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A comparison between current-based integral equations approaches for eddy current problems

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Abstract. In this paper, a comparison between two current-based Integral Equations approaches for eddy current problems is presented. In particular, the very well-known and widely adopted loop-current formulation (or electric vector potential formulation) is compared to the less common $\mathbf{J}\text{-}\varphi$ formulation. Pros and cons of the two formulations with respect to the problem size are discussed, as well as the adoption of low-rank approximation techniques. Although rarely considered in the literature, it is shown that the $\mathbf{J}\text{-}\varphi$ formulation may offer some useful advantages when large problems are considered. Indeed, for large-scale problems, while the computational efforts required by the two formulations are comparable, the $\mathbf{J}\text{-}\varphi$ formulation does not require any particular attention when non-simply connected domains are considered.

1. Introduction

In recent years, the interest in Integral Equation (IE) methods for the solution of 3-D electromagnetic (EM) problems has significantly increased. Indeed, with respect to the well-known and well established Finite Element Method (FEM), IE methods do not require the discretization of air domains (or, more generally, domains with the electromagnetic properties of vacuum), thus reducing the number of required unknowns [1]. Moreover, thanks to this feature, the construction of the mesh is also simplified, which is a particularly important task when complex geometries with small air gaps are analysed, such as large scale thermonuclear fusion devices [2, 3] or new generation electronic devices based on surface mounted technology [4, 5].

Depending on the frequency of the problem, different IE formulations can be used. For instance, when the frequency and the geometry of devices allow for exploiting the magneto-quasistatic assumption, formulations like the ones proposed in [6, 7, 8] can be used. Instead, for higher frequencies or, in general, when propagation effects can be neglected but capacitive effects must be considered, other formulations like the Partial Element Equivalent Circuit (PEEC) method can be used [9, 10, 11]. Finally, when the frequency is so high that propagation effects must be considered, depending on the kind of the problem, other IE formulations can be used [12, 13, 14].

The main problem of all IE methods is the generation of dense matrices representing the mutual interactions between the unknowns. Generally, the memory required for the storage of such dense matrices grows quadratically with the number of the unknowns, thus reducing the efficiency and applicability of IE methods for large scale problems if no remedy is applied. Also the time required for the generation of the dense matrices grows quadratically with the number of the unknowns, and thus cannot be completely ignored [15]. Moreover, the computational cost



required for the solution of the final IE system of equations is also significantly affected by the presence of fully populated matrices. Indeed, if direct solvers are adopted, the computational cost may grow with a $\mathcal{O}(N^3)$ complexity, where N is the number of unknowns. Instead, if an iterative solver is adopted, it is of fundamental importance to use an efficient preconditioner to limit the number of iterations; indeed, at each iteration, matrix-vector products involving the fully populated matrices are performed [16].

Fortunately, the intense research on low-rank compression techniques of the last years has led to several efficient techniques to solve this problem. Indeed, although they are dense, the fully-populated matrices arising from IE methods generally possess a Hierarchical Off-Diagonal Low-Rank (HODLR) structure, which can be exploited to represent them in data sparse formats [3, 2]. For instance, several methods based on, e.g., Hierarchical (\mathcal{H}) matrices and \mathcal{H} -Semi-Separable (HSS) matrices have been proposed to drastically reduce the memory required for the storage of such dense matrices, while keeping the accuracy high [17, 18] and also allowing for standard algebraic operations like matrix-matrix summation, matrix-vector product, and matrix inversion and factorization. The main idea of this kind of data-sparse formats is to represent the dense matrices with a hierarchical structure constructed on the basis of some specific criterion (i.e., algebraically and/or geometrically based). The hierarchical structure is obtained by successively partitioning rows and columns of the dense matrix, until the obtained matrix blocks satisfy an admissibility criterion, and therefore they are good candidates for the application of low-rank compression techniques such as Adaptive Cross Approximation (ACA) [19] or other more sophisticated methods. Thanks to low-rank compression techniques, several analyses by means of IE methods involving large scale devices [2, 20, 21] have been possible, which otherwise would have been computationally demanding and possibly prohibitive.

In this framework, the choice of the formulation is of paramount importance since it directly affects the efficiency of the solution strategy and also the performances of applying low-rank compression techniques. In this regard, the present paper aims at comparing two different IE formulations for eddy-current problems. The first one, very well-known, exploits the divergence free condition of the electric current by introducing an electric vector potential [16, 6]. Such formulation is also sometimes referred to the loop-currents formulation [22]. The other formulation, less frequently used, keeps the current density and the electric scalar potential as the problem unknowns [23], and can be seen as a particular form of the PEEC method where capacitive effects are neglected, i.e., the magneto-quasistatic assumption is exploited. The two formulations have pros and cons which are discussed in this paper considering the standard TEAM problem 7 [24] and a second test case consisting in an axisymmetric metallic conductor, resembling the typical geometry of a D-shaped vacuum vessel employed in thermonuclear fusion devices.

2. Basic Formulation

The starting point for both formulations is the well known Electric Field Integral Equation (EFIE) [25]

$$\mathbf{E}(\mathbf{r}) = -i\omega\mathbf{A}(\mathbf{r}) - \nabla\varphi(\mathbf{r}) + \mathbf{E}_{ext}(\mathbf{r}), \quad (1)$$

where \mathbf{E} is the electric field, \mathbf{A} is the magnetic vector potential, and φ is the scalar electric potential. In (1), \mathbf{r} is the field point, i is the imaginary unit, and ω is the angular frequency.

The magnetic vector potential, \mathbf{A} , is obtained from the current density vector distribution, \mathbf{J} , from the well-known integral expression [26]

$$\mathbf{A}(\mathbf{r}) = \mu_0 \int_{\Omega} \mathbf{J}(\mathbf{r}')g(\mathbf{r}, \mathbf{r}')d\Omega, \quad (2)$$

where \mathbf{r}' is the integration point, Ω is the conductive domain, and

$$g(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}'\|}, \quad (3)$$

is the static Green's function of vacuum [27].

Equation (1), is then complemented by the Ohm's law

$$\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{J}(\mathbf{r}), \quad (4)$$

which is a local relation between \mathbf{E} and \mathbf{J} , for $\mathbf{r} \in \Omega$, and ρ is the electric resistivity.

Since we are considering eddy-current problems, the magneto-quasistatic assumption can be exploited. Therefore, the following equation also holds:

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = 0. \quad (5)$$

The above equations are the starting point of the two formulations that are discussed and compared in the following. It is useful to provide a common discretization; thus, a mesh consisting of, e.g., tetrahedral, hexahedral, or polyhedral elements is now considered.

The electric current density \mathbf{J} can be then expanded by means of vector face shape functions:

$$\mathbf{J}(\mathbf{r}) = \sum_{k=1}^{N_f} \mathbf{w}_k(\mathbf{r})j_k, \quad (6)$$

where \mathbf{w}_k is the k th vector shape function, j_k is the flux of \mathbf{J} through the k th internal face of the mesh. Unknowns j_k , with $k = 1, \dots, N_f$, are stored in the array \mathbf{j} (N_f is the number of internal faces of the mesh). The electric scalar potential is also expanded as in [28] by using piece-wise constant basis functions. Therefore, the array Φ of size N_v (where N_v is the number of mesh elements) is also introduced, which stores the unknowns related to φ . Following a Galerkin scheme as in [22, 29], the array \mathbf{e}_{ext} corresponding to \mathbf{E}_{ext} is also introduced.

By using such expansions, equations (1)–(5) can be represented in equivalent matrix form:

$$(\mathbf{R} + i\omega\mathbf{L})\mathbf{j} + \mathbf{G}\Phi = \mathbf{e}_{ext}, \quad (7)$$

$$\mathbf{G}^T\mathbf{j} = \mathbf{0}, \quad (8)$$

where \mathbf{R} and \mathbf{L} are the equivalent (sparse) resistance and (dense) inductance matrices [22] of dimension $N_f \times N_f$, respectively, and \mathbf{G} is the volumes–faces incidence matrix [28] of dimension $N_f \times N_v$.

3. Loop–Currents Formulation

The very well-known loop-currents formulation (or electric vector potential formulation) strongly enforces the divergence-free condition of \mathbf{J} , i.e., (5), by expressing it as the linear combination of divergence-free loop-currents. As thoroughly discussed in [2], two main approaches can be adopted: 1) global loop-currents are identified by inspecting the equivalent dual graph of the problem (for instance using a reduced row echelon form, see [2] for more details), and 2) the current density is written as the curl of an electric vector potential, i.e., in terms of array of unknowns,

$$\mathbf{j} = \mathbf{C}\mathbf{t} + \mathbf{H}\mathbf{i}, \quad (9)$$

where \mathbf{t} is the array related to the electric vector potential \mathbf{T} , such as $\mathbf{J} = \nabla \times \mathbf{T}$, and \mathbf{C} is the curl faces–edges incidence matrix (where, according to [30], a set of edges forming a tree has been removed in order to impose a gauge on \mathbf{T}). The method based on (9) presents several advantages with respect to using global loop-currents as unknowns. Therefore, in the following, the formulation based on the electric vector potential is considered.

In (9), \mathbf{H} is a proper change of basis matrix which maps the extra global unknowns \mathbf{i} into \mathbf{j} for multiply-connected domains [31, 32]. Actually, \mathbf{i} is an array storing unknowns related to loop currents around holes of the multiply-connected domains (see [2] for more details).

By using (9), (8) is automatically satisfied. Then, by inserting (9) into (7) and projecting the resulting equations by using \mathbf{C}^T and \mathbf{H}^T , one finally obtains:

$$\begin{bmatrix} \mathbf{R}_{cc} + i\omega\mathbf{L}_{cc} & \mathbf{R}_{ch} + i\omega\mathbf{L}_{ch} \\ \mathbf{R}_{hc} + i\omega\mathbf{L}_{hc} & \mathbf{R}_{hh} + i\omega\mathbf{L}_{hh} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{i} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^T \mathbf{e}_{\text{ext}} \\ \mathbf{H}^T \mathbf{e}_{\text{ext}} \end{bmatrix}, \quad (10)$$

where

$$\mathbf{R}_{cc} = \mathbf{C}^T \mathbf{R} \mathbf{C}, \quad \mathbf{R}_{ch} = \mathbf{C}^T \mathbf{R} \mathbf{H}, \quad \mathbf{R}_{hc} = \mathbf{H}^T \mathbf{R} \mathbf{C}, \quad \mathbf{R}_{hh} = \mathbf{H}^T \mathbf{R} \mathbf{H}, \quad (11)$$

$$\mathbf{L}_{cc} = \mathbf{C}^T \mathbf{L} \mathbf{C}, \quad \mathbf{L}_{ch} = \mathbf{C}^T \mathbf{L} \mathbf{H}, \quad \mathbf{L}_{hc} = \mathbf{H}^T \mathbf{L} \mathbf{C}, \quad \mathbf{L}_{hh} = \mathbf{H}^T \mathbf{L} \mathbf{H}. \quad (12)$$

Matrices \mathbf{R}_{cc} and \mathbf{L}_{cc} are the resistance and inductance matrices related to \mathbf{t} (i.e., local loop-currents) and they can be constructed as shown in [6] without actually passing through \mathbf{R} and \mathbf{L} . Instead, \mathbf{R}_{hh} and \mathbf{L}_{hh} are matrices related to \mathbf{i} , i.e., loops around holes in multiply-connected domains. The remaining matrices in (11) and (12) represent mutual resistive and inductive coupling between \mathbf{t} and \mathbf{i} .

When the number of unknowns is small enough, (10) can be solved with a standard direct solver based on, e.g., the LU factorization. However, the memory required to store the system matrix (10) grows quadratically with the number of unknowns and the computation time for factorizing (10) grows with a $\mathcal{O}(N^3)$ complexity. Therefore, solving (10) by using a direct solver becomes prohibitive with the increase of the number of unknowns and different approaches based on iterative solvers should be preferred. However, the preconditioning of such iterative solvers applied to (10) is still a topic of active research and, usually, a simple preconditioner is constructed from (10) by considering only the resistive terms and the self and close-mutual inductive interactions, while neglecting all the other coefficients and thus leading to a sparse preconditioner. However, generally, the efficiency of this preconditioner worsens with the increase of the frequency since the importance of the mutual inductance's grows with the increase of the frequency.

Another more sophisticated approach consists on using \mathcal{H} -matrices coupled with ACA or Singular Value Decomposition (SVD) for representing (10) in a data-sparse format and exploiting \mathcal{H} -LU decomposition for speeding up the iterative solver, while keeping the memory requirement low [33, 19]. As shown in [2], such approach is efficient; however, undoubtedly, it is quite complex when non-simply connected domains are considered. Indeed, as thoroughly discussed in [2], the method adopted for constructing the \mathcal{H} -matrices must allow for independently treating matrix blocks related to \mathbf{t} and \mathbf{i} , since, in general, blocks related to \mathbf{i} may not be low-rank.

Although this is theoretically possible, such approach is quite complex and some kind of \mathcal{H} -matrices (e.g., HODLR and HSS) hardly treat such heterogeneous matrix structure. Moreover, in general, such heterogeneous structure negatively affects the compression performances and the efficiency of the \mathcal{H} -LU factorization.

4. J- φ Formulation

Starting from the discussion of Section 2, an alternative numerical method can be developed. Indeed, similarly to the PEEC formulation [22, 34], (7) and (8) can be directly used to set up a system of equations in the unknowns \mathbf{j} and Φ , i.e.,

$$\begin{bmatrix} \mathbf{R} + i\omega\mathbf{L} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{j} \\ \Phi \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{\text{ext}} \\ \mathbf{j}_{\text{inj}} \end{bmatrix}, \quad (13)$$

where \mathbf{j}_{inj} is the array corresponding to possible injected currents which can be easily considered in the proposed formulation. It is worth noting that a potential for each separated domain must be set to zero in order to give a reference to the electric scalar potential.

In the literature, (13) is rarely considered for solving magneto-quasistatic problems since, generally, the loop-currents formulation is preferred. Indeed, with respect to the formulation in Section 3 (i.e., system (10)), (13) requires a larger number of equations/unknowns (i.e., $N_f + N_v$ with respect to $N_e - N_n$ required by (10), where N_e is the number of edges of the mesh and N_n is the number of nodes). Moreover, (13) has an heterogeneous structure consisting of dense, sparse, and zero blocks, resulting into a saddle-point system.

Despite these drawbacks, differently from the loop-currents formulation, multiply-connected domains can be considered by (13) without any additional computational effort.

Thus, if the number of unknowns is small enough, (13) can be solved with a standard direct solver based on, e.g., the LU factorization. In this case, (13) can be preferable with respect to (10) since multiply-connected domains do not require any additional computational effort and the implementation of complex algorithms.

Instead, when larger problems are considered, an efficient procedure to solve (13) must be devised to keep the computational effort affordable. A procedure to efficiently solve (13) by means of an iterative solver (e.g., GMRES) and a proper preconditioner can be developed from the recent literature [23, 35].

Following [35], a preconditioner can be constructed by replacing the matrix block (1,1) of (13) with its diagonal and then exploiting the Schur complement technique as usually done in saddle-point problems (see [23] for more details).

As shown in the numerical results section, this preconditioning approach is very efficient (i.e., a limited number of iterations is required by GMRES to reach the convergence). However, when the Schur complement is performed at each preconditioned-GMRES iteration, the following system of equations must be solved:

$$\{\mathbf{G}^T [\text{diag}(\mathbf{R} + i\omega\mathbf{L})]^{-1} \mathbf{G}\} \mathbf{a} = \mathbf{b}, \quad (14)$$

where \mathbf{a} and \mathbf{b} are arrays of dimension N_v .

Since \mathbf{G} is very sparse and the inverse of the diagonal of $\mathbf{R} + i\omega\mathbf{L}$ is used, the resulting system (14) is also very sparse. However, when N_v is large, solving (14) can be still computationally demanding if a direct solver based on matrix-factorization is used. Fortunately, the structure of (14) is well-suited for using an Algebraic MultiGrid (AMG) solver [36, 37]. The main advantage of AMG is that it requires a reduced memory since it does not factorize the whole system matrix. Moreover, solving (14) by means of AMG requires two subsequent steps, i.e., 1) a pre-processing of the system matrix is performed in order to construct the multigrid levels and, 2) the system is solved by exploiting such AMG levels. Since AMG is used as preconditioner for solving (13), system (14) must be solved several times (as many as the number of iterations required by GMRES to converge) for different \mathbf{b} but the left-hand-side is always the same. Therefore, the construction of the multigrid levels (step 1) can be performed only once and then, at each GMRES iteration, system (14) is solved by exploiting the same multigrid construction, therefore significantly reducing the overall computational cost.

Thanks to this solution strategy, (13) can be efficiently solved. Moreover, when large problems are considered, the dense matrix \mathbf{L} can be also represented by using \mathcal{H} -matrices as previously discussed.

5. Loop-Currents Formulation vs \mathbf{J} - φ Formulation

Although the loop-currents formulation is generally more usually adopted for solving eddy-currents problems in the framework of IE methods, the two formulations presented in the previous sections have pros and cons and, depending on the problem, one should be preferable with respect to the other. In this short section, for the sake of clarity, the pros and cons of the two methods are briefly discussed.

The main advantage of the loop-currents formulation is that it requires a reduced number of unknowns with respect to the $\mathbf{J}\text{-}\varphi$ formulation. However, handling non-simply connected domains requires additional computational effort and particular attentions when \mathcal{H} -matrices are adopted [2]. Optimized algorithms for handling non-simply connected domains have been proposed [31], however, to be very robust, they require the implementation of interval arithmetic when the linking number between unknowns \mathbf{i} are computed [2], which is not an easy task.

Moreover, connections to ports and lumped excitation require to handle the boundary conditions and connections to external circuit components actually affect the topology of the problem; therefore the construction of matrix \mathbf{H} and the identification of unknowns \mathbf{i} should handle this situation. Namely, one should combine the formulation presented in [38] with the optimized algorithm to construct matrix \mathbf{H} of [2] and, if needed, \mathcal{H} -matrices.

The $\mathbf{J}\text{-}\varphi$ formulation requires a larger number of unknowns; therefore, for small size problems, the loop-currents formulation can be preferable since it requires a reduced computational effort. However, the $\mathbf{J}\text{-}\varphi$ formulation does not require any particular attention for non-simply connected domains and connections to ports or lumped excitation can be easily considered by following the same procedures used in the context of the PEEC method [39].

When the dimension of the problem is so large that a direct solver cannot be adopted and/or the system matrices are too large for the available computer memory by using both the formulations, the $\mathbf{J}\text{-}\varphi$ formulation may present some advantages with respect to the other one. Indeed, when non-simply connected domains are considered, \mathcal{H} -matrices can be more easily applied to the $\mathbf{J}\text{-}\varphi$ formulation to reduce the memory required for the storage of matrix \mathbf{L} and the solution strategy presented in Section 4, based on Schur complement and AMG preconditioner, results particularly efficient.

6. Numerical Results

In this section, the performances of the loop-currents and the $\mathbf{J}\text{-}\varphi$ formulations are compared by considering two test cases: 1) the TEAM problem 7 [24] and a second test case consisting in an axisymmetric metallic conductor, resembling the typical geometry of a D-shaped vacuum vessel employed in thermonuclear fusion devices. The simulations were run on a Linux machine equipped with twin Intel Xeon E5-2643 v4 processors (dual 6-core/12-thread, @3.40 GHz) and 512 GB of RAM.

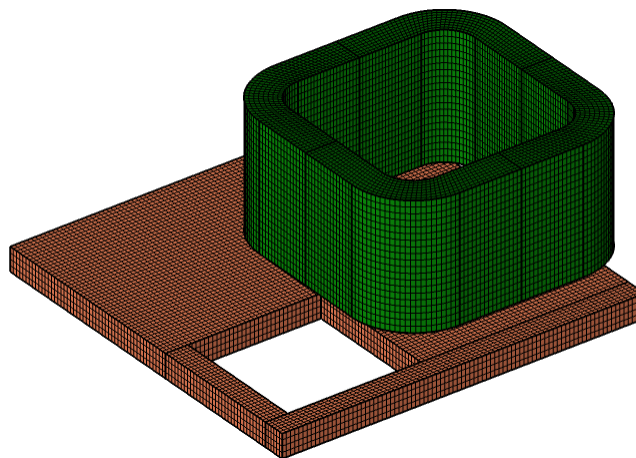


Figure 1. 3D view of model and mesh of the TEAM Workshop Problem 7. Green: exciting coil. Orange: asymmetrical conductor.

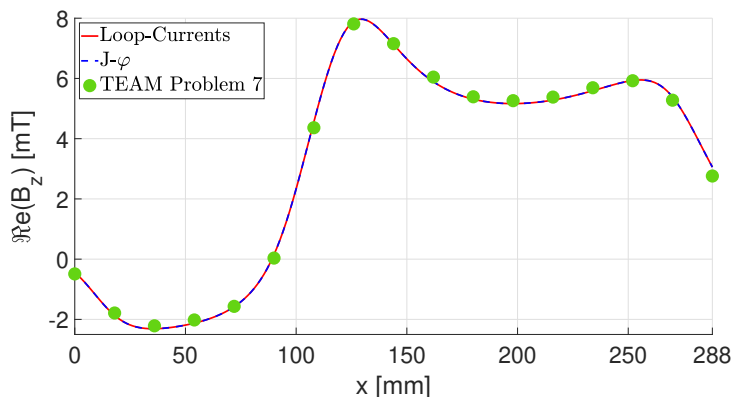


Figure 2. Comparison of the B_z -component (real part) with the two integral formulations and results of TEAM Workshop Problem 7, along a reference line.

Table 1. Computational performances of the loop-currents and $\mathbf{J}-\varphi$ formulations for the case of the TEAM Workshop Problem 7.

-	Memory [GB]	Assembling Time [s]	Solution Time [s]	GMRES Iterations [-]
Loop-Currents	22	400	35	13
$\mathbf{J}-\varphi$	40	405	46	13

6.1. TEAM Problem 7

The TEAM Workshop Problem 7 consists of an asymmetrical conductor of resistivity $\rho = 0.028 \mu\Omega m$ with a hole, excited by a coil with 2742 At, at 50 Hz (see figure 1). The model of the conductor consists of 26250 hexahedral elements, resulting into 48376 unknowns (i.e., loop-currents) for the loop-currents formulation, and 97293 unknowns (i.e., 71043 face currents and 26250 potentials) for the $\mathbf{J}-\varphi$ formulation. The exciting field is computed by using analytical formulas and no low-rank approximation is applied in this example.

The results of the two formulations in terms of the magnetic field along a reference line described in TEAM Workshop Problem 7 (i.e., line A1-B1) are compared with the ones provided by the TEAM Workshop Problem 7. This comparison is shown in figure 2. The computed values are in excellent agreement and they also agree with the trend of experimental points of the TEAM Workshop Problem 7.

Concerning the computational performances, table 1 shows the comparison of the two formulations. In both formulations, the problem is solved by using GMRES as iterative solver and the preconditioning techniques discussed in Section 3 and 4 are adopted. The tolerance for GMRES has been set to 10^{-6} . As discussed in Section 5, for this small/medium size test case with only one hole, the loop-currents formulation is preferable to the other one. Figure 3 shows the convergence plot of GMRES solver by using the $\mathbf{J}-\varphi$ formulation and the loop-currents formulation. As can be seen, the two formulations show a very similar convergence profile.

It is worth noting that, in addition to the assembling and solution time, the loop-currents formulation also requires to construct matrix \mathbf{H} which maps the extra global unknowns \mathbf{i} into \mathbf{j} for multiply-connected domains. However, in this case, the problem has a single hole and, by using the optimized algorithm presented in [2], the construction of \mathbf{H} is performed in only few seconds.

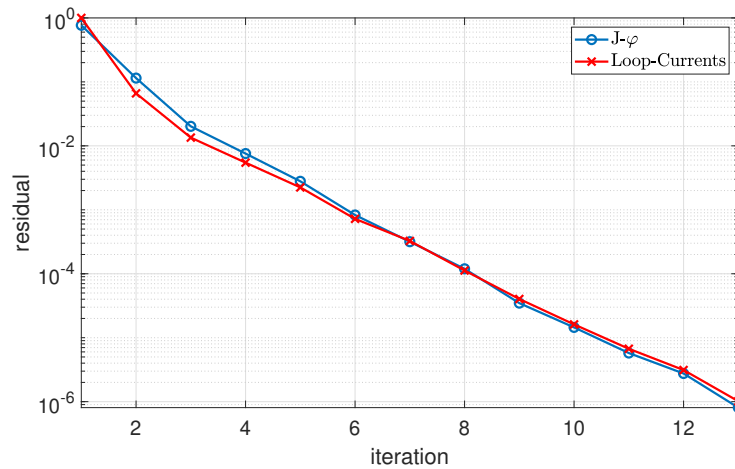


Figure 3. Convergence of GMRES solver for the TEAM Workshop Problem 7 by using the $\mathbf{J}\text{-}\varphi$ and the loop-currents formulations.

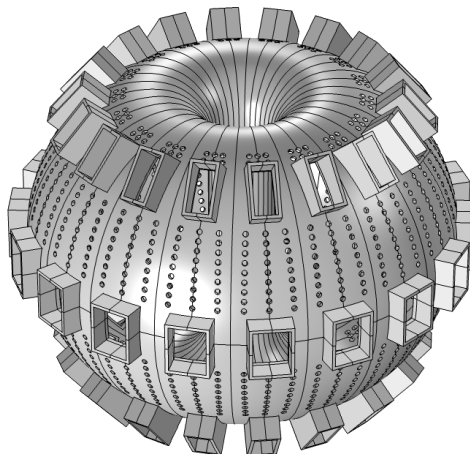


Figure 4. 3D view of the model of the D-shaped vacuum vessel employed in thermonuclear fusion devices.

6.2. D-shaped vacuum vessel

The second example consists on the model of a typical D-shaped VV used in thermonuclear fusion machines, in which a vertical field of 1 T (50 Hz) simulates the magnetizing field of the machine magnet system. The model is similar to the one considered in [2], however, the model used in this work has a greater number of ports (i.e., 1242), as shown in figure 4. In particular, the model consists on 240426 tetrahedral elements, resulting into 152748 unknowns for the loop-current formulation, and 224319 current unknowns and 240426 scalar potential unknowns for the $\mathbf{J}\text{-}\varphi$ formulation.

In this case, the reference solution has been obtained by solving the dense problem arising from the loop-currents formulation with a direct solver based on LU decomposition and using a Linux machine equipped with 16-core/32-thread processor (Intel Xeon Gold 6130 @2.10 GHz), for a total of 256 cores and 3 TB of RAM. The dense system (obtained by using the loop-currents formulation) required 373 GB of RAM. Low-rank compression techniques based on \mathcal{H} -matrices and ACA implemented in the highly optimized \mathcal{H} LibPro library [40] were applied to both the

Table 2. Computational performances of the loop-currents and $\mathbf{J}\text{-}\varphi$ formulations for the case of the D-shaped vacuum vessel.

-	Memory [GB]	Assembling Time [s]	Solution Time [s]	Generating \mathbf{H} [s]	GMRES Iterations [-]
Loop-Currents	34	7426	3316	9726	14
$\mathbf{J}\text{-}\varphi$	38	6812	3596	0	15

loop-currents and the $\mathbf{J}\text{-}\varphi$ formulations as described in Section 3 and Section 4, respectively.

The computational performances of the two formulations are reported in table 2. As can be seen, for this large-scale problem the two formulations show comparable performances, with the main difference that the loop-currents formulation require a non-negligible computation time to construct matrix \mathbf{H} (as suggested in [2], 50 basepoints have been used to construct \mathbf{H}). Indeed, although the systems arising from the two formulations have different number of unknowns, the one obtained from the $\mathbf{J}\text{-}\varphi$ formulation (which is the bigger one) can be efficiently compressed by using \mathcal{H} -matrices and ACA. This is because, differently from the $\mathbf{J}\text{-}\varphi$ formulation, the system of equations arising from the loop-current formulation has an heterogeneous structure due to the presence of the global unknowns \mathbf{i} which are required because the model is non-simply connected. Moreover, by imposing the same accuracy for the ACA, the mutual inductance coefficients between loop-currents (i.e., \mathbf{t}) are less compressible with respect to ones corresponding to face currents (i.e., \mathbf{j}) because of the different support and orientation of the shape functions. The accuracy of the two formulations with respect to the dense (reference) case was very good, i.e., errors below 1% in the system solution were obtained from both the formulations.

The comparison indicates that, for large scale problems and excluding the non-negligible time required to construct \mathbf{H} in the loop-currents formulation, the computational efforts required by the two formulations are comparable when \mathcal{H} -matrices are used. However, the loop-currents formulation requires a particular attention when non-simply connected domains are considered, especially when \mathcal{H} -matrices are adopted. Indeed, in this paper the optimized algorithm of [2] has been used.

Thus, we conclude that, aside the pre-processing time required by the loop-currents formulation when non-simply connected domains are involved, the two formulations show comparable results and they can be both adopted for the study of eddy currents problems. However, it is worth noting that connections to ports and lumped circuits can be easily considered with the $\mathbf{J}\text{-}\varphi$ formulation, while in the loop-currents formulation this would require to modify the algorithm for the construction of matrix \mathbf{H} , which is not a trivial task. Indeed, connections to ports may modify the topology of the problem and this must be considered when \mathbf{H} is construed.

7. Conclusions

In this paper, two current-based IE approaches for eddy current problems have been compared. In particular, the very well-known and widely adopted loop-current formulation (or electric vector potential formulation) is compared to the less common $\mathbf{J}\text{-}\varphi$ formulation. Pros and cons of the two formulations with respect to the problem size are discussed, as well as the adoption of low-rank approximation techniques. The loop-current formulation based on electric vector potential shows several advantages since it requires a minor number of unknowns. This is a particularly useful feature since, because of the full matrices, the memory requirement grows quadratically with the number of unknowns, whereas the computation time for the system solution grows with $\mathcal{O}(N^3)$ if no compression techniques are applied. The main drawback of such formulation is that non-simply connected domains must be carefully treated, and this is

particularly important when low-rank compression techniques are applied. Moreover, connections to ports and lumped excitation further complicate the use of low-rank compression techniques when non-simply connected domains are considered in the loop-currents formulations.

On the contrary, the main drawback of the $\mathbf{J}\text{-}\varphi$ formulation is that it requires a larger number of unknowns with respect to the loop-currents formulation. However, non-simply connected domains does not require any further computational effort and an efficient preconditioning strategy based on Schur complement and AMG solver recently proposed in the literature [23, 35] allows for reducing the overall computational cost required by the solution of the final system of equations. Finally, in the $\mathbf{J}\text{-}\varphi$ formulation connections to ports and lumped circuit can be easily considered and, thanks to the local support of the basis functions, low-rank compression techniques can be efficiently applied to compress the full inductance matrix, and thus reducing the overall memory requirement.

In this work, Volume Integral Equation methods have been considered. However, similar considerations hold for corresponding formulations developed by applying equivalent surface models [41].

8. References

- [1] P. Bettini, M. Passarotto, and R. Specogna. Coupling volume and surface integral formulations for eddy-current problems on general meshes. *IEEE Transactions on Magnetics*, 54(3):1–4, 2018.
- [2] D. Voltolina, R. Torchio, P. Bettini, R. Specogna, and P. Alotto. Optimized cycle basis in volume integral formulations for large scale eddy-current problems. *Computer Physics Communications*, 265:108004, 2021.
- [3] D. Voltolina, P. Bettini, P. Alotto, F. Moro, and R. Torchio. High-performance PEEC analysis of electromagnetic scatterers. *IEEE Transactions on Magnetics*, 55(6):1–4, 2019.
- [4] E. Vialardi, E. Clavel, O. Chadebec, J.-M. Guichon, and M. Lionet. Electromagnetic simulation of power modules via adapted modelling tools. In *Proceedings of 14th International Power Electronics and Motion Control Conference EPE-PEMC 2010*, pages T2–78–T2–83, 2010.
- [5] V. Ardon, J. Aime, O. Chadebec, E. Clavel, J.-M. Guichon, and E. Vialardi. EMC modeling of an industrial variable speed drive with an adapted PEEC method. *IEEE Transactions on Magnetics*, 46(8):2892–2898, 2010.
- [6] G. Rubinacci, A. Tamburrino, S. Ventre, and F. Villone. Fast computational methods for large-scale eddy-current computation. *IEEE Transactions on Magnetics*, 38(2):529–532, 2002.
- [7] G. Rubinacci, A. Tambunino, S. Ventre, and F. Villone. A fast algorithm for solving 3-d eddy current problems with integral formulations. *IEEE Transactions on Magnetics*, 37(5):3099–3103, 2001.
- [8] Q.-A. Phan, O. Chadebec, G. Meunier, J.-M. Guichon, and B. Bannwarth. 3D BEM formulations for eddy current problems with multiply connected domains and circuit coupling. *IEEE Transactions on Magnetics*, pages 1–1, 2020.
- [9] P. Baumgartner, T. Bauernfeind, O. Biro, C. Magele, W. Renhart, and R. Torchio. Synthesis of NFC antenna structure under multi-card condition. *Applied Computational Electromagnetics Society Journal*, 33(10):1161–1163, 2018.
- [10] J. Siau, G. Meunier, O. Chadebec, J.-M. Guichon, and R. Perrin-Bit. Volume integral formulation using face elements for electromagnetic problem considering conductors and dielectrics. *IEEE Transactions on Electromagnetic Compatibility*, 58(5):1587–1594, 2016.
- [11] T. Bauernfeind, P. Baumgartner, O. Bíró, A. Hackl, C. Magele, W. Renhart, and R. Torchio. Multi-objective synthesis of NFC-transponder systems based on PEEC method. *IEEE Transactions on Magnetics*, 54(3):1–4, 2018.
- [12] A. G. Polimeridis, J.F. Villena, L. Daniel, and J. K. White. Stable FFT-JVIE solvers for fast analysis of highly inhomogeneous dielectric objects. *Journal of Computational Physics*, 269:280–296, 2014.
- [13] G. Manara, A. Monorchio, and R. Reggiannini. A space-time discretization criterion for a stable time-marching solution of the electric field integral equation. *IEEE Transactions on Antennas and Propagation*, 45(3):527–532, 1997.
- [14] Y. Liu, A. C. Yücel, H. Bağcı, and E. Michielssen. A scalable parallel PWTD-accelerated SIE solver for analyzing transient scattering from electrically large objects. *IEEE Transactions on Antennas and Propagation*, 64(2):663–674, 2016.
- [15] P. Bettini, M. Passarotto, and R. Specogna. A volume integral formulation for solving eddy current problems on polyhedral meshes. *IEEE Transactions on Magnetics*, 53(6), 2017.

- [16] M. Passarotto, S. Ruben, and C. Geuzaine. Fast iterative schemes for the solution of eddy-current problems featuring multiple conductors by integral formulations. *IEEE Transactions on Magnetics*, 56(3):1–4, 2020.
- [17] S. Börm, L. Grasedyck, and W. Hackbusch. Introduction to hierarchical matrices with applications. *Engineering Analysis with Boundary Elements*, 27(5):405–422, 2003. Large scale problems using BEM.
- [18] J. Xia, S. Chandrasekaran, M. Gu, and X. S. Li. Fast algorithms for hierarchically semiseparable matrices. *Numerical Linear Algebra with Applications*, 17(6):953–976, 2010.
- [19] G. Antonini and D. Romano. Efficient frequency-domain analysis of peec circuits through multiscale compressed decomposition. *IEEE Transactions on Electromagnetic Compatibility*, 56(2):454–465, 2014.
- [20] V. Cirimele, R. Torchio, J. L. Villa, F. Freschi, P. Alotto, L. Codecasa, and L. Di Rienzo. Uncertainty quantification for SAE J2954 compliant static wireless charge components. *IEEE Access*, 8:171489–171501, 2020.
- [21] V. Cirimele, R. Torchio, A. Virgillito, F. Freschi, and P. Alotto. Challenges in the electromagnetic modeling of road embedded wireless power transfer. *Energies*, 12(14), 2019.
- [22] R. Torchio. A volume PEEC formulation based on the cell method for electromagnetic problems from low to high frequency. *IEEE Transactions on Antennas and Propagation*, 67(12):7452–7465, 2019.
- [23] A. C. Yucel, I. P. Georgakis, A. G. Polimeridis, H. Bağcı, and J. K. White. Voxhenry: FFT-accelerated inductance extraction for voxelized geometries. *IEEE Transactions on Microwave Theory and Techniques*, 66(4):1723–1735, 2018.
- [24] K. Fujiwara and T. Nakata. Results for benchmark problem 7 (asymmetrical conductor with a hole). *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, 9(3):137–154, Jan 1990.
- [25] R. Torchio, P. Bettini, and P. Alotto. PEEC-based analysis of complex fusion magnets during fast voltage transients with H-matrix compression. *IEEE Transactions on Magnetics*, 53(6):1–4, 2017.
- [26] D. Romano and G. Antonini. Quasi-static partial element equivalent circuit models of magneto-dielectric materials. *IET Microwaves, Antennas & Propagation*, 11(6):915–922, 2017.
- [27] T. Bauernfeind, P. Baumgartner, O. Bíró, C. Magele, K. Preis, and R. Torchio. PEEC-based multi-objective synthesis of non-uniformly spaced linear antenna arrays. *IEEE Transactions on Magnetics*, 53(6):1–4, 2017.
- [28] R. Torchio, F. Moro, G. Meunier, J.-M. Guichon, and O. Chadebec. An extension of unstructured-PEEC method to magnetic media. *IEEE Transactions on Magnetics*, 55(6):1–4, 2019.
- [29] D. Romano, G. Antonini, and A. E. Ruehli. Time-domain partial element equivalent circuit solver including non-linear magnetic materials. In *2015 IEEE MTT-S International Conference on Numerical Electromagnetic and Multiphysics Modeling and Optimization (NEMO)*, pages 1–3, 2015.
- [30] R. Albanese and G. Rubinacci. Finite element methods for the solution of 3D eddy current problems. volume 102 of *Advances in Imaging and Electron Physics*, pages 1–86. Elsevier, 1997.
- [31] P. Dłotko and R. Specogna. Physics inspired algorithms for (co)homology computations of three-dimensional combinatorial manifolds with boundary. *Computer Physics Communications*, 184(10):2257–2266, 2013.
- [32] P. Bettini and R. Specogna. A boundary integral method for computing eddy currents in thin conductors of arbitrary topology. *IEEE Transactions on Magnetics*, 51(3):1–4, 2015.
- [33] J. Siau, O. Chadebec, G. Meunier, J.-M. Guichon, and R. Perrin-Bit. Preconditioning of a low-frequency electric field integral equation formulation with circuit coupling using H-matrices. In *2016 IEEE Conference on Electromagnetic Field Computation (CEFC)*, pages 1–1, 2016.
- [34] D. Romano and G. Antonini. Augmented quasi-static partial element equivalent circuit models for transient analysis of lossy and dispersive magnetic materials. *IEEE Transactions on Magnetics*, 52(5):1–11, 2016.
- [35] P. Bettini, R. Torchio, F. Lucchini, D. Voltolina, and P. Alotto. Fast fourier transform-volume integral: a smart approach for the electromagnetic design of complex systems in large fusion devices. *Plasma Physics and Controlled Fusion*, 63(2):025010, dec 2020.
- [36] Y. Notay. AGMG software and documentation, <http://agmg.eu>.
- [37] A. Napov and Y. Notay. An algebraic multigrid method with guaranteed convergence rate. *SIAM Journal on Scientific Computing*, 34(2):A1079–A1109, 2012.
- [38] G. Rubinacci, A. Tamburrino, and F. Villone. Circuits/fields coupling and multiply connected domains in integral formulations. *IEEE Transactions on Magnetics*, 38(2):581–584, 2002.
- [39] R. Torchio, P. Alotto, P. Bettini, D. Voltolina, and F. Moro. A 3-D PEEC formulation based on the cell method for full-wave analyses with conductive, dielectric, and magnetic media. *IEEE Transactions on Magnetics*, 54(3):1–4, 2018.
- [40] R. Kriemann. Hlibpro library, <https://www.hlibpro.com/index.html>.
- [41] G. Meunier, Q.-A. Phan, O. Chadebec, J.-M. Guichon, B. Bannwarth, and R. Torchio. Unstructured PEEC method with the use of surface impedance boundary condition. *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, 39(5):1017–1030, Jan 2020.