Arch. Math. 118 (2022), 563–576

0003-889X/22/060563-14
published online April 22, 2022
https://doi.org/10.1007/s00013-022-01741-1

Archiv der Mathematik

On the connectivity of the non-generating graph

Andrea Lucchini and Daniele Nemmi

Abstract. Given a 2-generated finite group G, the non-generating graph of G has as vertices the elements of G and two vertices are adjacent if and only if they are distinct and do not generate G. We consider the graph $\Sigma(G)$ obtained from the non-generating graph of G by deleting the universal vertices. We prove that if the derived subgroup of G is not nilpotent, then this graph is connected, with diameter at most 5. Moreover, we give a complete classification of the finite groups G such that $\Sigma(G)$ is disconnected.

Mathematics Subject Classification. 20D60.

Keywords. Non-generating graph, Diameter, Connectivity.

1. Introduction. Let G be a finite group. The generating graph of G, written $\Gamma(G)$, is the graph in which the vertices are the elements of G and there is an edge between g_1 and g_2 if G is generated by g_1 and g_2. If G is not 2-generated, then there will be no edge in this graph. Thus, it is natural to assume that G is 2-generated. Quite a lot is known about this graph when G is a non-Abelian simple group; for example, Guralnick and Kantor [9] showed that there is no isolated vertex in $\Gamma(G)$ but the identity, and Breuer, Guralnick, Kantor [2] showed that the diameter of the subgraph of $\Gamma(G)$ induced by non-identity elements is 2 for all G. If G is an arbitrary finite group, then $\Gamma(G)$ could contain many isolated vertices. Let $\Delta(G)$ be the subgraph of $\Gamma(G)$ that is induced by all the vertices that are not isolated. In [5] and [11], it is proved that if G is a 2-generated soluble group, then $\Delta(G)$ is connected and $\mathrm{diam}(\Delta(G)) \leq 3$. The situation is different if the solubility assumption is dropped. It is an open problem whether or not $\Delta(G)$ is connected, but even when $\Delta(G)$ is connected, its diameter can be arbitrarily large. For example, if G is the largest 2-generated direct power of $\mathrm{SL}(2, 2^p)$ and p is a sufficiently large odd prime, then $\Delta(G)$ is connected but $\mathrm{diam}(\Delta(G)) \geq 2^{p-2} - 1$ (see [6, Theorem 5.4]).

The aim of this paper is to investigate the connectivity of the complement graph of $\Delta(G)$, denoted by $\Sigma(G)$. This graph can be described as follows: we take the non-generating graph of G, i.e. the graph whose vertices are the elements of G and where there is an edge between g_1 and g_2 if $\langle g_1, g_2\rangle \neq G$ and we remove the universal vertices (corresponding to the isolated vertices of the generating graph). We prove that $\Sigma(G)$ is connected, except for some families that can be completely described. In any case, if $\Sigma(G)$ is disconnected, then G is soluble and its derived subgroup is nilpotent.

Theorem 1. *Let G be a 2-generated finite group. Then $\Sigma(G)$ is connected if and only if none of the following occurs:*

(1) G is cyclic;
(2) G is a p-group;
(3) $G/\operatorname{Frat}(G) \cong (V_1 \times \cdots \times V_t) \rtimes H$, where $H \cong C_p$ for some prime p, and $V_1, \ldots, V_t$ are pairwise non-H-isomorphic non-trivial irreducible H-modules.
(4) $G/\operatorname{Frat}(G) \cong (V_1 \times \cdots \times V_t) \rtimes H$, where $H \cong C_p \times C_p$ for some prime p, $V_1, \ldots, V_t$ are pairwise non-H-isomorphic non-trivial irreducible H-modules and $C_H(V_1 \times \cdots \times V_t) \cong C_p$.

Moreover, if $\Sigma(G)$ is connected, then $\operatorname{diam}(\Sigma(G)) \leq 5$, *and* $\operatorname{diam}(\Sigma(G)) \leq 3$ *under the additional assumption that G is soluble.*

We do not know whether the bound on the diameter of $\Sigma(G)$ is the best possible. In any case, if B is the Baby Monster, then $\operatorname{diam}(\Sigma(\mathrm{B})) \geq 4$ (see the end of Section 3). On the other hand, for soluble groups, the bound $\operatorname{diam}(\Sigma(G)) \leq 3$ is the best possible. Consider for example $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, with $|a| = |b| = 2$ and $|c| = 3$. In this case, $1, c, c^2$ are the isolated vertices of the generating graph $\Gamma(G)$. Moreover, if g is adjacent to ac in $\Sigma(G)$, then $g \in \langle ac\rangle$, hence $g \in \{a, ac^2\}$. Similarly b and bc^2 are the only vertices of $\Sigma(G)$ that are adjacent to bc. Hence a shortest path in $\Sigma(G)$ between ac and bc is (ac, a, b, bc).

When $\Sigma(G)$ is disconnected, it is possible that it contains some isolated vertices. However this occurs only in few particular cases.

Proposition 2. *Let G be a 2-generated finite group. Then $\Sigma(G)$ has an isolated vertex if and only if*

(1) G is cyclic;
(2) $G \cong C_2 \times C_2$;
(3) $G \cong D_p$ is a dihedral group with $2p$ elements for p an odd prime.

The structure of the paper is as follows. In Section 2, we study $\Sigma(G)$ in the particular case when G is a primitive soluble group. In Section 3, we investigate $\Sigma(G)$ when G is a monolithic group whose socle is non-Abelian. Thanks to the fact that if N is a proper normal subgroup of G and $\Sigma(G/N)$ is connected, then $\Sigma(G)$ is also connected (see Proposition 15), using the results from Sections 2 and 3, we prove in Section 4 that if $\Sigma(G)$ is disconnected, then the derived subgroup of G is nilpotent. The case when the derived subgroup is nilpotent is analysed in Section 5. In the last section, we combine the partial results obtained in the previous sections to complete the proof of Theorem 1.

Although it is elementary and independent from the proof of Theorem 1, the proof of Proposition 2 is given at the end of Section 6.

2. Primitive soluble groups.

Definition 3. Let G be a finite group. We denote by $V(G)$ the subset of G consisting of the elements x with the property that $G = \langle x, y\rangle$ for some y.

Lemma 4 ([14, Proposition 2.2]). *Let G be a primitive soluble group. Let $N = \mathrm{soc}(G)$ and let H be a core-free maximal subgroup of G. Given $1 \neq h \in H$ and $n \in N$, $hn \in V(G)$ if and only if $h \in V(H)$. In particular, $V(H)\setminus\{1\} \subseteq V(G)$.*

Proposition 5. *Let G be a 2-generated primitive soluble group. Then $\Sigma(G)$ is disconnected if and only if either $G \cong C_p$ or $G/\mathrm{soc}(G) \cong C_p$. Moreover, if G is connected, then $\mathrm{diam}(\Sigma(G)) \leq 3$.*

Proof. If G is nilpotent, then $G \cong C_p$. In this case, $\Sigma(G)$ is a disconnected graph with p vertices and no edges. So we may assume that G is not nilpotent. Thus $G \cong V \rtimes H$, where $V = \mathrm{soc}(G)$ is a faithful irreducible H-module.

If $|H| = p$, with p a prime, then $\langle g, v\rangle = G$ for any $1 \neq v \in V$ and $g \notin V$. In particular, $V \setminus \{1\}$ is a connected component of $\Sigma(G)$.

So we may assume that $|H|$ is not a prime. Suppose that $g_1 = v_1h_1, g_2 = v_2h_2$ are two different elements of $V(G)$, with $v_1, v_2 \in V, h_1, h_2 \in H$. By Lemma 4, for $i \in \{1, 2\}$, either $h_i = 1$ (and therefore H is cyclic) or $h_i \in V(G)$. If neither h_1 nor h_2 is a generator of H, then $(v_1h_1, h_1, h_2, v_2h_2)$ is a path in $\Sigma(G)$. So we may assume $H = \langle h_1\rangle$. In this case, h_1 and g_1 are conjugate in G, so it is not restrictive to assume $v_1 = 1$. Since $|H|$ is not a prime, we can choose $1 \neq h \in H$ with $|h| < |H|$. Since all the complements of V in G are conjugate, there exists $v \in V$ such that $\langle g_2\rangle \leq \langle h_1^v\rangle$. But then (h_1, h, h^v, g_2) is a path in $\Sigma(G)$. □

3. Monolithic groups with non-Abelian socle. Let G be a 2-generated finite monolithic group, with a non-Abelian socle. The aim of this section is to prove that the graph $\Sigma(G)$ is connected, with diameter at most 5.

Assume $A = \mathrm{soc}(G) \cong S^n$, with S a finite non-Abelian simple group and $n \in \mathbb{N}$. We may identify G with a subgroup of $\mathrm{Aut}(S^n) = \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$, the wreath product of $\mathrm{Aut}(S)$ with the symmetric group of degree n. So the elements of G are of the kind $g = (\alpha_1, \dots, \alpha_n)\sigma$, with $\alpha_i \in \mathrm{Aut}(S)$ and $\sigma \in \mathrm{Sym}(n)$. For this section, we will refer to this identification and we will denote by π the homomorphism $\pi : \mathrm{Aut}(S)\wr\mathrm{Sym}(n) \to \mathrm{Sym}(n)$ mapping $(\alpha_1, \dots, \alpha_n)\sigma$ to σ.

We begin with two lemmas concerning some properties of $\mathrm{Aut}(S)$.

Lemma 6. *Let S be a finite non-Abelian simple group. There exist a subgroup H of $\mathrm{Aut}(S)$ and a prime divisor r of the order of S with the following properties:*

(1) $H \cap S < S$;
(2) $HS = \mathrm{Aut}(S)$;
(3) for every $h \in H$, we can find an element $s \in S\cap H$ such that $|h|_r \neq |hs|_r$.

Proof. First suppose that S is an alternating group of degree n (with $n \neq 6$) or a sporadic simple group. We claim that in this case we can take $r = 2$ and $H \in \mathrm{Syl}_2(\mathrm{Aut}(S))$. This is equivalent to saying that any coset $(S \cap H)h$ of $S \cap H$ in H contains x and y with $|x|_2 \neq |y|_2$. If $h \in S$, then it suffices to take $x = 1$ and y a non-trivial element of $H \cap S$. If $h \notin S$, then, since $|\mathrm{Aut}(S) : S| \leq 2$, we must have $|H : H \cap S| = |\mathrm{Aut}(S) : S| = 2$ and $(H \cap S)h = H \backslash H \cap S$. It is known that in this case $\mathrm{Aut}(S) \setminus S$ (and consequently $H \setminus H \cap S$) contains both an involution and an element of order 4 (see, for example, [1, Theorem 2]).

Now assume that S is a simple group of Lie type, defined over a field of characteristic p. The proof of the Lemma in [13, Section 2] implies that, except when $S = \mathrm{PSL}(2, q)$ and q is odd, we can take $r = p$ and $H = N_{\mathrm{Aut}\,(S)}(P)$ for $P \in \mathrm{Syl}_p(S)$. We remain with the case when $S = \mathrm{PSL}(2, q)$ and q is odd. In this case, let $r = 2$, $P \in \mathrm{Syl}\,_2(S)$, and $H = N_{\mathrm{Aut}\,(S)}(P)$. We may choose P so that the Frobenius automorphism σ belongs to H. Let $h \in H$. Up to multiplying with a suitable element of $S \cap H$, we may assume $h = y\sigma_1\sigma_2$, with $y \in H$, $|y| = 2$, $\sigma_1, \sigma_2 \in \langle \sigma \rangle$, $|\sigma_1|$ a 2-power, $|\sigma_2|$ odd, and $[y, \sigma] = 1$. Let $\tilde{q}$ be the size of the subfield of $GF(q)$ centralized by σ_2. By [1, Theorem 2], there exists $t \in \mathrm{PSL}(2, \tilde{q})\langle y\sigma_1 \rangle \cap H$ such that $|ty\sigma_1|_2 > |y\sigma_1|_2$. We have $|th|_2 > |h|_2$. □

Lemma 7. *Let S and H be as in the previous lemma. For any $h_1, h_2 \in H$, there exist $s \in S$, $t \in H \cap S$ such that $\langle h_1 s, h_2 t \rangle = \langle h_1, h_2 \rangle S$.*

Proof. It follows from Lemma 6 (3) that there exists $t \in H \cap S$ such that $\langle h_2 t \rangle \cap S \neq 1$. Indeed, if $\langle h_2 \rangle \cap S \neq 1$, then we can take $t = 1$. Otherwise take $t \in H \cap S$ such that $|h_2|_r \neq |h_2 t|_r$; then $\langle h_2 t \rangle \cap S$ contains a non-trivial r-element. Now let $1 \neq u \in \langle h_2 t \rangle \cap S$. By [3, Theorem 1], there exists $s \in S$ such that $\langle h_1 s, u \rangle = \langle h_1, S \rangle$. This implies $\langle h_1 s, h_2 t \rangle = \langle h_1, h_2 \rangle S$. □

In the next two lemmas, let $\tilde{H} = \{(\alpha_1, \ldots, \alpha_n)\sigma \in G \mid \alpha_1, \ldots, \alpha_n \in H\}$, where H is the subgroup of $\mathrm{Aut}(S)$ introduced in the statement of Lemma 6. Clearly, since $H \cap S < S$, $\tilde{H}$ is a proper subgroup of G. Moreover, in the proof of the next lemma, we will use a quasi-ordering relation on the set of the cyclic permutations which belong to the group $\mathrm{Sym}(n)$, whose definition depends on the choice of the prime r appearing in the statement of Lemma 6. Let $\sigma_1, \sigma_2 \in \mathrm{Sym}(n)$ be two cyclic permutations (including cycles of length 1); we define $\sigma_1 \leq \sigma_2$ if either $|\sigma_1|_r < |\sigma_2|_r$ or $|\sigma_1|_r = |\sigma_2|_r$ and $|\sigma_1| \leq |\sigma_2|$.

Lemma 8. *Suppose $\langle g_1, g_2 \rangle A = G$ and that one of the following holds:*

(1) g_1^π has a fixed point;
(2) g_2^π has a fixed point;
(3) $(g_1 g_2^i)^\pi$ is fixed-point-free for every $i \in \mathbb{Z}$.

Then there exist u_1, u_2 in A such that $\langle u_1 g_1, u_2 g_2 \rangle = G$ and $u_1 g_1 \in \tilde{H}$.

Proof. Since $SH = \mathrm{Aut}(S)$, it is not restrictive to assume $g_1, g_2 \in \tilde{H}$. By [13, Theorem 1.1] and its proof, there exist $u_1, u_2 \in A$ such that $\langle u_1 g_1, u_2 g_2 \rangle = G$. With a careful revision of the proof, it can be shown that u_1, u_2 can be chosen with the additional property that $u_1 \in A \cap H^n$. The proof of [13, Theorem 1.1] is constructive, but unfortunately long (7 pages) and intricate. Our aim here is not to repeat all the proof, but only to give a short sketch and indicate which parts of the proof require some adjustment in order to achieve the stronger formulation.

First we fix our notation. Since they are quite technical and it could be difficult to understand immediately their meaning, at any step we will indicate how our definitions work in the particular case when g_1^π has a fixed point.

We assume $g_1 = (\alpha_1, \dots, \alpha_n)\rho$, $g_2 = (\beta_1, \dots, \beta_n)\sigma$, with $\alpha_i, \beta_j \in \mathrm{Aut}(S)$ and $\rho, \sigma \in \mathrm{Sym}(n)$. Then we write $\rho = \rho_1 \cdots \rho_{s(\rho)}$ as products of disjoint cycles (including possibly cycles of length 1) in such a way that

$$\rho_1 \leq \cdots \leq \rho_{s(\rho)}.$$

For $1 \leq i \leq s(\rho)$, let $\rho_i = (m_{i,1}, \dots, m_{i,|\rho_i|})$ and set $m = m_{1,1}$. Notice that in the particular case when $g_1^\pi = \rho$ has at least one fixed point, m is one of these fixed points (i.e. $\rho_1 = (m)$).

The permutation σ has an orbit (possibly of length 1) containing m and acts on this orbit as a cycle $(m, n_2, \dots, n_t)$.

For a given pair $v_1 = (x_1, \dots, x_n)$, $v_2 = (y_1, \dots, y_n)$ of elements of $A = S^n$, we define

$$\bar{\alpha}_r = x_r\alpha_r, \quad \bar{\beta}_r = y_r\beta_r, \quad 1 \leq r \leq n,$$
$$a_i = \bar{\alpha}_{m_{i,1}} \cdots \bar{\alpha}_{m_{i,|\rho_i|}}, \; 1 \leq i \leq s(\rho).$$

Moreover, let $a = \alpha_m\bar{\alpha}_{m_{1,2}} \cdots \bar{\alpha}_{m_{1,|\rho_1|}}$, $b = \beta_m\bar{\beta}_{n_2} \cdots \bar{\beta}_{n_t}$ and consider $K = \langle a, b, S\rangle$. In the particular case when m is a fixed point of ρ, we just have $a = \alpha_m$.

Now we say that the $2n$-tuple $(x_1, \dots, x_n, y_1, \dots, y_n) \in S^{2n}$ is *good* if the following two conditions are satisfied:

(1) $\langle a_1, b_1\rangle = \langle x_m a, y_m b\rangle = K$.
(2) If $2 \leq i \leq s(\rho)$, then $a_i^{|\rho_1\cdots\rho_i|/|\rho_i|}$ is not conjugate to $a_1^{|\rho_1\cdots\rho_i|/|\rho_1|}$ in $\mathrm{Aut}(S)$.

The reason for this definition is the following. The first part of the proof of [13, Theorem 1.1] (ending with the last paragraph of page 177) shows that conditions (1) and (2) ensure that there exists a partition Φ of $\{1, \dots, n\}$ invariant for the action of $\langle \rho, \sigma\rangle$ such that $\langle v_1g_1, v_2g_2\rangle \cap S^n = \prod_{B\in\Phi} D_B$, where, for every block $B \in \Phi$, D_B is a full diagonal subgroup of $\prod_{j\in B} S_j$. Moreover, if B is the block of Φ containing m, then $B \subseteq \mathrm{supp}(\rho_1)$. In particular, if m is fixed by ρ, then $B = \{m\}$, hence $A = S^n \leq \langle v_1g_1, v_2g_2\rangle$ and therefore $G = \langle v_1g_1, v_2g_2\rangle$.

We claim that we can find a good $2n$-tuple with the additional property that $x_1, \dots, x_n \in H$. To construct such a $2n$-tuple, we start by choosing arbitrarily $x_1, \dots, x_n, y_1, \dots, y_n \in S \cap H$, and then we modify the elements x_m, y_m and the elements x_i for $i \in \{m_{i,1} \mid 2 \leq i \leq s(\rho)\}$. This can be done in two consecutive steps.

(i) First, by Lemma 7, we can find $x \in H \cap S$ and $y \in S$ such that $\langle xa, yb\rangle = K$. We substitute the original x_m, y_m by xx_m and yy_m.

(ii) Then let $i \in \{2, \dots, s(\rho)\}$. Since $\rho_1 \leq \cdots \leq \rho_i$, $|\rho_1 \cdots \rho_i|/|\rho_i|$ is coprime with r and therefore, by Lemma 6, there exists $s_i \in S \cap H$ such that $(s_i a_i)^{|\rho_1 \cdots \rho_i|/|\rho_i|}$ is not conjugate to $a_1^{|\rho_1 \cdots \rho_i|/|\rho_1|}$ in $\mathrm{Aut}(S)$. We substitute $x_{m_{i,1}}$ by $s_i x_{m_{i,1}}$.

Our first conclusion is that if ρ has at least a fixed point and $(x_1, \dots, x_n.y_1, \dots, y_n)$ is a good $2n$-tuple with the additional property ensured by the previous paragraph, then $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ satisfy the request of our statement.

The case when σ has a fixed point can be dealt with a similar argument, working with the ordered pair (g_2, g_1) instead of (g_1, g_2). However, if we repeat precisely the same procedure as above, we obtain $v_1, v_2 \in A$ such that $\langle v_1 g_2, v_2 g_1\rangle = G$ and $v_1 g_2 \in \tilde{H}$, while our request is to have $v_2 g_1 \in \tilde{H}$. On the other hand, when in step (i) we apply Lemma 7, we are free to choose x, y either with the property that $x \in H \cap S$ and $y \in S$ or with the property that $x \in S$ and $y \in H \cap S$, and the second choice solves the problem.

We remain with the case when $\rho\sigma^i$ is fixed-point-free for every $i \in \mathbb{Z}$. As we have seen before, we can find $v_1 = (x_1, \dots, x_n)$, $v_2 = (y_1, \dots, y_n) \in S^n$ with the properties that $(x_1, \dots, x_n, y_1, \dots, y_n)$ is a good $2n$-tuple with $x_1, \dots, x_n \in H \cap S$. This implies that $\langle v_1 g_1, v_2 g_2\rangle \cap S^n = \prod_{B \in \Phi} D_B$ for a suitable partition Φ of $\{1, \dots, n\}$. Clearly this is not enough to reach our conclusion and indeed this is the case for which the proof of [13, Theorem 1.1] requires more work. The crucial observation is that the conditions that make $(x_1, \dots, x_n, y_1, \dots, y_n)$ a good $2n$-tuple involve only the elements $x_1, \dots, x_n$ and the value of the product $y_m \beta_m y_{n_2} \beta_{n_2} \cdots y_{n_t} \beta_{n_t}$. This leaves a large freedom in the choice of elements $y_1, \dots, y_n$. Indeed, the second part of the proof of [13, Theorem 1.1], from page 177 to page 181, shows that we may choose $y_1, \dots, y_n$ so that $\prod_{B \in \Phi} D_B$ is normalized by $\langle v_1 g_1, v_2 g_2\rangle$ only if $\Phi = \{\{1\}, \dots, \{n\}\}$. With this choice, $\langle v_1 g_1, v_2 g_2\rangle = G$, as required. □

Corollary 9. *Let g be a vertex of $\Sigma(G)$. If $G/\mathrm{soc}(G)$ is not cyclic, then there exists a vertex $\tilde{g} \in \Sigma(G)$ such that $\tilde{g} \in \tilde{H}$ and the distance between g and $\tilde{g}$ in $\Sigma(G)$ is at most 2.*

Proof. Since $g \in \Sigma(G)$, there exists $g_2 \in G$ such that $\langle g, g_2\rangle = G$. If g and g_2 satisfy one of the three conditions in the statement of Lemma 8, then there exists u, u_2 in $\mathrm{soc}(G) = A$ such that $\langle ug, u_2 g_2\rangle = G$ and $\tilde{g} = ug \in \tilde{H}$. In particular, $\tilde{g} \in \Sigma(G)$ and $\langle g, \tilde{g}\rangle \leq \langle g\rangle A < G$ since we are assuming that G/A is not cyclic, and therefore g and $\tilde{g}$ are adjacent vertices of $\Sigma(G)$. Now assume that g^π and g_2^π are fixed-point-free but there exists $i \in \mathbb{Z}$ such that $g_3 = gg_2^i$ has a fixed point. If $\langle g, g_3\rangle = G$, then we repeat the previous argument using g_3 instead of g_2 and we find an element $\tilde{g}$ with the required properties. Suppose $\langle g, g_3\rangle \neq G$. In any case, $\langle g_2, g_3\rangle = \langle g_2, g\rangle = G$, so g_3 is a vertex of $\Sigma(G)$ which is adjacent to g. Moreover, we may apply Lemma 8 to the generating pair (g_3, g_2) in order to find $\tilde{g}_3 \in \tilde{H}$ which is adjacent in $\Sigma(G)$ to g_3. So $(g, g_3, \tilde{g}_3)$ is a path in $\Sigma(G)$ and we may take $\tilde{g} = \tilde{g}_3$. □

Proposition 10. *The graph* $\Sigma(G)$ *is connected, with diameter at most 5.*

Proof. We distinguish two cases:
a) $G/\operatorname{soc}(G)$ is not cyclic. Suppose that g_1, g_2 are two different vertices of $\Sigma(G)$. Choose $\tilde{g}_1, \tilde{g}_2$ as in the statement of Corollary 9. Since $\langle \tilde{g}_1, \tilde{g}_2\rangle \leq \tilde{H} < G$, $\tilde{g}_1, \tilde{g}_2$ are adjacent vertices of $\Sigma(G)$. So the distance in $\Sigma(G)$ between g_1 and g_2 is at most 5.
b) $G/\operatorname{soc}(G)$ is cyclic. Recall that the intersection graph $\mathrm{I}(G)$ of G is the graph whose vertices are the non-trivial proper subgroups of G and in which two vertices H and K are adjacent if and only if $H \cap K \neq 1$. By [3, Theorem 1], when $G/\operatorname{soc}(G)$ is cyclic, the vertex set $V(G)$ of $\Sigma(G)$ coincides with the set of the non-trivial elements of G. In particular, as explained in [4, Section 12], $(\Sigma(G), \mathrm{I}(G))$ is a dual pair of graphs, and therefore there is a natural bijection between connected components of $\Sigma(G)$ and connected components of $\mathrm{I}(G)$ with the property that corresponding components have diameters which are either equal or differ by 1. If G is neither soluble nor simple, then $\mathrm{I}(G)$ is connected with $\operatorname{diam}(\mathrm{I}(G)) \leq 4$ (see [16, Lemma 5]) hence $\operatorname{diam}(\Sigma(G)) \leq 5$. Freedman recently proved that also when G is a finite non-Abelian simple group, the graph $\Sigma(G)$ is connected with diameter at most 5 (see the remark after [4, Proposition 12]). □

We do not know whether the bound in the previous proposition is the best possible. However, Freedman [7] proved that $\operatorname{diam}(\mathrm{I}(G)) \leq 5$ for any finite non-Abelian simple group G and that the upper bound is attained only by the Baby Monster B and some unitary groups. In particular, since $(\mathrm{I}(\mathrm{B}), \Sigma(\mathrm{B}))$ is a dual pair of graphs, it follows that $\operatorname{diam}(\Sigma(\mathrm{B})) \geq 4$.

4. A reduction to the case when the derived subgroup is nilpotent. The main result of this section is the following proposition, which says in particular that if $\Sigma(G)$ is disconnected, then G is soluble and the derived subgroup of G is nilpotent.

Proposition 11. *Let G be a 2-generated group. If $\Sigma(G)$ is disconnected, then G is soluble. Moreover if G is soluble, then at least one of the following occurs.*

(1) $\Sigma(G)$ *is connected and* $\operatorname{diam}(\Sigma(G)) \leq 3$;
(2) The derived subgroup of G is nilpotent and $G/\operatorname{Frat}(G)$ has the following structure:

$$G/\operatorname{Frat}(G) \cong (V_1 \times \cdots \times V_t) \rtimes H,$$

where H is Abelian and $V_1, \dots, V_t$ are pairwise non-H-isomorphic non-trivial irreducible H-modules (including the possibility $t = 0$).

The proof requires some preliminary easy results. First we need to notice that $\Sigma(G)$ is disconnected if G is cyclic or has prime power order.

Lemma 12. *If G is a non-trivial cyclic group, then $\Sigma(G)$ is disconnected.*

Proof. If $\langle g\rangle = G$, then g is an isolated vertex of $\Sigma(G)$. □

Lemma 13. *If G is a 2-generated finite p-group, then $\Sigma(G)$ is disconnected.*

Proof. By the previous lemma, we may assume that G is not cyclic. Let $F = \mathrm{Frat}(G)$. Then G has precisely $p+1$ maximal subgroups $M_1, \dots, M_{p+1}$ and $V(G) = G \backslash F$. Moreover, two distinct vertices x and y of $\Sigma(G)$ are adjacent if and only if $x, y \in M_i$ for some $1 \leq i \leq p+1$. This implies that $\Sigma(G)$ is the union of $p+1$ complete graphs, with vertex sets $M_1 \setminus F, \dots, M_{p+1} \setminus F$. □

A crucial role in our proof will also be played by the following result, due to Gaschütz.

Proposition 14 ([8]). *Let N be a normal subgroup of a finite group G and suppose that $\langle g_1, \dots g_k \rangle N = G$. If $k \geq d(G)$, then there exist $n_1, \dots, n_k \in N$ so that $\langle g_1 n_1, \dots g_k n_k \rangle = G$.*

Using the previous proposition, we may prove the following basic observation.

Proposition 15. *Let N be a proper normal subgroup of G. If $\Sigma(G/N)$ is connected, then $\Sigma(G)$ is connected and* $\mathrm{diam}(\Sigma(G)) \leq \mathrm{diam}(\Sigma(G/N))$.

Proof. Let g_1, g_2 be two different vertices of $\Sigma(G)$. If $g_1 N = g_2 N$, then $\langle g_1, g_2 \rangle \leq \langle g_1, g_2 \rangle N < G$ since, by Lemma 12, G/N is not cyclic. In this case, g_1, g_2 are adjacent vertices of $\Sigma(G)$. So we may assume $g_1 N \neq g_2 N$. In this case, there exists a path $(g_1 N, y_1 N, \dots, y_r N, g_2 N)$ in $\Sigma(G/N)$. By Proposition 14, for any $1 \leq i \leq r$, $y_i n_i \in V(G)$ for some $n_i \in N$. Thus $(g_1, y_1 n_1, \dots, y_r n_r, g_2)$ is a path in $\Sigma(G)$. □

Corollary 16. *If G is non-soluble, then $\Sigma(G)$ is connected and* $\mathrm{diam}(\Sigma(G)) \leq 5$.

Proof. A non-soluble group G has an epimorphic image which is monolithic with non-Abelian socle. So the conclusion follows combining the previous proposition with Proposition 10. □

A chief factor $A = X/Y$ of a finite group G is said to be a non-Frattini chief factor if $X/Y \not\leq \mathrm{Frat}(G/Y)$.

Corollary 17. *Let G be a 2-generated soluble group. If there exists a non-Frattini and non-central chief factor A of G such that $G/C_G(A)$ is not cyclic of prime order, then G is connected and* $\mathrm{diam}(\Sigma(G)) \leq 3$.

Proof. If A is a non-Frattini chief factor of G, then G admits as an epimorphic image the semidirect product $A \rtimes G/C_G(A)$, so the conclusion follows combining Propositions 5 and 15. □

Proof of Proposition 11. By Corollary 16, we may assume that G is soluble. Denote by W and F, respectively, the Fitting subgroup and the Frattini subgroup of G. By [10, Theorem 10.6 (c)], W/F has a complement in G/F and it is a direct product of minimal normal subgroups of G/F. In particular, $G/F \cong (V_1 \times \cdots \times V_u) \rtimes K$, where V_j is an irreducible K-module for $1 \leq j \leq u$ and $\bigcap_{1 \leq j \leq u} C_K(V_j) = 1$. Now let $\mathcal{A}$ be the set of the non-trivial irreducible G-modules that are G-isomorphic to a non-Frattini chief factor. By Corollary 17, we may assume that $G/C_G(A)$ is cyclic of prime order for every

$A \in \mathcal{A}$. This implies in particular that, for $1 \le j \le u$, either $C_K(V_j) = K$ or $K/C_K(V_j) \cong C_{p_j}$ for some prime p_j. Moreover, $K \le \prod_{1\le j\le u} K/C_K(V_j)$ is Abelian. This implies in particular that G/W is Abelian, hence the derived subgroup G' of G is contained in W and therefore is nilpotent. We may assume that $V_j \le Z(G)$ if and only if $j > t$. So we have $G/F \cong (V_1 \times \cdots \times V_t) \rtimes H$, with $H = (V_{t+1} \times \cdots \times V_u) \times K$. If $1 \le j_1 < j_2 \le t$, then $(V_{j_1} \times V_{j_2}) \rtimes H$, being an epimorphic image of G, must be 2-generated, and this implies $V_{j_1} \not\cong_H V_{j_2}$. □

5. Groups with nilpotent derived subgroup. In this section, we investigate the connectivity of the graph $\Sigma(G)$ under the additional assumption that G is a finite soluble group whose derived subgroup is nilpotent. First we consider the particular case when G itself is nilpotent. The analysis of this case relies on Lemma 19, whose proof requires the following preliminary observation.

Lemma 18. *Let A and B be soluble groups without common epimorphic images. Then two elements (x_1, y_1), (x_2, y_2) of $A \times B$ generate $A \times B$ if and only if $\langle x_1, x_2\rangle = A$ and $\langle y_1, y_2\rangle = B$.*

Proof. Suppose $A = \langle x_1, x_2\rangle$ and $B = \langle y_1, y_2\rangle$. Since H is a subdirect product of A and B, by Goursat's lemma (see, for example, [15, 4.3.1]), we can have $H < G$ only if A and B have a common non-trivial epimorphic image. □

Lemma 19. *If $G = A \times B$ is a non-cyclic soluble group and A and B have no common Abelian epimorphic image, then $\Sigma(G)$ is connected and* $\mathrm{diam}(\Sigma(G)) \le 3$, *with equality if and only one of the following occurs:*

(1) A is cyclic and either $\Sigma(B)$ is disconnected or $\mathrm{diam}(\Sigma(B)) > 2$.
(2) B is cyclic and either $\Sigma(A)$ is disconnected or $\mathrm{diam}(\Sigma(A)) > 2$.

Proof. By Lemma 18, $V(G) = V(A) \times V(B)$. We may assume that A is not cyclic (if A and B are both cyclic, then either they have a common epimorphic image or $A \times B$ is cyclic). Suppose that $g_1 = (a_1, b_1)$, $g_2 = (a_2, b_2)$ are two different vertices of $\Sigma(G)$. If $B \ne \langle b_1\rangle$, then $((a_1, b_1), (a_2, b_1), (a_2, b_2))$ is a path in $\Sigma(G)$. Similarly, if $B \ne \langle b_2\rangle$, then $((a_1, b_1), (a_1, b_2), (a_2, b_2))$ is a path in $\Sigma(G)$. If $B = \langle b_1\rangle = \langle b_2\rangle$, then $((a_1, b_1), (a_1, 1), (a_2, 1), (a_2, b_2))$ is a path in $\Sigma(G)$. In the last case, if (a_1, a, a_2) is a path in $\Sigma(A)$, then $((a_1, b_1), (a, b_1), (a_2, b_2))$ is also a path in $\Sigma(G)$.

Finally assume that $B = \langle b\rangle$ is cyclic and there exist $a_1, a_2 \in V(A)$ without a common neighbour in $\Sigma(A)$. Then (a_1, b) and (a_2, b) do not have a common neighbour in $\Sigma(G)$ and therefore $\mathrm{diam}(\Sigma(G)) \ge 3$. □

Corollary 20. *Let G be a 2-generated finite nilpotent group. Then $\Sigma(G)$ is disconnected if and only if G is either a cyclic group or a p-group. Moreover, if G is neither a cyclic group nor a p-group, then* $\mathrm{diam}(\Sigma(G)) = 3$ *if G has only one non-cyclic Sylow subgroup,* $\mathrm{diam}(\Sigma(G)) = 2$ *otherwise.*

Proof. We decompose $G = P_1 \times \cdots \times P_t \times Q_1 \times \cdots \times Q_u$ where $P_1, \ldots, P_t$ are the cyclic Sylow subgroups of G and $Q_1, \ldots, Q_u$ are the remaining Sylow subgroups. If $u = 0$, then $\Sigma(G)$ is disconnected by Lemma 12. So we may assume $u \geq 1$. If $u = 1$ and $t = 0$, then $\Sigma(G)$ is disconnected by Lemma 13. If $u = 1$ and $t > 0$, then $G = X \times Q_1$ with $X = P_1 \times \cdots \times P_t$. So by Lemma 19, $\Sigma(G)$ is connected, and, since X is cyclic and $\Sigma(Q_1)$ is disconnected, $\mathrm{diam}(\Sigma(G)) = 3$. If $u \geq 2$, then $G = K \times Q_u$ with $K = P_1 \times \cdots \times P_t \times Q_1 \times \cdots \times Q_{u-1}$. Neither K nor Q_u is cyclic, so, again by Lemma 19, $\Sigma(G)$ is connected and $\mathrm{diam}(\Sigma(G)) = 2$. □

It remains to investigate the case when G is as described in Proposition 11 (2) and $t > 0$. By the following lemma it is not restrictive to assume $\mathrm{Frat}(G) = 1$.

Lemma 21. *Let G be a 2-generated finite group. Then $\Sigma(G)$ is connected if and only if $\Sigma(G/\mathrm{Frat}(G))$ is connected.*

Proof. Let $F = \mathrm{Frat}(G)$. Since G is cyclic if and only if G/F is cyclic, by Lemma 12, we may assume that G is not cyclic. By Proposition 15, we only have to prove that if $\Sigma(G/F)$ is disconnected, then $\Sigma(G)$ is also disconnected. So assume that Ω is a connected component of $\Sigma(G/F)$ and that there exists $yF \in V(G/F)\backslash\Omega$. Since $\langle g_1, g_2\rangle = G$ if and only if $\langle g_1F, g_2F\rangle = G/F$, it follows immediately that $\Omega^* = \{x \in G \mid xF \in \Omega\}$ is a connected component of $\Sigma(G)$ and $y \in V(G) \setminus \Omega^*$. □

Assume now that $\mathrm{Frat}(G) = 1$ and that G is a non-nilpotent group satisfying the description given in Proposition 11 (2). We have:

(∗) $G \cong (V_1 \times \cdots \times V_t) \rtimes H$, where H is Abelian and $V_1, \ldots, V_t$ are pairwise non-H-isomorphic non-trivial irreducible H-modules.

Notice that if G satisfies (∗) and M is a maximal subgroup of G, then either $M = (V_1 \times \cdots \times V_t)K$ for some maximal subgroup K of H or M contains a conjugate of H (and consequently it contains $H \cap Z(G)$). In particular, $Z(G) \cap \mathrm{Frat}(H) \leq \mathrm{Frat}(G) = 1$.

The following lemma describes the possibilities that can occur if G satisfies (∗).

Lemma 22. *Assume that G satisfies* (∗). *Then one of the following occurs:*

(1) $\Sigma(G)$ *is connected and* $\mathrm{diam}(G) \leq 3$;
(2) $H \cong C_p$ *for a suitable prime* p;
(3) $H \cong C_p^2$ *for a suitable prime* p.

Proof. Let $C_i = C_H(V_i)$ and $H_i = H/C_i$. The primitive soluble group $K_i = V_i \rtimes H_i$ is an epimorphic image of G. If $|H_i|$ is not a prime, then by Propositions 5 and 15, $\Sigma(G)$ is connected and $\mathrm{diam}(\Sigma(G)) \leq 3$. So we may assume $|H_i| = p_i$, with p_i a prime for $1 \leq i \leq t$. Suppose $p_i \neq p_j$ for some $1 \leq i < j \leq t$. Then $K_i \times K_j$ is an epimorphic image of G and it can be easily seen that $V(K_1 \times K_2) = V(K_1) \times V(K_2)$. Arguing as in the proof of Lemma 19, it can be deduced that $\Sigma(K_1 \times K_2)$, and consequently $\Sigma(G)$, is connected with diameter at most 3. So $p_i = p$ for $1 \leq i \leq t$. Moreover, $\bigcap_{1\leq i\leq t} C_i = Z(G)$, and therefore $H/Z(G)$ is an Abelian group of exponent p. Let A be a p'-Hall subgroup of $Z(G)$. Then A has a complement, say B, in H

and $G = ((V_1 \times \cdots \times V_t) \rtimes B) \times A$. Since A and $(V_1 \times \cdots \times V_t) \rtimes B$ have no common Abelian epimorphic image, it follows from Lemma 19 that $\Sigma(G)$ is connected with diameter at most 3. So H is a p-group and since $H^p \leq Z(G)$, it follows that $H^p \leq \mathrm{Frat}(H) \cap Z(G) = 1$. Since G is 2-generated, we conclude that $H \cong C_p^d$ with $d \leq 2$. □

In the notation introduced in $(*)$, let $F_i = \mathrm{End}_H(V_i)$. Since H is Abelian, $\dim_{F_i} V_i = 1$. We may identify V_i with the additive group of the field F_i. Moreover, if $h \in H$, then there exists $\alpha_i(h) \in F_i^*$ such that $v^h = \alpha_i(h)v$ for every $v \in V_i$. The following holds:

Lemma 23. *Assume that $g_1 = (v_{1,1}, \ldots, v_{1,t})h_1$ and $g_2 = (v_{2,1}, \ldots, v_{2,t})h_2$ are elements of G. For $1 \leq j \leq t$, consider the matrix*

$$A_j := \begin{pmatrix} 1 - \alpha_j(h_1) & 1 - \alpha_j(h_2) \\ v_{1,j} & v_{2,j} \end{pmatrix}.$$

Then $\langle g_1, g_2 \rangle = G$ if and only if the following hold:

(1) $\langle h_1, h_2 \rangle = H$;

(2) $\det(A_j) \neq 0$ *for every* $1 \leq j \leq t$.

Proof. See [12, Proposition 2.1 and Proposition 2.2]. □

With the help of the previous lemma, we may analyse the connectivity of $\Sigma(G)$ when G is as described in (2) or (3) of Lemma 22. There are three situations that can occur, as described in the following three lemmas.

Lemma 24. *Assume that G satisfies $(*)$. If $H \cong C_p$, then $\Sigma(G)$ is disconnected.*

Proof. Let $W = V_1 \times \cdots \times V_t$. By Lemma 23, if $w = (v_1, \ldots, v_t) \in W$, then $w \in V(G)$ if and only if $v_i \neq 0$ for $1 \leq i \leq t$. Consider $\Omega = W \cap V(G)$. If w_1, w_2 are two different elements of Ω, then they are adjacent in $\Sigma(G)$. Again by Lemma 23, if $g \in V(G) \backslash \Omega$, then $G = \langle g, w \rangle$ for any $w \in \Omega$. This implies that Ω is a proper connected component of $\Sigma(G)$. □

Lemma 25. *Assume that G satisfies $(*)$. If $H \cong C_p \times C_p$ and $Z(G) \neq 1$, then $\Sigma(G)$ is disconnected.*

Proof. In this case, $Z(G) = \langle h \rangle$ is a subgroup of G of order p. Let $W = V_1 \times \cdots \times V_t$. By Lemma 23, if $x = (v_1, \ldots, v_t)h^j \in W\langle h \rangle$, then $x \in V(G)$ if and only if $h^j \neq 1$ and $v_i \neq 0$ for $1 \leq i \leq t$. Consider $\Omega = W\langle h \rangle \cap V(G)$. If x_1, x_2 are two different elements of Ω, then they are adjacent in $\Sigma(G)$. Again by Lemma 23, if $g \in V(G) \backslash \Omega$, then $G = \langle g, x \rangle$ for any $x \in \Omega$. This implies that Ω is a proper connected component of $\Sigma(G)$. □

Lemma 26. *Assume that G satisfies $(*)$. If $H \cong C_p \times C_p$ and $Z(G) = 1$, then $\Sigma(G)$ is connected and* $\mathrm{diam}(\Sigma(G)) \leq 2$.

Proof. For $h \in H$, let $\Delta(h) = \{i \in \{1, \ldots, t\} \mid h \in C_H(V_i)\}$. Let $g = (v_1, \ldots, v_t)h \in G$. By Lemma 23, $g \in V(G)$ if and only if $h \neq 1$ and $v_j \neq 0$ for any $j \in \Delta(h)$. Suppose that $g_1 = (x_1, \ldots, x_t)h_1$, $g_2 = (y_1, \ldots, y_t)h_2$ are two distinct vertices of $\Sigma(G)$. We may assume $\langle g_1, g_2 \rangle = G$, otherwise g_1, g_2 are adjacent vertices of $\Sigma(G)$. Up to reordering, we may assume $\Delta(h_1) = \{1, \dots, r\}$ for some $r \in \{0, \dots, t\}$. Since $H = \langle h_1, h_2 \rangle$ and $Z(G) = \bigcap_{1 \leq j \leq t} C_H(V_j) = 1$, we must have $\Delta(h_2) \subseteq \{r+1, \dots, t\}$. Up to reordering, we may assume $\Delta(h_2) = \{r+1, \dots, r+s\}$ for some $s \in \{0, \dots, t-r\}$. Moreover, up to conjugation with a suitable element of $V_1 \times \cdots \times V_t$, we may assume $x_j = 0$ if $j > r$ and $y_k = 0$ if $k \leq r$.

If $r + s < t$, then $g = (0, \dots, 0, y_{r+1}, \dots, y_{r+s}, 0, \dots, 0)h_2 \in V(G)$. On the other hand, $\langle g_1, g \rangle$ is contained in $(V_1 \times \cdots \times V_{t-1}) \rtimes H$ and $\langle g_2, g \rangle$ is contained in $(V_1 \times \cdots \times V_t) \rtimes \langle h_2 \rangle$, so (g_1, g, g_2) is a path in $\Sigma(G)$.

Finally assume $r + s = t$. In this case, $\Delta(h_1 h_2) = \emptyset$, so $h_1 h_2 \in V(G)$. Moreover, $r > 0$, otherwise $\Delta(h_2) = \{1, \dots, t\}$ and $h_2 \in Z(G)$, and $r < t$, otherwise $\Delta(h_1) = \{1, \dots, t\}$ and $h_1 \in Z(G)$. Thus $\langle g_1, h_1 h_2 \rangle \leq (V_1 \times \cdots \times V_{t-1}) \rtimes H$ and $\langle g_2, h_1 h_2 \rangle \leq (V_2 \times \cdots \times V_t) \rtimes H$. But then $(g_1, h_1 h_2, g_2)$ is a path in $\Sigma(G)$. □

6. Proofs of Theorem 1 and Proposition 2.

Proof of Theorem 1. Suppose that G is a 2-generated finite group and that $\Sigma(G)$ is disconnected. By Proposition 11, G is soluble with a nilpotent derived subgroup. If G is nilpotent, then, by Corollary 20, $\Sigma(G)$ is disconnected if and only if G is a cyclic group or a p-group. Suppose that G is not nilpotent. By Lemma 21, $\Sigma(G)$ is connected if and only if $\Sigma(G/\operatorname{Frat}(G))$ is connected. Combining Proposition 11 and Lemma 22, it follows that if $\Sigma(G)$ is disconnected, then $G/\operatorname{Frat}(G) \cong (V_1 \times \cdots \times V_t) \rtimes H$, where either $H \cong C_p$ or $H \cong C_p \times C_p$ for some prime p, and $V_1, \dots, V_t$ are pairwise non-H-isomorphic non-trivial irreducible H-modules. If $H \cong C_p$, then $\Sigma(G)$ is disconnected by Lemma 24. If $H \cong C_p \times C_p$, then, by Lemmas 25 and 26, $\Sigma(G)$ is disconnected if and only if $Z(G/\operatorname{Frat}(G)) \neq 1$, or, equivalently, if and only if $C_H(V_1 \times \cdots \times V_t) \cong C_p$.

Finally, suppose $\Sigma(G)$ is connected. If G is non-soluble, Corollary 16 implies $\operatorname{diam}(\Sigma(G)) \leq 5$. If G is soluble, then $\operatorname{diam}(\Sigma(G)) \leq 3$: indeed, when G is nilpotent, the conclusion follows from Corollary 20 and when G is non-nilpotent, it follows by combining Proposition 11, Lemma 22, and Lemma 26. □

Proof of Proposition 2. First notice that generators of a cyclic group and involutions in $C_2 \times C_2$ and D_p are isolated vertices in the corresponding graphs. Conversely, let G be a 2-generated finite group and suppose that $g \in G$ is an isolated vertex of $\Sigma(G)$. If $g \neq g^{-1}$, then $G = \langle g, g^{-1} \rangle = \langle g \rangle$, so G is cyclic. Otherwise g is an involution. Suppose that this is the case and assume that G is not cyclic. Since $g \in V(G)$, the set $V_g = \{h \in G \mid \langle h, g \rangle = G\}$ is non-empty. Suppose $h \in V_g$. Then $G = \langle g, h \rangle = \langle gh^i, h \rangle$ for any $i \in \mathbb{Z}$. Hence $gh^i \in V(G)$ and therefore either $h^i = 1$ or $\langle g, gh^i \rangle = \langle g, h^i \rangle = G$. In other words, if g generates G together with h, then it generates G together with any non-trivial power of h. If $\langle g, h \rangle$ is Abelian, this is possible only if $|h|$ is a prime, and we must have $|h| = 2$ otherwise G would be cyclic. So if G is Abelian, then $G \cong C_2 \times C_2$. If G is non-Abelian and $h \in V_g$, then $g \neq g^h \in V(G)$, thus $G = \langle g, g^h \rangle \cong D_n$ is a dihedral group of order $2n$ with $n = |gg^h|$. In particular,

g generates G together with any non-trivial power of gg^h and this is possible only if n is a prime. □

Funding Open access funding provided by Università degli Studi di Padova within the CRUI-CARE Agreement.

References

[1] Aschbacher, M., Kleidman, P., Liebeck, M.: Exponents of almost simple groups and an application to the restricted Burnside problem. Math. Z. **208**(3), 401–409 (1991)

[2] Breuer, T., Guralnick, R.M., Kantor, W.M.: Probabilistic generation of finite simple groups II. J. Algebra **320**(2), 443–494 (2008)

[3] Burness, T., Guralnick, R., Harper, S.: The spread of a finite group. Ann. of Math. **193**(2), 619–687 (2021)

[4] Cameron, P.: Graphs defined on groups. Int. J. Group Theory **11**(2), 53–107 (2022)

[5] Crestani, E., Lucchini, A.: The generating graph of finite soluble groups. Israel J. Math. **198**(1), 63–74 (2013)

[6] Crestani, E., Lucchini, A.: The non-isolated vertices in the generating graph of a direct power of simple groups. J. Algebra. Combin. **37**(2), 249–263 (2013)

[7] Freedman, S.: The intersection graph of a finite simple group has diameter at most 5. Arch. Math. (Basel) **117**(1), 1–7 (2021)

[8] Gaschütz, W.: Zu einem von B. H. und H. Neumann gestellten Problem. Math. Nachr. **14**(4–6), 249–252 (1955)

[9] Guralnick, R., Kantor, W.: Probabilistic generation of finite simple groups. J. Algebra **234**(2), 743–792 (2000)

[10] Doerk, K., Hawkes, T.: Finite Soluble Groups. De Gruyter Expositions in Mathematics, 4. Walter de Gruyter & Co., Berlin (1992)

[11] Lucchini, A.: The diameter of the generating graph of a finite soluble group. J. Algebra **492**, 28–43 (2017)

[12] Lucchini, A., Maróti, A.: On the clique number of the generating graph of a finite group. Proc. Amer. Math. Soc. **137**(10), 3207–3217 (2009)

[13] Lucchini, A., Menegazzo, F.: Generators for finite groups with a unique minimal normal subgroup. Rend. Sem. Mat. Univ. Padova **98**, 173–191 (1997)

[14] Lucchini, A., Nemmi, D.: The non-$\mathfrak{F}$ graph of a finite group. Math. Nachr. **294**(10), 1912–1921 (2021)

[15] Scott, W.R.: Group Theory, 2nd edn. Dover Publications Inc., New York (1987)

[16] Shen, R.: Intersection graphs of subgroups of finite groups. Czechoslovak Math. J. **60**(4), 945–950 (2010)

Andrea Lucchini and Daniele Nemmi
Dipartimento di Matematica “Tullio Levi-Civita”
Università di Padova
Via Trieste 63
35121 Padova
Italy
e-mail: lucchini@math.unipd.it

Daniele Nemmi
e-mail: dnemmi@math.unipd.it

Received: 28 August 2021

Revised: 21 January 2022

Accepted: 15 March 2022.