcal{V}^\perp$, where $(\cdot)'$ and $(\cdot)''$ denote the components of $(\cdot)$ in $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively. Then since $z_0(\omega) \in \mathcal{V}$ and $x(t) \in \mathcal{V}$ for all $t \geqslant t_0$, it follows that $z_0''(\omega) = 0$ and $\dot{x}''(t_0) = 0$. Thus,

$$\begin{aligned}\dot{x}''(t_0) &= \int_0^\infty \mu_\alpha(\omega)(Ax(t_0) + Bu(t_0))''\mathrm{d}\omega \\ &= (Ax(t_0) + Bu(t_0))''\int_0^\infty \mu_\alpha(\omega)\mathrm{d}\omega.\end{aligned}$$

Because the integral above diverges and $\dot{x}''(t_0) = 0$, it must be that $(Ax(t_0)+Bu(t_0))'' = 0$. However, $(Ax(t_0)+Bu(t_0))'' \neq 0$ is immediate from the assumption $Ax(t_0) \notin \mathcal{V} + \mathrm{im}\, B$. This is a contradiction and thus $A\mathcal{V} \subseteq \mathcal{V} + \mathrm{im}\, B$.

The statement (ii) implies (iii) is a standard result, see for instance [22] for a proof of this.

Finally, to show (iii) implies (i), suppose there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$. Let $u(t) = Fx(t)$ and consider an arbitrary $z_0(\omega) \in \mathcal{V}$. The partial differential equation governing the evolution of $z(\omega,t)$ is of the form

$$\frac{\partial z}{\partial t}(\omega,t) = -\omega z(\omega,t) + (A+BF)x(t).$$

Applying Theorem 3.2, it follows that $x(t) \in \mathcal{V}$ for all $t \geqslant t_0$. Thus, $\mathcal{V}$ is controlled invariant. ■

If any control u exists that can keep the pseudo-state in $\mathcal{V}$ then one can always implement a pseudo-state feedback $u(t) = Fx(t)$ that can keep the pseudo-state trajectory in $\mathcal{V}$. A state-feedback matrix satisfying *(iii)* in Theorem 3.7 will be referred to as a *friend* of $\mathcal{V}$.

As mentioned before, in this case, unlike the result on invariance, the converse of condition (i) is an open problem. That is, the truth of the following implication is not known: if there exists a control u such that $x(t) \in \mathcal{V}$ for all $t \geqslant t_0$, then $z_0(\omega) \in \mathcal{V}$.

3.3. The converse implication

The above section showed that, similarly to LTI systems, fractional controlled invariant subspaces guarantee that, if the initialization function belongs to the controlled invariant, then the pseudo-state trajectory can always evolve on that controlled invariant subspace by means of a pseudo-state feedback. However, one might wonder whether there are instances where we can still evolve on a controlled invariant even if the initialization function does not belong to such a controlled invariant in a set of non-zero Lebesgue measure by using a more general feedforward control. This is, in fact, the case for another class of fractional systems defined using the Caputo fractional derivatives, see [25]. More formally, the problem at hand can be expressed as follows. Let $\mathcal{V}$ be a controlled invariant subspace and let $z_0(\omega)$ be such that

$$\int_0^\infty \mu_\alpha(\omega)\, z_0(\omega)\, \mathrm{d}\omega \in \mathcal{V}.$$

Under which conditions, if any, there exists a control u such that $x(t) \in \mathcal{V}$ for all $t \geq 0$?

Let u be a control such that

$$x(t) = \int_0^\infty \mu_\alpha(\omega)\, z(\omega, t)\, \mathrm{d}\omega \in \mathcal{V}. \tag{17}$$

Assume, without loss of generality, that u is a control in the form $u(t) = F\,x(t) + v(t)$, where F is a friend of $\mathcal{V}$, and v is an arbitrary (measurable) input signal. Using a change of coordinates in the pseudo-state space adapted to $\mathcal{V}$ and a change of coordinates in the input space adapted to $B^{-1}\mathcal{V}$ (see e.g. [22, p. 77]), we can write the differential equation in partitioned form as

$$\frac{\partial}{\partial t}\begin{bmatrix} z_1(\omega,t) \\ z_2(\omega,t) \end{bmatrix} = -\omega \begin{bmatrix} z_1(\omega,t) \\ z_2(\omega,t) \end{bmatrix} + \begin{bmatrix} A^F_{1,1} & A^F_{1,2} \\ 0 & A^F_{2,2} \end{bmatrix}\begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},$$

where the sub-matrices are defined conformably with the changes of basis. This differential equation is decoupled, in the sense that the evolution of $z_2(\omega,t)$ only depends on v_2, i.e., matrix $A^F_{2,2,}$ is irrelevant for the evolution of the system. We can therefore write the following equation

$$\frac{\partial}{\partial t} z_2(\omega,t) = -\omega\, z_2(\omega,t) + B_{2,2}\, v_2(t).$$

Since we have assumed that the control maintains the pseudo-state in $\mathcal{V}$, then (17) is equivalent to

$$\int_0^\infty \mu_\alpha(\omega)\, z_2(\omega,t)\, \mathrm{d}\omega = 0$$

for all $t \geq 0$. The important problem, which is currently only a conjecture, is that for the existence of a control v_2 such that this latter equality is satisfied, $z_2(\omega, 0)$ must be zero. This would imply that it is impossible to evolve on a controlled invariant subspace unless the entire initialization function belongs to the controlled invariant subspace almost everywhere. This seems to be a unique property of diffusive fractional systems. In fact, other infinite dimensional generalizations of the geometric control, including fractional systems defined by Caputo fractional derivative, do not enjoy this property. At the moment, it is only possible to prove this result under additional simplifying – but practically relevant – assumptions. We prove it, for the sake of argument, in the scalar case. The same argument can be adapted – *mutatis mutandis* – to the vector case.

Theorem 3.8. *Consider the differential equation*

$$\frac{\partial z(\omega,t)}{\partial t} = -\omega\, z(\omega,t) + u(t). \tag{18}$$

Let one of these assumptions hold:

(A1) *u is piecewise constant; or*

(A2) *u is a ramp; or*

(A3) *u is an exponential.*

Then, if

$$\int_0^\infty \mu_\alpha(\omega)\, z(\omega,t)\, \mathrm{d}\omega = 0,$$

then $z(\omega,0)$ is equal to zero.

Proof. Let us consider **(A1)**, so suppose that $u(t) = u_0$ for some constant $u_0 \in \mathbb{R}$. Then by (7), it follows that

$$z(\omega,t) = e^{-\omega t} z_0(\omega) + \frac{u_0}{\omega} - \frac{u_0 e^{-\omega t}}{\omega}. \tag{19}$$

We have

$$x(t) = \int_0^\infty \mu_\alpha(\omega) z(\omega,t)\mathrm{d}t = 0, \tag{20}$$

for all $t \geqslant 0$. This implies that $\dot{x}(t) = 0$ for all $t \geqslant 0$. Thus, by Lemma 3.5, we have

$$\dot{x}(t) = \int_0^\infty \mu_\alpha(\omega) \frac{\partial z}{\partial t} \mathrm{d}t = 0. \tag{21}$$

Now, from (19) we find $\frac{\partial z}{\partial t} = \left(\frac{u_0}{\omega^\alpha} - z_0(\omega)\omega^{1-\alpha} \right) e^{-\omega t}$. Substituting this equality into (21) we find

$$\begin{aligned} \dot{x}(t) &= \int_0^\infty \mu_\alpha(\omega) \left(\frac{u_0}{\omega^\alpha} - z_0(\omega)\omega^{1-\alpha} \right) e^{-\omega t} \mathrm{d}t \\ &= \int_0^\infty y(\omega) e^{-\omega t} \mathrm{d}t, \text{ where } y(\omega) \overset{\text{def}}{=} \mu_\alpha(\omega) \left(\frac{u_0}{\omega^\alpha} - z_0(\omega)\omega^{1-\alpha} \right). \end{aligned}$$

Viewing $\dot{x}(t)$ as the Laplace transform of $y(\omega)$ allows us to conclude that $y(\omega) = 0$ for almost all $\omega \in [0, +\infty)$. Therefore, $\frac{u_0}{\omega^\alpha} - z_0(\omega)\omega^{1-\alpha} = 0$, i.e., $z_0(\omega) = u_0/\omega$. This implies that $z(\omega, t) = u_0/\omega$ and for the integral $\int_0^\infty \mu_\alpha(\omega) \frac{u_0}{\omega} \mathrm{d}\omega$ to be zero, we must have $u_0 = 0$. Hence, $z_0(\omega) = 0$.

Let us now consider u to be a simple function, i.e., we define $u(t) = u_k$ for all $t \in [t_{k-1}, t_k]$, where $k = 0, \ldots, n$, and $t_{-1} = 0$. Let $t \in [t_{n-1}, t_n)$. We find

$$\begin{aligned} z(\omega, t) &= z_0(\omega)\, e^{-\omega t} + \int_0^t e^{-\omega(t-\tau)}\, u(\tau)\, d\tau \\ &= z_0(\omega)\, e^{-\omega t} + \int_0^{t_0} e^{-\omega(t-\tau)}\, u_0\, d\tau + \int_{t_0}^{t_1} e^{-\omega(t-\tau)}\, u_1\, d\tau + \cdots + \int_{t_{n-1}}^{t} e^{-\omega(t-\tau)}\, u_n\, d\tau \\ &= z_0(\omega)\, e^{-\omega t} + \frac{e^{-\omega t}}{\omega} \sum_{k=0}^{n-1} u_k\, (e^{\omega t_k} - e^{\omega t_{k-1}}) + \frac{u_n}{\omega}\, (1 - e^{-\omega(t - t_{n-1})}). \end{aligned}$$

Then, from $\frac{\partial z}{\partial t} = 0$ yields

$$z_0(\omega) = \frac{1}{\omega} \left(- \sum_{k=0}^{n-1} u_k\, (e^{\omega t_k} - e^{\omega t_{k-1}}) + u_n\, e^{\omega t_{n-1}} \right).$$

Substituting this into (20)

$$x(0) = \int_0^\infty \frac{1}{\omega^{\alpha+1}} \left(u_0\, (1 - e^{\omega t_0}) - \sum_{k=1}^{n-1} u_k\, (e^{\omega t_k} - e^{\omega t_{k-1}}) + u_n\, e^{\omega t_{n-1}} \right) d\omega = 0.$$

The integrand is asymptotically equivalent, as $\omega \to 0$, to

$$\frac{1}{\omega^\alpha} \left(u_0\, t_0 - \sum_{k=1}^{n-1} u_k\, (t_k - t_{k-1}) + u_n\, t_{n-1} \right) + \frac{u_n}{\omega^{\alpha+1}}.$$

The first addend is integrable but the second is not because $\alpha \in (0, 1)$, and therefore $u_n = 0$.

The same argument can be applied for any t. For example, we choose a time t smaller than t_{n-1} (say between t_{n-2} and t_{n-1}), then we can prove that u_{n-1} must be zero, and so forth.

Consider **(A2)**, so $u(t) = u_0 t$ for some $u_0 \in \mathbb{R}$. Then, by (7), we have

$$z(\omega, t) = e^{-\omega t} z_0(\omega) + \frac{u_0(\omega t + e^{-\omega t} - 1)}{\omega^2}.$$

The second partial derivative of z with respect to t is

$$\frac{\partial^2 z}{\partial t^2} = (\omega^2 z_0(\omega) + u_0) e^{-\omega t}.$$

Since $x(t) = 0$ for all $t \geqslant 0$, we have $\dot{x}(t) = 0$ and $\ddot{x}(t) = 0$ for all $t \geqslant 0$. Therefore, by Leibniz rule,[5]

$$\begin{aligned} \ddot{x}(t) &= \int_0^\infty \mu_\alpha(\omega) \frac{\partial^2 z}{\partial t^2} \mathrm{d}\omega \\ &= \int_0^\infty \mu_\alpha(\omega)(\omega^2 z_0(\omega) + u_0) e^{-\omega t} \mathrm{d}\omega \\ &= \mathcal{L}\left\{ \mu_\alpha(\omega)(\omega^2 z_0(\omega) + u_0) \right\} = 0. \end{aligned}$$

Hence, $\omega^2 z_0(\omega) + u_0 = 0$. So we have $z_0(\omega) = -u_0/\omega^2$. But for

$$x(0) = \int_0^\infty \mu_\alpha z_0(\omega) \mathrm{d}\omega = \int_0^\infty - \sin(\pi\alpha) \frac{u_0}{\omega^{\alpha+2}} \mathrm{d}\omega$$

[5] The first two conditions L1 and L2 are obviously satisfied. It is easy to adapt the argument presented in the previous case to show that the condition L3 is also satisfied by repeating the integration by parts twice.

to exist, we must have $u_0 = 0$. Hence $u(t) = 0$ and $z_0(\omega) = 0$.

Finally, consider **(A3)**. Let $u(t) = u_0\, e^{\beta t}$, where $u_0, \beta \in \mathbb{R}$ and $\beta \neq 0$. By (7),

$$z(\omega, t) = e^{-\omega t} z_0(\omega) + \frac{u_0}{\omega+\beta} e^{\beta t} - \frac{u_0}{\omega+\beta} e^{-\omega t}.$$

Hence,

$$\begin{aligned}\dot{x}(t) &= \int_0^\infty \mu_\alpha(\omega) \frac{\partial z}{\partial t} \mathrm{d}\omega \\ &= -\int_0^\infty \mu_\alpha(\omega) \omega e^{-\omega t} z_0(\omega) \mathrm{d}\omega + \int_0^\infty \mu_\alpha(\omega) \left(\frac{u_0 \beta}{\omega+\beta} e^{\beta t} + \frac{u_0 \omega}{\omega+\beta} e^{-\omega t} \right) \mathrm{d}\omega.\end{aligned}$$

Since the first integral in the right hand-side converges, the second integral also converges. If $\beta > 0$, then $\int_0^\infty \mu_\alpha(\omega) \frac{u_0\beta}{\omega+\beta} e^{\beta t} \mathrm{d}\omega$ converges and so $\int_0^\infty \mu_\alpha(\omega) \frac{u_0\omega}{\omega+\beta} e^{-\omega t} \mathrm{d}\omega$ must converge. This latter integral converges precisely when $u_0 = 0$. Therefore, u_0 must be zero. Now, if $\beta < 0$, then

$$\int_0^\infty \mu_\alpha(\omega) \left(\frac{u_0 \beta}{\omega+\beta} e^{\beta t} + \frac{u_0 \omega}{\omega+\beta} e^{-\omega t} \right) \mathrm{d}\omega = \int_0^\infty \frac{\mu_\alpha(\omega)}{\omega+\beta} \left(u_0 \beta e^{\beta t} + \omega e^{-\omega t} \right) \mathrm{d}\omega.$$

This integral diverges for all $t > 0$ unless $u_0 = 0$. Hence, $u_0 = 0$ and we conclude that $z_0(\omega) = 0$. ■

Since every Lebesgue measurable function can be obtained as the limit of a sequence of piecewise constant functions, Theorem 3.8 seems to suggest that the result is true for a much more general class of control inputs. However, at this stage this remains a conjecture, and a formal proof is yet to be developed.

4. Conditioned invariance

Consider the system

$$\begin{aligned}\frac{\partial z}{\partial t} &= -\omega z(\omega, t) + Ax(t) + Bu(t) \\ x(t) &= \int_0^\infty \mu_\alpha(\omega) z(\omega, t) \mathrm{d}t \\ y(t) &= Cx(t) + Du(t).\end{aligned} \tag{22}$$

Definition 4.1. Let $\mathcal{S} \subseteq \mathcal{X}$ be a subspace. An oracle observer for the system (22) modulo $\mathcal{S}$ is a system

$$\begin{aligned}\frac{\partial \hat{z}}{\partial t} &= -\omega \hat{z}(\omega, t) + P\hat{x}(t) + Qu(t) + R(y - \hat{y}) \\ \hat{x}(t) &= \int_0^\infty \mu_\alpha(\omega) \hat{z}(\omega, t) \mathrm{d}t \\ \hat{y}(t) &= C\hat{x}(t) + Du(t)\end{aligned}$$

such that for any pair of initialization functions $(z_0(\omega), \hat{z}_0(\omega))$ and any control $u(t)$, we have $\hat{z}_0(\omega) = z_0(\omega)/\mathcal{S}$ implies $\hat{z}(\omega, t) = z(\omega, t)/\mathcal{S}$ for all $t \geqslant t_0$, where $a = b/\mathcal{S}$ means that $a - b \in \mathcal{S}$.

Definition 4.2. Consider the system in (22). A subspace $\mathcal{S}$ is said to be conditioned invariant if there exists an oracle observer for (22) modulo $\mathcal{S}$.

Theorem 4.3. *Let $\mathcal{S}$ be a subspace of the pseudo-state space. Then the following statements are equivalent*

(i) $\mathcal{S}$ is conditioned invariant;
(ii) $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$;
(iii) There exists a matrix G such that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$;

Proof. We first show that (i) implies (ii). Let $\mathcal{S}$ be conditioned invariant and take an arbitrary $x_0 \in \mathcal{S} \cap \ker C$. Choose $z_0(\omega) \in \mathcal{S}$ such that $x(0) = x_0$. Let $\hat{z}_0(\omega) = 0$ and $u(t) = 0$. Then notice that $\hat{z}_0(\omega) - z_0(\omega) \in \mathcal{S}$ and thus, $\hat{z}(\omega, t) - z(\omega, t) \in \mathcal{S}$ for all $t \geqslant t_0$. Hence,

$$\frac{\partial \hat{z}}{\partial t} - \frac{\partial z}{\partial t} \in \mathcal{S}$$

for all $t \geqslant t_0$. Therefore,

$$\begin{aligned}\frac{\partial \hat{z}}{\partial t}(\omega, t_0) - \frac{\partial z}{\partial t}(\omega, t_0) &= -\omega \hat{z}_0(\omega) + P\hat{x}_0 + R(y - C\hat{x}_0) - (-\omega z_0(\omega) + Ax_0) \\ &= -\omega(\hat{z}_0(\omega) - z_0(\omega)) - Ax_0.\end{aligned}$$

It immediately follows that $Ax_0 \in \mathcal{S}$ since

$$\frac{\partial \hat{z}}{\partial t}(\omega, t_0) - \frac{\partial z}{\partial t}(\omega, t_0) \in \mathcal{S} \text{ and } \hat{z}_0(\omega) - z_0(\omega) \in \mathcal{S}.$$

The result (ii) implies (iii) is standard. See [22] for a proof.

Finally, we show that (iii) implies (i). Suppose there exists a matrix G such that $(A+GC)\mathcal{S} \subseteq \mathcal{S}$. Then we claim that the following system

$$\begin{aligned}\frac{\partial \hat{z}}{\partial t} &= -\omega\hat{z}(\omega,t) + A\hat{x}(t) + Bu(t) + G(y-\hat{y})\\ \hat{x}(t) &= \int_0^\infty \mu_\alpha(\omega)\hat{z}(\omega,t)\mathrm{d}t\\ \hat{y}(t) &= C\hat{x}(t) + Du(t),\end{aligned}$$

is an oracle observer for (22) modulo $\mathcal{S}$. To see why, consider arbitrary initialization functions $\hat{z}_0(\omega)$, and $z_0(\omega)$ and an arbitrary control $u(t)$. Further, suppose that $\hat{z}_0(\omega) - z_0(\omega) \in \mathcal{S}$. We now show that $\hat{z}(\omega,t) - z(\omega,t) \in \mathcal{S}$ for all $t \geqslant t_0$. Let us define $\epsilon(\omega,t) \stackrel{\text{def}}{=} \hat{z}(\omega,t) - z(\omega,t)$ and $e(t) \stackrel{\text{def}}{=} \hat{x}(t) - x(t)$. Then

$$\begin{aligned}\frac{\partial \epsilon}{\partial t}(\omega,t) &= -\omega\hat{z}(\omega,t) + A\hat{x}(t) + Bu(t) + G(Cx(t) + Du(t) - C\hat{x}(t) - Du(t))\\ &\quad - (-\omega z(\omega,t) + Ax(t) + Bu(t))\\ &= -\omega(\hat{z}(\omega,t) - z(\omega,t)) + A(\hat{x}(t) - x(t)) + GC(\hat{x}(t) - x(t))\\ &= -\omega\epsilon(\omega,t) + (A+GC)e(t)\\ e(t) &= \int_0^\infty \mu_\alpha(\omega)\epsilon(\omega,t)\mathrm{d}\omega.\end{aligned}$$

We know that $\epsilon_0(\omega) = \hat{z}_0(\omega) - z_0(\omega) \in \mathcal{S}$ and $(A+GC)\mathcal{S} \subseteq \mathcal{S}$ and thus, $\epsilon(\omega,t) \in \mathcal{S}$ for all $t \geqslant t_0$ by Theorem 3.2. Therefore, $\hat{z}(\omega,t) - z(\omega,t) \in \mathcal{S}$ for all $t \geqslant t_0$ and from which, it follows that $\mathcal{S}$ is a conditioned invariant. ■

5. Output nulling

We now parallel the theory of controlled invariance with the theory of output nullingness. Loosely speaking, an output nulling subspace is a controlled invariant subspace in which the pseudostate trajectory can evolve while maintaining the output at zero. Building on the theory of controlled invariance for fractional systems, the extension to output nullingness relies exclusively on the pseudostate-to-output map C and on the input-to-output map D, and therefore it follows along the same lines of the integer case and does not present major issues.

Definition 5.1. A subspace $\mathcal{V}$ is said to be an output nulling subspace if, for any initialization function $z_0(\omega) \in \mathcal{V}$, there exists a control function u such that the pseudostate trajectory generated by the system remains in $\mathcal{V}$ and the output remains identically at zero.

The following result adapts Theorem 3.7 to the case of output nulling subspaces; its proof requires only minor modifications.

Theorem 5.2. *Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. The following statements are equivalent:*

(a) *$\mathcal{V}$ is output nulling;*

(b) $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \mathcal{V} \subseteq (\mathcal{V} \oplus 0_{\mathcal{Y}}) + \operatorname{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$;

(c) *There exists $F \in \mathbb{R}^{m\times n}$ such that $\left[\begin{smallmatrix} A+BF \\ C+DF \end{smallmatrix}\right] \mathcal{V} \subseteq \mathcal{V} \oplus 0_{\mathcal{Y}}$.*

In this case, we say that F is an *output nulling friend* of $\mathcal{V}$. We denote by $\mathfrak{F}(\mathcal{V})$ the set of output nulling friends of $\mathcal{V}$. Obviously, an output nulling subspace is also controlled invariant, and an output nulling friend of $\mathcal{V}$ is also a friend of $\mathcal{V}$ (the converse is true if $D = 0$).

Even though the concepts of controlled invariant and output nulling subspaces are conceptually different in the fractional setting, the geometric condition characterizing these subspaces in terms of the matrices A, B, C and D is analogous to the classical one, thanks to the equivalences that we have established in Theorems 5.2 and 3.7. This enables us to define the concept of supremal output nulling subspace as a finite-dimensional subspace of $\mathcal{X}$ despite the infinite-dimensional nature of the system. Therefore, the supremal output nulling subspace, usually denoted by $\mathcal{V}^\star$, can be computed in finite terms by using a sequence of finite-dimensional subspaces derived form matrices A, B, C and D, which converges to $\mathcal{V}^\star$ in at most $n-1$ steps, see e.g. [22, p. 162].

In order to study the assignability of the closed-loop spectrum with respect to $\mathcal{V}^\star$, we need to make a preliminary observation on the reachability of a diffusive fractional system.

The set of points that can be reached from the origin for a diffusive fractional system is the same as for any fractional system characterized by the same pair of matrices (A, B). In fact, the input-to-state transfer function is independent from the particular type of fractional system considered, and the reachability is characterized by the controllability matrix $[\; B \quad AB \quad \dots \quad A^{n-1}B \;]$, see [13,18,27,32–34]. We observed in [25] that the image of the controllability matrix is the smallest A-invariant subspace containing the column-space of B. Theorem 3.2 guarantees that the reachable subspace is the same also for diffusive fractional systems, which we denote by $\langle A \mid \operatorname{im} B\rangle$.

From **(b-c)** in Theorem 5.2, we notice that, as in the integer case, we can define the set of points which are reachable with pseudo-state trajectories lying on $\mathcal{V}^\star$, and this set turns out to be a subspace of $\mathcal{V}^\star$ that only depends on the matrices A, B, C and D. In line with the integer case, this subspace will be denoted by $\mathcal{R}^\star$ and referred to as the largest reachability subspace, and from the previous discussion, we can write $\mathcal{R}^\star = \langle A+B\,F \mid \mathcal{V}^\star \cap B\,(\ker D)\rangle$, where F is an output-nulling friend of $\mathcal{V}^\star$. It can be proved, as in the integer case, that the subspace $\mathcal{R}^\star$ is independent from the particular output-nulling friend F of $\mathcal{V}^\star$. Hence, the closed-loop spectrum of $A+B\,F$, where F is an output-nulling friend of $\mathcal{V}^\star$, can be divided into two multi-sets: the eigenvalues of the mapping $A+B\,F \mid \mathcal{V}^\star$ and the eigenvalues of $A+B\,F \mid \frac{\mathcal{X}}{\mathcal{V}^\star}$. In turn, the eigenvalues of $A+B\,F \mid \mathcal{V}^\star$ can be divided into two multi-sets: the eigenvalues of $A+B\,F \mid \mathcal{R}^\star$ are all freely assignable with a suitable choice of F, whereas the eigenvalues of $A+B\,F \mid \frac{\mathcal{V}^\star}{\mathcal{R}^\star}$ are independent from F, and are the invariant zeros of the quadruple (A,B,C,D). Likewise, the eigenvalues of $A+B\,F \mid \frac{\mathcal{V}^\star+\langle A\,|\,\mathrm{im}\,B\rangle}{\mathcal{V}^\star}$ are all freely assignable with a suitable choice of F, whereas the eigenvalues of $A+B\,F \mid \frac{\mathcal{X}}{\mathcal{V}^\star+\langle A\,|\,\mathrm{im}\,B\rangle}$ are fixed for all F, and belongs to the set of uncontrollable modes of the system.

6. Disturbance decoupling problem by pseudo-state feedback

With the result established on invariance and controlled invariance, the disturbance decoupling problem can now be solved for fractional systems using the distributed frequency model.

Consider the following fractional system

$$\begin{aligned} \frac{\partial z}{\partial t}(\omega,t) &= -\omega z(\omega,t) + A\,x(t) + B\,u(t) + H\,w(t) \\ x(t) &= \int_0^\infty \mu_\alpha(\omega) z(\omega,t)\mathrm{d}\omega \\ y(t) &= C\,x(t) + D\,u(t) + G\,w(t), \end{aligned} \tag{23}$$

where w is the disturbance and H and G are matrices that distribute the disturbance into the pseudo-state and the output, respectively. In the system above, we did not consider the initial condition for the sake of readability, because the disturbance decoupling problem is concerned with the input–output behavior of the system, and as such the initial state is irrelevant. The solution to the disturbance decoupling problem involves finding the conditions under which the output y can be decoupled from the disturbance. The solution to this problem is presented in the theorem below.

Theorem 6.1. *Given the system in (23), the output can be decoupled from the disturbance if and only if $G = 0$ and*

$$\mathrm{im}\,H \subseteq \mathcal{V}^\star.$$

Proof (*If*).. Consider the control $u(t) = F\,x(t)$, where F is an output-nulling friend of $\mathcal{V}^\star$. The closed loop system becomes

$$\begin{aligned} \frac{\partial z}{\partial t}(\omega,t) &= -\omega z(\omega,t) + (A+B\,F)\,x(t) + H\,w(t) \\ y(t) &= (C+D\,F)\,x(t). \end{aligned} \tag{24}$$

Since $(A+BF)\mathcal{V}^\star \subseteq \mathcal{V}^\star = \mathcal{V}^\star + \mathrm{im}\,H$, by Theorem 3.7, it follows that $x(t) \in \mathcal{V}^\star$ for all $t \geqslant 0$. Thus, $y(t) = 0$ by Theorem 5.2.

(Only if). Now suppose that the output can be decoupled from the disturbance. From Theorem 2.1, we find the closed-loop transfer function between the disturbance and the output

$$\big((C+DF)(s^\alpha I - A - BF)\big)^{-1} H + G,$$

from which we conclude immediately that G must be zero. Therefore, for the problem to be solvable, we must have $((C+DF)(s^\alpha I - A - BF))^{-1} H = 0$. Since the problem is solvable, the set of all points reachable from the disturbance input must be in the kernel of $C+DF$. From the transfer function being the same irrespectively of the definition adopted for fractional systems, it follows that such a set is also independent from the definition of fractional systems and can be computed as $\mathrm{im}[\;H \quad (A+BF)H \quad (A+BF)^2H \quad \dots \quad (A+BF)^nH\;]$, which is the smallest $(A+BF)$-invariant containing the image of H, in symbols, $\langle A+BF|\,\mathrm{im}\,H\rangle$, see [25, Section 4]. Therefore, under the assumption of solvability, we find $(C+DF)\langle A+BF|\,\mathrm{im}\,H\rangle = \{0\}$. Hence, $\langle A+BF|\,\mathrm{im}\,H\rangle \subseteq \ker(C+DF)$ and $\mathcal{V}^\star$ is the largest $(A+BF)$-invariant contained in $\ker(C+DF)$, because $\mathcal{V}^\star$ is the supremal output nulling subspace. Hence, it follows that $\langle A+BF|\,\mathrm{im}\,H\rangle \subseteq \mathcal{V}^\star$, which implies that $\mathrm{im}\,H \subseteq \mathcal{V}^\star$. ■

Remark 2. It is not difficult to generalize the statement of Theorem 6.1 to the case where the disturbance is measurable, and the control function can take the form $u(t) = F\,x(t) + S\,w(t)$. In this case, the assumption $G = 0$ is not necessary, and the solvability condition becomes

$$\mathrm{im}\begin{bmatrix} H \\ G \end{bmatrix} \subseteq (\mathcal{V}^* \oplus 0_{\mathcal{Y}}) + \mathrm{im}\begin{bmatrix} B \\ D \end{bmatrix}.$$

The implication of this theorem is that if the system given in (23) can be decoupled with some control function, then there is a pseudo-state feedback that does the same job.

This forms a nice parallel with the classical theory. In fact, despite the fractional system being infinite dimensional, the desired disturbance-to-output transfer function can be obtained through a pseudo-state feedback that does not require taking into account the entire information on the infinite dimensional state – the $z(\omega,t)$ function – of the system.

Remark 3. A natural extension of the disturbance decoupling problem by static state feedback is the one where we impose the requirement of closed-loop stability. The theory developed for the problem without stability extends straightforwardly to the case with stability by replacing $\mathcal{V}^\star$ with the largest stabilizability output-nulling subspace, denoted by $\mathcal{V}^\star_{g,\alpha}$, plus the obvious requirement of the stabilizability of the pair (A,B). This subspace can be obtained as the limit of the sequence

$$\begin{cases} \mathcal{V}_0 &= \mathcal{E} \\ \mathcal{V}_{i+1} &= \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]^{-1}\left((\mathcal{V}_i \oplus 0_y) + \operatorname{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right) \cap \mathcal{E}, \quad i \in \mathbb{N}\setminus\{0\}, \end{cases}$$

which is monotonically non-increasing and converges in at most $n-1$ steps.[6] In this recursion, $\mathcal{E}$ represents the stabilizable subspace of the system, i.e., the set of points in the pseudo-state space for which a pseudo-state feedback can stabilize the system. Note that this concept is well posed in the fractional setting because $\mathcal{E}$ is an A-invariant subspace, and therefore Theorem 3.2 guarantees that, if $x(t^*) \in \mathcal{E}$, then $z(\omega,t) \in \mathcal{E}$ for almost all $\omega \in \mathbb{R}_+$ and for all $t \le t^*$. We can easily compute $\mathcal{E}$ as

$$\mathcal{E} = \langle A \mid \operatorname{im} B\rangle + \mathcal{V}_{\mathrm{u}},$$

where the first term is the reachable subspace of the system, and $\mathcal{V}_{\mathrm{u}}$ is the span of all eigenvectors associated with stable uncontrollable eigenvalues, [35]. Specifically, $\mathcal{V}_{\mathrm{u}} = \operatorname{im}\left[\begin{array}{ccc} V_1 & \dots & V_q \end{array}\right]$ where $\left[\begin{array}{c} V_i \\ W_i \end{array}\right]$ is a basis for $\ker\left[\begin{array}{c|c} A - \lambda_i I & B \end{array}\right]$ for every stable uncontrollable eigenvalue λ_i, with $i = 1,\dots,q$, partitioned conformably. In a fractional setting, the stability domain is no longer the left-half complex plane, but the region of the complex plane defined by the condition $|\operatorname{Arg}\lambda| > \alpha\pi/2$, and therefore, in general, the solvability of a disturbance decoupling problem with stability for a given set of matrices depends on the fractional order α: this is the reason why we use the subscript α in the symbol $\mathcal{V}^\star_{g,\alpha}$. We also note that the sequence above can be conveniently used to compute $\mathcal{V}^\star$ by simply replacing $\mathcal{E}$ with the entire pseudo state-space $\mathcal{X}$, and $\mathcal{R}^\star$ by replacing $\mathcal{E}$ with $\langle A \mid \operatorname{im} B\rangle$.

Example 6.2. Consider the following matrices:

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -8 \\ 7 \end{bmatrix}, \quad H = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & -1 & -9 \end{bmatrix}, \quad D = 0, \quad \text{and} \quad G = 0.$$

Using the sequence (7.9) in [22, p. 162] we find

$$\mathcal{V}^\star = \operatorname{im}\begin{bmatrix} 1 & 0 \\ 0 & -9 \\ 0 & 1 \end{bmatrix}.$$

It follows that the disturbance decoupling problem is solvable, see Fig. 1, but the presence of two invariant zeros at $(63 \pm 3\sqrt{20679}\,i)/110 \approx 0.57 \pm 3.92\,i$, means that it is not solvable with closed-loop asymptotic stability for any value of α. In particular, if $\alpha = 1$ the closed-loop system, for example using the output-nulling friend

$$F = \begin{bmatrix} -\frac{54}{55} & -\frac{1}{22} & -\frac{27}{110} \end{bmatrix}$$

of $\mathcal{V}^\star$ that assigns the remaining closed-loop eigenvalue at -1.5, is unstable as an integer system. However, if the dynamics is ruled by a diffusive system with, for example $\alpha = 0.75$, this feedback matrix ensures that the closed-loop system is stable. In other words, $\mathcal{V}^\star \neq \mathcal{V}^\star_{g,1}$ but $\mathcal{V}^\star = \mathcal{V}^\star_{g,0.75}$. In both the cases, the pseudo-state evolves on a controlled invariant subspace and the output is identically zero, see Figs. 1–3.

7. Input containing subspaces

Analogous to how controlled invariance gives rise to the notion of output-nulling subspaces, conditioned invariant subspaces can be used to define the dual of output-nulling subspaces, which are called input-containing subspaces.

Definition 7.1. Let $\mathcal{S} \subseteq \mathcal{X}$ be a subspace. An autonomous oracle observer for the system (22) modulo $\mathcal{S}$ is a system governed by

$$\frac{\partial \hat{z}}{\partial t} = -\omega\,\hat{z}(\omega,t) + P\,\hat{x}(t) + R\,(y - \hat{y})$$

[6] In this sequence, $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]^{-1}\left((\mathcal{V}_i \oplus 0_y) + \operatorname{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right)$ denotes the inverse image of $(\mathcal{V}_i \oplus 0_y) + \operatorname{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ under the mapping $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$, which is defined even if $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ is not invertible.

$$\hat{x}(t) = \int_0^\infty \mu_\alpha(\omega)\hat{z}(\omega,t)\mathrm{d}t$$
$$\hat{y}(t) = C\,\hat{x}(t)$$

such that for any pair of initialization functions $(z_0(\omega), \hat{z}_0(\omega))$ and any control $u(t)$, we have $\hat{z}_0(\omega) = z_0(\omega)/\mathcal{S}$ implies $\hat{z}(\omega,t) = z(\omega,t)/\mathcal{S}$ for all $t \geqslant t_0$, where $a = b/\mathcal{S}$ means that $a - b \in \mathcal{S}$.

We note that the difference between an autonomous oracle observer and an oracle observer is that the former does not require knowledge/access to the input to provide the estimate of the pseudo-state in the quotient space $\mathcal{X}/\mathcal{S}$.

Definition 7.2. Consider the system in (22). A subspace $\mathcal{S}$ is said to be input-containing if there exists an autonomous oracle observer for (22) modulo $\mathcal{S}$.

The following result is the dual of Theorem 5.2, and is presented without proof for the sake of brevity.

Theorem 7.3. *Let $\mathcal{S}$ be a subspace of the pseudo-state space. Then the following statements are equivalent*

(a) *$\mathcal{S}$ is input-containing;*

(b) $\begin{bmatrix} A & B \end{bmatrix}\big((\mathcal{S} \oplus \mathcal{U}) \cap \ker\begin{bmatrix} C & D \end{bmatrix}\big) \subseteq \mathcal{S}$;

(c) *There exists a matrix G such that* $\left[\begin{smallmatrix} A+G\,C \\ B+G\,D \end{smallmatrix}\right] (\mathcal{S} \oplus \mathcal{U}) \subseteq \mathcal{S}$.

It is possible to define $\mathcal{S}^\star$ as the dual subspace of $\mathcal{V}^\star$, i.e., the smallest input-containing subspace of the system, which can be computed in finite terms using a sequence dual to the one for the computation of $\mathcal{V}^\star$, see [22].

8. Disturbance decoupling problem by dynamic output feedback

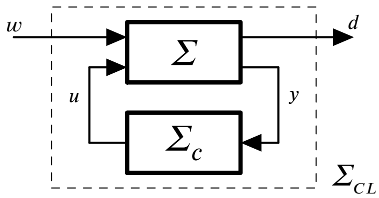
We consider the system Σ governed by

$$\Sigma : \begin{cases} \dfrac{\partial z}{\partial t}(\omega,t) &= -\omega z(\omega,t) + A\,x(t) + B\,u(t) + H\,w(t) \\ x(t) &= \displaystyle\int_0^\infty \mu_\alpha(\omega) z(\omega,t)\mathrm{d}\omega \\ y(t) &= C\,x(t) + D_y\,u(t) + G_y\,w(t) \\ d(t) &= E\,x(t) + D_z\,u(t) + G_z\,w(t), \end{cases} \tag{25}$$

where $w(t) \in \mathcal{W} = \mathbb{R}^q$ is the disturbance input, $y(t) \in \mathcal{Y} = \mathbb{R}^p$ is the measurement output and $d(t) \in \mathcal{D} = \mathbb{R}^r$ is the to-be-controlled output. We consider also the regulator Σ_C, with the same fractional order of the system, ruled by

$$\Sigma_C : \begin{cases} \dfrac{\partial \zeta}{\partial t}(\omega,t) &= -\omega\zeta(\omega,t) + A_c\,\xi(t) + B_c\,y(t) \\ \xi(t) &= \displaystyle\int_0^\infty \mu_\alpha(\omega)\zeta(\omega,t)\mathrm{d}\omega \\ u(t) &= C_c\,\xi(t) + D_c\,y(t), \end{cases} \tag{26}$$

where $\xi(t) \in \mathcal{X} = \mathbb{R}^n$ is the pseudo-state of the regulator. We want to control the system Σ with the regulator Σ_C such that in the closed-loop system the output d does not depend on the disturbance input w.

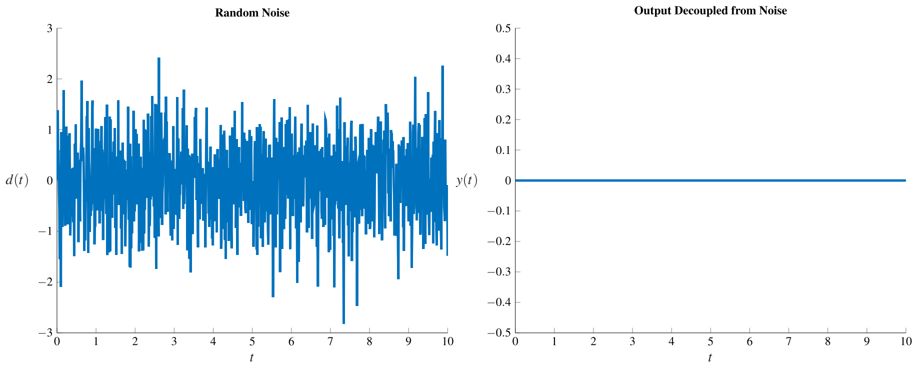


We say that the feedback interconnection of system Σ with the regulator Σ_C is *well posed* if the matrix $I - D_y\,D_c$ is non-singular, see [22, Chpt. 3]. In such case, the closed-loop system Σ_{CL} can be written in state-space form as

$$\Sigma_{CL} : \begin{cases} \dfrac{\partial \hat{z}}{\partial t}(\omega,t) &= -\omega\hat{z}(\omega,t) + \hat{A}\,\hat{x}(t) + \hat{H}\,w(t) \\ \hat{x}(t) &= \displaystyle\int_0^\infty \mu_\alpha(\omega)\hat{z}(\omega,t)\mathrm{d}\omega \\ d(t) &= \hat{C}\,\hat{x}(t) + \hat{G}\,w(t), \end{cases} \tag{27}$$

where $\hat{z}(\omega,t) = \left[\begin{smallmatrix} z(\omega,t) \\ \zeta(\omega,t) \end{smallmatrix}\right]$ is the extended initialization function, $\hat{x}(t) = \left[\begin{smallmatrix} x(t) \\ \xi(t) \end{smallmatrix}\right]$ is the extended pseudo-state, and the matrices in (27) are defined by

$$\hat{A} \overset{\text{def}}{=} \begin{bmatrix} A + B D_c W C & B\,C_c + B\,D_c\,W\,D_y\,C_c \\ B_c\,W\,C & A_c + B_c\,W\,D_y\,C_c \end{bmatrix}, \quad \hat{H} \overset{\text{def}}{=} \begin{bmatrix} H + B\,D_c\,W\,G_y \\ B_c\,W\,G_y \end{bmatrix},$$



(a) Random noise $w(t)$. (b) Output $y(t)$ is decoupled from the disturbance.

Fig. 1. White Gaussian noise and decoupled output with $\alpha = 1$.

$$\begin{aligned}\hat{C} &\stackrel{\text{def}}{=} \begin{bmatrix} E + D_z\, D_c\, W\, C & D_z\, C_c + D_z\, D_c\, W\, D_y\, C_c \end{bmatrix},\\ \hat{G} &\stackrel{\text{def}}{=} G_z + D_z\, D_c\, W\, G_y,\end{aligned} \tag{28}$$

where $W = (I - D_y\, D_c)^{-1}$. The transfer function of the closed-loop system Σ_{CL} is $G_{d,w}(s) = \hat{C}\,(s^\alpha\, I - \hat{A})^{-1}\,\hat{H} + \hat{G}$. The (full order) disturbance decoupling problem by dynamic output feedback is the problem of finding a compensator Σ_C for Σ, if it exists, such that the feedback interconnection of Σ with Σ_C is well posed and the transfer function matrix $G_{d,w}(s)$ of the closed-loop system Σ_{CL} is zero. Further, one can also introduce a stability requirement by constraining the eigenvalues of $\hat{A}$ to be stable, i.e., to satisfy the condition $|\operatorname{Arg}\lambda_i| > \alpha\,\pi/2$ for all eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ of $\hat{A}$. The equivalence between **(a)** and **(b-c)** in Theorem 5.2, and the dual equivalence between **(a)** and **(b-c)** in Theorem 7.3 allows us to express the solvability condition for this problem (without stability) as in the integer case in terms of the existence of a matrix $K \in \mathbb{R}^{m\times p}$ such that

(i) $\operatorname{im}\begin{bmatrix} H \\ G_z \end{bmatrix} \subseteq (\mathcal{V}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B \\ D_z \end{bmatrix}$;

(ii) $\ker\begin{bmatrix} E & G_z \end{bmatrix} \supseteq (\mathcal{S}^\star \oplus \mathcal{W}) \cap \ker\begin{bmatrix} C & G_y \end{bmatrix}$;

(iii) $\mathcal{S}^\star \subseteq \mathcal{V}^\star$;

(iv) $I + K\, D_y$ is non-singular, and K satisfies

$$\begin{bmatrix} A + B\,K\,C & H + B\,K\,G_y \\ E + D_z\,K\,C & G_z + D_z\,K\,G_y \end{bmatrix} (\mathcal{S}^\star \oplus \mathcal{W}) \subseteq \mathcal{V}^\star \oplus \{0\}, \tag{29}$$

where $\mathcal{V}^\star$ is associated with the quadruple (A, B, E, D_z) and $\mathcal{S}^\star$ with the quadruple (A, H, C, G_y), see [35]. Notice that **(i)-(iii)** guarantee the existence of K such that (29) holds. However, to solve the problem we need that at least one of the possibly infinite matrices satisfying (29) to render $I + K\,D_y$ non-singular.

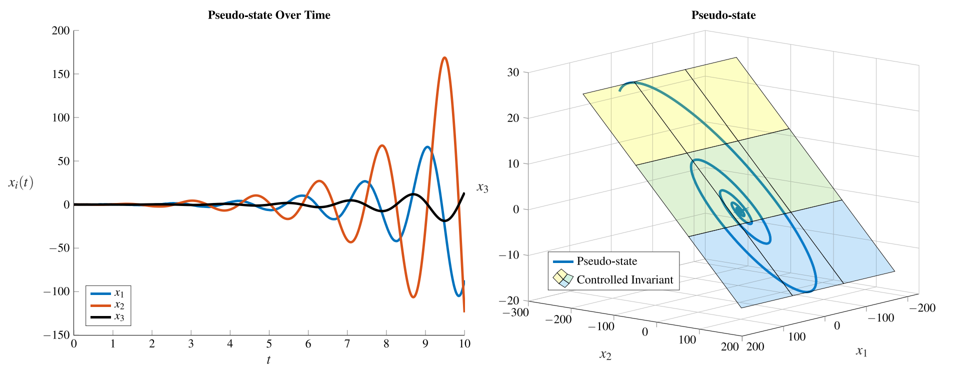
For the problem with stability, the conditions above are still necessary and sufficient for the solvability by replacing $\mathcal{V}^\star$ with $\mathcal{V}^\star_{g,\alpha}$ and $\mathcal{S}^\star$ with $\mathcal{S}^\star_{g,\alpha}$, plus the obvious requirement that (A, B) be stabilizable and (C, A) be detectable. The subspace $\mathcal{S}^\star_{g,\alpha}$ can be defined in a dual way as the smallest detectability subspace, i.e., the smallest externally stabilizable input-containing subspace (A, H, C, G_y).

Example 8.1. Consider the following matrices:

$$A = \begin{bmatrix} -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 10 & 0 & 0 \\ 0 & 13 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

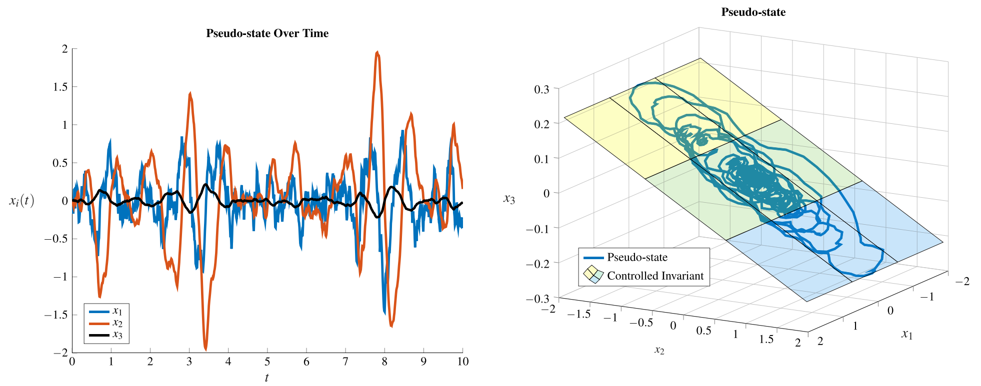
$$C = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -14 & 0 & 0 & 1 \end{bmatrix}, \quad D_y = \begin{bmatrix} -11 & 0 & 0 \\ -1 & 13 & -5 \\ 0 & 0 & -1 \end{bmatrix}, \quad G_y = \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & -20 \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}, \quad \text{and} \quad G_z = 1.$$



(a) Pseudo-state trajectory over time. (b) Pseudo-state trajectory on a controlled invariant.

Fig. 2. Evolution of the pseudo-state with $\alpha = 1$.



(a) Pseudo-state trajectory over time. (b) Pseudo-state trajectory on a controlled invariant.

Fig. 3. Evolution of the pseudo-state with $\alpha = 3/4$.

In this case, the subspaces $\mathcal{V}^\star$ of the quadruple (A, B, E, D_z) and $\mathcal{S}^\star$ with the quadruple (A, H, C, G_y) are, respectively $\mathbb{R}^4$ and the origin, and a matrix K that satisfies **(iv)** is

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -6 & 0 \end{bmatrix}.$$

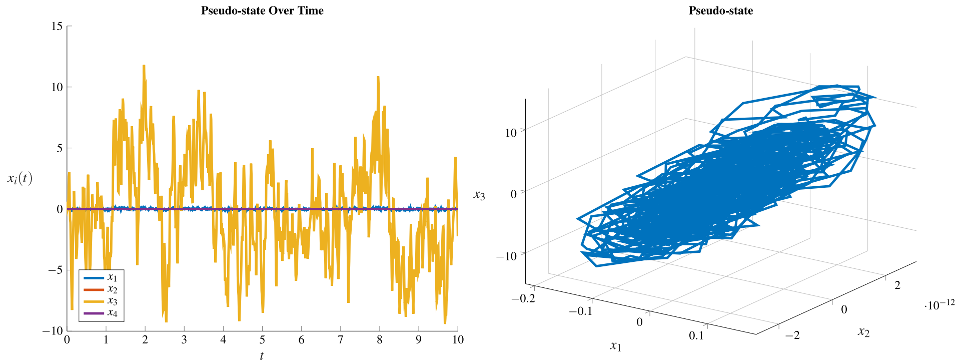
The quadruples (A, B, E, D_z) and (A, H, C, G_y) have invariant zeros at -20 and -1, respectively. The matrices F and G that assign the closed-loop eigenvalues -1.5, -2.5 and -3.5 are given by

$$F = \begin{bmatrix} 0 & -0.35 & 0 & 0 \\ 27.0259 & 0 & -0.002 & -1.5205 \\ 0 & 0 & 0 & -20 \end{bmatrix}, \quad G = \begin{bmatrix} -0.9330 & 3.6652 & -1.8661 \\ 1.8125 & -9.0625 & 0.125 \\ 0 & 0 & 0 \\ 1.6875 & -9.4375 & -0.125 \end{bmatrix},$$

so that the closed-loop spectrum contains the assigned eigenvalues with double multiplicity (one for $A + BF$ and one for $A + GC$) plus the invariant zeros, and it is therefore stable. Given the same noise shown in Fig. 1(a), the output is also identically zero and the state evolves on a controlled invariant, see Fig. 4.

Concluding remarks

In this paper we have developed a geometric control theory for fractional systems realized using a diffusive representation, which does not rely on a definition of fractional derivative. Instead, the diffusive representation uses a partial differential equation to cater for the infinite dimensional nature of fractional systems, thereby guaranteeing the semi group property. First, we have developed the concepts of invariance, controlled-invariance and conditioned-invariance. Then we have used such structural invariants to address



(a) Pseudo-state trajectory over time.

(b) Pseudo-state trajectory on a controlled invariant.

Fig. 4. Evolution of the pseudo-state with $\alpha = 1/2$.

the disturbance decoupling problem, by static-state feedback, and by dynamic output feedback, which are the fundamental problems in control theory. Interestingly, the conditions under which we can maintain the state-trajectory of a diffusive fractional system onto a controlled invariant (and the dual problem, for conditioned invariants) have the same structure as those for classical finite dimensional linear systems. In other words, differently from Caputo-type fractional systems studied in [25], using a control other than a feedback on the pseudo-state (an output injection, respectively) does not provide any benefit with the diffusive representation, which is surprising as the pseudo-state contains only a very limited part of the information contained in the state of a fractional system. We stress that these results have been possible thanks to a novel geometric theory that we developed in the framework of diffusive fractional system. To the best of our knowledge, this kind of theory was entirely missing in the literature.

The classic solvability conditions for the disturbance decoupling problem (see Theorem 6.1 and **(i)-(iv)** in Section 8) are, understandably, very stringent, because the task of decoupling the system output from a completely unknown deterministic disturbance is extremely ambitious. As for standard LTI systems, whenever the structural solvability conditions are not satisfied, one can relax the problem to that of finding the controller which minimizes some norm of the transfer function between the disturbance and the output. In particular, for standard LTI systems, if this minimization involves the H_2-norm, one can still use the classic geometric solution based on output-nulling subspaces, applied to the Hamiltonian system. The counterpart of this problem for distributed frequency models is currently under investigation by the authors.

CRediT authorship contribution statement

Vishnuram Arumugam: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Augusto Ferrante:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization. **Lorenzo Ntogramatzidis:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Fabrizio Padula:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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