Università degli Studi di Padova

Dipartimento di Matematica
Dipartimento di Matematica "Tullio Levi-Civita"

Sede Amministrativa: Università degli Studi di Padova
Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Dottorato in Scienze Matematiche
Curriculum Matematica Computazionale
XXXV ciclo

Four essays in between Probability Theory and Financial Mathematics

Tesi redatta con il contributo finanziario della Fondazione Cassa di Risparmio di Padova e Rovigo

Coordinatore
Prof. Giovanni Colombo
Università degli Studi di Padova

Supervisore
Prof. Markus Fischer
Università degli Studi di Padova

Co-supervisore
Prof. Giorgia Callegaro
Università degli Studi di Padova

Dottorando
Ofelia Bonesini
Matricola 1228416

Anno Accademico
2021/2022

Università degli Studi di Padova

Dipartimento Matematica
Dipartimento di Matematica "Tullio Levi-Civita"

Administrative Office: University of Padova
Department of Mathematics "Tullio Levi-Civita"

PhD Program in Mathematical Sciences
Computational Mathematics Curriculum
XXXV cycle

Four essays in between Probability Theory and Financial Mathematics

Thesis written with the financial support of Fondazione Cassa di Risparmio di Padova e Rovigo

Coordinator
Prof. Giovanni Colombo
University of Padova

Supervisor
Prof. Markus Fischer
University of Padova

Co-supervisor
Prof. Giorgia Callegaro
University of Padova

PhD Candidate
Ofelia Bonesini
ID 1228416

Academic Year
2021/2022

A tutte le donne che mi hanno ispirata, incoraggiata e plasmata

To all the women who have inspired me, encouraged me, and shaped me

Sommario

La seguente tesi di Dottorato consiste in quattro capitoli, corrispondenti a quattro diversi articoli che hanno costituito il fulcro della mia attività di ricerca durante il Ph.D. Tre dei quattro progetti sono il naturale proseguimento di quanto ho indagato durante gli studi magistrali e triennali. I primi due progetti trattano le tematiche dei *giochi a campo medio e dei giochi di tipo McKean-Vlasov*. In particolare, nel primo lavoro studiamo un gioco discreto ad N giocatori, per cui la nozione di equilibrio sottostante è quella di *equilibrio correlato*, e il suo limite di campo medio. Il secondo lavoro sviluppa un'applicazione dei giochi di tipo McKean-Vlasov ad un modello per lo studio dell'interazione tra un produttore e un consumatore che agiscono come speculatori nel mercato delle commodities. Gli ultimi due progetti vertono sullo studio della modellizzazione della volatilità nei modelli finanziari. Nel terzo capitolo, presentiamo un'applicazione della *quantizzazione funzionale*, una tecnica di discretizzazione per processi stocastici, ai cosiddetti *modelli a volatilità rough* con finalità di pricing: in particolare, ci focalizziamo sul prezzaggio di opzioni sul VIX e sulla volatilità realizzata. Nell'ultimo progetto, sviluppiamo uno studio teorico dettagliato di un modello a volatilità non Markoviana che è stato recentemente introdotto da Guyon.

Abstract

The following PhD Thesis consists of four chapters, corresponding to four different papers, which have been the core of my research activity during the Ph.D. Three out of four projects are the natural continuation of what I have been investigating during my Bachelor and Master's studies. The first couple of projects deals with *mean field and McKean-Vlasov games*. Indeed, in the first we study a discrete N-player game and its mean-field limit where the underlying notion of equilibrium is the one of *correlated equilibrium*. The second work develops an application of McKean Vlasov games to model the interaction between a producer and a consumer acting as speculators in commodity markets. The second couple of projects deals with volatility modellisation in financial models. In the third project, we present an application of *functional quantization*, a discretisation technique for stochastic processes, to *rough volatility models* with pricing purposes: in particular, we focus on pricing options on VIX and realised volatility. In the last project, we develop a detailed theoretical analysis of a path dependent volatility model that was recently introduced by Guyon.

Contents

List of Figures

List of Tables

Introduction

Fall in love with some activity, and do it! Nobody ever figures out what life is all about, and it doesn't matter. Explore the world. Nearly everything is really interesting if you go into it deeply enough. Work as hard and as much as you want to on the things you like to do the best. Don't think about what you want to be, but what you want to do. Keep up some kind of a minimum with other things so that society doesn't stop you from doing anything at all.

Richard P. Feynman

Disclaimer and educational background

As mentioned in the abstract, this is not the standard Ph.D. Thesis that develops around a single topic and in which, as a consequence, the chapters display different aspects of a unique problem. In my specific case, I had the possibility to pursue a Ph.D. program in the same University in which I had previously completed my Bachelor and Master's studies. Thus, I was given the chance to complete my path of study investigating in dept the topics that appealed me the most during the Bachelor and the Master programs. The first three chapters in the thesis result from this investigation. The last project can be framed in the same perspective of continuity and fluidity in the studies. Indeed, it was the result of a proposal to join a previously started project by one of Prof. Giorgia Callegaro's coauthors that I had the pleasure to meet while working on our very first work together.

Since my early Bachelor's studies I have focused my attention on Probability Theory and its potential applications to Finance. I have consequently decided to conclude this journey with a thesis whose title was *Optimal Quantization for the Minimisation of Capital Injection in an Insurance Company*. The advisors of this thesis were Prof. Wolfang Runggaldier and Prof. Giorgia Callegaro. This experience influenced deeply the following decisions I have made in my studies. Indeed, I was given the chance to meet Prof. Giorgia Callegaro but also to test myself on an open research problem. This experience has been challenging yet extremely stimulating and I had the opportunity to get acquainted with the (non-standard) random variables' discretisation technique which is known as optimal quantization.

I continued with a Master degree focused on Probability Theory and Finance but when I was approaching the end of my studies I wanted to concentrate on something more theoretical and not strictly applied to Finance. Prof. Markus Fischer was the advisor of my Master thesis. Together we studied *Existence and Uniqueness of Solutions for a Class of McKean-Vlasov SDEs*. Here, I have first met this particular type of SDEs where the parameters are not just depending on the process itself but also on its law. While investigating the underlying motivations for the introduction of this kind of equations, I opened the Pandora's box of McKean-Vlasov control problems and mean field games.

I was selected for the PhD program in Padova before submitting the Master's thesis. I accepted without hesitation, knowing that Padova would have been a fertile ground to pursue all the interests that I had discovered during my University studies. My two former advisors have been so crazy and brave to assume responsibility of me, supervising me together on different projects so that I could pursue all my inclinations.

Two main topics

After this long autobiographic digression that aims at providing a sort of disclaimer to justify the heterogeneity in the chapters, let us introduce the two macro-areas in which the four papers can be framed: mean field and McKean-Vlasov games and volatility modelisation in financial markets.

McKean-Vlasov and mean field games

Since their first appearance in the pioneering paper [115] by Lasry and Lions in 2006, *mean field games* have seen a slow but steady growth in interest until a few years ago when they literally bursted becoming a well-established field appealing not only theoretically inclined investigators but also applied mathematicians, engineers, and social scientists. For the sake of completeness, a similar approach for large games under the name of Nash Certainty Equivalence principle was investigated by Caines, Huang, and Malhamé in [99], but the natural appeal of the name *mean field games* is undeniable and this is how these models are known nowadays.

The essence of mean field games can be summarised via the following idea, which is as simple as brilliant: given the difficulty of computing equilibria for games where the players are particularly numerous, the number of players N is assumed to be so big that the game is well approximated by one with an infinite number of indistinguishable agents. Then, the analysis is limited to a control problem where a single player represents the whole system. In order for the passage to the limit to be meaningful, the representative player in the limit game should reflect in a consistent way the whole population. To this end, the starting N-player games possess the following characteristics. First of all, they are symmetric: the players are statistically indistinguishable or, equivalently, their joint law is invariant under permutations of its arguments. Furthermore, the interaction between the players is of *mean field* type, namely the influence of the action of a single agent vanishes as the number of players tends to infinity. Going more into details, it should be possible to model this interaction via the *empirical measure* associated to the system (a sort of centre of gravity of the system, equally weighted by $1/N$). Let us mention that the *measure flow* in the limit game acts as the limiting counterpart of the empirical measure.

The underlying idea is similar to what is studied in Mechanical Statistics under the name of *weakly-interacting particle systems*. The difference lays in the fact that here the systems are controlled. Indeed, the agents in the games are rational and so they want to optimise an objective functional. The fact that there is an underlying notion of equilibrium in the pre-limit game and that this has to be reflected in a consistent notion of solution for the limit game is crucial. Indeed, the distinction among mean field games and McKean-Vlasov control problems has its roots in the notion of equilibrium that is used in the pre-limit game. In general, mean field games are based on the concept of *Nash equilibrium* for the pre-limit N-player games, whereas McKean-Vlasov control problems originate from N-player games where a central planner has to face a population of players and thus the underlying notion of equilibrium there is the one of *Stackelberg equilibrium*.

As a consequence of the peculiar starting notion of equilibrium, the two limit problems display a different relationship between the system and the measure flow. Indeed, in mean field games the idea of *best response* (which is somehow the core of Nash equilibria) is translated into the so-called *consistency condition* in the definition of a solution: only at the equilibrium the measure flow is the law of the process itself. On the other side, for McKean-Vlasov control problems the measure flow, which appears in the dynamics and in the objective functional, is always the law of the state process itself. In analogy to this last one, the terminology McKean-Vlasov games refers to N-player games where the objective functional as well as the dynamics depend on the law of the processes.

The interested reader is referred to [33, 34] and [18], for an introduction to mean field games. Let us notice that the latter two works deal with optimal control problems of McKean-Vlasov type as well. Nevertheless, for a complete literature review, as well as a clear picture of the state of the Art in the two game formulations, we refer to the introductions of the specific chapters.

Volatility modelisation

Despite being so simple, classical financial models, where the volatility is assumed to be constant (e.g. Black-Scholes), are unable to fully describe the complexity and richness in market data. Motivated by the will, on one side, to reproduce empirically observed stylised facts, and, on the other side, to replicate the smile and skew behavior under a risk neutral probability measure, practitioners and academicians have introduced several sophisticated market models approaching the problem from different perspectives and with different tools. The stylised facts we are making reference to are essentially two: *volatility clustering* and the *persistence of volatility* in the prices' times series under the real-world probability measure.

In the perspective of option pricing, i.e. in the attempt to mimick the smile and skew behavior under a risk neutral probability measure, the approaches to volatility modelling can be essentially grouped into two. The first class is known under the name *Stochastic Volatility* (SV, in short) approach. As one can guess from the name itself, the volatility is assumed to be a stochastic process whose dynamics, and as a consequence the tractability of the corresponding model, varies with the specific assumptions of each model. This approach has a crucial feature in view of model calibration, that is to say it allows for a fast and efficient closed-form pricing of vanilla derivatives. Stein and Stein model [139], Heston model [92], the SABR model of Hagan et al. [88], Bergomi model [21] and their extensions are included is this broad class. Everything comes at a cost and even if these models succeed in reproducing the whole implied volatility surface in a very parsimonious way, they fail to reproduce the smile behaviour, and in

particular the steep implied volatility skew that characterises short maturities in equity markets. Furthermore, there is a side drawback. Indeed, the calibrated vol of vol parameters, as in the value of the mean reversion parameters, have to reach insanely high values to force the volatility process to stay in its natural range when historical volatility is very high as a consequence of a crisis. Dupire in [50] has given birth to a second class known with the name *Local Volatility* (LV, in short) approach. Here, the implied volatility surface is built by interpolation of the real prices so that the absence of arbitrage opportunities condition is preserved. The result is a partial differential equation for the option price efficiently solvable so that numerous routines are now available for the calibration of this model. This model has some weaknesses as well. First, the model requires frequent re-calibration, since the volatility is linked to the level of an asset and not its return. Furthermore, this approach is not suited for the pricing of options like forward starting options. Indeed, for local volatility models pricing results from the interpolation of known information. LV and SV models reveal the same limitations due to the inborn Markovian nature of both the approaches.
In a first attempt to let a model enjoy some memory overcoming the Markovian strictness as well as including the fact that volatility is persistent, Comte and Renault [40] introduced a fractional stochastic volatility model in which the volatility is a fractional Ornstein-Uhlenbeck process, driven by a fBM with Hurst parameter $H \geq 1/2$. This results in a continuous-time model with stochastic volatility that enjoys the long-memory property that characterises the fBM itself when its Hurst parameter is bigger than $1/2$. Later on, the academic community realised that the very irregular, i.e. rough, paths of a fBm with $H \leq 1/2$ could be exploited to reproduce the roughness in the volatility trajectories, see e.g. Bayer et al. [14] and Gatheral et al. [77]. In particular, the value of the H calibrated on market data is around 0.1, thus way smaller than 0.5 which is the case corresponding to standard BM. Rough volatility has rapidly gained a huge success and can now praise numerous descendants. The motivation for such a popularity lays in the fact that these models, despite being so parsimonious in the number of parameters (just one more than a standard stochastic volatility model), are able to portray not only the major stylised features of historical volatility time series but also SPX options smiles and skews. Nevertheless, the price one has to pay in order to handle these models is hidden in the Hurst parameter H itself. The smaller the value of H, the less regular the trajectories are but also the less tractable the problem is from a mathematical perspective, see e.g. [53, 55]. Indeed, in this new framework we do not only loose the semi-martingale property of volatility but also the Markovianity of the dynamics. From a computational viewpoint this implies that the well-known PDEs and Monte Carlo techniques are not readily available anymore and it means that for the extensions of classic models like Heston, SABR and Bergomi to the case where the volatility is rough finding an analytically tractable pricing technique is the very first challenge. Given the fact that practitioners are mostly concern with fast and efficient pricing, this is a serious weakness for these models. The main tool is therefore given by approximations via asymptotic expansions, see e.g. [53, 14, 73], with the sole exception of the rough Heston model [29]. A new yet very promising stream of literature is now focusing on signatures [43]: this approach is universal in the sense that it approximates arbitrarily well all classical models.

The take-home lesson we should learn from the study of rough volatility is the following:

despite being analytically tractable the Markovian framework is restrictive and allowing for memory in the volatility dynamics can prove itself to be extremely beneficial in reproducing most of market data stylised facts. In fact, implied volatility as well as future realised volatility are conditioned by the asset price trajectories in the recent past and the assumption that this should happen in a Markovian fashion, i.e. that the future should depend on the past only through the present, is unjustified. Indeed, recent studies have shown that financial markets reveal an evident path-dependent volatility pattern, see Guyon [86]. Another advantage of this last approach lays in its natural ability to disclose volatility in a solely endogenous way: no extra source of randomness is needed to generate a rich spot-vol dynamics. In turn, this yields the uniqueness of the corresponding prices under the only risk neutral measure. Under this approach, not just the current underlying asset prices but all the past ones store all the information exchanged by market participants. As a consequence, the volatility dynamics and the past asset returns are naturally linked and making the so-called Zumbach effect, see [142, 143, 44, 54, 78], reproducible.

Unquestionably, the natural setting for a volatility model is non Markovian. The use of a fractional process to introduce memory, as in the rough volatility approach, has prove itself to be a parsimonious and elegant way although lacking of analytical tractability. On the other hand, the presence of memory can be taken into account by adopting a path dependent volatility approach. These two approaches are deeply different and focus on different features of market data but the crucial point is the will to correctly introduce a dependence on the past in the volatility dynamics.

For a thorough literature review on rough volatility, respectively path dependent volatility, we refer the reader to the introduction at the beginning of the corresponding chapter.

Structure of the Ph.D. thesis

Now, we present the structure of the thesis. As mentioned above, each chapter represents a self-contained and autonomous paper. As a consequence, the notation changes chapter by chapter. The order in which we decided to present the work is not chronological but it is not casual either. Indeed, we begin by the most theoretical, namely a paper on a particular game theoretical problem, to proceed then with the second one, that provides an application of the so-called McKean-Vlasov games, to, finally, dive deeply into financial problems and associated applications of numerical discretisation techniques with the last couple of works.

Chapter 1 is devoted to paper [26]. This paper is a joint collaboration with my supervisor Prof. Markus Fischer and Prof. Luciano Campi (University of Milan). The paper was submitted for publication in December 2022 and it is available on arXiv. We focus on correlated equilibria (CE, in short) in game theory, which are a generalisation of Nash equilibria allowing for the possibility of a correlation between the strategies of the players. We study these equilibria in the context of N-player and mean field games, extending the results in [30] relaxing the hypothesis that the strategies used by the players are restricted. This generalisation is highly non-trivial and introduces several technical difficulties in the problem. In particular, we discuss the problem of constructing approximate equilibria when deviating players have access to the aggregate system state. An explicit example of a correlated mean field game solution not of Nash-type is provided as well. Finally, in the Appendix we present a direct proof for the existence of correlated solutions in mean field games for the case in restricted strategies.

In Chapter 2 we present paper [4]. This paper, written in collaboration with my Ph.D. co-supervisor Prof. Giorgia Callegaro and Profs. Renťe Aid (Université Paris Dauphine) and Luciano Campi (University of Milan), deals with commodity market prices manipulation. It was published in *Applied Mathematics and Optimization* in September 2022. We study a two-player Linear-Quadratic McKean-Vlasov stochastic differential game in which an energy producer affects the price dynamics of the good controlling drift and volatility of her production rate, while a consumer manipulates the price through his consumption rate in a similar way. Using methods based on the martingale optimality principle and BSDEs, we find a Nash equilibrium and characterise the corresponding strategies and payoffs in semi-explicit form. Numerical results are also provided.

Chapter 3 deals with paper [25]. This paper has been written in collaboration with Prof. Giorgia Callegaro and Prof. Antoine Jacquier (Imperial College London). It was submitted in July 2021 and its preliminary version is available on arXiv. It deals with a specific discretisation technique in the paths space of stochastic processes, namely product functional quantization, applied to the recent trendy field of rough processes. We focus, in particular, on the discretisation, in the trajectories space, of a family of Gaussian Volterra stochastic processes and the possible applications to the pricing of derivatives on the VIX volatility index and realised variance.

Finally, in Chapter 4 we discuss paper [27] which was submitted for publication in November 2022 and is available in the current version on arXiv. This project is a collaboration with Prof. Antoine Jacquier and Ph.D. Chloé Lacombe (Morgan Stanley). We provide a thorough analysis of the path-dependent volatility model introduced by Guyon in [86], proving existence and uniqueness of a strong solution, characterising its behaviour at boundary points, providing asymptotic closed-form option prices as well as deriving small-time behaviour estimates.

Correlated equilibria for mean field games with progressive strategies

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

Roger Myerson

This work is a joint collaboration with Prof. Luciano Campi and Prof. Markus Fischer. The corresponding paper was submitted in December 2022 and it is available on arXiv. In a discrete space and time framework, we study the mean field game limit for a class of symmetric N-player games based on the notion of correlated equilibrium. We give a definition of correlated solution that allows to construct approximate N-player correlated equilibria that are robust with respect to progressive deviations. We illustrate our definition by way of an example with explicit solutions. In the Appendix we display a detailed proof of chaos propagation as well as a direct proof for the existence of solutions for the mean field game in restricted strategies.

1.1 Introduction

Building on [30], we consider correlated equilibria for a simple class of symmetric finite horizon N-player games in discrete time and their natural mean field game counterpart as the number of players N goes to infinity.

MFGs is the acronym for *mean field games* and refers to a stream of literature in game theory extremely popular nowadays whose origins are quite recent. Indeed, MFGs were introduced nearly at the same time but independently by [99] and [115] in the mid 2000's. In a nutshell, MFGs are limit systems for symmetric stochastic N-player games with mean field interaction for $N \to \infty$. Thanks to the mean field interaction among the players, a kind of law of large numbers (known as propagation of chaos), one expects the empirical distribution of the players' states to converge as $N \to \infty$ to the law of some representative player. In the limit, the concept of Nash equilibrium translates into a two-step solution where (i) the representative player reacts optimally to the measure flow representing the distribution of the whole population, and (ii) the latter arises as aggregation of all such identical players' best responses at equilibrium. The reader interested in a broad yet detailed overview on the topic from a probabilistic viewpoint is referred to the two-volume book by Carmona and Delarue [33].

The connection between MFGs and their finite-player counterpart can be established in two ways. Crucial is the choice of the type of strategies the players are allowed to play. On one hand, a solution to the MFG can be exploited in order to build approximate Nash equilibria for N-player games; see, e.g., [32, 81, 99]. Existence of solutions can be established under general assumptions for various types of MFGs; see [110] and more recently [49]. On the other hand, approximate N-player Nash equilibria can be shown to converge to solutions of the corresponding MFG, as $N \to \infty$. Cardaliaguet, Delarue, Lasry and Lions in [31] gave an important contribution in this direction when the strategies are of closed loop type, exploiting the well-posedness of the so-called master equation, which implies uniqueness of MFG solutions. Later, Lacker in [111] was able to establish a general convergence result for non-degenerate diffusions, which he subsequently extended to the common noise case in the joint article [112] with Le Flem.

Correlated equilibria were first introduced for many-player games by Robert Aumann, see [10, 9]. His idea can be summarised in the following way: A *correlation device* or *mediator* (he) picks a strategy profile according to some probability distribution which is common knowledge among the players. Then, according to the selected profile, he privately suggests a strategy to each player, meaning that each player only knows the recommendation provided to him by the mediator. A correlated equilibrium (CE, for short) is a probability distribution on the space of strategy profiles such that no player is willing to unilaterally deviate from the mediator's suggestion. We notice that, when the distribution used by the mediator to generate his recommendations has a product form, then CE reduces to the usual notion of Nash equilibrium in mixed strategies. Traffic lights in routing games provide an intuitive example of a mediator in everyday life, e.g. [133, Section 13.1.4]. Other interpretations for such equilibria are available in the literature, we refer the interested reader to, e.g., [11].

The notion of CE was originally introduced for static games with complete information and it rapidly led to a massive research activity in game theory as well as in economic theory along many directions. The survey [62] provides a thorough analysis on several aspects of the more general notion of communication equilibrium within a wide range of games, such as stochastic games and games with incomplete information. In particular, for stochastic games we also refer to [136, 135, 137]. Many pleasant features of CE justify the scientific community interest towards it, for instance the fact that it may lead to higher payoffs than Nash equilibria, its lower computational complexity (see, e.g.,[80]), and also that CE are reachable by a wide range of learning procedures (see [89]).

CE in mean field games where first studied in [30], where the authors established approximation and convergence results for a class of symmetric finite horizon games in *restricted strategies*. After [30] two more papers on correlated equilibria in mean field games appeared, by Paul Müller and co-authors [126, 125], whose setting is very close to ours. Indeed, they, too, consider discrete time games with finite state and action spaces. The mean field interaction is modeled in the N-player games via the empirical measure of players' states. Players' strategies depend only on the player's individual states in a Markovian fashion. We stress that their definition of correlated equilibrium is different from the one we give in [30]. In particular, it does not require any explicit consistency condition for the flow of measures, which is obtained as a consequence of their definition. Nonetheless the most recent paper [125] has an interesting discussion on how to pass from our definition in [30] to theirs and vice-versa. Lastly, big parts of those papers are devoted to more computational issues focusing on learning algorithms approximating the equilibria.

Here, we consider correlated equilibria for a simple class of symmetric finite horizon N-player games and their natural MFG counterpart as $N \to \infty$. In the N-player setting, the state variables evolve in discrete time, both state space and the set of control actions are finite. The mediator recommends restricted strategies to the players, that is, feedback strategies that depend only on time and the corresponding individual state variable. This is the same framework as in [30]. As opposed to that work, and also to [126, 125], the deviating player is allowed to use (randomised) progressive strategies, that is, strategies that depend on the evolution of the entire system state up to current time; see Remark 1.4.4 below. We stress that the possibility for the players to deviate by playing progressive strategies make the analysis and the proofs much more delicate than in [30]. Our main results can be summarised as follows:

- We extend the notion of correlated solution for a mean field game to allow for progressive deviations. Two formulations are presented, one based on closed-loop controls, the other on stochastic open-loop controls.
- Starting from suitable correlated MFG solutions, we construct approximate N-player correlated equilibria that are robust against progressive deviations.
- We provide an explicit example for a mean field game possessing correlated solutions against progressive deviations that have non-deterministic flows of measures and satisfy all conditions of the approximation result.

The rest of the paper is structured as follows. In Section 1.2, we introduce the notation and state some preliminary definitions. In Section 1.3, we describe the underlying N-player games and give the definition of (approximate) correlated equilibrium against progressive deviations. Section 1.4 is dedicated to the corresponding mean field game. Correlated MFG solutions are first defined in feedback strategies with deviations that may directly depend on the possibly random flow of measures. In Section 1.5 we give an alternative definition of correlated MFG solution in stochastic open-loop strategies and establish an equivalence between the two formulations. Our main result is given in Section 1.6, where we show that suitable correlated MFG solutions yield approximate correlated solutions for the N-player game. An example of a correlated MFG with explicit solutions satisfying the assumptions of our approximation result is provided in Section 1.7. In Appendix 1.A, we collect some auxiliary results related to chaos propagation and in Appendix 1.B we provide a direct proof for the existence of solutions in restricted strategies.

1.2 Preliminaries and notation

We denote with $[\![m, M]\!]$ the set of natural numbers greater or equal to m and lower or equal to M, namely we set $[\![m, M]\!] := \{m, m+1, \dots, M-1, M\}$. A given $(T+1)$-dimensional vector, $(x_0, \dots, x_T)$, will be denoted with $(x_t)_{t=0}^T$ or just by x when its indices are clear from the context. Then, the $(t+1)$-dimensional vector of its first $t+1$ components is denoted with $x^{(t)} := (x_0, x_1, \dots, x_t)$. Similarly, for a T-dimensional vector, $(x_1, \dots, x_T)$, we introduce the notation $(x_t)_{t=1}^T$ (just x when the context is clear), and the t-dimensional vector of its first t components is denoted with $x^{(t)} := (x_1, \dots, x_t)$. Finally, let us fix a notation that is useful in the following. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random variable, X.

We consider the (discrete) time frame $[\![0, T]\!]$, with finite final time $T \in \mathbb{N}$. The individual states and the control actions lie in non-empty finite sets $\mathcal{X}$ and Γ, respectively. We mostly deal with finite sets and the sets of probability measures on them. Throughout the whole paper these sets are equipped with the discrete metric and the metric $\mathrm{dist}(\cdot, \cdot)$, respectively, making them *Polish spaces*. The metric $\mathrm{dist}(\cdot, \cdot)$ on the set $\mathcal{P}(E)$ of probability measures over a finite set E is defined as follows. For $\mu, \tilde{\mu} \in \mathcal{P}(E)$, set

$$\mathrm{dist}(\mu, \tilde{\mu}) := \frac{1}{2} \sum_{e \in E} |\mu(e) - \tilde{\mu}(e)|.$$

Notice that this metric is compatible with the weak convergence topology and, for measures over finite sets, weak convergence is equivalent to the convergence in total variation. The set $\mathcal{Z} = [0, 1]$ is the space of noise states. All the variables representing idiosyncratic noise are distributed according to ν, uniform distribution on $\mathcal{Z} = [0, 1]$.

The one-step individual state dynamics is given by the following system function:

$$\Psi \colon [\![0, T-1]\!] \times \mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma \times \mathcal{Z} \longrightarrow \mathcal{X}.$$

The running costs are specified through a function:

$$f \colon [\![0, T-1]\!] \times \mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma \longrightarrow \mathbb{R}.$$

The terminal costs are described by the following function:

$$F \colon \mathcal{X} \times \mathcal{P}(\mathcal{X}) \longrightarrow \mathbb{R}.$$

Consider the product space $[\![0, T-1]\!] \times \mathcal{X} \times \mathcal{P}(\mathcal{X})^T$. We equip this space with the product topology with respect to the topologies defined on each space, that are, respectively, discrete topology for $[\![0, T-1]\!]$ and $\mathcal{X}$, since they are finite sets, and the topology of weak convergence for the space $\mathcal{P}(\mathcal{X})$. Then, on the space $[\![0, T-1]\!] \times \mathcal{X} \times \mathcal{P}(\mathcal{X})^T$, we consider the σ-algebra:

$$\begin{aligned} \mathcal{B}\big([\![0, T-1]\!] \times \mathcal{X} \times \mathcal{P}(\mathcal{X})^T\big) &= \mathcal{B}([\![0, T-1]\!]) \otimes \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{P}(\mathcal{X})^T) \\ &= 2^{[\![0, T-1]\!]} \otimes 2^{\mathcal{X}} \otimes \mathcal{B}(\mathcal{P}(\mathcal{X}))^T, \end{aligned}$$

where 2^E denotes the power set of a finite set E. Notice that $\mathcal{B}(\mathcal{P}(\mathcal{X}))$ is the Borel σ-algebra induced by the topology of weak convergence, that in our case, where the state space $\mathcal{X}$ is finite, coincides with the one induced by the metric $\mathrm{dist}(\cdot, \cdot)$, on $\mathcal{P}(\mathcal{X})$. On the finite set Γ we consider the discrete topology and its Borel σ-algebra.

Let us define $\widehat{\mathcal{R}}$, the set of progressive feedback strategies:

$$\widehat{\mathcal{R}} := \Big\{\varphi : [\![0, T-1]\!] \times \mathcal{X}^T \times \mathcal{P}(\mathcal{X})^T \longrightarrow \Gamma, \quad \varphi \text{ progressively measurable}\Big\}.$$

As it is used several times in the following, we introduce another set of feedback strategies.

It corresponds to the Markov strategies that depend only on the individual player's state, see *restricted strategies* in [30]:

$$\mathcal{R} := \Big\{\varphi : [\![0, T-1]\!] \times \mathcal{X} \longrightarrow \Gamma\Big\}.$$

This space is equipped, as all finite sets in this paper, with the discrete topology. Notice that we have the natural inclusion $\mathcal{R} \subset \widehat{\mathcal{R}}$, and $\mathcal{R}$ is compact since it is finite.
Furthermore, for convenience of notation, for each $t \in [\![0, T-1]\!]$, we set

$$\begin{aligned}
\widehat{\mathcal{E}}_t &:= \Big\{\varphi : \mathcal{X}^{t+1} \times \mathcal{P}(\mathcal{X})^{t+1} \longrightarrow \Gamma, \quad \varphi \text{ Borel-measurable}\Big\}, \\
\widehat{\mathcal{E}}^{(t)} &:= \Big\{\varphi : [\![0, t]\!] \times \mathcal{X}^{t+1} \times \mathcal{P}(\mathcal{X})^{t+1} \longrightarrow \Gamma, \quad \varphi \text{ progressively measurable}\Big\},
\end{aligned}$$

and the corresponding restricted quantities

$$\begin{aligned}
\mathcal{E}_t = \mathcal{E} &:= \Big\{\varphi : \mathcal{X} \longrightarrow \Gamma, \quad \varphi \text{ Borel-measurable}\Big\}, \\
\mathcal{E}^{(t)} = \mathcal{E}^t &:= \Big\{\varphi : [\![0, t]\!] \times \mathcal{X} \longrightarrow \Gamma, \quad \varphi \text{ Borel-measurable}\Big\}.
\end{aligned}$$

When considering the N-player game, the set of progressively measurable feedback strategies corresponds to the following subset of $\widehat{\mathcal{R}}$

$$\widehat{\mathcal{R}}_N := \Big\{\varphi : [\![0, T-1]\!] \times \mathcal{X}^T \times (\mathcal{M}_N^{\mathcal{X}})^T \longrightarrow \Gamma, \quad \varphi \text{ progressively measurable}\Big\},$$

where $\mathcal{M}_N^{\mathcal{X}} := \{m \in \mathcal{P}(\mathcal{X}) : \text{ for any } x \in \mathcal{X}, m(x) = \frac{k}{N}, k \in [\![0, N]\!]\}$ is the set of empirical measures of N-samples. Notice that the set $\widehat{\mathcal{R}}_N$ is finite. Indeed this is a consequence of the finiteness of $\mathcal{M}_N^{\mathcal{X}}$, whose cardinality is $\frac{(N+|\mathcal{X}|-1)!}{N!(|\mathcal{X}|-1)!}$. Thus, we endow this set with the discrete topology. Analogously to what is done above, we set

$$\begin{aligned}
\widehat{\mathcal{E}}_{t,N} &:= \Big\{\varphi : \mathcal{X}^{t+1} \times (\mathcal{M}_N^{\mathcal{X}})^{t+1} \longrightarrow \Gamma, \quad \varphi \text{ Borel-measurable}\Big\}, \\
\widehat{\mathcal{E}}_N^{(t)} &:= \Big\{\varphi : [\![0, t]\!] \times \mathcal{X}^{t+1} \times (\mathcal{M}_N^{\mathcal{X}})^{t+1} \longrightarrow \Gamma, \quad \varphi \text{ progressively measurable}\Big\}.
\end{aligned}$$

Finally, set

$$\mathcal{D} := \{w : \mathcal{R} \to \mathcal{R}\}, \qquad \widehat{\mathcal{D}} := \{w : \widehat{\mathcal{R}} \to \widehat{\mathcal{R}}\},$$

which are respectively the sets of restricted and not strategies modifications. Notice again that the former set is clearly finite and the latter, when restricted to the N-player game, is finite and denoted by

$$\widehat{\mathcal{D}}_N := \{w : \widehat{\mathcal{R}}_N \to \widehat{\mathcal{R}}_N\}.$$

In the following we make extensive use of the concepts of regular conditional distribution and probability kernel, for which we refer to [105]. For all $N \in \mathbb{N}$, we define the set of flows of kernels

$$\mathcal{K}_N := \{\beta = (\beta_t)_{t=0}^{T-1} : \beta_t \text{ probability kernel from } (\widehat{\mathcal{R}}_N, \mathcal{B}(\widehat{\mathcal{R}}_N)) \text{ to } (\widehat{\mathcal{E}}_{t,N}, \mathcal{B}(\widehat{\mathcal{E}}_{t,N})),$$

for all $t \in [\![0,T]\!]\}$.

We can provide a natural interpretation for a flow of kernels $\beta \in \mathcal{K}_N$ in our context. It represents some procedure through which players in the N-player game select their strategies. Indeed, a player receives a $\widehat{\mathcal{R}}_N$-valued suggestion from the mediator at the beginning of the game and then, at each time step $t \in [\![0,T-1]\!]$, determines his $\widehat{\mathcal{E}}_{t,N}$-valued strategy as a function of the suggestion received and an additional independent randomisation factor (e.g. tossing a coin).

Finally, in the following, all σ-algebras and filtrations are assumed to be completed w.r.t. $\mathbb{P}$-null sets.

1.3 The N-player game

Consider a fixed number of players, $N \in \mathbb{N}$, and let $\mathfrak{m}^N \in \mathcal{P}(\mathcal{X}^N)$ represent the initial distribution of the N-player system. For any probability distribution $\gamma \in \mathcal{P}(\widehat{\mathcal{R}}_N)$, we define the set $\mathcal{N}_\gamma^N$ as

$$\mathcal{N}_\gamma^N := \Big\{\widetilde{\gamma} \in \mathcal{P}(\widehat{\mathcal{R}}_N \times \widehat{\mathcal{R}}_N) : \widetilde{\gamma}(\mathrm{d}\varphi, \mathrm{d}\psi) = \beta_0(\varphi, \mathrm{d}\psi_0)\ldots\beta_{T-1}(\varphi, \mathrm{d}\psi_{T-1})\gamma(\mathrm{d}\varphi), \\ \text{for some } \beta = (\beta_t)_{t=0}^{T-1} \in \mathcal{K}_N\Big\}.$$

where, for all $t \in [\![0,T-1]\!]$, ψ_t is the short form for $\psi(t,\cdot,\cdot)$. The elements of $\mathcal{N}_\gamma^N$ represent the joint distribution of the mediator's suggestion and players' strategy choices. In particular, if $\widetilde{\gamma} \in \mathcal{N}_\gamma^N$, then the first marginal of $\widetilde{\gamma}$ equals γ.

Finally, for a probability distribution $\gamma^N \in \mathcal{P}(\widehat{\mathcal{R}}_N^N)$, we denote its i-th marginal by

$$\gamma_i^N(\cdot) := \gamma^N(\widehat{\mathcal{R}}_N \times \cdots \times \cdot \times \cdots \times \widehat{\mathcal{R}}_N),$$

where $\cdot$ on the right-hand side above occupies the i-th coordinate.

Definition 1.3.1. *We call* correlated suggestion *any probability distribution $\gamma^N \in \mathcal{P}(\widehat{\mathcal{R}}_N^N)$. Then, consider a probability distribution $\widetilde{\gamma} \in \mathcal{N}_{\gamma_i^N}^N$ and call it a* strategy modification *for the i-th player. Let $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ be a complete probability space carrying $\mathcal{X}$-valued random variables $(X_t^{1,N},\ldots,X_t^{N,N})_{t=0}^T$, $\widehat{\mathcal{R}}_N$-valued random variables $\Phi_1,\ldots,\Phi_N,\widetilde{\Phi}_i$, and $\mathcal{Z}$-valued random variables $(\xi_t^{1,N},\ldots,\xi_t^{N,N})_{t=1}^T$ and $(\vartheta_t)_{t=0}^{T-1}$ such that the following properties hold:*

- ***i)*** $\mathbb{P}_N \circ (X_0^{1,N},\ldots,X_0^{N,N})^{-1} = \mathfrak{m}^N$;
  $\mathbb{P}_N \circ (\Phi_1,\ldots,\Phi_N)^{-1} = \gamma^N$;
- ***ii)*** $(\xi_t^{1,N},\ldots,\xi_t^{N,N})_{t=1}^T$ *are i.i.d. all distributed according to ν;*
- ***iii)*** $(\vartheta_t)_{t=0}^{T-1}$ *are i.i.d. all distributed according to ν;*
- ***iv)*** $(\xi_t^{1,N},\ldots,\xi_t^{N,N})_{t=1}^T$, $(X_0^{j,N})_{j=1}^N$, $(\vartheta_t)_{t=0}^{T-1}$, *and $(\Phi_j)_{j=1}^N$ are independent;*
- ***v)*** $\mathbb{P} \circ (\Phi_i,\widetilde{\Phi}_i)^{-1} = \widetilde{\gamma}$ *and, for any $t \in [\![0,T-1]\!]$, $\widetilde{\Phi}_i(t,\cdot,\cdot)$ is $\sigma(\Phi_i,\vartheta_t)$-measurable;*

vi) *for any* $t \in [\![0, T-1]\!]$,

$$X_{t+1}^{i,N} = \Psi\left(t, X_t^{i,N}, \mu_t^{i,N}, \widetilde{\Phi}_i(t, X^{i,N}, \mu^{i,N}), \xi_{t+1}^{i,N}\right),$$
$$X_{t+1}^{j,N} = \Psi\left(t, X_t^{j,N}, \mu_t^{j,N}, \Phi_j(t, X^{j,N}, \mu^{j,N}), \xi_{t+1}^{j,N}\right), \quad j \neq i, \qquad \mathbb{P}_N\text{-a.s.},$$

where $\mu_t^{l,N}$ *denotes the empirical measure of all N players' states but the l-th, i.e.* $\mu_t^{l,N} := \frac{1}{N-1}\sum_{j=1, j\neq l}^{N} \delta_{X_t^{j,N}}$*, and* $\mu^{l,N} := (\mu_t^{l,N})_{t=0}^{T} \in \mathcal{P}(\mathcal{X})^{T+1}$.

Any tuple $((\Omega_N, \mathcal{F}_N, \mathbb{P}_N), (\Phi_j)_{j=1}^N, (\vartheta_t)_{t=0}^{T-1}, (\xi_t^{1,N}, \ldots, \xi_t^{N,N})_{t=1}^T, \widetilde{\Phi}_i, (X_t^{1,N}, \ldots, X_t^{N,N})_{t=0}^T)$ *satisfying the conditions above is called a* realisation of the triple $(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma})$ for player $i \in [\![1, N]\!]$.

The correlated suggestion γ^N represents the known distribution, over the product set of the players' strategies, according to which the mediator gives his recommendations to the players, while $\widetilde{\gamma}$ represents the strategy modification for the deviating i-th player, encoded as the joint distribution of the suggestion received and the strategy he is actually taking into action. The fact that, for any $t \in [\![0, T-1]\!]$, $\widetilde{\Phi}_i(t, \cdot, \cdot)$ is $\sigma(\Phi_i, \vartheta_t)$-measurable yields that, at any time instant $t \in [\![0, T-1]\!]$, the i^{th} player can exploit an (independent) randomisation device to choose the strategy that he actually implements.

Remark 1.3.2. *Notice that, for any* $w \in \widehat{\mathcal{D}}_N$*, given a sequence of suggestions* $(\Phi_j)_{j=1}^N$*,* $\widetilde{\Phi}_i = w(\Phi_i)$ *satisfies assumption* **v)** *in Definition 1.3.1, with* $\sigma(\widetilde{\Phi}_i(t, \cdot, \cdot)) \subset \sigma(\Phi_i)$ *and* $\mathbb{P} \circ (\Phi_i, \widetilde{\Phi}_i)(\mathrm{d}\varphi, \mathrm{d}\psi) = \delta_{w(\varphi)}(\mathrm{d}\psi)\gamma_i^N(\mathrm{d}\varphi)$.

Remark 1.3.3. *We make the following useful remarks concerning (conditional) independence properties of a realisation.*

i) Notice that the following inclusion of σ*-algebras holds* $\sigma(\widetilde{\Phi}_i) \subseteq \sigma((\vartheta_t)_{t=0}^{T-1}, \Phi_i)$*, by definition. Indeed, we have*

$$\sigma(\widetilde{\Phi}_i) = \sigma((\widetilde{\Phi}_i(t, \cdot, \cdot))_{t=0}^{T-1}) = \bigvee_{t\in[\![0,T-1]\!]} \sigma(\widetilde{\Phi}_i(t, \cdot, \cdot)) \subseteq \bigvee_{t\in[\![0,T-1]\!]} \sigma(\Phi_i, \vartheta_t) = \sigma(\Phi_i, (\vartheta_t)_{t=0}^{T-1}),$$

The identities above hold since, for all $t \in [\![0, T-1]\!]$, $\widehat{\mathcal{E}}_{t,N}$ *are equipped with discrete topology (making them Polish spaces) so the Borel* σ*-algebra of the product space* $\widehat{\mathcal{R}}_N$ *coincides with the product of the Borel* σ*-algebras of* $\widehat{\mathcal{E}}_{t,N}$*. Thus, the* σ*-algebra generated by a* $\widehat{\mathcal{R}}_N$*-valued r.v. coincides with the one generated by its components in* $\widehat{\mathcal{E}}_{t,N}$*.*

ii) Notice that the assumptions in Definition 1.3.1, in particular ***iv)*** *and* ***v)****, imply that for a realisation of* $(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma})$ *as above*

$$(\xi_t^{1,N}, \ldots, \xi_t^{N,N})_{t=1}^T, (X_0^{j,N})_{j=1}^N \text{ and } (\widetilde{\Phi}_i, (\Phi_j)_{j=1}^N)) \text{ are independent.}$$

In fact, by ***v)****,* $\sigma(\widetilde{\Phi}_i, (\Phi_j)_{j=1}^N)) \subseteq \sigma((\vartheta_t)_{t=0}^{T-1}, (\Phi_j)_{j=1}^N)$ *and the* σ*-algebras* $\sigma((\vartheta_t)_{t=0}^{T-1}, (\Phi_j)_{j=1}^N)$*,* $\sigma((\xi_t^{1,N}, \ldots, \xi_t^{N,N})_{t=1}^T)$ *and* $\sigma((X_0^{j,N})_{j=1}^N)$ *are independent by* ***iv)****.*

iii) For a realisation of $(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma})$*, as above,*

$$(\Phi_j)_{j=1}^N \text{ and } \widetilde{\Phi}_i \text{ are conditionally independent given } \Phi_i.$$

We notice that $(\Phi_j)_j := (\Phi_j)_{j=1}^N$ *and* $\vartheta := (\vartheta_t)_{t=0}^{T-1}$ *are conditionally independent given* Φ_i*. Indeed, given* $A \in \mathcal{B}(\widehat{\mathcal{R}}_N^N)$*,* $B \in \mathcal{B}(\mathcal{Z}^{T+1})$*, we have,* $\mathbb{P}_N$*-a.s.,*

$$\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbf{1}_B(\vartheta)|\Phi_i] = \mathbb{E}_N[\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbf{1}_B(\vartheta)|(\Phi_1, \ldots, \Phi_N)]|\Phi_i]$$

$$
\begin{aligned}
&= \mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbb{E}_N[\mathbf{1}_B(\vartheta)|(\Phi_1,\ldots,\Phi_N)]|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbb{E}_N[\mathbf{1}_B(\vartheta)]|\Phi_i] = \mathbb{E}_N[\mathbf{1}_B(\vartheta)]\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_B(\vartheta)|\Phi_i]\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\Phi_i].
\end{aligned}
$$

*Then, for arbitrary sets $A \in \mathcal{B}(\widehat{\mathcal{R}}_N^N)$, $B \in \mathcal{B}(\widehat{\mathcal{R}}_N)$, exploiting **iv)** and **v)** and the conditional independence showed above, we see, $\mathbb{P}_N$-a.s.,*

$$
\begin{aligned}
\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbf{1}_B(\widetilde{\Phi}_i)|\Phi_i] &= \mathbb{E}_N[\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)\mathbf{1}_B(\widetilde{\Phi}_i)|\sigma(\vartheta,\Phi_i)]|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_B(\widetilde{\Phi}_i)\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\sigma(\vartheta,\Phi_i)]|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_B(\widetilde{\Phi}_i)\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\Phi_i]|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_B(\widetilde{\Phi}_i)|\Phi_i]\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\Phi_i] \\
&= \mathbb{E}_N[\mathbf{1}_B(\widetilde{\Phi}_i)|\Phi_i]\mathbb{E}_N[\mathbf{1}_A((\Phi_j)_j)|\Phi_i].
\end{aligned}
$$

Remark 1.3.4. *There is a strategy modification of particular interest for every correlated suggestion and every player. It reflects the case in which the player i, as all the other players, follows the suggestion he is given by the mediator. Exploiting the definition of realisation of a certain triple, this corresponds to $\Phi_i = \widetilde{\Phi}_i$, $\mathbb{P}_N$-a.s.. In particular, let $\mathbb{P}_N \circ \Phi_i^{-1} = \gamma$, we have*

$$
\mathbb{P}_N \circ (\Phi_i, \widetilde{\Phi}_i)^{-1}(\mathrm{d}\varphi, \mathrm{d}\psi) = \mathbb{P}_N \circ (\Phi_i, \Phi_i)^{-1}(\mathrm{d}\varphi, \mathrm{d}\psi) = \delta_\varphi(\mathrm{d}\psi)\gamma(\mathrm{d}\varphi).
$$

We denote this special strategy modification with $\iota_\gamma \in \mathcal{N}_\gamma^N$.
*Notice that property **v)** is obviously satisfied in this case and, viceversa, for a realisation of the triple $(\mathfrak{m}^N, \gamma^N, \iota_{\gamma_i^N})$, we have $\Phi_i = \widetilde{\Phi}_i$, $\mathbb{P}_N$-a.s. and $\widetilde{\Phi}_i(t,\cdot,\cdot)$ is $\sigma(\Phi_i, \vartheta_t)$-measurable, for any $t \in [\![0, T-1]\!]$.*

We associate to each triple $(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma}) \in \mathcal{P}(\mathcal{X}^N) \times \mathcal{P}(\widehat{\mathcal{R}}_N^N) \times \mathcal{P}(\widehat{\mathcal{R}}_N \times \widehat{\mathcal{R}}_N)$ a cost functional for player i, through the following expression that exploits the concept of realisation:

$$
J_i^N(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma}) := \mathbb{E}\left[\sum_{t=0}^{T-1} f\left(t, X_t^{i,N}, \mu_t^{i,N}, \widetilde{\Phi}_i\left(t, X^{i,N}, \mu^{i,N}\right)\right) + F\left(X_T^{i,N}, \mu_T^{i,N}\right)\right].
$$

By construction, the right-hand side of (1.3) does not depend on the particular realisation but only on $(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma})$. Indeed, $\widetilde{\gamma} \in \mathcal{N}_{\gamma_i^N}^N$ yields

$$
\widetilde{\gamma}(\mathrm{d}\varphi, \mathrm{d}\psi) = \beta_0^N(\varphi^{(0)}, \mathrm{d}\psi_0)\ldots\beta_T^N(\varphi^{(T)}, \mathrm{d}\psi_T)(\gamma_i^N)(\mathrm{d}\varphi),
$$

for some $\beta^N = (\beta_t^N)_{t\in[\![0,T]\!]} \in \mathcal{K}_N$. Thus, the cost functional above is well-posed and we write

$$
\begin{aligned}
&J_i^N(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma}) \\
&= \int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\widehat{\mathcal{R}}_N^N}\int_{\widehat{\mathcal{E}}_{0,N}}\ldots\int_{\widehat{\mathcal{E}}_{T,N}} G^N(x_1,\ldots,x_N,\varphi_0,\ldots,\varphi_{T-1},u_1,\ldots,u_N,z_1,\ldots,z_{NT}) \\
&\qquad \beta_T^N(u_i, \mathrm{d}\varphi_T)\cdots\beta_0^N(u_i,\mathrm{d}\varphi_0)\gamma^N(\mathrm{d}u_1,\ldots,\mathrm{d}u_N)\nu^{\otimes NT}(\mathrm{d}z_1,\ldots,\mathrm{d}z_{NT})\mathfrak{m}_0^{\otimes N}(\mathrm{d}x_1,\ldots,\mathrm{d}x_N),
\end{aligned}
$$

for some measurable function $G^N : \mathcal{X}^N \times \widehat{\mathcal{E}}_{0,N} \times \ldots \times \widehat{\mathcal{E}}_{T-1,N} \times \widehat{\mathcal{R}}_N^N \times \mathcal{Z}^{NT} \to \mathbb{R}$.

Since, for each $i \in [\![1, N]\!]$, the functional $J_i^N(\cdot)$ represents the costs that player i faces, his aim is to minimise it. As natural when dealing with several players, we deal with an equilibrium concept for optimality.

Definition 1.3.5. *Let $\varepsilon \geq 0$. We call a distribution $\gamma^N \in \mathcal{P}(\widehat{\mathcal{R}}_N^N)$ an ε-*correlated equilibrium *with initial distribution $\mathfrak{m}^N \in \mathcal{P}(\mathcal{X}^N)$ if we have*

$$J_i^N(\mathfrak{m}^N, \gamma^N, \iota_{\gamma_i^N}) \leq J_i^N(\mathfrak{m}^N, \gamma^N, \widetilde{\gamma}) + \varepsilon,$$

for every player $i \in [\![1, N]\!]$ and every strategy modification $\widetilde{\gamma} \in \mathcal{N}_{\gamma_i^N}$.
In particular, we call γ^N a correlated equilibrium, *denoted by CE, if $\varepsilon = 0$.*

Definition 1.3.5 is in line with the notion of correlated equilibrium present in the literature. We stress that, here, the deviating player has access to the entire history of the system and, in addition, is allowed to use a randomisation device.

1.4 The mean field game

Let $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$ be the initial distribution of our mean field system. In this model there is only one representative player in the mean field game because of the symmetry in the N-player game.

Definition 1.4.1. *Let $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ and call it a* correlated suggestion. *Call* strategy modification *a function $w \in \widehat{\mathcal{D}}$. Then, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting $\mathcal{X}$-valued process $(X_t)_{t=0}^T$, an $\mathcal{R}$-valued random variable Φ, a $\mathcal{P}(\mathcal{X})^{T+1}$ -valued random variable μ and $\mathcal{Z}$-valued random variables $(\xi_t)_{t=1}^T$, such that the following properties hold:*

- ***i)*** $\mathbb{P} \circ X_0^{-1} = \mathfrak{m}_0$;
- ***ii)*** $\mathbb{P} \circ (\Phi, (\mu_t)_{t=0}^T)^{-1} = \rho$;
- ***iii)*** $(\xi_t)_{t=1}^T$ *are i.i.d. all distributed according to ν;*
- ***iv)*** $(\xi_t)_{t=1}^T$, X_0 *and* $(\Phi, (\mu_t)_{t=0}^T)$ *are independent;*
- ***v)*** *the evolution of $(X_t)_{t=0}^T$ follows this dynamics: for any $t \in [\![0, T-1]\!]$,*

$$X_{t+1} = \Psi\left(t, X_t, \mu_t, w \circ \Phi(t, X, \mu), \xi_{t+1}\right), \qquad \mathbb{P}\text{-a.s.}. \tag{1.4.1}$$

We call any tuple $\big((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, w, (X_t)_{t=0}^T\big)$ satisfying the conditions above a realisation of the triple $(\mathfrak{m}_0, \rho, \widetilde{\rho})$.

The strategy modification w represents how the representative player decides to deviate from the suggestion he was given. Notice that, contrary to the N-player game where the i^{th} player can exploit a randomisation device when selecting the strategy to put in action, the choice here is a deterministic functional of the suggestion, Φ, provided by the mediator.

Remark 1.4.2. *As for the N-player game, we can characterise the form of a realisation for the case in which the representative player follows the suggestion provided to him.*
This is the case when the function w is just the identity. Indeed, we have $w \circ \varphi(t, x^{(t)}, m^{(t)}) = \varphi(t, x_t)$, for each $t \in [\![0, T-1]\!]$ and $\varphi \in \mathcal{R}$. We call this special modification ι.

The player in the mean field game faces costs associated to the triple $(\mathfrak{m}_0, \rho, w) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big) \times \widehat{\mathcal{D}}$ that are given by

$$J(\mathfrak{m}_0, \rho, w) := \mathbb{E}\left[\sum_{t=0}^{T-1} f\left(t, X_t, \mu_t, w \circ \Phi(t, X^{(t)}, \mu^{(t)})\right) + F\left(X_T, \mu_T\right)\right].$$

As noticed for the N-player game, we highlight that the cost functional above is well defined since the right-hand side does not depend on the realisation considered but only on $(\mathfrak{m}_0, \rho, w)$, and we may write

$$J(\mathfrak{m}_0, \rho, w) = \int_{\mathcal{X}} \int_{\mathcal{Z}^T} \int_{\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}} G_w(x, \varphi, z, m) \rho(\mathrm{d}\varphi, \mathrm{d}m) \nu^{\otimes T}(\mathrm{d}z) \mathfrak{m}_0(\mathrm{d}x),$$

for some function $G_w : \mathcal{X} \times \mathcal{R} \times \mathcal{Z}^T \times \mathcal{P}(\mathcal{X})^{T+1} \to \mathbb{R}$.

Definition 1.4.3. *We say that* $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ *is a* correlated solution for the mean field game *with initial distribution* $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$*, if the following two conditions hold:*

(Opt) *For each strategy modification* $w \in \widehat{\mathcal{D}}$*,*

$$J(\mathfrak{m}_0, \rho, \iota) \le J(\mathfrak{m}_0, \rho, w).$$

(Con) *For any realisation of* $(\mathfrak{m}_0, \rho, \iota)$*, namely* $\big((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t)_{t=0}^T\big)$*, setting* $\mathcal{F}^\mu := \sigma\big((\mu_t)_{t=0}^T\big)$*, we have*

$$\mu_t(\cdot) = \mathbb{P}(X_t \in \cdot \,|\mathcal{F}^\mu), \quad t \in [\![0, T]\!].$$

The first condition above is called *optimality condition*, the second is called *consistency condition*.

Remark 1.4.4. *A correlated solution according to Definition 1.4.3 is an element of* $\mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$*. The mediator thus suggests to play strategies that depend only on time and the representative player's current state (Markov open-loop or* restricted strategies *as in [30]). By the optimality condition, following the mediator's recommendations in those restricted strategies has to be optimal against progressive deviations, that is, strategies that may depend on the entire history of state and flow of measures up to current time. More precisely, if the representative player decides to deviate, then she chooses a strategy modification* w *(not equal to the identity on* $\mathcal{R}$*) that takes a (restricted) strategy recommended by the mediator and transforms it into a progressive feedback strategy, which is then applied to generate the state dynamics; see Equation (1.4.1).*

Remark 1.4.5. *In the consistency condition of Definition 1.4.3, we take conditional distribution with respect to* $\mathcal{F}^\mu$*, the* σ*-algebra generated by the entire flow of measures* μ *(up to terminal time* T*). This implies the generally weaker condition*

$$\mu_t(\cdot) = \mathbb{P}(X_t \in \cdot \,|\mathcal{F}_t^\mu), \quad t \in [\![0, T]\!], \tag{1.4.2}$$

where $\mathcal{F}_t^\mu := \sigma\big((\mu_s)_{s=0}^t\big)$ *is the* σ*-algebra generated by the flow of measures* μ *up to time* t*. The intuition behind conditioning on the entire flow of measures is the following. In choosing a correlated equilibrium,* *the mediator wants to induce a certain behavior of the population. That behavior is represented by the flow of measures μ. In equilibrium, the representative player accepts the mediator's recommendations. But those recommendations are potentially correlated with the flow of measures up to terminal time. As a consequence, the player's state X_t at any intermediate time t can be correlated with the flow of measures μ also at future times. In order to reproduce the population behavior given by μ, the representative player's state must therefore satisfy the consistency condition according to* **(Con)**, *not just* (1.4.2). *For further discussion also see Remark 4.2 in* [30].

1.5 The mean field game in open-loop strategies

Now, we formalise an alternative structure for the mean field game, extending the class of admissible control policies. We then prove in Section 1.5.2 that, under a mild assumption on the form of the correlated solution ρ, the value of the MFG remains the same.

1.5.1 The definition of the MFG in open-loop strategies

Let $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$ be the initial distribution of the mean field system.

Definition 1.5.1. *Let $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$.*
A tuple $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{G}_t\}_{t=0}^{T-1}, \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t)_{t=0}^{T-1})$ is said to be an open-loop control policy (open-loop strategy) *if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space supporting $\mathcal{X}$-valued random variables $X_t, t \in [\![0, T]\!]$, an $\mathcal{R}$-valued random variable Φ, a $\mathcal{P}(\mathcal{X})^{T+1}$ -valued random variable μ, $\mathcal{Z}$-valued random variables $(\xi_t)_{t=1}^{T}$ and Γ-valued random variables $u_t, t \in [\![0, T-1]\!]$, and $\{\mathcal{G}_t\}_{t=0}^{T-1}$ is a complete filtration such that*

- ***i)*** $\mathbb{P} \circ (X_0)^{-1} = \mathfrak{m}_0$;
- ***ii)*** $\mathbb{P} \circ (\Phi, (\mu_t)_{t=0}^{T})^{-1} = \rho$;
- ***iii)*** $(\xi_t)_{t=1}^{T}$ *are i.i.d. all distributed according* ν;
- ***iv)*** $(\xi_t)_{t=1}^{T}$, X_0 *and* $(\Phi, (\mu_t)_{t=0}^{T})$ *are independent;*
- ***iv')*** *for each* $t \in [\![0, T-1]\!]$:
  - ξ_t *is* $\mathcal{G}_t$*-measurable and* ξ_{t+k}, $k = 1, \ldots, T-t$, *are jointly independent of* $\mathcal{G}_t$,
  - $\mathcal{G}_t = \mathcal{H}_t \vee \sigma(\mu^{(t)}) \vee \sigma(\Phi) \vee \sigma(X_0)$, *with* $\mathcal{H}_t$ *independent of* $\sigma(\Phi, (\mu_t)_{t=0}^{T}, X_0)$,
  - u_t *is* $\mathcal{G}_t$*-measurable;*
- ***v)*** *for all* $t \in [\![0, T-1]\!]$,
$$X_{t+1} = \Psi\left(t, X_t, \mu_t, u_t, \xi_{t+1}\right), \qquad \mathbb{P}\text{-a.s.}.$$

We denote with $\mathcal{A}$ the set of all open-loop control policies and, with a slight abuse of notation, in the following we write $(u_t)_{t=0}^{T-1} \in \mathcal{A}$ for $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{G}_t\}_{t=0}^{T-1}, \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t)_{t=0}^{T-1}) \in \mathcal{A}$.
We call any tuple $\big((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{G}_t\}_{t=0}^{T-1}, \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t)_{t=0}^{T-1}, (X_t)_{t=0}^{T}\big)$ as above a realisation of the triple $(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1})$.

Remark 1.5.2. *Notice that this new setting includes the previous one. Indeed, setting, for* $t \in [\![0, T-1]\!]$, $u_t = w \circ \Phi(t, X^{(t)}, \mu^{(t)})$, *the recursive structure of the problem yields that* u_t *is* $\mathcal{G}_t$*-measurable with* $\mathcal{G}_t = \sigma(X_0) \vee \sigma(\Phi) \vee \sigma(\mu^{(t)}) \vee \sigma(\xi^{(t)})$, *that is* $\mathcal{H}_t = \sigma(\xi^{(t)})$, *and thus all the conditions in* ***iv')*** *hold. In particular, the closed-loop strategy* ι, *corresponding to the case in which the* i^{th}*-player follows the mediator's suggestion induces the open-loop admissible strategy*

$$u_t^{\iota} := \iota \circ \Phi(t, X^{(t)}, \mu^{(t)}) = \Phi(t, X_t), \qquad \Phi \in \mathcal{R}.$$

In this case the costs associated to the triple $(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1}) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big) \times \mathcal{A}$ are given by

$$\widehat{J}(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1}) := \mathbb{E}\left[\sum_{t=0}^{T-1} f\left(t, X_t, \mu_t, u_t\right) + F\left(X_T, \mu_T\right)\right].$$

In this definition of the costs, there is a little abuse of notation. Indeed, $(u_t)_{t=0}^{T-1} \in \mathcal{A}$ stands for $((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{G}_t\}_{t=0}^{T-1}, \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t)_{t=0}^{T-1}) \in \mathcal{A}$.

Definition 1.5.3. *We say that* $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ *is an* open-loop correlated solution for the mean field game *with initial distribution* $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$, *if the following two conditions hold:*

(Opt) *For each strategy modification* $(u_t)_{t=0}^{T-1} \in \mathcal{A}$,

$$\widehat{J}(\mathfrak{m}_0, \rho, (u_t^{\iota})_{t=0}^{T-1}) \le \widehat{J}(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1}).$$

(Con) *For any realisation of* $(\mathfrak{m}_0, \rho, (u_t^{\iota})_{t=0}^{T-1})$, *namely* $((\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t=0}^{T-1}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t^{\iota})_{t=0}^{T-1}, (X_t)_{t=0}^{T})$, *setting* $\mathcal{F}^{\mu} := \sigma\big((\mu_t)_{t=0}^{T}\big)$, *we have*

$$\mu_t(\cdot) = \mathbb{P}(X_t \in \cdot \,|\mathcal{F}^{\mu}), \quad t \in [\![0, T]\!].$$

1.5.2 The optimal value of the objective functional in the MFG

This section is devoted to proving that the value of the objective functional at equilibrium in the limit game remains the same if we enlarge the set of admissible strategies to include open-loop controls with the information structure given in Definition 1.5.1.

We start by showing that, under suitable technical assumptions needed to guarantee the well-posedness of all the conditional expectations involved, a conditional *Dynamic Programming Principle* holds for MFG solutions in the sense of Definition 1.4.3. Then, we prove by backward induction in time that the value of the MFG in closed-loop strategies is the same as the one in open-loop strategies and that, therefore, a closed-loop solution according to Definition 1.4.3 is also an open-loop solution according to Definition 1.5.3.

Our first assumption requires the state dynamics to be non-degenerate; more precisely:

(A1) For any $t \in [\![0, T-1]\!]$, any $m \in \mathcal{P}(\mathcal{X})$, any $x, y \in \mathcal{X}$ and any $u \in \Gamma$,

$$\mathbb{P}(\Psi(t, x, m, u, Z) = y) > 0,$$

where Z is a random variable with distribution ν.

In addition, we make a finiteness assumption on the structure of the correlated solution. To this end, let ρ be a solution of the MFG starting at $\mathfrak{m}_0$ according to Definition 1.4.3. Consider a realisation $\big((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, w, (X_t)_{t=0}^{T}\big)$ of $(\mathfrak{m}_0, \rho, w)$ according to Definition 1.4.1.

Given the fact that $\mathcal{R}$ is finite and limiting our analysis to the functions $\varphi \in \mathcal{R}$ such that $\mathbb{P}(\Phi = \varphi) > 0$, the induced conditional probability $\mathbb{P}_\varphi(\cdot) := \mathbb{P}(\cdot|\Phi = \varphi)$ is well-defined. The finiteness assumption on ρ is now:

(R1) If $(\Phi, (\mu_t)_{t=0}^T)$ is distributed according to ρ, then there exists, for any choice of $\varphi \in \mathcal{R}$ such that $\mathbb{P}(\Phi = \varphi) > 0$, a subset $\mathcal{P}_\varphi \subset \mathcal{P}(\mathcal{X})^{T+1}$ of finite cardinality such that $\mathbb{P}_\varphi(\mu^{(T)} \in \mathcal{P}_\varphi) = 1$ and, for any $m \in \mathcal{P}_\varphi$, $\mathbb{P}_\varphi(\mu^{(T)} = m) > 0$.

Remark 1.5.4. *The assumptions above are used to ensure the well-posedness of conditional probabilities of the form $\mathbb{P}_\varphi(\cdot|\mu^{(t)} = m^{(t)}, X^{(t)} = x^{(t)})$, for any $m^{(t)} \in \mathcal{P}_\varphi^{(t)}$, any $x^{(t)} \in \mathcal{X}^{t+1}$, where $\mathcal{P}_\varphi^{(t)} := \pi_{\mathcal{P}(\mathcal{X})^{t+1}}(\mathcal{P}_\varphi) = \{m \in \mathcal{P}(\mathcal{X})^{t+1}$ s.t. there exists $l \in \mathcal{P}(\mathcal{X})^{T-t}$ s.t. $(m, l) \in \mathcal{P}_\varphi\}$. Indeed, for this to hold it is enough to check that $\mathbb{P}_\varphi(\mu^{(T)} = m, X^{(T)} = x^{(T)}) > 0$, for any $x^{(T)} \in \mathcal{X}^{T+1}$ and any $m \in \mathcal{P}_\varphi$. First, exploiting disintegration we write*

$$\mathbb{P}_\varphi(\mu^{(T)} = m, X^{(T)} = x^{(T)}) = \mathbb{P}_\varphi(X^{(T)} = x^{(T)}|\mu^{(T)} = m) \cdot \mathbb{P}_\varphi(\mu^{(T)} = m), \tag{1.5.1}$$

*where the second term in the product on the right is clearly strictly positive by Assumption **(R1)**. Then, another round of disintegration yields*

$$\begin{aligned} &\mathbb{P}_\varphi(X^{(T)} = x^{(T)}|\mu^{(T)} = m) \\ &\quad = \mathbb{P}_\varphi(X_0 = x_0|\mu^{(T)} = m) \prod_{t=0}^{T-1} \mathbb{P}_\varphi(X_{t+1} = x_{t+1}|\mu^{(T)} = m, X^{(t)} = x^{(t)}) \\ &\quad = \mathfrak{m}_0(\{x_0\}) \prod_{t=0}^{T-1} \mathbb{P}_\varphi(X_{t+1} = x_{t+1}|\mu^{(T)} = m, X^{(t)} = x^{(t)}). \end{aligned} \tag{1.5.2}$$

Now, exploiting the iterative dynamics of the state in the game, we have that, for any fixed $t \in [\![0, T-1]\!]$,

$$\begin{aligned} &\mathbb{P}_\varphi(X_{t+1} = x_{t+1}|\mu^{(T)} = m, X^{(t)} = x^{(t)}) \\ &= \mathbb{P}_\varphi(\Psi(t, x_t, m_t, u_t, \xi_{t+1}) = x_{t+1}|\mu^{(T)} = m, X^{(t)} = x^{(t)}) \\ &= \sum_{\gamma \in \Gamma} \mathbb{P}_\varphi(\Psi(t, x_t, m_t, \gamma, \xi_{t+1}) = x_{t+1})\mathbb{P}_\varphi(u_t = \gamma|\mu^{(T)} = m, X^{(t)} = x^{(t)}) > 0. \end{aligned} \tag{1.5.3}$$

Hence, putting together Equations (1.5.1), (1.5.2) and (1.5.3), we get

$$\mathbb{P}_\varphi(\mu^{(T)} = m, X^{(T)} = x^{(T)}) \geq \mathbb{P}_\varphi(\mu^{(T)} = m)\mathfrak{m}_0(\{x_0\}) \prod_{t=0}^{T-1} \mathbb{P}_\varphi(X_{t+1} = x_{t+1}|\mu^{(T)} = m, X^{(t)} = x^{(t)}) > 0.$$

Finally notice that the very same proof can be carried out replacing u_t with $w \circ \varphi(t, x^{(t)}, m^{(t)})$ and so the result holds, in particular, for the MFG in Definition 1.4.1.

Let ρ be a solution of the MFG starting at $\mathfrak{m}_0$ and satisfying **(R1)**. Consider a realisation $((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, w, (X_t)_{t=0}^T)$ of $(\mathfrak{m}_0, \rho, w)$. Set $\rho_2(\cdot|\varphi) = \mathbb{P}(\mu \in \cdot|\Phi = \varphi)$. Such a realisation then has the following properties, conditionally on the event $\{\Phi = \varphi\}$:

i)$_\varphi$ $\mathbb{P}_\varphi \circ (X_0)^{-1} = \mathfrak{m}_0$;

ii)$_\varphi$ $\mathbb{P}_\varphi \circ (\Phi, (\mu_t)_{t=0}^T)^{-1} = \mathbb{P}_\varphi \circ (\varphi, (\mu_t)_{t=0}^T)^{-1} = \delta_\varphi \otimes \rho_2(\cdot|\varphi)$;

iii)$_\varphi$ $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according to $\mathbb{P}_\varphi \cdot (\xi_t)^{-1} = \nu$;

iv)$_\varphi$ $(\xi_t)_{t=1}^T$, X_0 and $(\mu_t)_{t=0}^T$ are independent w.r.t. $\mathbb{P}_\varphi$;

v)$_\varphi$ for all $t \in [\![0, T-1]\!]$,

$$X_{t+1} = \Psi\left(t, X_t, \mu_t, w \circ \varphi(t, X^{(t)}, \mu^{(t)}), \xi_{t+1}\right), \qquad \mathbb{P}_\varphi\text{-a.s.}.$$

Notice that properties **i)**$_\varphi$, **ii)**$_\varphi$, **iii)**$_\varphi$, **iv)**$_\varphi$ and **v)**$_\varphi$ are a consequence of the corresponding properties in the unconditional setting and the independence in property **iv)**.

Hence, the (conditional) costs associated to the triple $(\mathfrak{m}_0, \rho, w) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}) \times \widehat{\mathcal{D}}$ are

$$J_\varphi(\mathfrak{m}_0, \rho, w) := \mathbb{E}_\varphi\left[\sum_{t=0}^{T-1} f\left(t, X_t, \mu_t, w \circ \varphi(t, X^{(t)}, \mu^{(t)})\right) + F\left(X_T, \mu_T\right)\right],$$

where $\mathbb{E}_\varphi[\cdot] := \mathbb{E}[\cdot|\Phi = \varphi]$.

We set

$$J_\varphi(t, x^{(t)}, m^{(t)}, w) = \mathbb{E}_\varphi\left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, w \circ \varphi(s, X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right].$$

and thus, in particular,

$$J_\varphi(T, x^{(T)}, m^{(T)}, w) = F(x_T, m_T).$$

Notice that, for any fixed $t \in [\![0, T-1]\!]$, $X^{(t)}$ and $(\mu_s)_{s=t+1}^T$ are $\mathbb{P}_\varphi$-conditionally independent given $\mu^{(t)}$. Indeed, consider a fixed $t \in [\![0, T-1]\!]$ and let $x^{(t)} \in \mathcal{X}^{t+1}$ and $m^{[t+1]} := (m_s)_{s=t+1}^T \in \mathcal{P}_\varphi^{[T-t]}$, with $\mathcal{P}_\varphi^{[T-t]} := \pi_{\mathcal{P}(\mathcal{X})^{T-t}}(\mathcal{P}_\varphi) = \{m \in \mathcal{P}(\mathcal{X})^{T-t}$ s.t. there exists $l \in \mathcal{P}(\mathcal{X})^{t+1}$ s.t. $(l, m) \in \mathcal{P}_\varphi\}$. Exploiting tower property and measurability, we have

$$\begin{aligned}
\mathbb{P}_\varphi(X^{(t)} = x^{(t)}, (\mu_s)_{s=t+1}^T = m^{[t+1]} | \mu^{(t)}) &= \mathbb{E}_\varphi[\mathbb{E}_\varphi[\mathbf{1}_{\{x^{(t)}\}}(X^{(t)})\mathbf{1}_{\{m^{[t+1]}\}}((\mu_s)_{s=t+1}^T) | X_0, \mu^{(t)}, \xi^{(t)}] | \mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{x^{(t)}\}}(X^{(t)})\mathbb{E}_\varphi[\mathbf{1}_{\{m^{[t+1]}\}}((\mu_s)_{s=t+1}^T) | X_0, \mu^{(t)}, \xi^{(t)}] | \mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{x^{(t)}\}}(X^{(t)})\mathbb{E}_\varphi[\mathbf{1}_{\{m^{[t+1]}\}}((\mu_s)_{s=t+1}^T) | \mu^{(t)}] | \mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{m^{[t+1]}\}}((\mu_s)_{s=t+1}^T) | \mu^{(t)}]\mathbb{E}_\varphi[\mathbf{1}_{\{x^{(t)}\}}(X^{(t)}) | \mu^{(t)}] \\
&= \mathbb{P}_\varphi(X^{(t)} = x^{(t)} | \mu^{(t)})\mathbb{P}_\varphi((\mu_s)_{s=t+1}^T = m^{[t+1]} | \mu^{(t)}).
\end{aligned}$$

As a consequence of the conditional independence stated above, if $w \circ \varphi(u, \cdot) = \widetilde{w} \circ \varphi(u, \cdot)$, for $u \geq t$, $J_\varphi(t, x^{(t)}, m^{(t)}, w) = J_\varphi(t, x^{(t)}, m^{(t)}, \widetilde{w})$. Indeed, take $w, \widetilde{w} \in \widehat{\mathcal{D}}$ such that $w \circ \varphi(u, \cdot) = \widetilde{w} \circ \varphi(u, \cdot)$, for $u \geq t$. We have

$$\begin{aligned}
&J_\varphi(t, x^{(t)}, m^{(t)}, \widetilde{w}) \qquad (1.5.4) \\
&= \mathbb{E}_\varphi\left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, \widetilde{w} \circ \varphi(s, X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right] \\
&=: \mathbb{E}_\varphi\left[G_t(x^{(t)}, m^{(t)}, (\mu_s)_{s=t+1}^T, (\xi_s)_{s=t+1}^T, (\widetilde{w} \circ \varphi(s, \cdot))_{s=t}^T) | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{Z}^{T-t}} \sum_{m^{[t+1]} \in \mathcal{P}_\varphi^{T-t}} G_t(x^{(t)}, m^{(t)}, m^{[t+1]}, (z_s)_{s=t+1}^T, (\widetilde{w} \circ \varphi(s, \cdot))_{s=t}^T) \mathbb{P}_\varphi((\mu_s)_{s=t+1}^T = m^{[t+1]} | \mu^{(t)}) \nu^{\otimes(T-t-1)}(\mathrm{d}z) \\
&= \int_{\mathcal{Z}^{T-t}} \sum_{m^{[t+1]} \in \mathcal{P}_\varphi^{T-t}} G_t(x^{(t)}, m^{(t)}, m^{[t+1]}, (z_s)_{s=t+1}^T, (w \circ \varphi(s, \cdot))_{s=t}^T) \mathbb{P}_\varphi((\mu_s)_{s=t+1}^T = m^{[t+1]} | \mu^{(t)}) \nu^{\otimes(T-t-1)}(\mathrm{d}z) \\
&= J_\varphi(t, x^{(t)}, m^{(t)}, w),
\end{aligned}
$$

where we have used the notation $\mathrm{d}z = \mathrm{d}z_{t+1}, \ldots, \mathrm{d}z_T$ and in the third identity we have exploited the fact that, for any fixed $t \in [\![0, T-1]\!]$, $X^{(t)}$ and $(\mu_s)_{s=t+1}^T$ are $\mathbb{P}_\varphi$-conditionally independent given $\mu^{(t)}$.
Then, we write $J_\varphi(t, x^{(t)}, m^{(t)}, (w_s)_{s=t}^T) = J_\varphi(t, x^{(t)}, m^{(t)}, w)$. Thus, the optimal value function is defined as

$$
\begin{aligned}
&V_\varphi(t, x^{(t)}, m^{(t)}) \\
&\quad = \inf_{w_t \in \widehat{\mathcal{R}}_t} \mathbb{E}_\varphi \left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, w_t(s, X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)} \right],
\end{aligned}
$$

where $\widehat{\mathcal{R}}_t := \{w : [\![t, T]\!] \times \mathcal{X}^T \times \mathcal{P}_\varphi \to \Gamma, \text{ progressively measurable}\}$.

Our aim, now, is to show that, even in this non-Markovian setting, the following DPP holds.

Proposition 1.5.5. *For any $t \in [\![0, T-1]\!]$,*

$$
\begin{aligned}
&V_\varphi(t, x^{(t)}, m^{(t)}) \\
&\quad = \inf_{\gamma \in \Gamma} \mathbb{E}_\varphi \left[f(t, x_t, m_t, \gamma) + V_\varphi\left(t, (x^{(t)}, \Psi(t, x_t, m_t, \gamma, \xi_{t+1})), (m^{(t)}, \mu_{t+1})\right) \middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)} \right].
\end{aligned}
$$

Proof. By construction and measurability properties, it holds

$$
\begin{aligned}
&V_\varphi(t, x^{(t)}, m^{(t)}) \\
&\quad = \inf_{w_t \in \widehat{\mathcal{R}}_t} \mathbb{E}_\varphi \left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, w_t(s, X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) \middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)} \right] \\
&\quad = \inf_{w_t \in \widehat{\mathcal{R}}_t} \mathbb{E}_\varphi \Bigg[f(t, x_t, m_t, w_t(t, x^{(t)}, m^{(t)})) + \sum_{s=t+1}^{T-1} f(s, X_s, \mu_s, w_t(s, X^{(s)}, \mu^{(s)})) \\
&\qquad\quad + F(X_T, \mu_T) \Bigg| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)} \Bigg] \\
&\quad = \inf_{w_t \in \widehat{\mathcal{R}}_t} \Bigg\{ f(t, x_t, m_t, w_t(t, x^{(t)}, m^{(t)})) \\
&\qquad + \sum_{(y,l) \in \mathcal{X} \times \mathcal{P}_\varphi} \mathbb{P}_\varphi(\Psi(t, x_t, m_t, w_t(t, x^{(t)}, m^{(t)}), \xi_{t+1}) = y, \mu_{t+1} = l | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \cdot \\
&\qquad \cdot \mathbb{E}_\varphi \Big[\sum_{s=t+1}^{T-1} f(s, X_s, \mu_s, w_t(s, X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) \Big| X^{(t+1)} = (x^{(t)}, y), \mu^{(t+1)} = (m^{(t)}, l) \Big] \Bigg\}
\end{aligned}
$$

Now, exploiting the fact that $\mathcal{P}_\varphi$ is finite (to exchange the inf and the summation) and the conditional independence property shown above (together with the consequent identity in

Equation (1.5.4)), we have

$$
\begin{aligned}
&V_\varphi(t, x^{(t)}, m^{(t)}) \\
&\quad = \inf_{\gamma\in\Gamma} \inf_{w_{t+1}\in\widehat{\mathcal{R}}_{t+1}} \Bigg\{ f(t, x_t, m_t, \gamma) \\
&\qquad + \sum_{(y,l)\in\mathcal{X}\times\mathcal{P}_\varphi} \mathbb{P}_\varphi(\Psi(t, x_t, m_t, \gamma, \xi_{t+1}) = y, \mu_{t+1} = l | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \cdot \\
&\qquad \cdot \mathbb{E}_\varphi\Big[\sum_{s=t+1}^{T-1} f(s, X_s, \mu_s, w_s(X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) \Big| X^{(t+1)} = (x^{(t)}, y), \mu^{(t+1)} = (m^{(t)}, l) \Big] \Bigg\} \\
&\quad = \inf_{\gamma\in\Gamma} \Bigg\{ f(t, x_t, m_t, \gamma) \\
&\qquad + \sum_{(y,l)\in\mathcal{X}\times\mathcal{P}_\varphi} \inf_{w_{t+1}\in\widehat{\mathcal{R}}_{t+1}} \Bigg\{ \mathbb{P}_\varphi(\Psi(t, x_t, m_t, \gamma, \xi_{t+1}) = y, \mu_{t+1} = l | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \cdot \\
&\qquad \cdot \mathbb{E}_\varphi\Big[\sum_{s=t+1}^{T-1} f(s, X_s, \mu_s, w_s(X^{(s)}, \mu^{(s)})) + F(X_T, \mu_T) \Big| X^{(t+1)} = (x^{(t)}, y), \mu^{(t+1)} = (m^{(t)}, l) \Big] \Bigg\} \Bigg\} \\
&\quad = \inf_{\gamma\in\Gamma} \Bigg\{ f(t, x_t, m_t, \gamma) + \sum_{(y,l)\in\mathcal{X}\times\mathcal{P}_\varphi} \Bigg\{ V_\varphi(t+1, (x^{(t+1)}, y), (m^{(t)}, l)) \cdot \\
&\qquad \cdot \mathbb{P}_\varphi(\Psi(t, x_t, m_t, \gamma, \xi_{t+1}) = y, \mu_{t+1} = l | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \Bigg\} \Bigg\} \\
&\quad = \inf_{\gamma\in\Gamma} \Bigg\{ \mathbb{E}_\varphi\Big[f(t, x_t, m_t, \gamma) + V_\varphi(t, (x^{(t)}, \Psi(t, x_t, m_t, \gamma, \xi_{t+1})), (m^{(t)}, \mu_{t+1})) | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)} \Big] \Bigg\}.
\end{aligned}
$$

□

Thus, we have shown the DPP and we can proceed with the second step.

Proposition 1.5.6. *Assume* **(A1)***. Let $\rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$ be a correlated solution of the MFG in closed-loop strategies starting at $\mathfrak{m}_0$ according to Definition 1.4.3. If ρ satisfies* **(R1)***, then ρ is a solution for the mean field game in open-loop strategies, as in Definition 1.5.3, too. In particular, for any $\varphi \in \mathcal{R}$, $\widehat{V}_\varphi(t, x^{(t)}, m^{(t)})$ and $V_\varphi(t, x^{(t)}, m^{(t)})$ coincide.*

Remark 1.5.7. *Notice that, since the consistency conditions in Definitions 1.4.3 and 1.5.3 are the same and the set of closed-loop strategies is included in the set of open-loop strategies, a solution of the correlated MFG in open-loop strategies, $\rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$, is automatically a solution for the corresponding game in closed-loop strategies.*

Proof. We have already discussed the form of the objective functional for the MFG in closed-loop controls when showing the DPP. Regarding the relaxed MFG, conditionally on the suggestion received by the representative player (that is on the event $\{\Phi = \varphi\}$), a realisation of $(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1})$, i.e. a tuple $\big((\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t=0}^{T-1}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^{T}, X_0, (\xi_t)_{t=1}^{T}, (u_t)_{t=0}^{T-1}, (X_t)_{t=0}^{T}\big)$, satisfies the following:

i)$_\varphi$ $\mathbb{P}_\varphi \circ (X_0)^{-1} = \mathfrak{m}_0$;

ii)$_\varphi$ $\mathbb{P}_\varphi \circ (\Phi, (\mu_t)_{t=0}^T)^{-1} = \delta_\varphi \otimes \rho_2(\cdot|\varphi)$;

iii)$_\varphi$ $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according $\mathbb{P}_\varphi \circ (\xi_t)^{-1} = \nu$;

iv)$_\varphi$ $(\xi_t)_{t=1}^T$, X_0 and $(\mu_t)_{t=0}^T$ are independent w.r.t. $\mathbb{P}_\varphi$;

iv')$_\varphi$ For each $t \in [\![0, T-1]\!]$,

- ξ_t is $\mathcal{G}_t$-measurable and ξ_{t+k}, $k = 1, \dots, T-t$, are jointly independent of $\mathcal{G}_t$ w.r.t. $\mathbb{P}_\varphi$,
- $\mathcal{G}_t = \mathcal{H}_t \vee \sigma(\mu^{(t)}) \vee \sigma(\Phi) \vee \sigma(X_0)$, with $\mathcal{H}_t$ independent of $\sigma((\mu_t)_{t=0}^T, X_0, \Phi)$ w.r.t. $\mathbb{P}_\varphi$,
- u_t is $\mathcal{G}_t$-measurable,

v')$_\varphi$ for any $t \in [\![0, T-1]\!]$,

$$X_{t+1} = \Psi\left(t, X_t, \mu_t, u_t, \xi_{t+1}\right), \qquad \mathbb{P}_\varphi\text{-a.s.}.$$

Let's quickly review how we check the properties in **iv')**$_\varphi$. Notice that the other properties are trivial. Let us first recall that measurability properties concern σ-algebras and not the specific probability measure on them, hence we have to exhibit proofs only for the independence properties. For any arbitrary fixed $t \in [\![0, T-1]\!]$, we have

- $(\xi_{t+k})_{k=1}^{T-t}$, are jointly independent of $\mathcal{G}_t$ w.r.t. $\mathbb{P}_\varphi$. Indeed, let $A \in \mathcal{G}_t$ and $(B_k)_{k=1}^{T-t} \in \mathcal{B}(\mathcal{Z})$, exploiting the tower property, the fact that $\sigma(\Phi) \subset \mathcal{G}_t$ and the fact that $\mathcal{G}_t$ and $(\xi_{t+k})_{k=1}^{T-t}$ are independent w.r.t. $\mathbb{P}$, we obtain

$$\begin{aligned}
&\mathbb{P}_\varphi(A \cap \{\xi_{t+1} \in B_1\} \cap \dots \cap \{\xi_T \in B_{T-t}\}) = \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A\mathbf{1}_{B_1}(\xi_{t+1})\dots\mathbf{1}_{B_{T-t}}(\xi_T)]}{\mathbb{P}(\Phi = \varphi)} \\
&= \frac{\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A\mathbf{1}_{B_1}(\xi_{t+1})\dots\mathbf{1}_{B_{T-t}}(\xi_T)|\mathcal{G}_t]]}{\mathbb{P}(\Phi = \varphi)} = \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A\mathbb{E}[\mathbf{1}_{B_1}(\xi_{t+1})\dots\mathbf{1}_{B_{T-t}}(\xi_T)|\mathcal{G}_t]]}{\mathbb{P}(\Phi = \varphi)} \\
&= \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A \prod_{k=1}^{T-t}\mathbb{P}(\xi_{t+k} \in B_k)]}{\mathbb{P}(\Phi = \varphi)} = \prod_{k=1}^{T-t}\mathbb{P}(\xi_{t+k} \in B_k)\mathbb{P}(A|\Phi = \varphi).
\end{aligned}$$

- $\mathcal{G}_t = \mathcal{H}_t \vee \sigma(\mu^{(t)}) \vee \sigma(\Phi) \vee \sigma(X_0)$, with $\mathcal{H}_t$ independent of $\sigma(\mu, X_0, \Phi)$ w.r.t. $\mathbb{P}_\varphi$. By assumption, $\mathcal{H}_t$, $\sigma(X_0)$ and $\sigma(\Phi, \mu)$ are independent w.r.t. $\mathbb{P}$. Take $A \in \mathcal{H}_t$, $B \in \sigma(\mu)$ and $C \in \sigma(X_0)$, we get

$$\begin{aligned}
\mathbb{P}_\varphi(A \cap B \cap C) &= \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C]}{\mathbb{P}(\Phi = \varphi)} = \frac{\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C|\Phi, \mu]]}{\mathbb{P}(\Phi = \varphi)} = \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_B\mathbb{E}[\mathbf{1}_A\mathbf{1}_C|\Phi, \mu]]}{\mathbb{P}(\Phi = \varphi)} \\
&= \frac{\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_B\mathbb{E}[\mathbf{1}_A\mathbf{1}_C]]}{\mathbb{P}(\Phi = \varphi)} = \frac{\mathbb{P}(A)\mathbb{P}(C)\mathbb{E}[\mathbf{1}_{\{\varphi\}}(\Phi)\mathbf{1}_B]}{\mathbb{P}(\Phi = \varphi)} = \mathbb{P}_\varphi(A)\mathbb{P}_\varphi(C)\mathbb{P}_\varphi(B).
\end{aligned}$$

The (conditional) costs associated to the triple $(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1}) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big) \times \mathcal{A}$ are

$$\widehat{J}_\varphi(\mathfrak{m}_0, \rho, (u_t)_{t=0}^{T-1}) := \mathbb{E}_\varphi\left[\sum_{t=0}^{T-1} f\left(t, X_t, \mu_t, u_t)\right) + F\left(X_T, \mu_T\right)\right].$$

Then,

$$\widehat{V}_\varphi(t, x^{(t)}, m^{(t)}) = \inf_{(u_t)_{t=0}^{T-1} \in \mathcal{A}} \mathbb{E}_\varphi\left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, u_s) + F(X_T, \mu_T)|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right],$$

and so, in particular, at the terminal time $T \in \mathbb{N}$, we have

$$\widehat{V}_\varphi(T, x^{(T)}, m^{(T)}) = F(X_T, m_T).$$

We want to prove that $\widehat{V}_\varphi = V_\varphi$. One side of the inequality is straightforward. Indeed, closed-loop controls as in Definition 1.4.1 induce admissible open-loop controls in the sense of Definition 1.5.1, through

$$u_t := w \circ \varphi(t, X^{(t)}, \mu^{(t)}), \qquad t \in [\![0, T-1]\!].$$

Thus, it holds $\widehat{V}_\varphi \leq V_\varphi$. We show that $\widehat{V}_\varphi \geq V_\varphi$, by backward induction on t. We have $\widehat{V}_\varphi(T, x^{(T)}, m^{(T)}) = F(X_T, m_T) = V_\varphi(T, x^{(T)}, m^{(T)})$. Now, as an induction hypothesis, assume that $\widehat{V}_\varphi(t+1, x^{(t+1)}, m^{(t+1)}) = V_\varphi(t+1, x^{(t+1)}, m^{(t+1)})$. To prove that $\widehat{V}_\varphi(t, x^{(t)}, m^{(t)}) = V_\varphi(t, x^{(t)}, m^{(t)})$, it is enough to check that $\widehat{J}_\varphi(t, x^{(t)}, m^{(t)}, (u_t)_{t=0}^{T-1}) \geq V_\varphi(t, x^{(t)}, m^{(t)})$, for any admissible sequence of controls $u \in \mathcal{A}$. Exploiting the definitions and induction hypothesis, we see

$$\begin{aligned}
&\widehat{J}_\varphi(t, x^{(t)}, m^{(t)}, (u_t)_{t=0}^{T-1}) = \mathbb{E}_\varphi\left[\sum_{s=t}^{T-1} f(s, X_s, \mu_s, u_s) + F(X_T, \mu_T)\middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right] \\
&= \mathbb{E}_\varphi\left[f(t, x_t, m_t, u_t)\middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right] \\
&\quad + \int_{X \times \mathcal{P}(X)} \mathbb{P}_\varphi(X_{t+1} = X_{t+1}, \mu_{t+1} = m_{t+1} | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \cdot \\
&\quad \cdot \mathbb{E}_\varphi\left[\sum_{s=t+1}^{T-1} f(s, X_s, \mu_s, u_s) + F(X_T, \mu_T)\middle| X^{(t+1)} = x^{(t+1)}, \mu^{(t+1)} = m^{(t+1)}\right] \\
&= \mathbb{E}_\varphi\left[f(t, x_t, m_t, u_t)\middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right] + \int_{X \times \mathcal{P}(X)} \widehat{J}_\varphi(t+1, x^{(t+1)}, m^{(t+1)}, u) \cdot \\
&\quad \cdot \mathbb{P}_\varphi(X_{t+1} = x_{t+1}, \mu_{t+1} = m_{t+1} | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \\
&\geq \mathbb{E}_\varphi\left[f(t, x_t, m_t, u_t)\middle| X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\right] + \int_{X \times \mathcal{P}(X)} V_\varphi(t+1, x^{(t+1)}, m^{(t+1)}) \cdot \\
&\quad \cdot \mathbb{P}_\varphi(X_{t+1} = x_{t+1}, \mu_{t+1} = m_{t+1} | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}).
\end{aligned}$$

Now, exploiting, in sequence, the fact that ξ_{t+1} is jointly independent of $X^{(t)}, u_t$ and $\mu^{(t+1)}$, the tower property, the fact that u_t is $\mathcal{G}_t$-measurable, the fact that μ_{t+1} and $\mathcal{G}_t$ are $\mathbb{P}_\varphi$-conditionally independent given $\mu^{(t)}$ and the measurability properties of conditional expectations, we obtain, for any $A \in \mathcal{B}(\mathcal{P}(\mathcal{X}))$, $B \in \mathcal{B}(\mathcal{Z})$ and $C \in \mathcal{B}(\Gamma)$,

$$\begin{aligned}
&\mathbb{P}_\varphi(\mu_{t+1} \in A, \xi_{t+1} \in B, u_t \in C | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \qquad (1.5.5) \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \mathbb{P}_\varphi(\mu_{t+1} \in A, u_t \in C | X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B) \frac{\mathbb{E}[(\mathbf{1}_A(\mu_{t+1}) \mathbf{1}_C(u_t))(\mathbf{1}_{\{x^{(t)}\}}(X^{(t)}) \mathbf{1}_{\{m^{(t)}\}}(\mu^{(t)}) \mathbf{1}_{\{\varphi\}}(\Phi))]}{\mathbb{P}(X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}, \Phi = \varphi)} \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B) \frac{\mathbb{E}[\mathbb{E}[\mathbf{1}_A(\mu_{t+1}) | \mathcal{G}_t] \mathbf{1}_C(u_t) \mathbf{1}_{\{x^{(t)}\}}(X^{(t)}) \mathbf{1}_{\{m^{(t)}\}}(\mu^{(t)}) \mathbf{1}_{\{\varphi\}}(\Phi)]}{\mathbb{P}(X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}, \Phi = \varphi)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}_\varphi(\xi_{t+1} \in B) \frac{\mathbb{E}[\mathbb{E}[\mathbf{1}_A(\mu_{t+1})|\Phi, \mu^{(t)}]\mathbf{1}_C(u_t)\mathbf{1}_{\{x^{(t)}\}}(X^{(t)})\mathbf{1}_{\{m^{(t)}\}}(\mu^{(t)})\mathbf{1}_{\{\varphi\}}(\Phi)]}{\mathbb{P}(X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}, \Phi = \varphi)} \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B)\mathbb{E}[\mathbf{1}_C(u_t)\mathbb{E}[\mathbf{1}_A(\mu_{t+1})|\Phi, \mu^{(t)}]|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}, \Phi = \varphi] \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B)\mathbb{E}[\mathbf{1}_A(\mu_{t+1})|\Phi = \varphi, \mu^{(t)} = m^{(t)}]\mathbb{E}[\mathbf{1}_C(u_t)|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}, \Phi = \varphi] \\
&= \mathbb{P}_\varphi(\xi_{t+1} \in B)\mathbb{P}_\varphi(\mu_{t+1} \in A|\mu^{(t)} = m^{(t)})\mathbb{P}_\varphi(u_t \in C|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}) \\
&= \nu(B)\mathbb{P}_\varphi(\mu_{t+1} \in A|\mu^{(t)} = m^{(t)})\lambda_t(C),
\end{aligned}
$$

where $\lambda_t(C) := \mathbb{P}_\varphi(u_t \in C|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)})$. Then, exploiting the iterative dynamics of the state and Equation (1.5.5), we have

$$
\begin{aligned}
&\widehat{J}_\varphi(t, x^{(t)}, m^{(t)}, (u_t)_{t=0}^{T-1}) \geq \mathbb{E}_\varphi\Big[f(t, x, m_t, u_t) \\
&\qquad + V_\varphi(t+1, (x^{(t)}, \Psi(t, x_t, m_t, u_t, \xi_{t+1})), (m^{(t)}, \mu_{t+1}))|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\Big] \\
&= \mathbb{E}_\varphi\Big[f(t, x, m_t, u_t) + \int_{\mathcal{Z}} V_\varphi(t+1, (x^{(t)}, \Psi(t, x_t, m_t, u_t, z)), (m^{(t)}, \mu_{t+1}))\nu(\mathrm{d}z)|X^{(t)} = x^{(t)}, \mu^{(t)} = m^{(t)}\Big] \\
&= \int_\Gamma \Big\{ f(t, x, m_t, \gamma) + \int_{\mathcal{Z}} \mathbb{E}_\varphi\Big[V_\varphi(t+1, (x^{(t)}, \Psi(t, x_t, m_t, \gamma, z)), (m^{(t)}, \mu_{t+1}))|\mu^{(t)} = m^{(t)}\Big]\nu(\mathrm{d}z)\Big\}\lambda_t(\mathrm{d}\gamma) \\
&\geq \inf_{\gamma \in \Gamma} \Big\{ f(t, x, m_t, \gamma)) + \int_{\mathcal{Z}} \mathbb{E}_\varphi\Big[V_\varphi(t+1, (x^{(t)}, \Psi(t, x_t, m_t, \gamma, z)), (m^{(t)}, \mu_{t+1}))|\mu^{(t)} = m^{(t)}\Big]\nu(\mathrm{d}z)\Big\} \\
&= V_\varphi(t, x^{(t)}, m^{(t)}),
\end{aligned}
$$

where the last identity follows from the DPP in Proposition 1.5.5. Finally, to conclude it is sufficient to integrate with respect to $\rho_1(\mathrm{d}\varphi)$. Hence, we have shown that under Assumptions **(A1)** and **(R1)** the optimal value of the two mean field games is the same.

□

1.6 Approximate N-player correlated equilibria

Here, we show how to construct approximate N-player correlated equilibria starting from a suitable solution of the MFG. We make the following additional assumptions on dynamics and costs:

(A2) *Continuity* of $\Psi\colon [\![0, T-1]\!] \times \mathcal{X} \times \Gamma \times \mathcal{Z} \to \mathcal{X}$:

1) For every $(t, x, \gamma) \in [\![0, T-1]\!] \times \mathcal{X} \times \Gamma$ and for all $m, \widetilde{m} \in \mathcal{P}(\mathcal{X})$,

$$\nu(\{z\colon \Psi(t, x, m, \gamma, z) \neq \Psi(t, x, \widetilde{m}, \gamma, z)\}) \leq \mathfrak{W}(\mathrm{dist}(m, \widetilde{m})),$$

where $\mathfrak{W}\colon [0, +\infty) \to [0, 1]$ is some non-decreasing function with $\lim_{s \to 0^+} \mathfrak{W}(s) = 0$.

2) For any $t \in [\![0, T-1]\!]$, $\Psi(t, \cdot)$ is $\tau \otimes \nu$-almost everywhere continuous, for every $\tau \in \mathcal{P}(\mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma)$.

(A3) The functions f and F, running cost and terminal cost, are Lipschitz continuous with the same Lipschitz constant L.

For an illustration of the continuity assumption **(A2)** on the dynamics, see [30, Remark 6.1]. Assumption **(A3)** is rather standard; in our finite setting, it is a true restriction only with respect to the measure argument of f and F.

The correlated suggestion ρ we start with must satisfy, in addition to **(R1)**, the following condition on its information structure:

(R2) If $(\Phi, (\mu_t)_{t=0}^T)$ is distributed according to ρ, then there exist $\alpha_t : [0,1] \times \mathcal{P}(X)^{t+2} \to \mathcal{E}$, $t \in [\![0, T-1]\!]$, Borel-measurable functions and a uniformly distributed random variable $Z \overset{d}{\sim} \nu$, independent of μ s.t. $\Phi(t,\cdot) := \alpha_t(Z, \mu^{(t+1)})(\cdot)$, for all $t \in [\![0, T-1]\!]$.

Remark 1.6.1. *If ρ satisfies **(R2)**, then it admits a decomposition of the form*

$$\begin{aligned}\rho(C_0 \times \cdots \times C_{T-1} \times B) &= \int_B \rho_1(C_0 \times \cdots \times C_{T-1}|m)\rho_2(\mathrm{d}m) \\ &= \int_B \int_{\mathcal{Z}} \otimes_{t=0}^{T-1} \delta_{\alpha_t(z, m^{(t+1)})}(C_t)\nu(\mathrm{d}z)\rho_2(\mathrm{d}m),\end{aligned}$$

for any $C_t \in \mathcal{B}(\mathcal{E})$, $t \in [\![0, T-1]\!]$ and $B \in \otimes^{T+1}\mathcal{B}(\mathcal{P}(\mathcal{X}))$ and where, for $t \in [\![0, T-1]\!]$, $\alpha_t : [0,1] \times \mathcal{P}(\mathcal{X})^{t+2} \to \mathcal{E}$ are Borel functions.

*Finally let us notice that, if $(\Phi, (\mu_t)_{t=0}^T)$ is distributed according to ρ that satisfies **(R2)**, then, for each $t \in [\![0, T-1]\!]$, $\Phi^{(t)}$ and μ are conditionally independent given $\mu^{(t+1)}$. The example presented in Section 1.7 seems to suggest that the two conditions are equivalent.*

Theorem 1.6.2. *Let $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$, and suppose **(A1)–(A3)** hold. Let $\rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$ be a correlated solution of the mean field game starting at $\mathfrak{m}_0$ and satisfying **(R1)–(R2)**. For $N \in \mathbb{N}$, define $\gamma^N \in \mathcal{P}(\mathcal{R}^N)$ by*

$$\gamma^N(C_1 \times \cdots \times C_N) := \int_{\mathcal{P}(\mathcal{X})^{T+1}} \prod_{j=1}^N \rho_1(C_j|m)\rho_2(\mathrm{d}m).$$

Then, for all $N \in \mathbb{N}$, γ^N is an ε_N-correlated equilibrium for the N-player game with initial distribution $\mathfrak{m}_0^{\otimes N}$ and the sequence $\{\varepsilon_N\}_{N\in\mathbb{N}} \subseteq [0, +\infty)$ is such that $\lim_{N\to\infty} \varepsilon_N = 0$.

Remark 1.6.3. *Let $(Z_j)_{j=1}^N$ be i.i.d. r.v.s distributed according to ν, also independent of μ, and define, for $j \in [\![1, N]\!]$, Φ_j through $\Phi_j(t,\cdot) := \alpha_t(Z_j, \mu^{(t+1)})(\cdot)$, $t \in [\![0, T-1]\!]$, with α as in **(R2)**. Then we have $\mathbb{P} \circ (\Phi_1, \ldots, \Phi_N)^{-1} = \gamma^N := \int_{\mathcal{P}(\mathcal{X})^{T+1}} \prod_{j=1}^N \rho_1(\cdot|m)\rho_2(\mathrm{d}m)$.*

Proof. We prove the result only for strategy modifications of the first player. Then, the general result is a consequence of the symmetry in the problem. With a small abuse of notation, we simply write ι for ι_γ, when its clear from the context the distribution that it refers to. We also use this same symbol ι for both the N-player game and the mean field game. Consider the correlated suggestion $\gamma^N \in \mathcal{P}(\mathcal{R}^N)$ defined in the statement of the theorem. For each $N \in \mathbb{N}$, γ^N is an ε_N-correlated equilibrium for the initial distribution $\mathfrak{m}_0^{\otimes N}$, once the sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ is defined as

$$\varepsilon_N := J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota) - \inf_{\widetilde{\beta}^N \in \mathcal{N}_{\gamma_1^N}} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\beta}^N), \qquad \text{for all } N \in \mathbb{N}.$$

By definition of infimum, it is possible to find a sequence of strategy modifications, $\{\widetilde{\gamma}^N\}_{N\in\mathbb{N}} \subseteq \mathcal{N}_{\gamma_1^N}$, such that

$$J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) \leq \inf_{\widetilde{\beta}^N \in \mathcal{N}_{\gamma_1^N}} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\beta}^N) + \frac{1}{N}, \quad N \in \mathbb{N}. \tag{1.6.1}$$

Thence, to complete the proof of the theorem, so showing that $\varepsilon_N \overset{N\to\infty}{\longrightarrow} 0$, it suffices to prove the following:

$$\lim_{N\to\infty} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota_{\gamma_1^N}) = J(\mathfrak{m}_0, \rho, \iota), \tag{1.6.2}$$

$$\liminf_{N\to\infty} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) \geq J(\mathfrak{m}_0, \rho, \iota). \tag{1.6.3}$$

Proof of (1.6.2). First of all, let us notice that the following equation holds

$$J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota_{\gamma_1^N}) = \int_{\mathcal{P}(\mathcal{X})^{T+1}} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota_{\gamma_{m,1}^N})\rho_2(dm),$$

where, for each $N \in \mathbb{N}$ and for each $m \in \mathcal{P}(\mathcal{X})^{T+1}$, $\gamma_m^N := \otimes^N \rho_1(\cdot|m)$. In fact, it holds $\gamma_1^N = \gamma^N \circ \pi_1^{-1} = \left(\int_{\mathcal{P}(\mathcal{X})^{T+1}} \rho_1(\cdot|m)^{\otimes N}\rho_2(\mathrm{d}m)\right) \circ \pi_1^{-1} = \int_{\mathcal{P}(\mathcal{X})^{T+1}} \rho_1(\cdot|m)\rho_2(\mathrm{d}m)$, and $\gamma_{m,1}^N = \gamma_m^N \circ \pi_1^{-1} = \rho_1(\cdot|m)^{\otimes N} \circ \pi_1^{-1} = \rho_1(\cdot|m)$.
Indeed, thanks to the particular structure of the cost functional and the fact that $\iota_{\gamma_1^N}(\mathrm{d}\varphi, \mathrm{d}u) = \delta_u(\mathrm{d}\varphi)\gamma_1^N(\mathrm{d}u)$, we write

$$\begin{aligned}
&J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota_{\gamma_1^N}) \\
&= \int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\mathcal{R}^N}\int_{\mathcal{R}} G^N(x, \varphi, (u_j)_{j=2}^N, z)\delta_{u_1}(\mathrm{d}\varphi)\gamma^N(\mathrm{d}u_1, \dots, \mathrm{d}u_N)\nu^{\otimes NT}(\mathrm{d}z)\mathfrak{m}_0^{\otimes N}(\mathrm{d}x) \\
&= \int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\mathcal{R}^N}\int_{\mathcal{R}} G^N(x, (u_j)_{j=1}^N, z)\gamma^N(\mathrm{d}u_1, \dots, \mathrm{d}u_N)\nu^{\otimes NT}(\mathrm{d}z)\mathfrak{m}_0^{\otimes N}(\mathrm{d}x) \\
&= \int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\mathcal{P}(\mathcal{X})^{T+1}}\int_{\mathcal{R}^N} G^N(x, (u_j)_{j=1}^N, z)\gamma_m^N(\mathrm{d}u_1, \dots, \mathrm{d}u_N)\rho_2(\mathrm{d}m)\nu^{\otimes NT}(\mathrm{d}z)\mathfrak{m}_0^{\otimes N}(\mathrm{d}x) \\
&= \int_{\mathcal{P}(\mathcal{X})^{T+1}}\left(\int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\mathcal{R}^N} G^N(x, (u_j)_{j=1}^N, z)\gamma_m^N(\mathrm{d}u_1, \dots, \mathrm{d}u_N)\nu^{\otimes NT}(\mathrm{d}z)\mathfrak{m}_0^{\otimes N}(\mathrm{d}x)\right)\rho_2(\mathrm{d}m) \\
&= \int_{\mathcal{P}(\mathcal{X})^{T+1}}\left(\int_{\mathcal{X}^N}\int_{\mathcal{Z}^{NT}}\int_{\mathcal{R}^{N+1}} G^N(x, \varphi, (u_j)_{j=2}^N, z)\delta_{u_1}(\mathrm{d}\varphi)\gamma_m^N(\mathrm{d}u)\nu^{\otimes NT}(\mathrm{d}z)\mathfrak{m}_0^{\otimes N}(\mathrm{d}x)\right)\rho_2(\mathrm{d}m) \\
&= \int_{\mathcal{P}(\mathcal{X})^{T+1}} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota_{\gamma_{m,1}^N})\rho_2(\mathrm{d}m).
\end{aligned}$$

Notice that, here we have implicitly exploited the conditional independence and independence properties proved in Remark 1.3.3, points ii) and iii). Indeed, assume $\mathbb{P}_N \circ (\widetilde{\Phi}_1, \Phi_1, \dots, \Phi_N)^{-1} = \lambda$ and denote with λ_i the measure projected on it i^{th} component(s), that is $\lambda_i := \lambda \circ (\pi_i)^{-1}$. Let $A, B_1, \dots, B_N \in \mathcal{B}(\widehat{\mathcal{R}}_N)$. Exploiting Remark 1.3.3 (iii), we get

$$\begin{aligned}
&\mathbb{P}_N(\widetilde{\Phi}_1 \in A, \Phi_1 \in B_1, \dots, \Phi_N \in B_N) \\
&= \int_{\widehat{\mathcal{R}}_N} \mathbf{1}_{B_1}(\varphi_1)\int_{\widehat{\mathcal{R}}_N\times\widehat{\mathcal{R}}_N^{N-1}} \mathbf{1}_A(\mathrm{d}\psi)\prod_{j=2}^N \mathbf{1}_{B_j}(\varphi_j)\lambda_{1,3,\dots,N+1}(\mathrm{d}\psi, \mathrm{d}\varphi_2, \dots, \mathrm{d}\varphi_N|\varphi_1)\lambda_2(\mathrm{d}\varphi_1) \\
&= \int_{\widehat{\mathcal{R}}_N} \mathbf{1}_{B_1}(\varphi_1)\int_{\widehat{\mathcal{R}}_N} \mathbf{1}_A(\mathrm{d}\psi)\lambda_1(\mathrm{d}\psi|\varphi_1)\int_{\widehat{\mathcal{R}}_N^{N-1}}\prod_{j=2}^N \mathbf{1}_{B_j}(\varphi_j)\lambda_{3,\dots,N+1}(\mathrm{d}\varphi_2, \dots, \mathrm{d}\varphi_N|\varphi_1)\lambda_2(\mathrm{d}\varphi_1)
\end{aligned}$$

$$= \int_{\widehat{\mathcal{R}}_N \times \widehat{\mathcal{R}}_N^N} \mathbf{1}_A(\mathrm{d}\psi) \prod_{j=1}^{N} \mathbf{1}_{B_j}(\varphi_j) \lambda_1(\mathrm{d}\psi|\varphi_1) \lambda_{3,\dots,N+1}(\mathrm{d}\varphi_2, \dots, \mathrm{d}\varphi_N|\varphi_1) \lambda_2(\mathrm{d}\varphi_1)$$

$$= \int_{\widehat{\mathcal{R}}_N \times \widehat{\mathcal{R}}_N^N} \mathbf{1}_A(\mathrm{d}\psi) \prod_{j=1}^{N} \mathbf{1}_{B_j}(\varphi_j) \delta_{\varphi_1}(\mathrm{d}\psi) \gamma^N(\mathrm{d}\varphi_1, \dots, \mathrm{d}\varphi_N),$$

where the last step is a consequence of the fact that $\mathbb{P}_N \circ (\Phi_1, \dots, \Phi_N)^{-1} = \gamma^N$ and $\mathbb{P}_N \circ (\widetilde{\Phi}_1, \Phi_1)^{-1} = \iota_{\gamma_i^N}$. Thus, we have $\lambda(\mathrm{d}\psi, \mathrm{d}\varphi_1, \dots, \mathrm{d}\varphi_N) = \delta_{\varphi_1}(\mathrm{d}\psi)\gamma^N(\mathrm{d}\varphi_1, \dots, \mathrm{d}\varphi_N)$.

Furthermore, for the mean field game, we have

$$J(\mathfrak{m}_0, \rho, \iota) = \int_{\mathcal{P}(\mathcal{X})^{T+1}} J(\mathfrak{m}_0, \rho_1(\cdot|m) \otimes \delta_m, \iota) \rho_2(\mathrm{d}m),$$

where $\rho_1(\cdot|m)$ as in the statement of the theorem. In fact, with similar computations as above for J_1^N, we get

$$\begin{aligned}
J(\mathfrak{m}_0, \rho, \iota) &= \int_{\mathcal{X}} \int_{\mathcal{Z}^T} \int_{\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}} G_\iota(x, \varphi, z, m) \rho(\mathrm{d}\varphi, \mathrm{d}m) \nu^{\otimes T}(\mathrm{d}z) \mathfrak{m}_0(\mathrm{d}x) \\
&= \int_{\mathcal{X}} \int_{\mathcal{Z}^T} \int_{\mathcal{P}(\mathcal{X})^{T+1}} \int_{\mathcal{R}} G_\iota(x, \varphi, z, m) \rho_1(\mathrm{d}\varphi|m) \rho_2(\mathrm{d}m) \nu^{\otimes T}(\mathrm{d}z) \mathfrak{m}_0(\mathrm{d}x) \\
&= \int_{\mathcal{P}(\mathcal{X})^{T+1}} \int_{\mathcal{X}} \int_{\mathcal{Z}^T} \int_{\mathcal{R}} G_\iota(x, \varphi, z, m) \rho_1(\mathrm{d}\varphi|m) \nu^{\otimes T}(\mathrm{d}z) \mathfrak{m}_0(\mathrm{d}x) \rho_2(\mathrm{d}m) \\
&= \int_{\mathcal{P}(\mathcal{X})^{T+1}} J(\mathfrak{m}_0, \rho_1(\cdot|m) \otimes \delta_m, \iota) \rho_2(\mathrm{d}m),
\end{aligned}$$

and this ends the proof of the identity.

In the proof of (1.6.2), that is the case in which all the players follow the mediator's suggestion, computations simplify considerably. Indeed, since the recommendation γ^N belongs to $\mathcal{P}(\mathcal{R}^N)$, for any $N \in \mathbb{N}$, we can proceed as in the proof of [30, Theorem 5.1 and Theorem 6.1], that is through the following three steps:

1. We show that, for any fixed $m \in \mathcal{P}(\mathcal{X})^{T+1}$, there exists a subsequence of indices such that
$$\lim_{k \to \infty} J_1^{N_k}(\mathfrak{m}_0^{\otimes N_k}, \gamma_m^{N_k}, \iota) = J(\mathfrak{m}_0, \rho_m, \iota_{\rho_m}),$$
for some $\rho_m \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}$), with $\gamma_m^N = \rho_1(\cdot|m)^{\otimes N}$.

2. We prove a result of chaos propagation that enables us to deduce that, in the limit, for all $m \in \mathcal{P}(\mathcal{X})^{T+1}$, we have
$$\mathbb{P}_m \circ (X_t^m, \mu_t^m)^{-1} = \widehat{m}_t^m \otimes \delta_{\widehat{m}_t^m}, \quad \text{for all } t \in [\![0, T]\!],$$
for some $\widehat{m}_t^m \in \mathcal{P}(\mathcal{X})$.

3. We show that, for ρ_2-almost every $m \in \mathcal{P}(\mathcal{X})^{T+1}$, $(\widehat{m}_t^m)_{t=0}^T = (m_t)_{t=0}^T$, independently of the convergent subsequence considered, and conclude by integrating in $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$ w.r.t. $\rho_2(\mathrm{d}m)$.

Step 1

Fix a flow of measure $m \in \mathcal{P}(\mathcal{X})^{T+1}$. Consider the sequence of triples $\{(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota)\}_{N\in\mathbb{N}}$. For each $N \in \mathbb{N}$, consider the tuple $((\Omega_{N,m}, \mathcal{F}_{N,m}, \mathbb{P}_{N,m}), (\Phi_j^{N,m})_{j=1}^N, (\vartheta_t^{N,m})_{t=0}^{T-1}, (\xi_t^{1,N,m}, \ldots, \xi_t^{N,N,m})_{t=1}^T, \widetilde{\Phi}_1^{N,m}, (X_t^{1,N,m}, \ldots, X_t^{N,N,m})_{t=0}^T)$, a realisation of $(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota)$. Since we are proving (1.6.2), w.l.o.g. we assume that $\Phi_1^{N,m} = \widetilde{\Phi}_1^{N,m}$, $\mathbb{P}_{N,m}$-a.s. Set, for any $N \in \mathbb{N}$,

$$\eta_m^N := \mathbb{P}_{N,m} \circ (\Phi_1^{N,m}, (\mu_t^{1,N,m})_{t=0}^T, (\xi_t^{1,N,m})_{t=1}^T, \widetilde{\Phi}_1^{N,m}, (X_t^{1,N,m})_{t=0}^T)^{-1}.$$

Since, for any $N \in \mathbb{N}$, η_m^N belongs to the compact set $\mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \times \mathcal{Z}^T \times \mathcal{R} \times \mathcal{X}^{T+1})$, the sequence $\{\eta_m^N\}_{N\in\mathbb{N}}$ admits a convergent subsequence, $\{\eta_m^{N_k}\}_{k\in\mathbb{N}}$, with limit η_m. On a suitable probability space, $(\Omega_m, \mathcal{F}_m, \mathbb{P}_m)$, we consider a $\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \times \mathcal{Z}^T \times \mathcal{R} \times \mathcal{X}^{T+1}$- valued random vector, $(\Phi^m, (\mu_t^m)_{t=0}^T, (\xi_t^m)_{t=1}^T, \widetilde{\Phi}^m, (X_t^m)_{t=0}^T)$, such that

$$\eta_m := \mathbb{P}_m \circ (\Phi^m, (\mu_t^m)_{t=0}^T, (\xi_t^m)_{t=1}^T, \widetilde{\Phi}^m, (X_t^m)_{t=0}^T)^{-1}$$

and set

$$\rho_m := \mathbb{P}_m \circ (\Phi^m, (\mu_t^m)_{t=0}^T)^{-1} \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}),$$
$$\beta_m := \mathbb{P}_m \circ (\widetilde{\Phi}^m, \Phi^m, (\mu_t^m)_{t=0}^T)^{-1} \in \mathcal{P}(\mathcal{R} \times \mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}).$$

Then, the *limit variables*, $(\Phi^m, (\mu_t^m)_{t=0}^T, (\xi_t^m)_{t=1}^T, \widetilde{\Phi}^m, (X_t^m)_{t=0}^T)$, satisfy the following properties:

i) By the continuous mapping theorem and the fact that, by hypothesis, $X_0^{1,N,m} \overset{d}{\sim} \mathfrak{m}_0$, for all $N \in \mathbb{N}$, we get
$$\mathbb{P}_m \circ (X_0^{\mathfrak{m}})^{-1} = \mathfrak{m}_0.$$

ii) $\rho_m = \mathbb{P}_m \circ (\Phi^m, (\mu_t^m)_{t=0}^T)^{-1} \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$, by definition.

iii) As a consequence of the independence of the variables $(\xi_t^{1,N,m})_{t=1}^T$, $X_0^{1,N,m}$, and $(\Phi_1^{N,m}, \widetilde{\Phi}_1^{N,m})$ and the fact that they jointly converge in distribution, together with the continuous mapping theorem and the fact that $\xi_t^{1,N,m} \overset{d}{\sim} \nu$, $N \in \mathbb{N}$, $t \in [\![1, T]\!]$, we have
$$\xi_t^m \overset{d}{\sim} \nu, \qquad \text{for any } t \in [\![1, T]\!].$$

iv) Since, for any $N \in \mathbb{N}$, $\Phi_1^{N,m} = \widetilde{\Phi}_1^{N,m}$, $\mathbb{P}_{N,m}$-a.s., we get $\Phi^m = \widetilde{\Phi}^m$, $\mathbb{P}_m$-a.s. Then, we have $\widetilde{\Phi}^m(t, x^{(t)}, m^{(t)}) = \Phi^m(t, x^{(t)}, m^{(t)}) = \iota_t(\Phi^m, x^{(t)}, m^{(t)})$.
Furthermore, since $(\Phi_j^{N_k,m})_{j=1}^{N_k}$, $\widetilde{\Phi}_1^{N_k,m}$, as well as Φ^m and $\widetilde{\Phi}^m$, are $\mathcal{R}$-valued variables, reasoning as in the Step 3 of the proof of [30, Theorem 5.1], we get that $(\xi_t^m)_{t=1}^T$, $X_0^{\mathfrak{m}}$ and $(\Phi^m, (\mu_t^m)_{t=0}^T)$ are independent.

v) Furthermore, proceeding as in the Step 3. of the proof of [30, Theorem 5.1], it is possible to prove that $(X_t^m)_{t=0}^T$ follows the dynamics: for any $t \in [\![0, T-1]\!]$,
$$X_{t+1}^m = \Psi\left(t, X_t^m, \Phi^m\left(t, X^{m,(t)}, \mu^{m,(t)}\right), \xi_{t+1}^m\right), \qquad \mathbb{P}_m\text{-a.s.}$$
$$= \Psi\left(t, X_t^m, \Phi^m\left(t, X_t^m\right), \xi_{t+1}^m\right).$$

These features correspond to properties **i)-v)** in Definition 1.4.1. We have proved that the tuple $((\Omega_m, \mathcal{F}_m, \mathbb{P}_m), \Phi^m, (\mu_t^m)_{t=0}^T, X_0^{\mathfrak{m}}, (\xi_t^m)_{t=1}^T, \widetilde{\Phi}^m, (X_t^m)_{t=0}^T)$ is a realisation of the triple $(\mathfrak{m}_0, \rho_m, \iota)$.

Furthermore, since, for any $N \in \mathbb{N}$, $\mathbb{P}_{N,m} \circ (\Phi_1^{N,m})^{-1} = \rho_1(\cdot|m)$, we get $\mathbb{P}_m \circ (\Phi^m)^{-1} = \rho_1(\cdot|m)$. Furthermore, we have

$$\lim_{k\to\infty} J_1^{N_k}(\mathfrak{m}_0^{\otimes N_k}, \gamma_m^{N_k}, \iota) = J(\mathfrak{m}_0, \rho_m, \iota). \tag{1.6.4}$$

Equation (1.6.4) follows from the joint convergence in distribution of the variables that form a realisation together with hypothesis **(A3)** and the dominated convergence theorem. Notice that, here, the fact that $\Phi_1^{N,m}$, as well as Φ_m, are $\mathcal{R}$-valued is crucial.

Step 2

The symmetry and independence among the players in the pre-limit game enable us to prove a result of chaos propagation for the convergent subsequence associated to $(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota)$. We are not going to show this property directly but exploiting an equivalent characterisation of propagation of chaos, namely [82, Theorem 4.2](see Theorem 1.A.1, in the Appendix).
In fact, we can work iteratively to show that chaos propagates from t to $t+1$ for each $t \in [\![0, T-1]\!]$. This fact implies

$$\mathbb{P}_m \circ (X_t^m, \mu_t^m)^{-1} = \widehat{m}_t^m \otimes \delta_{\widehat{m}_t^m}, \qquad t \in [\![0, T]\!].$$

for some deterministic flow of measures $(\widehat{m}_t^m)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$, with $\widehat{m}_0^{\mathrm{m}} = m_0$.
We show that propagation of chaos holds, for our specific structure, for the first time step. This same reasoning can be immediately extended to the other time steps implying our thesis. We notice that this is possible only because the variables $\{\Phi_j^{N,m}\}_{j=1}^N$ take values in $\mathcal{R}$. For a detailed proof see Appendix 1.A.

Reframing the result (1.6) of chaos propagation in the dynamics described in Equation (1.6), we get, $\mathbb{P}_m$-a.s.,

$$\begin{cases} X_{t+1}^m = \Psi\left(t, X_t^m, \widehat{m}_t^m, \Phi^m\left(t, X_t^m\right), \xi_{t+1}^m\right), \\ \mathbb{P}_m \circ (X_t^m)^{-1} = \widehat{m}_t^m, \qquad t \in [\![0, T]\!]. \end{cases}$$

Notice that, in the variable Φ^m, we are omitting the dependence on the measure $\widehat{m}$. We are allowed to do this because this variable takes values in $\mathcal{R}$, being distributed according to $\rho_1(\cdot|m)$. The system in (1.6) has a unique solution. It is a consequence of the iterative definition of the process $(X_t^m)_{t=0}^T$ and of properties **i)-v)** of the limit realisation. Thence, the flow of measures $(\widehat{m}_t^m)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$, corresponding to this system, is uniquely determined for each $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$.

Step 3

Now, our aim is to prove that $(\widehat{m}_t^m)_{t=0}^T = (m_t)_{t=0}^T$, for ρ_2-almost all $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$. Let ρ be the correlated solution for the mean field game starting at $\mathfrak{m}_0$, as in the statement of the theorem, and consider a realisation of $(\mathfrak{m}_0, \rho, \iota)$, i.e. $\left((\Omega^*, \mathcal{F}^*, \mathbb{P}^*), \Phi^*, (\mu_t^*)_{t=0}^T, X_0^*, (\xi_t^*)_{t=1}^T, \iota, (X_t^*)_{t=0}^T\right)$. By definition of realisation, such a tuple satisfies properties **i)-v)** in Definition 1.4.1. In particular, without loss of generality, we set

- **i)** $\mathbb{P}^* \circ (X_0^*)^{-1} = \mathfrak{m}_0$;
- **ii)** $\mathbb{P}^* \circ (\Phi^*, (\mu_t^*)_{t=0}^T)^{-1} = \rho$;

iii) $(\xi^*_t)_{t=1}^T$ i.i.d. with $\xi^*_t \overset{d}{\sim} \nu$;

iv) $(\xi^*_t)_{t=1}^T$, X^*_0 and $(\Phi^*, (\mu^*_t)_{t=0}^T)$ independent;

iv') $\iota(\Phi^*) = \Phi^*$, $\mathbb{P}^*$-a.s.;

v) for all $t \in [\![0, T-1]\!]$,

$$X^*_{t+1} = \Psi\left(t, X^*_t, \mu^*_t, \iota \circ \Phi^*\left(t, X^*, \mu^*\right), \xi^*_{t+1}\right) = \Psi\left(t, X^*_t, \mu^*_t, \Phi^*\left(t, X^*_t\right), \xi^*_{t+1}\right), \quad \mathbb{P}^*\text{-a.s.}$$

The fact that ρ is a correlated solution for the mean field game (consistency condition) and the definition of $\rho_1(\cdot|m)$ imply, respectively, that, for ρ_2-almost all $m \in \mathcal{P}(\mathcal{X})^{T+1}$, we have:

- $\mathbb{P}^*\left(X^*_t \in \cdot \middle| (\mu^*_t)_{t=0}^T = (m_t)_{t=0}^T\right) = m_t, \qquad t \in [\![0, T]\!]$;
- $\mathbb{P}^*\left(\Phi^* \in \cdot \middle| (\mu^*_t)_{t=0}^T = (m_t)_{t=0}^T\right) = \rho_1(\cdot|m)$;

Then, setting $\mathbb{Q}^m(\cdot) = \mathbb{P}^*(\cdot|(\mu^*_t)_{t=0}^T = (m_t)_{t=0}^T)$, we get:

- $\mathbb{Q}^m \circ (X^*_t)^{-1} = m_t, \qquad t \in [\![0, T]\!]$;
- $\mathbb{Q}^m \circ (\Phi^*)^{-1} = \rho_1(\cdot|m)$;
- $\mathbb{Q}^m \circ (\xi^*_t)^{-1} = \nu, \qquad t \in [\![1, T]\!]$;

where the last item is a consequence of the fact that $(\xi^*_t)_{t=1}^T$ is jointly independent of $(\mu^*_t)_{t=0}^T$, by property **iv)** above. Hence, for ρ_2-almost all $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$, $\mathbb{Q}^m$-almost surely, for any $t \in [\![0, T-1]\!]$, we have

$$\begin{cases} X^*_{t+1} = \Psi\left(t, X^*_t, m_t, \Phi^*\left(t, X^*_t\right), \xi^*_{t+1}\right) \\ \mathbb{Q}^m \circ (X^*_t)^{-1} = m_t, \qquad t \in [\![0, T]\!]. \end{cases}$$

This means that the tuple $\left((\Omega^*, \mathcal{F}^*, \mathbb{Q}^m), \Phi^*, (\mu^*_t)_{t=0}^T, X^*_0, (\xi^*_t)_{t=1}^T, (X^*_t)_{t=0}^T\right)$ is a solution for the system (1.6). Finally, exploiting the uniqueness of solution for this system, we obtain the following identities, that hold for ρ_2-almost all $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$:

$$(\widehat{m}^m_t)_{t=0}^T = (m_t)_{t=0}^T, \qquad \rho_m = \mathbb{P}_m \circ (\Phi^m, \mu_m)^{-1} = \mathbb{Q}^m \circ (\Phi^*, m)^{-1} = \rho_1(\cdot|m) \otimes \delta_m.$$

Notice that the second equation is a consequence of the fact that $\mathbb{P}_m \circ (\Phi^m)^{-1} = \rho_1(\cdot|m)$. In particular, we rewrite the equation in (1.6.4) as

$$\lim_{k\to\infty} J_1^{N_k}(\mathfrak{m}_0^{\otimes N_k}, \gamma_m^{N_k}, \iota) = J(\mathfrak{m}_0, \rho_1(\cdot|m) \otimes \delta_m, \iota).$$

Notice that the limit above does not depend on the subsequence considered and so we can deduce that the whole sequence converges to this limit. Now, an application of the dominated convergence theorem, together with the identities (1.6) and (1.6), yields

$$\begin{aligned} \lim_{N\to\infty} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota) &= \lim_{N\to\infty} \int_{\mathcal{P}(\mathcal{X})^{T+1}} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota)\rho_2(\mathrm{d}m) \\ &= \int_{\mathcal{P}(\mathcal{X})^{T+1}} \lim_{N\to\infty} J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma_m^N, \iota)\rho_2(\mathrm{d}m) \\ &= \int_{\mathcal{P}(\mathcal{X})^{T+1}} J(\mathfrak{m}_0, \rho_1(\cdot|m) \otimes \delta_m, \iota)\rho_2(\mathrm{d}m) = J(\mathfrak{m}_0, \rho, \iota). \end{aligned}$$

This ends the proof of (1.6.2).

Proof of (1.6.3). Consider the minimising sequence of strategy modifications $\{\widetilde{\gamma}^N\}_{N\in\mathbb{N}} \subseteq \mathcal{N}_{\gamma_1^N}$, defined in (1.6.1). Now, set

$$\begin{aligned}
\overline{\overline{\gamma}}_N(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N,\mathrm{d}m) &:= \left(\bigotimes_{j=1}^N \rho_1(\mathrm{d}\varphi_j|m)\right)\rho_2(\mathrm{d}m)\\
&= \left(\bigotimes_{j=1}^N \left(\int_{\mathcal{Z}} \bigotimes_{t=0}^{T-1} \delta_{\alpha_t(z_j,m^{(t+1)})}(\mathrm{d}\varphi_j(t,\cdot))\nu(\mathrm{d}z_j)\right)\right)\rho_2(\mathrm{d}m)\\
&= \gamma_m^N(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N)\rho_2(\mathrm{d}m) \in \mathcal{P}(\mathcal{R}^N \times \mathcal{P}(\mathcal{X})^{T+1}),
\end{aligned}$$

where α_t has been chosen according to **(R2)**, see Remark 1.6.1 and 1.6.3. The peculiar form of the starting MFG solution ρ that satisfies assumption **(R2)**, and the consequent form of the correlated suggestion in the N-player game, will be crucial to give an interpretation to any N-player game realisation in the mean-field sense. Now, we want to build a sequence of realisations of $\{(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N)\}_{N\in\mathbb{N}}$. For a fixed $N \in \mathbb{N}$, let $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ be a complete probability space. On this probability space, we set:

- **i)** $(X_0^j)_{j=1}^N$, $\mathcal{X}$-valued random variables i.i.d. according to $\mathfrak{m}_0$;
  $(\Phi_j)_{j=1}^N$, $\mathcal{R}$-valued random variables, such that
  $$\Phi_j(t,\cdot) := \alpha_t(Z_j, \mu^{(t+1)}), \qquad j = 1,\dots,N, \quad t = 0,\dots,T-1,$$
  with $(Z_j)_{j=1}^N$ i.i.d. $\overset{d}{\sim} \nu$ and independent of $\mu \overset{d}{\sim} \rho_2$.
  In particular, this implies that $\mathbb{P}_N\circ((\Phi_j)_{j=1}^N, \mu)^{-1}(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N,\mathrm{d}m) = \overline{\overline{\gamma}}_N(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N,\mathrm{d}m)$ and so that $\mathbb{P}_N \circ ((\Phi_j)_{j=1}^N)^{-1}(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N) = \gamma_N(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N)$;
- **ii)** $(\xi_t^1,\dots,\xi_t^N)_{t=0}^T$, $\mathcal{Z}$-valued random variables i.i.d. all distributed according to ν;
- **iii)** $(\vartheta_t)_{t=0}^T$, $\mathcal{Z}$-valued random variables i.i.d. all distributed according to ν;
- **iv)** $(\xi_t^1,\dots,\xi_t^N)_{t=0}^T$, $(X_0^j)_{j=1}^N$, $((\mu_t)_{t=0}^T, (Z_j)_{j=1}^N)$ and $(\vartheta_t)_{t=0}^{T-1}$ are independent;
- **v)** $\widetilde{\Upsilon}_1^N$, $\widehat{\mathcal{R}}$-valued random variable s.t. $\widetilde{\Upsilon}_1^N(t,\cdot) = w_t^N(\vartheta_t,\Phi_1)(\cdot)$, with $w_t^N : [0,1]\times\mathcal{R} \to \widehat{\mathcal{E}}_{t,N}$ Borel function, for any $t \in [\![0,T-1]\!]$, and $\mathbb{P}_N \circ (\Phi_1, \widetilde{\Upsilon}_1^N)^{-1} = \widetilde{\gamma}_N$[1].

We set the following dynamics for the $\mathcal{X}$-valued processes, $(X_t^{j,N})_{t=0}^T$, $j \in [\![1,N]\!]$, for $t \in [\![0,T-1]\!]$,

$$X_{t+1}^{j,N} = \Psi\left(t, X_t^{j,N}, \mu_t^{j,N}, \Phi_j(t,X_t^j), \xi_{t+1}^j\right), \quad \mathbb{P}_N\text{-a.s.},$$

where, for all $t \in [\![0,T]\!]$ and $j \in [\![1,N]\!]$, $\mu_t^{j,N} := \frac{1}{N-1}\sum_{k\neq j,k=1}^N \delta_{X_t^{k,N}}$ and $\mu^{j,N} := (\mu_t^{j,N})_{t=0}^T$. This corresponds to the case where all the players stick to the suggestion given by the mediator.

[1]The existence of these Borel functions is a consequence of the measurability condition in **v)** in Definition 1.3.1 and of Doob's Lemma (see [105, Lemma 1.13]).

Then, we define another sequence of processes, $(\widetilde{X}_t^{j,N})_{t=0}^T$, $j \in [\![1,N]\!]$, setting $\widetilde{X}_0^{j,N} := X_0^j$ and, for $t \in [\![0,T-1]\!]$,

$$\begin{aligned}\widetilde{X}_{t+1}^{j,N} &= \Psi\left(t, \widetilde{X}_t^{j,N}, \widetilde{\mu}_t^{j,N}, \Phi_j(t, \widetilde{X}_t^{j,N}), \xi_{t+1}^j\right), \\ \widetilde{X}_{t+1}^{1,N} &= \Psi\left(t, \widetilde{X}_t^{1,N}, \widetilde{\mu}_t^{1,N}, \widetilde{\Upsilon}^{1,N}(t, \widetilde{X}^{1,N}, \widetilde{\mu}^{1,N}), \xi_{t+1}^1\right), \quad \mathbb{P}_N\text{-a.s.,}\end{aligned} \tag{1.6.5}$$

where, for all $t \in [\![0,T]\!]$ and $j \in [\![1,N]\!]$, $\widetilde{\mu}_t^{j,N} := \frac{1}{N-1}\sum_{k\neq j,k=1}^N \delta_{\widetilde{X}_t^{k,N}}$ and $\widetilde{\mu}^{j,N} := (\widetilde{\mu}_t^{j,N})_{t=0}^T$. This represents the case in which only the first player is deviating from the suggestion according to the minimising sequence of strategy modifications introduced in the beginning of the proof. Hence, $((\Omega_N, \mathcal{F}_N, \mathbb{P}_N), (\Phi_j)_{j=1}^N, (\vartheta_t)_{t=0}^{T-1}, (\xi_t^1, \dots, \xi_t^N)_{t=1}^T, \Phi_1, (X_t^{1,N}, \dots, X_t^{N,N})_{t=0}^T)$ and $((\Omega_N, \mathcal{F}_N, \mathbb{P}_N), (\Phi_j)_{j=1}^N, (\vartheta_t)_{t=0}^{T-1}, (\xi_t^1, \dots, \xi_t^N)_{t=1}^T, \widetilde{\Upsilon}_1^N, (\widetilde{X}_t^{1,N}, \dots, \widetilde{X}_t^{N,N})_{t=0}^T)$ are, respectively, a realisation of the triple $(\mathfrak{m}_0^{\otimes N}, \gamma^N, \iota)$ and $(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N)$ for the first player. Indeed, we notice that, with this construction, also condition **v)** in Definition 1.3.1 is satisfied.
Then, we define another sequence of processes, $(\widehat{X}_t^{j,N})_{t=0}^T$, $j \in [\![2,N]\!]$, setting $\widehat{X}_0^{j,N} := X_0^j$ and, for $t \in [\![0,T-1]\!]$,

$$\widehat{X}_{t+1}^{j,N} = \Psi\left(t, \widehat{X}_t^{j,N}, \widehat{\mu}_t^{1,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right), \quad \mathbb{P}_N\text{-a.s.,}$$

where, for all $t \in [\![0,T]\!]$, $\widehat{\mu}_t^{1,N} := \frac{1}{N-1}\sum_{k=2}^N \delta_{\widehat{X}_t^{k,N}}$ and $\widehat{\mu}^{1,N} := (\widehat{\mu}_t^{1,N})_{t=0}^T$. These processes describe the evolution of the system excluding the first player.
Finally, we define the process, $(\overline{X}_t^{1,N})_{t=0}^T$, setting $\overline{X}_0^{1,N} := X_0^1$ and, for $t \in [\![0,T-1]\!]$,

$$\overline{X}_{t+1}^{1,N} = \Psi\left(t, \overline{X}_t^{1,N}, \mu_t, \widetilde{\Upsilon}^{1,N}(t, \widetilde{X}^{1,N}, \widetilde{\mu}^{1,N}), \xi_{t+1}^1\right), \quad \mathbb{P}_N\text{-a.s..} \tag{1.6.6}$$

This last one is an auxiliary process whose utility will be made clear in the following. From now on, for simplicity of notation, for $t \in [\![0,T-1]\!]$, we write $\widetilde{u}_t^{1,N}$ for $\widetilde{\Upsilon}^{1,N}(t, \widetilde{X}^{1,N}, \widetilde{\mu}^{1,N})$.
First of all, we focus on $((\Omega_N, \mathcal{F}_N, \mathbb{P}_N), \Phi_1, (\mu_t)_{t=0}^T, (\mu_t^{1,N})_{t=0}^T, (\vartheta_t)_{t=0}^{T-1}, (\xi_t^1)_{t=1}^T, (X_t^{1,N})_{t=0}^T)$. For all $t \in [\![0,T]\!]$, an application of the tower property yields

$$\mathbb{E}_N[\text{dist}(\mu_t^{1,N}, \mu_t)] = \int_{\mathcal{P}(\mathcal{X})^{T+1}} \mathbb{E}_N[\text{dist}(\mu_t^{1,N}, \mu_t) | \mu = m] \rho_2(\mathrm{d}m).$$

Conditionally on the event $\{(\mu_t)_{t=0}^T = (m_t)_{t=0}^T\}$, we have already seen that $(\mu_t^{1,N})_{t=0}^T$ converges weakly to $(m_t)_{t=0}^T$, as N goes to infinity. Since $(m_t)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$ is deterministic, the convergence result above holds in probability, that is, for any fixed $\varepsilon > 0$, $\mathbb{P}_N^m(\text{dist}(\mu_t^{1,N}, \mu_t) > \varepsilon) \overset{N\to\infty}{\longrightarrow} 0$. Then, we have

$$\begin{aligned}\mathbb{E}_N^m[\text{dist}(\mu_t^{1,N}, \mu_t)] &\leq \mathbb{P}_N^m(\text{dist}(\mu_t^{1,N}, \mu_t) > \varepsilon) + \varepsilon \mathbb{P}_N^m(\text{dist}(\mu_t^{1,N}, \mu_t) \leq \varepsilon) \\ &\leq \mathbb{P}_N^m(\text{dist}(\mu_t^{1,N}, \mu_t) > \varepsilon) + \varepsilon \overset{N\to\infty}{\longrightarrow} \varepsilon,\end{aligned}$$

and we obtain by the arbitrariness of $\varepsilon > 0$ that $\mathbb{E}_N^m[\text{dist}(\mu_t^{1,N}, \mu_t)] \overset{N\to\infty}{\longrightarrow} 0$, for any $t \in [\![0,T]\!]$.

Finally, by disintegration, an application of the dominated convergence theorem yields

$$\lim_{N\to\infty} \mathbb{E}_N[\mathrm{dist}(\mu_t^{1,N},\mu_t)] = \int_{\mathcal{P}(\mathcal{X})^{T+1}} \lim_{N\to\infty} \mathbb{E}_N[\mathrm{dist}(\mu_t^{1,N},\mu_t)|\mu = m]\rho_2(\mathrm{d}m) = 0, \text{ for all } t \in [\![0,T]\!],$$

and consequently

$$\lim_{N\to\infty} \mathbb{E}_N[\mathrm{dist}_T(\mu^{1,N},\mu)] = 0. \tag{1.6.7}$$

Now, we prove the following claim.

Claim 1. *For any* $\lambda = (\lambda_t)_{t=0}^T \in \{\widetilde{\mu}_t^{j,N}, j \in [\![1,N]\!]\} \cup \{\mu_t^{j,N}, j \in [\![1,N]\!]\}$

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{dist}_T(\lambda,\widehat{\mu}^{1,N})\right] = 0.$$

Proof of Claim 1. We prove the claim for $\lambda = \widetilde{\mu}^{j,N}$, the proof for $\lambda = \mu^{j,N}$ being similar. Since by definition of dist_T, we have

$$\mathbb{E}\left[\mathrm{dist}_T(\widetilde{\mu}^{j,N},\widehat{\mu}^{1,N})\right] = \mathbb{E}\left[\sum_{t=0}^T \mathrm{dist}(\widetilde{\mu}_t^{j,N},\widehat{\mu}_t^{1,N})\right] = \sum_{t=0}^T \mathbb{E}\left[\mathrm{dist}(\widetilde{\mu}_t^{j,N},\widehat{\mu}_t^{1,N})\right],$$

it suffices to prove that, for any $j \in [\![1,N]\!]$, and any $t \in [\![0,T]\!]$,

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{dist}(\widetilde{\mu}_t^{j,N},\widehat{\mu}_t^{1,N})\right] = 0. \tag{1.6.8}$$

We notice that the definition of the distance dist together with the upper bound for empirical measures in (2.1) in [30] implies that, for all $j \in [\![1,N]\!]$, $t \in [\![0,T]\!]$,

$$\mathbb{E}\left[\mathrm{dist}(\widehat{\mu}_t^{1,N},\widetilde{\mu}_t^{j,N})\right] \le \frac{1}{N-1} + \frac{1}{N-1}\sum_{l=2}^N \mathbb{P}\left(\widetilde{X}_t^{l,N} \ne \widehat{X}_t^{l,N}\right). \tag{1.6.9}$$

In fact, for $j = 1$, we have

$$\mathbb{E}\left[\mathrm{dist}(\widehat{\mu}_t^{1,N},\widetilde{\mu}_t^{j,N})\right] \overset{(2.1)}{\le} \mathbb{E}\left[\frac{1}{N-1}\sum_{l=2}^N \mathbf{1}_{\widetilde{X}_t^{l,N}\ne\widehat{X}_t^{l,N}}\right] = \frac{1}{N-1}\sum_{l=2}^N \mathbb{P}\left(\widetilde{X}_t^{l,N} \ne \widehat{X}_t^{l,N}\right).$$

Whereas, for $j \in [\![2,N]\!]$, we get

$$\begin{aligned}
\mathbb{E}\left[\mathrm{dist}(\widehat{\mu}_t^{1,N},\widetilde{\mu}_t^{j,N})\right] &\overset{(2.1)}{\le} \mathbb{E}\left[\frac{1}{N-1}\sum_{l=2,l\ne j}^N \mathbf{1}_{\widetilde{X}_t^{l,N}\ne\widehat{X}_t^{l,N}} + \frac{1}{N-1}\mathbf{1}_{\widetilde{X}_t^{1,N}\ne\widehat{X}_t^{j,N}}\right] \\
&\le \mathbb{E}\left[\frac{1}{N-1}\sum_{l=2}^N \mathbf{1}_{\widetilde{X}_t^{l,N}\ne\widehat{X}_t^{l,N}} + \frac{1}{N-1}\right] \\
&= \frac{1}{N-1}\sum_{l=2}^N \mathbb{P}\left(\widetilde{X}_t^{l,N} \ne \widehat{X}_t^{l,N}\right) + \frac{1}{N-1}.
\end{aligned}$$

Furthermore, we prove that, for all $t \in [\![0,T]\!]$, we have the following convergence, as $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N-1} \sum_{j=2}^{N} \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) = 0.$$

In fact, (1.6), with $t = 0$, follows from the fact that, for all $N \in \mathbb{N}$,

$$\sum_{j=2}^{N} \mathbb{P}\left(\widetilde{X}_0^{j,N} \neq \widehat{X}_0^{j,N}\right) = 0.$$

which is a consequence of the fact that, by construction, for all $N \in \mathbb{N}$, $j \in [\![1,N]\!]$, $\widetilde{X}_0^{j,N} = X_0^j$, $\mathbb{P}_N$-a.s. and for all $N \in \mathbb{N}$, $j \in [\![2,N]\!]$, $\widehat{X}_0^{j,N} = X_0^j$, $\mathbb{P}_N$-a.s.. Then, we prove (1.6) for a generic time, reasoning by induction. Let us assume that (1.6) holds for t, for all $j \in [\![2,N]\!]$, we have

$$\begin{aligned}\mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}\right) &= \mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}, \widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}, \widetilde{X}_t^{j,N} = \widehat{X}_t^{j,N}\right) \\ &\leq \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}, \widetilde{X}_t^{j,N} = \widehat{X}_t^{j,N}\right) =: \star,\end{aligned}$$

where we have exploited disintegration. Using the iterative definition of the processes $(\widetilde{X}_t^{j,N})_{t=0}^T$ and $(\widehat{X}_t^{j,N})_{t=0}^T$ through Ψ and the fact that Φ_j, by construction, takes values in $\mathcal{R}$ we get

$$\begin{aligned}\star &= \mathbb{P}\Big(\Psi\left(t, \widetilde{X}_t^{j,N}, \widetilde{\mu}_t^{j,N}, \Phi_j(t, \widetilde{X}_t^{j,N}), \xi_{t+1}^j\right) \neq \Psi\left(t, \widehat{X}_t^{j,N}, \widehat{\mu}_t^{1,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right), \widetilde{X}_t^{j,N} = \widehat{X}_t^{j,N}\Big) \\ &\quad + \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) \\ &= \mathbb{P}\Big(\Psi\left(t, \widehat{X}_t^{j,N}, \widetilde{\mu}_t^{j,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right) \neq \Psi\left(t, \widehat{X}_t^{j,N}, \widehat{\mu}_t^{1,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right), \widetilde{X}_t^{j,N} = \widehat{X}_t^{j,N}\Big) \\ &\quad + \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) \\ &= \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \mathbb{P}\Big(\Psi\left(t, \widehat{X}_t^{j,N}, \widetilde{\mu}_t^{j,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right) \neq \Psi\left(t, \widehat{X}_t^{j,N}, \widehat{\mu}_t^{1,N}, \Phi_j(t, \widehat{X}_t^{j,N}), \xi_{t+1}^j\right)\Big)\end{aligned}$$

Then, an application of Fubini's Theorem, together with the independence properties of $(\xi_{t+1}^j)_{j=2}^N$, yields

$$\begin{aligned}\mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}\right) &= \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \mathbb{E}\left[\int_{\mathcal{Z}} \mathbf{1}_{\left\{\Psi\left(t,\widehat{X}_t^{j,N},\widetilde{\mu}_t^{j,N},\Phi_j(t,\widehat{X}_t^{j,N}),z\right) \neq \Psi\left(t,\widehat{X}_t^{j,N},\widehat{\mu}_t^{1,N},\Phi_j(t,\widehat{X}_t^{j,N}),z\right)\right\}} \nu(\mathrm{d}z)\right] \\ &\leq \mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \mathbb{E}\left[\mathfrak{W}(\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N})\right],\end{aligned}$$

where the inequality in the last row follows from Assumption **(A2)** 1).
Now, notice that

$$\lim_{N\to\infty} \max_{j\in[\![2,N]\!]} \mathbb{E}\left[\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N})\right] \overset{(1.6.9)}{\leq} \lim_{N\to\infty} \left\{\frac{1}{N-1} + \frac{1}{N-1} \sum_{l=2}^{N} \mathbb{P}\left(\widetilde{X}_t^{l,N} \neq \widehat{X}_t^{l,N}\right)\right\} = 0,$$

because of the induction hypothesis. Thence, with the notation $\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N}) = \delta_j^N$, for any $\varepsilon > 0$, we have

$$\begin{aligned}
\max_{j\in[\![2,N]\!]} \mathbb{E}\left[\mathfrak{W}(\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N}))\right] &= \max_{j\in[\![2,N]\!]} \mathbb{E}\left[\mathfrak{W}(\delta_j^N)\right] \\
&\le \max_{j\in[\![2,N]\!]} \left\{\mathbb{E}\left[\mathfrak{W}(\delta_j^N)\middle|\delta_j^N \ge \varepsilon\right]\mathbb{P}(\delta_j^N \ge \varepsilon) + \mathbb{E}\left[\mathfrak{W}(\delta_j^N)\middle|\delta_j^N < \varepsilon\right]\mathbb{P}(\delta_j^N < \varepsilon)\right\} \\
&\le \max_{j\in[\![2,N]\!]} \left\{||\mathfrak{W}||_\infty \mathbb{P}(\delta_j^N \ge \varepsilon) + \mathbb{E}\left[\mathfrak{W}(\delta_j^N)\middle|\delta_j^N < \varepsilon\right]\right\} \\
&\le \max_{j\in[\![2,N]\!]} \left\{||\mathfrak{W}||_\infty \mathbb{P}(\delta_j^N \ge \varepsilon) + \mathfrak{W}(\varepsilon)\right\} \le \mathfrak{W}(\varepsilon) + ||\mathfrak{W}||_\infty \max_{j\in[\![2,N]\!]} \frac{\mathbb{E}[\delta_j^N]}{\varepsilon} \\
&\le \mathfrak{W}(\varepsilon) + \frac{||\mathfrak{W}||_\infty}{\varepsilon} \max_{j\in[\![2,N]\!]} \mathbb{E}[\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N})] \overset{N\to\infty}{\longrightarrow} \mathfrak{W}(\varepsilon),
\end{aligned}$$

where we have made use of disintegration, the fact that $\mathfrak{W}$ is bounded, Markov's inequality and the convergence result in (1.6).

The fact that $\mathfrak{W}$ converges to 0 as its argument goes to zero and the arbitrariness of $\varepsilon > 0$ therefore implies

$$\lim_{N\to\infty} \left\{\max_{j\in[\![2,N]\!]} \mathbb{E}\left[\mathfrak{W}(\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N}))\right]\right\} = 0. \tag{1.6.10}$$

Applying once more the induction hypothesis to the inequality in (1.6), we get

$$\lim_{N\to\infty} \frac{1}{N-1}\mathbb{P}\left(\widetilde{X}_{t+1}^{j,N} \neq \widehat{X}_{t+1}^{j,N}\right) \le \lim_{N\to\infty} \left\{\frac{1}{N-1}\mathbb{P}\left(\widetilde{X}_t^{j,N} \neq \widehat{X}_t^{j,N}\right) + \max_{j\in[\![2,N]\!]} \mathbb{E}\left[\mathfrak{W}(\mathrm{dist}(\widetilde{\mu}_t^{j,N}, \widehat{\mu}_t^{1,N})\right]\right\} = 0.$$

Thus, we have shown (1.6), which, together with (1.6.9), implies (1.6.8) and so our claim.

Then, by the triangular inequality and the monotonicity of expectation, Equation (1.6.7) together with the statement in Claim 1 yields

$$\mathbb{E}_N[\mathrm{dist}_T(\widetilde{\mu}^{1,N}, \mu)] \le \mathbb{E}_N[\mathrm{dist}_T(\widetilde{\mu}^{1,N}, \widehat{\mu}^{1,N})] + \mathbb{E}_N[\mathrm{dist}_T(\widehat{\mu}^{1,N}, \mu^{1,N})] + \mathbb{E}_N[\mathrm{dist}_T(\mu^{1,N}, \mu)] \overset{N\to\infty}{\longrightarrow} 0. \tag{1.6.11}$$

Now, set

$$\widetilde{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) := \mathbb{E}_N\left[\sum_{t=0}^T f(t, \widetilde{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N}) + F(\widetilde{X}_T^{1,N}, \mu_T)\right],$$

and

$$\overline{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) := \mathbb{E}_N\left[\sum_{t=0}^T f(t, \overline{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N}) + F(\overline{X}_T^{1,N}, \mu_T)\right],$$

with processes $\widetilde{X}^{1,N}$ and $\overline{X}^{1,N}$ defined in Equations (1.6.5) and (1.6.6).

Now, consider a real valued sequence $\{f_n\}_{n\in\mathbb{N}}$ s.t., for any $n \in \mathbb{N}$, $f_n = h_n + g_n + h$ with

$\lim_{n\to\infty} h_n = 0$, $g_n \geq 0$, for all $n \in \mathbb{N}$. Then,

$$\liminf_{n\to\infty} f_n \geq h. \tag{1.6.12}$$

In order to prove Equation (1.6.3), we want to exploit the inequality in Equation (1.6.12) with $g_N = \overline{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) - J(\mathfrak{m}_0, \rho, \iota)$, $h_N = J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N)$ and $h = J(\mathfrak{m}_0, \rho, \iota)$. First of all, **(A3)** and the convergence in Equation (1.6.11) imply

$$\begin{aligned}
&|J_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) - \widetilde{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N)| \qquad (1.6.13)\\
&\leq \mathbb{E}_N\left[\sum_{t=0}^{T} |f(t, \widetilde{X}_t^{1,N}, \widetilde{\mu}_t^{1,N}, \widetilde{u}_t^{1,N}) - f(t, \widetilde{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N})| + |F(\widetilde{X}_T^{1,N}, \widetilde{\mu}_T^{1,N}) - F(\widetilde{X}_T^{1,N}, \mu_T)|\right]\\
&\leq \mathbb{E}_N\left[\sum_{t=0}^{T} L\mathrm{dist}(\widetilde{\mu}_t^{1,N}, \mu_t) + L\mathrm{dist}(\widetilde{\mu}_T^{1,N}, \mu_T)\right] = L\mathbb{E}\left[\mathrm{dist}_T(\widetilde{\mu}^{1,N}, \mu)\right] \overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Furthermore, for all $t \in [\![0, T]\!]$,

$$\lim_{N\to\infty} \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) = 0.$$

We show this by induction on $t \in [\![0, T]\!]$. Indeed, for $t = 0$, $\mathbb{P}_N(\widetilde{X}_0^{1,N} \neq \overline{X}_0^{1,N}) = 0$, being $\widetilde{X}_0^{1,N} = \overline{X}_0^{1,N} = X_0^1$, $\mathbb{P}_N$-a.s., by construction. Now, suppose that $\lim_{N\to\infty} \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) = 0$, for some $t \in [\![0, T]\!]$. Then, exploiting Assumption **(A2)** 1), we obtain

$$\begin{aligned}
\mathbb{P}_N(\widetilde{X}_{t+1}^{1,N} \neq \overline{X}_{t+1}^{1,N}) &\leq \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) + \mathbb{P}_N(\widetilde{X}_{t+1}^{1,N} \neq \overline{X}_{t+1}^{1,N}, \widetilde{X}_t^{1,N} = \overline{X}_t^{1,N})\\
&\leq \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N})\\
&\quad + \mathbb{P}_N\left(\Psi\left(t, \widetilde{X}_t^{1,N}, \widetilde{\mu}_t^{1,N}, \widetilde{u}_t^{1,N}, \xi_{t+1}^1\right) \neq \Psi\left(t, \overline{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N}, \xi_{t+1}^1\right), \widetilde{X}_t^{1,N} = \overline{X}_t^{1,N}\right)\\
&\leq \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) + \mathbb{P}_N\left(\Psi\left(t, \widetilde{X}_t^{1,N}, \widetilde{\mu}_t^{1,N}, \widetilde{u}_t^{1,N}, \xi_{t+1}^1\right) \neq \Psi\left(t, \widetilde{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N}, \xi_{t+1}^1\right)\right)\\
&\leq \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) + \mathbb{E}_N\left[\mathfrak{W}\left(\mathrm{dist}(\widetilde{\mu}_t^{1,N}, \mu_t)\right)\right],
\end{aligned}$$

and the last term on the right goes to zero as N goes to infinity by the induction assumption and the convergence in Equation (1.6.11), reasoning in a similar way as in the proof of Equation (1.6.10). As a consequence, we see

$$\begin{aligned}
&|\widetilde{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N}, \gamma^N, \widetilde{\gamma}^N)| \qquad (1.6.14)\\
&\leq \mathbb{E}_N\left[\sum_{t=0}^{T} |f(t, \widetilde{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N}) - f(t, \overline{X}_t^{1,N}, \mu_t, \widetilde{u}_t^{1,N})| + |F(\widetilde{X}_T^{1,N}, \mu_T) - F(\overline{X}_T^{1,N}, \mu_T)|\right]\\
&\leq 2\|f\|_\infty \sum_{t=0}^{T} \mathbb{P}_N(\widetilde{X}_t^{1,N} \neq \overline{X}_t^{1,N}) + 2\|F\|_\infty \mathbb{P}_N(\widetilde{X}_T^{1,N} \neq \overline{X}_T^{1,N}) \overset{N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have exploited the fact that f and F being L-Lipschitz continuous real-valued function on a compact domain are bounded. The convergences in Equations (1.6.13) and (1.6.14) implies

$$\begin{aligned}|h_N| &= |J_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)| \\ &\leq |J_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)| + |J_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)| \overset{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus, an application of the inequality in (1.6.12) with

$$g_N = \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - J(\mathfrak{m}_0,\rho,\iota),$$

$$h_N = J_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)$$

and

$$h = J(\mathfrak{m}_0,\rho,\iota),$$

yields (1.6.3) provided that $g_N = \overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) - J(\mathfrak{m}_0,\rho,\iota) \geq 0$. This is a consequence of the fact that $\overline{J}_1^N(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)$ can be interpreted as the value of the MFG when the representative player implements the strategy $\widetilde{u}_t^{1,N} = \widetilde{\Upsilon}_1^N(t,\widetilde{X}_t^{1,N},\widetilde{\mu}_1^N)$, $t \in [\![0,T-1]\!]$. Indeed, the realisation of the triple $(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N)$ for the first player on the previously defined complete probability space $(\Omega_N,\mathcal{F}_N,\mathbb{P}_N)$ can be seen as a tuple $\big((\Omega_N,\mathcal{F}_N,\{\mathcal{G}_t^N\}_{t=0}^{T-1},\mathbb{P}_N),\Phi_1,(\mu_t)_{t=0}^T,X_0,(\xi_t^1)_{t=1}^T,(\widetilde{u}_t^{1,N})_{t=0}^{T-1},$ $(\overline{X}_t^{1,N})_{t=0}^T\big)$ such that

- **i)** $\mathbb{P}_N \circ (X_0^1)^{-1} = \mathfrak{m}_0$;
- **ii)** $\mathbb{P}_N \circ (\Phi_1,(\mu_t)_{t=0}^T)^{-1} = \rho$;
- **iii)** $(\xi_t^1)_{t=1}^T$, $\mathcal{Z}$-valued random variables i.i.d. all distributed according to ν;
- **iv)** $X_0^1,(\xi_t^1)_{t=1}^T,(\Phi_1,(\mu_t)_{t=0}^T)$ are independent;
- **iv')** For each $t \in [\![0,T-1]\!]$,
  - ξ_t^1 is $\mathcal{G}_t^N$-measurable and $(\xi_{t+k}^1)_{k=1}^T$ are jointly independent of $\mathcal{G}_t^N$,
  - $\mathcal{G}_t^N = \mathcal{H}_t^N \vee \sigma(\mu^{(t)}) \vee \sigma(\Phi_1) \vee \sigma(X_0^1)$, with $\mathcal{H}_t^N$ independent of $\sigma(\Phi_1,(\mu_t)_{t=0}^T,X_0^1)$,
  - $\widetilde{u}_t^{1,N}$ is $\mathcal{G}_t^N$-measurable,
- **v)** Finally, for $t \in [\![0,T-1]\!]$, the state dynamics for the first player is given by
$$\overline{X}_{t+1}^{1,N} = \Psi\left(t,\overline{X}_t^{1,N},\mu_t,\widetilde{u}_t^{1,N},\xi_{t+1}^1\right),\quad \mathbb{P}_N\text{-a.s.}$$

Above we have exploited the fact that, by definition, the sequence of control actions $(\widetilde{u}_t^{1,N})_{t=0}^{T-1}$,

$$\widetilde{u}_t^{1,N} = \widetilde{\Upsilon}_1^N(t,\widetilde{X}^{1,N},\widetilde{\mu}^{1,N}) = w_t^N(\vartheta_t,\Phi_1)((\widetilde{X}^{1,N})^{(t)},(\widetilde{\mu}^{1,N})^{(t)}),$$

is adapted to the filtration $\{\mathcal{G}_t^N\}_{t=0}^{T-1}$, defined as

$$\mathcal{G}_t^N := \sigma((X_0^j)_{j=1}^N,(\xi_s^1,\ldots,\xi_s^N)_{s=1}^t,\Phi_1,(\vartheta_s)_{s=0}^t,(Z_j)_{j=2}^N,\mu^{(t)}) = \mathcal{H}_t^N \vee \sigma(\mu^{(t)}) \vee \sigma(\Phi_1) \vee \sigma(X_0^1),$$

with $\mathcal{H}_t^N := \sigma((X_0^j)_{j=2}^N,(Z_j)_{j=2}^N,(\xi_s^1,\ldots,\xi_s^N)_{s=1}^t,\vartheta^{(t)})$.
Notice that, for all $t \in [\![1,T]\!]$, ξ_t^1 is $\mathcal{G}_t^N$-measurable and, in turn, $\mathcal{G}_t^N$ is jointly independent of

$(\xi^1_{t+k})_{k=1}^{T-t}$. Furthermore, for all $t \in [\![0,T]\!]$, $\mathcal{H}^N_t$, $\sigma(X^1_0)$ and $\sigma(\Phi_1,(\mu_t)_{t=0}^T)$ are independent. Hence, the tuple $((\Omega_N,\mathcal{F}_N,\{\mathcal{G}^N_t\}_{t=0}^{T-1},\mathbb{P}_N),\Phi_1,(\mu_t)_{t=0}^T,X^1_0,(\xi^1_t)_{t=1}^T,(\widetilde{u}^{1,N}_t)_{t=0}^{T-1},(\overline{X}^{1,N}_t)_{t=0}^T)$ represents a realisation of the triple $(\mathfrak{m}_0,\rho,(\widetilde{u}^{1,N}_t)_{t=0}^{T-1})$ for the open-loop MFG, with costs given by

$$\widehat{J}(\mathfrak{m}_0,\rho,(\widetilde{u}^{1,N}_t)_{t=0}^{T-1}) = \overline{J}^N_1(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N).$$

Now, ρ is a solution of the correlated MFG according to Definition 1.4.3 and the values of the objective functionals at the equilibrium for the correlated MFGs in open-loop and closed-loop strategies are the same (see Proposition 1.5.6). Thus, by the optimality condition in Definition 1.5.3, we get $\overline{J}^N_1(\mathfrak{m}_0^{\otimes N},\gamma^N,\widetilde{\gamma}^N) \geq J(\mathfrak{m}_0,\rho,\iota) \geq 0$ and this ends our proof.

□

1.7 A toy example

In order to further motivate the definition of mean field game solution given in Section 1.4, we consider the two-state example introduced in [30] and show that it possesses correlated solutions with non-deterministic flow of measures also in the sense of Definition 1.4.3. Moreover, assumptions **(A1)** – **(A3)** as well as conditions **(R1)** – **(R2)** on the correlated solution will be seen to hold.

Let us recall the setting. Let $T = 2$, $\mathcal{X} = \{-1,1\}$, and $\Gamma = \{0,1\}$. Let the system function and the cost functional, respectively, be given by

$$\begin{aligned}\Psi(x,\gamma,z) = \Psi(t,x,\gamma,z) &= x[\mathbf{1}_{\{0\}}(\gamma)(\mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{(\frac{1}{2},1]})(z) + \mathbf{1}_{\{1\}}(\gamma)(\mathbf{1}_{[0,\frac{3}{4}]} - \mathbf{1}_{(\frac{3}{4},1]})(z)]\\ &= x[(1-\gamma)(\mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{(\frac{1}{2},1]})(z) + \gamma(\mathbf{1}_{[0,\frac{3}{4}]} - \mathbf{1}_{(\frac{3}{4},1]})(z)],\end{aligned}$$

and

$$\begin{aligned}f(t,x,\gamma,m) &= c_0(1-t)\gamma + t(c_1\gamma - x\mathrm{M}(m)),\\ F(x,m) &= -x\mathrm{M}(m),\end{aligned}$$

with $c_0, c_1 > 0$.

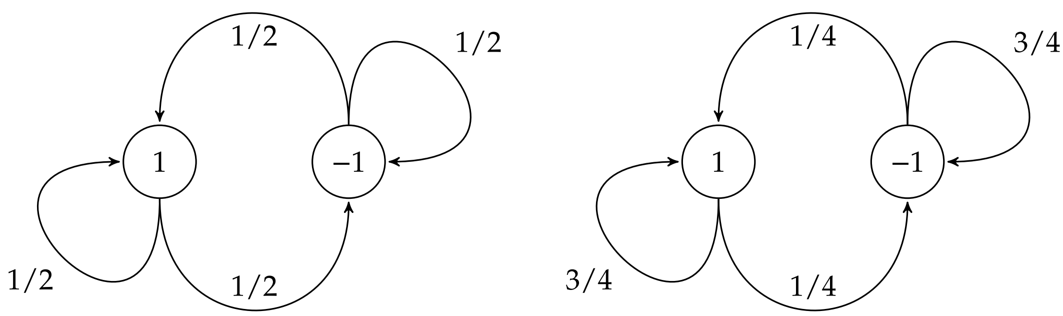

Figure 1.1: States and corresponding transition probabilities for the action $\gamma = 0$ (left) and $\gamma = 1$ (right).

Now, we consider the following *candidate* correlated solution for the game

$$\rho = +\beta_1(\delta_{(\varphi_+,m_+)} + \delta_{(\varphi_-,m_-)}) + \beta_2(\delta_{(\varphi_0,m_+)} + \delta_{(\varphi_0,m_-)}) \\ + \beta_3(\delta_{(\widehat{\varphi}_+,\widehat{m}_+)} + \delta_{(\widehat{\varphi}_-,\widehat{m}_-)}) + \beta_4(\delta_{(\varphi_0,\widehat{m}_+)} + \delta_{(\varphi_0,\widehat{m}_-)}),$$

where

$$\varphi_0(t,x) := 0, \qquad \varphi_+(t,x) := \mathbf{1}_{\{1\}}(x) = \frac{1+x}{2}, \qquad \varphi_-(t,x) := \mathbf{1}_{\{-1\}}(x) = \frac{1-x}{2},$$
$$\widehat{\varphi}_+(t,x) = \mathbf{1}_{\{0\}}(t)\mathbf{1}_{\{1\}}(x) = \frac{(1-t)(1+x)}{2}, \qquad \widehat{\varphi}_-(t,x) := \mathbf{1}_{\{0\}}(t)\mathbf{1}_{\{-1\}}(x) = \frac{(1-t)(1-x)}{2}$$

and

$$m_+ := (m_0, m_1^+, m_2^+), \quad m_+ := (m_0, m_1^-, m_2^-), \quad \widehat{m}_+ := (m_0, m_1^+, m_0), \quad \widehat{m}_- := (m_0, m_1^-, m_0),$$

with

$$m_0 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1},$$
$$m_1^+ = \frac{5\beta_1 + 4\beta_2}{8(\beta_1+\beta_2)}\delta_1 + \frac{3\beta_1 + 4\beta_2}{8(\beta_1+\beta_2)}\delta_{-1}, \qquad m_1^- = \frac{3\beta_1 + 4\beta_2}{8(\beta_1+\beta_2)}\delta_1 + \frac{5\beta_1 + 4\beta_2}{8(\beta_1+\beta_2)}\delta_{-1},$$
$$m_2^+ = \frac{21\beta_1 + 16\beta_2}{32(\beta_1+\beta_2)}\delta_1 + \frac{11\beta_1 + 16\beta_2}{32(\beta_1+\beta_2)}\delta_{-1}, \qquad m_2^- = \frac{11\beta_1 + 16\beta_2}{32(\beta_1+\beta_2)}\delta_1 + \frac{21\beta_1 + 16\beta_2}{32(\beta_1+\beta_2)}\delta_{-1},$$

and $\beta_i > 0$, $i \in [\![1,4]\!]$, $\sum_{i=1}^4 \beta_i = \frac{1}{2}$.

Let $((\Omega,\mathcal{F},\mathbb{P}),\Phi,\iota,(X_0,X_1,X_2),(\mu_0,\mu_1,\mu_2),(\xi_1,\xi_2))$ be a realisation of (m_0,ρ,ι). First of all, let's check that this example satisfies the additional assumptions we have set for this extended framework.

(A1) Fix $t \in \{0,1\}$, $x,y \in \{-1,1\}$ and $\gamma \in \{0,1\}$ and let Z be a r.v. distributed according to ν defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We have

$$\mathbb{P}(\Psi(x,\gamma,Z) = y) = \mathbb{P}(x[(1-\gamma)(\mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{(\frac{1}{2},1]})(Z) + \gamma(\mathbf{1}_{[0,\frac{3}{4}]} - \mathbf{1}_{(\frac{3}{4},1]})(Z)] = y)$$

and so

- for $x = y \in \{-1,1\}$ and $\gamma = 0$:
$$\mathbb{P}\left(\Psi(x,\gamma,Z) = y\right) = \mathbb{P}\left((\mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{(\frac{1}{2},1]})(Z) = 1\right) = \mathbb{P}\left(Z \in \left[0,\frac{1}{2}\right]\right) = \frac{1}{2};$$
- for $x = y \in \{-1,1\}$ and $\gamma = 1$:
$$\mathbb{P}\left(\Psi(x,\gamma,Z) = y\right) = \mathbb{P}\left((\mathbf{1}_{[0,\frac{3}{4}]} - \mathbf{1}_{(\frac{3}{4},1]})(Z) = 1\right) = \mathbb{P}\left(Z \in \left[0,\frac{3}{4}\right]\right) = \frac{3}{4};$$
- for $x \neq y \in \{-1,1\}$ and $\gamma = 0$:
$$\mathbb{P}\left(\Psi(x,\gamma,Z) = y\right) = \mathbb{P}\left((\mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{(\frac{1}{2},1]})(Z) = -1\right) = \mathbb{P}\left(Z \in \left(\frac{1}{2},1\right]\right) = \frac{1}{2};$$

– for $x \neq y \in \{-1,1\}$ and $\gamma = 1$:

$$\mathbb{P}\left(\Psi(x,\gamma,Z) = y\right) = \mathbb{P}\left((\mathbf{1}_{[0,\frac{3}{4}]} - \mathbf{1}_{(\frac{3}{4},1]})(Z) = -1\right) = \mathbb{P}\left(Z \in \left(\frac{3}{4},1\right]\right) = \frac{1}{4}.$$

Thus, for any $t \in \{0,1\}$, $x, y \in \{-1,1\}$ and $\gamma \in \{0,1\}$,

$$\mathbb{P}\left(\Psi(x,\gamma,Z) = y\right) \geq \frac{1}{4} > 0.$$

(R1) Notice that in the example $\mathbb{P}(\Phi = \varphi) > 0$ if and only if $\varphi \in \{\varphi_0, \varphi_+, \varphi_-, \widehat{\varphi}_+, \widehat{\varphi}_-\} =: \mathfrak{F}$. Thus, if $\varphi \in \mathfrak{F} \setminus \{\varphi_0\}$, the conditions in **(R1)** are obviously satisfied. Indeed, the corresponding set $\mathcal{P}_\varphi$ reduces to a singleton: in particular, we have $\mathcal{P}_{\varphi_+} = \{m_+\}$, $\mathcal{P}_{\varphi_-} = \{m_-\}$, $\mathcal{P}_{\widehat{\varphi}_+} = \{\widehat{m}_+\}$ and $\mathcal{P}_{\widehat{\varphi}_-} = \{\widehat{m}_-\}$. When $\{\Phi = \varphi_0\}$, we have $\mathcal{P}_{\varphi_0} = \{m_+, m_-, \widehat{m}_+, \widehat{m}_-\}$ and:

1. $|\mathcal{P}_{\varphi_0}| = 4$;
2. $\mathbb{P}_{\varphi_0}(\mu \in \mathcal{P}_{\varphi_0}) = 1$;
3. $\mathbb{P}_{\varphi_0}(\mu = m) \geq \min\{\frac{\beta_2}{2(\beta_2+\beta_4)}, \frac{\beta_4}{2(\beta_2+\beta_4)}\}$, for any $m \in \mathfrak{M} := \{m_+, m_-, \widehat{m}_+, \widehat{m}_-\}$.

Further notice that, in this case, we have

$$\mathcal{P}^{(0)}_{\varphi_0} = \{m_0\}, \qquad \mathcal{P}^{(1)}_{\varphi_0} = \{m^{(1)}_+, m^{(1)}_-\} = \{(m_0, m_1^+), (m_0, m_1^-)\}.$$

(R2) In order to guarantee the validity of this assumption, we have to set a new condition on the parameters of the model, that is $\beta_1 = \beta_3 = \beta$ and $\beta_2 = \beta_4 = \gamma$ (so that $\beta + \gamma = \frac{1}{4}$). It is sufficient to notice that, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a couple of independent random variables $\mu \sim \rho_2$, with $\rho_2 = \rho \circ \pi^{-1}_{\mathcal{P}(\mathcal{X})} = \frac{1}{4}(\delta_{m_-} + \delta_{m_+} + \delta_{\widehat{m}_-} + \delta_{\widehat{m}_+})$, and $W \sim \nu$ and setting

$$\begin{aligned}\Phi = \;& \mathbf{1}_{\{m_+\}}(\mu)(\mathbf{1}_{[0,4\beta]}(W)\varphi_+ + \mathbf{1}_{(4\beta,1]}(W)\varphi_0) + \mathbf{1}_{\{m_-\}}(\mu)(\mathbf{1}_{[0,4\beta]}(W)\varphi_- + \mathbf{1}_{(4\beta,1]}(W)\varphi_0)\\ &+ \mathbf{1}_{\{\widehat{m}_+\}}(\mu)(\mathbf{1}_{[0,4\beta]}(W)\widehat{\varphi}_+ + \mathbf{1}_{(4\beta,1]}(W)\varphi_0)\\ &+ \mathbf{1}_{\{\widehat{m}_+\}}(\mu)(\mathbf{1}_{[0,4\beta]}(W)\widehat{\varphi}_+ + \mathbf{1}_{(4\beta,1]}(W)\varphi_0) \qquad (1.7.1)\\ =:\;& \alpha_1(W, \mu^{(2)}),\end{aligned}$$

we have:

– $\mathbb{P} \circ (\Phi, \mu)^{-1} = \rho$. Indeed, exploiting the fact that μ is distributed according to ρ_2 and that Φ is defined via Equation (1.7.1), for $(\varphi, \widetilde{m}) \in \mathfrak{F} \times \mathfrak{M}$, we have

$$\begin{aligned}&\mathbb{P}((\Phi,\mu) = (\varphi, \widetilde{m}))\\ &= \sum_{m \in \mathfrak{M}} \mathbf{1}_{\{m\}}(\widetilde{m})\mathbb{P}(\mu = m)\mathbb{P}(\Phi = \varphi | \mu = m)\\ &= \frac{1}{4}\Big\{\mathbf{1}_{\{m_+\}}(\widetilde{m})(4\beta\mathbf{1}_{\{\varphi_+\}}(\varphi) + 4\gamma\mathbf{1}_{\{\varphi_0\}}(\varphi)) + \mathbf{1}_{\{m_-\}}(\widetilde{m})(4\beta\mathbf{1}_{\{\varphi_-\}}(\varphi) + 4\gamma\mathbf{1}_{\{\varphi_0\}}(\varphi))\\ &\qquad + \mathbf{1}_{\{\widehat{m}_+\}}(\widetilde{m})(4\beta\mathbf{1}_{\{\widehat{\varphi}_+\}}(\varphi) + 4\gamma\mathbf{1}_{\{\varphi_0\}}(\varphi)) + \mathbf{1}_{\{\widehat{m}_-\}}(\widetilde{m})(4\beta\mathbf{1}_{\{\widehat{\varphi}_-\}}(\varphi) + 4\gamma\mathbf{1}_{\{\varphi_0\}}(\varphi))\Big\}\\ &= \rho(\varphi, \widetilde{m}).\end{aligned}$$

– It holds that

$$\begin{aligned}\Phi(0,\cdot) &= \mathbf{1}_{\{m_+\}}(\mu)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{1\}} + \mathbf{1}_{\{m_-\}}(\mu)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{-1\}}\\ &\quad + \mathbf{1}_{\{\widehat{m}_+\}}(\mu)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{1\}} + \mathbf{1}_{\{\widehat{m}_-\}}(\mu)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{-1\}}\\ &= \mathbf{1}_{\{m_1^+\}}(\mu_1)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{1\}} + \mathbf{1}_{\{m_1^-\}}(\mu_1)\mathbf{1}_{[0,4\beta]}(W)\mathbf{1}_{\{-1\}}\\ &= \alpha_0(W, \mu_1),\end{aligned}$$

with $\alpha_0 : \mathcal{Z} \times \mathcal{P}(\mathcal{X})^2 \to \mathcal{E}$, measurable function defined as

$$\alpha_0(w, m) := \mathbf{1}_{\{m_1^+\}}(m)\mathbf{1}_{[0,4\beta]}(w)\mathbf{1}_{\{1\}} + \mathbf{1}_{\{m_1^-\}}(m)\mathbf{1}_{[0,4\beta]}(w)\mathbf{1}_{\{-1\}},$$

Hence, the conditional independence property holds being equivalent to the existence of a $Z \sim \nu$ independent of μ s.t. $\Phi(0, \cdot) = u(Z, \mu^{(1)})$, with $u : \mathcal{Z} \times \mathcal{P}(\mathcal{X})^2 \to \mathcal{E}$, measurable function (see [105, Proposition 6.13]).

(A2) This is omitted being the same as in [30].

(A3) Let us start by checking the Lipschitzianity of f.

-$t = 0$: for any $x_1, x_2 \in \mathcal{X}$, $\gamma_1, \gamma_2 \in \Gamma$ and $m_1, m_2 \in \mathcal{P}(\mathcal{X})$,

$$|f(0, x_1, \gamma_1, m_1) - f(0, x_2, \gamma_2, m_2)| = c_0|\gamma_1 - \gamma_2| = c_0 \mathrm{d}(\gamma_1, \gamma_2);$$

-$t = 1$: for any $x_1, x_2 \in \mathcal{X}$, $\gamma_1, \gamma_2 \in \Gamma$ and $m_1, m_2 \in \mathcal{P}(\mathcal{X})$,

$$|f(1, x_1, \gamma_1, m_1) - f(1, x_2, \gamma_2, m_2)| \le c_1 \mathrm{d}(\gamma_1, \gamma_2) + 2\mathrm{dist}(m_1, m_2) + 2\mathrm{d}(x_1, x_2).$$

Now, for any $x_1, x_2 \in \mathcal{X}$ and $m_1, m_2 \in \mathcal{P}(\mathcal{X})$,

$$|F(x_1, m_1) - F(x_2, m_2)| \le 2\mathrm{d}(x_1, x_2) + 2\mathrm{dist}(m_1, m_2).$$

Hence, the validity of the last assumption follows from the choice $L = \max\{c_0, 4, c_1 + 4\} = \max\{c_0, c_1 + 4\}$.

Now, let us write down some identities specific for the example that we are going to exploit in the following. Concerning the *means* associated to the measure flows, we have

$$\mathrm{M}(m_0) = 0, \qquad \mathrm{M}(m_1^+) = -\mathrm{M}(m_1^-) = \frac{\beta_1}{4(\beta_1 + \beta_2)} = \beta,$$
$$\mathrm{M}(m_2^+) = -\mathrm{M}(m_2^-) = \frac{5\beta_1}{16(\beta_1 + \beta_2)} = \frac{5}{8}\beta.$$

Then, set $\mathbb{P}_0(\cdot) := \mathbb{P}(\cdot|\Phi = \varphi_0)$ and, analogously, $\mathbb{E}_0[\cdot] := \mathbb{E}[\cdot|\Phi = \varphi_0]$. The distribution of the measure flow conditionally on the event $\{\Phi = \varphi_0\}$ can be computed explicitly and it is given by

$$\mathbb{P}_0\left(\mu^{(2)} = m_+\right) = \mathbb{P}_0\left(\mu^{(2)} = m_-\right) = \frac{\beta_2}{2(\beta_2 + \beta_4)} = \frac{1}{4},$$
$$\mathbb{P}_0\left(\mu^{(2)} = \widehat{m}_-\right) = \mathbb{P}_0\left(\mu^{(2)} = \widehat{m}_+\right) = \frac{\beta_4}{2(\beta_2 + \beta_4)} = \frac{1}{4},$$

and, setting $m_+^{(1)} := (m_0, m_1^+)$ and $m_-^{(1)} := (m_0, m_1^-)$, we have

$$\mathbb{P}_0\left(\mu^{(1)} = m_+^{(1)}\right) = \mathbb{P}_0\left(\mu^{(1)} = m_-^{(1)}\right) = \frac{1}{2}.$$

Then, we compute the distribution of $\mu^{(2)}$ conditionally on $\mu^{(1)}$:

$$\mathbb{P}_0\left(\mu^{(2)} = m_+|\mu^{(1)} = m_+^{(1)}\right) = \mathbb{P}_0\left(\mu^{(2)} = m_-|\mu^{(1)} = m_-^{(1)}\right) = \frac{\beta_2}{\beta_2 + \beta_4} = \frac{1}{2},$$
$$\mathbb{P}_0\left(\mu^{(2)} = \widehat{m}_+|\mu^{(1)} = m_+^{(1)}\right) = \mathbb{P}_0\left(\mu^{(2)} = \widehat{m}_-|\mu^{(1)} = m_-^{(1)}\right) = \frac{\beta_4}{\beta_2 + \beta_4} = \frac{1}{2}.$$

The conditions on parameters ensuring the optimality of ρ are presented in the following result.

Proposition 1.7.1. *Consider the MFG setting described above. Then,*

$$\rho = \beta(\delta_{(\varphi_+,m_+)} + \delta_{(\varphi_-,m_-)} + \delta_{(\widehat{\varphi}_+,\widehat{m}_+)} + \delta_{(\widehat{\varphi}_-,\widehat{m}_-)}) + \gamma(\delta_{(\varphi_0,m_+)} + \delta_{(\varphi_0,m_-)} + \delta_{(\varphi_0,\widehat{m}_+)} + \delta_{(\varphi_0,\widehat{m}_-)}), \tag{1.7.2}$$

is optimal provided that

i) $\beta, \gamma \in [0,1]$ *and* $\beta + \gamma = \frac{1}{4}$,

ii) $0 < c_0 < \frac{\beta}{2}$,

ii) $\frac{5}{32}\beta < c_1 < \frac{5}{16}\beta$.

Remark 1.7.2. *Under the assumption that* $\beta_1 = \beta_3 = \beta$ *and* $\beta_2 = \beta_4 = \gamma$*, which we have previously set to ensure the validity of* ***(R2)****, the consistency property is automatically satisfied. Furthermore, under the stronger conditions in the Proposition above, there are still infinitely many correlated solutions but we loose a degree of freedom w.r.t. the result in [30].*

Proof. In this simplified context the set of strategy modifications maps the set $\mathfrak{F}$ into

$$\widehat{\mathcal{R}} = \{\psi : \{0,1\} \times \mathcal{X}^3 \times \mathfrak{M} \to \Gamma, \text{ progressively measurable}\},$$

that is, for any $w \in \widehat{\mathcal{D}}$ and for any $\varphi \in \mathfrak{F}$, $w(\varphi)(0,(x_0,x_1,x_2),(m_0,m_1,m_2)) = w(\varphi)(0,x_0,m_0)$ and $w(\varphi)(1,(x_0,x_1,x_2),(m_0,m_1,m_2)) = w(\varphi)(1,(x_0,x_1),(m_0,m_1))$. In order to find the conditions on the parameters in the definition of ρ in Equation (1.7.2) ensuring that it is a solution in the MFG, we rewrite the cost functional exploiting desintegration over sets of the form $\{\Phi = \varphi\}$, with $\varphi \in \mathfrak{F}$,

$$\begin{aligned}
J(m_0,\rho,w) &= \mathbb{E}\Big[c_0 w(\Phi)(0,X_0,m_0) + c_1 w(\Phi)(1,(X_0,X_1),(m_0,\mu_1)) - X_1 \mathrm{M}(\mu_1) - X_2 \mathrm{M}(\mu_2)\Big] \\
&= \beta\Big\{c_0 \mathbb{E}_+\big[w(\varphi_+)(0,X_0,m_0)\big] + c_1 \mathbb{E}_+\big[w(\varphi_+)(1,(X_0,X_1),(m_0,m_1^+))\big] \\
&\quad - \mathbb{E}_+\big[X_1\big]\mathrm{M}(m_1^+) - \mathbb{E}_+\big[X_2\big]\mathrm{M}(m_2^+)\Big\} + \beta\Big\{c_0 \widehat{\mathbb{E}}_+\big[w(\widehat{\varphi}_+)(0,X_0,m_0)\big] \\
&\quad + c_1 \widehat{\mathbb{E}}_+\big[w(\widehat{\varphi}_+)(1,(X_0,X_1),(m_0,m_1^+))\big] - \widehat{\mathbb{E}}_+\big[\widehat{X}_1\big]\mathrm{M}(m_1^+) - \widehat{\mathbb{E}}_+\big[\widehat{X}_2\big]\mathrm{M}(m_0)\Big\} \\
&\quad + \beta\Big\{c_0 \mathbb{E}_-\big[w(\varphi_-)(0,X_0,m_0)\big] + c_1 \mathbb{E}_-\big[w(\varphi_-)(1,(X_0,X_1),(m_0,m_1^-))\big] \\
&\quad - \mathbb{E}_-\big[X_1\big]\mathrm{M}(m_1^-) - \mathbb{E}_-\big[X_2\big]\mathrm{M}(m_2^-)\Big\} + \beta\Big\{c_0 \widehat{\mathbb{E}}_-\big[w(\widehat{\varphi}_-)(0,X_0,m_0)\big] \\
&\quad + c_1 \widehat{\mathbb{E}}_-\big[w(\widehat{\varphi}_-)(1,(X_0,X_1),(m_0,m_1^-))\big] - \widehat{\mathbb{E}}_-\big[X_1\big]\mathrm{M}(m_1^-) - \widehat{\mathbb{E}}_-\big[X_2\big]\mathrm{M}(m_0)\Big\} \\
&\quad + 4\gamma\Big\{c_0 \mathbb{E}_0\big[w(\varphi_0)(0,X_0,m_0)\big] + c_1 \mathbb{E}_0\big[w(\varphi_0)(1,(X_0,X_1),(m_0,\mu_1))\big]
\end{aligned}$$

$$- \mathbb{E}_0\left[X_1 \mathrm{M}(\mu_1)\right] - \mathbb{E}_0\left[X_2 \mathrm{M}(\mu_2)\right]\Bigg\},$$

where we have exploited the fact that the conditioning on $\{\Phi = \varphi\}$, with $\varphi \in \{\varphi_+, \varphi_-, \widehat{\varphi}_+, \widehat{\varphi}_-\}$, completely determines the measure flow as well. Notice that the notation $\mathbb{E}_+$ (resp. $\mathbb{E}_-$, $\widehat{\mathbb{E}}_+$, $\widehat{\mathbb{E}}_-$ and $\mathbb{E}_0$) was introduced to denote conditional expectation w.r.t. the event $\{\Phi = \varphi_+\}$ (resp. $\varphi_-, \widehat{\varphi}_+, \widehat{\varphi}_-$ and φ_0). Before proceeding with the study of the different cases we make a useful remark.

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\varphi)$, where $\mathbb{P}_\varphi(\cdot) = \mathbb{P}(\cdot|\Phi = \varphi)$, with $\varphi \in \mathfrak{F}$. For any $t \in [\![0, T-1]\!]$, $X^{(t)}$ and $(\mu_{t+1}, \dots, \mu_T)$ are conditionally independent given $\mu^{(t)}$. Indeed, for any $m \in \mathcal{P}(\mathcal{X})^{T-t}$, $x \in \mathcal{X}^{t+1}$, exploiting in sequence the tower property, the measurability of $X^{(t)}$ w.r.t. $\sigma(X_0, \xi_1, \dots, \xi_t, \Phi, \mu^{(t)})$, the joint independence of μ from X_0 and $\xi_1, \dots, \xi_T$, and the measurability of conditional expectations, we have

$$\begin{aligned}
\mathbb{P}_\varphi((\mu_{t+1}, \dots, \mu_T) = m, X^{(t)} = x|\mu^{(t)}) &= \mathbb{E}_\varphi[\mathbf{1}_{\{m\}}(\mu_{t+1}, \dots, \mu_T)\mathbf{1}_{\{x\}}(X^{(t)})|\mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{x\}}(X^{(t)})\mathbb{E}_\varphi[\mathbf{1}_{\{m\}}(\mu_{t+1}, \dots, \mu_T)|\mu^{(t)}, X_0, \xi_1, \dots, \xi_t]|\mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{x\}}(X^{(t)})\mathbb{E}_\varphi[\mathbf{1}_{\{m\}}(\mu_{t+1}, \dots, \mu_T)|\mu^{(t)}]|\mu^{(t)}] \\
&= \mathbb{E}_\varphi[\mathbf{1}_{\{m\}}(\mu_{t+1}, \dots, \mu_T)|\mu^{(t)}]\mathbb{E}_\varphi[\mathbf{1}_{\{x\}}(X^{(t)})|\mu^{(t)}] \\
&= \mathbb{P}_\varphi((\mu_{t+1}, \dots, \mu_T) = m|\mu^{(t)})\mathbb{P}_\varphi(X^{(t)} = x|\mu^{(t)}).
\end{aligned}$$

Now, let's start by discussing the first case, that is when the suggestion is $\{\Phi = \varphi_+\}$. We proceed exploiting the DPP (Proposition 1.5.5). In the following we omit the dependency on the measure flow being it identically equal to a single element and we introduce the following simplified notations: $V_+ := V_{\varphi_+}$, $V_- := V_{\varphi_-}$, $\widehat{V}_+ := V_{\widehat{\varphi}_+}$, $\widehat{V}_- := V_{\widehat{\varphi}_-}$ and $V_0 := V_{\varphi_0}$.

- For $t = 2$, $x \in \{-1, 1\}^3$,

$$\begin{aligned}
V_+(2, (x_0, x_1, 1)) &= F(1, m_2^+) = -\mathrm{M}(m_2^+) = -\frac{5}{4}\beta, \\
V_+(2, (x_0, x_1, -1)) &= F(-1, m_2^+) = +\mathrm{M}(m_2^+) = \frac{5}{4}\beta,
\end{aligned}$$

- For $t = 1$, $x \in \{-1, 1\}^2$,

$$\begin{aligned}
V_+(1, (x_0, -1)) &= \min_{\gamma \in \{0,1\}} \left\{ c_1\gamma + \mathrm{M}(m_1^+) + \mathbb{E}_+\left[V_+\left(2, (x_0, -1, \Psi(-1, \gamma, \xi_2))\right)\right] \right\} \\
&= \min_{\gamma \in \{0,1\}} \left\{ c_1\gamma + \mathrm{M}(m_1^+) + \mathrm{M}(m_2^+)\left[\mathbb{P}_+\left(\Psi(-1, \gamma, \xi_2) = -1\right) - \mathbb{P}_+\left(\Psi(-1, \gamma, \xi_2) = 1\right)\right] \right\} \\
&= \mathrm{M}(m_1^+) + \min\left\{ \mathrm{M}(m_2^+)\left(\frac{1}{2} - \frac{1}{2}\right), c_1 + \mathrm{M}(m_2^+)\left(-\frac{1}{4} + \frac{3}{4}\right) \right\} \\
&= \beta + \min\left\{0, c_1 + \frac{5}{16}\beta\right\}.
\end{aligned}$$

This implies that, at time $t = 1$ in state $(x_0, -1)$, $\gamma = 0$ is optimal which corresponds to φ_+

evaluated at $t = 1, x = -1$. Analogously,

$$V_+(1,(x_0,1)) = \min_{\gamma\in\{0,1\}} \left\{ c_1\gamma - \mathrm{M}(m_1^+) + \mathbb{E}_+ \left[V_+\left(2,(x_0,1,\Psi(1,\gamma,\xi_2))\right)\right] \right\}$$
$$= -\beta + \min\left\{0, c_1 - \frac{5}{16}\beta\right\}.$$

This implies that $\gamma = 1$ (and so φ_+) is optimal at time $t = 1$ and state $(x_0, 1)$ if and only if $c_1 - \frac{5}{16}\beta < 0$, that is

$$0 < c_1 < \frac{5}{16}\beta. \tag{1.7.3}$$

- For $t = 0$, $x \in \{-1, 1\}$,

$$V_+(0,-1) = \min_{\gamma\in\{0,1\}} \left\{ c_0\gamma + \mathbb{E}_+ \left[V_+\left(1,(-1,\Psi(-1,\gamma,\xi_2))\right)\right] \right\}$$
$$= \min_{\gamma\in\{0,1\}} \left\{ c_0\gamma + \left(-\beta + c_1 - \frac{5}{16}\beta\right)\mathbb{P}_+\left(\Psi(-1,\gamma,\xi_2) = 1\right) + \beta\mathbb{P}_+\left(\Psi(-1,\gamma,\xi_2) = -1\right) \right\}$$
$$= \min\left\{0 + \frac{1}{2}\left(-\beta + c_1 - \frac{5}{16}\beta\right) + \frac{1}{2}\beta, c_0 + \left(-\beta + c_1 - \frac{5}{16}\beta\right)\frac{1}{4} + \beta\frac{3}{4}\right\}$$
$$= \min\left\{\frac{1}{2}\left(c_1 - \frac{5}{16}\beta\right), c_0 + \left(c_1 - \frac{5}{16}\beta\right)\frac{1}{4} + \beta\frac{1}{2}\right\}.$$

Since $c_1 - \frac{5}{16}\beta < 0$ and all the parameters are positive, at time $t = 0$ in state $x_0 = -1$, $\gamma = 0$ is optimal which corresponds to φ_+ evaluated at $t = 0, x = -1$. Analogously,

$$V_+(0,1) = \min_{\gamma\in\{0,1\}} \left\{ c_0\gamma + \mathbb{E}_+ \left[V_+\left(1,(1,\Psi(1,\gamma,\xi_2))\right)\right] \right\}$$
$$= \min\left\{\frac{1}{2}\left(c_1 - \frac{5}{16}\beta\right), c_0 + \left(c_1 - \frac{5}{16}\beta\right)\frac{3}{4} - \beta\frac{1}{2}\right\}.$$

This implies that $\gamma = 1$ (and so φ_+) is optimal at time $t = 0$ and state 1 if and only if $\frac{1}{2}\left(c_1 - \frac{5}{16}\beta\right) > c_0 + \left(c_1 - \frac{5}{16}\beta\right)\frac{3}{4} - \beta\frac{1}{2}$. Since we have already set $c_1 < \frac{5}{16}\beta$, we set the following stronger condition that guarantees the validity of the inequality above

$$0 < c_0 < \frac{1}{2}\beta. \tag{1.7.4}$$

Hence, we have shown that, conditionally on the event $\{\Phi = \varphi_+\}$, φ_+ is optimal.

The case $\{\Phi = \varphi_-\}$ is completely analogous and leads to the same constraints on the coefficients.

Now, let's discuss in details the case in which the suggestion is $\{\Phi = \widehat{\varphi}_+\}$.

- For $t = 2$, $x \in \{-1, 1\}^3$,

$$\widehat{V}_+(2,(x_0,x_1,x_2)) = F(x_2, m_0) = -x_2\mathrm{M}(m_0) = 0,$$

- For $t = 1$, $x \in \{-1, 1\}^2$,

$$\widehat{V}_+(1,(x_0,x_1)) = \min_{\gamma\in\{0,1\}} \left\{ c_1\gamma - x_1\mathrm{M}(m_1^+) + \widehat{\mathbb{E}}_+ \left[\widehat{V}_+\left(2,(x_0,x_1,\Psi(x_1,\gamma,\xi_2))\right)\right] \right\}$$

$$= \min_{\gamma \in \{0,1\}} \left\{ c_1 \gamma - x_1 \mathrm{M}(m_1^+) \right\}$$
$$= -x_1 \mathrm{M}(m_1^+) + c_1 \min \left\{ 0, \gamma \right\} = -x_1 \mathrm{M}(m_1^+).$$

This implies that at time $t = 1$, in any state (x_0, x_1), $\gamma = 0$ is optimal which corresponds to $\widehat{\varphi}_+$ evaluated at $t = 1$.

- For $t = 0$, $x \in \{-1, 1\}$,

$$\widehat{V}_+(0, -1) = \min_{\gamma \in \{0,1\}} \left\{ c_0 \gamma + \widehat{\mathbb{E}}_+ \left[\widehat{V}_+ \left(1, (-1, \Psi(-1, \gamma, \xi_2))\right) \right] \right\}$$
$$= \min_{\gamma \in \{0,1\}} \left\{ c_0 \gamma + (-\beta) \widehat{\mathbb{E}}_+ \left[\Psi(-1, \gamma, \xi_2) \right] \right\}$$
$$= \min \left\{ 0, c_0 + \frac{\beta}{2} \right\}.$$

At time $t = 0$ in state $x_0 = -1$, $\gamma = 0$ is optimal which corresponds to $\widehat{\varphi}_+$ evaluated at $t = 0, x = -1$. Analogously,

$$\widehat{V}_+(0, 1) = \min_{\gamma \in \{0,1\}} \left\{ c_0 \gamma + \widehat{\mathbb{E}}_+ \left[\widehat{V}_+ \left(1, (1, \Psi(1, \gamma, \xi_2))\right) \right] \right\} = \min \left\{ 0, c_0 - \frac{\beta}{2} \right\}.$$

The condition that we have set in Equation (1.7.4) yields that $\gamma = 1$ (and so $\widehat{\varphi}_+$) is optimal at time $t = 0$ and state 1. Hence, we have checked that, conditionally on the event $\{\Phi = \widehat{\varphi}_+\}$, $\widehat{\varphi}_+$ is optimal.

The computations for the case $\{\Phi = \widehat{\varphi}_-\}$ are analogous and lead to the same constraints.

The last case, namely $\{\Phi = \varphi_0\}$, is the most complicated. Indeed, in this case we have to handle a random measure flow and consequently different flows of measure and different outcomes when evaluating the strategies of the representative player. This is done exploiting again the dynamic programming principle.

- For $t = 2$, $x \in \{1, -1\}^3$, $(m_0, m_1, m_2) \in \{m_+, \widehat{m}_+, m_-, \widehat{m}_-\} = \mathcal{D}_{\varphi_0}$,

$$V_0(2, (x_0, x_1, x_2), (m_0, m_1, m_2)) = -x_2 \mathrm{M}(m_2).$$

In particular, we have

$$V_0(2, (x_0, x_1, 1), m_+) = V_0(2, (x_0, x_1, -1), m_-) = -\mathrm{M}(m_2^+) = -\frac{5}{8}\beta,$$
$$V_0(2, (x_0, x_1, -1), m_+) = V_0(2, (x_0, x_1, 1), m_-) = \mathrm{M}(m_2^+) = \frac{5}{8}\beta,$$
$$V_0(2, (x_0, x_1, x_2), (m_0, m_1, m_0)) = 0.$$

- For $t = 1$, $x \in \{1, -1\}^2$, $(m_0, m_1) \in \{m_+^{(1)}, m_-^{(1)}\} = \mathcal{D}_{\varphi_0}^{(1)}$,

$$V_0(1, (x_0, x_1), (m_0, m_1)) = \min_{\gamma \in \{0,1\}} \Big\{ c_1 \gamma - x \mathrm{M}(m_1)$$
$$+ \mathbb{E}_0 \left[V_0(2, (x_0, x_1, \Psi(x_1, \gamma, \xi_2), (m_0, m_1, \mu_2))) | \mathrm{X}^{(1)} = (x_0, x_1), \mu^{(1)} = (m_0, m_1) \right] \Big\}.$$

Exploiting the computations at the previous step, the fact that ξ_2 and $(\Phi, \mu, \mathrm{X}_0, \xi_1)$ are independent, and the fact that, on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{\varphi_0})$, $\mathrm{X}^{(1)}$ and μ_2 are conditionally independent given $\mu^{(1)}$, we have

$$
\begin{aligned}
V_0(1,(x_0,1),m_+^{(1)}) &= \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma - \mathrm{M}(m_1^+) \\
&\quad + \mathbb{E}_0\Big[V_0(2,(x_0,1,\Psi(1,\gamma,\xi_2)),(m_+^{(1)},\mu^{(2)}))|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_+^{(1)}\Big]\Big\} \\
&= -\mathrm{M}(m_1^+) + \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \mathrm{M}(m_2^+)\Big[\mathbb{P}_0(\Psi(1,\gamma,\xi_2)=-1,\mu=m_2^+|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_+^{(1)}) \\
&\quad - \mathbb{P}_0(\Psi(1,\gamma,\xi_2)=1,\mu=m_2^+|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_+^{(1)})\Big]\Big\} \\
&= -\mathrm{M}(m_1^+) + \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \mathrm{M}(m_2^+)\Big[\mathbb{P}_0(\Psi(1,\gamma,\xi_2)=-1)\mathbb{P}_0(\mu=m_2^+|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_+^{(1)}) \\
&\quad - \mathbb{P}_0(\Psi(1,\gamma,\xi_2)=1)\mathbb{P}_0(\mu=m_2^+|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_+^{(1)})\Big]\Big\} \\
&= -\beta + \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \frac{5}{8}\beta\Big[\mathbb{P}_0(\Psi(1,\gamma,\xi_2)=-1)\mathbb{P}_0(\mu=m_2^+|\mu^{(1)}=m_+^{(1)}) \\
&\quad - \mathbb{P}_0(\Psi(1,\gamma,\xi_2)=1)\mathbb{P}_0(\mu=m_2^+|\mu^{(1)}=m_+^{(1)})\Big]\Big\} \\
&= -\beta + \min\left\{0+\frac{5}{16}\beta\left[\frac{1}{2}-\frac{1}{2}\right], c_1+\frac{5}{16}\beta\left[\frac{1}{4}-\frac{3}{4}\right]\right\} = -\beta + \min\left\{0, c_1-\frac{5}{32}\beta\right\}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
V_0(1,(x_0,-1),m_-^{(1)}) &= \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \mathrm{M}(m_1^-) \\
&\quad + \mathbb{E}_0\left[V_0(2,(x_0,-1,\Psi(-1,\gamma,\xi_2)),(m_-^{(1)},\mu^{(2)}))|\mathrm{X}^{(1)}=(x_0,-1),\mu^{(1)}=m_-^{(1)}\right]\Big\} \\
&= -\beta + \min\left\{0+\frac{5}{16}\beta\left[\frac{1}{2}-\frac{1}{2}\right], c_1+\frac{5}{16}\beta\left[\frac{1}{4}-\frac{3}{4}\right]\right\} = -\beta + \min\left\{0, c_1-\frac{5}{32}\beta\right\}.
\end{aligned}
$$

This yields that $\gamma = 0$ (and so φ_0) is optimal at time $t = 0$ when $(x, m) \in \{((x_0, 1), (m_0, m_1^+)), ((x_0, -1), (m_0, m_1^-))\}$ if and only if $c_1 - \frac{5}{32}\beta > 0$. Thus, we set the condition

$$
\frac{5}{32}\beta < c_1. \tag{1.7.5}
$$

Analogously, we compute

$$
\begin{aligned}
V_0(1,(x_0,-1),m_+^{(1)}) &= \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \mathrm{M}(m_1^+) \\
&\quad + \mathbb{E}_0\Big[V_0(2,(x_0,-1,\Psi(-1,\gamma,\xi_2)),(m_+^{(1)},\mu^{(2)}))|\mathrm{X}^{(1)}=(x_0,-1),\mu^{(1)}=m_+^{(1)}\Big]\Big\} \\
&= \beta + \min\left\{0+\frac{5}{16}\beta\left[\frac{1}{2}-\frac{1}{2}\right], c_1+\frac{5}{16}\beta\left[\frac{3}{4}-\frac{1}{4}\right]\right\} = \beta + \min\left\{0, c_1+\frac{5}{32}\beta\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
V_0(1,(x_0,1),m_-^{(1)}) &= \min_{\gamma\in\{0,1\}}\Big\{c_1\gamma + \mathrm{M}(m_1^-) \\
&\quad + \mathbb{E}_0\left[V_0(2,(x_0,1,\Psi(1,\gamma,\xi_2)),(m_-^{(1)},\mu^{(2)}))|\mathrm{X}^{(1)}=(x_0,1),\mu^{(1)}=m_-^{(1)}\right]\Big\}
\end{aligned}
$$

$$= \beta + \min\left\{0 + \frac{5}{16}\beta\left[\frac{1}{2} - \frac{1}{2}\right], c_1 + \frac{5}{16}\beta\left[\frac{3}{4} - \frac{1}{4}\right]\right\} = \beta + \min\left\{0, c_1 + \frac{5}{32}\beta\right\}$$

Thus, $\gamma = 0$ (and so φ_0) is optimal at time $t = 0$ when $(x, m) \in \{((x_0, -1), (m_0, m_1^+)), ((x_0, 1), (m_0, m_1^-))\}$, without the need of any further constraint.

- For $t = 0$, $x_0 \in \{1, -1\}$,

$$\begin{aligned} V_0(0, x_0) &= V_0(0, x_0, m_0) \\ &= \min_{\gamma\in\{0,1\}} \left\{c_0\gamma + \mathbb{E}_0\left[V_0(1, (x_0, \Psi(x_0, \gamma, \xi_1), (m_0, \mu_1))|X_0 = x_0, \mu_0 = m_0\right]\right\}. \end{aligned}$$

Finally, we study the initial time step in detail, exploiting the fact that X_0, ξ_1 and μ_1 are independent on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$:

$$\begin{aligned} V_0(0, 1) &= \min_{\gamma\in\{0,1\}} \left\{c_0\gamma + \mathbb{E}_0\left[V_0(1, (1, \Psi(1, \gamma, \xi_1), (m_0, \mu_1))|X_0 = 1, \mu_0 = m_0\right]\right\} \\ &= \min_{\gamma\in\{0,1\}} \left\{c_0\gamma + \mathbb{E}_0\left[V_0(1, (1, \Psi(1, \gamma, \xi_1), (m_0, \mu_1))|X_0 = 1\right]\right\} \\ &= \min_{\gamma\in\{0,1\}} \left\{c_0\gamma + \frac{\beta}{2}\Big[\mathbb{P}_0\left(\Psi(1, \gamma, \xi_1) = -1\right) + \mathbb{P}_0\left(\Psi(1, \gamma, \xi_1) = 1\right)\Big]\right. \\ &\qquad \left. - \frac{\beta}{2}\Big[\mathbb{P}_0\left(\Psi(1, \gamma, \xi_1) = 1\right) + \mathbb{P}_0\left(\Psi(1, \gamma, \xi_1) = -1\right)\Big]\right\} \\ &= \min\left\{0 + \frac{\beta}{2}\left[\left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}\right)\right], c_0 + \frac{\beta}{2}\left[\left(\frac{1}{4} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{4}\right)\right]\right\} \\ &= \min\{0, c_0\} \end{aligned}$$

and, similarly,

$$\begin{aligned} V_0(0, -1) &= \min_{\gamma\in\{0,1\}} \left\{c_0\gamma + \mathbb{E}_0\left[V_0(1, (-1, \Psi(-1, \gamma, \xi_1), (m_0, \mu_1))|X_0 = -1, \mu_0 = m_0\right]\right\} \\ &= \min\left\{0 + \frac{\beta}{2}\left[\left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}\right)\right], c_0 + \frac{\beta}{2}\left[\left(\frac{1}{4} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{4}\right)\right]\right\} \\ &= \min\{0, c_0\}. \end{aligned}$$

Hence, at time $t = 0$, $\gamma = 0$ (and so φ_0) is optimal at any state.

Thus, we have proved that, conditionally on the event $\{\Phi = \varphi_0\}$, the strategy φ_0 is optimal, completing the analysis of the various cases. Now, putting together the conditions in Equations (1.7.3), (1.7.4) and (1.7.5), we obtain the statement of the theorem.

□

Appendix

1.A Propagation of chaos

First of all, let us recall some basic definitions, for which we refer to [82]. We denote with Π_n the set of permutations over n elements, namely over $[\![1,n]\!]$. Consider a probability measure $p \in \mathcal{P}(\mathcal{X})$ and a sequence of symmetric probability measures $\{p_n\}_{n\in\mathbb{N}}$, with $p_n \in \mathcal{P}(\mathcal{X}^n)$, for each $n \in \mathbb{N}$. We call the sequence of probability measures $(p_n)_{n\in\mathbb{N}}$ *p-chaotic* if for any choice of $k \in \mathbb{N}$ continuous and bounded functions on $\mathcal{X}$, $g_1, \dots, g_k$, we have

$$\lim_{n\to\infty} \int_{\mathcal{X}^n} g_1(s_1)\dots g_k(s_k) p_n(\mathrm{d}s_1, \dots, \mathrm{d}s_n) = \prod_{j=1}^{k} \int_{\mathcal{X}} g_j(s) p(\mathrm{d}s).$$

Then, we call a sequence of symmetric probability measures $(p_n)_{n\in\mathbb{N}}$ *chaotic*, if there exists a probability measure $p \in \mathcal{P}(\mathcal{X})$ s.t. $(p_n)_{n\in\mathbb{N}}$ is *p-chaotic*. Let $(\beta_n(\cdot,\cdot))_{n\in\mathbb{N}}$ be a sequence of probability kernels such that, for any $n \in \mathbb{N}$, $\beta_n : \mathcal{X}^n \times \mathcal{B}(\mathcal{X})^n \to [0,1]$ satisfies the following (symmetry) condition:

$$\beta_n(x,B) = \beta_n(\pi x, \pi B), \qquad \text{for any } \pi \in \Pi_n.$$

We say that *propagation of chaos* holds for the sequence $(\beta_n(\cdot,\cdot))_{n\in\mathbb{N}}$ if $(Up_n)_{n\in\mathbb{N}}$ is chaotic for any chaotic sequence $(p_n)_{n\in\mathbb{N}}$, where, for any $n \in \mathbb{N}$,

$$Up_n(B) := \int_{\mathcal{X}^n} \beta_n(x,B) p_n(\mathrm{d}x), \qquad \text{for all } B \in \mathcal{B}(\mathcal{X})^n.$$

We are going to show that propagation of chaos holds in our case via the following equivalent characterisation.

Theorem 1.A.1 (Theorem 4.2, in [82]). *Consider a couple of complete and separable metric spaces, $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$. For each $n \in \mathbb{N}$, let Π_n denote the set of permutations over $[\![1,n]\!]$. Let $\beta_n : \mathcal{X}^n \times \mathcal{B}(\mathcal{Y}^n) \to [0,1]$ be a sequence of Markovian transition functions (probability kernels), i.e. for $x_n \in \mathcal{X}^n$ and $B \in \mathcal{B}(\mathcal{Y}^n)$, $\beta_N(x_n, B)$ is the probability that the state of the n-particle system lies in B, given that the initial state was x_n. Suppose that the transition functions satisfy the following condition:*

$$\beta_n(x_n, B) = \beta_n(\pi x_n, \pi B), \quad \textit{for all } \pi \in \Pi_n, \textit{ for all } x_n \in \mathcal{X}^n \textit{ and for all } B \in \mathcal{B}(\mathcal{Y}^n).$$

Then, $\{\beta_n\}_{n\in\mathbb{N}}$ propagates chaos if and only if, whenever $\mu_n(x_n) := \frac{1}{n}\sum_{j=1}^n \delta_{(x_n)^j} \to p$ in $\mathcal{P}(\mathcal{X})$ with $x_n \in \mathcal{X}^n$, then $\{\widetilde{\beta}_n(x_n,\cdot)\}_{n\in\mathbb{N}}$ is $F(p)$-chaotic, where $F : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$, is a continuous function w.r.t. weak topologies and $\widetilde{\beta}_n$ is defined as

$$\widetilde{\beta}_n(x_n, B) = \frac{1}{n!} \sum_{\pi\in\Pi_n} \beta_n(x_n, \pi B).$$

Now, we should reframe the general definitions above in our context. Consider $x^N \in \mathcal{X}^N$ (initial conditions) and $B \in \mathcal{B}(\mathcal{X}^N)$. In our case, for an arbitrary fixed $N \in \mathbb{N}$, the probability kernel is given by

$$\begin{aligned}\beta_N(x^N, B) &= \mathbb{P}_{N,m} \circ (X_1^{1,N,m}, \dots, X_1^{N,N,m})^{-1}(B) \\ &= \mathbb{P}_{N,m}\left(\left(\Psi(0, x_j^N, \frac{1}{N-1}\sum_{k\neq j}\delta_{x_k^N}, \Phi_j^{N,m}(0, x_j^N), \xi_1^{j,N,m})\right)_{j=1}^N \in B\right),\end{aligned} \tag{1.A.1}$$

where, in the second line, we have exploited the fact that $\Phi_1^{N,m} = \widetilde{\Phi}_1^{N,m}$, $\mathbb{P}_{N,m}$-a.s., and that, since $\gamma_m^N = \rho_1(\cdot|m)^{\otimes N}$, $\Phi_j^{N,m}$ takes values in $\mathcal{R}$, for each $j \in [\![1, N]\!]$.

We have the following propagation of chaos result:

Claim 2. *Propagation of chaos holds for the first time step of our model, i.e.* $(\beta_N(\cdot,\cdot))_{N\in\mathbb{N}}$*, as defined in Equation* (1.A.1)*, propagates chaos.*

Proof of Claim 2. First of all, we need to prove that condition (1.A.1) in Theorem 1.A.1 holds. We denote with π a generic permutation of $[\![1, N]\!]$. For any $x^N \in \mathcal{X}^N$ and $B = B_1 \times \ldots \times B_N \in \mathcal{B}(\mathcal{X}^N)$, with $\pi B = B_{\pi(1)} \times \ldots \times B_{\pi(N)}$, we have

$$\begin{aligned}&\beta_N(\pi x^N, \pi B) \\ &= \mathbb{P}_{N,m}\left(\left(\Psi(0, x_{\pi(j)}^N, \frac{1}{N-1}\sum_{k\neq j}\delta_{x_{\pi(k)}^N}, \Phi_j^{N,m}(0, x_{\pi(j)}^N), \xi_1^{j,N,m})\right)_{j=1}^N \in \pi B\right) = \star.\end{aligned}$$

Since $(\Phi_j^{N,m})_{j=1}^N \overset{d}{\sim} \rho_1(\cdot|m)^{\otimes N}$ and $(\xi_1^{j,N,m})_{j=1}^N \overset{d}{\sim} \nu^{\otimes N}$ are independent, we reorder the terms to get

$$\begin{aligned}\star &= \mathbb{P}_{N,m}\left(\left(\Psi(0, x_{\pi(j)}^N, \frac{1}{N-1}\sum_{k\neq j}\delta_{x_{\pi(k)}^N}, \Phi_{\pi(j)}^{N,m}(0, x_{\pi(j)}^N), \xi_1^{\pi(j),N,m})\right)_{j=1}^N \in \pi B\right) \\ &= \mathbb{P}_{N,m}\left(\left(\Psi(0, x_j^N, \frac{1}{N-1}\sum_{k\neq j}\delta_{x_k^N}, \Phi_j^{N,m}(0, x_j^n), \xi_1^{j,N,m})\right)_{j=1}^N \in B\right) = \beta_N(x^N, B).\end{aligned}$$

Thus, we have shown that condition (1.A.1) holds. Now, to conclude that $(\beta_N(\cdot,\cdot))_{N\in\mathbb{N}}$ propagates chaos we need to prove that, for any given sequence $x^N \in \mathcal{X}^N$, $N \in \mathbb{N}$, such that $\mu_N(x^N) := \frac{1}{N}\sum_{j=1}^N \delta_{x_j^N} \to p$ in $\mathcal{P}(\mathcal{X})$, the sequence $(\widetilde{\beta}_N(x^N,\cdot))_{N=1}^\infty$, with $\widetilde{\beta}_N$ defined as

$$\widetilde{\beta}_N(x^N, B) = \frac{1}{N!}\sum_{\pi\in\Pi_N}\beta_N(x_N, \pi B), \qquad x^N \in \mathcal{X}^N, B \in \mathcal{B}(\mathcal{X})^N,$$

is $F(p)$-chaotic, where $F : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is a suitable continuous function.
Suppose that $\mu_N(x^N) = \frac{1}{N}\sum_{j=1}^N \delta_{x_j^N} \to p$ in $\mathcal{P}(\mathcal{X})$, let us consider $g_1, \dots, g_l \in C_b(\mathcal{X})$, $l \in \mathbb{N}$, exploiting property (1.A.1) we have

$$\int_{\mathcal{X}^N} g_1(y_1)\dots g_l(y_l)\widetilde{\beta}_N(x^N, dy_1 \dots dy_N) = \frac{1}{N!}\sum_{\pi\in\Pi_N}\int_{\mathcal{X}^N} g_1(y_1)\dots g_l(y_l)\beta_N(x^N, dy_{\pi(1)} \dots dy_{\pi(N)})$$

$$= \frac{1}{N!} \sum_{\pi \in \Pi_N} \int_{\mathcal{X}^N} g_1(y_1) \dots g_l(y_l) \beta_N(\pi x^N, \mathrm{d}y_1 \dots \mathrm{d}y_N) =: \star$$

Now, we exploit the definition of $\beta_N(\cdot,\cdot)$ to gather terms together in order to get

$$\star = \frac{1}{N!} \sum_{\pi \in \Pi_N} \int_{\mathcal{R}^N} \int_{\mathcal{Z}^N} \prod_{j=1}^{l} g_j(\Psi(0, x^N_{\pi(j)}, \frac{1}{N-1} \sum_{k \neq j} \delta_{x^N_{\pi(k)}}, \varphi_j(0, x^N_{\pi(j)}), z_j)) \nu^{\otimes N}(\mathrm{d}z_1, \dots, \mathrm{d}z_N) \gamma^N_m(\mathrm{d}\varphi)$$

$$= \frac{1}{N!} \sum_{\pi \in \Pi_N} \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, x^N_{\pi(j)}, \frac{1}{N-1} \sum_{k \neq j} \delta_{x^N_{\pi(k)}}, \varphi(0, x^N_{\pi(j)}), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi | m)$$

$$= \frac{1}{N!} \sum_{\pi \in \Pi_N} \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, x^N_{\pi(j)}, \frac{N}{N-1} \mu_N(x^N) - \frac{1}{N-1} \delta_{x^N_{\pi(j)}}, \varphi(0, x^N_{\pi(j)}), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi | m)$$

$$= \frac{(N-l)!}{N!} \sum_{\lambda \in \mathcal{I}_{N:l}} \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, x^N_{\lambda(j)}, \frac{N}{N-1} \mu_N(x^N) - \frac{1}{N-1} \delta_{x^N_{\lambda(j)}}, \varphi(0, x^N_{\lambda(j)}), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi | m)$$

$$=: \diamond,$$

where $\mathcal{I}_{N:l}$ denotes the set of injections from $[\![1, l]\!]$ to $[\![1, N]\!]$.
Set $\mu_{N:l}$ to be, for a vector $x^N \in \mathcal{X}^N$, the symmetric probability measure given by

$$\mu_{N:l}(x^N) = \frac{(N-l)!}{N!} \sum_{\lambda \in \mathcal{I}_{N:l}} \delta_{(x^N_{\lambda(1)}, \dots, x^N_{\lambda(l)})}.$$

It is possible to show, see [82] pg. 29, that $\mu_N(x^N) \underset{N \to \infty}{\longrightarrow} p$ implies $\mu_{N:l}(x^N) \underset{N \to \infty}{\longrightarrow} p^{\otimes l}$. We have

$$\diamond = \frac{(N-l)!}{N!} \sum_{\lambda \in \mathcal{I}_{N:l}} \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, x^N_{\lambda(j)}, \frac{N}{N-1} \mu_N(x^N) - \frac{1}{N-1} \delta_{x^N_{\lambda(j)}}, \varphi(0, x^N_{\lambda(j)}), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi | m)$$

$$= \int_{\mathcal{X}^l} \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, \frac{N}{N-1} \mu_N(x^N) - \frac{1}{N-1} \delta_{y_j}, \varphi(0, y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi | m) \mu_{N:l}(x^N)(\mathrm{d}y_1, \dots, \mathrm{d}y_l)$$

$$= \int_{\mathcal{X}^l} \mu_{N:l}(x^N)(\mathrm{d}y) \Big\{ \prod_{j=1}^{l} \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi | m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) g_j(\Psi(0, y_j, \frac{N}{N-1} \mu_N(x^N) - \frac{1}{N-1} \delta_{y_j}, \varphi(0, y_j), z)) \Big\}$$

$$\underset{N \to \infty}{\longrightarrow} \int_{\mathcal{X}^l} p^{\otimes l}(\mathrm{d}y) \Big\{ \prod_{j=1}^{l} \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi | m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) g_j(\Psi(0, y_j, p, \varphi(0, y_j), z)) \Big\}$$

$$= \prod_{j=1}^{l} \int_{\mathcal{X}} p(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) g_j(\Psi(0,y,p,\varphi(0,y),z)) = \prod_{j=1}^{l} \int_{\mathcal{X}} g_j(x) q(p)(\mathrm{d}x),$$

where $q(p)$ is the image of $(p, \rho_1(\cdot|m), \nu)$ via the mapping $(y,\varphi,z) \mapsto \Psi(0,y,p,\varphi(0,y),z)$. In particular, the convergence in the fourth line is proved as follows, exploiting a generalisation of the continuous mapping theorem, namely [22, Theorem I.5.5]. In the notation of [22, Theorem I.5.5], we have $\mathbf{P}_N = \mu_{N:l}(x^N) \overset{N\to\infty}{\longrightarrow} p^{\otimes l}$, by assumption. Furthermore, we consider the following functions $h_N : \mathcal{X}^l \to [-\prod_{j=1}^{l} \|g_j\|_\infty, \prod_{j=1}^{l} \|g_j\|_\infty]$, for all $N \in \mathbb{N}$, and $h : \mathcal{X}^l \to [-\prod_{j=1}^{l} \|g_j\|_\infty, \prod_{j=1}^{l} \|g_j\|_\infty]$, defined, for $y \in \mathcal{X}$, by

$$h_N(y) := \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, \frac{N}{N-1}\mu_N(x^N) - \frac{1}{N-1}\delta_{y_j}, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m),$$

$$h(y) := \prod_{j=1}^{l} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, p, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m).$$

We have that both previous functions are measurable, since the finite set $\mathcal{X}^l$ is equipped with the discrete metric. Finally, we show that, for any $y \in \mathcal{X}^l$, $h_N(y) \to h(y)$, as $N \to \infty$. We prove this for $l = 2$, but the result can be extended to any $l \in \mathbb{N}$. In the following we exploit the notation $\bar{\varepsilon}_{N,j} := \frac{N}{N-1}\mu_N(x^N) - \frac{1}{N-1}\delta_{y_j}$. Exploiting the fact that $g_1, g_2 \in C_b(\mathcal{X})$ and **(A2)**, we have

$$\begin{aligned}
&|h_N(y) - h(y)| \\
&= \Big| \prod_{j=1}^{2} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, \bar{\varepsilon}_{N,j}, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) - \prod_{j=1}^{2} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, p, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\le \Big| \prod_{j=1}^{2} \int_{\mathcal{R}} \int_{\mathcal{Z}} g_j(\Psi(0, y_j, \bar{\varepsilon}_{N,j}, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \\
&\quad - \int_{\mathcal{R}} \int_{\mathcal{Z}} g_1(\Psi(0, y_1, \bar{\varepsilon}_{N,1}, \varphi(0,y_1), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{R}} \int_{\mathcal{Z}} g_2(\Psi(0, y_2, p, \varphi(0,y_2), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\quad + \Big| \int_{\mathcal{R}} \int_{\mathcal{Z}} g_1(\Psi(0, y_1, \bar{\varepsilon}_{N,1}, \varphi(0,y_1), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{R}} \int_{\mathcal{Z}} g_2(\Psi(0, y_2, p, \varphi(0,y_2), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \\
&\quad - \prod_{j=1}^{2} \int_{\mathcal{R}} \int_{\mathcal{Z}} \varphi_j(\Psi(0, y_j, p, \varphi(0,y_j), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\le \|g_1\|_\infty \Big| \int_{\mathcal{R}} \int_{\mathcal{Z}} g_2(\Psi(0, y_2, \bar{\varepsilon}_{N,2}, \varphi(0,y_2), z)) - g_2(\Psi(0, y_2, p, \varphi(0,y_2), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\quad + \|g_2\|_\infty \Big| \int_{\mathcal{R}} \int_{\mathcal{Z}} g_1(\Psi(0, y_1, \bar{\varepsilon}_{N,1}, \varphi(0,y_1), z)) - g_1(\Psi(0, y_1, p, \varphi(0,y_1), z)) \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\le 2\|g_1\|_\infty \|g_2\|_\infty \Big| \int_{\mathcal{R}} \int_{\mathcal{Z}} \mathbf{1}_{\Psi(0,y_2,\bar{\varepsilon}_{N,2},\varphi(0,y_2),z) \neq \Psi(0,y_2,p,\varphi(0,y_2),z)} \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\quad + 2\|g_1\|_\infty \|g_2\|_\infty \Big| \int_{\mathcal{R}} \int_{\mathcal{Z}} \mathbf{1}_{\Psi(0,y_1,\bar{\varepsilon}_{N,1},\varphi(0,y_1),z) \neq \Psi(0,y_1,p,\varphi(0,y_1),z)} \nu(\mathrm{d}z) \rho_1(\mathrm{d}\varphi|m) \Big| \\
&\le 2\|g_1\|_\infty \|g_2\|_\infty (\mathfrak{W}(\mathrm{dist}(\bar{\varepsilon}_{N,1}, p)) + \mathfrak{W}(\mathrm{dist}(\bar{\varepsilon}_{N,2}, p))) \overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Indeed, $\lim_{s\to 0^+} \mathfrak{W}(s) = 0$ and, for any $j \in \{1,2\}$, $\mathrm{dist}(\bar{\varepsilon}_{N,j}, p) \le \mathrm{dist}(\bar{\varepsilon}_{N,j}, \mu_N(x^N)) + \mathrm{dist}(\mu_N(x^N), p)$.

The second term on the right vanishes as $N \to \infty$ by assumption and

$$\begin{aligned}\operatorname{dist}\left(\frac{N}{N-1}\mu_N(x^N) - \frac{1}{N-1}\delta_{y_j}, \mu_N(x^N)\right) &= \frac{1}{2}\sum_{z\in\mathcal{X}} \left|\frac{N}{N-1}\mu_N(x^N)(z) - \frac{1}{N-1}\delta_{y_j}(z) - \mu_N(x^N)(z)\right| \\ &= \frac{1}{2(N-1)}\sum_{z\in\mathcal{X}} |\mu_N(x^N)(z) - \delta_{y_j}(z)| \leq \frac{1}{N-1} \to 0.\end{aligned}$$

Thence, an application of [22, Theorem I.5.5] yields the desired convergence. To conclude we need to show that the function $q : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$, defined, for $p \in \mathcal{P}(\mathcal{X})$, as the image of $(p, \rho_1(\cdot|m), \nu)$ via the mapping $(y, \varphi, z) \mapsto \Psi(0, y, p, \varphi(0, y), z)$, is a continuous function of p. This function $q(p)$ corresponds to the function $F(p)$ in the statement of Theorem 1.A.1. Let's consider a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(\mathcal{X})$, such that $p_n \underset{N\to\infty}{\longrightarrow} p$ weakly and let $B \in \mathcal{B}(\mathcal{X})$. Exploiting hypothesis **(A2)**, we are able to deduce

$$\begin{aligned}&|q(p_n)(B) - q(p)(B)| \\ &\leq \Bigg| \int_{\mathcal{X}} p_n(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \mathbf{1}_B(\Psi(0, y, p_n, \varphi(0, y), z)) \\ &\qquad - \int_{\mathcal{X}} p(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \mathbf{1}_B(\Psi(0, y, p, \varphi(0, y), z)) \Bigg| \\ &\leq \left| \int_{\mathcal{X}} p_n(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \Big\{ \mathbf{1}_B(\Psi(0, y, p_n, \varphi(0, y), z)) - \mathbf{1}_B(\Psi(0, y, p, \varphi(0, y), z)) \Big\} \right| \\ &\qquad + \left| \int_{\mathcal{X}} (p_n - p)(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \mathbf{1}_B(\Psi(0, y, p, \varphi(0, y), z) \right| \\ &\leq \int_{\mathcal{X}} p_n(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \mathbf{1}_{\{\Psi(0,y,p_n,\varphi(0,y),z) \neq \Psi(0,y,p,\varphi(0,y),z)\}} \\ &\qquad + \left| \int_{\mathcal{X}} (p_n - p)(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \int_{\mathcal{Z}} \nu(\mathrm{d}z) \right| \\ &\leq \int_{\mathcal{X}} p_n(\mathrm{d}y) \int_{\mathcal{R}} \rho_1(\mathrm{d}\varphi|m) \mathfrak{W}(\operatorname{dist}(p_n, p)) + \int_{\mathcal{X}} |p_n - p|(\mathrm{d}y) \\ &\leq \mathfrak{W}(\operatorname{dist}(p_n, p)) + \operatorname{dist}(p_n, p).\end{aligned}$$

This fact, in particular, implies

$$\begin{aligned}\operatorname{dist}(q(p_n), q(p)) = d_{TV}(q(p_n), q(p)) &= \sup_{B\in\mathcal{B}(\mathcal{X})} |q(p_n)(B) - q(p)(B)| \\ &\leq \mathfrak{W}(\operatorname{dist}(p_n, p)) + \operatorname{dist}(p_n, p) \underset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

where d_{TV} denotes the distance in total variation, that coincides with the distance $\operatorname{dist}(\cdot, \cdot)$, compatible with weak topology, because the set $\mathcal{X}$ is finite. So, we get the continuity of q and conclude the proof of chaos propagation. □

1.B Existence of solutions for the MFG in restricted strategies

We prove existence of correlated solutions for mean field games in the setting described in [30]. In fact, in the aforementioned paper the existence of correlated solutions for the MFG is a consequence of the existence of correlated solutions in the N-player games together with the convergence of N-player solutions to the mean field one.

1.B.1 The mean field game in restricted strategies

Let us start by recalling the structure of the MFG in restricted strategies. Let $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$ be the initial distribution of our mean field system.

Definition 1.B.1. *Let $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ and call it a* correlated suggestion. *Call* strategy modification *a function, $w \in \mathcal{D}$. Then, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting an $\mathcal{X}$-valued process $(X_t)_{t=0}^T$, an $\mathcal{R}$-valued random variable Φ, a $\mathcal{P}(\mathcal{X})^{T+1}$ -valued random variable μ and $\mathcal{Z}$-valued random variables $(\xi_t)_{t=1}^T$, such that the following properties hold:*

i) $\mathbb{P} \circ X_0^{-1} = \mathfrak{m}_0$*;*

ii) $\mathbb{P} \circ (\Phi, (\mu_t)_{t=0}^T)^{-1} = \rho$*;*

iii) $(\xi_t)_{t=1}^T$ *are i.i.d. all distributed according to* ν*;*

iv) $(\xi_t)_{t=1}^T$*,* X_0 *and* $(\Phi, (\mu_t)_{t=0}^T)$ *are independent;*

v) *for any* $t \in [\![0, T-1]\!]$*,*
$$X_{t+1} = \Psi\left(t, X_t, \mu_t, w \circ \Phi(t, X_t), \xi_{t+1}\right), \qquad \mathbb{P}\text{-a.s.}.$$

We call any tuple $\big((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, w, (X_t)_{t=0}^T\big)$ satisfying the conditions above a realisation of the triple $(\mathfrak{m}_0, \rho, w)$.

The strategy modification w represents how the representative player decides to deviate from the suggestion he was given. Notice that the deviation is a deterministic functional of the suggestion, Φ, provided by the mediator. Furthermore, here we limit ourselves to restricted strategies in the sense that the deviations live in $\mathcal{R}$, as well, that is they do not depend on the measure flow.

Remark 1.B.2. *As for the N-players game and generalised MFG, we can characterise the form of a realisation for the case in which the representative player follows the suggestion he was given.*
This is the case when the function w is just the identity, which we denote by ι by analogy to the previous sections.

The player in this mean field game faces costs associated to the triple $(\mathfrak{m}_0, \rho, w) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big) \times \mathcal{D}$ that are given by

$$J(\mathfrak{m}_0, \rho, w) := \mathbb{E}\left[\sum_{t=0}^{T-1} f\left(t, X_t, \mu_t, w \circ \Phi(t, X_t)\right) + F\left(X_T, \mu_T\right)\right].$$

As noticed in the previous sections, we highlight that the cost functional above is well defined since the right-hand side does not depend on the realisation considered but only on $(\mathfrak{m}_0, \rho, w)$.

Definition 1.B.3. *We say that* $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ *is a* correlated solution for the mean field game in restricted strategies *with initial distribution* $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$*, if the following two conditions hold:*

(Opt) *For each strategy modification* $w \in \mathcal{D}$*,*

$$J(\mathfrak{m}_0, \rho, \iota) \leq J(\mathfrak{m}_0, \rho, w).$$

(Con) *For any realisation of* $(\mathfrak{m}_0, \rho, \iota)$*, namely* $\big((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t)_{t=0}^T\big)$*, setting* $\mathcal{F}^\mu := \sigma\big((\mu_t)_{t=0}^T\big)$*, we have*

$$\mu_t(\cdot) = \mathbb{P}(X_t \in \cdot\, | \mathcal{F}^\mu), \quad t \in [\![0, T]\!].$$

The first condition is called *optimality condition*, the second is called *consistency condition*.

1.B.2 The result and its proof

We prove our results under the following set of assumptions (which is the same as in [30])

(A2′) "Continuity" of $\Psi : [\![0, T-1]\!] \times \mathcal{X} \times \Gamma \times \mathcal{Z} \to \mathcal{X}$:

1) For every $(t, x, \gamma) \in [\![0, T-1]\!] \times \mathcal{X} \times \Gamma$ and for all $m, \widetilde{m} \in \mathcal{P}(\mathcal{X})$,
$$\nu\left(\{z : \Psi(t, x, m, \gamma, z) \neq \Psi(t, x, \widetilde{m}, \gamma, z)\}\right) \leq \mathfrak{W}(\mathrm{dist}(m, \widetilde{m})),$$
where $\mathfrak{W} : [0, +\infty) \to [0, 1]$ is a non-decreasing measurable function with
$$\lim_{s \to 0^+} \mathfrak{W}(s) = 0.$$

2) For any $t \in [\![0, T-1]\!]$, $\Psi(t, \cdot)$ is $\tau \otimes \nu$-almost everywhere continuous, for every $\tau \in \mathcal{P}(\mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma)$.

(A3′) The functions f and F, running cost and terminal cost, are continuous.

Proposition 1.B.4. *Let Assumptions* ***(A2′)*** *and* ***(A3′)*** *hold and let* $\mathfrak{m}_0 \in \mathcal{P}(\mathcal{X})$*. Then there exists a correlated solution in restricted strategies for the mean field game with initial distribution* $\mathfrak{m}_0$*.*

Proof. Consider the set

$$\mathcal{A}_{\mathfrak{m}_0} := \{\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big) : \rho \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0} \text{ and the corresponding consistency condition holds}\},$$

with $\pi_2 : \mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \to \mathcal{P}(\mathcal{X})$, $\pi_2(\varphi, (m_t)_{t=0}^T) = m_0$.

The proof proceeds through the following two steps:

1. We show that the set $\mathcal{A}_{\mathfrak{m}_0}$ is a non-empty, compact and convex subset of $\mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$.
2. We see the original game as a two-player zero-sum game and reason as in [90] to show that this game possesses a solution.

Step 1: Properties of the set $\mathcal{A}_{\mathfrak{m}_0}$ Let us start by showing that the set $\mathcal{A}_{\mathfrak{m}_0}$ is non-empty. Consider an arbitrary fixed $\varphi \in \mathcal{R}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a $\mathcal{X}$-valued random variable X_0, such that $\mathbb{P} \circ X_0^{-1} = \mathfrak{m}_0$, and $\mathcal{Z}$-valued i.i.d. random variables $(\xi_t)_{t=1}^T$ $(\xi_t)_{t=1}^T$ all distributed according to ν and jointly independent of X_0. Then, we iteratively define a $\mathcal{X}$-valued random process, $(X_t)_{t=0}^T$, setting, $\mathbb{P}$-a.s.,

$$
\begin{aligned}
&X_1 = \Psi\left(0, X_0, \mathfrak{m}_0, \varphi(0, X_0), \xi_1\right) && m_1^\varphi = \mathbb{P} \circ X_1^{-1} \\
&X_2 = \Psi\left(1, X_1, m_1^\varphi, \varphi(1, X_1), \xi_2\right) && m_2^\varphi = \mathbb{P} \circ X_2^{-1} \\
&\vdots && \vdots \\
&X_{t+1} = \Psi\left(t, X_t, m_t^\varphi, \varphi(t, X_t), \xi_{t+1}\right) && m_{t+1}^\varphi = \mathbb{P} \circ X_{t+1}^{-1} \\
&\vdots && \vdots \\
&X_T = \Psi\left(T-1, X_{T-1}, m_{T-1}^\varphi, \varphi(T-1, X_{T-1}), \xi_T\right) && m_T^\varphi = \mathbb{P} \circ X_T^{-1}.
\end{aligned}
$$

By construction, $\left((\Omega, \mathcal{F}, \mathbb{P}), \varphi, (m_t^\varphi)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t)_{t=0}^T\right)$ is a realisation of $(\mathfrak{m}_0, \rho^\varphi, \iota)$, with $\rho^\varphi := \delta_\varphi \otimes \delta_{(m_t^\varphi)_{t=0}^T}$. Notice, in particular, that $\rho^\varphi \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0}$. Furthermore, for any $t \in [\![0, T]\!]$,

$$
\mathbb{P}(X_t \in \cdot | (m_t^\varphi)_{t=0}^T) = \mathbb{P}(X_t \in \cdot) = m_t^\varphi, \quad \mathbb{P}\text{-a.s.}
$$

and so ρ^φ belongs to $\mathcal{A}_{\mathfrak{m}_0}$, which is non-empty.

Now, we prove the convexity of $\mathcal{A}_{\mathfrak{m}_0}$. Let ρ_1 and ρ_2 in $\mathcal{A}_{m_0}$, $\lambda \in [0,1]$ and set $\rho := \lambda\rho_1 + (1-\lambda)\rho_2$. It clearly holds $\rho \in \mathcal{P}\left(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\right)$ and, by linearity, we also have

$$
\rho \circ \pi_2^{-1} = [\lambda\rho_1 + (1-\lambda)\rho_2] \circ \pi_2^{-1} = \lambda\rho_1 \circ \pi_2^{-1} + (1-\lambda)\rho_2 \circ \pi_2^{-1} = \lambda\delta_{\mathfrak{m}_0} + (1-\lambda)\delta_{\mathfrak{m}_0} = \delta_{\mathfrak{m}_0}.
$$

To conclude, we have to prove that the consistency condition is satisfied by an arbitrary realisation of $(\mathfrak{m}_0, \rho, \iota)$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider three $\mathcal{X}$-valued processes $(X_t^1)_{t=0}^T$, $(X_t^2)_{t=0}^T$ and $(X_t)_{t=0}^T$, a couple of $\mathcal{R}$-valued random variables Φ^1 and Φ^2, a couple of $\mathcal{P}(\mathcal{X})^{T+1}$ -valued random variables μ^1 and μ^2, $\mathcal{Z}$-valued random variables $(\xi_t)_{t=1}^T$ and a $\{0,1\}$-valued random variable Z, such that the following properties hold:

- **i)** $\mathbb{P} \circ X_0^{-1} = \mathfrak{m}_0$ and $X_0 = X_0^1 = X_0^2$, $\mathbb{P}$-a.s.;
- **ii)** $\mathbb{P} \circ (\Phi^1, (\mu_t^1)_{t=0}^T)^{-1} = \rho_1$ and $\mathbb{P} \circ (\Phi^2, (\mu_t^2)_{t=0}^T)^{-1} = \rho_2$;
- **ii')** $\Phi = Z\Phi^1 + (1-Z)\Phi^2$ and $\mu = Z\mu^1 + (1-Z)\mu^2$;
- **iii)** $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according to ν;
- **iii')** Z is a Bernoulli random variable of parameter λ;
- **iv)** $(\xi_t)_{t=1}^T$, X_0 and $(\Phi^1, (\mu_t^1)_{t=0}^T)$, $(\Phi^2, (\mu_t^2)_{t=0}^T)$ and Z are independent;
- **v)** The evolution of $(X_t^1)_{t=0}^T$, $(X_t^2)_{t=0}^T$ and $(X_t)_{t=0}^T$ follows these dynamics.
  For any $t \in [\![0, T-1]\!]$, $\mathbb{P}$-a.s.,

$$
\begin{aligned}
X_{t+1}^1 &= \Psi\left(t, X_t^1, \mu_t^1, \Phi^1(t, X_t^1), \xi_{t+1}\right), \\
X_{t+1}^2 &= \Psi\left(t, X_t^2, \mu_t^2, \Phi^2(t, X_t^2), \xi_{t+1}\right),
\end{aligned}
$$

$$X_{t+1} = \Psi\left(t, X_t, \mu_t, \Phi(t, X_t), \xi_{t+1}\right).$$

First of all, notice that $\mathbb{P} \circ (\Phi, \mu)^{-1} = \rho$. Indeed, for any $A \in \mathcal{B}(\mathcal{R})$ and any $B \in \mathcal{B}(\mathcal{P}(\mathcal{X})^{T+1})$, exploiting disintegration and the fact that $(\Phi^1, (\mu_t^1)_{t=0}^T)$, $(\Phi^2, (\mu_t^2)_{t=0}^T)$ and Z are independent, we have

$$\begin{aligned}\mathbb{P} \circ (\Phi, \mu)^{-1}(A \times B) &= \mathbb{P}(\Phi \in A, \mu \in B) = \mathbb{P}(Z\Phi^1 + (1-Z)\Phi^2 \in A, Z\mu^1 + (1-Z)\mu^2 \in B) \\ &= \mathbb{P}(\Phi^1 \in A, \mu^1 \in B)\mathbb{P}(Z = 1) + \mathbb{P}(\Phi^2 \in A, \mu^2 \in B)\mathbb{P}(Z = 0) \\ &= \rho_1(A \times B)\lambda + \rho_2(A \times B)(1 - \lambda) = \rho(A \times B).\end{aligned}$$

Thus, we have

1. $\left((\Omega, \mathcal{F}, \mathbb{P}), \Phi^1, (\mu_t^1)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t^1)_{t=0}^T\right)$ is a realisation of $(\mathfrak{m}_0, \rho_1, \iota)$,
2. $\left((\Omega, \mathcal{F}, \mathbb{P}), \Phi^2, (\mu_t^2)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t^2)_{t=0}^T\right)$ is a realisation of $(\mathfrak{m}_0, \rho_2, \iota)$,
3. $\left((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (\mu_t)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t)_{t=0}^T\right)$ is a realisation of $(\mathfrak{m}_0, \rho, \iota)$.

Exploiting 1. and 2. and the fact that $\rho_1, \rho_2 \in \mathcal{A}_{m_0}$, we have, for all $t \in [\![0, T]\!]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\mu_t^1(\cdot) &= \mathbb{P}(X_t^1 \in \cdot | \mu^1), \\ \mu_t^2(\cdot) &= \mathbb{P}(X_t^2 \in \cdot | \mu^2).\end{aligned} \tag{1.B.1}$$

Then, to deduce that $\rho \in \mathcal{A}_{m_0}$, we have to prove that, for any $x \in \mathcal{X}$,

$$\mu_t(x) = \mathbb{P}(X_t = x | \mu), \quad \mathbb{P}\text{-a.s.}$$

Consider an arbitrary fixed $x \in \mathcal{X}$. By Equation (1.B.1), $\mathbb{P}$-a.s. on the event $\{Z = 1\}$,

$$\mu_t(x) = \mu_t^1(x) = \mathbb{P}(X_t^1 = x | \mu^1) = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t^1) | \mu^1]. \tag{1.B.2}$$

Then, an application of [105, Lemma 6.2] with $\mathcal{F} = \sigma(\mu^1, Z)$, $\mathcal{G} = \sigma(\mu, Z)$, $A = \{Z = 1\}$, $\xi = \mathbf{1}_{\{x\}}(X_t^1)$ and $\eta = \mathbf{1}_{\{x\}}(X_t)$, yields

$$\mathbb{E}[\mathbf{1}_{\{x\}}(X_t^1) | \mu^1, Z] = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t) | \mu, Z], \quad \mathbb{P}\text{-a.s. on } \{Z = 1\}. \tag{1.B.3}$$

Thus, exploiting Equation (1.B.3), the fact that (μ^1, X_t^1) and Z are independent together with Equation (1.B.2) and the fact that $\rho_1 \in \mathcal{A}_{m_0}$, we have, $\mathbb{P}$-a.s. on $\{Z = 1\}$,

$$\mathbb{P}(X_t = x | \mu, Z) = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t) | \mu, Z] = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t^1) | \mu^1, Z] = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t^1) | \mu^1] = \mu_t^1(x) = \mu_t(x).$$

This is equivalent to

$$\mathbb{E}[\mathbf{1}_{\{Z=1\}} \mathbf{1}_{\{\mu_t(x) \neq \mathbb{P}(X_t = x | \mu, Z)\}}] = 0. \tag{1.B.4}$$

Reasoning in a similar way, it is possible to prove

$$\mathbb{E}[\mathbf{1}_{\{Z=0\}} \mathbf{1}_{\{\mu_t(x) \neq \mathbb{P}(X_t = x | \mu, Z)\}}] = 0, \tag{1.B.5}$$

and so summing Equations (1.B.4) and (1.B.5), we obtain

$$0 = \mathbb{E}[(\mathbf{1}_{\{Z=1\}} + \mathbf{1}_{\{Z=0\}})\mathbf{1}_{\{\mu_t(x)\neq \mathbb{P}(X_t=x|\mu,Z)\}}] = \mathbb{E}[\mathbf{1}_{\{\mu_t(x)\neq \mathbb{P}(X_t=x|\mu,Z)\}}] = \mathbb{P}(\mu_t(x) \neq \mathbb{P}(X_t = x|\mu, Z)).$$

Hence, $\mu_t(x) = \mathbb{P}(X_t = x|\mu, Z)$, $\mathbb{P}$-a.s. Now, applying $\mathbb{E}[\cdot|\mu]$ to both sides and exploiting the tower property and the fact that $\mu_t(x)$ is $\sigma(\mu)$-measurable, we get

$$\mu_t(x) = \mathbb{E}[\mu_t(x)|\mu] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{x\}}(X_t)|\mu, Z]|\mu] = \mathbb{E}[\mathbf{1}_{\{x\}}(X_t)|\mu] = \mathbb{P}(X_t = x|\mu).$$

Finally, we conclude by the arbitrariness of x in the finite set $\mathcal{X}$.

To complete the first step, we show that the set $\mathcal{A}_{\mathfrak{m}_0}$ is compact in $\mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$. Since $\mathcal{A}_{m_0} \subset \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$ and the latter set, endowed with weak topology, is a compact, we have just to check that $\mathcal{A}_{m_0}$ is closed therein. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_{m_0}$ converging to some $\rho \in \mathcal{P}\big(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}\big)$, weakly. We have to check that ρ belongs to $\mathcal{A}_{m_0}$, too. First of all, by the continuous mapping theorem, $\rho \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0}$. Indeed, $\rho_n \overset{n\to\infty}{\longrightarrow} \rho$ weakly, implies that $\rho_n \circ \pi_2^{-1} \overset{n\to\infty}{\longrightarrow} \rho \circ \pi_2^{-1}$ weakly, and, since $\rho_n \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0}$, for all $n \in \mathbb{N}$, it should be $\rho \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0}$, as well. In order to conclude that $\rho \in \mathcal{A}_{m_0}$, we have to show the validity of *consistency* property for the limit ρ. For each $n \in \mathbb{N}$, on a suitable probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ consider a realisation of the triple $(\mathfrak{m}_0, \rho_n, \iota)$, namely $((\Omega_n, \mathcal{F}_n, \mathbb{P}_n), \Phi_n, (\mu_t^n)_{t=0}^T, X_0^n, (\xi_t^n)_{t=1}^T, \iota, (X_t^n)_{t=0}^T)$. It satisfies:

i) $\mathbb{P}_n \circ (X_0^n)^{-1} = \mathfrak{m}_0$;

ii) $\mathbb{P}_n \circ (\Phi_n, (\mu_t^n)_{t=0}^T)^{-1} = \rho_n$;

iii) $(\xi_t^n)_{t=1}^T$ are i.i.d. all distributed according to ν;

iv) $(\xi_t^n)_{t=1}^T$, X_0^n and $(\Phi_n, (\mu_t^n)_{t=0}^T)$ are independent;

v) The evolution of $(X_t^n)_{t=0}^T$ follows this dynamics: for any $t \in [\![0, T-1]\!]$,
$$X_{t+1}^n = \Psi\left(t, X_t^n, \mu_t^n, \Phi_n(t, X_t^n), \xi_{t+1}^n\right), \qquad \mathbb{P}_n\text{-a.s.}.$$

It should satisfy the consistency condition, being $\rho_n \in \mathcal{A}_{m_0}$, that is, for all $t \in [\![0, T]\!]$,

$$\mu_t^n(\cdot) = \mathbb{P}(X_t^n \in \cdot|\mu^n), \quad \mathbb{P}_n\text{-a.s.}$$

Set $\eta_n := \mathbb{P}_n \circ (\Phi_n, (\mu_t^n)_{t=0}^T, (\xi_t^n)_{t=1}^T, (X_t^n)_{t=0}^T)^{-1} \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \times \mathcal{Z}^T \times \mathcal{X}^{T+1})$. Being the set $\mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \times \mathcal{Z}^T \times \mathcal{X}^{T+1})$ compact w.r.t. weak topology, passing in case to a suitable subsequence, $\eta_n \overset{n\to\infty}{\longrightarrow} \eta$, with $\eta \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1} \times \mathcal{Z}^T \times \mathcal{X}^{T+1})$. On a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we set $\eta := \mathbb{P} \circ (\Phi, (\mu_t)_{t=0}^T, (\xi_t)_{t=1}^T, (X_t)_{t=0}^T)^{-1}$. Reasoning as in [30], exploiting the continuous mapping theorem, we see

i) $\mathbb{P} \circ (X_0)^{-1} = \mathfrak{m}_0$;

iii) $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according to ν;

iv) $(\xi_t)_{t=1}^T$, X_0 and $(\Phi, (\mu_t)_{t=0}^T)$ are independent.

Furthermore, since by construction $\eta_n \overset{n\to\infty}{\longrightarrow} \eta$, exploiting once more the continuous mapping theorem, we have $\rho_n = \eta_n \circ \pi_{\mathcal{R}\times\mathcal{P}(\mathcal{X})^{T+1}}^{-1} \overset{n\to\infty}{\longrightarrow} \eta \circ \pi_{\mathcal{R}\times\mathcal{P}(\mathcal{X})^{T+1}}^{-1}$, weakly. Since, by assumption,

$\rho_n \overset{n\to\infty}{\longrightarrow} \rho$ weakly, we have $\mathbb{P}\circ(\Phi,(\mu_t)_{t=0}^T)^{-1} = \eta\circ\pi^{-1}_{\mathcal{R}\times\mathcal{P}(\mathcal{X})^{T+1}} = \rho$. Finally, we notice that property **v)** in Definition 1.B.1, that is the iterative dynamics, can be shown exploiting some of the arguments used in [30]. Define functions $G_t : \mathcal{R}\times\mathcal{X}\times\mathcal{P}(\mathcal{X})\times\mathcal{Z}\to\mathcal{X}$, $t\in[\![0,T]\!]$, $G_t(\varphi,x,m,z) = \Psi(t,x,m,\varphi(t,x),z)$. By assumption **(A2')** 2) and the finiteness of $\mathcal{X}$ and $\mathcal{R}$, the function G_t is $\sigma\otimes\nu$-a.e. continuous for every given $\sigma\in\mathcal{P}(\mathcal{R}\times\mathcal{X}\times\mathcal{P}(\mathcal{X}))$. Now, the independence properties **iii)** and **iv)**, shown above, together with the aforementioned continuity and the joint convergence in distribution yields the thesis by an application of [30, Lemma A.1]. Indeed, via the mapping theorem, we have the convergence in distribution of $(X^n_{t+1},(\Phi_n,X^n_t,\mu^n_t,\xi^n_{t+1}))$ to $(X_{t+1},(\Phi,X_t,\mu_t,\xi_{t+1}))$, as $n\to\infty$. Thus, we have shown that the tuple $((\Omega,\mathcal{F},\mathbb{P}),\Phi,(\mu_t)_{t=0}^T,X_0,(\xi_t)_{t=1}^T,\iota,(X_t)_{t=0}^T)$ is a realisation of $(\mathfrak{m}_0,\rho,\iota)$. Finally, exploiting [30, Lemma A.3] with $\kappa_n = \mu^n_t = \mathbb{P}_n(X^n_t\in\cdot|\mu^n)$, $Y_n = X^n_t$ and $Z_n = \mu^n_t$, $n\in\mathbb{N}$, since $(X^n_t,(\mu^n_t)_{t=0}^T,\mu^n_t) \overset{n\to\infty}{\longrightarrow} (X_t,(\mu_t)_{t=0}^T,\mu_t)$ in distribution, by the continuous mapping theorem, for any $t\in[\![0,T]\!]$, and since $\mu_t(A)$ is $\sigma((\mu_t)_{t=0}^T)$-measurable, for any $A\in\mathcal{B}(\mathcal{X})$, we get

$$\mathbb{P}(X_t\in B,(\mu_t)_{t=0}^T\in C) = \mathbb{E}[\mu_t(B)\mathbf{1}_C((\mu_t)_{t=0}^T)],$$

for any $B\in\mathcal{B}(\mathcal{X})$ and $C\in\mathcal{B}(\mathcal{P}(\mathcal{X})^{T+1})$, and μ_t is a regular conditional distribution of X_t given $\mu=(\mu_t)_{t=0}^T$, as in the *consistency* condition. This completes the proof of the compactness of $\mathcal{A}_{m_0}$.

Step 2: Two-player zero-sum game interpretation Now, we reframe the problem similarly to [90]. A distribution $\rho\in\mathcal{A}_{m_0}$ is a correlated solution for the MFG if for any $w\in\mathcal{D}$,

$$0 \le J(\mathfrak{m}_0,\rho,w) - J(\mathfrak{m}_0,\rho,\iota), \tag{1.B.6}$$

where $J(\mathfrak{m}_0,\rho,w) = \sum_{\varphi\in\mathcal{R}}\int_{\mathcal{P}(\mathcal{X})^{T+1}} J(\mathfrak{m}_0,\delta_{(\varphi,m)},w)\rho(\varphi,\mathrm{d}m)$, that means

$$J(\mathfrak{m}_0,\delta_{(\varphi,m)},w) = \mathbb{E}\left[\sum_{t=0}^{T-1} f\left(t,X_t,m_t,w\circ\varphi(t,X_t)\right) + F\left(X_T,m_T\right)\right],$$

with

- **i)** $\mathbb{P}\circ X_0^{-1} = \mathfrak{m}_0$;
- **ii)** $\mathbb{P}\circ(\Phi,(\mu_t)_{t=0}^T)^{-1} = \delta_{(\varphi,m)}$;
- **iii)** $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according to ν;
- **iv)** $(\xi_t)_{t=1}^T$ and X_0 and are independent;
- **v)** The evolution of $(X_t)_{t=0}^T$ follows this dynamics: for any $t\in[\![0,T-1]\!]$,

$$X_{t+1} = \Psi\left(t,X_t,m_t,w\circ\varphi(t,X_t),\xi_{t+1}\right), \qquad \mathbb{P}\text{-a.s.}.$$

Remark 1.B.5. *This way of rewriting the objective functional is possible because, by definition of realisation in the MFG, (Φ,μ), $(\xi_t)_{t=1}^T$, X_0 and are independent.*

Thus, w.l.o.g. we write

$$J(\mathfrak{m}_0, \rho, w) = \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} I(w \circ \varphi, (m_t)_{t=0}^T) \rho(\varphi, \mathrm{d}m),$$

with $I(w \circ \varphi, (m_t)_{t=0}^T) := J(\mathfrak{m}_0, \delta_{(\varphi, m)}, w)$. Now, notice that condition (1.B.6) is equivalent to

$$0 \leq \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(w \circ \varphi, (m_t)_{t=0}^T) - I(\varphi, (m_t)_{t=0}^T)] \rho(\varphi, \mathrm{d}m), \quad \text{for all } w \in \mathcal{D}.$$

Since $\mathcal{R}$ is a finite set, this is equivalent to the following condition

$$\int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(\psi, (m_t)_{t=0}^T) - I(\varphi, (m_t)_{t=0}^T)] \rho(\varphi, \mathrm{d}m) \geq 0, \quad \text{for all } \varphi, \psi \in \mathcal{R}. \tag{1.B.7}$$

To show the existence of a correlated solution for the MFG, it is then sufficient to show the existence of a $\rho \in \mathcal{A}_{m_0}$ satisfying (1.B.7).

Now, consider the following auxiliary two-player zero-sum game, that in pure strategies reads as follows:

- player I(the maximiser) chooses a couple $(\varphi, m) \in \mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}$;
- player II(the minimiser) chooses a couple $(\eta, \psi) \in \mathcal{R}^2$.

The payoff (II to I) is $I(\psi, m) - I(\varphi, m)$ if $\eta = \varphi$ and 0 otherwise. Notice that in what follows, for convexity reasons, we work with the corresponding set of mixed strategies and a non empty compact convex subset of it. Then, a strategy $\rho \in \mathcal{A}_{m_0}$ satisfying (1.B.7) corresponds to a strategy of player I (hence belonging to $\mathcal{A}_{m_0}$, as well) yielding a non-negative payoff in the game described above. Indeed, such a strategy corresponds to an $x \in \mathcal{A}_{m_0} \subset \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$ such that

$$\begin{aligned} 0 &\leq \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(\psi, (m_t)_{t=0}^T) - I(\eta, (m_t)_{t=0}^T)] \delta_\varphi(\eta) x(\varphi, \mathrm{d}m) \\ &= \int_{\mathcal{P}(\mathcal{X})^{T+1}} \sum_{\varphi \in \mathcal{R}} [I(\psi, (m_t)_{t=0}^T) - I(\eta, (m_t)_{t=0}^T)] \delta_\varphi(\eta) x(\varphi, \mathrm{d}m) \\ &= \int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(\psi, (m_t)_{t=0}^T) - I(\varphi, (m_t)_{t=0}^T)] x(\varphi, \mathrm{d}m). \end{aligned}$$

By the *MinMax Theorem* such a strategy exists if, for every given strategy of player II, there exists a strategy of player I (again in $\mathcal{A}_{m_0}$) yielding a non-negative payoff. Now, let $(y(\eta, \psi))_{\eta, \psi \in \mathcal{R}}$ be a strategy of player II. By construction, the payoff associated to the couple of strategies $(x, y) \in \mathcal{A}_{m_0} \times \mathcal{P}(\mathcal{R}^2)$ is given by

$$\begin{aligned} &\sum_{\eta, \psi \in \mathcal{R}} \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(\psi, m) - I(\eta, m)] \delta_\varphi(\eta) x(\varphi, \mathrm{d}m) y(\eta, \psi) \\ &= \sum_{\varphi, \psi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} [I(\psi, m) - I(\varphi, m)] x(\varphi, \mathrm{d}m) y(\varphi, \psi) \end{aligned}$$

$$= \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} x(\varphi, \mathrm{d}m) \sum_{\psi \in \mathcal{R}} y(\varphi, \psi)[I(\psi, m) - I(\varphi, m)],$$

where $m = (m_t)_{t=0}^T$ and we want it to be non negative.

Thus, we have to prove that, for any fixed $y \in \mathcal{P}(\mathcal{R}^2)$, there exists a $\rho \in \mathcal{A}_{m_0}$ such that

$$\sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} \rho(\varphi, \mathrm{d}m) \sum_{\psi \in \mathcal{R}} y(\varphi, \psi)[I(\psi, m) - I(\varphi, m)] \geq 0. \tag{1.B.8}$$

We exploit the following proposition.

Proposition 1.B.6. *For any $p \in \mathcal{P}(\mathcal{R})$, there exists $m^p := (m_t^p)_{t=0}^T \in \mathcal{P}(\mathcal{X})^{T+1}$, such that $\rho_p := p \otimes \delta_{m^p}$ belongs to $\mathcal{A}_{m_0}$.*

Proof. Consider an arbitrary fixed $p \in \mathcal{P}(\mathcal{R})$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with an $\mathcal{X}$-valued random variable X_0, a $\mathcal{R}$-valued random variable Φ and $\mathcal{Z}$-valued random variables $(\xi_t)_{t=1}^T$, such that $\mathbb{P} \circ X_0^{-1} = \mathfrak{m}_0$, $\mathbb{P} \circ \Phi^{-1} = p$, $(\xi_t)_{t=1}^T$ are i.i.d. all distributed according to ν and jointly independent of X_0 and Φ which are independent as well. Then, we iteratively define an $\mathcal{X}$-valued random process, $(X_t)_{t=0}^T$, setting, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&X_1 = \Psi(0, X_0, \mathfrak{m}_0, \Phi(0, X_0), \xi_1) && m_1^p = \mathbb{P} \circ X_1^{-1} \\
&X_2 = \Psi\left(1, X_1, m_1^p, \Phi(1, X_1), \xi_2\right) && m_2^p = \mathbb{P} \circ X_2^{-1} \\
&\vdots && \vdots \\
&X_{t+1} = \Psi\left(t, X_t, m_t^p, \Phi(t, X_t), \xi_{t+1}\right) && m_{t+1}^p = \mathbb{P} \circ X_{t+1}^{-1} \\
&\vdots && \vdots \\
&X_T = \Psi\left(T-1, X_{T-1}, m_{T-1}^p, \Phi(T-1, X_{T-1}), \xi_T\right) && m_T^p = \mathbb{P} \circ X_T^{-1}.
\end{aligned}$$

By construction, $\left((\Omega, \mathcal{F}, \mathbb{P}), \Phi, (m_t^p)_{t=0}^T, X_0, (\xi_t)_{t=1}^T, \iota, (X_t)_{t=0}^T\right)$ is a realisation of $(\mathfrak{m}_0, \rho_p, \iota)$, with $\rho_p := p \otimes \delta_{(m_t^p)_{t=0}^T}$. Notice, in particular, that $\rho_p \circ \pi_2^{-1} = \delta_{\mathfrak{m}_0}$ Furthermore, since, for any $t \in [\![0, T]\!]$, $\mathbb{P}$-a.s.

$$\mathbb{P}(X_t \in \cdot | (m_t^p)_{t=0}^T) = \mathbb{P}(X_t \in \cdot) = m_t^p,$$

and so ρ_p belongs to $\mathcal{A}_{\mathfrak{m}_0}$. □

Now, for any fixed $p \in \mathcal{P}(\mathcal{R})$, the value of the objective functional associated to the couple of strategies (ρ_p, y) is given by

$$\begin{aligned}
&\sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} \rho_p(\varphi, \mathrm{d}m) \sum_{\psi \in \mathcal{R}} y(\varphi, \psi)[I(\psi, m) - I(\varphi, m)] \\
&= \sum_{\varphi \in \mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} p(\varphi)\delta_{m^p}(\mathrm{d}m) \sum_{\psi \in \mathcal{R}} y(\varphi, \psi)[I(\psi, m) - I(\varphi, m)] \\
&= \sum_{\varphi \in \mathcal{R}} p(\varphi) \sum_{\psi \in \mathcal{R}} y(\varphi, \psi)[I(\psi, m^p) - I(\varphi, m^p)].
\end{aligned}$$

Now, [90, Lemma](up to a change of sign since here they are considering the corresponding maximisation problem) guarantees that, given a probability distribution $(y(\eta,\psi))_{\eta,\psi\in\mathcal{R}}$, there exists a probability vector $\overline{p}^y = (\overline{p}^y(\varphi))_{\varphi\in\mathcal{R}}$ such that, for any vector $(v_\psi)_{\psi\in\mathcal{R}}$,

$$\sum_{\varphi\in\mathcal{R}} \overline{p}^y(\varphi) \sum_{\psi\in\mathcal{R}} y(\varphi,\psi)[v_\psi - v_\varphi] = 0. \tag{1.B.9}$$

Equation (1.B.9) holds, in particular, for $v_\psi = I(\psi, m^{\overline{p}^y})$, $\psi \in \mathcal{R}$, and consequently

$$\begin{aligned} 0 &= \sum_{\varphi\in\mathcal{R}} \overline{p}^y(\varphi) \sum_{\psi\in\mathcal{R}} y(\varphi,\psi)[I(\psi, m^{\overline{p}^y}) - I(\varphi, m^{\overline{p}^y})] \\ &= \sum_{\varphi\in\mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} \overline{p}^y(\varphi)\delta_{m^{\overline{p}^y}}(\mathrm{d}m) \sum_{\psi\in\mathcal{R}} y(\varphi,\psi)[I(\psi,m) - I(\varphi,m)] \\ &= \sum_{\varphi\in\mathcal{R}} \int_{\mathcal{P}(\mathcal{X})^{T+1}} \rho_{\overline{p}^y}(\varphi,m) \sum_{\psi\in\mathcal{R}} y(\varphi,\psi)[I(\psi,m) - I(\varphi,m)], \end{aligned}$$

and so $\rho_{\overline{p}^y}$ satisfies Equation (1.B.8) and we conclude.

□

A McKean-Vlasov game of commodity production, consumption and trading

All models are wrong, but some are useful.

George E. P. Box and Norman R. Draper

This chapter deals with a model for prices manipulation in commodity market. The corresponding paper, written in collaboration with my Ph.D. co-supervisor Prof. Giorgia Callegaro and Profs. René Aid (Université Paris Dauphine) and Luciano Campi (University of Milan), was published in *Applied Mathematics and Optimization* in September 2022. We focus on a two-player Linear-Quadratic McKean-Vlasov stochastic differential game where an energy producer and a consumer both affect the price dynamics of the good controlling drift and volatility of production rate and consumption rate, respectively. We compute a Nash equilibrium and characterise the corresponding strategies and payoffs in semi-explicit form. We illustrate our results via numerical simulations, showing that the model is consistent with economic intuition.

2.1 Introduction

In this paper, we develop an economic model of a commodity market where a representative producer interacts with a representative processor who buys the commodity and transforms it into a final product sold to the retail market (think of crude oil production transformed into gasoline or wheat transformed into bread). For the sake of simplicity, the processor will be referred to as consumer from now on. In our model, the production and the consumption rates are described as Itô processes driven each by an independent Brownian motion and whose coefficients are controlled by, respectively, the producer and the consumer. We stress that in our model the producer can control, in particular, the volatility of the production rate (by investing in devices making the production more reliable), and similarly the consumer can control the one of the consumption rate (by investing in storage devices, for instance). Further, the players are risk-averse (see below for details) and they are linked by a financial derivative in the commodity, a plain forward agreement on price and volume exchanged. For some motivations on the control of volatility, we refer the reader to the paper by [5], which focuses on the interaction between a producer controlling the drift of the spot price and a trader controlling the volatility, and exchanging a quadratic derivative. In that paper, it was shown that when the trader is short in the derivative, he would increase the volatility of the spot price in order to get a higher price of the derivative sold to the producer. In the present setting, we are interested in the joint effect of the costs of controlling the volatility of production or consumption rates and the players' risk aversion parameters on the "agreement indifference price". Indeed, when only one player has market power, the effect of the parameters on the forward price is clear. On the other hand, when the two players interact, the joint effect is not obvious. In this paper, we are interested in the outcome of the combined effect on the forward price of the relative risk aversions and the volatility control costs of the producer and the consumer.

Both players have market power on the spot price of the commodity: the spot price depends linearly on production and consumption rates so that the higher the rate of production, the lower the spot price and the higher the rate of consumption, the higher the spot price. Furthermore, they agree to exchange a forward contract with finite maturity T over a certain quantity λ of the commodity that will be determined at equilibrium together with its price F. This setting is inspired from the seminal papers of [6] and [7], where the authors establish the mitigating effect of forward agreement on the exercise of producers market power.

In our framework, since production and consumption rates are driven by two independent Brownian motions and there is only one tradable risky asset, i.e. the commodity spot price, the market is incomplete. Therefore, we define the forward price in the spirit of the indifference pricing approach (see the paper [91] for an overview and [20] for an application to power markets). The players' goal is to maximise their respective objective functionals, which are expectations of the following main components: the profit from selling, the sourcing costs (only for the consumer), the costs from exerting the controls, the forward contract payoff and, finally, the integrated variance of the market price of the derivative.

The latter component describes the risk aversion both players have towards their financial position. More precisely, in this context where the agents can control the volatility of their state variable, the modelling of their risk aversion using utility functions (e.g. exponential utility) would lead to non-linear PDEs which are difficult to handle. Hence, for technical convenience we turn to a sort of dynamic mean-variance criterion leading to the objective functionals described above. Mathematically speaking we are dealing with a two-player stochastic differential game with objective functionals of McKean-Vlasov type, i.e. depending on the laws of the state variables. Economically speaking, it means that both players act as speculators on the forward market, as they disconnect their forward position from their production or transformation profit. Although this feature of our model originates from a computational limitation induced by the linear-quadratic McKean-Vlasov game setting, there exists some evidence, documented by a stream of the economic literature, that large commodity players can act as speculators on their markets (see [36] for such evidence and references on the subject of financiarisation of commodity markets).

This modeling approach for the risk aversion has been already investigated and used for portfolio selection by [140] and more recently by [102] and [116]. Moreover, due to the fast development of mean-field games as a new framework to study stochastic differential games for a large number of players since the seminal papers by [113, 114, 115] and [99] (see also [28] for a survey), there has been a regain of interest for control problems of McKean-Vlasov dynamics. The latter, also known as mean field control, corresponds in some way to the limit of a sequence of stochastic control problems for a regulator willing to optimise the average expected payoff of a group of agents interacting through the empirical distribution of their states (see [109] and the two-volume book [33]). In particular, the linear-quadratic case has been treated in [83, 19] and [12]. Recently, stochastic differential games with both state dynamics and objective functionals of McKean-Vlasov type has been addressed in, e.g., [122, 42] and also [70] for a Stackelberg game arising from an optimal portfolio liquidation problem. Although a large number of applications in economics and finance have been developed with mean field games and mean field control, the applications of games with finitely many players and McKean-Vlasov dynamics and objective functionals in economics is much more recent, hence less developed (see, e.g., [3]).

We will analyse the model along the following program: first we will find a Nash equilibrium for a fixed quantity λ of the commodity exchanged through the forward contract with fixed price F; second, we will compute the indifference prices of the forward contract for the two players separately (they are going to depend on λ); third, we will compute the quantity λ such that the two prices are equal, hence making the exchange compatible with the equilibrium found in the first step. This price will be called *agreement indifference price*.

This framework makes it possible to analyse the formation of the risk premium defined as the difference between the (unitary) agreement indifference price and the expected spot price of the commodity. The question of the determinants of the risk premium on commodity markets goes back (at least) to Keynes's *Treatise on Money*, (1930). Keynes formulated the *normal backwardation theory*, i.e. the claim that forward prices should be lower than expected spot prices because risk-averse producers are willing to sell forward at a premium to avoid price risk. Presently, the *hedging pressure theory* (see [45, 95, 94, 93]) provides explanation of the sign of the risk premium depending on the relative size of population types in the market (producers, storers, speculators) and their risk-aversion (see [52] for a complete equilibrium model with mean-variance utility players explaining the different possible sign of the premia).

Mathematical results. The main mathematical contribution of the paper (ref. Theorem 2.3.1) consists in a complete description of a Nash equilibrium in open loop strategies of the two-player stochastic differential game arising from the interaction model described above. More in detail, we adopt the following resolution approach: first, we prove a suitable version of a verification theorem exploiting the weak martingale optimality principle; second, the verification theorem and the linear-quadratic structure of the game allows to provide a semi-explicit form for the best response map; third, a Nash equilibrium is found as a fixed point of the best response map with closed-form expressions for the equilibrium strategies and payoffs of both players up to solving numerically a Riccati system of ODEs. Once we have a Nash equilibrium at our disposal, computing the corresponding agreement indifference price together with the exchanged quantity at equilibrium is a pretty straightforward task.

Economic insights. First, we find that the forward agreement indifference price is higher (resp. lower) than the expected spot price when the producer is more (resp. less) risk-averse than the consumer. Because in our model, the players act as speculators on the forward market, a seller requires a higher forward price to enter in the agreement and a buyer asks for a lower price. The presence of market power of both players allows for the formation of an equilibrium. In that sense, our model is consistent with the economic intuition of the hedging pressure theory applied to a market populated with producers and consumers acting as speculators. Second, we observe that producers can achieve the same agreement indifference price and the same trading volume either by having high risk aversion and a low volatility control cost, or a low risk aversion and a high volatility control cost. This effect manifests itself whatever the relative risk aversion of the producer and the consumer or the relative costs of volatility control. Nevertheless, it is more apparent when the volatility control costs are low. Thus, to the list of determinants of the sign of the risk premium of forward commodity price, one could add the costs of reducing the production uncertainty. For commodity where storage is utmost costly like electricity, reducing production uncertainty is highly costly and thus, leads to higher risk premium.

Organisation of the paper. The paper is organised in the following way. The model is described in Section 2.2.1 together with the definition of a forward agreement indifference price and quantity in Section 2.2.2. The main result on the existence of a Nash equilibrium is given in Section 2.3. The proof of the main result is given is Section 2.4. Numerical results on the comparative static of the risk premium and the joint effect of risk aversion and volatility control costs are given in Section 2.5.

Notations. We denote by $\mathbb{R}_+$ (respectively $\mathbb{R}_-$) the closed semi-interval $[0,+\infty)$ (respectively $(-\infty,0]$). Given a function $f:\mathbb{R}\to S$, with S a regular space, we denote its first derivative by f'. The expected value of a random variable X will be equivalently denoted by $\mathbb{E}[X]$, as usual, or by $\bar{X}$, for brevity. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Given a positive integer d, a strictly positive time horizon T and a filtration $\mathbb{F}:=(\mathcal{F}_t)_{t\in[0,T]}$, we set

$$L^2([0,T],\mathbb{R}^d) := \left\{\varphi:[0,T]\to\mathbb{R}^d,\ \text{s.t. } \varphi \text{ is measurable and } \int_0^T |\varphi_t|^2 \mathrm{d}t < \infty\right\},$$

$$L^\infty([0,T],\mathbb{R}^d) := \left\{\varphi:[0,T]\to\mathbb{R}^d,\ \text{s.t. } \varphi \text{ is measurable and } \sup_{t\in[0,T]} |\varphi_t| < \infty\right\},$$

$$L^2_{\mathcal{F}_T}(\Omega,\mathbb{R}^d) := \left\{\psi:\Omega\to\mathbb{R}^d,\ \text{s.t. } \psi \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{E}\left[|\psi|^2\right] < \infty\right\},$$

$$L^2_{\mathbb{F}}(\Omega\times[0,T],\mathbb{R}^d) := \left\{\eta:\Omega\times[0,T]\to\mathbb{R}^d,\ \text{s.t. } \eta \text{ is } \mathbb{F}\text{-adapted and } \mathbb{E}\left[\int_0^T |\eta_t|^2 \mathrm{d}t\right] < \infty\right\},$$

$$S^2_{\mathbb{F}}(\Omega\times[0,T],\mathbb{R}^d) := \left\{\eta:\Omega\times[0,T]\to\mathbb{R}^d,\ \text{s.t. } \eta \text{ is } \mathbb{F}\text{-adapted and } \mathbb{E}\left[\sup_{t\in[0,T]} |\eta_t|^2\right] < \infty\right\}.$$

2.2 The model

We consider a stochastic game between a representative producer and a representative consumer. While the producer produces a good at a certain rate, the consumer buys the commodity and transforms it into a final good sold in the retail market.

2.2.1 Market model

We consider a finite time window $[0, T]$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a two-dimensional Brownian motion $(W, B) = \{(W_t, B_t)\}_{t\in[0,T]}$ and its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ augmented with the $\mathbb{P}$-null sets in $\mathcal{F}$. The *production rate* of the producer $\{q_t\}_{t\in[0,T]}$ evolves according to a dynamics given by

$$\mathrm{d}q_t = u_t\mathrm{d}t + z_t\mathrm{d}W_t, \quad q_0 > 0,$$

where $\{u_t\}_{t\in[0,T]}$ and $\{z_t\}_{t\in[0,T]}$ are the producer's strategies. The associated instantaneous costs are $\frac{k_p}{2}u_t^2$ and $\frac{\ell_p}{2}(z_t - \sigma_p)^2$, respectively, with $k_p, \ell_p \geq 0$ and where $\sigma_p > 0$ represents the nominal uncertainty in production without dedicated effort of the producer to reduce it. In a similar way, the *consumption rate* (or selling rate to the retail market) of the consumer, $\{c_t\}_{t\in[0,T]}$, has dynamics given by

$$\mathrm{d}c_t = v_t\mathrm{d}t + y_t\mathrm{d}B_t, \quad c_0 > 0.$$

Here, $\{v_t\}_{t\in[0,T]}$ and $\{y_t\}_{t\in[0,T]}$ are the *consumer's strategies*, and the associated instantaneous costs are, respectively, $\frac{k_c}{2}v_t^2$ and $\frac{\ell_c}{2}(y_t - \sigma_c)^2$, with $k_c, \ell_c \geq 0$ and $\sigma_c > 0$. We assume a linear impact on the *observed market price*, $\{S_t\}_{t\in[0,T]}$, namely $\{S_t\}_{t\in[0,T]}$ evolves according to

$$S_t := s_0 - \rho_p q_t + \gamma\rho_c c_t, \qquad s_0 > 0$$

with $\rho_p, \rho_c > 0$ and $\gamma > 0$ (the role of γ will be clear in a few lines). The instantaneous profits at time t of the producer π_t^p and of the consumer π_t^c are given by:

$$\begin{aligned}
\pi_t^p &:= q_t S_t - \frac{k_p}{2}u_t^2 - \frac{\ell_p}{2}(z_t - \sigma_p)^2, \\
\pi_t^c &:= c_t(p_0 + p_1 S_t) - \gamma c_t(S_t + \delta) - \frac{k_c}{2}v_t^2 - \frac{\ell_c}{2}(y_t - \sigma_c)^2,
\end{aligned}$$

where $c_t(p_0 + p_1 S_t)$ is the income from selling the quantity c_t at the retail price $p_0 + p_1 S_t$, a linear function of the commodity price, with $p_0, p_1 > 0$ and $\gamma c_t(S_t + \delta)$ represents the sourcing cost of buying the quantity γc_t (which is used to obtain c_t to be sold) at price S_t plus the transformation cost δ, with $\gamma, \delta > 0$. We assume $\gamma > p_1$ to ensure the concavity of the objective functional of the consumer (i.e. the processor cannot charge increasing prices to final consumers without seeing the demand decreasing).

Remark 2.2.1. *Our producer and consumer are large players as their actions have an effect on market prices. This is the reason why we did not impose any constraint on the relation between consumption and production: there could be other small producers and consumers present and so the consumption c_t might, in principle, be greater than q_t. Moreover, we consider a commodity for which storage has a little effect on the price and in our framework we do not include neither capacity constraints nor consumption/production constraints for technical reasons.*

The producer and the consumer exchange a forward contract of λ units of the commodity at a fixed amount of money $F \in \mathbb{R}$. Both players aim at maximising their respective objective functionals, which have two components: an expected profit term and a penalisation term modelling the player risk aversion (more comments below). In formulae, they are given by

$$
\begin{aligned}
J_p^{\lambda,F}(u,z;v,y) &:= \mathbb{E}\big[P_T^p\big] - \eta_p \int_0^T \mathbb{V}\left[\lambda S_t\right] \mathrm{d}t, \qquad \eta_p > 0, \\
J_c^{\lambda,F}(v,y;u,z) &:= \mathbb{E}\big[P_T^c\big] - \eta_c \int_0^T \mathbb{V}\left[\lambda S_t\right] \mathrm{d}t, \qquad \eta_c > 0,
\end{aligned}
\tag{2.2.1}
$$

where $\mathbb{V}$ stands for the variance and the process P_T^p (resp. P_T^c) represents the cumulative profit over the time period $[0,T]$ of the producer (resp. the consumer), i.e.

$$
P_T^p := \int_0^T \pi_t^p \mathrm{d}t + F - \lambda S_T, \quad P_T^c := \int_0^T \pi_t^c \mathrm{d}t - F + \lambda S_T.
$$

The set of admissible strategies for the players is given by $\mathcal{A}^2 := \mathcal{A} \times \mathcal{A}$, where $\mathcal{A} = L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}^2)$.

The way risk aversion is modelled and the choice of the derivative require two comments. First, a more standard way to take into account the players' risk aversion would consist in using utility functions. In our case and with an exponential utility function, where players can control the volatility of their production and consumption rates, this approach would lead to Monge-Ampère PDEs, which are difficult to handle. For this reason, we turn to a different way to model risk aversion, which is reminiscent of what is done in mean-variance optimal dynamic portfolio choice (see [140] and more recently by [102] and [116]). A similar approach was also previously used for distributed renewable energy development in [3]. Second, we observe that the variance penalisation term involves only the derivative and not the profit from production or transformation. As already stated in the introduction, this representation of risk aversion transforms players into speculators on the forward market. Indeed, players only care about the variance of their financial position $\lambda S_t - F$, not about their production or consumption profits. This modeling is motivated by the desire to remain in a framework where tractable solutions can be exhibited. Its sole consequence would be to reverse the sign of the risk premium: producers wish to sell at a lower price than the expected spot price whereas speculators want to sell at a higher price. For the sake of simplicity, we have chosen to consider only a static hedging position with a simple forward contract in order to analyse the risk premium between the forward "agreement indifference price" and the expected price at maturity (see Section 2.2.2 for a definition of the forward agreement indifference price).

To sum up, we deal with a two-player stochastic differential game of McKean-Vlasov linear-quadratic type. Hence, it is natural to look for *Nash equilibria* according to the following definition.

Definition 2.2.2. *We call the couple* $\big((u^*,z^*)^\top,(v^*,y^*)^\top\big) \in \mathcal{A} \times \mathcal{A}$ *a* Nash equilibrium *if*

$$
\begin{aligned}
J_p^{\lambda,F}(u^*,z^*;v^*,y^*) &\geq J_p^{\lambda,F}(u,z;v^*,y^*), \qquad \text{for all } (u,z)^\top \in \mathcal{A}, \\
J_c^{\lambda,F}(v^*,y^*;u^*,z^*) &\geq J_c^{\lambda,F}(v,y;u^*,z^*), \qquad \text{for all } (v,y)^\top \in \mathcal{A}.
\end{aligned}
$$

2.2.2 Equilibrium forward agreement

For a Nash equilibrium $(v^*, y^*; u^*, z^*)$, we denote by

$$J_c^*(\lambda, F) = J_c^{\lambda,F}(v^*, y^*; u^*, z^*), \qquad J_p^*(\lambda, F) = J_p^{\lambda,F}(u^*, z^*; v^*, y^*),$$

the corresponding equilibrium payoffs of consumer and producer, respectively. They depend on the number of units λ, on which the forward contract is written, and the respective forward price F. Both players determine their prices using the *indifference pricing approach*, namely the consumer computes $F_c^{\lambda,*}$ as solution of $J_c^*(\lambda, F) = J_c^*(0,0)$ and analogously for the producer, leading to a price $F_p^{\lambda,*}$ as a solution of $J_p^*(\lambda, F) = J_p^*(0,0)$. By linearity of the payoffs with respect to F, we get

$$J_c^*(\lambda, F) = J_c^*(\lambda, 0) - F \quad \text{and} \quad J_p^*(\lambda, F) = J_p^*(\lambda, 0) + F,$$

yielding

$$F_c^{\lambda,*} = J_c^*(\lambda, 0) - J_c^*(0,0), \quad \text{and} \quad F_p^{\lambda,*} = J_p^*(0,0) - J_p^*(\lambda, 0).$$

Thus, $F_c^{\lambda,*}$ represents the maximum amount the consumer is willing to pay, while $F_p^{\lambda,*}$ is the minimum amount the producer is willing to accept for selling a forward contract on λ units of the underlying. As a consequence, trading is possible if and only if

$$F_p^{\lambda,*} \leq F_c^{\lambda,*}.$$

We conclude this section with the definition of agreement indifference price.

Definition 2.2.3. *Let λ^* be the number of units of the underlying for which the two parties agree on the forward price, namely $F_p^{\lambda^*,*} = F_c^{\lambda^*,*}$. We define the* agreement indifference price *as*

$$F_{\lambda^*}^* := F_p^{\lambda^*,*} = F_c^{\lambda^*,*}.$$

In Section 2.5, we will provide some numerical illustrations on how the risk aversion parameters and the volatility control costs of the players might affect the quantity λ^* as well as the corresponding agreement indifference price $F_{\lambda^*}^*$.

2.3 Nash equilibrium

In this section we state and comment the main result of the paper. In particular we show that a Nash equilibrium exists and we characterise the corresponding strategies and payoffs in a semi-explicit way. Its proof will be given in full detail in the next section.

2.3.1 Main result

Let us start with some useful notation: for $t \in [0, T]$,

$$K_p(t) = -\frac{k_p}{2}\sqrt{\frac{2(\rho_p + \eta_p \lambda^2 \rho_p^2)}{k_p}} \tanh\left(\sqrt{\frac{2(\rho_p + \eta_p \lambda^2 \rho_p^2)}{k_p}}(T - t)\right), \tag{2.3.1}$$

$$\Lambda_p(t) = -\frac{k_p}{2}\sqrt{\frac{2\rho_p}{k_p}}\tanh\left(\sqrt{\frac{2\rho_p}{k_p}}(T-t)\right),$$
$$K_c(t) = -\frac{k_c}{2}\sqrt{\frac{2(\gamma\rho_c(\gamma-p_1)+\eta_c\lambda^2\gamma^2\rho_c^2)}{k_c}}\tanh\left(\sqrt{\frac{2(\gamma\rho_c(\gamma-p_1)+\eta_c\lambda^2\gamma^2\rho_c^2)}{k_c}}(T-t)\right),$$
$$\Lambda_c(t) = -\frac{k_c}{2}\sqrt{\frac{2\gamma\rho_c(\gamma-p_1)}{k_c}}\tanh\left(\sqrt{\frac{2\gamma\rho_c(\gamma-p_1)}{k_c}}(T-t)\right),$$

$$\Xi = \begin{pmatrix} 0 & -\rho_p\gamma\rho_c\eta_p\lambda^2 - \frac{\gamma\rho_c}{2} \\ -\rho_p\gamma\rho_c\eta_c\lambda^2 - \frac{\rho_p(\gamma-p_1)}{2} & 0 \end{pmatrix}, \quad \widehat{\Xi} = \begin{pmatrix} 0 & -\frac{\gamma\rho_c}{2} \\ -\frac{\rho_p(\gamma-p_1)}{2} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} -\frac{2}{k_p} & 0 \\ 0 & -\frac{2}{k_c} \end{pmatrix},$$

$$\Phi(t) = \begin{pmatrix} -\frac{2}{k_p}K_p(t) & 0 \\ 0 & -\frac{2}{k_c}K_c(t) \end{pmatrix}, \quad \widehat{\Phi}(t) = \begin{pmatrix} -\frac{2}{k_p}\Lambda_p(t) & 0 \\ 0 & -\frac{2}{k_c}\Lambda_c(t) \end{pmatrix}, \quad \Psi = \begin{pmatrix} -s_0/2 \\ -\frac{p_0+p_1s_0-\gamma(\delta+s_0)}{2} \end{pmatrix}.$$

Furthermore, let us introduce the following system of ODEs defined on $t \in [0,T]$:

$$\begin{cases} \pi'(t) = \Xi + \Phi(t)\pi(t) + \pi(t)\Phi(t) + \pi(t)R\pi(t), & \pi(T) = 0, \\ \widehat{\pi}'(t) = \widehat{\Xi} + \widehat{\Phi}(t)\widehat{\pi}(t) + \widehat{\pi}(t)\widehat{\Phi}(t) + \widehat{\pi}(t)R\widehat{\pi}(t), & \widehat{\pi}(T) = 0, \end{cases} \tag{2.3.2}$$

$$\mathrm{d}h(t) = \left\{\left[\widehat{\pi}(t)R + \widehat{\Phi}(t)\right]h(t) + \Psi\right\}\mathrm{d}t, \quad h(T) = \tfrac{1}{2}\lambda(\rho_p, \gamma\rho_c)^\top, \tag{2.3.3}$$

and let us denote by T_{max} the right end of the maximal interval where the system (2.3.2) admits a unique solution according to Picard-Lindelöf Theorem (see, e.g., [39], Ch. I, Theorem 2.3, which can be applied by standard time-inversion).

Theorem 2.3.1. *Assume that the following conditions hold:*

(A1) $T < T_{max}$,

(A2) $\ell_p - 2(K_p(t) + \pi_{11}(t)) > 0$ *and* $\ell_c - 2(K_c(t) + \pi_{22}(t)) > 0$, *for all* $t \in [0,T]$.

Then,

1. *there exists a Nash equilibrium* $((u^*, z^*)^\top, (v^*, y^*)^\top) \in \mathcal{A}^2$ *in the following feedback form*

$$u_t^* = \frac{2}{k_p}\left[(K_p(t)+\pi_{11}(t))(q_t - \bar{q}_t) + \pi_{12}(t)(c_t - \bar{c}_t) + (\Lambda_p(t) + \widehat{\pi}_{11}(t))\bar{q}_t + \widehat{\pi}_{12}(t)\bar{c}_t + h_1(t)\right],$$

$$z^*(t) = \frac{\sigma_p\ell_p}{\ell_p - 2(K_p(t)+\pi_{11}(t))},$$

$$v_t^* = \frac{2}{k_c}\left[(K_c(t)+\pi_{22}(t))(c_t - \bar{c}_t) + \pi_{21}(t)(q_t - \bar{q}_t) + (\Lambda_c(t) + \widehat{\pi}_{22}(t))\bar{c}_t + \widehat{\pi}_{21}(t)\bar{q}_t + h_2(t)\right],$$

$$y^*(t) = \frac{\sigma_c\ell_c}{\ell_c - 2(K_c(t)+\pi_{22}(t))}.$$

2. *The equilibrium payoffs satisfy*

$$J_p^*(\lambda, F) = \Lambda_p(0)q_0^2 + 2\bar{Y}_0^p q_0 + R_p(0) + F - \lambda s_0 - \frac{1}{2}\ell_p\sigma_p^2 T, \tag{2.3.4}$$

$$J_c^*(\lambda, F) = \Lambda_c(0)c_0^2 + 2\bar{Y}_0^c c_0 + R_c(0) - F + \lambda s_0 - \frac{1}{2}\ell_c \sigma_c^2 T, \tag{2.3.5}$$
$$\bar{Y}_t^p = \widehat{\pi}_{11}\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t + h_1, \quad \bar{Y}_t^c = \widehat{\pi}_{22}\bar{c}_t + \widehat{\pi}_{21}\bar{q}_t + h_2,$$

where

$$R_p(0) = R_p^{(\lambda)}(0) = \int_0^T \left[\frac{2}{k_p}\mathbb{E}[(Y_u^p)^2] - \eta_p \lambda^2 \gamma^2 \rho_c^2 \mathbb{V}[c_u] + \frac{2\big(\pi_{11}(u) z_u^* + \frac{\ell_p \sigma_p}{2}\big)^2}{\ell_p - 2K_p(u)} \right] \mathrm{d}u - \lambda\gamma\rho_c \mathbb{E}[c_T], \tag{2.3.6}$$

$$R_c(0) = R_c^{(\lambda)}(0) = \int_0^T \left[\frac{2}{k_c}\mathbb{E}[(Y_u^c)^2] - \eta_c \lambda^2 \rho_p^2 \mathbb{V}[q_u] + \frac{2\big(\pi_{22}(u) y_u^* + \frac{\ell_c \sigma_c}{2}\big)^2}{\ell_c - 2K_c(u)} \right] \mathrm{d}u - \lambda \rho_p \mathbb{E}[q_T].$$

See Appendix 2.C for the details on the computations of the quantities involved in the definition of R_p and R_c.

2.3.2 Comments

1. Although our model is close to the one presented in [122], it is not possible to directly exploit their results, since their hypotheses **(H2)(a)** and **(H2)(d)** are not satisfied in our case. Therefore, in order to be self contained, we decided to prove a suitable verification theorem from scratch.

2. We observe that the functions Λ_i, $i \in \{p, c\}$, do not depend on λ. It is also the case for the functions $\widehat{\pi}_{ij}$. Furthermore, the functions h_i, $i = 1, 2$, are linear in λ because they depend on it only by their terminal conditions. Besides, they are also non-decreasing in λ. Thus, the average production and consumption rates, $\bar{q}_t$ and $\bar{c}_t$, which satisfy

$$\mathrm{d}\bar{q}_t = \bar{u}_t^* \mathrm{d}t = \frac{2}{k_p}\Big[\big(\Lambda_p(t) + \widehat{\pi}_{11}(t)\big)\bar{q}_t + \widehat{\pi}_{12}(t)\bar{c}_t + h_1(t)\Big]\mathrm{d}t,$$
$$\mathrm{d}\bar{c}_t = \bar{v}_t^* \mathrm{d}t = \frac{2}{k_c}\Big[\big(\Lambda_c(t) + \widehat{\pi}_{22}(t)\big)\bar{c}_t + \widehat{\pi}_{21}(t)\bar{q}_t + h_2(t)\Big]\mathrm{d}t,$$

   are increasing in λ. As the terminal conditions of h_i, $i \in \{p, c\}$, depend only on λ and on the parameters ρ_p and $\gamma\rho_c$, the resulting effect on the average spot price $\bar{S}_t = s_0 - \rho_p \bar{q}_t + \gamma\rho_c \bar{c}_t$ only depends on the relative market power of the producer and the consumer. Thus, if $\gamma\rho_c < \rho_p$ (resp. $\rho_p < \gamma\rho_c$), when the quantity of the commodity λ of the producer increases, the average spot price decreases (resp. increases).

3. The functions Λ_p, Λ_c and the $\widehat{\pi}_{ij}$ do not depend on the risk aversion parameters η_p and η_c, therefore the average production and consumption rates do not depend on them either, as one could expect. Regarding the volatilities, while it is clear that K_p and K_c are non-decreasing in η_p and η_c, respectively, it is not so obvious what to expect for π_{11} and π_{22}, and thus to deduce the effect of risk-aversion on z^* and y^*. However, one can find numerically that the higher the risk aversions of the players, the lower the volatilities, even in the absence of forward agreement. Nevertheless, it is possible to provide more insight on this issue when the producer has no market power, i.e. $\rho_p = 0$, and the consumer does have some, i.e. $\gamma\rho_c > 0$. In this case, the price process appears as exogenously driven for the producer and as a controlled variable for the consumer. Hence $K_p = \Lambda_p = 0$ and $K_c < 0$, $\Lambda_c < 0$. Further, if $\rho_p = 0$, then $\pi_{21} = 0$, leading to $\pi_{11} = 0$ due to $K_p = 0$, and it holds also that $\pi_{22} = 0$ and $\widehat{\pi}_{11} = \widehat{\pi}_{21} = 0$. Thus, $z^* = \sigma_p$ and the producer does not reduce her volatility. On the other hand, the production does covariate with consumption. Indeed, in Theorem 2.3.1, the Nash equilibrium consumer's strategies depend on the state variables only via $c_t - \bar{c}_t$ and $\bar{c}_t$:

$$u_t^* = \frac{2}{k_p}\left\{\pi_{12}(t)(c_t - \bar{c}_t) + \widehat{\pi}_{12}(t)\bar{c}_t + h_1(t)\right\}, \quad z_t^* = \sigma_p,$$

$$v_t^* = \frac{2}{k_c}\left\{K_c(t)(c_t - \bar{c}_t) + \Lambda_c(t)\bar{c}_t + h_2(t)\right\}, \quad y_t^* = \frac{\sigma_c \ell_c}{\ell_c - 2K_c(t)} < \sigma_c.$$

Finally, since $K_c(t)$ is non-increasing in λ, the higher the exposure to the financial risk coming from the forward contract, the more the consumer reduces his volatility, as the intuition predicts.

4. Exploiting Theorem 2.3.1-2., we can specify more precisely the non-linear equations giving the forward agreement values $F^*_{\lambda^*}$ and λ^*. Indeed, it holds that (see equations (2.3.4) and (2.3.5))

$$J_p^*(\lambda, F) = \Lambda_p(0)q_0^2 + 2\bar{Y}_0^{p(\lambda)} q_0 + R_p^{(\lambda)}(0) + F - \lambda s_0 - \frac{1}{2}\ell_p \sigma_p^2 T,$$
$$J_c^*(\lambda, F) = \Lambda_c(0)c_0^2 + 2\bar{Y}_0^{c(\lambda)} c_0 + R_c^{(\lambda)}(0) - F + \lambda s_0 - \frac{1}{2}\ell_c \sigma_c^2 T,$$

where the superscript (λ) is used to emphasise the dependency on the number of options traded. We can isolate the parts $j_p(\lambda, F)$ and $j_c(\lambda, F)$ depending on λ and F, defined as

$$j_p(\lambda, F) = 2h_1^{(\lambda)}(0)q_0 + R_p^{(\lambda)}(0) + F - \lambda s_0,$$
$$j_c(\lambda, F) = 2h_2^{(\lambda)}(0)c_0 + R_c^{(\lambda)}(0) - F + \lambda s_0.$$

Thus, for a fixed λ the indifference prices $F_p^{\lambda,*}$ and $F_c^{\lambda,*}$ are given by

$$2h_1^{(0)}(0)q_0 + R_p^{(0)}(0) = 2h_1^{(\lambda)}(0)q_0 + R_p^{(\lambda)}(0) + F_p^{\lambda,*} - \lambda s_0,$$
$$2h_2^{(0)}(0)c_0 + R_c^{(0)}(0) = 2h_2^{(\lambda)}(0)c_0 + R_c^{(\lambda)}(0) - F_c^{\lambda,*} + \lambda s_0.$$

Thus, if it exists, the equilibrium price should be given by $F_p^{\lambda^*,*} = F_c^{\lambda^*,*} = F^*_{\lambda^*}$, i.e.,

$$2(h_1^{(0)}(0) - h_1^{(\lambda^*)}(0))q_0 + R_p^{(0)}(0) - R_p^{(\lambda^*)}(0) = 2(h_2^{(\lambda^*)}(0) - h_2^{(0)}(0))c_0 + R_c^{(\lambda^*)}(0) - R_c^{(0)}(0),$$

or, equivalently,

$$2h_1^{(0)}(0)q_0 + 2h_2^{(0)}(0)c_0 + R_c^{(0)}(0) + R_p^{(0)}(0) = 2h_1^{(\lambda^*)}(0)q_0 + 2h_2^{(\lambda^*)}(0)c_0 + R_c^{(\lambda^*)}(0) + R_p^{(\lambda^*)}(0),$$

with $R_p^{(\lambda^*)}(0)$ and $R_c^{(\lambda^*)}(0)$ defined in Equation (2.3.6) and $h^{(\lambda^*)}$ in Equation (2.3.3).

The last remark speeds up considerably the computations for the plots that appear in the Section 2.5. Indeed, all the quantities that we need to compute can be obtained by solving numerically the ODEs presented in Appendix 2.C.

2.4 Proof of Theorem 2.3.1

2.4.1 The solution approach

We prove Theorem 2.3.1 following a methodology based on a combination of a suitable *Verification Theorem* and of the *weak Martingale Optimality Principle*. As already stressed in the first comment below Theorem 2.3.1, despite our model is very close to the class of games studied in [122], their results cannot be applied directly here, therefore we had to adapt the methodology to our framework. We proceed through the following steps:

1) we compute the best response maps of both players;
2) we check that the system coming from the best response computations has a unique solution;
3) we get a Nash equilibrium as a fixed point of the best response map;

4) we verify that there exists a unique solution to the system characterising the fixed point found in step 3).

2.4.2 Preliminary reformulation of the problem

For convenience, we introduce the following vector notation for the players' strategies:

$$\alpha = \left((\alpha^p)^\top, (\alpha^c)^\top\right)^\top \in \mathcal{A}^2, \quad \alpha^p := \begin{pmatrix} u \\ z \end{pmatrix} = \left\{\begin{pmatrix} u_t \\ z_t \end{pmatrix}\right\}_{t\in[0,T]} \quad \text{and} \quad \alpha^c := \begin{pmatrix} v \\ y \end{pmatrix} = \left\{\begin{pmatrix} v_t \\ y_t \end{pmatrix}\right\}_{t\in[0,T]},$$

so that the dynamics of the state variables can be rewritten as

$$\begin{aligned} \mathrm{d}q_t &= e_1^\top \alpha_t^p \mathrm{d}t + e_2^\top \alpha_t^p \mathrm{d}W_t, \\ \mathrm{d}c_t &= e_1^\top \alpha_t^c \mathrm{d}t + e_2^\top \alpha_t^c \mathrm{d}B_t, \qquad t \in [0,T], \end{aligned} \tag{2.4.1}$$

with $e_1^\top = (1,0)$ and $e_2^\top = (0,1)$.

The following identity is exploited to get a suitable reformulation of our problem: using the dynamics of S_t and applying Fubini's theorem, it is easy to see that

$$\int_0^T \mathbb{V}[S_t]\,\mathrm{d}t = \mathbb{E}\left[\int_0^T \left\{\rho_p^2(q_t - \mathbb{E}[q_t])^2 + \gamma^2\rho_c^2(c_t - \mathbb{E}[c_t])^2 - 2\rho_p\gamma\rho_c(q_t - \mathbb{E}[q_t])(c_t - \mathbb{E}[c_t])\right\} \mathrm{d}t\right]. \tag{2.4.2}$$

Rearranging the terms in the expressions of the producer objective functional, we obtain

$$J_p^{\lambda,F}(u,z;v,y) = \widetilde{J}_p^\lambda(u,z;v,y) + F - \lambda s_0 - \frac{\ell_p\sigma_p^2 T}{2},$$

where

$$\begin{aligned} \widetilde{J}_p^\lambda(u,z;v,y) :=& \mathbb{E}\Bigg[\int_0^T \Bigg(-(\rho_p + \eta_p\lambda^2\rho_p^2)(q_t - \mathbb{E}[q_t])^2 - \rho_p\mathbb{E}[q_t]^2 + [s_0 + \gamma\rho_c c_t \\ &+ 2\rho_p\gamma\rho_c\eta_p\lambda^2(c_t - \mathbb{E}[c_t])]q_t - \frac{k_p}{2}u_t^2 - \frac{\ell_p}{2}z_t^2 + \ell_p\sigma_p z_t - \eta_p\lambda^2\gamma^2\rho_c^2(c_t - \mathbb{E}[c_t])^2\Bigg)\mathrm{d}t \\ &+ \lambda\rho_p q_T - \lambda\gamma\rho_c c_T\Bigg]. \end{aligned}$$

Then, neglecting the constant terms, we can study without loss of generality the equivalent formulation in which the producer aims at maximising $\widetilde{J}_p^\lambda(u,z;v,y)$.

Remark 2.4.1. *Fixing a strategy α^p for the producer (resp. α^c for the consumer) is equivalent, from the perspective of the competitor, to fixing the corresponding state q^{α^p} (resp. c^{α^c}). Thus, with some abuse of notation we will write simply q (resp. c) when the strategy used is clear from the context. Moreover, to ease the notation, we will also omit the dependence on $\bar{c}$ and $\bar{q}$.*

For a given consumption process $\{c_t\}_{t\in[0,T]}$, we write

$$\widetilde{J}_p^\lambda(\alpha^p;\alpha^c) = \widetilde{J}_p^\lambda(\alpha^p;c) := \mathbb{E}\left[\int_0^T f_p(t,q_t,\mathbb{E}[q_t],\alpha_t^p,\mathbb{E}[\alpha_t^p];c)\mathrm{d}t + g_p(q_T,\mathbb{E}[q_T];c)\right], \text{ with}$$
$$f_p(t,q,\bar{q},a_p,\bar{a}_p;c) = Q_p(q-\bar{q})^2 + (Q_p+\widetilde{Q}_p)\bar{q}^2 + 2M^p(c)_t q + a_p^\top N_p a_p + 2H_p^\top a_p + T^p(c)_t,$$
$$g_p(q,\bar{q};c) = 2L_p q + \widetilde{T}^p(c),$$

where

$$Q_p := -\rho_p - \eta_p\lambda^2\rho_p^2, \quad \widetilde{Q}_p := \eta_p\lambda^2\rho_p^2, \quad M^p(c)_t := \frac{s_0}{2} + \frac{\gamma\rho_c}{2}c_t + \rho_p\gamma\rho_c\eta_p\lambda^2(c_t - \mathbb{E}[c_t]),$$
$$N_p := \begin{pmatrix} -\frac{k_p}{2} & 0 \\ 0 & -\frac{\ell_p}{2}\end{pmatrix}, \quad H_p := \begin{pmatrix} 0 \\ \frac{\sigma_p\ell_p}{2}\end{pmatrix}, \quad T^p(c)_t := -\eta_p\lambda^2\gamma^2\rho_c^2(c_t-\mathbb{E}[c_t])^2, \quad L_p := \frac{\rho_p\lambda}{2}, \tag{2.4.3}$$
$$\text{and} \quad \widetilde{T}^p(c) := -\lambda\gamma\rho_c c_T.$$

Now, let us turn to the objective functional of the consumer. From (2.2.1) and (2.4.2), we have

$$J_c^{\lambda,F}(v,y;u,z) = \widetilde{J}_c^\lambda(v,y;u,z) - F + \lambda s_0 - \frac{\ell_c\sigma_c^2 T}{2},$$

where

$$\widetilde{J}_c^\lambda(v,y;u,z) := \mathbb{E}\Bigg[\int_0^T\Big(-[\gamma\rho_c(\gamma-p_1)+\eta_c\lambda^2\gamma^2\rho_c^2](c_t-\mathbb{E}[c_t])^2 - \gamma\rho_c(\gamma-p_1)\mathbb{E}[c_t]^2$$
$$+ [(p_0+s_0p_1-\gamma\delta-\gamma s_0)+\rho_p(\gamma-p_1)q_t + 2\rho_p\gamma\rho_c\eta_c\lambda^2(q_t-\mathbb{E}[q_t])]c_t$$
$$-\frac{k_c}{2}v_t^2 - \frac{\ell_c}{2}y_t^2 + \sigma_c\ell_c y_t - \eta_c\lambda^2\rho_p^2(q_t-\mathbb{E}[q_t])^2\Big)\mathrm{d}t - \lambda\rho_p q_T + \lambda\gamma\rho_c c_T\Bigg].$$

Analogously as above, let $\{q_t\}_{t\in[0,T]}$ be a given production rate. We can write

$$\widetilde{J}_c^\lambda(\alpha^c;\alpha^p) = \widetilde{J}_c^\lambda(\alpha^c;q) := \mathbb{E}\left[\int_0^T f_c\big(t,c_t,\mathbb{E}[c_t],\alpha_t^c,\mathbb{E}[\alpha_t^c];q\big)\mathrm{d}t + g_c(c_T,\mathbb{E}[c_T];q)\right], \text{ with}$$
$$f_c(t,c,\bar{c},a_c,\bar{a}_c;q) = Q_c(c-\bar{c})^2 + (Q_c+\widetilde{Q}_c)\bar{c}^2 + 2M^c(q)_t c + a_c^\top N_c a_c + 2H_c^\top a_c + T^c(q)_t,$$
$$g_c(c,\bar{c};q) = 2L_c c + \widetilde{T}^c(q),$$

and

$$Q_c := -\gamma\rho_c(\gamma-p_1) - \eta_c\lambda^2\gamma^2\rho_c^2, \quad \widetilde{Q}_c := \eta_c\lambda^2\gamma^2\rho_c^2, \quad N_c := \begin{pmatrix} -\frac{k_c}{2} & 0 \\ 0 & -\frac{\ell_c}{2}\end{pmatrix},$$
$$M^c(q)_t := \frac{p_0+p_1s_0-\gamma(s_0+\delta)}{2} + \frac{\rho_p(\gamma-p_1)}{2}q_t + \rho_p\gamma\rho_c\eta_c\lambda^2(q_t-\mathbb{E}[q_t]),$$
$$H_c := \begin{pmatrix} 0 \\ \frac{\sigma_c\ell_c}{2}\end{pmatrix}, \quad T^c(q)_t := -\eta_c\lambda^2\rho_p^2(q_t-\mathbb{E}[q_t])^2, \quad L_c := \frac{\lambda\gamma\rho_c}{2}, \quad \text{and} \quad \widetilde{T}^c(q) := -\lambda\rho_p q_T.$$

Finally, we set

$$
\begin{aligned}
V_p^\lambda(\alpha^c) &:= \sup_{\alpha^p \in \mathcal{A}} \widetilde{J}_p^\lambda(\alpha^p;\alpha^c), \qquad \alpha^c \in \mathcal{A},\\
V_c^\lambda(\alpha^p) &:= \sup_{\alpha^c \in \mathcal{A}} \widetilde{J}_c^\lambda(\alpha^c;\alpha^p), \qquad \alpha^p \in \mathcal{A}.
\end{aligned}
$$

2.4.3 First step: computation of the best response maps

The first step is focused on the computation of the best response map of each player. This is done by exploiting the following version of the *Verification Theorem*:

Theorem 2.4.2 (Verification Theorem)**.** *Fix a couple of strategies $\beta^p, \beta^c \in \mathcal{A}$ for the producer and the consumer, respectively. Let $\mathcal{W}_t^{p,\alpha^p}$ and $\mathcal{W}_t^{c,\alpha^c}$ be defined as*

$$
\mathcal{W}_t^{p,\alpha^p} = w_t^p(q_t^{\alpha^p}, \mathbb{E}[q_t^{\alpha^p}]), \quad \mathcal{W}_t^{c,\alpha^c} = w_t^c(c_t^{\alpha^c}, \mathbb{E}[c_t^{\alpha^c}]), \quad t \in [0,T], \quad \alpha^p, \alpha^c \in \mathcal{A},
$$

where the $\mathbb{F}$-adapted random fields $\{w_t^p(q,\bar{q}), t \in [0,T], q, \bar{q} \in \mathbb{R}\}$ and $\{w_t^c(c,\bar{c}), t \in [0,T], c, \bar{c} \in \mathbb{R}\}$ satisfy the following growth conditions: for all $t \in [0,T]$, for all $x, \bar{x} \in \mathbb{R}$,

$$
|w_t^p(x,\bar{x})| \le C_p(\nu_t^p + |x|^2 + |\bar{x}|^2), \qquad |w_t^c(x,\bar{x})| \le C_c(\nu_t^c + |x|^2 + |\bar{x}|^2), \tag{2.4.4}
$$

for some constants $C_p, C_c > 0$ and for some non-negative processes ν^p and ν^c such that

$$
\sup_{t \in [0,T]} \mathbb{E}\left[\nu_t^p + \nu_t^c\right] < \infty.
$$

Furthermore, we assume that the following conditions are fulfilled:

i) $\mathbb{E}[w_T^p(q_T^{\alpha^p}, \bar{q_T}^{\alpha^p})] = \mathbb{E}[g_p(q_T^{\alpha^p}, \bar{q_T}^{\alpha^p}; c^{\beta^c})]$ and $\mathbb{E}[w_T^c(c_T^{\alpha^c}, \bar{c_T}^{\alpha^c})] = \mathbb{E}[g_c(c_T^{\alpha^c}, \bar{c_T}^{\alpha^c}; q^{\beta^p})]$, for any $\alpha^p, \alpha^c \in \mathcal{A}$.

ii) The application $[0,T] \ni t \mapsto \mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]\Big($resp. $\mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]\Big)$ is well-defined and non-increasing, for any $\alpha^p \in \mathcal{A}$ (resp. for any $\alpha^c \in \mathcal{A}$), where:

$$
\begin{aligned}
\mathcal{S}_t^{p,\alpha^p} &= \mathcal{W}_t^{p,\alpha^p} + \int_0^t f_p(s, q_s^{\alpha^p}, \bar{q}_s^{\alpha^p}, \alpha_s^p, \bar{\alpha}_s^p; c^{\beta^c}) \mathrm{d}s,\\
\mathcal{S}_t^{c,\alpha^c} &= \mathcal{W}_t^{c,\alpha^c} + \int_0^t f_c(s, c_s^{\alpha^c}, \bar{c}_s^{\alpha^c}, \alpha_s^c, \bar{\alpha}_s^c; q^{\beta^p}) \mathrm{d}s.
\end{aligned} \tag{2.4.5}
$$

iii) For some $\alpha^{p,\star} \in \mathcal{A}$ and $\alpha^{c,\star} \in \mathcal{A}$, the application $[0,T] \ni t \mapsto \mathbb{E}[\mathcal{S}_t^{p,\alpha^{p,\star}}]\Big($resp. $\mathbb{E}[\mathcal{S}_t^{c,\alpha^{c,\star}}]\Big)$ is constant.

Then, the control $\alpha^\star = (\alpha^{p,\star}, \alpha^{c,\star})$ is the best response *to the control (β^p, β^c) meaning that*

$$
\alpha^{p,\star} = \mathbf{B}_p(\beta^c) := \arg\max_{\alpha^p \in \mathcal{A}} \widetilde{J}_p^\lambda(\alpha^p;\beta^c), \quad \alpha^{c,\star} = \mathbf{B}_c(\beta^p) := \arg\max_{\alpha^c \in \mathcal{A}} \widetilde{J}_c^\lambda(\alpha^c;\beta^p),
$$

and

$$
\widetilde{J}_p^\lambda(\alpha^{p,\star}; c^{\beta^c}) = V_p^\lambda(\beta^c) = \mathbb{E}[\mathcal{W}_0^{P,\alpha^p}] \quad and \quad \widetilde{J}_c^\lambda(\alpha^{c,\star}; q^{\beta^p}) = V_c^\lambda(\beta^p) = \mathbb{E}[\mathcal{W}_0^{C,\alpha^c}].
$$

Finally, if $\widetilde{\alpha} = (\widetilde{\alpha}^p, \widetilde{\alpha}^c)$ is another best response to the control (β^p, β^c), then condition iii) holds also for $\widetilde{\alpha}^p$ and $\widetilde{\alpha}^c$.

We define the best response map $\mathbf{B} : \mathcal{A}^2 \to \mathcal{A}^2$ as $\mathbf{B} := (\mathbf{B}_p, \mathbf{B}_c)$. The Nash equilibrium we find will be a fixed point of this map.

Once we have fixed the strategies β^p and β^c in $\mathcal{A}$, the first step can be divided into four sub-steps:

1.1 Since the players objective functionals are quadratic, we propose a suitable candidate $(\mathcal{W}_t^{p,\alpha^p}, \mathcal{W}_t^{c,\alpha^c})$ in feedback form.

1.2 Applying Itô's formula, we compute $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]$ corresponding to the candidate $(\mathcal{W}_t^{p,\alpha^p}, \mathcal{W}_t^{c,\alpha^c})$.

1.3 We postulate that the conditions of Theorem 2.4.2 are satisfied and get a system of backward SDEs involving the coefficients of the candidate $(\mathcal{W}_t^{p,\alpha^p}, \mathcal{W}_t^{c,\alpha^c})$.

1.4 We compute each player's best response by looking for strategies cancelling the expectation of the drifts of the processes $\mathcal{S}_t^{p,\alpha^p}$ and $\mathcal{S}_t^{c,\alpha^c}$.

Sub-step 1.1 Given the quadratic nature of our objective functional, it seems natural to look for a family of processes $(\mathcal{W}_t^{p,\alpha^p}, \mathcal{W}_t^{c,\alpha^c})_{t\in[0,T]}$ of the following form: $\mathcal{W}_t^{p,\alpha^p} = w_t^p(q_t^{\alpha^p}, \mathbb{E}[q_t^{\alpha^p}])$ and $\mathcal{W}_t^{c,\alpha^c} = w_t^c(c_t^{\alpha^c}, \mathbb{E}[c_t^{\alpha^c}])$, for some parametric adapted random field $\{w_t^i(x,\bar{x}), t \in [0,T], x, \bar{x} \in \mathbb{R}\}, i \in \{p,c\}$, such that

$$w_t^i(x,\bar{x}) = K_i(t)(x-\bar{x})^2 + \Lambda_i(t)\bar{x}^2 + 2Y_t^i x + R_i(t),$$

with $(K_i, \Lambda_i, Y^i, R_i) \in L^\infty([0,T], \mathbb{R}_-)^2 \times S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}) \times L^\infty([0,T], \mathbb{R})$, $i \in \{p,c\}$, solving the systems of ODEs and SDEs:

$$\begin{cases} dK_p(t) &= K_p'(t)\mathrm{d}t, & K_p(T) = 0, \\ \mathrm{d}\Lambda_p(t) &= \Lambda_p'(t)\mathrm{d}t, & \Lambda_p(T) = 0, \\ \mathrm{d}Y_t^p &= Y_t^{p\prime}\mathrm{d}t + Z_t^{p,B}\mathrm{d}B_t + Z_t^{p,W}\mathrm{d}W_t, & Y_T^p = \frac{\lambda\rho_p}{2}, \\ dR_p(t) &= R_p'(t)\mathrm{d}t, & R_p(T) = -\lambda\gamma\rho_c\mathbb{E}[c_T], \end{cases}$$

$$\begin{cases} dK_c(t) &= K_c'(t)\mathrm{d}t, & K_c(T) = 0, \\ \mathrm{d}\Lambda_c(t) &= \Lambda_c'(t)\mathrm{d}t, & \Lambda_c(T) = 0, \\ \mathrm{d}Y_t^c &= Y_t^{c\prime}\mathrm{d}t + Z_t^{c,B}\mathrm{d}B_t + Z_t^{c,W}\mathrm{d}W_t, & Y_T^c = \frac{\lambda\gamma\rho_c}{2}, \\ dR_c(t) &= R_c'(t)\mathrm{d}t, & R_c(T) = -\lambda\rho_p\mathbb{E}[q_T], \end{cases}$$

for some deterministic processes K_i', Λ_i', R_i' and for some $\mathbb{F}$-adapted processes $Y^{i\prime}, Z^{i,W}, Z^{i,B}$, $i \in \{p,c\}$.

Sub-step 1.2 For the sake of simplicity, from now on, we explicitly develop only the producer case. The consumer problem can be studied in same way. Let $t \in [0,T]$ and $\alpha^p \in \mathcal{A}$. As in (2.4.5) in Theorem 2.4.2 (Verification Theorem), we set

$$\mathcal{S}_t^{p,\alpha^p} = w_t^p(q_t^{\alpha^p}, \mathbb{E}[q_t^{\alpha^p}]) + \int_0^t f_p(u, q_u^{\alpha^p}, \mathbb{E}[q_u^{\alpha^p}], \alpha_u^p, \mathbb{E}[\alpha_u^p]; c^{\beta^c})\mathrm{d}u.$$

In the following, we write simply c instead of c^{β^c} (resp. q instead of q^{β^p}), when the strategies are clear from the context (see Remark 2.4.1). After some computations (see Appendix 2.B for details), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}] = \mathbb{E}\Big[&(K_p'(t) + Q_p)(q_t - \mathbb{E}[q_t])^2 + (\Lambda_p'(t) + Q_p + \widetilde{Q}_p)\mathbb{E}[q_t]^2 + 2(Y_t^{p\prime} + M^p(c)_t)q_t \\ &+ R_p'(t) + T^p(c)_t + \chi_t^p(\alpha_t^p)\Big],\end{aligned} \tag{2.4.6}$$

where, for any $t \in [0,T]$, we have set

$$\begin{cases} \chi_t^p(\alpha_t^p) := (\alpha_t^p)^\top S_p(t)\alpha_t^p + 2[U_p(t)(q_t - \mathbb{E}[q_t]) + V_p(t)q_t + \xi_t^p + \bar{\xi}_t^p + O_p(t)]^\top \alpha^p(t) \\ S_p(t) := N_p + e_2 K_p(t) e_2^\top \\ U_p(t) := K_p(t)e_1 \\ V_p(t) := \Lambda_p(t)e_1 \\ O_p(t) := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}] \\ \xi_t^p := H_p + e_1 Y_t^p + e_2 Z_t^{p,W} \\ \bar{\xi}_t^p := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}], \end{cases} \tag{2.4.7}$$

where $Q_p, \widetilde{Q}_p, M^p(c), N_p, H_p$ and $T^p(c)$ are defined in Equation (2.4.3).

Sub-step 1.3 Now, we find conditions granting that assumptions i), ii) and iii) of Theorem 2.4.2, involving $\mathcal{S}^{p,\alpha^p}$, hold. Suppose that the matrix $S_p(t)$ is negative definite and thus invertible. We check this later, verifying that $K_p(t) \leq 0$, for all $t \in [0,T]$ (see Remark 2.4.5). We complete the squares and rewrite the equation (2.4.6) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}] = \mathbb{E}\Big[&\big(K_p'(t) + Q_p - U_p(t)^\top S_p(t)^{-1}U_p(t)\big)(q_t - \mathbb{E}[q_t])^2 \\ &+ \big(\Lambda_p'(t) + Q_p + \widetilde{Q}_p - V_p(t)^\top S_p(t)^{-1}V_p(t)\big)\mathbb{E}[q_t]^2 \\ &+ 2\big[Y_t^{p\prime} + M^p(c)_t - U_p(t)^\top S_p(t)^{-1}(\xi_t^p - \bar{\xi}_t^p) - V_p(t)^\top S_p(t)^{-1}O_p(t)\big]q_t \\ &+ R_p'(t) + T^p(c)_t - (\xi_t^p - \bar{\xi}_t^p)^\top S_p(t)^{-1}(\xi_t^p - \bar{\xi}_t^p) - O_p(t)^\top S_p(t)^{-1}O_p(t) \\ &+ (\alpha_t^p - \eta_t^p)^\top S_p(t)^{-1}(\alpha_t^p - \eta_t^p)\Big],\end{aligned}$$

where, for all $t \in [0,T]$, we have defined

$$\eta_t^p := -S_p(t)^{-1}\left[U_p(t)(q_t - \mathbb{E}[q_t]) + V_p(t)\mathbb{E}[q_t] + (\xi_t^p - \bar{\xi}_t^p) + O_p(t)\right].$$

Choosing processes K_p, Λ_p, Y^p and R_p, whose existence is shown in the next sub-step, that solve the following system of BSDEs

$$\begin{cases} K_p'(t) + Q_p - U_p(t)^\top S_p(t)^{-1} U_p(t) = 0, \quad K_p(T) = 0, \\ \Lambda_p'(t) + Q_p + \widetilde{Q}_p - V_p(t)^\top S_p(t)^{-1} V_p(t) = 0, \quad \Lambda_p(T) = 0, \\ \mathrm{d}Y_t^p = \quad \left[-M^p(c)_t + U_p(t)^\top S_p(t)^{-1}(\xi_t^p - \overline{\xi}_t^p) + V_p(t)^\top S_p(t)^{-1} O_p(t)\right] \mathrm{d}t \\ \qquad\qquad + Z_t^{p,B} \mathrm{d}B_t + Z_t^{p,W} \mathrm{d}W_t, \\ Y_T^p = L_p, \\ R_p'(t) + \mathbb{E}[T^p(c)_t - (\xi_t^p - \overline{\xi}_t^p)^\top S_p(t)^{-1}(\xi_t^p - \overline{\xi}_t^p) - O_p(t)^\top S_p(t)^{-1} O_p(t)] = 0, \\ R_p(T) = \mathbb{E}[\widetilde{T}^p(c)], \end{cases}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}] = \mathbb{E}\Big[(\alpha_t^p - \eta_t^p)^\top S_p(t)^{-1}(\alpha_t^p - \eta_t^p)\Big], \tag{2.4.8}$$

which is clearly non-positive for all $t \in [0, T]$, since $S_p(t)$ (defined in Equation (2.4.7)) is negative definite for all $t \in [0, T]$.

Remark 2.4.3. *We stress the fact that the processes $Y^p, Z^{p,W}, Z^{p,B}$ and R_p depend only on the strategy of the consumer through the state process $\{c_t\}_{t\in[0,T]}$, with $c_t = c_t^{\beta^c}, t \in [0, T]$, which is controlled only by β^c. Thus, the feedback best response control are functions of different state variables, namely the best response for the producer is feedback in q and its expectation, whereas the best response for the consumer is feedback in c and its expectation.*

Sub-step 1.4 Now we combine the results in the previous steps in order to get the best response maps.

Proposition 2.4.4. *The best response maps are given by*

$$\begin{aligned} \mathbf{B}_p(\beta^c)_t &= -(N_p + e_2 K_p(t) e_2^\top)^{-1}[e_1 K_p(t)(q_t - \mathbb{E}[q_t]) + e_1 \Lambda_p(t)\mathbb{E}[q_t] + e_1 Y_t^p + e_2 Z_t^{p,W} + H_p], \\ \mathbf{B}_c(\beta^p)_t &= -(N_c + e_2 K_c(t) e_2^\top)^{-1}[e_1 K_c(t)(c_t - \mathbb{E}[c_t]) + e_1 \Lambda_c(t)\mathbb{E}[c_t] + e_1 Y_t^c + e_2 Z_t^{c,B} + H_c], \end{aligned} \tag{2.4.9}$$

where the processes $(K_p, \Lambda_p, Y^p, R_p)$ and $(K_c, \Lambda_c, Y^c, R_c)$ above solve the following systems of backward ODEs and SDEs, given $c_t = c_t^{\beta^c}$ (respectively, given $q_t = q_t^{\beta^p}$), $t \in [0, T]$:

$$\begin{cases} K_p'(t) = -\frac{2}{k_p} K_p(t)^2 + \rho_p + \eta_p \lambda^2 \rho_p^2, \quad K_p(T) = 0, \\ \Lambda_p'(t) = -\frac{2}{k_p} \Lambda_p(t)^2 + \rho_p, \quad \Lambda_p(T) = 0, \\ \mathrm{d}Y_t^p \quad = -\left\{\frac{s_0}{2} + \frac{\gamma\rho_c}{2} c_t + \rho_p \gamma \rho_c \eta_p \lambda^2 (c_t - \mathbb{E}[c_t]) + \frac{2}{k_p}\Big[K_p(t)\left(Y_t^p - \mathbb{E}[Y_t^p]\right) + \Lambda_p(t)\mathbb{E}[Y_t^p]\Big]\right\} \mathrm{d}t \\ \qquad\qquad + Z_t^{p,B} \mathrm{d}B_t + Z_t^{p,W} \mathrm{d}W_t, \\ Y_T^p \quad = \frac{\lambda \rho_p}{2}, \\ R_p'(t) = \eta_p \lambda^2 \gamma^2 \rho_c^2 \mathbb{V}[c_t] - \frac{2}{k_p}(\mathbb{V}[Y_t^p] + \mathbb{E}[Y_t^p]^2) - \frac{2}{\ell_p - 2K_p(t)}(\mathbb{V}[Z_t^{p,W}] + (\mathbb{E}[Z_t^{p,W}] + \frac{\ell_p \sigma_p}{2})^2), \\ R_p(T) = -\lambda \gamma \rho_c \mathbb{E}[c_T], \end{cases} \tag{2.4.10}$$

and

$$\begin{cases} K_c'(t) = -\frac{2}{k_c}K_c(t)^2 + \gamma\rho_c(\gamma - p_1) + \eta_c\lambda^2\gamma^2\rho_c^2 = 0, \quad K_c(T) = 0, \\ \Lambda_c'(t) = -\frac{2}{k_c}\Lambda_c(t)^2 + \gamma\rho_c(\gamma - p_1), \quad \Lambda_c(T) = 0, \\ \mathrm{d}Y_t^c \ \ = -\Big\{\frac{p_0+p_1s_0-\gamma(s_0+\delta)}{2} + \frac{\rho_p(\gamma-p_1)}{2}q_t + \rho_p\gamma\rho_c\eta_c\lambda^2(q_t - \mathbb{E}[q_t]) + \frac{2}{k_c}\Big[K_c(t)(Y_t^c - \mathbb{E}[Y_t^c]) \\ \qquad +\Lambda_c(t)\mathbb{E}[Y_t^c]\Big]\Big\}\mathrm{d}t + Z_t^{c,B}\mathrm{d}B_t + Z_t^{c,W}\mathrm{d}W_t, \\ Y_T^c \ \ = \frac{\lambda\gamma\rho_c}{2}, \\ R_c'(t) = \ \ \eta_c\lambda^2\rho_p^2\mathbb{V}[q_t] - \frac{2}{k_c}(\mathbb{V}[Y_t^c] + \mathbb{E}[Y_t^c]^2) - \frac{2}{\ell_c-2K_c(t)}[\mathbb{V}[Z_t^{c,B}] + (\mathbb{E}[Z_t^{c,B}] + \frac{\ell_c\sigma_c}{2})^2], \\ R_c(T) = \ \ -\lambda\rho_p\mathbb{E}[q_T]. \end{cases} \tag{2.4.11}$$

So, we have

$$\widetilde{J}_p^\lambda(\mathbf{B}_p(\beta^c);\beta^c) = V_p^\lambda(\beta^c) \text{ and } \widetilde{J}_c^\lambda(\mathbf{B}_c(\beta^p);\beta^p) = V_c^\lambda(\beta^p).$$

Moreover, we have an explicit expression for the Nash equilibrium values which are given by

$$V_p^\lambda(\beta^c) = \Lambda_p(0)q_0^2 + 2\mathbb{E}[Y_0^p]q_0 + R_p(0) \quad \text{and}$$
$$V_c^\lambda(\beta^p) = \Lambda_c(0)c_0^2 + 2\mathbb{E}[Y_0^c]c_0 + R_c(0).$$

Remark 2.4.5. *Notice that the first two equations in the systems (2.4.10) and (2.4.11) are one-dimensional Riccati differential equations, for which it is known that there exists a unique global solution given by Equation (2.3.1). In the following we face more complicated Riccati equations (non-symmetric matrix Riccati equations) for which existence of solutions is not guaranteed. The fact that $K_p(t)$ and $K_c(t)$ are given by a hyperbolic tangent with a positive argument multiplied by a negative constant yields that $K_p(t) \leq 0$ and $K_c(t) \leq 0$, granting that $S_p(t) = N_p + e_2K_p(t)e_2^\top$ and $S_c(t) = N_c + e_2K_c(t)e_2^\top$ are negative definite for all $t \in [0,T]$, hence matching the assumptions made at the beginning of Sub-step 1.3.*

Proof. To prove the proposition we need to apply Theorem 2.4.2. So, let us check that its hypotheses are fulfilled. Fix a couple of strategies $\beta^p, \beta^c \in \mathcal{A}$. First of all, condition i) is a consequence of the terminal conditions of systems (2.4.10) and (2.4.11). Furthermore, we notice that assumption ii) is verified, for any $\alpha^p \in \mathcal{A}$ (resp. for any $\alpha^c \in \mathcal{A}$), because the fact that the processes $(K_p, \Lambda_p, Y^p, R_p)$ and $(K_c, \Lambda_c, Y^c, R_c)$ solve the systems (2.4.10) and (2.4.11) yields that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]$ are negative and so the monotonicity of the functions $[0,T] \ni t \mapsto E[\mathcal{S}_t^{p,\alpha^p}]$(resp. $\mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]$). Then, by (2.4.8), we notice that, given $\beta^c \in \mathcal{A}$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}] = 0$, for all $t \in [0,T]$, if and only if, for all $t \in [0,T]$, we have

$$\alpha_t^p = \eta_t^p = -S_p(t)^{-1}\Big[U_p(t)(q_t - \mathbb{E}[q_t]) - V_p(t)\mathbb{E}[q_t] - (\xi_t^p - \bar{\xi}_t^p) - O_p(t)\Big], \quad \mathbb{P}\text{-a.s.},$$

and analogously, given $\beta^p \in \mathcal{A}$, $0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]$, for all $t \in [0,T]$, if and only if, for all $t \in [0,T]$, we have

$$\alpha_t^c = \eta_t^c = -S_c(t)^{-1}\left[U_c(t)(c_t - \mathbb{E}[c_t]) - V_c(t)\mathbb{E}[c_t] - (\xi_t^c - \bar{\xi}_t^c) - O_c(t)\right], \quad \mathbb{P}\text{-a.s.}$$

Hence, the strategies in (2.4.9) satisfy iii) as well.

Finally, let us check the admissibility of the strategies $\mathbf{B}_p(\beta^c)$ and $\mathbf{B}_c(\beta^p)$, i.e. $\mathbf{B}_p(\beta^c) \in \mathcal{A}$ and $\mathbf{B}_c(\beta^p) \in \mathcal{A}$. We need to verify their square-integrability. Let us check it for $\mathbf{B}_p(\beta^c)$, the same can be done for $\mathbf{B}_c(\beta^p)$. The state variable $q = \{q_t\}_{t\in[0,T]} = \{q^{\mathbf{B}_p(\beta^c)}(t)\}_{t\in[0,T]}$ is the solution of a linear SDE and so it satisfies $\mathbb{E}[\sup_{t\in[0,T]} |q_t|^2] < \infty$. Furthermore, S_p, U_p, V_p, defined in (2.4.7), are bounded, being continuous matrix-valued functions over a finite time-interval, and the process (O_p, ξ^p) belongs to $L^2([0,T], \mathbb{R}^2) \times L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}^2)$. This implies that the feedback control $\mathbf{B}_p(\beta^c) \in L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}^2)$. □

2.4.4 Second step: well-posedness of the best response map system

This subsection provides the proof of existence and uniqueness of solutions to the systems in (2.4.10) and (2.4.11),

$$K_p, K_c, \Lambda_p \text{ and } \Lambda_c \in L^\infty([0,T], \mathbb{R}_-), \quad R_p \text{ and } R_c \in L^\infty([0,T], \mathbb{R}),$$

$$(Y^p, Z^{p,W}, Z^{p,B}) \text{ and } (Y^c, Z^{c,W}, Z^{c,B}) \in S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}) \times L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R})^2,$$

given the state controlled by the other player.

The fact that there exist unique $K_p, K_c, \Lambda_p, \Lambda_c \in L^\infty([0,T], \mathbb{R}_+)$ is straightforward (see Remark 2.4.5). We also have explicit formulae for them (see Equation (2.3.1)). Moreover the non-positivity of K_p and K_c implies that the matrices S_p and S_c, defined in (2.4.7), are negative definite.

Now, consider the mean-field BSDE associated to the processes $(Y^p, Z^{p,W}, Z^{p,B})$, given K_p and Λ_p :

$$\begin{cases} \mathrm{d}Y^p_t = -\left\{\frac{s_0}{2} + \frac{\gamma\rho_c}{2} c_t + \rho_p \gamma \rho_c \eta_p \lambda^2 (c_t - \mathbb{E}[c_t]) + \frac{2}{k_p}\left(K_p(t)(Y^p_t - \mathbb{E}[Y^p_t]) + \Lambda_p(t)\mathbb{E}[Y^p_t]\right)\right\} \mathrm{d}t \\ \qquad +Z^{p,B}_t \mathrm{d}B_t + Z^{p,W}_t \mathrm{d}W_t, \\ Y^p_T = \frac{\lambda\rho_p}{2}. \end{cases}$$

Existence and uniqueness of the solution $(Y^p, Z^{p,W}, Z^{p,B}) \in S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}) \times L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R})^2$ is a consequence of [117, Theorem 2.1] and the fact that $c \in S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R})$ by the admissibility of the associated control β^c.

Finally, given $(K_p, \Lambda_p, (Y^p, Z^{p,W}, Z^{p,B}))$, the linear ODE associated to R_p in system (2.4.10) has a unique solution given by

$$R_p(t) = -\lambda\gamma\rho_c\mathbb{E}[c_T] + \int_t^T \Bigg[-\eta_p\lambda^2\gamma^2\rho_c^2\mathbb{V}[c_u] + \frac{2}{k_p}\left(\mathbb{V}[Y^p_u] + \mathbb{E}[Y^p_u]^2\right) \\ + \frac{2}{\ell_p - 2K_p(u)}\left(\mathbb{V}[Z^{p,W}_u] + \left(\mathbb{E}[Z^{p,W}_u] + \frac{\ell_p\sigma_p}{2}\right)^2\right)\Bigg] \mathrm{d}u.$$

The same arguments are used to prove existence and uniqueness for the processes $(Y^c, Z^{c,W}, Z^{c,B})$ in $S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}) \times L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R})^2$ and $R_c \in L^\infty([0,T], \mathbb{R})$. This ends the proof of existence and uniqueness for systems (2.4.10) and (2.4.11).

2.4.5 Third step: fixed point of the best response map

Here, we prove the existence of a fixed point of the best response maps in order to get a Nash equilibrium. First of all, for convenience of notation, we rewrite the two-dimensional state variable as $X_t := (q_t, c_t)^\top$, for all $t \in [0,T]$, and so its linear dynamics is given by the following SDE

$$\mathrm{d}X_t = \begin{pmatrix} \mathrm{d}q_t \\ \mathrm{d}c_t \end{pmatrix} = \begin{pmatrix} u_t \\ v_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} z_t \\ 0 \end{pmatrix} \mathrm{d}W_t + \begin{pmatrix} 0 \\ y_t \end{pmatrix} \mathrm{d}B_t,$$

with a deterministic initial condition $X_0 = (q_0, c_0)^\top \in \mathbb{R}^2_+$. Then, we have

$$\mathrm{d}X_t = b\alpha_t \mathrm{d}t + \sigma^W \alpha_t \mathrm{d}W_t + \sigma^B \alpha_t \mathrm{d}B_t,$$

with

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma^W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we rewrite explicitly the form that a candidate equilibrium feedback control $\alpha^* = ((\alpha^{*,P})^\top, (\alpha^{*,C})^\top)^\top$ should have, together with the backward dynamics of the corresponding process $Y = ((Y^p)^\top, (Y^c)^\top)^\top$ (we write Z^W for $((Z^{p,W})^\top, (Z^{c,W})^\top)^\top$, respectively Z^B for $((Z^{p,B})^\top, (Z^{c,B})^\top)^\top$),[1]

$$\begin{aligned} \alpha^*_t - \bar{\alpha}^*_t &= \Delta(t)\left(X_t - \bar{X}_t\right) + \Gamma\left(Y_t - \bar{Y}_t\right) + H^W(t)\left(Z^W_t - \bar{Z}^W_t\right) + H^B(t)\left(Z^B_t - \bar{Z}^B_t\right), \\ \bar{\alpha}^*_t &= \widehat{\Delta}(t)\bar{X}_t + \Gamma\bar{Y}_t + H^W(t)\bar{Z}^W_t + H^B(t)\bar{Z}^B_t + \Theta(t), \end{aligned} \tag{2.4.12}$$

$$\mathrm{d}Y_t = \left[\Xi\left(X_t - \bar{X}_t\right) + \Phi(t)\left(Y_t - \bar{Y}_t\right)\right]\mathrm{d}t + \left[\widehat{\Xi}\bar{X}_t + \widehat{\Phi}(t)\bar{Y}_t + \Psi\right]\mathrm{d}t + Z^B_t \mathrm{d}B_t + Z^W_t \mathrm{d}W_t, \tag{2.4.13}$$

with

$$\Delta(t) = \begin{pmatrix} \frac{2}{k_p}K_p(t) & 0 \\ 0 & 0 \\ 0 & \frac{2}{k_c}K_c(t) \\ 0 & 0 \end{pmatrix}, \quad \widehat{\Delta}(t) = \begin{pmatrix} \frac{2}{k_p}\Lambda_p(t) & 0 \\ 0 & 0 \\ 0 & \frac{2}{k_c}\Lambda_c(t) \\ 0 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \frac{2}{k_p} & 0 \\ 0 & 0 \\ 0 & \frac{2}{k_c} \\ 0 & 0 \end{pmatrix},$$

$$\Theta(t) = \begin{pmatrix} 0 \\ \sigma_p(1 - 2\frac{K_p(t)}{\ell_p})^{-1} \\ 0 \\ \sigma_c(1 - 2\frac{K_c(t)}{\ell_c})^{-1} \end{pmatrix}, \quad H^W(t) = \begin{pmatrix} 0 & 0 \\ \frac{2}{\ell_p - 2K_p(t)} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^B(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{2}{\ell_c - 2K_c(t)} \end{pmatrix},$$

[1] Here, we have omitted, in all the processes but α, the superscript $*$ in order to have a simpler notation.

and Ξ, $\widehat{\Xi}$, $\Phi(t)$, $\widehat{\Phi}(t)$ and Ψ as defined at the beginning of Section 2.3.1.

Now, as an ansatz for Y, we assume Y linear in the state:

$$Y_t = \pi(t)(X_t - \bar{X}_t) + \widehat{\pi}(t)\bar{X}_t + \zeta_t, \tag{2.4.14}$$

with $\pi, \widehat{\pi}$ deterministic $\mathbb{R}^{2\times 2}$-valued processes and $\zeta \in S^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}^2)$ satisfying the SDE

$$d\zeta_t = \psi_t dt + \phi^W_t dW_t + \phi^B_t dB_t, \quad \zeta_T = \frac{1}{2}\lambda(\rho_p, \gamma\rho_c)^\top, \tag{2.4.15}$$

for some ψ, ϕ^B, ϕ^W in suitable spaces. The affine term in the expression (2.4.14) allows Y to have some extra stochasticity apart from the linear dependency on the state. Furthermore, the terminal condition in (2.4.15) guarantees that Y satisfies its terminal condition.
An application of Itô's formula to the ansatz (2.4.14) yields

$$\begin{aligned} dY_t =& [\pi'(t)(X_t - \bar{X}_t) + \pi(t)b(\alpha^*_t - \bar{\alpha}^*_t) + \psi_t - \bar{\psi}_t]dt + (\widehat{\pi}'(t)\bar{X}_t + \widehat{\pi}(t)b\bar{\alpha}^*_t + \bar{\psi}_t)dt \\ &+ (\pi(t)\sigma^W\alpha^*_t + \phi^W_t)dW_t + (\pi(t)\sigma^B\alpha^*_t + \phi^B_t)dB_t. \end{aligned} \tag{2.4.16}$$

If we match the two dynamics of Y in Equations (2.4.13) and (2.4.16), and then replace Y with its ansatz (2.4.14) and α^* with its feedback form (2.4.12), we get the following system of equations:

$$\begin{cases} \pi'(t)(X_t - \bar{X}_t) + \pi(t)b(\mathbf{I} - H^W(t)\pi(t)\sigma^W - H^B(t)\pi(t)\sigma^B)^{-1}[(\Delta(t) \\ \quad +\Gamma\pi(t))(X_t - \bar{X}_t) + \Gamma(\zeta_t - \bar{\zeta}_t) + H^W(t)(\phi^W_t - \bar{\phi}^W_t) + H^B(t)(\phi^B_t - \bar{\phi}^B_t)] \\ \quad +\psi_t - \bar{\psi}_t = \Xi(X_t - \bar{X}_t) + \Phi(t)(\pi(t)(X_t - \bar{X}_t) + \zeta_t - \bar{\zeta}_t) \\ \\ \widehat{\pi}'(t)\bar{X}_t + \widehat{\pi}(t)b(\mathbf{I} - H^W(t)\pi(t)\sigma^W - H^B(t)\pi(t)\sigma^B)^{-1}[(\widehat{\Delta}(t) + \Gamma\widehat{\pi}(t))\bar{X}_t + \Gamma\bar{\zeta}_t \\ \quad +\Theta(t) + H^W(t)\bar{\phi}^W_t + H^B(t)\bar{\phi}^B_t] + \bar{\psi}_t = \widehat{\Xi}\bar{X}_t + \widehat{\Phi}(t)(\widehat{\pi}(t)\bar{X}_t + \bar{\zeta}_t) + \Psi \\ \\ Z^B_t = \pi(t)\sigma^W\alpha^*_t + \phi^W_t \\ Z^W_t = \pi(t)\sigma^B\alpha^*_t + \phi^B_t. \end{cases} \tag{2.4.17}$$

Finally, exploiting the fact that:

$$b\left(\mathbf{I} - H^W(t)\pi(t)\sigma^W - H^B(t)\pi(t)\sigma^B\right)^{-1} = b,$$

we find the equations that the coefficients $(\pi, \widehat{\pi}, \psi, \phi^W, \phi^B)$ in the ansatz for Y should solve in order to provide a fixed point of the best response map:

$$\begin{cases} \pi'(t) = \Xi + \Phi(t)\pi(t) + \pi(t)\Phi(t) + \pi(t)R\pi(t), \quad \pi(T) = 0, \\ \widehat{\pi}'(t) = \widehat{\Xi} + \widehat{\Phi}(t)\widehat{\pi}(t) + \widehat{\pi}(t)\widehat{\Phi}(t) + \widehat{\pi}(t)R\widehat{\pi}(t), \quad \widehat{\pi}(T) = 0, \\ d\zeta_t = \psi_t dt + \phi^W_t dW_t + \phi^B_t dB_t, \quad \zeta_T = \frac{1}{2}\lambda(\rho_p, \gamma\rho_c)^\top, \\ \psi_t = \psi_t - \bar{\psi}_t + \bar{\psi}_t = \Big(\pi(t)R + \Phi(t)\Big)(\zeta_t - \bar{\zeta}_t) + \Big(\widehat{\pi}(t)R + \widehat{\Phi}(t)\Big)\bar{\zeta}_t + \Psi, \end{cases} \tag{2.4.18}$$

where $R = \begin{pmatrix} -2/k_p & 0 \\ 0 & -2/k_c \end{pmatrix}$. In fact, inserting (Y, Z) from the ansatz and Equation (2.4.17) into the best response given by Equations (2.4.12) provides an equilibrium strategy α^* in feedback form which is computed in details in the next step.

Remark 2.4.6. *To obtain explicit expressions for α^* and Z, we have used Assumption **(A2)** in Theorem 2.3.1. Indeed, such a condition is needed for the invertibility of the matrices $D(t) := (\mathbf{I} - H^W(t)\pi(t)\sigma^W - H^B(t)\pi(t)\sigma^B)$, $t \in [0, T]$, that appear in*

$$Z_t^W = \phi_t^W + \pi(t)\sigma^W \alpha_t^*, \qquad Z_t^B = \phi_t^B + \pi(t)\sigma^B \alpha_t^*,$$

where

$$\alpha_t^* = D(t)^{-1}[(\Delta(t) + \Gamma\pi(t))(X_t - \bar{X}_t) + (\widehat{\Delta}(t) + \Gamma\widehat{\pi}(t))\bar{X}_t + \Gamma\zeta_t + H^W(t)\phi_t^W + H^B(t)\phi_t^B + \Theta(t)].$$

2.4.6 Fourth step: Nash equilibrium strategies

In order to complete the proof of the main theorem, we are left with showing that the system (2.4.18) has a unique solution over the finite time interval $[0, T]$. The equations associated to $t \mapsto (\pi(t), \widehat{\pi}(t))$ are non-symmetric matrix Riccati equations for which there is no general condition ensuring the global existence of solutions. Nevertheless, the regularity of the coefficients and the Picard-Lindelöf Theorem ensure the local existence and uniqueness of solutions over a compact interval $[0, T_{max}]$[2]. Thus, we recover the existence and uniqueness condition in Assumption **(A1)** of Theorem 2.3.1 choosing a time horizon T small enough, namely $T < T_{max}$. Then, for a given $(\pi, \widehat{\pi})$, the process $(\zeta, \psi, \phi^W, \phi^B)$ evolves according to the following linear mean field BSDE:

$$\begin{aligned} &\mathrm{d}\zeta_t = \psi_t \mathrm{d}t + \phi_t^W \mathrm{d}W_t + \phi_t^B \mathrm{d}B_t, \quad \zeta_T = \frac{1}{2}\lambda(\rho_p, \gamma\rho_c)^\top, \\ &\psi_t = \psi_t - \bar{\psi}_t + \bar{\psi}_t = (\pi(t)R + \Phi(t))(\zeta_t - \bar{\zeta}_t) + (\widehat{\pi}(t)R + \widehat{\Phi}(t))\bar{\zeta}_t + \Psi. \end{aligned} \tag{2.4.19}$$

Exploiting once more [117, Theorem 2.1], we have a unique solution $(\zeta, \phi^W, \phi^B) \in S^2_{\mathbb{F}}(\Omega \times [0, T], \mathbb{R}^2) \times L^2_{\mathbb{F}}(\Omega \times [0, T], \mathbb{R}^2)^2$. Furthermore, we notice that the drift ψ in the system (2.4.19) does not depend on ϕ^W and ϕ^B and all the coefficients involved in the second line of (2.4.19) are deterministic. Moreover, the terminal condition is also deterministic. Thus, the unique solution (ζ, ϕ^W, ϕ^B) to this system is given by $(h, 0, 0)$, where $h : [0, T] \to \mathbb{R}^2$ is the unique (deterministic) solution to the following backward linear ODE:

$$\begin{cases} \mathrm{d}h(t) = \left\{ \left[\widehat{\pi}(t)R + \widehat{\Phi}(t) \right] h(t) + \Psi \right\} \mathrm{d}t, \\ h(T) = \frac{1}{2}\lambda(\rho_p, \gamma\rho_c)^\top. \end{cases}$$

[2]Despite it is not possible in general to obtain an explicit characterisation of T_{max}, we notice that we did not observe any explosion for all typical values of the parameters we have considered in the numerical experiments (ref. Section 2.5).

So, the system of ODEs and SDEs in (2.4.18) reduces to the one made up of Equations (2.3.2) and (2.3.3).

We write the Nash equilibrium strategies $\alpha^* = ((\alpha^{*,P}), (\alpha^{*,C}))^\top = ((u^*, z^*)^\top, (v^*, y^*)^\top)^\top$ explicitly as

$$\alpha_t^* = D(t)^{-1}(\Delta(t) + \Gamma\pi(t))(X_t - \bar{X}_t) + D(t)^{-1}(\widehat{\Delta}(t) + \Gamma\widehat{\pi}(t))\bar{X}_t + D(t)^{-1}(\Gamma h(t) + \Theta(t)),$$

that is

$$\begin{aligned}
u_t^* &= \frac{2}{k_p}\left[(K_p(t) + \pi_{11}(t))(q_t - \bar{q}_t) + \pi_{12}(t)(c_t - \bar{c}_t) + (\Lambda_p(t) + \widehat{\pi}_{11}(t))\bar{q}_t + \widehat{\pi}_{12}(t)\bar{c}_t + h_1(t)\right], \\
z^*(t) &= \frac{\sigma_p \ell_p}{\ell_p - 2(K_p(t) + \pi_{11}(t))}, \\
v_t^* &= \frac{2}{k_c}\left[(K_c(t) + \pi_{22}(t))(c_t - \bar{c}_t) + \pi_{21}(t)(q_t - \bar{q}_t) + (\Lambda_c(t) + \widehat{\pi}_{22}(t))\bar{c}_t + \widehat{\pi}_{21}(t)\bar{q}_t + h_2(t)\right], \\
y^*(t) &= \frac{\sigma_c \ell_c}{\ell_c - 2(K_c(t) + \pi_{22}(t))},
\end{aligned}$$

where $K_p, K_c, \Lambda_p, \Lambda_c$ are defined in (2.3.1) and $\pi, \widehat{\pi}$ and h are respectively the solutions to the systems (2.3.2), (2.3.3).

Finally, we derive the corresponding equilibrium dynamics for the state

$$\begin{aligned}
\mathrm{d}X_t = &\left\{ \begin{pmatrix} \frac{2}{k_p}(K_p(t) + \pi_{11}(t)) & \frac{2}{k_p}\pi_{12}(t) \\ \frac{2}{k_c}\pi_{21}(t) & \frac{2}{k_c}(K_c(t) + \pi_{22}(t)) \end{pmatrix} (X_t - \bar{X}_t) \right. \\
&\left. + \begin{pmatrix} \frac{2}{k_p}(\Lambda_p(t) + \widehat{\pi}_{11}(t)) & \frac{2}{k_p}\widehat{\pi}_{12}(t) \\ \frac{2}{k_c}\widehat{\pi}_{21}(t) & \frac{2}{k_c}(\Lambda_c(t) + \widehat{\pi}_{22}(t)) \end{pmatrix} \bar{X}_t + \begin{pmatrix} \frac{2}{k_p}h_1(t) \\ \frac{2}{k_c}h_2(t) \end{pmatrix} \right\} \mathrm{d}t \\
&+ \begin{pmatrix} \frac{\sigma_p \ell_p}{\ell_p - 2(K_p(t) + \pi_{11}(t))} \\ 0 \end{pmatrix} \mathrm{d}W_t + \begin{pmatrix} 0 \\ \frac{\sigma_c \ell_c}{\ell_c - 2(K_c(t) + \pi_{22}(t))} \end{pmatrix} \mathrm{d}B_t, \quad t \in [0, T],
\end{aligned}$$

which is a linear mean-field SDE, hence admitting a unique solution.

2.5 Numerics

We consider the following parameters setting $T = 1$, $k_p = k_c = 5$, $\sigma_p = \sigma_c = 10$, $q_0 = c_0 = 100$, $s_0 = 50$, $\rho_p = \gamma\rho_c = 0.5$ and $\gamma = 1.2$, $\delta = 5$, $p_0 = 2s_0 + \gamma\delta$, and $p_1 = \gamma - 1$. With this parametrisation, the players are symmetric in the sense that they have the same absolute effect on the price and they share the same costs of average production rate or consumption rate. Moreover, if they shared the same risk aversion parameters ($\eta_p = \eta_c$) and the same costs of volatility control ($\ell_p = \ell_c$), then the strategies of the producer (u^*, z^*) and of the consumer (v^*, y^*) would be identical. The initial conditions have been chosen to be close to a long-run stationary equilibrium that we observe when we take large T, which allows avoiding potential transitory effects.

In the next sub-sections, we illustrate first the effect of the risk aversion parameters on the forward agreement indifference price when every other parameter is fixed. Second, we show how different combinations of risk aversions and volatility control costs can lead to the same forward agreement indifference price and volume.

2.5.1 The effect of risk aversion

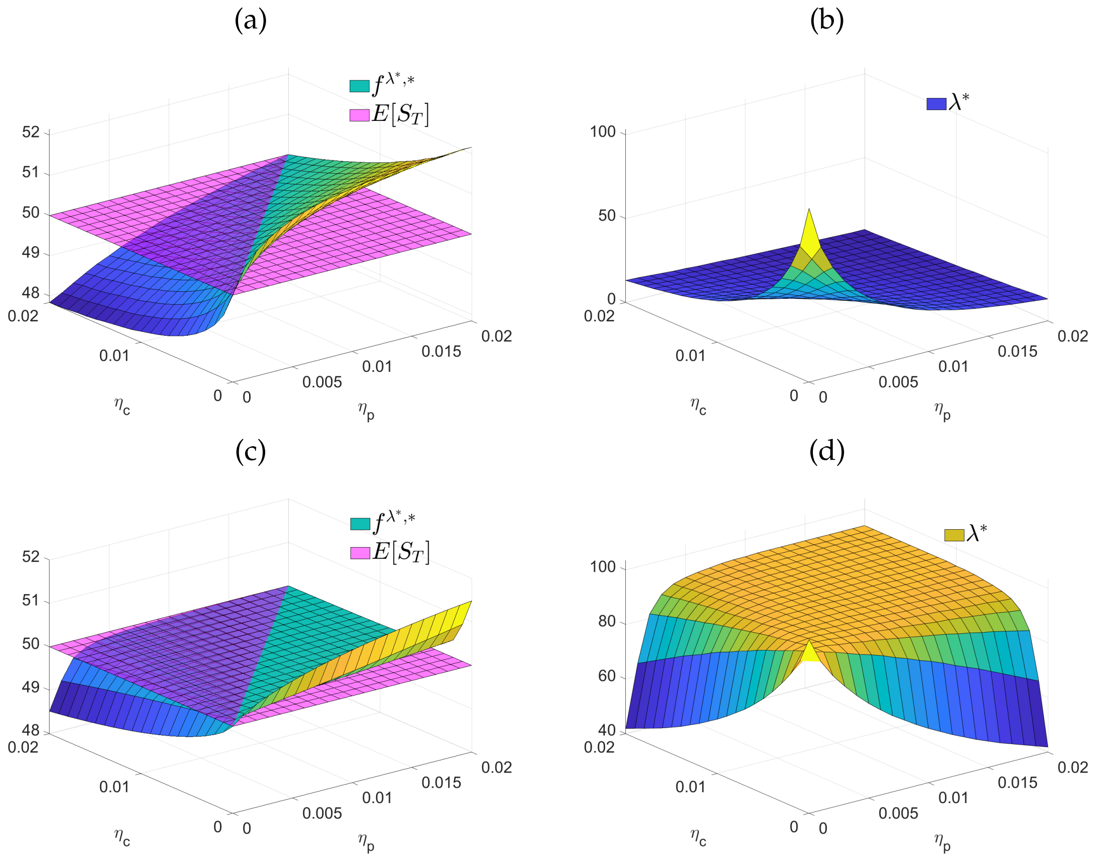

Figure 2.5.1: (a) and (b) $\ell_p = \ell_c = 5$, (c) and (d) $\ell_p = \ell_c = 0.7$.

Figure 2.5.1 presents the unitary forward agreement indifference price $f^{\lambda^*,*} := F^*_{\lambda^*}/\lambda^*$ and the volume that the players agreed upon when the costs of volatility control are high (Figure 2.5.1 (a) and (b)) and when they are low (Figure 2.5.1 (c) and (d)). We find that $f^{\lambda^*,*}$ is higher (resp. lower) than the expected spot price when the producer is more (resp. less) risk-averse than the consumer, which is consistent with both the economic intuition and the hedging pressure theory, once recalled that in our model players act as speculators on the forward market. In hedging pressure theory (see [45] and [52]), the risk premium is determined by the relations between risk aversions of producers, consumers, storers and speculators. It extends Keynes's normal backwardation theory which claims that in commodity markets, the forward price should be lower than the expected spot price because the producer would be ready to pay a premium to avoid being exposed to price risk on his production. In our case, the most risk-averse speculator obtains the appropriate premium to enter into the agreement. This property holds whatever the level of volatility control costs. We see on Figure 2.5.1 that the producer is requiring a positive premium to accept the risk coming from his financial position. Regarding the exchanged volume, we observe that it can be both non-increasing or non-decreasing in the risk aversion parameters of the players, depending on the costs of volatility control. When the volatility manipulation costs are high for both players, there is a low trading volume even when both players have a high risk aversion. On the other side, when the volatility manipulation costs are low, there is a low trading volume when only one of the player has a high risk aversion but the trading volume is huge when both players have a high risk aversion. This could be explained by the fact that in the latter case the players can act on their volatilities (almost costlessly) to stabilise the spot price and hence they would be willing to trade more.

2.5.2 Joint effect of risk aversion and volatility control cost

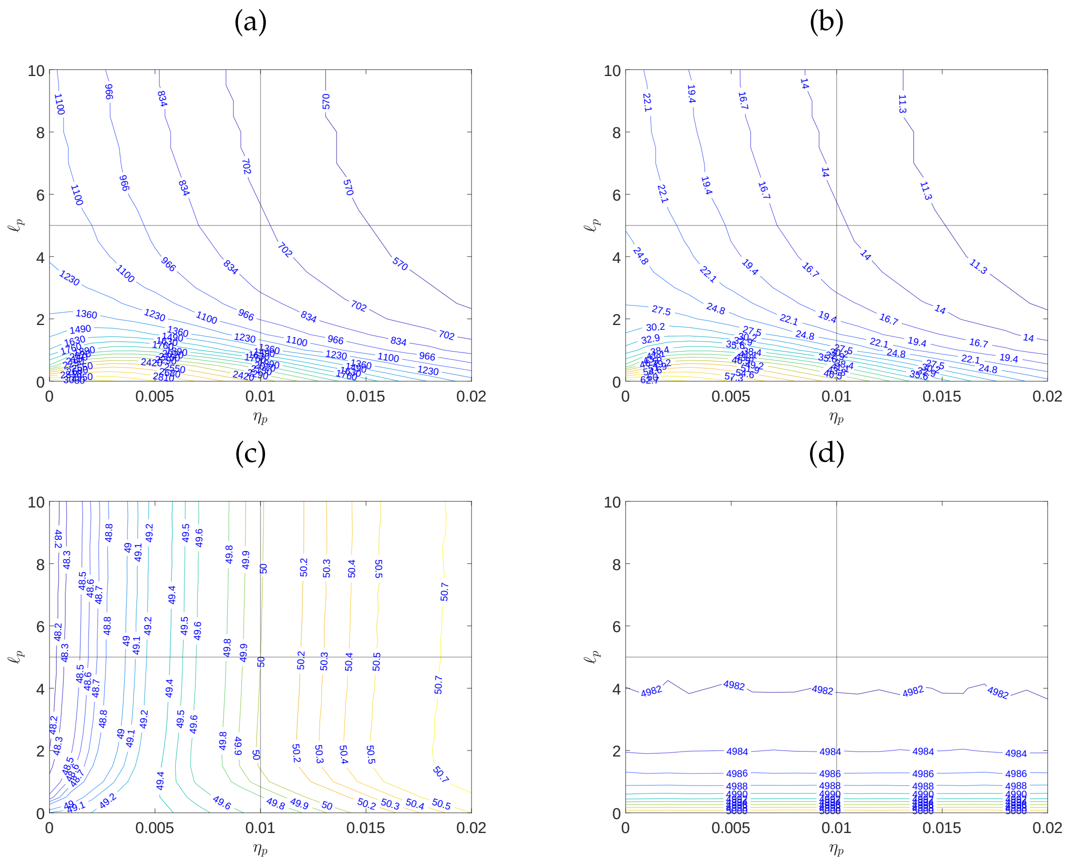

Figure 2.5.2: Level lines of (a) the forward agreement price $F^*_{\lambda^*}$, (b) the traded quantity λ^*, (c) the per unit agreement price $f^{\lambda^*,*} = F^*_{\lambda^*}/\lambda^*$, (d) the value of the producer's equilibrium payoff $J^*_p(\lambda^*, F^*_{\lambda^*})$.

We freeze now the risk aversion parameter and the cost for controlling the volatility of the consumer at $\eta_c = 0.01$ and $\ell_c = 5$, and observe the agreement price, the traded volume, the per unit agreement indifference price and the equilibrium payoff at the agreement of the producer. Results are provided in Figure 2.5.2, when the producer's risk aversion parameter η_p and his volatility manipulation cost ℓ_p vary. The vertical and horizontal lines in each graph are set to the values of ℓ_c and η_c.

We observe a sort of "substitution effect" between η_p and ℓ_p in the sense that for a producer with a given combination of risk aversion and volatility control cost, we can find another producer *trading at the same agreement price* with a higher risk aversion and a low volatility control cost (Figure 2.5.2 (a)). We observe that this phenomenon occurs also for the traded quantity (Figure 2.5.2 (b)). This substitution makes sense in our model where volatility represents a cost for the producer that can be mitigated either by requiring a payment to bear this volatility or by paying the cost to reduce it. We note that for a fixed value of η_p, the lower the value of ℓ_p, the larger the forward agreement price *and* the traded volume. The Figure 2.5.2 (c) gives the resulting unitary agreement forward price. The volatility control cost has little effect on the per unit forward price compared to the risk aversion parameter. This figure is a way of showing that when the volatility control costs are high, the producer has little alternative than asking for a premium to enter in forward agreement, and thus, the price is basically determined by his risk-aversion.

To conclude, we note that the producer's equilibrium payoff is independent of the value of η_p (Figure 2.5.2 (d)) because, by definition of the agreement forward price, it is always equal to $J_p^*(0,0)$, which is independent of η_p.

Appendix

2.A Proof of Theorem 2.4.2 (Verification Theorem)

For any $\alpha^p \in \mathcal{A}$ (resp. for any $\alpha^c \in \mathcal{A}$), the map $[0,T] \ni t \mapsto \mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]\Big(\text{resp. } \mathbb{E}[\mathcal{S}_t^{c,\alpha^c}]\Big)$ is well-defined (it does not explode in finite time), because of the condition (2.4.4) and the linear structure of the SDEs for the state variables (2.4.1).
Assumptions i) and ii) yields that: for any $\alpha^p \in \mathcal{A}^p$,

$$\mathbb{E}[w_0^p(q_0,\bar{q}_0)] = \mathbb{E}[\mathcal{S}_0^{p,\alpha^p}] \overset{\text{ii)}}{\geq} \mathbb{E}[\mathcal{S}_T^{p,\alpha^p}] = \mathbb{E}\left[\mathcal{W}_T^{p,\alpha^p} + \int_0^T f_p(s, q_s^{\alpha^p}, \mathbb{E}[q_s^{\alpha^p}], \alpha_s^p, \mathbb{E}[\alpha_s^p]; c^{\beta^c})\mathrm{d}s\right]$$
$$\overset{\text{i)}}{=} \mathbb{E}\left[g_p(q_T^{\alpha^p}, \mathbb{E}[q_T^{\alpha^p}]; c^{\beta^c}) + \int_0^T f_p(s, q_s^{\alpha^p}, \mathbb{E}[q_s^{\alpha^p}], \alpha_s^p, \mathbb{E}[\alpha_s^p]; c^{\beta^c})\mathrm{d}s\right] = \widetilde{J}_p^\lambda(\alpha^p; c^{\beta^c})$$
$$= \widetilde{J}_p^\lambda(\alpha^p; \beta^c).$$

Then, the arbitrariness of $\alpha^p \in \mathcal{A}$ implies that $\mathbb{E}[w_0^p(q_0,\bar{q}_0)] \geq \sup_{\alpha^p\in\mathcal{A}} \widetilde{J}_p^\lambda(\alpha^p; c^{\beta^c}) = V_p^\lambda(\beta^c)$.

Performing the same computations with $\alpha^{p,\star}$ instead of α^p, by condition iii), we get: $\mathbb{E}[w_0^p(q_0,\bar{q}_0)] = \widetilde{J}_p^\lambda(\alpha^{p,\star}; \beta^c)$. Then, we have showed that $\alpha^{p,\star} = \mathbf{B}_p(\beta^c)$ is the best response to β^c. The fact that $\alpha^{c,\star} = \mathbf{B}_c(\beta^p)$ is the best response to β^p is proved analogously.

Now, take $\widetilde{\alpha}^p \in \mathcal{A}$ to be another best response to β^c. We have

$$\mathbb{E}[\mathcal{S}_0^{P,\widetilde{\alpha}^p}] = \mathbb{E}[w_0^p(q_0,\bar{q}_0)] = V_p^\lambda(\beta^c) = \widetilde{J}_p^\lambda(\widetilde{\alpha}^p, \beta^c) = \mathbb{E}[\mathcal{S}_T^{P,\widetilde{\alpha}^p}].$$

Then, we conclude that the map $[0,T] \ni t \mapsto \mathbb{E}[\mathcal{S}_t^{P,\widetilde{\alpha}^p}]$ is constant, since it is non-increasing and it takes the same value at its extremal points. This reasoning, with a few modifications, can be replicated for $\widetilde{\alpha}^c$, hence concluding the proof.

2.B Computations of the best response maps

As we have done in Section 2.4.3 (Sub-step 1.2), we develop here only the computations for the best response of the producer. The best response of the consumer is obtained following very similar computations. In this section we show that, setting $w_t^p(q,\bar{q}) = K_p(t)(q-\bar{q})^2 + \Lambda_p(t)\bar{q}^2 + 2Y_t^p q + R_p(t)$, with $(K_p, \Lambda_p, Y^p, R_p) \in L^\infty([0,T],\mathbb{R}_-)^2 \times S_{\mathbb{F}}^2(\Omega\times[0,T],\mathbb{R}) \times L^\infty([0,T],\mathbb{R})$, once $\mathcal{S}^{p,\alpha^p}$ is defined as in the Verification Theorem in Theorem 2.4.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}] = \mathbb{E}\Big[(K_p'(t)+Q_p)(q_t-\mathbb{E}[q_t])^2 + (\Lambda_p'(t)+Q_p+\widetilde{Q}_p)\mathbb{E}[q_t]^2 + 2(Y_t^{p\prime} + M_t^{p,c})q_t$$
$$+ R_p'(t) + T_t^{p,c} + \chi_t^p(\alpha_t^p)\Big],$$

where, for all $t \in [0, T]$, we have set

$$\begin{cases} \chi_t^p(\alpha^p(t)) := (\alpha_t^p)^\top S_p(t)\alpha_t^p \\ \qquad +2[U_p(t)(q_t - \mathbb{E}[q_t]) + V_p(t)q_t + \xi_t^p + \bar{\xi}_t^p + O_p(t)]^\top \alpha_t^p \\ S_p(t) := N_p + e_2 K_p(t) e_2^\top \\ U_p(t) := K_p(t)e_1 \\ V_p(t) := \Lambda_p(t)e_1 \\ O_p(t) := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}] \\ \xi_t^p := H_p + e_1 Y_t^p + e_2 Z_t^{p,W} \\ \bar{\xi}_t^p := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}]. \end{cases}$$

First of all, we notice

$$\frac{\mathrm{d}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]}{\mathrm{d}t} = \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[w_t^p(q_t^{\alpha^p}, \mathbb{E}[q_t^{\alpha^p}])] + f_P(t, q_t^{\alpha^p}, \mathbb{E}[q_t^{\alpha^p}], \alpha_t^p, \mathbb{E}[\alpha_t^p]; c^{\beta^c})\right].$$

The dynamics of the state variable controlled by the producer is rewritten as

$$\begin{aligned} \mathrm{d}\bar{q}_t^{\alpha^p} &= e_1^\top \bar{\alpha}_t^p \mathrm{d}t, \\ \mathrm{d}(q_t^{\alpha^p} - \bar{q}_t^{\alpha^p}) &= e_1^\top(\alpha_t^p - \bar{\alpha}_t^p)\mathrm{d}t + e_2^\top \alpha_t^p \mathrm{d}W_t, \end{aligned}$$

From now on, we write q_t for $q_t^{\alpha^p}$ to simplify the notation. Applying Itô's formula to $w_t^p(q_t, \mathbb{E}[q_t])$, we get

$$\begin{aligned} \mathrm{d}w_t^p(q_t, \mathbb{E}[q_t]) &= K_p'(t)(q_t - \bar{q}_t)^2\mathrm{d}t + K_p(t)[2(q_t - \bar{q}_t)\mathrm{d}(q_t - \bar{q}_t) + (e_2^\top\alpha_t^p)^2\mathrm{d}t] + \Lambda_p'(t)(\bar{q}_t)^2\mathrm{d}t \\ &\quad + 2\Lambda_p(t)\bar{q}_t d\bar{q}_t + 2q_t\mathrm{d}Y_t^p + 2Y_t^p\mathrm{d}q_t + Z_t^{p,W} e_2^\top\alpha_t^p\mathrm{d}t + R_p'(t)\mathrm{d}t \\ &= K_p'(t)(q_t - \bar{q}_t)^2\mathrm{d}t + K_p(t)\{2(q_t - \bar{q}_t)[e_1^\top(\alpha_t^p - \bar{\alpha}_t^p)\mathrm{d}t + e_2^\top\alpha_t^p\mathrm{d}W_t] + (e_2^\top\alpha_t^p)^2\mathrm{d}t\} \\ &\quad + \Lambda_p'(t)(\bar{q}_t)^2\mathrm{d}t + 2\Lambda_p(t)\bar{q}_t e_1^\top\bar{\alpha}_t^p\mathrm{d}t + 2q_t(Y_t^{p\prime}\mathrm{d}t + Z_t^{p,W}\mathrm{d}W_t + Z_p^B\mathrm{d}B_t) \\ &\quad + 2Y_t^p(e_1^\top\alpha_t^p\mathrm{d}t + e_2^\top\alpha_t^p\mathrm{d}W_t) + Z_t^{p,W}e_2^\top\alpha_t^p\mathrm{d}t + R_p'(t)\mathrm{d}t \\ &= [K_p'(t)(q_t - \bar{q}_t)^2 + 2K_p(t)(q_t - \bar{q}_t)e_1^\top(\alpha_t^p - \bar{\alpha}_t^p) + K_p(t)(e_2^\top\alpha_t^p)^2 + \Lambda_p'(t)(\bar{q}_t)^2 \\ &\quad + 2\Lambda_p(t)\bar{q}_t e_1^\top\bar{\alpha}_t^p + 2Y_t^{p\prime}q_t + 2Y_t^p e_1^\top\alpha_t^p + 2Z_t^{p,W}e_2^\top\alpha_t^p + R_p'(t)]\mathrm{d}t \\ &\quad + 2[K_p(t)(q_t - \bar{q}_t)e_2^\top\alpha_t^p + Z_t^{p,W} + Y_t^p e_2^\top\alpha_t^p]\mathrm{d}W_t + 2Z_t^{p,B}\mathrm{d}B_t \end{aligned}$$

Then, taking its expected value, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[w_t^p(q_t, \bar{q}_t)] = \frac{\mathbb{E}[\mathrm{d}w_t^p(q_t, \mathbb{E}[q_t])]}{\mathrm{d}t} &= \mathbb{E}\Big[K_p'(t)(q_t - \bar{q}_t)^2 + 2K_p(t)(q_t - \bar{q}_t)e_1^\top(\alpha_t^p - \bar{\alpha}_t^p) \\ &\quad + K_p(t)(e_2^\top\alpha_t^p)^2 + \Lambda_p'(t)(\bar{q}_t)^2 + 2\Lambda_p(t)\bar{q}_t e_1^\top\bar{\alpha}_t^p + 2Y_t^{p\prime}q_t + 2Y_t^p e_1^\top\alpha_t^p \\ &\quad + R_p'(t) + 2Z_t^{p,W}e_2^\top\alpha_t^p\Big] \qquad (2.B.1) \\ &= \mathbb{E}\Big[K_p'(t)(q_t - \bar{q}_t)^2 + 2K_p(t)(q_t - \bar{q}_t)e_1^\top\alpha_t^p + K_p(t)(e_2^\top\alpha_t^p)^2 + \Lambda_p'(t)(\bar{q}_t)^2 \\ &\quad + 2\Lambda_p(t)\bar{q}_t e_1^\top\alpha_t^p + 2Y_t^{p\prime}q_t + 2Y_t^p e_1^\top\alpha_t^p + R_p'(t) + 2Z_t^{p,W}e_2^\top\alpha_t^p\Big] \end{aligned}$$

$$= \mathbb{E}\Big[K_p'(t)(q_t - \bar{q}_t)^2 + \Lambda_p'(t)(\bar{q}_t)^2 + 2Y_t^{p\prime} q_t + R_p'(t) + K_p(t)(e_2^\top \alpha_t^p)^2 \\ + \Big[2(K_p(t)(q_t - \bar{q}_t) + \Lambda_p(t)\bar{q}_t + Y_t^p)e_1 + 2Z_t^{p,W} e_2\Big]^\top \alpha_t^p\Big],$$

where we have used the following simplifications: $\mathbb{E}[2\Lambda_p(t)e_1^\top \bar{\alpha}_t^p \bar{q}_t] = \mathbb{E}[2\Lambda_p(t)\bar{q}_t e_1^\top \alpha_t^p]$ and $\mathbb{E}[2K_p(t)e_1^\top \bar{\alpha}_t^p (q_t - \bar{q}_t)] = 2K_p(t)e_1^\top \bar{\alpha}_t^p \mathbb{E}[q_t - \bar{q}_t] = 0$. Moreover, since

$$\mathbb{E}[f_p(t, q_t, \bar{q}_t, \alpha_t^p, \bar{\alpha}_t^p; c^{\beta^c})] = \mathbb{E}[Q_p(q_t - \bar{q}_t)^2 + (Q_p + \widetilde{Q}_p)\bar{q}_t^2 + 2M_t^{p,c} q_t + (\alpha_t^p)^\top N_p \alpha_t^p + 2H_p^\top \alpha_t^p + T_t^{p,c}], \tag{2.B.2}$$

by adding up (2.B.1) and (2.B.2), we get

$$\frac{\mathrm{d}\mathbb{E}[\mathcal{S}_t^{p,\alpha^p}]}{\mathrm{d}t} = \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[w_t^p(q_t, \bar{q}_t)] + f_p(q_t, \bar{q}_t, \alpha_t^p, \bar{\alpha}_t^p; c^{\beta^c})\right] \\ = \mathbb{E}\Big[(K_p'(t) + Q_p)(q_t - \bar{q}_t)^2 + (\Lambda_p'(t) + Q_p + \widetilde{Q}_p)(\bar{q}_t)^2 + 2(Y_t^{p\prime} + M^p(c)_t)q_t \\ + R_p'(t) + T^p(c)_t + \chi_t^p(\alpha_t^p)\Big],$$

where we have set

$$\chi_t^p(\alpha_t^p) := K_p(t)(e_2^\top \alpha_t^p)^2 + \Big\{2[K_p(t)(q_t - \bar{q}_t) + \Lambda_p(t)\bar{q}_t + Y_t^p]e_1 + 2Z_t^{p,W} e_2\Big\}^\top \alpha_t^p \\ + (\alpha_t^p)^\top N_p \alpha_t^p + 2H_p^\top \alpha_t^p \\ = \Big\{2[K_p(t)(q_t - \bar{q}_t) + \Lambda_p(t)\bar{q}_t + Y_t^p]\ell_p + 2Z_t^{p,W} e_2 + 2H_p\Big\}^\top \alpha_t^p \\ + (\alpha_t^p)^\top (N_p + e_2 K_p(t) e_2^\top)\alpha_t^p \\ = 2[U_p(t)(q_t - \mathbb{E}[q_t]) + V_p(t)q_t + \xi_t^p + \bar{\xi}_t^p + O_p(t)]^\top \alpha_t^p \\ + (\alpha_t^p)^\top S_p(t)\alpha_t^p,$$

with

$$\begin{cases} S_p(t) := N_p + e_2 K_p(t) e_2^\top \\ U_p(t) := K_p(t)e_1 \\ V_p(t) := \Lambda_p(t)e_1 \\ O_p(t) := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}] \\ \xi_t^p := H_p + e_1 Y_t^p + e_2 Z_t^{p,W} \\ \bar{\xi}_t^p := H_p + e_1\mathbb{E}[Y_t^p] + e_2\mathbb{E}[Z_t^{p,W}]. \end{cases}$$

2.C Computations of the equilibrium payoffs

In this section we perform some computations to get a more explicit formula for the objective functionals at the equilibrium in Theorem 2.3.1. In particular, we find explicit expressions for $R_p(0)$ and $R_c(0)$. In all the following computations we are using the optimal strategies but we are suppressing the stars in the notation for the sake readability (e.g. we write u_t instead of u_t^* and so on). For the same reason we are suppressing the dependency on time when clear from the context.

Proposition 2.C.1. *It holds that*

$$R_p^{(\lambda)}(0) = \int_0^T \left[\frac{2}{k_p}\mathbb{E}[(Y_u^p)^2] - \eta_p\lambda^2\gamma^2\rho_c^2\mathbb{V}[c_u] + \frac{2\big(\pi_{11}(u)z_u + \frac{\ell_p\sigma_p}{2}\big)^2}{\ell_p - 2K_p(u)} \right] \mathrm{d}u - \lambda\gamma\rho_c\bar{c}_T,$$

$$R_c^{(\lambda)}(0) = \int_0^T \left[\frac{2}{k_c}\mathbb{E}[(Y_u^c)^2] - \eta_c\lambda^2\rho_p^2\mathbb{V}[q_u] + \frac{2\big(\pi_{22}(u)y_u + \frac{\ell_c\sigma_c}{2}\big)^2}{\ell_c - 2K_c(u)} \right] \mathrm{d}u - \lambda\rho_p\bar{q}_T,$$

where

$$\begin{aligned}
\mathrm{d}\bar{c}_t &= \frac{2}{k_c}\Big[\big(\Lambda_c + \widehat{\pi}_{22}\big)\bar{c}_t + \widehat{\pi}_{21}\bar{q}_t + h_2\Big]\mathrm{d}t,\\
\mathrm{d}\bar{q}_t &= \frac{2}{k_p}\Big[\widehat{\pi}_{12}\bar{c}_t + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{q}_t + h_1\Big]\mathrm{d}t,\\
\mathrm{d}\mathbb{E}[c_t^2] &= \frac{4}{k_c}\Big[(K_c + \pi_{22})\left(\mathbb{E}[c_t^2] - \bar{c}_t^2\right) + \pi_{21}\big(\mathbb{E}[c_tq_t] - \bar{c}_t\bar{q}_t\big) + \big[\Lambda_c + \widehat{\pi}_{22}\big]\bar{c}_t^2 + \widehat{\pi}_{21}\bar{c}_t\bar{q}_t + h_2\bar{c}_t\Big]\mathrm{d}t + y_t^2\mathrm{d}t,\\
\mathrm{d}\mathbb{E}[q_t^2] &= \frac{4}{k_p}\Big[\left(K_p + \pi_{11}\right)\left(\mathbb{E}[q_t^2] - \bar{q}_t^2\right) + \pi_{12}\big(\mathbb{E}[c_tq_t] - \bar{c}_t\bar{q}_t\big) + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{q}_t^2 + \widehat{\pi}_{12}\bar{c}_t\bar{q}_t + h_1\bar{q}_t\Big]\mathrm{d}t + z_t^2\mathrm{d}t,\\
\mathrm{d}\mathbb{E}[c_tq_t] &= \frac{2}{k_p}\Big[\big(K_p + \pi_{11}\big)\big(\mathbb{E}[c_tq_t] - \bar{c}_t\bar{q}_t\big) + \pi_{12}\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{c}_t\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t^2 + h_1\bar{c}_t\Big]\mathrm{d}t\\
&\quad + \frac{2}{k_c}\Big[\big(K_c + \pi_{22}\big)\big(\mathbb{E}[c_tq_t] - \bar{c}_t\bar{q}_t\big) + \pi_{21}\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \big(\Lambda_c + \widehat{\pi}_{22}\big)\bar{c}_t\bar{q}_t + \widehat{\pi}_{21}\bar{q}_t^2 + h_2\bar{q}_t\Big]\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\mathbb{E}[(Y_t^p)^2] &= -2\Big\{\frac{1}{2}s_0\bar{Y}_t^p + \frac{1}{2}\gamma\rho_c\mathbb{E}[Y_t^pc_t] + \rho_p\gamma\rho_c\eta_p\lambda^2\big(\mathbb{E}[Y_t^pc_t] - \bar{Y}_t^p\bar{c}_t\big) + \frac{2}{k_p}\Big[K_p\mathbb{E}\big[(Y_t^p)^2 - (\bar{Y}_t^p)^2\big]\\
&\quad + \Lambda_p(\bar{Y}_t^p)^2\Big]\Big\}\mathrm{d}t + \big(\pi_{11}^2z_t^2 + \pi_{12}^2y_t^2\big)\mathrm{d}t,\\
\mathrm{d}\mathbb{E}[(Y_t^c)^2] &= -2\Big\{\frac{p_0 + p_1s_0 - \gamma(s_0+\delta)}{2}\bar{Y}_t^c + \frac{\rho_p(\gamma - p_1)}{2}\mathbb{E}[Y_t^cq_t] + \rho_p\gamma\rho_c\eta_c\lambda^2\big(\mathbb{E}[Y_t^cq_t] - \bar{Y}_t^c\bar{q}_t\big)\\
&\quad + \frac{2}{k_c}\left[K_c\mathbb{E}\big[(Y_t^c)^2 - (\bar{Y}_t^c)^2\big] + \Lambda_c(\bar{Y}_t^c)^2\right]\Big\}\mathrm{d}t + \big(\pi_{21}^2z_t^2 + \pi_{22}^2y_t^2\big)\mathrm{d}t,
\end{aligned}$$

with

$$\begin{aligned}
&\bar{Y}_t^p = \widehat{\pi}_{11}\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t + h_1, \quad \bar{Y}_t^c = \widehat{\pi}_{21}\bar{q}_t + \widehat{\pi}_{22}\bar{c}_t + h_2,\\
&\mathbb{E}[Y_t^pc_t] = \pi_{11}\big(\mathbb{E}[c_tq_t] - \bar{c}_t\bar{q}_t\big) + \pi_{12}\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \widehat{\pi}_{11}\bar{c}_t\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t^2 + h_1\bar{c}_t,\\
&\mathbb{E}[Y_t^cq_t] = \pi_{21}\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \pi_{22}\big(\mathbb{E}[q_tc_t] - \bar{c}_t\bar{q}_t\big) + \widehat{\pi}_{21}\bar{q}_t^2 + \widehat{\pi}_{22}\bar{c}_t\bar{q}_t + h_2\bar{q}_t,
\end{aligned}$$

and terminal conditions

$$(Y_T^p)^2 = \frac{\lambda^2\rho_p^2}{4}, \quad (Y_T^c)^2 = \frac{\lambda^2\gamma^2\rho_c^2}{4}.$$

Proof. For the terms $\bar{q}_t = \mathbb{E}[q_t]$ and $\bar{c}_t = \mathbb{E}[c_T]$ we have

$$\mathrm{d}\bar{q}_t = \bar{u}_t\mathrm{d}t = \frac{2}{k_p}\Big[\big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t + h_1\Big]\mathrm{d}t, \quad \mathrm{d}\bar{c}_t = \bar{v}_t\mathrm{d}t = \frac{2}{k_c}\Big[\big(\Lambda_c + \widehat{\pi}_{22}\big)\bar{c}_t + \widehat{\pi}_{21}\bar{q}_t + h_2\Big]\mathrm{d}t,$$

so we have a 2-dimensional ODE giving $\bar{c}_T$ and $\bar{q}_T$.

For the terms $\mathbb{V}[q_t]$ and $\mathbb{V}[c_t]$, we have

$$\mathbb{V}[c_t] = \mathbb{E}[c_t^2] - \bar{c}_t^2, \quad \mathrm{d}\mathbb{E}[c_t^2] = \big(2\mathbb{E}[c_t v_t] + (y_t)^2\big)\mathrm{d}t,$$
$$\mathbb{V}[q_t] = \mathbb{E}[q_t^2] - \bar{q}_t^2, \quad \mathrm{d}\mathbb{E}[q_t^2] = \big(2\mathbb{E}[q_t u_t] + (z_t)^2\big)\mathrm{d}t,$$

because z_t and y_t are deterministic. Further,

$$\mathbb{E}[c_t v_t] = \frac{2}{k_c}\Big[(K_c + \pi_{22})\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \pi_{21}\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \big(\Lambda_c + \widehat{\pi}_{22}\big)\bar{c}_t^2 + \widehat{\pi}_{21}\bar{c}_t\bar{q}_t + h_2\bar{c}_t\Big],$$
$$\mathrm{d}\mathbb{E}[c_t^2] = \frac{4}{k_c}\Big[(K_c + \pi_{22})\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \pi_{21}\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \big[\Lambda_c + \widehat{\pi}_{22}\big]\bar{c}_t^2 + \widehat{\pi}_{21}\bar{c}_t\bar{q}_t + h_2\bar{c}_t\Big]\mathrm{d}t + y_t^2\mathrm{d}t,$$
$$\mathbb{E}[q_t u_t] = \frac{2}{k_p}\Big[(K_p + \pi_{11})\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \pi_{12}\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{q}_t^2 + \widehat{\pi}_{12}\bar{c}_t\bar{q}_t + h_1\bar{q}_t\Big],$$
$$\mathrm{d}\mathbb{E}[q_t^2] = \frac{4}{k_p}\Big[(K_p + \pi_{11})\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \pi_{12}\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{q}_t^2 + \widehat{\pi}_{12}\bar{c}_t\bar{q}_t + h_1\bar{q}_t\Big]\mathrm{d}t + z_t^2\mathrm{d}t,$$

and we have for $\mathbb{E}[c_t q_t]$, that $\mathrm{d}\mathbb{E}[c_t q_t] = \mathbb{E}\big[c_t u_t + q_t v_t\big]\mathrm{d}t$, so that

$$\begin{aligned}\mathrm{d}\mathbb{E}[c_t q_t] =& \frac{2}{k_p}\Big[\big(K_p + \pi_{11}\big)\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \pi_{12}\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \big(\Lambda_p + \widehat{\pi}_{11}\big)\bar{c}_t\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t^2 + h_1\bar{c}_t\Big]\mathrm{d}t \\ &+ \frac{2}{k_c}\Big[\big(K_c + \pi_{22}\big)\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \pi_{21}\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \big(\Lambda_c + \widehat{\pi}_{22}\big)\bar{c}_t\bar{q}_t + \widehat{\pi}_{21}\bar{q}_t^2 + h_2\bar{q}_t\Big]\mathrm{d}t.\end{aligned}$$

For the term $\mathbb{V}[Y_t^p]$, we have $\mathbb{V}[Y_t^p] + \mathbb{E}[Y_t^p]^2 = \mathbb{E}[(Y_t^p)^2]$, where

$$\begin{aligned}&Y_t^p = \pi_{11}(q_t - \bar{q}_t) + \pi_{12}(c_t - \bar{c}_t) + \widehat{\pi}_{11}\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t + h_1(t), \quad \bar{Y}_t^p = \widehat{\pi}_{11}\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t + h_1, \\ &\mathbb{E}[Y_t^p c_t] = \pi_{11}\big(\mathbb{E}[c_t q_t] - \bar{c}_t\bar{q}_t\big) + \pi_{12}\big(\mathbb{E}[c_t^2] - \bar{c}_t^2\big) + \widehat{\pi}_{11}\bar{c}_t\bar{q}_t + \widehat{\pi}_{12}\bar{c}_t^2 + h_1\bar{c}_t, \\ &\mathrm{d}\mathbb{E}[(Y_t^p)^2] = -2\Big\{\frac{1}{2}s_0\bar{Y}_t^p + \frac{1}{2}\gamma\rho_c\mathbb{E}[Y_t^p c_t] + \rho_p\gamma\rho_c\eta_p\lambda^2\big(\mathbb{E}[Y_t^p c_t] - \bar{Y}_t^p\bar{c}_t\big) + \frac{2}{k_p}\Big[K_p\mathbb{E}\big[(Y_t^p)^2 - (\bar{Y}_t^p)^2\big] \\ &\qquad + \Lambda_p(\bar{Y}_t^p)^2\Big]\Big\}\mathrm{d}t + \big(\pi_{11}^2 z_t^2 + \pi_{12}^2 y_t^2\big)\mathrm{d}t, \quad (Y_T^p)^2 = \frac{1}{4}\lambda^2\rho_p^2,\end{aligned}$$

where we have exploited the representation of Y^p in Equation (2.4.10). Analogously, exploiting the representation of Y^c in Equation (2.4.11), we get

$$\begin{aligned}&Y_t^c = \pi_{21}(q_t - \bar{q}_t) + \pi_{22}(c_t - \bar{c}_t) + \widehat{\pi}_{21}\bar{q}_t + \widehat{\pi}_{22}\bar{c}_t + h_2(t), \quad \bar{Y}_t^c = \widehat{\pi}_{21}\bar{q}_t + \widehat{\pi}_{22}\bar{c}_t + h_2, \\ &+ \frac{2}{k_c}\Big[K_c\mathbb{E}\big[(Y_t^c)^2 - (\bar{Y}_t^c)^2\big] + \Lambda_c(\bar{Y}_t^c)^2\Big]\Big\}\mathrm{d}t, \\ &\mathbb{E}[Y_t^c q_t] = \pi_{21}\big(\mathbb{E}[q_t^2] - \bar{q}_t^2\big) + \pi_{22}\big(\mathbb{E}[q_t c_t] - \bar{c}_t\bar{q}_t\big) + \widehat{\pi}_{21}\bar{q}_t^2 + \widehat{\pi}_{22}\bar{c}_t\bar{q}_t + h_2\bar{q}_t, \\ &\mathrm{d}\mathbb{E}[(Y_t^c)^2] = -2\Big\{\frac{p_0 + p_1 s_0 - \gamma(s_0 + \delta)}{2}\bar{Y}_t^c + \frac{\rho_p(\gamma - p_1)}{2}\mathbb{E}[Y_t^c q_t] + \rho_p\gamma\rho_c\eta_c\lambda^2\big(\mathbb{E}[Y_t^c q_t] - \bar{Y}_t^c\bar{q}_t\big) \\ &\qquad + \frac{2}{k_c}\Big[K_c\mathbb{E}\big[(Y_t^c)^2 - (\bar{Y}_t^c)^2\big] + \Lambda_c(\bar{Y}_t^c)^2\Big]\Big\}\mathrm{d}t + \big(\pi_{21}^2 z_t^2 + \pi_{22}^2 y_t^2\big)\mathrm{d}t, \quad (Y_T^p)^2 = \frac{1}{4}\lambda^2\gamma^2\rho_c^2,\end{aligned}$$

Summing up, we have obtained a backward ODE for $\mathbb{E}[(Y_t^p)^2]$ and $\mathbb{E}[(Y_t^c)^2]$. Finally, we have

$$Z_t^{p,W} = \pi_{11}z_t, \quad \mathbb{V}[Z_t^{p,W}] = 0, \quad \mathbb{E}[Z_t^{p,W}] = \pi_{11}z_t,$$

and

$$Z_t^{c,B} = \pi_{22} y_t, \quad \mathbb{V}[Z_t^{c,B}] = 0, \quad \mathbb{E}[Z_t^{c,B}] = \pi_{22} y_t.$$

Recalling that

$$\begin{aligned} R_p^{(\lambda)}(t) = &- \lambda\gamma\rho_c\mathbb{E}[c_T] + \int_t^T \Bigg[- \eta_p\lambda^2\gamma^2\rho_c^2\mathbb{V}[c_u] + \frac{2}{k_p}\left(\mathbb{V}[Y_u^p] + \mathbb{E}[Y_u^p]^2\right) \\ &+ \frac{2}{\ell_p - 2K_p(u)}\left(\mathbb{V}[Z_u^{p,W}] + \left(\mathbb{E}[Z_u^{p,W}] + \frac{\ell_p\sigma_p}{2}\right)^2\right)\Bigg]\mathrm{d}u, \end{aligned}$$

and analogously

$$\begin{aligned} R_c^{(\lambda)}(t) = &- \lambda\rho_p\mathbb{E}[q_T] + \int_t^T \Bigg[- \eta_c\lambda^2\rho_p^2\mathbb{V}[q_u] + \frac{2}{k_c}\left(\mathbb{V}[Y_u^c] + \mathbb{E}[Y_u^c]^2\right) \\ &+ \frac{2}{\ell_c - 2K_c(u)}\left(\mathbb{V}[Z_u^{c,B}] + \left(\mathbb{E}[Z_u^{c,B}] + \frac{\ell_c\sigma_c}{2}\right)^2\right)\Bigg]\mathrm{d}u, \end{aligned}$$

the results follow. □

Functional quantization of rough volatility and applications to volatility derivatives

We believe that FBMs provide useful models for a host of natural time series and that their curious properties deserve to be presented to scientists, engineers and statisticians.

Benoit B. Mandelbrot and John W. Van Ness

This chapter covers paper [25], which is a joint collaboration with Prof. Giorgia Callegaro and Prof. Antoine Jacquier. Its preliminary version is available on arXiv and it was submitted in July 2021. Prof. Jacquier is an expert in the popular field of rough volatility, while Prof. Callegaro has been investigating the discretisation technique called quantization and its wide range of applications throughout her career. Together, we consider here product functional quantization, i.e. quantization of stochastic processes in their paths' space, and we develop it in a rough framework. In particular, we obtain a discretisation in the trajectories space of a family of Gaussian Volterra stochastic processes and we exploit it for the pricing of derivatives on the VIX volatility index and realised variance. The results obtained are illustrated via numerical simulations comparing our technique with the state of the art methodology as a benchmark when a closed formula is not available.

3.1 Introduction

Gatheral, Jaisson and Rosenbaum [77] recently introduced a new framework for financial modelling. To be precise — according to the reference website `https://sites.google.com/site/roughvol/home` — almost twenty-four hundred days have passed since instantaneous volatility was shown to have a rough nature, in the sense that its sample paths are α-Hölder-continuous with $\alpha < \frac{1}{2}$. Many studies, both empirical [17, 74, 72] and theoretical [71, 8], have confirmed this, showing that these so-called rough volatility models are a more accurate fit to the implied volatility surface and to estimate historical volatility time series.

On equity markets, the quality of a model is usually measured by its ability to calibrate not only to the SPX implied volatility but also VIX Futures and the VIX implied volatility. The market standard models had so far been Markovian, in particular the double mean-reverting process [76, 100], Bergomi's model [21] and, to some extent, jump models [35, 108]. However, they each suffer from several drawbacks, which the new generation of rough volatility models seems to overcome. For VIX Futures pricing, the rough version of Bergomi's model was thoroughly investigated in [103], showing accurate results. Nothing comes for free though and the new challenges set by rough volatility models lie on the numerical side, as new tools are needed to develop fast and accurate numerical techniques. Since classical simulation tools for fractional Brownian motions are too slow for realistic purposes, new schemes have been proposed to speed it up, among which the Monte Carlo hybrid scheme [17, 121], a tree formulation [97], quasi Monte-Carlo methods [13] and Markovian approximations [1, 141].

We suggest here a new approach, based on product functional quantization [128]. Quantization was originally conceived as a discretisation technique to approximate a continuous signal by a discrete one [134], later developed at Bell Laboratory in the 1950s for signal transmission [79]. It was however only in the 1990s that its power to compute (conditional) expectations of functionals of random variables [84] was fully understood. Given an $\mathbb{R}^d$-valued random vector on some probability space, optimal vector quantization investigates how to select an $\mathbb{R}^d$-valued random vector $\widehat{X}$, supported on at most N elements, that best approximates X according to a given criterion (such as the L^r-distance, $r \geq 1$). Functional quantization is the infinite-dimensional version, approximating a stochastic process with a random vector taking a finite number of values in the space of trajectories for the original process. It has been investigated precisely [119, 128] in the case of Brownian diffusions, in particular for financial applications [129]. However, optimal functional quantizers are in general hard to compute numerically and instead product functional quantizers provide a rate-optimal (so, in principle, sub-optimal) alternative often admitting closed-form expressions [120, 129].

In Section 3.2 we briefly review important properties of *Gaussian Volterra processes*, displaying a series expansion representation, and paying special attention to the *Riemann-Liouville* case in Section 3.2.2. This expansion yields, in Section 3.3, a product functional quantization of the processes, that shows an L^2-error of order $\log(N)^{-H}$, with N the number of paths and H a regularity index. We then show, in Section 3.3.1, that these functional quantizers, although sub-optimal, are stationary. We specialise our setup to the generalised rough Bergomi model in Section 3.4 and show how product functional quantization applies to the pricing of VIX Futures and VIX options, proving in particular precise rates of convergence. Finally, Section 3.5 provides a numerical confirmation of the quality of our approximations for VIX Futures and Call Options on the VIX in the rough Bergomi model, benchmarked against other existing schemes. In this Section, we also discuss how product functional quantization of the Riemann-Liouville process itself can be exploited to price options on realised variance.

We set $\mathbb{N}$ as the set of strictly positive natural numbers. We denote by $\mathcal{C}[0,1]$ the space of real-valued continuous functions over $[0,1]$ and by $L^2[0,1]$ the Hilbert space of real-valued square integrable functions on $[0,1]$, with inner product $\langle f, g\rangle_{L^2[0,1]} := \int_0^1 f(t)g(t)\mathrm{d}t$, inducing the norm $\|f\|_{L^2[0,1]} := (\int_0^1 |f(t)|^2\mathrm{d}t)^{1/2}$, for each $f, g \in L^2[0,1]$. $L^2(\mathbb{P})$ denotes the space of square integrable (with respect to $\mathbb{P}$) random variables.

3.2 Gaussian Volterra processes on $\mathbb{R}_+$

For clarity, we restrict ourselves to the time interval $[0,1]$. Let $\{W_t\}_{t\in[0,1]}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$, with $\{\mathcal{F}_t\}_{t\in[0,1]}$ its natural filtration. On this probability space we introduce the Volterra process

$$Z_t := \int_0^t K(t-s)\mathrm{d}W_s, \qquad t \in [0,1], \tag{3.2.1}$$

and we consider the following assumptions for the kernel K:

Assumption 3.2.1. *There exist $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$ and $L : (0,1] \to (0,\infty)$ continuously differentiable, slowly varying at 0, that is, for any $t > 0$, $\lim_{x\downarrow 0} \frac{L(tx)}{L(x)} = 1$, and bounded away from 0 function with $|L'(x)| \leq C(1 + x^{-1})$, for $x \in (0,1]$, for some $C > 0$, such that*

$$K(x) = x^{\alpha} L(x), \qquad x \in (0,1].$$

This implies in particular that $K \in L^2[0,1]$, so that the stochastic integral (3.2.1) is well defined. The Gamma kernel, with $K(u) = \mathrm{e}^{-\beta u} u^{\alpha}$, for $\beta > 0$ and $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, is a classical example satisfying Assumption 3.2.1. Straightforward computations show that the covariance function of Z reads

$$R_Z(s,t) = \int_0^{t\wedge s} K(t-u)K(s-u)\mathrm{d}u, \quad s,t \in [0,1].$$

Under Assumption 3.2.1, Z is a Gaussian process admitting a version which is ε-Hölder continuous for any $\varepsilon < \frac{1}{2} + \alpha = H$ and hence also admits a continuous version [17, Proposition 2.11].

3.2.1 Series expansion

We introduce a series expansion representation for the centred Gaussian process Z in (3.2.1), which will be key to develop its functional quantization. Inspired by [120], introduce the stochastic process

$$Y_t := \sum_{n\geq 1} \mathcal{K}[\psi_n](t)\xi_n, \qquad t \in [0,1], \tag{3.2.2}$$

where $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. standard Gaussian random variables, $\{\psi_n\}_{n\geq 1}$ denotes the orthonormal basis of $L^2[0,1]$:

$$\psi_n(t) = \sqrt{2}\cos\left(\frac{t}{\sqrt{\lambda_n}}\right), \quad \text{with } \lambda_n = \frac{4}{(2n-1)^2\pi^2}, \tag{3.2.3}$$

and the operator $\mathcal{K} : L^2[0,1] \to C[0,1]$ is defined for $f \in L^2[0,1]$ as

$$\mathcal{K}[f](t) := \int_0^t K(t-s)f(s)\mathrm{d}s, \qquad \text{for all } t \in [0,1]. \tag{3.2.4}$$

Remark 3.2.2. *The stochastic process Y in (3.2.2) is defined as a weighted sum of independent centred Gaussian variables, so for every $t \in [0,1]$ the random variable Y_t is a centred Gaussian random variable and the whole process Y is Gaussian with zero mean.*

We set the following assumptions on the functions $\{\mathcal{K}[\psi_n]\}_{n\in\mathbb{N}}$:

Assumption 3.2.3. *There exists $H \in (0, \frac{1}{2})$ such that*

(A) there is a constant $C_1 > 0$ for which, for any $n \geq 1$, $\mathcal{K}[\psi_n]$ is $(H + \frac{1}{2})$-Hölder continuous, with

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathcal{K}[\psi_n](t) - \mathcal{K}[\psi_n](s)|}{|t-s|^{H+\frac{1}{2}}} \leq C_1 n;$$

(B) there exists a constant $C_2 > 0$ such that

$$\sup_{t\in[0,1]} |\mathcal{K}[\psi_n](t)| \leq C_2 n^{-(H+\frac{1}{2})}, \quad \text{for all } n \geq 1.$$

Notice that under these assumptions, the series (3.2.2) converges both almost surely and in $L^2(\mathbb{P})$ for each $t \in [0, 1]$ by Khintchine-Kolmogorov Convergence Theorem [38, Theorem 1, Section 5.1].

It is natural to wonder whether Assumption 3.2.1 implies Assumption 3.2.3 given the basis functions (3.2.3). This is far from trivial in our general setup and we provide examples and justifications later on for models of interest. Similar considerations with slightly different conditions can be found in [120]. We now focus on the variance-covariance structure of the Gaussian process Y.

Lemma 3.2.4. *For any $s, t \in [0, 1]$, the covariance function of Y is given by*

$$R_Y(s,t) := \mathbb{E}[Y_s Y_t] = \int_0^{t\wedge s} K(t-u)K(s-u)\mathrm{d}u.$$

Proof. Exploiting the definition of Y in (3.2.2), the definition of $\mathcal{K}$ in (3.2.4) and the fact that the random variable ξ_n's are i.i.d. standard Normal, we obtain

$$\begin{aligned}
R_Y(s,t) &= \mathbb{E}[Y_s Y_t] = \mathbb{E}\left[\Big(\sum_{n\geq1} \mathcal{K}[\psi_n](s)\xi_n\Big)\Big(\sum_{m\geq1}\mathcal{K}[\psi_m](t)\xi_m\Big)\right] = \sum_{n\geq1}\mathcal{K}[\psi_n](s)\mathcal{K}[\psi_n](t) \\
&= \sum_{n\geq1}\Big(\int_0^1 K(s-u)\mathbf{1}_{[0,s]}(u)\psi_n(u)\mathrm{d}u \int_0^1 K(t-r)\mathbf{1}_{[0,t]}(r)\psi_n(r)\mathrm{d}r\Big) \\
&= \sum_{n\geq1}\langle K(s-\cdot)\mathbf{1}_{[0,s]}(\cdot), \psi_n\rangle_{L^2[0,1]} \cdot \langle K(t-\cdot)\mathbf{1}_{[0,t]}(\cdot), \psi_n\rangle_{L^2[0,1]} \\
&= \sum_{n\geq1}\Big\langle K(t-\cdot)\mathbf{1}_{[0,t]}(\cdot), \langle K(s-\cdot)\mathbf{1}_{[0,s]}(\cdot), \psi_n\rangle_{L^2[0,1]}\psi_n\Big\rangle_{L^2[0,1]} \\
&= \Big\langle K(t-\cdot)\mathbf{1}_{[0,t]}(\cdot), \sum_{n\geq1}\langle K(s-\cdot)\mathbf{1}_{[0,s]}(\cdot), \psi_n\rangle_{L^2[0,1]}\psi_n\Big\rangle_{L^2[0,1]} \\
&= \langle K(t-\cdot)\mathbf{1}_{[0,t]}(\cdot), K(s-\cdot)\mathbf{1}_{[0,s]}(\cdot)\rangle_{L^2[0,1]} \\
&= \int_0^1 K(s-u)\mathbf{1}_{[0,s]}(u)K(t-u)\mathbf{1}_{[0,t]}(u)\mathrm{d}u = \int_0^{t\wedge s} K(t-u)K(s-u)\mathrm{d}u.
\end{aligned}$$

□

Remark 3.2.5. *Notice that the centred Gaussian stochastic process Y admits a continuous version, too. Indeed, we have shown that Y has the same mean and covariance function as Z and, consequently, that the increments of the two processes share the same distribution. Thus, [17, Proposition 2.11] applies* *to Y as well, yielding that the process admits a continuous version. This last key property of Y can be alternatively proved directly as done in Appendix 3.A.2.*

Lemma 3.2.4 implies that $\mathbb{E}[Y_s Y_t] = \mathbb{E}[Z_s Z_t]$, for all $s, t \in [0,1]$. Both Z and Y are continuous, centred, Gaussian with the same covariance structure, so from now on we will work with Y, using

$$Z = \sum_{n\geq 1} \mathcal{K}[\psi_n]\xi_n, \quad \mathbb{P}\text{-a.s.} \tag{3.2.5}$$

3.2.2 The Riemann - Liouville case

For $K(u) = u^{H-\frac{1}{2}}$, with $H \in (0, \frac{1}{2})$, the process (3.2.1) takes the form

$$Z_t^H := \int_0^t (t-s)^{H-\frac{1}{2}} \mathrm{d}W_s, \qquad t \in [0,1],$$

where we add the superscript H to emphasise its importance. It is called a *Riemann-Liouville process* (henceforth RL) (also known as *Type II fractional Brownian motion* or *Lévy fractional Brownian motion*), as it is obtained by applying the Riemann-Liouville fractional operator to the standard Brownian motion, and is an example of a Volterra process. This process enjoys properties similar to those of the fractional Brownian motion (fBM), in particular being H-self-similar and centred Gaussian. However, contrary to the fractional Brownian motion, its increments are not stationary. For a more detailed comparison between the fBM and Z^H we refer to [131, Theorem 5.1]. In the RL case, the covariance function $R_{Z^H}(\cdot,\cdot)$ is available [104, Proposition 2.1] explicitly as

$$R_{Z^H}(s,t) = \frac{1}{H+\frac{1}{2}} (s\wedge t)^{H+\frac{1}{2}} (s \vee t)^{H-\frac{1}{2}} \, {}_2F_1\left(1, \frac{1}{2} - H; 2H+1; \frac{s\wedge t}{s\vee t}\right), \quad s,t \in [0,1],$$

where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function [127, Chapter 5, Section 9]. More generally, [127, Chapter 5, Section 11], the generalised hypergeometric functions ${}_pF_q(z)$ are defined as

$${}_pF_q(z) = {}_pF_q(a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_q; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(c_1)_k (c_2)_k \cdots (c_q)_k} \frac{z}{k!}, \tag{3.2.6}$$

with the Pochammer's notation $(a)_0 := 1$ and $(a)_k := a(a+1)(a+2)\cdots(a+k-1)$, for $k \geq 1$, where none of the c_k are negative integers or zero. For $p \leq q$ the series (3.2.6) converges for all z and when $p = q+1$ convergence holds for $|z| < 1$ and the function is defined outside this disk by analytic continuation. Finally, when $p > q+1$ the series diverges for nonzero z unless one of the a_k's is zero or a negative integer.

Regarding the series representation (3.2.2), we have, for $t \in [0,1]$ and $n \geq 1$,

$$\begin{aligned} \mathcal{K}_H[\psi_n](t) :&= \sqrt{2} \int_0^t (t-s)^{H-\frac{1}{2}} \cos\left(\frac{s}{\sqrt{\lambda_n}}\right) \mathrm{d}s \\ &= \frac{2\sqrt{2}}{1+2H} t^{H+\frac{1}{2}} \, {}_1F_2\left(1; \frac{3}{4} + \frac{H}{2}, \frac{5}{4} + \frac{H}{2}; -\frac{t^2}{4\lambda_n}\right). \end{aligned} \tag{3.2.7}$$

Assumption 3.2.3 holds in the RL case here using [120, Lemma 4] (identifying $\mathcal{K}_H[\psi_n]$ to f_n from [120, Equation (3.7)]). Assumption 3.2.3 **(B)** implies that, for all $t \in [0,1]$,

$$\sum_{n\geq 1} \mathcal{K}_H[\psi_n](t)^2 \leq \sum_{n\geq 1} \left(\sup_{t\in[0,1]} |\mathcal{K}_H[\psi_n](t)| \right)^2 \leq C_2^2 \sum_{n\geq 1} \frac{1}{n^{1+2H}} < \infty,$$

and therefore the series (3.2.2) converges both almost surely and in $L^2(\mathbb{P})$ for each $t \in [0,1]$ by Khintchine-Kolmogorov Convergence Theorem [38, Theorem 1, Section 5.1].

Remark 3.2.6. *The expansion* (3.2.2) *is in general not a Karhunen-Loève decomposition [129, Section 4.1.1]. In the RL case, it can be numerically checked that the basis* $\{\mathcal{K}_H[\psi_n]\}_{n\in\mathbb{N}}$ *is not orthogonal in* $L^2[0,1]$ *and does not correspond to eigenvectors for the covariance operator of the Riemann-Liouville process. In his PhD Thesis [41], Corlay exploited a numerical method to obtain approximations of the first terms in the K-L expansion of processes for which an explicit form is not available.*

3.3 Functional quantization and error estimation

Optimal (quadratic) vector quantization was conceived to approximate a square integrable random vector $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$ by another one $\widehat{X}$, taking at most a finite number N of values, on a grid $\Gamma^N := \{x_1^N, x_2^N, \ldots, x_N^N\}$, with $x_i^N \in \mathbb{R}^d$, $i = 1, \ldots, N$. The quantization of X is defined as $\widehat{X} := \mathrm{Proj}_{\Gamma^N}(X)$, where $\mathrm{Proj}_{\Gamma^N} : \mathbb{R}^d \to \Gamma^N$ denotes the nearest neighbour projection. Of course the choice of the N-quantizer Γ^N is based on a given optimality criterion: in most cases Γ^N minimises the distance $\mathbb{E}[|X - \widehat{X}|^2]^{1/2}$. We recall basic results for one-dimensional standard Gaussian, which shall be needed later, and refer to [84] for a comprehensive introduction to quantization.

Definition 3.3.1. *Let ξ be a one-dimensional standard Gaussian on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For each $n \in \mathbb{N}$, we define the* optimal quadratic n-quantization of ξ *as the random variable* $\widehat{\xi}^n := \mathrm{Proj}_{\Gamma^n}(\xi) = \sum_{i=1}^n x_i^n 1_{C_i(\Gamma^n)}(\xi)$, *where* $\Gamma^n = \{x_1^n, \ldots, x_n^n\}$ *is the unique optimal quadratic n-quantizer of ξ, namely the unique solution to the minimisation problem*

$$\min_{\Gamma^n \subset \mathbb{R}, \mathrm{Card}(\Gamma^n)=n} \mathbb{E}[|\xi - \mathrm{Proj}_{\Gamma^n}(\xi)|^2],$$

and $\{C_i(\Gamma^n)\}_{i\in\{1,\ldots,n\}}$ *is a Voronoi partition of $\mathbb{R}$, that is a Borel partition of $\mathbb{R}$ that satisfies*

$$C_i(\Gamma^n) \subset \left\{ y \in \mathbb{R} : |y - x_i^n| = \min_{1\leq j\leq n} |y - x_j^n| \right\} \subset \overline{C}_i(\Gamma^n),$$

where the right-hand side denotes the closure of the set in $\mathbb{R}$.

The unique optimal quadratic n-quantizer $\Gamma^n = \{x_1^n, \ldots, x_n^n\}$ and the corresponding quadratic error are available online, at `http://www.quantize.maths-fi.com/gaussian_database` for $n \in \{1, \ldots, 5999\}$.

Given a stochastic process, viewed as a random vector taking values in its trajectories space, such as $L^2[0,1]$, functional quantization does the analogue to vector quantization in an infinite-dimensional setting, approximating the process with a finite number of trajectories. In this section, we focus on product functional quantization of the centred Gaussian process Z from (3.2.1) of order N (see [128, Section 7.4] for a general introduction to product functional quantization). Recall that we are working with the continuous version of Z in the series (3.2.5). For any $m, N \in \mathbb{N}$, we introduce the following set, which will be of key importance all throughout the paper:

$$\mathcal{D}_m^N := \left\{ \mathbf{d} \in \mathbb{N}^m : \prod_{i=1}^m d(i) \leq N \right\}. \tag{3.3.1}$$

Definition 3.3.2. *A* product functional quantization of Z of order N *is defined as*

$$\widehat{Z}_t^{\mathbf{d}} := \sum_{n=1}^m \mathcal{K}[\psi_n](t) \widehat{\xi}_n^{d(n)}, \qquad t \in [0,1], \tag{3.3.2}$$

where $\mathbf{d} \in \mathcal{D}_m^N$, for some $m \in \mathbb{N}$, and for every $n \in \{1, \dots, m\}$, $\widehat{\xi}_n^{d(n)}$ is the (unique) optimal quadratic quantization of the standard Gaussian random variable ξ_n of order $d(n)$, according to Definition 3.3.1.

Remark 3.3.3. *The condition $\prod_{i=1}^m d(i) \leq N$ in Equation (3.3.1) motivates the wording 'product' functional quantization. Clearly, the optimality of the quantizer also depends on the choice of m and $\mathbf{d}$, for which we refer to Proposition 3.3.6 and Section 3.5.1.*

Before proceeding, we need to make precise the explicit form for the product functional quantizer of the stochastic process Z:

Definition 3.3.4. *The* product functional $\mathbf{d}$-quantizer of Z *is defined as*

$$\chi_{\underline{i}}^{\mathbf{d}}(t) := \sum_{n=1}^m \mathcal{K}[\psi_n](t)\, x_{i_n}^{d(n)}, \qquad t \in [0,1], \quad \underline{i} = (i_1, \dots, i_m),$$

for $\mathbf{d} \in \mathcal{D}_m^N$ and $1 \leq i_n \leq d(n)$ for each $n = 1, \dots, m$.

Remark 3.3.5. *Intuitively, the quantizer is chosen as a Cartesian product of grids of the one-dimensional standard Gaussian random variables. So, we also immediately find the probability associated to every trajectory $\chi_{\underline{i}}^{\mathbf{d}}$: for every $\underline{i} = (i_1, \dots, i_m) \in \prod_{n=1}^m \{1, \dots, d(n)\}$,*

$$\mathbb{P}(\widehat{Z}^{\mathbf{d}} = \chi_{\underline{i}}^{\mathbf{d}}) = \prod_{n=1}^m \mathbb{P}(\xi_n \in C_{i_n}(\Gamma^{d(n)})),$$

where $C_j(\Gamma^{d(n)})$ is the j-th Voronoi cell relative to the $d(n)$-quantizer $\Gamma^{d(n)}$ in Definition 3.3.1.

The following, proved in Appendix 3.A.1, deals with the quantization error estimation and its minimisation and provides hints to choose $(m, \mathbf{d})$. A similar result on the error can be obtained applying [120, Theorem 2] to the first example provided in the reference. For completeness we preferred to prove the result in an autonomous way in order to further characterise the explicit expression of the rate optimal parameters. Indeed, we then compare these rate optimal parameters with the (numerically computed) optimal ones in Section 3.5.1. The symbol $\lfloor \cdot \rfloor$ denotes the lower integer part.

Proposition 3.3.6. *Under Assumption 3.2.3, for any $N \geq 1$, there exist $m^*(N) \in \mathbb{N}$ and $C > 0$ such that*

$$\mathbb{E}\left[\left\|\widehat{Z}^{\mathbf{d}_N^*} - Z\right\|_{L^2[0,1]}^2\right]^{\frac{1}{2}} \leq C \log(N)^{-H},$$

where $\mathbf{d}_N^* \in \mathcal{D}_{m^*(N)}^N$ and with, for each $n = 1, \dots, m^*(N)$,

$$d_N^*(n) = \left\lfloor N^{\frac{1}{m^*(N)}} n^{-(H+\frac{1}{2})} (m^*(N)!)^{\frac{2H+1}{2m^*(N)}} \right\rfloor.$$

Furthermore $m^*(N) = \mathcal{O}(\log(N))$.

Remark 3.3.7. *In the RL case, the trajectories of $\widehat{Z}^{H,\mathbf{d}}$ are easily computable and they are used in the numerical implementations to approximate the process Z^H. In practice, the parameters m and $\mathbf{d} = (d(1), \dots, d(m))$ are chosen as explained in Section 3.5.1.*

3.3.1 Stationarity

We now show that the quantizers we are using are stationary. The use of stationary quantizers is motivated by the fact that their expectation provides a lower bound for the expectation of convex functionals of the process (Remark 3.3.9) and they yield a lower (weak) error in cubature formulae [128, page 26]. We first recall the definition of stationarity for the quadratic quantizer of a random vector [128, Definition 1].

Definition 3.3.8. *Let X be an $\mathbb{R}^d$-valued random vector on $(\Omega, \mathcal{F}, \mathbb{P})$. A quantizer Γ for X is* stationary *if the nearest neighbour projection $\widehat{X}^\Gamma = \mathrm{Proj}_\Gamma(X)$ satisfies*

$$\mathbb{E}\left[X|\widehat{X}^\Gamma\right] = \widehat{X}^\Gamma. \tag{3.3.3}$$

Remark 3.3.9. *Taking expectation on both sides of (3.3.3) yields*

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\widehat{X}^\Gamma]] = \mathbb{E}[\widehat{X}^\Gamma].$$

Furthermore, for any convex function $f : \mathbb{R}^d \to \mathbb{R}$, the identity above, the conditional Jensen's inequality and the tower property yield

$$\mathbb{E}[f(\widehat{X}^\Gamma)] = \mathbb{E}[f(\mathbb{E}[X|\widehat{X}^\Gamma])] \leq \mathbb{E}[\mathbb{E}[f(X)|\widehat{X}^\Gamma]] = \mathbb{E}[f(X)].$$

While an optimal quadratic quantizer of order N of a random vector is always stationary [128, Proposition 1(c)], the converse is not true in general. We now present the corresponding definition for a stochastic process.

Definition 3.3.10. *Let $\{X_t\}_{t\in[T_1,T_2]}$ be a stochastic process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[T_1,T_2]}, \mathbb{P})$. We say that an N-quantizer $\Lambda^N := \{\lambda_1^N, \cdots, \lambda_N^N\} \subset L^2[T_1, T_2]$, inducing the quantization $\widehat{X} = \widehat{X}^{\Lambda^N}$, is* stationary *if $\mathbb{E}[X_t|\widehat{X}_t] = \widehat{X}_t$, for all $t \in [T_1, T_2]$.*

Remark 3.3.11. *To ease the notation, we omit the grid Λ^N in $\widehat{X}^{\Lambda^N}$, while the dependence on the dimension N remains via the superscript $\mathbf{d} \in \mathcal{D}_m^N$ (recall (3.3.2)).*

As was stated in Section 3.2.1, we are working with the continuous version of the Gaussian Volterra process Z given by the series expansion (3.2.5). This will ease the proof of stationarity below (for a similar result in the case of the Brownian motion [128, Proposition 2]).

Proposition 3.3.12. *The product functional quantizers inducing $\widehat{Z}^{\mathbf{d}}$ in (3.3.2) are stationary.*

Proof. For any $t \in [0,1]$, by linearity, we have the following chain of equalities:

$$\mathbb{E}\left[Z_t|\{\widehat{\xi}_n^{d(n)}\}_{1\le n\le m}\right] = \mathbb{E}\left[\sum_{k\ge1}\mathcal{K}[\psi_k](t)\xi_k\Big|\{\widehat{\xi}_n^{d(n)}\}_{1\le n\le m}\right] = \sum_{k\ge1}\mathcal{K}[\psi_k](t)\mathbb{E}\left[\xi_k\Big|\{\widehat{\xi}_n^{d(n)}\}_{1\le n\le m}\right].$$

Since the $\mathcal{N}(0,1)$-Gaussian ξ_n's are i.i.d., by definition of optimal quadratic quantizers (hence stationary), we have $\mathbb{E}[\xi_k|\widehat{\xi}_i^{d(i)}] = \delta_{ik}\widehat{\xi}_i^{d(i)}$, for all $i,k \in \{1,\dots,m\}$, and therefore

$$\mathbb{E}\left[\xi_k\Big|\{\widehat{\xi}_n^{d(n)}\}_{1\le n\le m}\right] = \mathbb{E}\left[\xi_k\Big|\widehat{\xi}_k^{d(k)}\right] = \widehat{\xi}_k^{d(k)}, \text{ for all } k \in \{1,\dots,m\}.$$

Thus, we obtain

$$\mathbb{E}\left[Z_t\Big|\{\widehat{\xi}_n^{d(n)}\}_{1\le n\le m}\right] = \sum_{k\ge1}\mathcal{K}[\psi_k](t)\widehat{\xi}_k^{d(k)} = \widehat{Z}_t^{\mathbf{d}}.$$

Finally, exploiting the tower property and the fact that the σ-algebra generated by $\widehat{Z}_t^{\mathbf{d}}$ is included in the σ-algebra generated by $\{\widehat{\xi}_n^{d(n)}\}_{n\in\{1,\dots,m\}}$ by Definition 3.3.2, we obtain

$$\mathbb{E}\left[Z_t\Big|\widehat{Z}_t^{\mathbf{d}}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_t\Big|\{\widehat{\xi}_n^{d(n)}\}_{n\in\{1,\dots,m\}}\right]\Big|\widehat{Z}_t^{\mathbf{d}}\right] = \mathbb{E}\left[\widehat{Z}_t^{\mathbf{d}}\Big|\widehat{Z}_t^{\mathbf{d}}\right] = \widehat{Z}_t^{\mathbf{d}},$$

which concludes the proof. □

3.4 Application to VIX derivatives in rough Bergomi

We now specialise the setup above to the case of rough volatility models. These models are extensions of classical stochastic volatility models, introduced to better reproduce the market implied volatility surface. The volatility process is stochastic and driven by a rough process, by which we mean a process whose trajectories are H-Hölder continuous with $H \in (0, \frac{1}{2})$. The empirical study [77] was the first to suggest such a rough behaviour for the volatility, and ignited tremendous interest in the topic. The website https://sites.google.com/site/roughvol/home contains an exhaustive and up-to-date review of the literature on rough volatility. Unlike continuous Markovian stochastic volatility models, which are not able to fully describe the steep implied volatility skew of short-maturity options in equity markets, rough volatility models have shown accurate fit for this crucial feature. Within rough volatility, the rough Bergomi model [14] is one of the simplest, yet decisive frameworks to harness the power of the roughness for pricing purposes. We show how to adapt our functional quantization setup to this case.

3.4.1 The generalised Bergomi model

We work here with a slightly generalised version of the rough Bergomi model, defined as

$$\begin{cases} X_t &= -\frac{1}{2}\int_0^t \mathcal{V}_s \mathrm{d}s + \int_0^t \sqrt{\mathcal{V}_s}\mathrm{d}B_s, & X_0 = 0, \\ \mathcal{V}_t &= v_0(t)\exp\left\{\gamma Z_t - \frac{\gamma^2}{2}\int_0^t K(t-s)^2 \mathrm{d}s\right\}, & \mathcal{V}_0 > 0, \end{cases}$$

where X is the log-stock price, $\mathcal{V}$ the instantaneous variance process driven by the Gaussian Volterra process Z in (3.2.1), $\gamma > 0$ and B is a Brownian motion defined as $B := \rho W + \sqrt{1-\rho^2}W^{\perp}$ for some correlation $\rho \in [-1,1]$ and $W, W^{\perp}$ orthogonal Brownian motions. The filtered probability space is therefore taken as $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^{W^{\perp}}$, $t \geq 0$. This is a non-Markovian generalization of Bergomi's second generation stochastic volatility model [21], letting the variance be driven by a Gaussian Volterra process instead of a standard Brownian motion. Here, $v_T(t)$ denotes the forward variance for a remaining maturity t, observed at time T. In particular, v_0 is the initial forward variance curve, assumed to be $\mathcal{F}_0$-measurable. Indeed, given market prices of variance swaps $\sigma_T^2(t)$ at time T with remaining maturity t, the forward variance curve can be recovered as $v_T(t) = \frac{d}{\mathrm{d}t}\left(t\sigma_T^2(t)\right)$, for all $t \geq 0$, and the process $\{v_s(t-s)\}_{0\leq s\leq t}$ is a martingale for all fixed $t > 0$.

Remark 3.4.1. *With $K(u) = u^{H-\frac{1}{2}}$, $\gamma = 2\nu C_H$, for $\nu > 0$, and $C_H := \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$, we recover the standard rough Bergomi model [14].*

3.4.2 VIX Futures in the generalised Bergomi

We consider the pricing of VIX Futures (www.cboe.com/tradable_products/vix/) in the rough Bergomi model. They are highly liquid Futures on the Chicago Board Options Exchange Volatility Index, introduced on March 26, 2004, to allow for trading in the underlying VIX. Each VIX Future represents the expected implied volatility for the 30 days following the expiration date of the Futures contract itself. The continuous version of the VIX at time T is determined by the continuous-time monitoring formula

$$\begin{aligned} \mathrm{VIX}_T^2 :&= \mathbb{E}_T\left[\frac{1}{\Delta}\int_T^{T+\Delta} \mathrm{d}\langle X_s, X_s\rangle\right] = \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}[\mathcal{V}_s|\mathcal{F}_T]\mathrm{d}s \qquad (3.4.1) \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}_T\left[v_0(s)\mathrm{e}^{\gamma Z_s - \frac{\gamma^2}{2}\int_0^s K(s-u)^2\mathrm{d}u}\right]\mathrm{d}s \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} v_0(s)\mathrm{e}^{\gamma\int_0^T K(s-u)\mathrm{d}W_u - \frac{\gamma^2}{2}\int_0^s K(s-u)^2\mathrm{d}u}\mathbb{E}_T\left[\mathrm{e}^{\gamma\int_T^s K(s-u)\mathrm{d}W_u}\right]\mathrm{d}s \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} v_0(s)\mathrm{e}^{\gamma\int_0^T K(s-u)\mathrm{d}W_u - \frac{\gamma^2}{2}\int_0^s K(s-u)^2\mathrm{d}u}\mathrm{e}^{\frac{\gamma^2}{2}\int_T^s K(s-u)^2\mathrm{d}u}\mathrm{d}s, \end{aligned}$$

similarly to [103], where Δ is equal to 30 days, and we write $\mathbb{E}_T[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_T]$ (dropping the subscript when $T = 0$). Thus, the price of a VIX Future with maturity T is given by

$$\mathcal{P}_T := \mathbb{E}\left[\mathrm{VIX}_T\right] = \mathbb{E}\left[\left(\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\mathrm{e}^{\gamma Z_t^{T,\Delta}+\frac{\gamma^2}{2}\left(\int_0^{t-T}K(s)^2\mathrm{d}s-\int_0^t K(s)^2\mathrm{d}s\right)}\mathrm{d}t\right)^{\frac{1}{2}}\right],$$

where the process $(Z_t^{T,\Delta})_{t\in[T,T+\Delta]}$ is given by

$$Z_t^{T,\Delta} = \int_0^T K(t-s)\mathrm{d}W_s, \qquad t \in [T, T+\Delta].$$

To develop a functional quantization setup for VIX Futures, we need to quantize the process $Z^{T,\Delta}$, which is close, yet slightly different, from the Gaussian Volterra process Z in (3.2.1).

3.4.3 Properties of Z^T

To retrieve the same setting as above, we normalise the time interval to $[0,1]$, that is $T+\Delta = 1$. Then, for T fixed, we define the process $Z^T := Z^{T,1-T}$ as

$$Z_t^T := \int_0^T K(t-s)\mathrm{d}W_s, \qquad t \in [T,1],$$

which is well defined by the square integrability of K. By definition, the process Z^T is centred Gaussian and Itô isometry gives its covariance function as

$$R_{Z^T}(t,s) = \int_0^T K(t-u)K(s-u)\mathrm{d}u, \qquad t,s \in [T,1].$$

Proceeding as previously, we introduce a Gaussian process with same mean and covariance as those of Z^T, represented as a series expansion involving standard Gaussian random variables; from which product functional quantization follows. It is easy to see that the process Z^T has continuous trajectories. Indeed, $(Z_t^T - Z_s^T)^2 \le \mathbb{E}[|Z_t - Z_s|^2|\mathcal{F}_T^W]$, by conditional Jensen's inequality since $Z_t^T = \mathbb{E}[Z_t|\mathcal{F}_T^W]$. Then, applying tower property, for any $T \le s < t \le 1$,

$$\mathbb{E}\left[\left|Z_t^T - Z_s^T\right|^2\right] \le \mathbb{E}\left[|Z_t - Z_s|^2\right],$$

and therefore the H-Hölder regularity of Z (Section 3.2) implies that of Z^T.

Series expansion

Let $\{\xi_n\}_{n\ge1}$ be an i.i.d. sequence of standard Gaussian and $\{\psi_n\}_{n\ge1}$ the orthonormal basis of $L^2[0,1]$ from (3.2.3). Denote by $\mathcal{K}^T(\cdot)$ the operator from $L^2[0,1]$ to $C[T,1]$ that associates to each $f \in L^2[0,1]$,

$$\mathcal{K}^T[f](t) := \int_0^T K(t-s)f(s)\mathrm{d}s, \quad t \in [T,1]. \tag{3.4.2}$$

We define the process Y^T as (recall the analogous (3.2.2)):

$$Y_t^T := \sum_{n\geq 1} \mathcal{K}^T[\psi_n](t)\xi_n, \qquad t \in [T,1].$$

The lemma below follows from the corresponding results in Remark 3.2.2 and Lemma 3.2.4:

Lemma 3.4.2. *The process Y^T is centred, Gaussian and with covariance function*

$$R_{Y^T}(s,t) := \mathbb{E}\left[Y_s^T Y_t^T\right] = \int_0^T K(t-u)K(s-u)\mathrm{d}u, \qquad \textit{for all } s,t \in [T,1].$$

To complete the analysis of Z^T, we require an analogue version of Assumption 3.2.3.

Assumption 3.4.3. *Assumption 3.2.3 holds for the sequence $(\mathcal{K}^T[\psi_n])_{n\geq 1}$ on $[T,1]$ with the constants C_1 and C_2 depending on T.*

3.4.4 The truncated RL case

We again pay special attention to the RL case, for which the operator (3.4.2) reads, for each $n \in \mathbb{N}$,

$$\mathcal{K}_H^T[\psi_n](t) := \int_0^T (t-s)^{H-\frac{1}{2}}\psi_n(s)\mathrm{d}s, \qquad \text{for all } t \in [T,1],$$

and satisfies the following, proved in Appendix 3.A.4:

Lemma 3.4.4. *The functions $\{\mathcal{K}_H^T[\psi_n]\}_{n\geq 1}$ satisfy Assumption 3.4.3.*

A key role in this proof is played by an intermediate lemma, proved in Appendix 3.A.3, which provides a convenient representation for the integral $\int_0^T (t-u)^{H-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi u}\mathrm{d}u$, $t \geq T \geq 0$, in terms of the generalised hypergeometric function ${}_1F_2(\cdot)$.

Lemma 3.4.5. *For any $t \geq T \geq 0$, the representation*

$$\int_0^T (t-u)^{H-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi u}\mathrm{d}u = \mathrm{e}^{\mathrm{i}\pi t}\left[\Big(\zeta_{\frac{1}{2}}(t,h_1) - \zeta_{\frac{1}{2}}((t-T),h_1)\Big) - \mathrm{i}\pi\Big(\zeta_{\frac{3}{2}}(t,h_2) - \zeta_{\frac{3}{2}}((t-T),h_2)\Big)\right]$$

holds, where $h_1 := \frac{1}{2}(H+\frac{1}{2})$ and $h_2 = \frac{1}{2} + h_1$, $\chi(z) := -\frac{1}{4}\pi^2 z^2$ and

$$\zeta_k(z,h) := \frac{z^{2h}}{2h}\,{}_1F_2\left(h;k,1+h;\chi(z)\right), \qquad \textit{for } k \in \left\{\frac{1}{2},\frac{3}{2}\right\}. \tag{3.4.3}$$

Remark 3.4.6. *The representation in Lemma 3.4.5 can be exploited to obtain an explicit formula for $\mathcal{K}_H^T[\psi_n](t)$, $t \in [T,1]$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
\mathcal{K}_H^T[\psi_n](t) &= \frac{\sqrt{2}}{m^{H+\frac{1}{2}}}\int_0^{mT}(mt-u)^{H-\frac{1}{2}}\cos(\pi u)\mathrm{d}u = \frac{\sqrt{2}}{m^{H+\frac{1}{2}}}\Re\left\{\int_0^{mT}(mt-u)^{H-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi u}\mathrm{d}u\right\}\\
&= \frac{\sqrt{2}}{m^{H+\frac{1}{2}}}\Re\left\{\mathrm{e}^{\mathrm{i}\pi mt}\left[\Big(\zeta_{\frac{1}{2}}(mt,h_1) - \zeta_{\frac{1}{2}}(m(t-T),h_1)\Big) - \mathrm{i}\pi\Big(\zeta_{\frac{3}{2}}(mt,h_2) - \zeta_{\frac{3}{2}}(m(t-T),h_2)\Big)\right]\right\}\\
&= \frac{\sqrt{2}}{m^{H+\frac{1}{2}}}\left\{\cos(mt\pi)\Big(\zeta_{\frac{1}{2}}(mt,h_1) - \zeta_{\frac{1}{2}}(m(t-T),h_1)\Big) + \pi\sin(mt\pi)\Big(\zeta_{\frac{3}{2}}(mt,h_2) - \zeta_{\frac{3}{2}}(m(t-T),h_2)\Big)\right\},
\end{aligned}$$

with $m := n - \frac{1}{2}$ *and* $\zeta_{\frac{1}{2}}(\cdot)$, $\zeta_{\frac{3}{2}}(\cdot)$ *in* (3.4.3). *We shall exploit this in our numerical simulations.*

3.4.5 VIX Derivatives Pricing

We can now introduce the quantization for the process $Z^{T,\Delta}$, similarly to Definition 3.3.2, recalling the definition of the set $\mathcal{D}_m^N$ in (3.3.1):

Definition 3.4.7. *A* product functional quantization for $Z^{T,\Delta}$ of order N *is defined as*

$$\widehat{Z}_t^{T,\Delta,\mathbf{d}} := \sum_{n=1}^{m} \mathcal{K}^{T,\Delta}[\psi_n^{T,\Delta}](t)\widehat{\xi}_n^{d(n)}, \qquad t \in [T, T+\Delta],$$

where $\mathbf{d} \in \mathcal{D}_m^N$*, for some* $m \in \mathbb{N}$*, and for every* $n \in \{1, \dots, m\}$, $\widehat{\xi}_n^{d(n)}$ *is the (unique) optimal quadratic quantization of the Gaussian variable* ξ_n *of order* $d(n)$.

The sequence $\{\psi_n^{T,\Delta}\}_{n\in\mathbb{N}}$ denotes the orthonormal basis of $L^2[0, T+\Delta]$ given by

$$\psi_n^{T,\Delta}(t) = \sqrt{\frac{2}{T+\Delta}} \cos\left(\frac{t}{\sqrt{\lambda_n}(T+\Delta)}\right), \quad \text{with } \lambda_n = \frac{4}{(2n-1)^2\pi^2},$$

and the operator $\mathcal{K}^{T,\Delta} : L^2[0, T+\Delta] \to C[T, T+\Delta]$ is defined for $f \in L^2[0, T+\Delta]$ as

$$\mathcal{K}^{T,\Delta}[f](t) := \int_0^T K(t-s)f(s)\mathrm{d}s, \qquad t \in [T, T+\Delta].$$

Adapting the proof of Proposition 3.3.12 it is possible to prove that these quantizers are stationary, too.

Remark 3.4.8. *The dependence on* Δ *is due to the fact that the coefficients in the series expansion depend on the time interval* $[T, T+\Delta]$.

In the RL case for each $n \in \mathbb{N}$, we can write, using Remark 3.4.6, for any $t \in [T, T+\Delta]$:

$$\begin{aligned}
\mathcal{K}_H^{T,\Delta}[\psi_n^{T,\Delta}](t) &= \sqrt{\frac{2}{T+\Delta}} \int_0^T (t-s)^{H-\frac{1}{2}} \cos\left(\frac{s}{\sqrt{\lambda_n}(T+\Delta)}\right)\mathrm{d}s, \\
&= \frac{\sqrt{2}(T+\Delta)^H}{(n-1/2)^{H+\frac{1}{2}}} \int_0^{\frac{(n-1/2)}{T+\Delta}T} \left(\frac{(n-1/2)}{T+\Delta}t - u\right)^{H-\frac{1}{2}} \cos(\pi u)\mathrm{d}u \\
&= \frac{\sqrt{2}(T+\Delta)^H}{(n-\frac{1}{2})^{H+\frac{1}{2}}} \left\{ \cos\left(\frac{(n-\frac{1}{2})}{T+\Delta}t\pi\right)\left(\zeta_{\frac{1}{2}}\left(\frac{(n-\frac{1}{2})}{T+\Delta}t, h_1\right) - \zeta_{\frac{1}{2}}\left(\frac{(n-\frac{1}{2})}{T+\Delta}(t-T), h_1\right)\right)\right. \\
&\quad \left. + \pi \sin\left(\frac{(n-\frac{1}{2})}{T+\Delta}t\pi\right)\left(\zeta_{\frac{3}{2}}\left(\frac{(n-\frac{1}{2})}{T+\Delta}t, h_2\right) - \zeta_{\frac{3}{2}}\left(\frac{(n-\frac{1}{2})}{T+\Delta}(t-T), h_2\right)\right)\right\}.
\end{aligned}$$

We thus exploit $\widehat{Z}^{T,\Delta,\mathbf{d}}$ to obtain an estimation of VIX_T and of VIX Futures through the following

$$\widehat{\mathrm{VIX}}_T^{\mathbf{d}} := \left(\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t) \exp\left\{\gamma \widehat{Z}_t^{T,\Delta,\mathbf{d}} + \frac{\gamma^2}{2}\left(\int_0^{t-T} K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)\right\}\mathrm{d}t\right)^{\frac{1}{2}}, \quad (3.4.4)$$

$$\widehat{\mathcal{P}}_T^{\mathbf{d}} := \mathbb{E}\left[\left(\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\exp\left\{\gamma\widehat{Z}_t^{T,\Delta,\mathbf{d}} + \frac{\gamma^2}{2}\left(\int_0^{t-T} K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)\right\}\mathrm{d}t\right)^{\frac{1}{2}}\right].$$

Remark 3.4.9. *The expectation above reduces to the following deterministic summation, making its computation immediate:*

$$\begin{aligned}\widehat{\mathcal{P}}_T^{\mathbf{d}} &= \mathbb{E}\left[\left(\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\mathrm{e}^{\gamma\sum_{n=1}^m \mathcal{K}^{T,\Delta}[\psi_n^{T,\Delta}](t)\widehat{\xi}_n^{d(n)} + \frac{\gamma^2}{2}\left(\int_0^{t-T}K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)}\mathrm{d}t\right)^{\frac{1}{2}}\right]\\ &= \sum_{\underline{i}\in I^d}\left(\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\mathrm{e}^{\gamma\sum_{n=1}^m \mathcal{K}^{T,\Delta}[\psi_n^{T,\Delta}](t)x_{i_n}^{d(n)} + \frac{\gamma^2}{2}\left(\int_0^{t-T}K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)}\mathrm{d}t\right)^{\frac{1}{2}}\\ &\quad\cdot\prod_{n=1}^m \mathbb{P}(\xi_n \in C_{i_n}(\Gamma^{d(n)})),\end{aligned}$$

where $\widehat{\xi}_n^{d(n)}$ *is the (unique) optimal quadratic quantization of* ξ_n *of order* $d(n)$*,* $C_j(\Gamma^{d(n)})$ *is the* j*-th Voronoi cell relative to the* $d(n)$*-quantizer (Definition 3.3.1), with* $j = 1,\dots,d(n)$ *and* $\underline{i} = (i_1,\dots,i_m) \in \prod_{j=1}^m\{1,\dots,d(j)\}$*. In the numerical illustrations displayed in Section 3.5, we exploited Simpson rule to evaluate these integrals. In particular, we used* simps *function from* scipy.integrate *with* 300 *points.*

3.4.6 Quantization error of VIX Derivatives

The following L^2-error estimate is a consequence of Assumption 3.4.3 **(B)** and its proof is omitted since it is analogous to that of Proposition 3.3.6:

Proposition 3.4.10. *Under Assumption 3.4.3, for any* $N \geq 1$*, there exist* $m_T^*(N) \in \mathbb{N}$*,* $C > 0$ *such that*

$$\mathbb{E}\left[\left\|\widehat{Z}^{T,\Delta,\mathbf{d}^*_{T,N}} - Z^{T,\Delta}\right\|^2_{L^2[T,T+\Delta]}\right]^{\frac{1}{2}} \leq C\log(N)^{-H},$$

for $\mathbf{d}^*_{T,N} \in \mathcal{D}^N_{m_T^*(N)}$ *and with, for each* $n = 1,\dots,m_T^*(N)$*,*

$$d^*_{T,N}(n) = \left\lfloor N^{\frac{1}{m_T^*(N)}} n^{-(H+\frac{1}{2})}\left(m_T^*(N)!\right)^{\frac{2H+1}{2m_T^*(N)}}\right\rfloor.$$

Furthermore $m_T^*(N) = \mathcal{O}(\log(N))$*.*

As a consequence, we have the following error quantification for European options on the VIX:

Theorem 3.4.11. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a globally Lipschitz-continuous function and* $\mathbf{d} \in \mathbb{N}^m$ *for some* $m \in \mathbb{N}$*. There exists* $\mathfrak{K} > 0$ *such that*

$$\left|\mathbb{E}\left[F\left(\mathrm{VIX}_T\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathrm{VIX}}_T^{\mathbf{d}}\right)\right]\right| \leq \mathfrak{K}\,\mathbb{E}\left[\left\|Z^{T,\Delta} - \widehat{Z}^{T,\Delta,\mathbf{d}}\right\|^2_{L^2([T,T+\Delta])}\right]^{\frac{1}{2}}. \tag{3.4.5}$$

Furthermore, for any $N \geq 1$, *there exist* $m_T^*(N) \in \mathbb{N}$ *and* $\mathfrak{C} > 0$ *such that, with* $\mathbf{d}_{T,N}^* \in \mathcal{D}_{m_T^*(N)}^N$,

$$\left|\mathbb{E}\left[F\left(\mathrm{VIX}_T\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathrm{VIX}}_T^{\mathbf{d}_{T,N}^*}\right)\right]\right| \leq \mathfrak{C}\log(N)^{-H}. \tag{3.4.6}$$

The upper bound in (3.4.6) is an immediate consequence of (3.4.5) and Proposition 3.4.10. The proof of (3.4.5) is much more involved and is postponed to Appendix 3.A.5.

Remark 3.4.12.

- *When* $F(x) = 1$, *we obtain the price of VIX Futures and the quantization error*

$$\left|\mathcal{P}_T - \widehat{\mathcal{P}}_T^{\mathbf{d}}\right| \leq \mathfrak{K}\,\mathbb{E}\left[\left\|Z^{T,\Delta} - \widehat{Z}^{T,\Delta,\mathbf{d}}\right\|^2_{L^2([T,T+\Delta])}\right]^{\frac{1}{2}},$$

  and, for any $N \geq 1$, *Theorem 3.4.11 yields the existence of* $m_T^*(N) \in \mathbb{N}$, $\mathfrak{C} > 0$ *such that*

$$\left|\mathcal{P}_T - \widehat{\mathcal{P}}_T^{\mathbf{d}_{T,N}^*}\right| \leq \mathfrak{C}\log(N)^{-H}.$$

- *Since the functions* $F(x) := (x-K)_+$ *and* $F(x) := (K-x)_+$ *are globally Lipschitz continuous, the same bounds apply for European Call and Put options on the VIX.*

3.5 Numerical results for the RL case

We now test the quality of the quantization on the pricing of VIX Futures in the standard rough Bergomi model, considering the RL kernel in Remark 3.4.1.

3.5.1 Practical considerations for m and $\mathbf{d}$

Proposition 3.3.6 provides, for any fixed $N \in \mathbb{N}$, some indications on $m^*(N)$ and $\mathbf{d}_N^* \in \mathcal{D}_m^N$ (see (3.3.1)), for which the rate of convergence of the quantization error is $\log(N)^{-H}$. We present now a numerical algorithm to compute the optimal parameters. For a given number of trajectories $N \in \mathbb{N}$, the problem is equivalent to finding $m \in \mathbb{N}$ and $\mathbf{d} \in \mathcal{D}_m^N$ such that $\mathbb{E}[\|Z^H - \widehat{Z}^{H,\mathbf{d}}\|^2_{L^2[0,1]}]$ is minimal. Starting from (3.A.1) and adding and subtracting the quantity $\sum_{n=1}^m(\int_0^1 \mathcal{K}_H[\psi_n](t)^2\mathrm{d}t)$, we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|Z^H - \widehat{Z}^{H,\mathbf{d}}\right\|^2_{L^2[0,1]}\right] &= \sum_{n=1}^m\left(\int_0^1 \mathcal{K}_H[\psi_n](t)^2\mathrm{d}t\right)[\varepsilon^{d(n)}(\xi_n)]^2 + \sum_{k\geq m+1}\int_0^1 \mathcal{K}_H[\psi_k](t)^2\mathrm{d}t \\ &= \sum_{n=1}^m\left(\int_0^1 \mathcal{K}_H[\psi_n](t)^2\mathrm{d}t\right)\left\{\left[\varepsilon^{d(n)}(\xi_n)\right]^2 - 1\right\} + \sum_{k\geq 1}\int_0^1 \mathcal{K}_H[\psi_k](t)^2\mathrm{d}t,\end{aligned} \tag{3.5.1}$$

where $\varepsilon^{d(n)}(\xi_n)$ denotes the optimal quadratic quantization error for the quadratic quantizer of order $d(n)$ of the standard Gaussian random variable ξ_n (see Appendix 3.A.1 for more details). Notice that the last term on the right-hand side of (3.5.1) does not depend on m, nor on $\mathbf{d}$. We therefore simply look for m and $\mathbf{d}$ that minimise

$$A(m,\mathbf{d}) := \sum_{n=1}^m\left(\int_0^1 \mathcal{K}_H[\psi_n]^2(t)\mathrm{d}t\right)\left([\varepsilon^{d(n)}(\xi_n)]^2 - 1\right).$$

This can be easily implemented: the functions $\mathcal{K}_H[\psi_n]$ can be obtained numerically from the hypergeometric function and the quadratic errors $\varepsilon^{d(n)}(\xi_n)$ are available at www.quantize.maths-fi.com/gaussian_database, for $d(n) \in \{1, \dots, 5999\}$. The algorithm therefore reads as follows

(i) fix m;

(ii) minimise $A(m, \mathbf{d})$ over $\mathbf{d} \in \mathcal{D}_m^N$ and call it $\widetilde{A}(m)$;

(iii) minimise $\widetilde{A}(m)$ over $m \in \mathbb{N}$.

Table 3.5.1: Optimal parameters.

N	$\overline{m}(N)$	$\overline{\mathbf{d}}_N$	$\overline{N}_{traj}$
10	2	5 - 2	10
10^2	4	8 - 3 - 2 - 2	96
10^3	6	10 - 4 - 3 - 2 - 2 - 2	960
10^4	8	10 - 5 - 4 - 3 - 2 - 2 - 2 - 2	9600
10^5	10	14 - 6 - 4 - 3 - 3 - 2 - 2 - 2 - 2 - 2	96768
10^6	12	14 - 6 - 5 - 4 - 3 - 3 - 2 - 2 - 2 - 2 - 2 - 2	967680

Table 3.5.2: Rate-optimal parameters.

N	$m^*(N) = \lfloor\log(N)\rfloor$	**Relative error**	$\mathbf{d}_N^*$	N_{traj}^*
10	2	2.75%	3 - 2	6
10^2	4	1.30%	5 - 3 - 2 - 2	60
10^3	6	1.09%	6 - 4 - 3 - 2 - 2 - 2	576
10^4	9	3.08%	6 - 4 - 3 - 2 - 2 - 2 - 2 - 1 - 1	1152
10^5	11	3.65%	7 - 4 - 3 - 3 - 2 - 2 - 2 - 2 - 1 - 1 - 1	4032
10^6	13	2.80%	8 - 5 - 4 - 3 - 3 - 2 - 2 - 2 - 2 - 2 - 1 - 1 - 1	46080

N	$m^*(N) = \lfloor\log(N)\rfloor - 1$	**Relative error**	$\mathbf{d}_N^*$	N_{traj}^*
10	1	2.78%	10	10
10^2	3	1.13%	6 - 4 - 3	72
10^3	5	1.22%	7 - 4 - 3 - 3 - 2	504
10^4	8	1.35%	7 - 4 - 3 - 3 - 2 - 2 - 2 - 2	4032
10^5	10	2.29%	7 - 5 - 4 - 3 - 2 - 2 - 2 - 2 - 2 - 1	13440
10^6	12	2.25%	8 - 5 - 4 - 3 - 3 - 2 - 2 - 2 - 2 - 2 - 2 - 1	92160

N	$m^*(N) = \lfloor\log(N)\rfloor - 2$	**Relative error**	$\mathbf{d}_N^*$	N_{traj}^*
10^2	2	2.53%	12 - 8	96
10^3	4	1.44%	9 - 5 - 4 - 3	540
10^4	7	1.46%	7 - 5 - 4 - 3 - 2 - 2 - 2	3360
10^5	9	1.57%	8 - 5 - 4 - 3 - 3 - 2 - 2 - 2 - 2	23040
10^6	11	1.48%	9 - 6 - 4 - 3 - 3 - 3 - 2 - 2 - 2 - 2 - 2	186624

The results of the algorithm for some reference values of $N \in \mathbb{N}$ are available in Table 3.5.1, where $\overline{N}_{traj} := \prod_{i=1}^{\overline{m}(N)} \overline{d}_N(i)$ represents the number of trajectories actually computed in the optimal case. In Table 3.5.2, we compute the rate optimal parameters derived in Proposition 3.3.6: the column 'Relative error' contains the normalised difference between the L^2-quantization error made with the optimal choice of $\overline{m}(N)$ and $\overline{\mathbf{d}}_N$ in Table 3.5.1 and the L^2-quantization error made with $m^*(N)$ and $\mathbf{d}_N^*$ of the corresponding line of the table, namely $\frac{\left| \|Z^H - \widehat{Z}^{H,\overline{\mathbf{d}}_N}\|_{L^2[0,1]} - \|Z^H - \widehat{Z}^{H,\mathbf{d}_N^*}\|_{L^2[0,1]} \right|}{\|Z^H - \widehat{Z}^{H,\overline{\mathbf{d}}_N}\|_{L^2[0,1]}}$. In the column $N_{traj}^* := \prod_{i=1}^{m^*(N)} d_N^*(i)$ we display the number of trajectories actually computed in the rate-optimal case. The optimal quadratic vector quantization of a standard Gaussian of order 1 is the random variable identically equal to zero and so when $d(i) = 1$ the corresponding term is uninfluential in the representation.

3.5.2 The functional quantizers

The computations in Section 3.2 and 3.3 for the RL process, respectively the ones in Section 3.4.3 and 3.4.4 for $Z^{H,T}$, provide a way to obtain the functional quantizers of the processes.

Quantizers of the RL process

For the RL process, Definition 3.3.4 shows that its quantizer is a weighted Cartesian product of grids of the one-dimensional standard Gaussian random variables. The time-dependent weights $\mathcal{K}_H[\psi_n](\cdot)$ are computed using (3.2.7), and for a fixed number of trajectories N, suitable $\overline{m}(N)$ and $\overline{\mathbf{d}}_N \in \mathcal{D}^N_{\overline{m}(N)}$ are chosen according to the algorithm in Section 3.5.1. Not surprisingly, Figures 3.5.1 show that as the paths of the process get smoother (H increases) the trajectories become less fluctuating and shrink around zero. For $H = 0.5$, where the RL process reduces to the standard Brownian motion, we recover the well-known quantizer from [128, Figures 7-8]. This is consistent as in that case $\mathcal{K}_H[\psi_n](t) = \sqrt{\lambda_n}\sqrt{2}\sin\left(\frac{t}{\sqrt{\lambda_n}}\right)$, and so Y^H is the Karhuenen-Loève expansion for the Brownian motion [128, Section 7.1].

Quantizers of $Z^{H,T}$

A quantizer for $Z^{H,T}$ is defined analogously to that of Z^H using Definition 3.3.4. The weights $\mathcal{K}_H^T[\psi_n](\cdot)$ in the summation are available in closed form, as shown in Remark 3.4.6. It is therefore possible to compute the N-product functional quantizer, for any $N \in \mathbb{N}$, as Figure 3.5.2 displays.

3.5.3 Pricing and comparison with Monte Carlo

In this section we show and comment some plots related to the estimation of prices of derivatives on the VIX and realised variance. We set the values $H = 0.1$ and $\nu = 1.18778$ for the parameters and investigate three different initial forward variance curves $v_0(\cdot)$, as in [103]:

Scenario 1. $v_0(t) = 0.234^2$;

Scenario 2. $v_0(t) = 0.234^2(1+t)^2$;

Scenario 3. $v_0(t) = 0.234^2\sqrt{1+t}$.

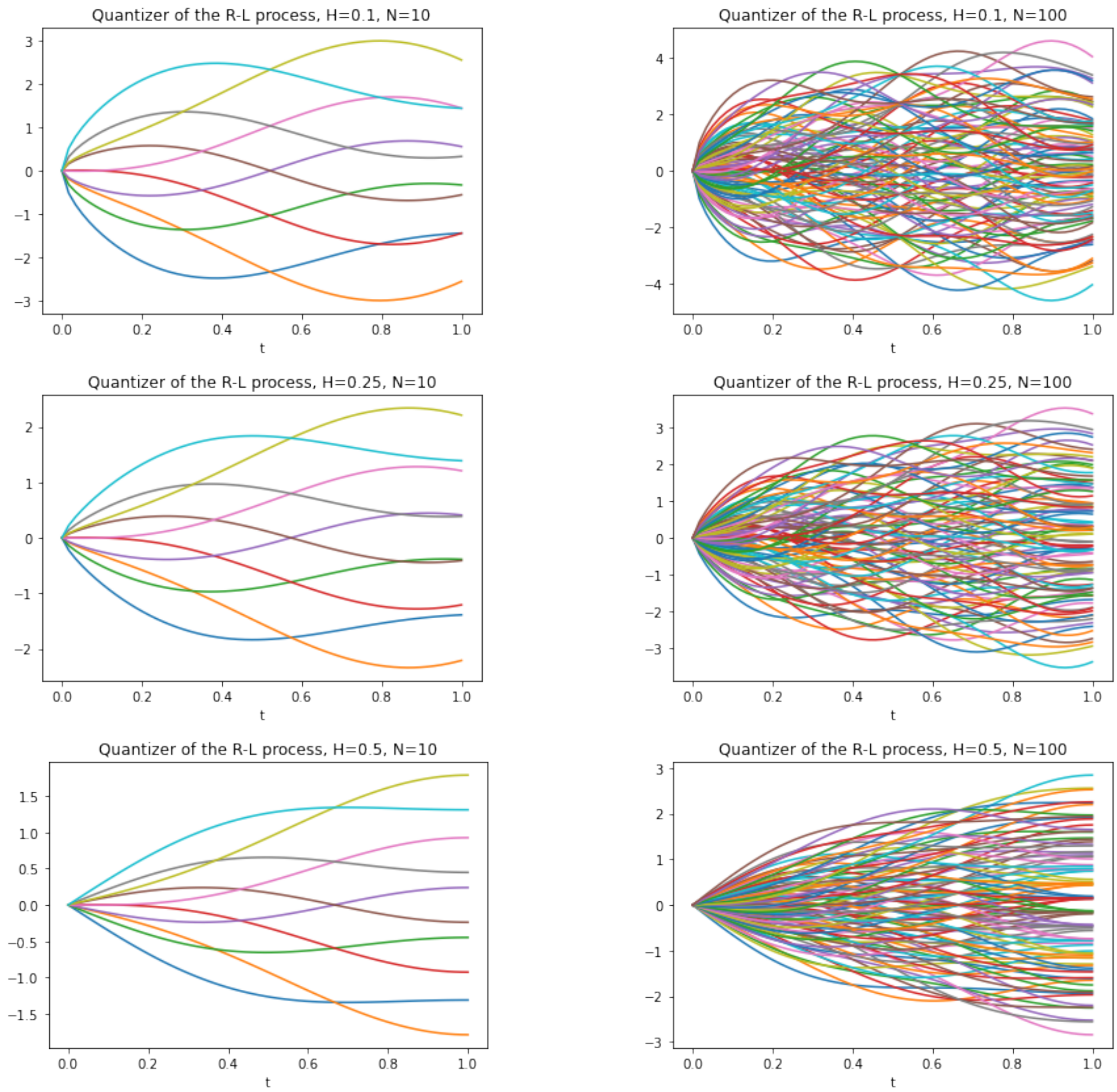

Figure 3.5.1: Product functional quantization of the RL process with N-quantizers, for $H \in \{0.1, 0.25, 0.5\}$, for $N = 10$ and $N = 100$.

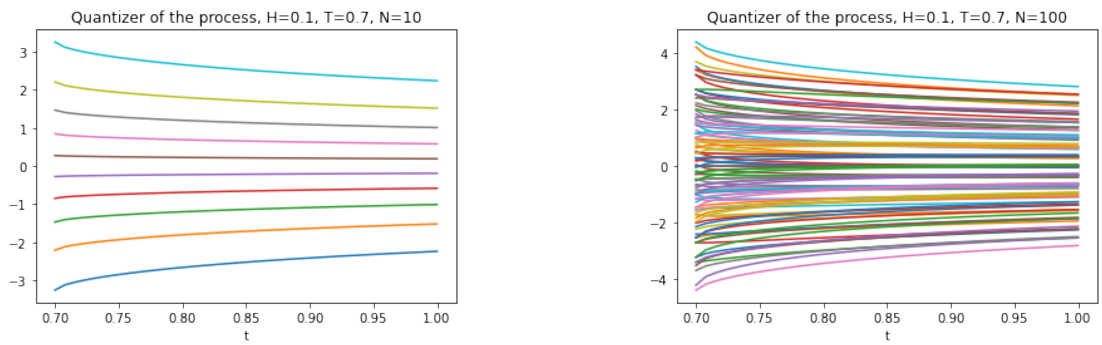

Figure 3.5.2: Product functional quantization of $Z^{H,T}$ via N-quantizers, with $H = 0.1$, $T = 0.7$, for $N \in \{10, 100\}$.

The choice of such ν is a consequence of the choice $\eta = 1.9$, consistently with [17], and of the relationship $\nu = \eta \frac{\sqrt{2H}}{2C_H}$. In all these cases, v_0 is an increasing function of time, whose value at zero is close to the square of the reference value of 0.25.

VIX Futures Pricing

One of the most recent and effective way to compute the price of VIX Futures is a Monte-Carlo-simulation method based on Cholesky decomposition, for which we refer to [103, Section 3.3.2]. It can be considered as a good approximation of the true price when the number M of computed paths is large. In fact, in [103] the authors tested three simulation-based methods (Hybrid scheme + forward Euler, Truncated Cholesky, SVD decomposition) and 'all three methods seem to approximate the prices similarly well'. We thus consider the truncated Cholesky approach as a benchmark and take $M = 10^6$ trajectories and 300 equidistant point for the time grid.

In Figure 3.5.3, we plot the VIX Futures prices as a function of the maturity T, where T ranges in $\{1, 2, 3, 6, 9, 12\}$ months (consistently with actual quotations) on the left, and the corresponding relative error w.r.t. the Monte Carlo benchmark on the right. It is clear that the quantization approximates the benchmark from below and that the accuracy increases with the number of trajectories.
We highlight that the quantization scheme for VIX Futures can be sped up considerably by storing ahead the quantized trajectories for $Z^{H,T,\Delta}$, so that we only need to compute the integrations and summations in Remark 3.4.9, which are extremely fast.

Table 3.5.3: Grid organisation times (in seconds) as a function of the maturity (rows, in months) and of the number of trajectories (columns).

Grid organisation time					
	10^2	10^3	10^4	10^5	10^6
1	0.474	0.491	0.99	4.113	37.183
2	0.476	0.487	0.752	4.294	39.134
3	0.617	0.536	0.826	4.197	37.744
6	0.474	0.475	0.787	4.432	37.847
9	0.459	0.6	0.858	3.73	41.988
12	0.498	0.647	1.016	3.995	38.045

Furthermore, the grid organisation time itself is not that significant. In Table 3.5.3 we display the grid organisation times (in seconds) as a function of the maturity (rows) expressed in months and of the number of trajectories (columns). From this table one might deduce that the time needed for the organisation of the grids is suitable to be performed once per day (say every morning) as it should be for actual pricing purposes. It is interesting to note that the estimations obtained with quantization (which is an exact method) are consistent in that they mimic the trend of benchmark prices over time even for very small values of N. However, as a consequence of the variance in the estimations, the Monte Carlo prices are almost useless for small values of M. Moreover, improving the estimations with Monte Carlo requires to increase the number of points in the time grid with clear impact on computational time, while this is

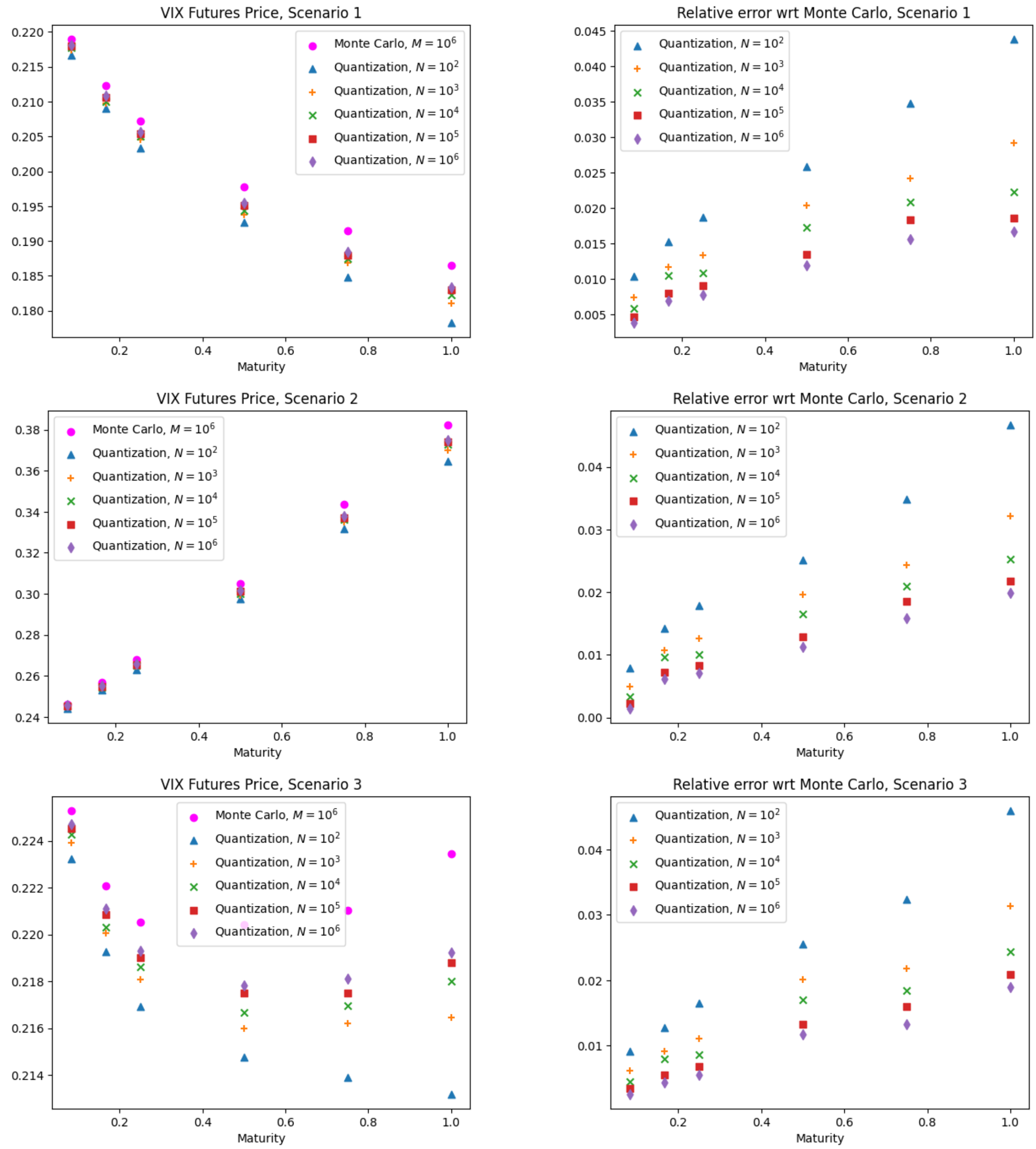

Figure 3.5.3: VIX Futures prices (left) and relative error (right) computed with quantization and with Monte-Carlo as a function of the maturity T, for different numbers of trajectories, for each forward variance curve scenario.

not the case with quantization since the trajectories in the quantizers are smooth. Indeed, the trajectories in the quantizers are not only smooth but also almost constant over time, hence reducing the number of time steps to get the desired level of accuracy. Notice that here we may refer also to the issue of complexity related to discretisation: a quadrature formula over n points has a cost $O(n)$, while the simulation with a Cholesky method over the same grid has cost $O(n^2)$. Finally, our quantization method does not require much RAM. Indeed, all the simulations performed with quantization can be easily run on a personal laptop[1], while this is not the case for the Monte Carlo scheme proposed here[2]. For the sake of completeness, we also recall that combining Monte Carlo pricing of VIX futures/options with an efficient control variate speeds up the computations significantly [98].

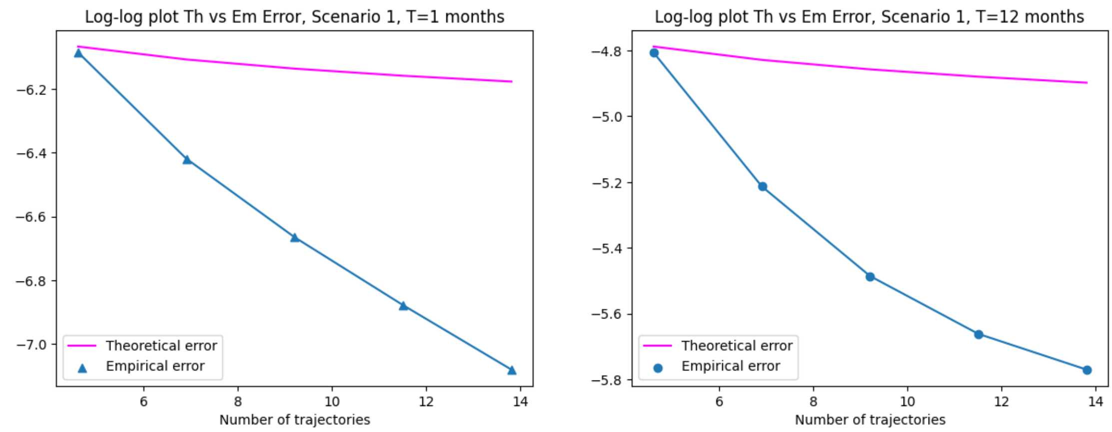

Figure 3.5.4: Log-log (natural logarithm) plot of the empirical absolute error with the theoretically predicted one for Scenario 1, with $T \in \{1, 12\}$ months.

In Figure 3.5.4, we show some plots comparing the behaviour of the empirical error with the theoretically predicted one. We have decided to display only a couple of maturities for the first scenario since the other plots are very similar. The figures display in a clear way that the order of convergence of the empirical error should be bigger than the theoretically predicted one: in particular, we expect it to be $O(\log(N)^{-1})$.

VIX Options Pricing

To complete the discussion on VIX Options pricing, we present in Figure 3.5.5 the approximation of the prices of ATM Call Options on the VIX obtained via quantization as a function of the maturity T and for different numbers of trajectories against the same price computed via Monte Carlo simulations with $M = 10^6$ trajectories, as a benchmark. Each plot represents a different scenario for the initial forward variance curve. For all scenarios, as the number N of trajectories goes to infinity, the prices in Figure 3.5.5 are clearly converging, and the limiting curve is increasing in the maturity, as it should be.

[1]The personal computer used to run the quantization codes has the following technical specifications: RAM: 8.00 GB, SSD memory: 512 GB, Processor: AMD Ryzen 7 4700U with Radeon Graphics 2.00 GHz.

[2]The computer used to run the Monte Carlo codes is a virtual machine (OpenStack/Nova/KVM/Qemu, www.openstack.org) with the following technical specifications: RAM: 32.00 GB, CPU: 8 virtual cores, Hypervisor CPU:

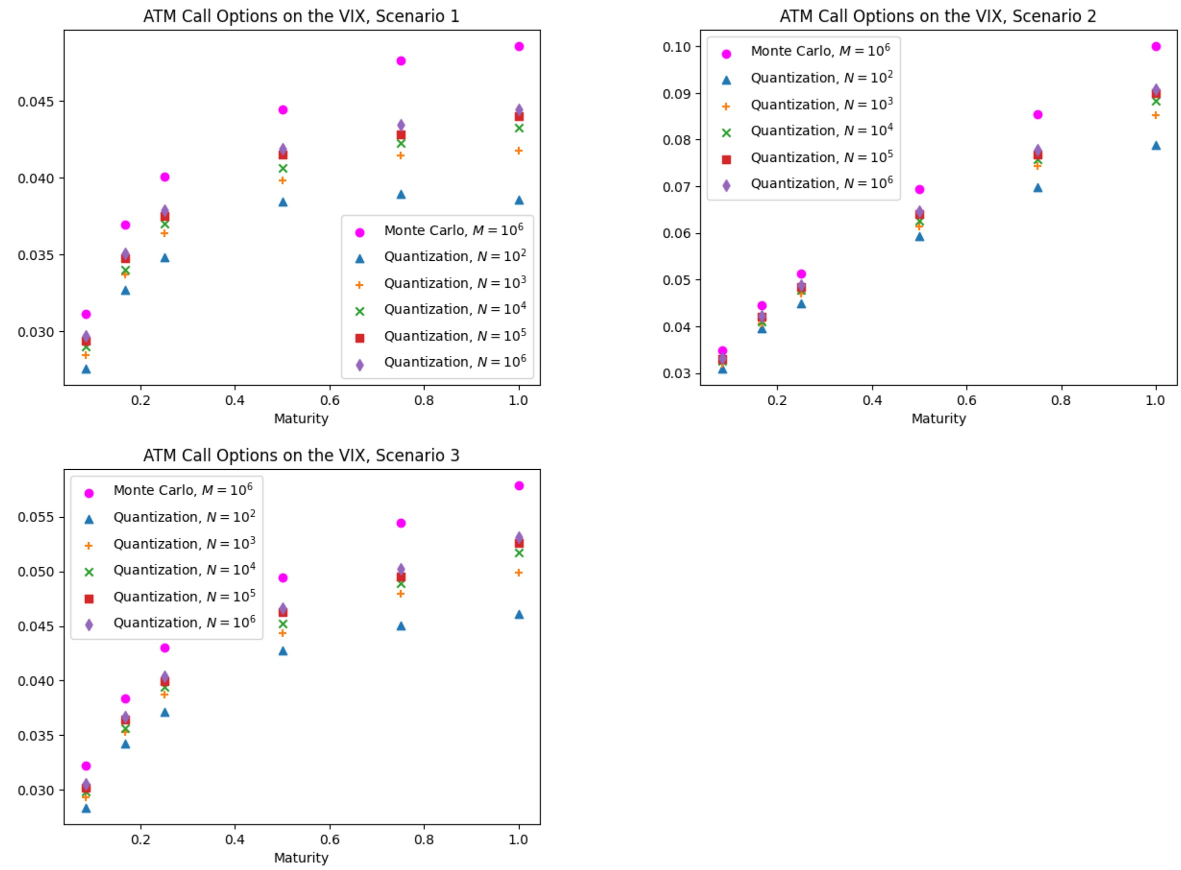

Figure 3.5.5: Prices of ATM Call Options on the VIX via quantization.

Pricing of Continuously Monitored Options on Realised Variance

Product functional quantization of the process $(Z_t^H)_{t\in[0,T]}$ can be exploited for (meaningful) pricing purposes, too. We first price variance swaps, whose price is given by the following expression

$$\mathfrak{S}_T := \mathbb{E}\left[\frac{1}{T}\int_0^T \mathcal{V}_t \mathrm{d}t \middle| \mathcal{F}_0\right].$$

Let us recall that, in the rough Bergomi model,

$$\mathcal{V}_t = v_0(t)\exp\left(2\nu C_H Z_t^H - \frac{\nu^2 C_H^2}{H} t^{2H}\right),$$

where $C_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$, $\nu > 0$ is an endogenous constant and $v_0(t)$ being the initial forward variance curve. Thus, exploiting the fact that, for any fixed $t \in [0,T]$, Z_t^H is distributed according to a centred Gaussian random variable with variance $\int_0^t (t-s)^{2H-1}\mathrm{d}s = \frac{t^{2H}}{2H}$, the quantity $\mathfrak{S}_T$ can be explicitly computed:

$$\mathfrak{S}_T = \frac{1}{T}\int_0^T v_0(t)\mathrm{d}t.$$

This is particularly handy and provides us a simple benchmark. The price $\mathfrak{S}_T$ is, then, approx-

Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz, RAM 128GB, Storage: CEPH cluster (www.ceph.com).

imated via quantization through

$$\widehat{\mathfrak{S}}^{\mathbf{d}}_T = \sum_{\underline{i}\in I^d} \left(\frac{1}{T}\int_0^T v_0(t)\exp\left(2\nu C_H \sum_{n=1}^{m} \mathcal{K}_H[\psi_n](t)\, x^{d(n)}_{i_n} - \frac{\nu^2 C_H^2}{H} t^{2H}\right)\mathrm{d}t\right)\prod_{n=1}^{m} \mathbb{P}(\xi_n \in C_{i_n}(\Gamma^{d(n)})).$$

True price	0.0548
Quantization, $N = 10^2$	0.0230
Quantization, $N = 10^3$	0.0246
Quantization, $N = 10^4$	0.0257
Quantization, $N = 10^5$	0.0266
Quantization, $N = 10^6$	0.0273

Figure 3.5.6: Prices and errors of variance swaps.

Numerical results are presented in Figure 3.5.6. On the left-hand side we display a table with the approximations (depending on N, the number of trajectories) of the price of a swap on the realised variance in Scenario 1, for $T = 1$, and the true value computed by integration. On the right-hand side a log-log (natural logarithm) plot of the error against the function $c\log(N)^{-H}$, with c being a suitable positive constant. For variance swaps the error is not performing very well. It is indeed very close to the upper bound $c\log(N)^{-H}$ that we have computed theoretically. One possible theoretical motivation for this behaviour lies in the difference between *strong* and *weak* error rates. Weak error and strong error do not necessarily share the same order of convergence, being the weak error faster in general. See [15, 16, 75] for recent developments on the topic in the rough volatility framework. For pricing purposes, we are interested in weak error rates. Indeed, the pricing error should in principle have the following form $\mathbb{E}[f(W^H)] - \mathbb{E}[f(\widehat{W}^H)]$, where $\widehat{W}^H$ is the process that we are using to approximate the original W^H and f is a functional that comes from the payoff function and that we can interpret as a test function. Thus, the functional f has a smoothing effect. On the other hand, the upper bound for the quantization error we have computed is a strong error rate. This theoretical discrepancy motivates the findings in Figure 3.5.4 when pricing VIX Futures and other options on the VIX: the empirical error seems to converge with order $\mathcal{O}(\log(N)^{-1})$, while the predicted order is $\mathcal{O}(\log(N)^{-H})$. When pricing variance swaps, there is no functional f involved, so we expect a lower discrepancy between the two errors. Moreover, the different empirical rates that are seen in Figure 3.5.4 for VIX futures (roughly $\mathcal{O}(\log(N)^{-1})$)) and in Figure 3.5.6 for variance swaps (much closer to $\mathcal{O}(\log(N)^{-H})$) could be also related to the different degree of pathwise regularity of the processes Z and Z^T . While $t \to Z_t = \int_0^t K(t-s)\mathrm{d}W_s$ is a.s. $(H-\varepsilon)$-Hölder, for fixed T, the trajectories $t \to Z_t^T = \int_0^T K(t-s)\mathrm{d}W_s$ of Z^T are much smoother when $t \in (T, T+\Delta)$ and t is bounded away from T. When pricing VIX derivatives, we are quantizing almost everywhere a smooth Gaussian process (hence error rate of order $\log(N)^{-1}$), while when pricing derivatives on realised variance, we are applying quantization to a rough Gaussian process (hence error rate of order $\mathcal{O}(\log(N)^{-H})$), resulting in a deteriorated accuracy for the prices of realised volatility derivatives such as the variance swaps in Figure 3.5.6.

Furthermore, it can be easily shown that, for any $\mathbf{d} \in \mathcal{D}_m^N$ and for any $m, N \in \mathbb{N}$, with $m < N$, $\widehat{\mathfrak{S}}_T^{\mathbf{d}}$ always provides a lower bound for the true price $\mathfrak{S}_T$. Indeed, since the quantizers $\widehat{Z}^{H,\mathbf{d}}$ of the process Z^H are stationary (cfr. Proposition 3.3.12), an application of Remark 3.3.9 to the convex function $f(x) = \exp(2\nu C_H x)$ together with the positivity of $v_0(t)\exp(-\frac{\nu^2 C_H^2 t^{2H}}{H})$, for any $t \in [0, T]$, yields

$$\begin{aligned}
\widehat{\mathfrak{S}}_T^{\mathbf{d}} &= \mathbb{E}\left[\frac{1}{T}\int_0^T v_0(t)\exp\left(-\frac{\nu^2 C_H^2 t^{2H}}{H}\right)\exp\left(2\nu C_H \widehat{Z}_T^{H,\mathbf{d}}\right)\mathrm{d}t \middle| \mathcal{F}_0\right] \\
&= \frac{1}{T}\int_0^T v_0(t)\exp\left(-\frac{\nu^2 C_H^2 t^{2H}}{H}\right)\mathbb{E}_0\left[\exp\left(2\nu C_H \widehat{Z}_T^{H,\mathbf{d}}\right)\right]\mathrm{d}t \\
&\le \frac{1}{T}\int_0^T v_0(t)\exp\left(-\frac{\nu^2 C_H^2 t^{2H}}{H}\right)\mathbb{E}_0\left[\exp\left(2\nu C_H Z_T^{H}\right)\right]\mathrm{d}t = \mathfrak{S}_T.
\end{aligned}$$

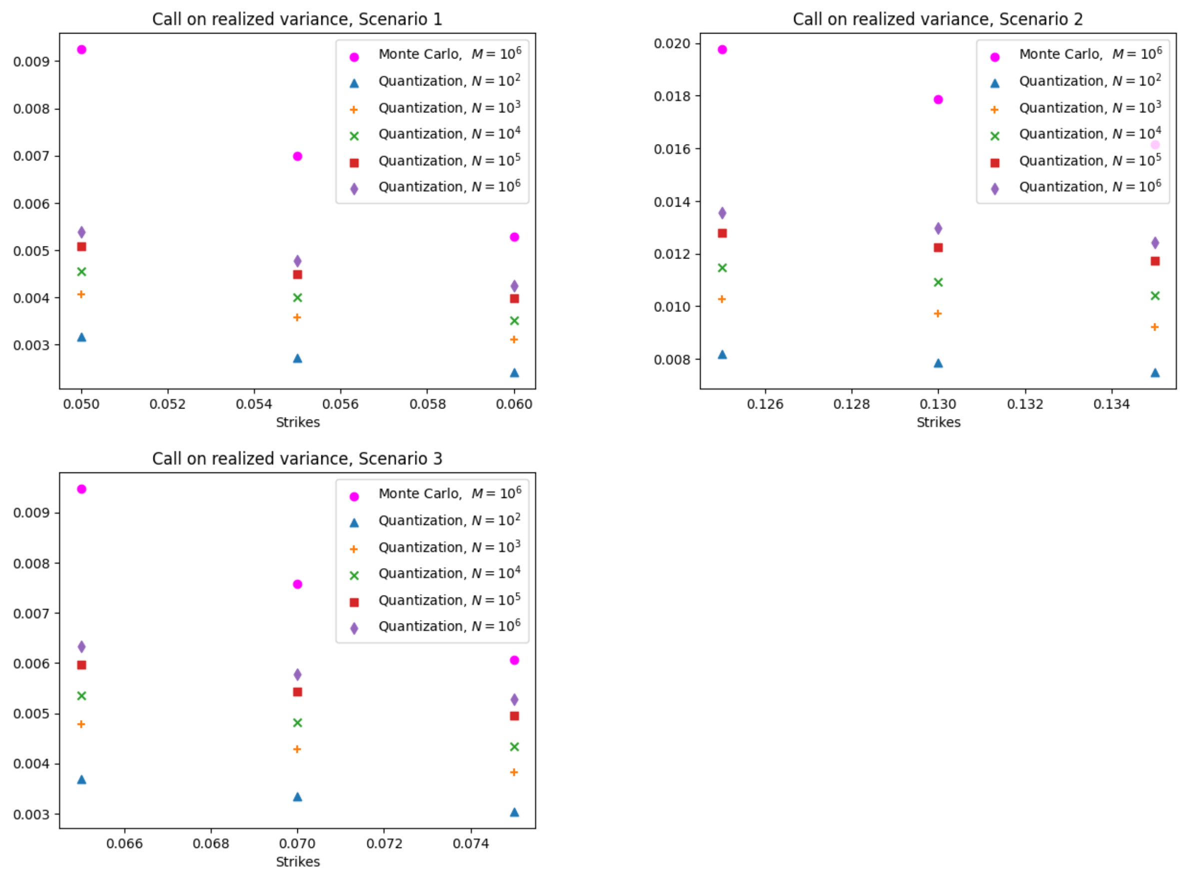

Figure 3.5.7: Prices of European Call Option on realised variance as a function of K, via Monte Carlo with $M = 10^6$ trajectories and via quantization with $N \in \{10^2, 10^3, 10^4, 10^5, 10^6\}$ trajectories.

To complete this section, we plot in Figure 3.5.7 approximated prices of European Call Options on the realised variance via quantization with $N \in \{10^2, 10^3, 10^4, 10^5, 10^6\}$ trajectories and via Monte Carlo with $M = 10^6$ trajectories, as a benchmark. In order to take advantage of the trajectories obtained, we compute the price of a realised variance Call option with strike K and maturity $T = 1$ as

$$C(K,T) = \mathbb{E}\left[\left(\frac{1}{T}\int_0^T \mathcal{V}_t \mathrm{d}t - K\right)_+ \middle| \mathcal{F}_0\right],$$

and we approximate it via quantization through

$$\widehat{C}^{\mathbf{d}}(K,T) = \sum_{\underline{i}\in I^d}\left(\frac{1}{T}\int_0^T v_0(t)\exp\left(2\nu C_H\sum_{n=1}^m \mathcal{K}_H[\psi_n](t)x_{i_n}^{d(n)} - \frac{\nu^2 C_H^2}{H}t^{2H}\right)\mathrm{d}t - K\right)_+ \prod_{n=1}^m \mathbb{P}(\xi_n \in C_{i_n}(\Gamma^{d(n)})).$$

The three plots in Figure 3.5.7 display the behaviour of the price of a European Call on the realised variance as a function of the strike price K (close to the ATM value) for the three scenarios considered before.

Quantization and MC comparison

In order to make a fair comparison between quantization and Monte Carlo simulations, we present a figure to display, for each methodology, the computational work needed for a given error tolerance for the pricing of VIX Futures. The plots in Figure 3.5.8 should be read as follows. First, for any $M, N \in \{10^2, 10^3, 10^4, 10^5, 10^6\}$, we have computed the corresponding pricing errors: $\varepsilon^{MC}(M) := |\text{Price}^{MC}(M) - \text{RefPrice}|$ and $\varepsilon^{Q}(N) := |\text{Price}^{Q}(N) - \text{RefPrice}|$ where $\text{Price}^{MC}(M)$ is the Monte Carlo price obtained via truncated Cholesky with M trajectories, $\text{Price}^{Q}(N)$ is the price computed via quantization with N trajectories and RefPrice comes from the lower-bound in Equation (3.4) in [103] and the associated computational time in seconds $t^{MC}(M)$ and $t^{Q}(N)$, respectively for Monte Carlo simulation and quantization. Then, each point in the plot is associated either to a value of M in case of Monte Carlo (the circles in Figure 3.5.8), or N in case of quantization (the triangles in Figure 3.5.8), and its x-coordinate provides the absolute value of the associated pricing error, while its y-coordinate represents the associated computational cost in seconds.

These plots lead to the following observations:

- For quantization, which is an exact method, the error is strictly monotone in the number of trajectories.
- When a small number of trajectories is considered, quantization provides a lower error with respect to Monte Carlo, at a comparable cost.
- For large numbers of trajectories Monte Carlo overcomes quantization both in terms of accuracy and of computational time.

To conclude, quantization can always be run with an arbitrary number of trajectories and furthermore for $N \in \{10^2, 10^3, 10^4\}$ it leads to a lower error with respect to Monte Carlo, at a comparable computational cost, as it is visible from Figure 3.5.8. This makes quantization particularly suitable to be used when dealing with standard machines, i.e., laptops with a RAM memory smaller or equal to 16GB.

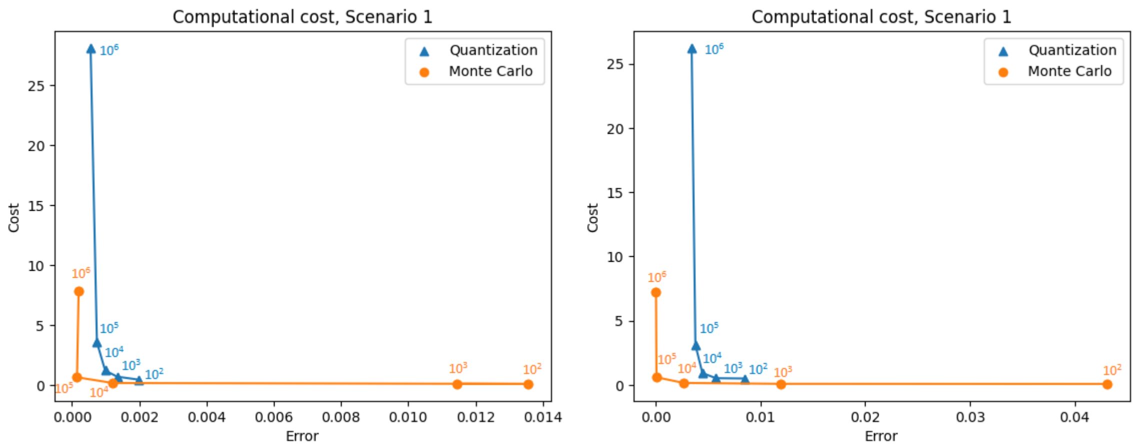

Figure 3.5.8: Computational costs for quantization vs Monte Carlo for Scenario 1, with $T = 1$ month (left-hand side) and $T = 12$ months (right-hand side). The number of trajectories, M for Monte Carlo and N for quantization, corresponding to a specific dot is displayed above it.

3.6 Conclusion

In this paper we provided, on the theoretical side, a precise and detailed result on the convergence of product functional quantizers of *Gaussian Volterra processes*, showing that the L^2-error is of order $\log(N)^{-H}$, with N the number of trajectories and H the regularity index.

Furthermore, we explicitly characterised the rate optimal parameters, m_N^* and $\mathbf{d}_N^*$ and we compared them with the corresponding optimal parameters, $\overline{m}_N$ and $\overline{\mathbf{d}}_N$, computed numerically.

In the rough Bergomi model, we applied product functional quantization to the pricing of VIX options, with precise rates of convergence, and of options on realised variance, comparing those – whenever possible – to standard Monte Carlo methods. Product functional quantization has proved to be rather flexible and suitable to price a wide range of options on different underlyings, in the context of rough stochastic volatility models. Moreover, this methodology achieves its best performances when pricing path-dependent options, which opens the gate to future investigations.

It is also worth mentioning that product functional quantization, being an exact method, could be exploited to obtain a control variate to reduce the variance in Monte Carlo simulations. Another direction to investigate is the comparison with product functional quantization obtained starting from alternative series representation of the Riemann-Liouville process. Finally, it would be interesting to apply this technique to alternative rough volatility models such as rough Heston.

Appendix

3.A Proofs

3.A.1 Proof of Proposition 3.3.6

Consider a fixed $N \geq 1$ and $(m, \mathbf{d})$ for $\mathbf{d} \in \mathcal{D}_m^N$. We have

$$\begin{aligned}
\mathbb{E}\left[\left\|Z - \widehat{Z}^{\mathbf{d}}\right\|^2_{L^2[0,1]}\right] &= \mathbb{E}\left[\left\|\sum_{n\geq 1}\mathcal{K}[\psi_n](\cdot)\xi_n - \sum_{n=1}^{m}\mathcal{K}[\psi_n](\cdot)\widehat{\xi}_n^{d(n)}\right\|^2_{L^2[0,1]}\right] \\
&= \mathbb{E}\left[\left\|\sum_{n=1}^{m}\mathcal{K}[\psi_n](\cdot)(\xi_n - \widehat{\xi}_n^{d(n)}) + \sum_{k\geq m+1}\mathcal{K}[\psi_k](\cdot)\xi_k\right\|^2_{L^2[0,1]}\right] \\
&= \mathbb{E}\left[\int_0^1\left|\sum_{n=1}^{m}\mathcal{K}[\psi_n](t)(\xi_n - \widehat{\xi}_n^{d(n)}) + \sum_{k\geq m+1}\mathcal{K}[\psi_k](t)\xi_k\right|^2 \mathrm{d}t\right] \\
&= \int_0^1\left(\sum_{n=1}^{m}\mathcal{K}[\psi_n]^2(t)\mathbb{E}\left[|\xi_n - \widehat{\xi}_n^{d(n)}|^2\right] + \sum_{k\geq m+1}\mathcal{K}[\psi_k]^2(t)\right)\mathrm{d}t \\
&= \int_0^1\left(\sum_{n=1}^{m}\mathcal{K}[\psi_n]^2(t)\varepsilon^{d(n)}(\xi_n)^2 + \sum_{k\geq m+1}\mathcal{K}[\psi_k]^2(t)\right)\mathrm{d}t, \qquad (3.A.1)
\end{aligned}$$

using Fubini's Theorem and the fact that $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. Gaussian and where $\varepsilon^{d(n)}(\xi_n) := \inf_{(\alpha_1,\dots,\alpha_{d(n)})\in\mathbb{R}^{d(n)}}\sqrt{\mathbb{E}[\min_{1\leq i\leq d(n)}|\xi_n - \alpha_i|^2]}$. The Extended Pierce Lemma [128, Theorem 1(b)] ensures that $\varepsilon^{d(n)}(\xi_n) \leq \frac{L}{d(n)}$ for a suitable positive constant L. Exploiting this error bound and the property **(B)** for $\mathcal{K}[\psi_n]$ in Assumption 3.2.3, we obtain

$$\begin{aligned}
\mathbb{E}\left[\|Z - \widehat{Z}^{\mathbf{d}}\|^2_{L^2[0,1]}\right] &= \sum_{n=1}^{m}\left(\int_0^1\mathcal{K}[\psi_n]^2(t)\mathrm{d}t\right)\varepsilon^{d(n)}(\xi_n)^2 + \sum_{k\geq m+1}\int_0^1\mathcal{K}[\psi_k]^2(t)\mathrm{d}t \qquad (3.A.2) \\
&\leq C_2^2\left\{\sum_{n=1}^{m}n^{-(2H+1)}\varepsilon^{d(n)}(\xi_n)^2 + \sum_{k\geq m+1}k^{-(2H+1)}\right\} \\
&\leq C_2^2\left\{\sum_{n=1}^{m}n^{-(2H+1)}\frac{L^2}{d(n)^2} + \sum_{k\geq m+1}k^{-(2H+1)}\right\} \\
&\leq \widetilde{C}\left(\sum_{n=1}^{m}\frac{1}{n^{2H+1}d(n)^2} + \sum_{k\geq m+1}k^{-(2H+1)}\right),
\end{aligned}$$

with $\widetilde{C} = \max\{L^2C_2^2, C_2^2\}$. Inspired by [119, Section 4.1], we now look for an "optimal" choice of $m \in \mathbb{N}$ and $\mathbf{d} \in \mathcal{D}_m^N$. This reduces the error in approximating Z with a product quantization of the form in (3.3.2). Define the optimal product functional quantization $\widehat{Z}^{N,\star}$ of order N as the $\widehat{Z}^{\mathbf{d}}$ which realises the minimal error:

$$\mathbb{E}\left[\left\|Z - \widehat{Z}^{N,\star}\right\|^2_{L^2[0,1]}\right] = \min\left\{\mathbb{E}\left[\left\|Z - \widehat{Z}^{\mathbf{d}}\right\|^2_{L^2[0,1]}\right], m \in \mathbb{N}, \mathbf{d} \in \mathcal{D}_m^N\right\}.$$

From (3.A.2) we deduce

$$\mathbb{E}\left[\left\|Z - \widehat{Z}^{N,\star}\right\|^2_{L^2[0,1]}\right] \le \widetilde{C} \inf_{m\in\mathbb{N}} \left\{ \sum_{k\ge m+1} \frac{1}{k^{2H+1}} + \inf\left\{ \sum_{n=1}^{m} \frac{1}{n^{2H+1}d(n)^2}, \mathbf{d} \in \mathcal{D}_m^N \right\}\right\}. \tag{3.A.3}$$

For any fixed $m \in \mathbb{N}$ we associate to the internal minimisation problem the one we get by relaxing the hypothesis that $d(n) \in \mathbb{N}$:

$$\mathfrak{I} := \inf\left\{ \sum_{n=1}^{m} \frac{1}{n^{2H+1}z(n)^2}, \{z(n)\}_{n=1,\dots,m} \in (0,\infty) : \prod_{n=1}^{m} z(n) \le N\right\}.$$

For this infimum, we derive a simple solution exploiting the arithmetic-geometric inequality using Lemma 3.B.2. Setting $\widetilde{z}(n) := \gamma_{N,m} n^{-(H+\frac{1}{2})}$, with $\gamma_{N,m} := N^{\frac{1}{m}}\left(\prod_{j=1}^{m} j^{-(2H+1)}\right)^{-\frac{1}{2m}}$, $n = 1,\dots,m$, we get

$$\mathfrak{I} = \sum_{n=1}^{m} \frac{1}{n^{2H+1}\widetilde{z}(n)^2} = N^{-\frac{2}{m}} m \left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}},$$

and notice that the sequence $\{\widetilde{z}(n)\}$ is decreasing. Since ultimately the vector $\mathbf{d}$ consists of integers, we use $\widetilde{d}(n) = \lfloor \widetilde{z}(n) \rfloor$, $n = 1,\dots,m$. In fact, this choice guarantees that

$$\prod_{n=1}^{m} \widetilde{d}(n) = \prod_{n=1}^{m} \lfloor \widetilde{z}(n) \rfloor \le \prod_{n=1}^{m} \widetilde{z}(n) = N.$$

Furthermore, setting $\widetilde{d}(j) = \lfloor \widetilde{z}(j) \rfloor$ for each $j \in \{1,\dots,m\}$, we obtain

$$\frac{\widetilde{d}(j)+1}{(j^{-(2H+1)})^{\frac{1}{2}}} = j^{H+\frac{1}{2}}(\lfloor \widetilde{z}(j) \rfloor + 1) \ge j^{H+\frac{1}{2}}\widetilde{z}(j) = \frac{j^{H+\frac{1}{2}} N^{\frac{1}{m}}}{j^{H+\frac{1}{2}}} \left\{\prod_{n=1}^{m} \frac{1}{n^{2H+1}}\right\}^{-\frac{1}{2m}} = N^{\frac{1}{m}} \left\{\prod_{n=1}^{m} \frac{1}{n^{2H+1}}\right\}^{-\frac{1}{2m}}.$$

Ordering the terms, we have $(\widetilde{d}(j)+1)^2 N^{-\frac{2}{m}} \left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}} \ge j^{-(2H+1)}$, for each $j \in \{1,\dots,m\}$. From this we deduce the following inequality (notice that the left-hand side term is defined only if $\widetilde{d}(1),\dots,\widetilde{d}(m) > 0$):

$$\begin{aligned}\sum_{j=1}^{m} j^{-(2H+1)}\widetilde{d}(j)^{-2} &\le \sum_{j=1}^{m} \left(\frac{\widetilde{d}(j)+1}{\widetilde{d}(j)}\right)^2 N^{-\frac{2}{m}} \left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}} \\ &= N^{-\frac{2}{m}} \left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}} \sum_{j=1}^{m} \left(\frac{\widetilde{d}(j)+1}{\widetilde{d}(j)}\right)^2 \\ &\le 4mN^{-\frac{2}{m}} \left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}}.\end{aligned} \tag{3.A.4}$$

Hence, we are able to make a first error estimation, placing in the internal minimisation of the right-hand side of (3.A.3) the result of inequality in (3.A.4).

$$\begin{aligned}\mathbb{E}\left[\left\|Z-\widehat{Z}^{N,\star}\right\|^2_{L^2[0,1]}\right] &\leq \widetilde{C}\inf\left\{\sum_{k\geq m+1}\frac{1}{k^{2H+1}}+4mN^{-\frac{2}{m}}\left(\prod_{n=1}^{m}n^{-(2H+1)}\right)^{\frac{1}{m}}, m\in I(N)\right\} \qquad (3.A.5)\\ &\leq C'\inf\left\{\sum_{k\geq m+1}\frac{1}{k^{2H+1}}+mN^{-\frac{2}{m}}\left(\prod_{n=1}^{m}n^{-(2H+1)}\right)^{\frac{1}{m}}, m\in I(N)\right\},\end{aligned}$$

where $C'=4\widetilde{C}$ and the set

$$I(N):=\{m\in\mathbb{N}: N^{\frac{2}{m}}m^{-(2H+1)}\Big(\prod_{n=1}^{m}n^{-(2H+1)}\Big)^{-\frac{1}{m}}\geq 1\}, \qquad (3.A.6)$$

which represents all m's such that all $\widetilde{d}(1),\dots,\widetilde{d}(m)$ are positive integers. This is to avoid the case where $\prod_{i=1}^{m}\widetilde{d}(i)\leq N$ holds only because one of the factors is zero. In fact, for all $n\in\{1,\dots,m\}$, $\widetilde{d}(n)=\lfloor\widetilde{z}(n)\rfloor$ is a positive integer if and only if $\widetilde{z}(n)\geq 1$. Thanks to the monotonicity of $\{z(n)\}_{n=1,\dots,m}$, we only need to check that

$$\widetilde{z}(m)=N^{\frac{1}{m}}m^{-(H+\frac{1}{2})}\Big(\prod_{n=1}^{m}n^{-(2H+1)}\Big)^{-\frac{1}{2m}}\geq 1.$$

First, let us show that $I(N)$, defined in (3.A.6) for each $N\geq 1$, is a non-empty finite set with maximum given by $m^*(N)$ of order $\log(N)$. We can rewrite it as $I(N)=\{m\geq 1: a_m\leq\log(N)\}$, where

$$a_n=\frac{1}{2}\log\left(\prod_{j=1}^{n}\frac{n^{2H+1}}{j^{2H+1}}\right).$$

We can now verify that the sequence a_n is increasing in $n\in\mathbb{N}$:

$$\begin{aligned}&a_n\leq a_{n+1}\\ \iff\ &\sum_{j=1}^{n}\log\left(j^{-(2H+1)}\right)-n\log\left(n^{-(2H+1)}\right)\leq\sum_{j=1}^{n+1}\log\left(j^{-(2H+1)}\right)-(n+1)\log\left((n+1)^{-(2H+1)}\right)\\ \iff\ &-n\log\left(n^{-(2H+1)}\right)\leq\log\left((n+1)^{-(2H+1)}\right)-(n+1)\log\left((n+1)^{-(2H+1)}\right)\\ \iff\ &\log\left(n^{-(2H+1)}\right)\geq\log\left((n+1)^{-(2H+1)}\right),\end{aligned}$$

which is obviously true. Furthermore the sequence $(a_n)_n$ diverges to infinity since

$$\prod_{j=1}^{n}\frac{n^{(2H+1)}}{j^{(2H+1)}}=n^{(2H+1)n}\prod_{j=1}^{n}\frac{1}{j^{(2H+1)}}\geq n^{(2H+1)n}\prod_{j=2}^{n}\frac{1}{j^{(2H+1)}}\geq n^{(2H+1)n}\frac{1}{n^{(2H+1)(n-1)}}\geq n^{(2H+1)}.$$

and $H\in(0,\frac{1}{2})$. We immediately deduce that $I(N)$ is finite and, since $\{1\}\subset I(N)$, it is also non-empty. Hence $I(N)=\{1,\dots,m^*(N)\}$. Moreover, for all $N\geq 1$, $a_{m^*(N)}\leq\log(N)<a_{m^*(N)+1}$, which implies that $m^*(N)=\mathcal{O}(\log(N))$.

Now, the error estimation in (3.A.5) can be further simplified exploiting the fact that, for each $m \in I(N)$,

$$mN^{-\frac{2}{m}}\left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{\frac{1}{m}} = m m^{-(2H+1)}\left(m^{-(2H+1)}N^{\frac{2}{m}}\left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{-\frac{1}{m}}\right)^{-1} \leq m^{-2H}.$$

The last inequality is a consequence of the fact that $\left(\prod_{n=1}^{m} n^{-(2H+1)}\right)^{-\frac{1}{m}} \geq 1$ by definition. Hence,

$$\mathbb{E}\left[\|Z - \widehat{Z}^{N,\star}\|^2_{L^2[0,1]}\right] \leq C' \inf\left\{\sum_{k\geq m+1} \frac{1}{k^{2H+1}} + m^{-2H}, m \in I(N)\right\}, \tag{3.A.7}$$

for some suitable constant $C' > 0$.

Consider now the sequence $\{b_n\}_{n\in\mathbb{N}}$, given by $b_n = \sum_{k\geq n+1} \frac{1}{k^{2H+1}} + n^{-2H}$. For $n \geq 1$,

$$b_{n+1} - b_n = \sum_{k\geq n+2} \frac{1}{k^{2H+1}} + \frac{1}{(n+1)^{2H}} - \left[\sum_{k\geq n+1} \frac{1}{k^{2H+1}} + \frac{1}{n^{2H}}\right] = -\frac{1}{(n+1)^{2H}} + \frac{1}{(n+1)^{2H+1}} - \frac{1}{n^{2H}} \leq 0,$$

so that the sequence is decreasing and the infimum in (3.A.7) is attained at $m = m^*(N)$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\|Z - \widehat{Z}^{N,\star}\|^2_{L^2[0,1]}\right] &\leq C' \inf\left\{\sum_{k\geq m+1} \frac{1}{k^{2H+1}} + m^{-2H}, m \in I(N)\right\} \\
&= C'\left(\sum_{k\geq m^*(N)+1} \frac{1}{k^{2H+1}} + m^*(N)^{-2H}\right) \leq C'\left(m^*(N)^{-2H-1+1} + m^*(N)^{-2H}\right) \\
&= 2C' m^*(N)^{-2H} \leq C \log(N)^{-2H}.
\end{aligned}$$

3.A.2 Proof of Remark 3.2.5

This can be proved specialising the computations done in [120, page 656]. Consider an arbitrary index $n \geq 1$. For all $t, s \in [0,1]$, exploiting Assumption 3.2.3, we have that, for any $\rho \in [0,1]$,

$$\begin{aligned}
\left|\mathcal{K}[\psi_n](t) - \mathcal{K}[\psi_n](s)\right| &= \left|\mathcal{K}[\psi_n](t) - \mathcal{K}[\psi_n](s)\right|^{\rho}\left|\mathcal{K}[\psi_n](t) - \mathcal{K}[\psi_n](s)\right|^{1-\rho} \\
&\leq \left(\sup_{u,v\in[0,1],u\neq v} \frac{|\mathcal{K}[\psi_n](u) - \mathcal{K}[\psi_n](v)|}{|u-v|^{H+\frac{1}{2}}}|t-s|^{H+\frac{1}{2}}\right)^{\rho}\left(2\sup_{t\in[0,1]} \mathcal{K}[\psi_n](t)\right)^{1-\rho} \\
&\leq (C_1 n)^{\rho}(2C_2 n^{-(H+\frac{1}{2})})^{1-\rho}|t-s|^{\rho(H+\frac{1}{2})} = C_\rho n^{\rho(H+\frac{3}{2})-(H+\frac{1}{2})}|t-s|^{\rho(H+\frac{1}{2})},
\end{aligned}$$

where $C_\rho := C_1^{\rho}(2C_2)^{1-\rho} < \infty$. Therefore

$$\left[\mathcal{K}[\psi_n]\right]_{\rho(H+\frac{1}{2})} = \sup_{t\neq s\in[0,1]} \frac{|\mathcal{K}[\psi_n](t) - \mathcal{K}[\psi_n](s)|}{|t-s|^{\rho(H+\frac{1}{2})}} \leq C_\rho n^{\rho(H+\frac{3}{2})-(H+\frac{1}{2})}. \tag{3.A.8}$$

Notice that $\rho(H+\frac{3}{2})-(H+\frac{1}{2})<-\frac{1}{2}$ when $\rho\in[0,\frac{H}{H+3/2}]$ so that (3.A.8) implies

$$\sum_{n=1}^{\infty}\left[\mathcal{K}[\psi_n]\right]^2_{\rho(H+\frac{1}{2})}\leq C_\rho^2\sum_{n=1}^{\infty}n^{2\rho(H+\frac{3}{2})-2(H+\frac{1}{2})}\leq C_\rho^2\sum_{n=1}^{\infty}n^{-(1+\varepsilon)}=K<\infty.$$

In particular,

$$\mathbb{E}\left[|Y_t-Y_s|^2\right]=\sum_{n=1}^{\infty}\left|\mathcal{K}[\psi_n](t)-\mathcal{K}[\psi_n](s)\right|^2\leq\sum_{n=1}^{\infty}\left[\mathcal{K}[\psi_n]\right]^2_{\rho(H+\frac{1}{2})}|t-s|^{2\rho(H+\frac{1}{2})}\leq K|t-s|^{2\rho(H+\frac{1}{2})}.$$

As noticed in Remark 3.2.2 the process Y is centred Gaussian. Hence, for each $t,s\in[0,1]$ so is Y_t-Y_s. Proposition 3.B.1 therefore implies that, for any $r\in\mathbb{N}$,

$$\mathbb{E}\left[|Y_t-Y_s|^{2r}\right]=\mathbb{E}\left[|Y_t-Y_s|^2\right]^r(2r-1)!!\leq K'|t-s|^{2r\rho(H+\frac{1}{2})},$$

where $K'=K^r(2r-1)!!$, yielding existence of a continuous version of Y since choosing $r\in\mathbb{N}$ such that $2r\rho(H+\frac{1}{2})>1$, Kolmogorov continuity theorem [105, Theorem 3.23] applies directly.

3.A.3 Proof of Lemma 3.4.5

Let $H_+ := H+\frac{1}{2}$. Using [107, Corollary 1, Equation (12)] (with $\psi=b_2+b_1-a>1/2$), the identity

$${}_1F_2(a,b_1,b_2,-r)=\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a)\sqrt{\pi}}\int_0^1 G^{2,0}_{2,2}\left([b_1,b_2],\left[a,\frac{1}{2}\right],u\right)\cos\left(2\sqrt{ru}\right)\frac{\mathrm{d}u}{u},$$

holds for all $r>0$, where G denotes the Meijer-G function, generally defined through the so-called Mellin-Barnes type integral [118, Equation (1), Section 5.2]) as

$$G^{m,n}_{p,q}([a_1,\dots,a_p],[b_1,\dots,b_q],z)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}z^s\,\mathrm{d}s.$$

This representation holds if $z\neq 0$, $0\leq m\leq q$ and $0\leq n\leq p$, for integers m,n,p,q, and $a_k-b_j\neq 1,2,3,\dots$, for $k=1,2,\dots,n$ and $j=1,2,\dots,m$. The last constraint is set to prevent any pole of any $\Gamma(b_j-s)$, $j=1,2,\dots,m$, from coinciding with any pole of any $\Gamma(1-a_k+s)$, $k=1,2,\dots,n$. With $a>0$, $b_2=1+a$ and $b_1=\frac{1}{2}$, since $G^{2,0}_{2,2}\left(\left[\frac{1}{2},a+1\right],\left[a,\frac{1}{2}\right],u\right)=u^a$, we can therefore write

$$\int_0^1 u^{a-1}\cos\left(2\sqrt{ru}\right)\mathrm{d}u=\frac{1}{a}{}_1F_2\left(a;\frac{1}{2},a+1;-r\right). \tag{3.A.9}$$

Similarly, using integration by parts and properties of generalised hypergeometric functions,

$$\begin{aligned}\int_0^1 u^{a-1}\sin\left(2\sqrt{ru}\right)\mathrm{d}u&=\frac{\sin(2\sqrt{r})}{a}-\frac{\sqrt{r}}{a}\int_0^1 u^{a-\frac{1}{2}}\cos(2\sqrt{ru})\mathrm{d}u\\&=\frac{\sin(2\sqrt{r})}{a}-\frac{\sqrt{r}}{a(a+\frac{1}{2})}{}_1F_2\left(a+\frac{1}{2};\frac{1}{2},a+\frac{3}{2};-r\right)\end{aligned} \tag{3.A.10}$$

$$= \frac{2\sqrt{r}}{a+\frac{1}{2}} {}_1F_2\left(a+\frac{1}{2}; \frac{3}{2}, a+\frac{3}{2}; -r\right),$$

where the last step follows from the definition of generalised sine function $\sin(z) = z\,{}_0F_1(\frac{3}{2}, -\frac{1}{4}z^2)$. Indeed, exploiting (3.2.6), we have

$$\begin{aligned}
\frac{\sin(2\sqrt{r})}{a} &- \frac{\sqrt{r}}{a(a+\frac{1}{2})} {}_1F_2\left(a+\frac{1}{2}; \frac{1}{2}, a+\frac{3}{2}; -r\right) \\
&= \frac{2\sqrt{r}}{a} {}_0F_1\left(\frac{3}{2}, -r\right) - \frac{\sqrt{r}}{a(a+\frac{1}{2})} {}_1F_2\left(a+\frac{1}{2}; \frac{1}{2}, a+\frac{3}{2}; -r\right) \\
&= \frac{2\sqrt{r}}{a\left(a+\frac{1}{2}\right)} \left[\left(a+\frac{1}{2}\right) {}_0F_1\left(\frac{3}{2}; -r\right) - \frac{1}{2} {}_1F_2\left(a+\frac{1}{2}; \frac{1}{2}, a+\frac{3}{2}; -r\right)\right] \\
&= \frac{2\sqrt{r}}{a\left(a+\frac{1}{2}\right)} \left[\left(a+\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-r)^k}{k!(3/2)_k} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(a+1/2)_k}{k!(1/2)_k(a+3/2)_k}(-r)^k\right] \\
&= \frac{2\sqrt{r}}{a\left(a+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{(a+1/2)}{(3/2)_k} - \frac{1/2(a+1/2)_k}{(1/2)_k(a+3/2)_k}\right](-r)^k \\
&= \frac{2\sqrt{r}}{a\left(a+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{a(a+1/2)_k}{(3/2)_k(a+3/2)_k}\right](-r)^k \\
&= \frac{2\sqrt{r}}{\left(a+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a+1/2)_k}{(3/2)_k(a+3/2)_k}(-r)^k = \frac{2\sqrt{r}}{\left(a+\frac{1}{2}\right)} {}_1F_2\left(a+\frac{1}{2}; \frac{3}{2}, a+\frac{3}{2}; -r\right).
\end{aligned}$$

Letting $\alpha := H - \frac{1}{2}$, $\tau := t - T$, and mapping $v := t - u$, $w := \frac{v}{t}$ and $y := w^2$, we write

$$\begin{aligned}
\int_0^T (t-u)^{\alpha} \mathrm{e}^{\mathrm{i}\pi u} \mathrm{d}u &= \mathrm{e}^{\mathrm{i}\pi t} \int_{(t-T)}^{t} v^{\alpha} \mathrm{e}^{-\mathrm{i}\pi v} \mathrm{d}v = \mathrm{e}^{\mathrm{i}\pi t} \left[\int_0^t v^{\alpha} \mathrm{e}^{-\mathrm{i}\pi v} \mathrm{d}v - \int_0^{\tau} v^{\alpha} \mathrm{e}^{-\mathrm{i}\pi v} \mathrm{d}v\right] \\
&= \mathrm{e}^{\mathrm{i}\pi t} \left[t^{1+\alpha} \int_0^1 w^{\alpha} \mathrm{e}^{-\mathrm{i}\pi w t} \mathrm{d}w - \tau^{1+\alpha} \int_0^1 w^{\alpha} \mathrm{e}^{-\mathrm{i}\pi w \tau} \mathrm{d}w\right] \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2} \left[t^{1+\alpha} \int_0^1 y^{\frac{\alpha-1}{2}} \mathrm{e}^{-\mathrm{i}\pi t\sqrt{y}} \mathrm{d}y - \tau^{1+\alpha} \int_0^1 y^{\frac{\alpha-1}{2}} \mathrm{e}^{-\mathrm{i}\pi y \tau \sqrt{y}} \mathrm{d}y\right] \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2} \left[I(t) - I(\tau)\right], \qquad (3.A.11)
\end{aligned}$$

where $I(z) := z^{1+\alpha} \int_0^1 v^{\frac{\alpha-1}{2}} \mathrm{e}^{-\mathrm{i}\pi z\sqrt{v}} \mathrm{d}v$.
We therefore write, for $z \in \{t, \tau\}$, using (3.A.9)-(3.A.10), $\pi z = 2\sqrt{r}$, and identifying $a - 1 = \frac{\alpha-1}{2}$,

$$\begin{aligned}
I(z) = z^{1+\alpha} \int_0^1 v^{\frac{\alpha-1}{2}} \mathrm{e}^{-\mathrm{i}\pi z\sqrt{v}} \mathrm{d}v &= z^{1+\alpha} \int_0^1 v^{\frac{\alpha-1}{2}} \cos(\pi z\sqrt{v}) \mathrm{d}v - \mathrm{i} z^{1+\alpha} \int_0^1 v^{\frac{\alpha-1}{2}} \sin(\pi z\sqrt{v}) \mathrm{d}v \\
= \frac{2z^{1+\alpha}}{H_+} {}_1F_2\left(\frac{H_+}{2}; \frac{1}{2}, 1+\frac{H_+}{2}; -r\right) &- \mathrm{i} z^{H_+} \frac{4\sqrt{r}}{1+H_+} {}_1F_2\left(\frac{1}{2}+\frac{H_+}{2}; \frac{3}{2}, \frac{3}{2}+\frac{H_+}{2}; -r\right) \\
= \frac{z^{H_+}}{h_1} {}_1F_2\left(h_1; \frac{1}{2}, 1+h_1; -\frac{\pi^2 z^2}{4}\right) &- \mathrm{i} \frac{\pi z^{1+H_+}}{h_2} {}_1F_2\left(h_2; \frac{3}{2}, 1+h_2; -\frac{\pi^2 z^2}{4}\right),
\end{aligned}$$

since $\alpha = H - \frac{1}{2} = H_+ - 1$, $h_1 = \frac{H_+}{2}$ and $h_2 = \frac{1}{2} + h_1$. Plugging these into (3.A.11), we obtain

$$\begin{aligned}
\int_0^T (t-u)^\alpha \mathrm{e}^{\mathrm{i}\pi u}\mathrm{d}u &= \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2}\left[I(t) - I(\tau)\right] \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2}\left[\frac{z^{H_+}}{h_1}{}_1F_2\left(h_1;\frac{1}{2},1+h_1;-\frac{\pi^2 z^2}{4}\right) - \mathrm{i}\frac{\pi z^{1+H_+}}{h_2}{}_1F_2\left(h_2;\frac{3}{2},1+h_2;-\frac{\pi^2 z^2}{4}\right)\right]_{z=t} \\
&\quad - \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2}\left[\frac{z^{H_+}}{h_1}{}_1F_2\left(h_1;\frac{1}{2},1+h_1;-\frac{\pi^2 z^2}{4}\right) - \mathrm{i}\frac{\pi z^{1+H_+}}{h_2}{}_1F_2\left(h_2;\frac{3}{2},1+h_2;-\frac{\pi^2 z^2}{4}\right)\right]_{z=\tau} \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi t}}{2h_1}\left[(t)^{H_+}{}_1F_2\left(h_1;\frac{1}{2},1+h_1;-\frac{\pi^2 t^2}{4}\right) - (\tau)^{H_+}{}_1F_2\left(h_1;\frac{1}{2},1+h_1;-\frac{\pi^2 \tau^2}{4}\right)\right] \\
&\quad - \mathrm{i}\frac{\pi\mathrm{e}^{\mathrm{i}\pi t}}{2h_2}\left[(t)^{1+H_+}{}_1F_2\left(h_2;\frac{3}{2},1+h_2;-\frac{\pi^2 t^2}{4}\right) - (\tau)^{1+H_+}{}_1F_2\left(h_2;\frac{3}{2},1+h_2;-\frac{\pi^2 \tau^2}{4}\right)\right] \\
&= \mathrm{e}^{\mathrm{i}\pi t}\left[\zeta_{\frac{1}{2}}(t,h_1) - \zeta_{\frac{1}{2}}(\tau,h_1) - \mathrm{i}\pi\left(\zeta_{\frac{3}{2}}(t,h_2) - \zeta_{\frac{3}{2}}(\tau,h_2)\right)\right],
\end{aligned}$$

where $\chi(z) := -\frac{1}{4}\pi^2 z^2$ and $\zeta_{\frac{1}{2}}$ and $\zeta_{\frac{3}{2}}$ as defined in the lemma.

3.A.4 Proof of Lemma 3.4.4

We first prove **(A)**. For each $n \in \mathbb{N}$ and all $t \in [T,1]$, recall that

$$\mathcal{K}_H^T[\psi_n](t) = \sqrt{2}\int_0^T (t-u)^{H-\frac{1}{2}}\cos\left(\frac{u}{\sqrt{\lambda_n}}\right)\mathrm{d}u = \sqrt{2}\int_{t-T}^t v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v,$$

with the change of variables $v = t-u$. Assume $T \le s < t \le 1$. Two situations are possible:

- If $0 \le s-T < t-T \le s < t \le 1$, we have

$$\begin{aligned}
\left|\mathcal{K}_H^T[\psi_n](t) - \mathcal{K}_H^T[\psi_n](s)\right| &= \sqrt{2}\left|\int_{t-T}^t v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v - \int_{s-T}^s v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right| \\
&\le \sqrt{2}\left(\left|\int_{t-T}^s v^{H-\frac{1}{2}}\left(\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right) - \cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\right)\mathrm{d}v\right|\right. \\
&\quad \left. + \left|\int_s^t v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right| + \left|\int_{s-T}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right|\right) \\
&\le \sqrt{2}\left(\int_{t-T}^s v^{H-\frac{1}{2}}\left|\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right) - \cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\right|\mathrm{d}v\right. \\
&\quad \left. + \int_s^t v^{H-\frac{1}{2}}\mathrm{d}v + \int_{s-T}^{t-T} v^{H-\frac{1}{2}}\mathrm{d}v\right) \\
&\le \sqrt{2}\left(\int_{t-T}^s v^{H-\frac{1}{2}}\left|\frac{t-s}{\sqrt{\lambda_n}}\right|\mathrm{d}v + K|t-s|^{H+\frac{1}{2}} + K|t-s|^{H+\frac{1}{2}}\right) \\
&\le \sqrt{2}\left(\frac{|t-s|}{\sqrt{\lambda_n}}\int_{t-T}^s v^{H-\frac{1}{2}}\mathrm{d}v + 2K|t-s|^{H+\frac{1}{2}}\right) \\
&\le \sqrt{2}\left(\frac{|t-s|}{\sqrt{\lambda_n}}\|(\cdot)^{H-\frac{1}{2}}\|_{L^1[0,1]} + 2K|t-s|^{H+\frac{1}{2}}\right) \le \widetilde{C}_1^T|t-s|^{H+\frac{1}{2}},
\end{aligned}$$

with $\widetilde{C}_1^T = \max\left\{2\sqrt{2}K, \sqrt{\frac{2}{\lambda_n}}\|(\cdot)^{H-\frac{1}{2}}\|_{L^1[0,1]}\right\} = \max\left\{2\sqrt{2}K, \frac{\sqrt{2}(2n-1)\pi}{2}\|(\cdot)^{H-\frac{1}{2}}\|_{L^1[0,1]}\right\}$, since cos(·) is Lipschitz on any compact and $\int_0^{\cdot} v^{H-\frac{1}{2}}\mathrm{d}v$ is $(H+\frac{1}{2})$-Hölder continuous.

- If $0 \le s-T \le s \le t-T \le t \le 1$,

$$\begin{aligned}
\left|\mathcal{K}_H^T[\psi_n](t) - \mathcal{K}_H^T[\psi_n](s)\right| &= \sqrt{2}\left|\int_{t-T}^{t} v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v - \int_{s-T}^{s} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right| \\
&= \sqrt{2}\left|\int_{t-T}^{t} v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v - \int_{s-T}^{s} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right. \\
&\quad + \int_{s}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v - \int_{s}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v \\
&\quad \left. + \int_{s}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v - \int_{s}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right| \\
&\le \sqrt{2}\left(\left|\int_{s}^{t-T} v^{H-\frac{1}{2}}\left(\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right) - \cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\right)\mathrm{d}v\right|\right. \\
&\quad \left. + \left|\int_{s}^{t} v^{H-\frac{1}{2}}\cos\left(\frac{t-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right| + \left|\int_{s-T}^{t-T} v^{H-\frac{1}{2}}\cos\left(\frac{s-v}{\sqrt{\lambda_n}}\right)\mathrm{d}v\right|\right) \\
&\le \cdots \le \widetilde{C}_1^T |t-s|^{H+\frac{1}{2}},
\end{aligned}$$

  where the dots correspond to the same computations as in the previous case and leads to the same estimation with the same constant $\widetilde{C}_1^T$.

This proves **(A)**.

To prove **(B)**, recall that, for $T \in [0,1]$ and $n \in \mathbb{N}$, the function $\mathcal{K}_H^T[\psi_n] : [T,1] \to \mathbb{R}$ reads

$$\begin{aligned}
\mathcal{K}_H^T[\psi_n](t) &= \sqrt{2}\int_0^T (t-s)^{H-\frac{1}{2}}\cos\left(\left(n-\frac{1}{2}\right)\pi s\right)\mathrm{d}s \\
&= \frac{\sqrt{2}}{m^{H+\frac{1}{2}}}\int_0^{mT}(mt-u)^{H-\frac{1}{2}}\cos(\pi u)\,\mathrm{d}u =: \Phi_m(t). \qquad (3.A.12)
\end{aligned}$$

with the change of variable $u = (n-\frac{1}{2})s =: ms$. Denote from now on $\widetilde{\mathbb{N}} := \{m = n - \frac{1}{2}, n \in \mathbb{N}\}$. From (3.A.12), we deduce, for each $m \in \widetilde{\mathbb{N}}$ and $t \in [T,1]$,

$$m^{H+\frac{1}{2}}\Phi_m(t) = \sqrt{2}\int_0^{mT}(mt-u)^{H-\frac{1}{2}}\cos(\pi u)\,\mathrm{d}u =: \sqrt{2}\phi_m(t). \qquad (3.A.13)$$

To end the proof of **(B)**, it therefore suffices to show that $(\phi_m(t))_{m\in\widetilde{\mathbb{N}},t\in[T,1]}$ is uniformly bounded since, in that case we have

$$\begin{aligned}
\|\mathcal{K}_H^T[\psi_n]\|_\infty &= \sup_{t\in[T,1]}|\mathcal{K}_H^T[\psi_n](t)| = \sup_{t\in[T,1]}|\Phi_{n-\frac{1}{2}}(t)| = \frac{\sqrt{2}}{(n-\frac{1}{2})^{H+\frac{1}{2}}}\sup_{t\in[T,1]}|\phi_{n-\frac{1}{2}}(t)| \\
&\le \frac{\sqrt{2}}{(n-\frac{1}{2})^{H+\frac{1}{2}}}\sup_{t\in[T,1],m\in\widetilde{\mathbb{N}}}|\phi_m(t)| \le \frac{\sqrt{2}}{(n-\frac{1}{2})^{H+\frac{1}{2}}}C \le C_2^T n^{-(H+\frac{1}{2})},
\end{aligned}$$

for some $C_2^T > 0$, proving **(B)**. The following guarantees the uniform boundedness of ϕ_x in (3.A.13).

Proposition 3.A.1. *For any $T \in [0,1]$, there exists $C > 0$ such that $|\phi_x(t)| \le C$ for all $x \ge 0$, $t \in [T,1]$.*

Proof. For $x > 0$, we write

$$\phi_x(t) = \int_0^{xT} (xt-u)^{H-\frac{1}{2}} \cos(\pi u)\, \mathrm{d}u = \Re\left\{\int_0^{xT} (xt-u)^{H-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi u} \mathrm{d}u\right\}.$$

Using the representation in Lemma 3.4.5, we are thus left to prove that the maps $\zeta_{\frac{1}{2}}(\cdot, h_1)$ and $\zeta_{\frac{3}{2}}(\cdot, h_2)$, defined in (3.4.3), are bounded on $[0,\infty)$ by, say $L_{\frac{1}{2}}$ and $L_{\frac{3}{2}}$. Indeed, in this case,

$$\begin{aligned}
\sup_{x>0,t\in[T,1]} |\phi_x(t)| &= \sup_{x>0,t\in[T,1]} \left| \int_0^{xT} (xt-u)^{H-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi u} \mathrm{d}u \right| \\
&\le \sup_{x>0,t\in[T,1]} \left| \frac{\mathrm{e}^{\mathrm{i}\pi xt}}{2} \left[\left(\zeta_{\frac{1}{2}}(xt,h_1) - \zeta_{\frac{1}{2}}(x(t-T),h_1)\right) - \mathrm{i}\pi\left(\zeta_{\frac{3}{2}}(xt,h_2) - \zeta_{\frac{3}{2}}(x(t-T),h_2)\right)\right]\right| \\
&\le \frac{1}{2} \sup_{y,z\in[0,\infty)} \left| \left(\zeta_{\frac{1}{2}}(y,h_1) - \zeta_{\frac{1}{2}}(z,h_1)\right) - \mathrm{i}\pi\left(\zeta_{\frac{3}{2}}(y,h_2) - \zeta_{\frac{3}{2}}(z,h_2)\right)\right| \\
&\le \pi \left\{ \sup_{y\in[0,\infty)} \left|\zeta_{\frac{1}{2}}(y,h_1)\right| + \sup_{y\in[0,\infty)} \left|\zeta_{\frac{3}{2}}(y,h_2)\right| \right\} \le L_{\frac{1}{2}} + L_{\frac{3}{2}} = C < +\infty.
\end{aligned}$$

The maps $\zeta_{\frac{1}{2}}(\cdot, h_1)$ and $\zeta_{\frac{3}{2}}(\cdot, h_2)$ are both clearly continuous. Moreover, as z tends to infinity $\zeta_k(z,h)$ converges to a constant c_k, for $(k,h) \in (\{\frac{1}{2},\frac{3}{2}\},\{h_1,h_2\})$. The identities

$$\frac{{}_1F_2\left(h;\frac{1}{2},1+h;-x\right)}{h} = \int_0^1 \frac{\cos(2\sqrt{xu})}{u^{1-h}}\mathrm{d}u \quad \text{and} \quad \frac{{}_1F_2\left(h;\frac{3}{2},1+h;-x\right)}{h} = \frac{1}{2\sqrt{x}}\int_0^1 \frac{\sin(2\sqrt{xu})}{u^{3/2-h}}\mathrm{d}u$$

hold (this can be checked with `Wolfram Mathematica` for example) and therefore,

$$\begin{aligned}
\zeta_{\frac{1}{2}}(z,h_1) &= \frac{z^{2h_1}}{2h_1} {}_1F_2\left(h_1;\frac{1}{2},1+h_1;-\frac{\pi^2 z^2}{4}\right) = \frac{z^{2h_1}}{2}\int_0^1 u^{h_1-1}\cos(\pi z\sqrt{u})\mathrm{d}u \\
&= \frac{z^{2h_1}}{2}\int_0^{\pi z} \frac{x^{2(h_1-1)}}{(\pi z)^{2(h_1-1)}}\cos(x)\frac{2x}{\pi^2 z^2}\mathrm{d}x = \frac{1}{\pi^{2h_1}}\int_0^{\pi z} x^{2h_1-1}\cos(x)\mathrm{d}x,
\end{aligned}$$

where, in the second line, we used the change of variables $x = \pi z\sqrt{u}$. In particular, as z tends to infinity, this converges to $\pi^{-2h_1}\int_0^{+\infty} x^{2h_1-1}\cos(x)\mathrm{d}x = \frac{\cos(\pi h_1)}{\pi^{2h_1}}\Gamma(2h_1) =: c_{1/2} \approx 0.440433$. Analogously, for $k = \frac{3}{2}$,

$$\begin{aligned}
\zeta_{\frac{3}{2}}(z,h_2) &= \frac{z^{2h_2}}{2h_2} {}_1F_2\left(h_2;\frac{3}{2},1+h_2;-\frac{\pi^2 z^2}{4}\right) = \frac{z^{2h_2}}{2\pi z}\int_0^1 u^{h_2-3/2}\sin(\pi z\sqrt{u})\mathrm{d}u \\
&= \frac{z^{2h_2-1}}{2\pi}\int_0^{\pi z} \frac{x^{2h_2-3)}}{(\pi z)^{2h_2-3)}}\sin(x)\frac{2x}{\pi^2 z^2}\mathrm{d}x = \frac{1}{\pi^{2h_2}}\int_0^{\pi z} x^{2(h_2-1)}\sin(x)\mathrm{d}x,
\end{aligned}$$

with the same change of variables as before. This converges to $\pi^{-2h_2}\int_0^{+\infty} x^{2h_2-2}\sin(x)\mathrm{d}x = \frac{-\cos(\pi h_2)}{\pi^{2h_2}}\Gamma(2h_2-1) =: c_{3/2} \approx 0.193$ as z tends to infinity. For $k > 0$, $\zeta_k(z,h) = z^{2h}(1+\mathcal{O}(z^2))$ at zero. Since $H \in (0,\frac{1}{2})$, the two functions are continuous and bounded and the proposition follows. □

3.A.5 Proof of Theorem 3.4.11

We only provide the proof of (3.4.5) since, as already noticed, that of (3.4.6) follows immediately. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with constant M. By Definitions (3.4.1) and (3.4.4), we have

$$
\begin{aligned}
&\left|\mathbb{E}\left[F\left(\mathrm{VIX}_T\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathrm{VIX}}_T^{\mathbf{d}}\right)\right]\right| \\
= &\left|\mathbb{E}\left[F\left(\left|\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\exp\left\{\gamma Z_t^{T,\Delta} + \frac{\gamma^2}{2}\left(\int_0^{t-T} K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)\right\}\mathrm{d}t\right|^{\frac{1}{2}}\right)\right]\right. \\
&\left.-\mathbb{E}\left[F\left(\left|\frac{1}{\Delta}\int_T^{T+\Delta} v_0(t)\exp\left\{\gamma \widehat{Z}_t^{T,\Delta,\mathbf{d}} + \frac{\gamma^2}{2}\left(\int_0^{t-T} K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)\right\}\mathrm{d}t\right|^{\frac{1}{2}}\right)\right]\right|.
\end{aligned}
$$

For clarity, let $Z := Z^{T,\Delta}$, $\widehat{Z} := \widehat{Z}^{T,\Delta,\mathbf{d}}$, $\mathfrak{H} := \int_T^{T+\Delta} h(t)\mathrm{e}^{\gamma Z_t}\mathrm{d}t$ and $\widehat{\mathfrak{H}} := \int_T^{T+\Delta} h(t)\mathrm{e}^{\gamma \widehat{Z}_t}\mathrm{d}t$, with

$$
h(t) := \frac{v_0(t)}{\Delta}\exp\left\{\frac{\gamma^2}{2}\left(\int_0^{t-T} K(s)^2\mathrm{d}s - \int_0^t K(s)^2\mathrm{d}s\right)\right\}, \qquad \text{for } t \in [T, T+\Delta].
$$

We can therefore write, using the Lipschitz property of F (with constant M) and Lemma 3.B.3,

$$
\begin{aligned}
\left|\mathbb{E}\left[F\left(\mathrm{VIX}_T\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathrm{VIX}}_T^{\mathbf{d}}\right)\right]\right| &= \left|\mathbb{E}\left[F\left(\mathfrak{H}^{\frac{1}{2}}\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathfrak{H}}^{\frac{1}{2}}\right)\right]\right| \leq \mathbb{E}\left[\left|F\left(\mathfrak{H}^{\frac{1}{2}}\right) - F\left(\widehat{\mathfrak{H}}^{\frac{1}{2}}\right)\right|\right] \\
&\leq M\mathbb{E}\left[\left|\mathfrak{H}^{\frac{1}{2}} - \widehat{\mathfrak{H}}^{\frac{1}{2}}\right|\right] \leq M\mathbb{E}\left[\left(\frac{1}{\mathfrak{H}} + \frac{1}{\widehat{\mathfrak{H}}}\right)\left|\mathfrak{H} - \widehat{\mathfrak{H}}\right|\right] \\
&=: M\mathbb{E}\left[A\left|\mathfrak{H} - \widehat{\mathfrak{H}}\right|\right] \leq M\mathbb{E}\left[A\int_T^{T+\Delta} h(t)\left|\mathrm{e}^{\gamma Z_t} - \mathrm{e}^{\gamma \widehat{Z}_t}\right|\mathrm{d}t\right] \\
&\leq M\mathbb{E}\left[A\int_T^{T+\Delta} h(t)\gamma\left(\mathrm{e}^{\gamma Z_t} + \mathrm{e}^{\gamma \widehat{Z}_t}\right)\left|Z_t - \widehat{Z}_t\right|\mathrm{d}t\right].
\end{aligned}
$$

Now, an application of Hölder's inequality yields

$$
\begin{aligned}
\left|\mathbb{E}\left[F\left(\mathrm{VIX}_T\right)\right] - \mathbb{E}\left[F\left(\widehat{\mathrm{VIX}}_T^{\mathbf{d}}\right)\right]\right| &\leq M\mathbb{E}\left[\gamma A\left|\int_T^{T+\Delta} h(t)^2\left(\mathrm{e}^{\gamma Z_t} + \mathrm{e}^{\gamma \widehat{Z}_t}\right)^2\mathrm{d}t\right|^{\frac{1}{2}}\left|\int_T^{T+\Delta}\left|Z_t - \widehat{Z}_t\right|^2\mathrm{d}t\right|^{\frac{1}{2}}\right] \\
&\leq M\mathbb{E}\left[(\gamma A)^2\int_T^{T+\Delta} h(t)^2\left(\mathrm{e}^{\gamma Z_t} + \mathrm{e}^{\gamma \widehat{Z}_t}\right)^2\mathrm{d}t\right]^{\frac{1}{2}}\mathbb{E}\left[\int_T^{T+\Delta}\left|Z_t - \widehat{Z}_t\right|^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&= \mathfrak{K}\,\mathbb{E}\left[\int_T^{T+\Delta}\left|Z_t - \widehat{Z}_t\right|^2\mathrm{d}t\right]^{\frac{1}{2}},
\end{aligned}
$$

where $\mathfrak{K} := M\mathbb{E}[\gamma^2 A^2\int_T^{T+\Delta} h(t)^2(\mathrm{e}^{\gamma Z_t} + \mathrm{e}^{\gamma \widehat{Z}_t})^2\mathrm{d}t]^{\frac{1}{2}}$. It remains to show that $\mathfrak{K}$ is a strictly positive finite constant. This follows from the fact that $\{Z_t\}_{t\in[T,T+\Delta]}$ does not explode in finite time (and so does not its quantization $\widehat{Z}$ either). The identity $(a+b)^2 \leq 2(a^2+b^2)$ and Hölder's inequality imply

$$\begin{aligned}
\mathfrak{K}^2 &\leq 4M^2\gamma^2\mathbb{E}\left[\left(\frac{1}{\mathfrak{H}}+\frac{1}{\widehat{\mathfrak{H}}}\right)\int_T^{T+\Delta} h(t)^2\left(\mathrm{e}^{2\gamma Z_t}+\mathrm{e}^{2\gamma\widehat{Z}_t}\right)\mathrm{d}t\right]\\
&\leq 4M^2\gamma^2\mathbb{E}\left[\left|\frac{1}{\mathfrak{H}}+\frac{1}{\widehat{\mathfrak{H}}}\right|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\int_T^{T+\Delta} h(t)^2\left(\mathrm{e}^{2\gamma Z_t}+\mathrm{e}^{2\gamma\widehat{Z}_t}\right)\mathrm{d}t\right|^2\right]^{\frac{1}{2}}\\
&\leq 16M^2\gamma^2\mathbb{E}\left[\frac{1}{\mathfrak{H}^2}+\frac{1}{\widehat{\mathfrak{H}}^2}\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\int_T^{T+\Delta} h(t)^2\mathrm{e}^{2\gamma Z_t}\mathrm{d}t\right|^2+\left|\int_T^{T+\Delta} h(t)^2\mathrm{e}^{2\gamma\widehat{Z}_t}\mathrm{d}t\right|^2\right]^{\frac{1}{2}}\\
&=: 16M^2\gamma^2(A_1+A_2)^{\frac{1}{2}}(B_1+B_2)^{\frac{1}{2}}.
\end{aligned}$$

We only need to show that A_1, A_2, B_1 and B_2 are finite. Since h is a positive continuous function on the compact interval $[T, T+\Delta]$, we have

$$\begin{aligned}
\mathfrak{H} &\geq \int_T^{T+\Delta}\inf_{s\in[T,T+\Delta]}\left(h(s)\mathrm{e}^{\gamma Z_s}\right)\mathrm{d}t \geq \Delta\inf_{s\in[T,T+\Delta]} h(s)\mathrm{e}^{\gamma Z_s} \\
&\geq \Delta\inf_{t\in[T,T+\Delta]} h(t)\inf_{s\in[T,T+\Delta]}\mathrm{e}^{\gamma Z_s} \geq \Delta\widetilde{h}\exp\left\{\gamma\inf_{s\in[T,T+\Delta]} Z_s\right\},
\end{aligned} \tag{3.A.14}$$

with $\widetilde{h} := \inf_{t\in[T,T+\Delta]} h(t) > 0$. The inequality (3.A.14) implies

$$\begin{aligned}
A_1 &= \mathbb{E}\left[\mathfrak{H}^{-2}\right] \leq \frac{\mathbb{E}\left[\exp\left\{-2\gamma\inf_{s\in[T,T+\Delta]} Z_s\right\}\right]}{\Delta^2\widetilde{h}^2} = \frac{\mathbb{E}\left[\exp\left\{2\gamma\sup_{s\in[T,T+\Delta]}(-Z_s)\right\}\right]}{\Delta^2\widetilde{h}^2}\\
&= \frac{1}{\Delta^2\widetilde{h}^2}\mathbb{E}\left[\exp\left\{2\gamma\sup_{s\in[T,T+\Delta]} Z_s\right\}\right],
\end{aligned}$$

since $-Z$ and Z have the same law. The process $Z = (Z_t)_{t\in[T,T+\Delta]}$ is a continuous centred Gaussian process defined on a compact set. Thus, by Theorem 1.5.4 in [2], it is almost surely bounded there. Furthermore, exploiting Lemma 3.B.4 and *Borel-TIS* inequality [2, Theorem 2.1.1], we have

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{2\gamma\sup_{s\in[T,T+\Delta]} Z_s}\right] =: \mathbb{E}\left[\mathrm{e}^{2\gamma\|Z\|}\right] = \int_0^{+\infty}\mathbb{P}\left(\mathrm{e}^{2\gamma\|Z\|} > u\right)\mathrm{d}u = \int_0^{+\infty}\mathbb{P}\left(\|Z\| > \frac{\log(u)}{2\gamma}\right)\mathrm{d}u\\
&= \int_0^{\mathrm{e}^{2\gamma\mathbb{E}[\|Z\|]}}\mathrm{d}u + \int_{\mathrm{e}^{2\gamma\mathbb{E}[\|V\|]}}^{+\infty}\mathbb{P}\left(\|Z\| > \frac{\log(u)}{2\gamma}\right)\mathrm{d}u = \mathrm{e}^{2\gamma\mathbb{E}[\|Z\|]} + \int_{\mathrm{e}^{2\gamma\mathbb{E}[\|V\|]}}^{+\infty}\mathrm{e}^{-\frac{1}{2}\left(\frac{\frac{1}{2\gamma}\log(u)-\mathbb{E}[\|Z\|]}{\sigma_T}\right)^2}\mathrm{d}u\\
&\leq \mathrm{e}^{2\gamma\mathbb{E}[\|Z\|]} + \int_0^{+\infty}\mathrm{e}^{-\frac{1}{2}\left(\frac{\frac{1}{2\gamma}\log(u)-\mathbb{E}[\|Z\|]}{\sigma_T}\right)^2}\mathrm{d}u,
\end{aligned} \tag{3.A.15}$$

with $\|Z\| := \sup_{s\in[T,T+\Delta]} Z_s$ and $\sigma_T^2 := \sup_{t\in[T,T+\Delta]} \mathbb{E}[Z_t^2]$. The change of variable $\frac{\log(u)}{2\gamma} = v$ in the last term in (3.A.15) yields

$$\int_0^{+\infty} \mathrm{e}^{-\frac{1}{2}\left(\frac{\frac{1}{2\gamma}\log(u)-\mathbb{E}[\|Z\|]}{\sigma_T}\right)^2} \mathrm{d}u = 2\gamma \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\left(\frac{v-\mathbb{E}[\|Z\|]}{\sigma_T}\right)^2} \mathrm{e}^{2\gamma v}\mathrm{d}v = \sqrt{2\pi}2\gamma\mathbb{E}[\mathrm{e}^{2\gamma Y}],$$

since $Y \sim \mathcal{N}(\mathbb{E}[\|Z\|], \sigma_T)$, and hence A_1 is finite. Now, notice that, in analogy to the last line of the proof of Proposition 3.3.12, for any $t \in [T, T+\Delta]$, we have

$$\mathbb{E}\left[Z_t \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_t \middle| \{\widehat{\xi}_n^{d(n)}\}_{n=1,\dots,m}\right] \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right] = \mathbb{E}\left[\widehat{Z}_t \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right] = \widehat{Z}_t, \tag{3.A.16}$$

since the sigma-algebra generated by $(\widehat{Z}_s)_{s\in[T,T+\Delta]}$ is included in the sigma-algebra generated by $\{\widehat{\xi}_n^{d(n)}\}_{n=1,\dots,m}$. Now, exploiting, in sequence, (3.A.16), the conditional version of $\sup_{t\in[T_1,T_2]} \mathbb{E}[f_t] \le \mathbb{E}[\sup_{t\in[T_1,T_2]} f_t]$, conditional Jensen's inequality together with the convexity of $x \mapsto \mathrm{e}^{\gamma x}$, for $\gamma > 0$ and the tower property, we obtain

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} \widehat{Z}_t\right\}\right] &= \mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} \mathbb{E}\left[Z_t \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right]\right\}\right] \\
&\le \mathbb{E}\left[\exp\left\{\gamma \mathbb{E}\left[\sup_{t\in[T,T+\Delta]} Z_t \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right]\right\}\right] \\
&\le \mathbb{E}\left[\mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} Z_t\right\} \middle| (\widehat{Z}_s)_{s\in[T,T+\Delta]}\right]\right] \\
&= \mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} Z_t\right\}\right].
\end{aligned}$$

Thus, we have

$$A_2 = \mathbb{E}\left[\widehat{\mathfrak{H}}^{-2}\right] \le \frac{1}{\Delta^2 \widetilde{h}^2}\mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} \widehat{Z}_t\right\}\right] \le \frac{1}{\Delta^2 \widetilde{h}^2}\mathbb{E}\left[\exp\left\{\gamma \sup_{t\in[T,T+\Delta]} Z_t\right\}\right],$$

which is finite because of the proof of the finiteness of A_1, above.

Exploiting Fubini's theorem we rewrite B_1 as

$$B_1 = \mathbb{E}\left[\left(\int_T^{T+\Delta} h(t)^2 \mathrm{e}^{2\gamma Z_t}\mathrm{d}t\right)^2\right] = \int_T^{T+\Delta}\int_T^{T+\Delta} h(t)^2 h(s)^2 \mathbb{E}\left[\mathrm{e}^{2\gamma(Z_t+Z_s)}\right]\mathrm{d}t\mathrm{d}s.$$

Since $(Z_t)_{t\in[T,T+\Delta]}$ is centred Gaussian with covariance $\mathbb{E}[Z_t Z_s] = \int_0^T K(t-u)K(s-u)\mathrm{d}u$, then $(Z_t + Z_s) \sim \mathcal{N}(0, g(t,s))$, with $g(t,s) := \mathbb{E}[(Z_t+Z_s)^2] = \int_0^T (K(t-u)+K(s-u))^2\mathrm{d}u$ and therefore

$$B_1 = \int_T^{T+\Delta}\int_T^{T+\Delta} h(t)^2 h(s)^2 \mathrm{e}^{2\gamma^2 g(t,s)}\mathrm{d}t\mathrm{d}s$$

is finite since both h and g are continuous on compact intervals. Finally, for B_2 we have

$$\begin{aligned} B_2 &= \mathbb{E}\left[\left(\int_T^{T+\Delta} h(t)^2 \mathrm{e}^{2\gamma\widehat{Z}_t}\mathrm{d}t\right)^2\right] = \int_T^{T+\Delta}\int_T^{T+\Delta} h(t)^2 h(s)^2 \mathbb{E}\left[\mathrm{e}^{2\gamma(\widehat{Z}_t+\widehat{Z}_s)}\right]\mathrm{d}t\mathrm{d}s \\ &\leq \int_T^{T+\Delta}\int_T^{T+\Delta} h(t)^2 h(s)^2 \mathbb{E}\left[\mathrm{e}^{2\gamma(Z_t+Z_s)}\right]\mathrm{d}t\mathrm{d}s = B_1, \end{aligned}$$

where we have used the fact that for all $t, s \in [T, T+\Delta]$, $(\widehat{Z}_t + \widehat{Z}_s)$ is a stationary quantizer for $(Z_t + Z_s)$ and so $\mathbb{E}[\mathrm{e}^{2\gamma(\widehat{Z}_t+\widehat{Z}_s)}] \leq \mathbb{E}[\mathrm{e}^{2\gamma(Z_t+Z_s)}]$ since $f(x) = \mathrm{e}^{2\gamma x}$ is a convex function (see Remark 3.3.9 in Section 3.3.1). Therefore B_2 is finite and the proof follows.

3.B Some useful results

We recall some important results used throughout the text. Straightforward proofs are omitted.

Proposition 3.B.1. *For a Gaussian random variable $Z \sim \mathcal{N}(\mu, \sigma)$,*

$$\mathbb{E}\left[|Z-\mu|^p\right] = \begin{cases} (p-1)!!\sigma^p, & \text{if } p \text{ is even}, \\ 0, & \text{if } p \text{ is odd}. \end{cases}$$

We recall [138, Problem 8.5], correcting a small error, used in the proof of Proposition 3.3.6:

Lemma 3.B.2. *Let $m, N \in \mathbb{N}$ and $p_1, \dots, p_m$ positive real numbers. Then*

$$\inf\left\{\sum_{n=1}^m \frac{p_n}{x_n^2} : \quad x_1,\dots,x_m \in (0,\infty), \quad \prod_{n=1}^m x_n \leq N\right\} = mN^{-\frac{2}{m}}\left(\prod_{j=1}^m p_j\right)^{\frac{1}{m}},$$

where the infimum is attained for $x_n = N^{\frac{1}{m}} p_n^{\frac{1}{2}}\left(\prod_{j=1}^m p_j\right)^{-\frac{1}{2m}}$, for all $n \in \{1,\dots,m\}$.

Proof. The general arithmetic-geometric inequalities imply

$$\frac{1}{m}\sum_{n=1}^m \frac{p_n}{x_n^2} \geq \left(\prod_{n=1}^m \frac{p_n}{x_n^2}\right)^{\frac{1}{m}} = \left(\prod_{n=1}^m p_n\right)^{\frac{1}{m}}\left(\prod_{n=1}^m \frac{1}{x_n^2}\right)^{\frac{1}{m}} \geq \left(\prod_{n=1}^m p_n\right)^{\frac{1}{m}} N^{-\frac{2}{m}},$$

since $\prod_{n=1}^m x_n \geq N$ by assumption. The right-hand side does not depend on $x_1, \dots, x_m$, so

$$\inf\left\{\sum_{n=1}^m \frac{p_n}{x_n^2} : \quad x_1,\dots,x_m \in (0,\infty), \quad \prod_{n=1}^m x_n \leq N\right\} \geq m\left(\prod_{n=1}^m p_n\right)^{\frac{1}{m}} N^{-\frac{2}{m}}.$$

Choosing $\widetilde{x}_n = N^{\frac{1}{m}} p_n^{\frac{1}{2}}\left(\prod_{j=1}^m p_j\right)^{-\frac{1}{2m}}$, for all $n \in \{1,\dots,m\}$, we obtain

$$m\left(\prod_{n=1}^m \frac{p_n}{N^2}\right)^{\frac{1}{m}} = \sum_{n=1}^m \frac{p_n}{\widetilde{x}_n^2} \geq \inf\left\{\sum_{n=1}^m \frac{p_n}{x_n^2} : x_1,\dots,x_m \in (0,\infty), \prod_{n=1}^m x_n \leq N\right\} \geq m\left(\prod_{n=1}^m \frac{p_n}{N^2}\right)^{\frac{1}{m}},$$

which concludes the proof. □

Lemma 3.B.3. *The following hold:*

(i) For any $x, y > 0$, $|\sqrt{x} - \sqrt{y}| \leq \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right)|x - y|$.

(ii) Set $C > 0$. *For any* $x, y \in \mathbb{R}$, $|\mathrm{e}^{Cx} - \mathrm{e}^{Cy}| \leq C\left(\mathrm{e}^{Cx} + \mathrm{e}^{Cy}\right)|x - y|$.

Lemma 3.B.4. *For a positive random variable* X *on* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > u)\mathrm{d}u$.

A theoretical analysis of Guyon's toy volatility model

Whenever a theory appears to you as the only possible one, take this as a sign that you have neither understood the theory nor the problem which it was intended to solve.

Karl Popper

Here we discuss paper [27], available on arXiv. This project results from a collaboration with Prof. A. Jacquier and Ph.D. Chloé Lacombe and it was submitted for publication in November 2022. In 2014, Guyon introduced a toy model with path dependent volatility, see [86]. In the aforementioned paper, we carry a detailed theoretical analysis of this model. In particular, we first prove existence and uniqueness of a strong solution and characterise its behaviour at boundary points. Then, leveraging on these results, we provide asymptotic closed-form option prices and we derive small-time behaviour estimates as well.

4.1 Introduction

Stochastic volatility models have been used extensively over the past three decades in order to reproduce particular features of market data, on Equities, FX and Fixed Income markets, both under the historical measure and for pricing purposes. Most of them are based on a Markovian assumption for the underlying process, essentially for mathematical convenience, as PDE techniques and Monte Carlo schemes are more readily available then. However, recent models have departed from this Markovian confinement and have shown to provide extremely accurate fit to market data. One approach considers instantaneous volatility driven by fractional Brownian motion, giving rise to the rough volatility generation and its numerous descendants [8, 14, 53, 61, 71, 77, 85]. A less stridden, yet very intuitive, path, originally introduced by Engle [56] and Bollerslev [24] in the early 1980s suggested to consider models where volatility depends on the past history of the stock price process. Their approach, though, was under the historical measure, and Duan [48] investigated these discrete-time models in the context of option pricing. With this in mind, Hobson and Rogers [96] extended this approach to continuous time, proposing that instantaneous volatility should depend on exponentially weighted moments of the stock price. Contrary to stochastic volatility models, the market here is complete. Hobson and Rogers [96] showed that such models generate implied volatility smiles and skews consistent with market data. Further results investigated some theoretical properties of these models, in particular [124] proving existence and uniqueness of strong solutions. This path has recently been given new highlights by Guyon [86], who assumed that the underlying stock price process behaves as

$$\frac{\mathrm{d}S_t}{S_t} = \sigma(t, S_t, Y_t)l(t, S_t)\mathrm{d}W_t, \quad S_0 := s_0 > 0,$$

where W is a standard Brownian motion, Y an adapted process and $l(\cdot)$ a leverage function ensuring that European options are fully recovered. Inspired by Hobson and Rogers [96], Guyon [87] suggested to choose Y as an exponentially weighted moving average of S. Not only does this model calibrate perfectly to the observed smile, but the diffusion map $\sigma(\cdot)$ can be chosen in such a way that joint calibration with VIX data becomes feasible, a notoriously difficult task.

Motivated by his empirical results, we investigate the theoretical properties of this model. We provide a full characterisation of the behaviour of the volatility process at its boundaries, together with its ergodic behaviour, and derive closed-form asymptotics for the corresponding option prices in small time. In Section 4.2, we set the notations and present Guyon's model. Section 4.3 gathers the main theoretical results, proving existence and uniqueness of a strong solution (Section 4.3.2), deriving the stationary distribution (Section 4.3.3), which we use to obtain an expansion of the option price in Section 4.3.4. We finally provide small-time option price and implied volatility asymptotics for this model in Section 4.3.5. We gather all (lengthy) proofs in the appendix.

This project arises as an empirical analysis carried out by Guyon [87] (see also [86]) to describe the relationship between the VIX index and the VVIX, a volatility of volatility index. Figure 4.1.1 below shows a scatter plot of one versus the other over a five-year period. The approximate linear relationship highlighted by the least-square regression fit was first noted by Guyon [87], and we follow his recommendations here.

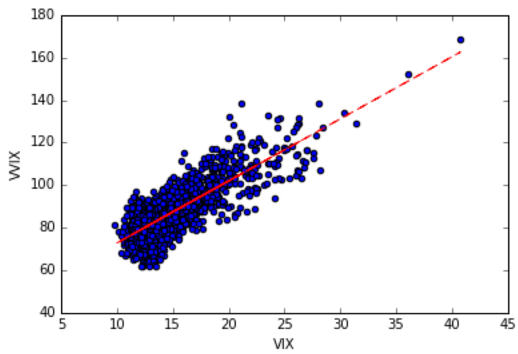

Figure 4.1.1: Historical VVIX vs historical VIX (13/4/12-8/5/17). *Source: CBOE.*

4.2 Set up and notations

The underlying process S, describing the evolution of the S&P index follows the general dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \sigma(Y_t)\mathrm{d}W_t, \quad S_0 = s_0 > 0,$$

for some given Brownian motion W generating a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\sigma : \mathbb{R}_+^* \to \mathbb{R}$ is non anticipative. Following Guyon [87] and Hobson and Rogers [96], we assume that the process Y is adapted to $\mathcal{F}$ and is a function of the past history of the stock S, making the latter non-Markovian, in the sense

$$Y_t := \frac{S_t}{\overline{S}_t^h}, \quad \text{for } t \in [0, T], \qquad \text{where } \overline{S}_t^h := \frac{1}{h} \int_{-\infty}^{t} \exp\left\{-\frac{t-u}{h}\right\} S_u \mathrm{d}u$$

is the exponentially weighted moving average (EWMA) of the stock price process. Here, the time horizon is set to be T. The constant $h > 0$, denoting the length of the time window, is left unspecified for now. Using Itô's formula and denoting $X := \log(S)$, we can summarise the dynamics for the couple (X, Y) as

$$\begin{cases} \mathrm{d}X_t &= -\frac{1}{2}\sigma(Y_t)^2 \mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \quad X_0 = x_0 := \log(s_0), \\ \mathrm{d}Y_t &= b(Y_t)\mathrm{d}t + \widetilde{\sigma}(Y_t)\mathrm{d}W_t, \qquad Y_0 = y_0 > 0, \end{cases} \tag{4.2.1}$$

with $b(y) := \frac{1}{h}y(1-y)$ and $\widetilde{\sigma}(y) := y\sigma(y)$, for $y > 0$ and some $h > 0$. Guyon [87] showed that, for the linear relationship between the VIX and the VVIX to hold, one needs to consider a diffusion coefficient of the form

$$\sigma(y) := -\frac{\alpha}{\beta} + \gamma y^{-\beta},$$

with $\alpha, \beta, \gamma > 0$. In that case, $\widetilde{\sigma}$ is null at $y_\sigma := \left(\frac{\beta\gamma}{\alpha}\right)^{1/\beta}$, and

$$\widetilde{\sigma}(0) = \begin{cases} \text{not defined}, & \text{if } \beta > 1, \\ 0, & \text{if } \beta < 1, \\ \gamma, & \text{if } \beta = 1. \end{cases}$$

We note that, for $y \in (y_\sigma, \infty)$, $\sigma(y) < 0$ and therefore $\widetilde{\sigma}(y) < 0$ as well. While this may appear odd, it is not however an issue as Brownian increments are symmetric around the origin. While Figure 4.1.1 provides strong empirical arguments in favour of such a model, a theoretical analysis thereof is however needed in order to investigate further its practical benefits. For example, since $\widetilde{\sigma}(y)^2 \sim \gamma^2 y^{2(1-\beta)}$ as y approaches zero, the map $\widetilde{\sigma}$ is square integrable around the origin if and only if $\beta < \frac{3}{2}$, and theoretical issues will arise if this is not satisfied (therefore ruling out such values out of calibration). That said, as we will show below, our main interest will be on the behaviour of the process on (y_σ, ∞), and therefore this restriction on β will not be enforced. Here and in the following, given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we shall write $f(y) \sim g(y)$ as y tends to a (possible infinite) point $\bar{y}$ whenever $\lim_{y \to \bar{y}} f(y)/g(y) = 1$. We write $\mathbb{P}_y(\cdot)$ for the conditional probability $\mathbb{P}_y(\cdot|Y_0 = y)$ and consequently $\mathbb{E}_y[\cdot]$ for $\mathbb{E}_y[\cdot|Y_0 = y]$.
We now step into this theoretical analysis by first concentrating on the existence and uniqueness of a strong solution for (4.2.1), then by deriving a precise classification of the special points 0, y_σ and ∞, before diving into the asymptotic behaviour of the process and the corresponding option prices.

4.3 Main results

4.3.1 Existence and uniqueness of strong solutions

Following [101, Definition 2.1], the definition of a strong solution allows for explosion in finite time. It is enough to check existence and uniqueness of solutions for the one-dimensional equation associated to the process Y since the process $X = \log(S)$ is uniquely determined as a function of Y and W. A localised version of [101, Corollary to Theorem 3.2] which, according to the authors, can be proved similarly to Theorem 3.1 therein, yields the existence of a unique strong solution, provided that the drift $b(\cdot)$ is locally Lipschitz and the volatility $\widetilde{\sigma}(\cdot)$ is $\frac{1}{2}$-Hölder (as mentioned in [101, page 184], this condition is in some sense maximal). Alternatively, one can exploit [47, Proposition 2.3], since the conditions therein are a direct consequence of local Lipschitzianity and local Hölderianity of b and σ. Notice that the fact that we are just focusing here on the positive half-line and not on $\mathbb{R}$ can be overcome just by setting σ and b identically equal to zero for negative arguments. The local Lipschitz property of the drift is straightforward: for any $N \in \mathbb{N}$ and any $x, y \in [-N, N]$, we have

$$|b(x) - b(y)| = \frac{\left|x - x^2 - (y - y^2)\right|}{h} \leq \frac{|x-y| + \left|x^2 - y^2\right|}{h} \leq \frac{|x-y| + 2N|x-y|}{h} \leq K_N|x-y|,$$

from which the local Lipschitz property with constant $K_N := \frac{2N+1}{h}$ follows. Now, the volatility function is α-Hölder with $\alpha \geq \frac{1}{2}$ if and only if $0 < \beta \leq \frac{1}{2}$ or $\beta = 1$: for $\beta = 1$, $\widetilde{\sigma}(y) = -\frac{\alpha}{\beta}y + \gamma$ is affine hence globally Lipschitz. Now, for any $N \in \mathbb{N}$ and $x, y \in [-N, N]$,

$$\begin{aligned}\left|\widetilde{\sigma}(x) - \widetilde{\sigma}(y)\right| &= \left|-\frac{\alpha}{\beta}x + \gamma x^{1-\beta} + \frac{\alpha}{\beta}y - \gamma y^{1-\beta}\right| \leq \frac{\alpha}{\beta}|x-y| + \gamma\left|x^{1-\beta} - y^{1-\beta}\right| \\ &\leq 2\max\left\{\frac{\alpha}{\beta}, \gamma\right\}|x-y|^{1-\beta},\end{aligned}$$

which is locally $\frac{1}{2}$-Hölder continuous if and only if $\beta \leq \frac{1}{2}$. We will below consider the process Y, not on the positive half line, but on the open interval (y_σ, ∞), on which it enjoys nice ergodic properties. There, the regularity of $\widetilde{\sigma}$ is improved since, for any $x, y \in (y_\sigma, \infty)$,

$$\begin{aligned}\left|\widetilde{\sigma}(x) - \widetilde{\sigma}(y)\right| &= \left|\frac{\alpha}{\beta}(y-x) + \gamma\left(x^{1-\beta} - y^{1-\beta}\right)\right| \leq \frac{\alpha}{\beta}|x-y| + \frac{\gamma(1-\beta)}{y_\sigma^\beta}|x-y| \\ &\leq \left(\frac{\alpha}{\beta} + \gamma(1-\beta)\frac{\alpha}{\beta\gamma}\right)|x-y| = \frac{\alpha(2-\beta)}{\beta}|x-y|.\end{aligned}$$

4.3.2 Boundary classification

Now, we need to analyse its behaviour in its domain and in particular at the boundary of the latter. To do so, we follow the boundary classification in [106, Chapter 15, Section 6]. The reason for this choice is that it seems the most suitable reference here. First, it includes both *Feller* and *Russian* boundary classifications, therefore allowing for a precise comparison. Second, it only requires the volatility coefficient to be non-null in the interior of the domain considered. On the contrary, the treatise in [37], although more complete in some sense, requires the volatility process to be non null everywhere in $\mathbb{R}$. Consider a *regular* (in the sense of [106]) diffusion process $Y = \{Y_t\}_{t\geq 0}$, on a domain $\mathfrak{D} \subset \mathbb{R}$, with left and right boundaries l and r:

$$\mathrm{d}Y_t = \mu(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \qquad Y_0 = y_0 \in \mathfrak{D}.$$

For any point y in the interior of $\mathfrak{D}$, namely $y \in (l, r)$, we assume that the drift and variance coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ are continuous and that $\sigma(y) > 0$ for all y in the interior of $\mathfrak{D}$. For any $x, y \in \mathbb{R}$, introduce the hitting times $\tau_x := \inf\{t \geq 0 : Y_t = x\}$ and $\tau_{x,y} := \min\{\tau_x, \tau_y\}$.

We study the left boundary l, the discussion for the right boundary r being similar. To provide a precise description, we recall some standard notions. The *scale function* $S : \mathfrak{D} \to \mathbb{R}$ is defined in terms of the so-called the *scale density* $s : \mathfrak{D} \to \mathbb{R}$ via

$$s(\xi) := \exp\left(-\int_{\xi_0}^{\xi} \frac{2\mu(v)}{\sigma^2(v)}\mathrm{d}v\right), \qquad S(x) := \int_{x_0}^{x} s(\xi)\mathrm{d}\xi,$$

where $\xi_0, x_0 \in (l, r)$ are arbitrary fixed points. The particular choice of these points has no importance for the boundary discussion [106, Chapter 15, Section 3]. For any closed interval $I := [a, b] \subset (l, r)$, we also introduce the *scale measure*, namely the map $S : I \mapsto S(b) - S(a)$. Then, we define the *speed density* m and *speed measure* M:

$$m(\xi) := \frac{1}{\sigma^2(\xi)s(\xi)}, \qquad M[I] = M[a, b] := \int_a^b m(x)\mathrm{d}x.$$

Notice that both S and M are positive and finite on their domain. Finally,

$$N(l) := \int_l^x S[\eta, x]\mathrm{d}M(\eta) = \int_l^x M(l, \xi]\mathrm{d}S(\xi) = \int_l^x S[\eta, x]\frac{\mathrm{d}\eta}{\sigma^2(\eta)s(\eta)}.$$

Since we are only interested in whether the integrals are finite or not, the upper bound x is irrelevant, explaining why we omit it from the notations. Through the quantity $M(l, x]$, we can in some sense estimate the velocity of the process near l and with the quantity $N(l)$ we can approximately quantify how long it takes to hit a point $x \in (l, r)$ starting at the left boundary l. Now, we are ready to give the first classification:

Definition 4.3.1.

- *The left boundary l is attracting if $S(l, x_0] < \infty$ for some x_0 in (l, r). Then,*
$$\mathbb{P}(\tau_{l_+} \leq \tau_b | Y_0 = x) > 0, \qquad \text{for any } l < x < b < r.$$
- *The left boundary l is unattracting when $S(l, x_0] = \infty$ for some x_0 in (l, r). Then,*
$$\mathbb{P}(\tau_{l_+} < \tau_b | Y_0 = x) = 0, \qquad \text{for any } l < x < b < r.$$

A left boundary l is therefore attracting when there is a positive probability that the process reaches l prior to the arbitrary state b (not necessarily in finite time), when its initial condition is $x < b$. To complete our discussion of boundary classification we introduce the quantity $\Sigma(l) := \int_l^x S(l, \xi]\frac{\mathrm{d}\xi}{\sigma^2(\xi)s(\xi)}$, where again the upper bound x in the integration is irrelevant. Roughly speaking, $\Sigma(l)$ determines the time required by the process, starting from an interior point x, to reach the boundary l or another interior point $b > x$.

Definition 4.3.2. *The boundary l is attainable if $\Sigma(l) < \infty$; otherwise it is unattainable.*

A straightforward argument shows that if l is attainable, then it is attracting. Indeed, $S(l, x_0] < \infty$ whenever $\Sigma(l) < \infty$. This is in contrast to unattainable boundaries that may or may not be attracting. For an attracting attainable boundary l, for any $l < x < b < r$,

$$\mathbb{P}(\tau_{l_+} < \infty | y_0 = x) > 0 \qquad \text{and} \qquad \mathbb{E}[\tau_{l_+} \wedge \tau_b | y_0 = x] < \infty.$$

Table 6.1 in [106] provides a complete portrait of Feller and Russian characterisations in terms of $S(l, x]$, $M(l, x]$, $\Sigma(l)$ and $N(l)$. We give here a short description in words of Feller's:

- **Regular boundary:** A regular boundary is attracting and attainable. A diffusion process can enter but also leave from such a boundary point.
- **Exit boundary:** An exit boundary is attracting and attainable too, but when the initial point gets closer to it, the process cannot reach any interior point b regardless how close b is to l. Indeed, in this case it should hold: $\lim_{b \searrow l} \lim_{x \searrow l} \mathbb{P}(\tau_b < t | Y_0 = x) = 0$, for any $t > 0$. No continuous sample path can exit l after touching it.
- **Entrance boundary:** An entrance boundary is unattracting and unattainable. A process starting from any point in the interior of the domain $\mathfrak{D}$ can not reach the entrance boundary. Nevertheless, one can consider a process starting at the entrance boundary l: in this case, the process moves to the interior of the domain and never comes back to the boundary.
- **Natural (Feller) boundary:** A point is a natural boundary when it is unattainable (it can be attracting or not). In general, such boundaries are discarded from the state space of the process since a diffusion process cannot start from nor reach it in finite time.

The following theorem, proved in Appendix 4.A, provides a detailed analysis of the behaviour of the process Y in Equation (4.2.1) at the boundaries of its domain.

Theorem 4.3.3.

- *Consider the process Y in (4.2.1) over the domain $\mathfrak{D} = (y_\sigma, \infty)$. The right boundary $r = \infty$ is entrance (unattracting, unattainable) while the left boundary $l = y_\sigma$ is*

Left boundary y_σ	*Feller*	*Russian*
$y_\sigma > 1$	*exit-trap-absorbing*	*attracting attainable*
$y_\sigma = 1$	*natural*	*attracting unattainable*
$y_\sigma < 1$	*entrance*	*unattracting unattainable*

- *If the process Y in (4.2.1) is defined over $\mathfrak{D} = (0, y_\sigma)$, then the left boundary $l = 0$ is*

Left boundary 0	*Feller*	*Russian*
$\beta < \frac{1}{2}$	*regular*	*attracting attainable*
$\beta \geq \frac{1}{2}$	*exit-trap-absorbing*	*attracting attainable*

while the right boundary $r = y_\sigma$ is

Right boundary y_σ	*Feller*	*Russian*
$y_\sigma > 1$	*entrance*	*unattracting unattainable*
$y_\sigma = 1$	*natural*	*attracting unattainable*
$y_\sigma < 1$	*exit-trap-absorbing*	*attracting attainable*

Remark 4.3.4. *As a consequence of this classification we limit our discussion to the domain* (y_σ, ∞) *with* $y_\sigma < 1$. *On* $(0, y_\sigma)$ *the strict positivity of* Y *is not guaranteed as the origin is attracting and attainable. Moreover, the case where* $\mathfrak{D} = (y_\sigma, \infty)$ *with* $y_\sigma \geq 1$, *should be ruled out as well since* y_σ *is attracting and attainable, so that* Y *may then exit it to enter* $(0, y_\sigma)$ *with strictly positive probability. An application of [101, Theorem 3.2, Section 4, Chapter 4] guarantees that* Y *does not explode in finite time, or more precisely that*

$$\mathbb{P}\left(\inf\left\{t \geq 0 : Y_t \in \{y_\sigma, +\infty\}\right\} = \infty | Y_0 = y_0\right) = 1, \qquad \textit{for any } y_0 \in (y_\sigma, \infty).$$

4.3.3 Ergodic behaviour and stationary distribution

Ergodic behaviour

We now discuss the ergodic behaviour of the process Y in (4.2.1) through the following theorem proved in Appendix 4.B.1. To do so, introduce the probabilities

$$\mathrm{P}(z) := \mathbb{P}_{y_0}\left(\lim_{t \uparrow \tau_{\mathfrak{D}}} Y_t = z\right), \qquad \text{for } z \in \{l, r\},$$

where $\tau_{\mathfrak{D}}$ denotes the lifetime of the process in $\mathfrak{D} = (l, r)$. Following [132, Section 2.7-2.8], transience of the process then corresponds to $\mathrm{P}(l) + \mathrm{P}(r) = 1$.

Theorem 4.3.5. *The ergodic behaviour of the process* Y *in* (4.2.1) *is as follows:*

	$\mathfrak{D} = (0, y_\sigma)$	$\mathfrak{D} = (y_\sigma, \infty)$
$y_\sigma < 1$	Y *transient and* $\mathrm{P}(y_\sigma)$ *in* (4.3.1)	Y *recurrent*
$y_\sigma = 1$	Y *transient and* $\mathrm{P}(y_\sigma)$ *in* (4.3.1)	Y *transient and* $\mathrm{P}(y_\sigma) = 1$
$y_\sigma > 1$	Y *transient and* $\mathrm{P}(0) = 1$	Y *transient and* $\mathrm{P}(y_\sigma) = 1$

with

$$\mathrm{P}(y_\sigma) = \frac{\int_0^{y_0} \exp\left\{-\int_x^y \frac{2b(s)}{\tilde{\sigma}^2(s)} \mathrm{d}s\right\} \mathrm{d}y}{\int_0^{y_\sigma} \exp\left\{-\int_x^y \frac{2b(s)}{\tilde{\sigma}^2(s)} \mathrm{d}s\right\} \mathrm{d}y} \quad \textit{and} \quad \mathrm{P}(0) = 1 - \mathrm{P}(y_\sigma), \quad \textit{for any } x \in (0, y_\sigma). \tag{4.3.1}$$

An immediate consequence of the fact that Y is recurrent when $y_\sigma < 1$ on the domain (y_σ, ∞) is that Y does not explode in finite time with probability one. This will thus be the case of interest, for which a stationary distribution is available (Proposition 4.3.7).

Stationary distribution over the domain $\mathfrak{D} = (y_\sigma, \infty)$

We now investigate the ergodic properties of the process Y in Equation (4.2.1) over the domain $\mathfrak{D} = (y_\sigma, \infty)$. Recall that its infinitesimal generator is defined, for any $y \in \mathfrak{D}$, as

$$(\mathcal{L}_Y \varphi)(y) := \lim_{t \downarrow 0} \frac{\mathbb{E}[\varphi(Y_t) | Y_0 = y] - \varphi(y)}{t},$$

for all functions φ such that the limit is finite for all $y \in \mathfrak{D}$. We recall [67, Section 3.2] that a process $(Y_t)_{t>0}$ is ergodic if it admits a unique, stationary distribution Π, and for any measurable bounded function ϕ, the almost sure limit

$$\lim_{t\uparrow\infty}\frac{1}{t}\int_0^t \phi(Y_s)\mathrm{d}s = \int_{\mathfrak{D}} \phi(y)\Pi(\mathrm{d}y)$$

holds. If this limit exists, an ergodic solution must satisfy $\mathcal{L}_Y^*\Pi = 0$, where $\mathcal{L}_Y^*$ is the adjoint of the infinitesimal generator $\mathcal{L}_Y$, defined in [67, Section 1.5.3] via the identity

$$\int g(\xi)\mathcal{L}_Y f(\xi)\mathrm{d}\xi = \int f(\xi)\mathcal{L}_Y^* g(\xi)\mathrm{d}\xi, \tag{4.3.2}$$

for any rapidly decaying smooth test functions f and g. The generator and its adjoint are available explicitly here:

Proposition 4.3.6. *For any $y \in \mathfrak{D}$, we have*

$$(\mathcal{L}_Y f)(y) = \frac{1}{h}y(1-y)\partial_y f(y) + \frac{1}{2}y^2\sigma^2(y)\partial_y^2 f(y),$$
$$(\mathcal{L}_Y^* g)(y) = -\frac{1}{h}\partial_y\Big(y(1-y)g(y)\Big) + \frac{1}{2}\partial_y^2\Big(y^2\sigma^2(y)g(y)\Big).$$

Proof. The expression for $\mathcal{L}_Y$ is standard and the one for $\mathcal{L}_Y^*$ follows using (4.3.2) and integration by parts. Given $y \in \mathfrak{D}$ and $f, g : \mathfrak{D} \to \mathbb{R}$ twice continuously differentiable functions with bounded derivatives, and such that the two functions and their derivatives tend to zero fast enough at the boundaries, we have

$$\begin{aligned}
\langle f, \mathcal{L}_Y^* g\rangle = \langle \mathcal{L}_Y f, g\rangle &= \int_{\mathfrak{D}}\left[\frac{y(1-y)}{h}\partial_y f(y) + \frac{1}{2}y^2\sigma^2(y)\partial_y^2 f(y)\right]g(y)\mathrm{d}y\\
&= \frac{1}{h}\int_{\mathfrak{D}}\partial_y f(y)y(1-y)g(y)\mathrm{d}y + \frac{1}{2}\int_{\mathfrak{D}}\partial_y^2 f(y)y^2\sigma^2(y)g(y)\mathrm{d}y\\
&= \frac{1}{h}\int_{\mathfrak{D}}\partial_y f(y)y(1-y)g(y)\mathrm{d}y - \frac{1}{2}\int_{\mathfrak{D}}\partial_y f(y)\partial_y\left(y^2\sigma^2(y)g(y)\right)\mathrm{d}y\\
&= -\int_{\mathfrak{D}}\partial_y f(y)\left\{-\frac{1}{h}y(1-y)g(y) + \frac{1}{2}\partial_y\left(y^2\sigma^2(y)g(y)\right)\right\}\mathrm{d}y\\
&= \int_{\mathfrak{D}} f(y)\left[-\partial_y\left(\frac{1}{h}y(1-y)g(y)\right) + \frac{1}{2}\partial_y^2\left(y^2\sigma^2(y)g(y)\right)\right]\mathrm{d}y,
\end{aligned}$$

and the proposition follows. □

For $f : \mathfrak{D} \to \mathbb{R}$, finding the explicit solution of the Poisson equation is tedious. Indeed, $(\mathcal{L}_Y^* f)(y) = 0$ is equivalent to

$$\begin{aligned}
&\frac{1}{2}y^2 f''(y)\left[\left(\frac{\alpha}{\beta}\right)^2 - \frac{2\alpha\gamma}{\beta}y^{-\beta} + \gamma^2 y^{-2\beta}\right]\\
&+ yf'(y)\left[2\left\{\left(\frac{\alpha}{\beta}\right)^2 - \frac{\alpha\gamma}{\beta}(2-\beta)y^{-\beta} + (1-\beta)\gamma^2 y^{-2\beta}\right\} - \frac{1}{h}(1-y)\right]\\
&+ f(y)\left[\left(\frac{\alpha}{\beta}\right)^2 - \frac{\alpha\gamma}{\beta}(1-\beta)(2-\beta)y^{-\beta} + \gamma^2(1-\beta)(1-2\beta)y^{-2\beta} - \frac{1}{h}(1-2y)\right] = 0,
\end{aligned}$$

with the constraint $\int_{\mathfrak{D}} f(y)\mathrm{d}y = 1$. This is a highly non-linear problem, which does not admit any obvious explicit solution. However, using the probabilistic tools developed in [106, page 242], such a closed-form expression can be derived as in the following proposition, proved in Appendix 4.B.2).

Proposition 4.3.7. *If $y_\sigma < 1$, $\mathfrak{D} = (y_\sigma, \infty)$, the unique stationary distribution reads*

$$\Pi(\mathrm{d}y) = \left(\int_{y_\sigma}^{\infty} \frac{\mathrm{d}\xi}{\widetilde{\sigma}^2(\xi)s(\xi)}\right)^{-1} \frac{\mathrm{d}y}{\widetilde{\sigma}^2(y)s(y)}. \tag{4.3.3}$$

4.3.4 Pricing PDE and expansion

Pricing options on the stock price given in (4.2.1) can obviously be done with Monte Carlo simulations. However, through Feynman-Kac, PDE techniques are (when available) often faster and may also (as we shall see below) provide closed-form expressions. Consider an option with payoff $h(X_T)$ at expiry T, and denote its price $P(t, X_t, Y_t)$ at time $t \le T$. Introduce the operators

$$\mathcal{L}_1 := y\sigma^2(y)\partial_{xy} \qquad \text{and} \qquad \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} := \partial_t + \frac{\sigma^2(y)}{2}\partial_x^2 - \frac{\sigma^2(y)}{2}\partial_x. \tag{4.3.4}$$

and recall that $\mathcal{L}_Y$ is defined in Proposition 4.3.6, while the operator $\mathcal{L}_{\mathrm{BS}}^{\sigma(y)}$ is nothing else than the Black-Scholes infinitesimal generator with volatility $\sigma(y)$.

Proposition 4.3.8. *Under the risk-neutral measure, the pricing PDE associated to (4.2.1) is*

$$\left(\mathcal{L}_Y + \mathcal{L}_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}\right) P(t, x, y) = 0,$$

for all $t \in [0, T)$, $x \in \mathbb{R}$ and $y \in \mathfrak{D} = (y_\sigma, \infty)$, with terminal condition $P(T, x, y) = h(x)$.

Note that the PDE is stated in the domain $\mathfrak{D} = (y_\sigma, \infty)$ and not on the whole positive half-line in the y-dimension. On $\mathfrak{D}$, the drift is quadratic (so that this representation follows from [23] for example),while the diffusion coefficient is at most of linear growth. Note that since $\sigma(\cdot)$ is not bounded away from zero, the operator $\mathcal{L}_{\mathrm{BS}}^{\sigma(y)}$ is not strictly elliptic, but only hypoelliptic. Unfortunately, this pricing PDE does not admit an obvious explicit solution. However, approximate solutions can be found by expanding the solution using perturbation methods, as developed in [67]. A key ingredient is the (unique) stationary distribution of the ergodic process Y, which we proved above for the case $\mathfrak{D} = (y_\sigma, \infty)$. This perturbation analysis relies on a few other items that we need to tackle. In particular, we assume that the pricing PDE admits a unique classical solution.

Following [57, 58, 64, 66, 65, 67], consider a 'fast' version of the original process Y, defined as

$$\begin{cases} \mathrm{d}X_t &= -\frac{1}{2}\sigma(Y_t)^2\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}, \\ \mathrm{d}Y_t &= \frac{1}{\varepsilon}b(Y_t)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\widetilde{\sigma}(Y_t)\mathrm{d}W_t, \quad Y_0 = y_0 > 0, \end{cases}$$

for $\varepsilon > 0$. Proposition 4.3.8 then implies that the option price P^ε, with payoff h, satisfies

$$\left[\frac{1}{\varepsilon}\mathcal{L}_Y + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}\right] P^\varepsilon(t,x,y) = 0, \tag{4.3.5}$$

for all $t \in [0,T)$, $x \in \mathbb{R}$ and $y \in \mathfrak{D}$, with boundary condition $P^\varepsilon(T,x,y) = h(x)$. Inspired by [64, 66, 67], we now provide an approximation for the price P^ε, proved in Appendix 4.B.3:

Proposition 4.3.9. *If the payoff h is smooth, then the equality*

$$P^\varepsilon(t,x,y) = P_0(t,x) + \sqrt{\varepsilon}P_1(t,x) + \mathcal{O}(\varepsilon)$$

holds pointwise in $(t,x,y) \in [0,T)\times\mathbb{R}\times\mathfrak{D}$ as ε tends to zero, where P_0 corresponds to the Black-Scholes price of the option having payoff h with volatility $\boldsymbol{\varkappa} := \sqrt{\langle\sigma^2,\Pi\rangle}$ and

$$P_1(t,x) = -\frac{T-t}{2}\langle\varpi,\Pi\rangle\left(\partial_x^3 - \partial_x^2\right)P_0(t,x),$$

for all $(x,t) \in \mathbb{R}\times[0,T)$ with boundary condition $P_1(T,x) = 0$ and with

$$\varpi(y) := y\sigma^2(y)\psi'(y). \tag{4.3.6}$$

Finally ψ is the unique solution to

$$\mathcal{L}_Y\psi(y) = \sigma^2(y) - \boldsymbol{\varkappa}^2, \qquad \text{for all } y \in (y_\sigma,\infty). \tag{4.3.7}$$

Remark 4.3.10. *The assumption of a smooth payoff follows that in [64, 67]. Using mollification arguments, it could be relaxed to include standard European Call and Put options, but we leave this subtlety for later.*

4.3.5 Small-time asymptotics

We finally investigate the small-time behaviour of the solution to (4.2.1) using large deviations techniques, leading to closed-form asymptotics for option prices and implied volatilities. We refer the reader to [69] for an overview of this topic. For $\varepsilon > 0$, $t \in [0,T]$, introduce the small-time rescaling $(X_t^\varepsilon, Y_t^\varepsilon) := (X_{\varepsilon t}, Y_{\varepsilon t})$, which satisfies

$$\begin{cases} \mathrm{d}X_t^\varepsilon &= -\frac{\varepsilon}{2}\sigma^2(Y_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\,\sigma(Y_t^\varepsilon)\mathrm{d}W_t, & X_0^\varepsilon := x_0 \in \mathbb{R},\\ \mathrm{d}Y_t^\varepsilon &= \varepsilon b(Y_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\,\widetilde{\sigma}(Y_t^\varepsilon)\mathrm{d}W_t, & Y_0^\varepsilon = y_0 > 0.\end{cases} \tag{4.3.8}$$

Let $\overline{\mathcal{H}}$ denote the space of absolutely continuous functions starting at the origin, with square integrable derivatives, such that

$$\overline{\mathcal{H}} := \left\{f : [0,T] \to \mathbb{R} \text{ with } f = \int g(s)\mathrm{d}s \text{ for some } g \in L^2([0,T]), \text{ and } \inf_{t\in[0,T]} f_t \geq \frac{1}{\alpha}\log\left(1 - y_0^\beta\frac{\alpha}{\beta\gamma}\right)\right\}.$$

Remark 4.3.11. *When $y_0 \geq y_\sigma$, the condition*

$$\inf_{t\in[0,T]} f_t \geq \frac{1}{\alpha}\log\left(1 - y_0^\beta \frac{\alpha}{\beta\gamma}\right), \tag{4.3.9}$$

is automatically satisfied and $\overline{\mathcal{H}}$ is the usual Cameron-Martin space. When $y_0 < y_\sigma$, (4.3.9) is needed to ensure that the solution of the controlled ODE introduced below is positive.

We now state and prove (in Appendix 4.C.2) a pathwise large deviations principle for the log-stock price process. With $\mathrm{x}_0 := (x_0, y_0)$, introduce the map $\mathrm{I}^{X,Y}$ on $\mathcal{C}([0,T], \mathbb{R}\times\mathbb{R}_+^*)$ by

$$\mathrm{I}^{X,Y}(\mathrm{g}) := \inf\left\{\Lambda(f), f \in \overline{\mathcal{H}}, \mathcal{S}^{\mathrm{x}_0}(f) = \mathrm{g}\right\},$$

where Λ is the usual rate function driving the large deviations of the Brownian motion:

$$\Lambda(f) := \begin{cases} \frac{1}{2}\int_0^T \left\|\dot{f}_t\right\|^2 \mathrm{d}t, & \text{if } f \in \overline{\mathcal{H}}, \\ \infty, & \text{otherwise,} \end{cases}$$

and $\mathcal{S}^{\mathrm{x}_0}(f)$ on $[0,T]$ is the solution to the controlled ODE $\dot{\mathrm{g}}_t = \dot{f}_t\left(\underline{\sigma}(\mathrm{g}_t), \underline{\widetilde{\sigma}}(\mathrm{g}_t)\right)^\top$, with $\underline{\sigma}(x,y) = \sigma(y)$ and $\underline{\widetilde{\sigma}}(x,y) = \widetilde{\sigma}(y)$, starting from $\mathrm{g}_0 = \mathrm{x}_0$.

Theorem 4.3.12. *The rescaled* log*-stock price process X^ε in (4.3.8) satisfies a pathwise large deviations principle on $\mathcal{C}([0,T],\mathbb{R})$ as ε tends to zero with speed ε and rate function*

$$\mathrm{I}^X(g) := \inf\left\{\mathrm{I}^{X,Y}(\mathrm{h}), \mathrm{h} := (g, l), l \in \mathcal{C}([0,T], \mathbb{R}_+^*), l_0 = y_0\right\},$$

The proof of the theorem relies on first obtaining a large deviations principle for the rescaled process Y^ε, which we state below (and defer its proof to Appendix 4.C.1). Similarly to above, denote $\mathcal{S}_2^y(f)$ the solution to the controlled ODE $\dot{g}_t = \widetilde{\sigma}(g_t)\dot{f}_t$, with $g_0 = y_0$.

Proposition 4.3.13. *The rescaled process Y^ε satisfies a pathwise large deviations principle on $\mathcal{C}([0,T],\mathbb{R}_+^*)$ as ε tends to zero with speed ε and rate function*

$$\mathrm{I}^Y(g) := \inf\left\{\Lambda(f), f \in \overline{\mathcal{H}}, \mathcal{S}_2^{y_0}(f) = g\right\}.$$

Large deviations have been used extensively in Mathematical Finance to derive asymptotic behaviours of the implied volatility and we refer the reader to the monograph [69] for a thorough overview. The latter, $\Sigma_t(k)$, is the unique non-negative solution to $C_{\mathrm{BS}}(t, \mathrm{e}^k, \Sigma_t(k)) = C_{obs}(t, \mathrm{e}^k)$, with $C_{obs}(t, \mathrm{e}^k)$ an observed (or computed) Call option price with maturity t and strike e^k, and C_{BS} is the corresponding Call price in the Black-Scholes model. A large deviations principle is the first step to understand the short-time behaviour of the process, and going from there to the corresponding behaviour of the implied volatility requires a few small steps that we follow in Appendix 4.C.3.

Corollary 4.3.14. *For $y_0 \in \mathfrak{D} = (y_\sigma, \infty)$, with $y_\sigma < 1$, small-time out-of-the-money options behave as*

$$\begin{array}{ll} \lim_{t\downarrow 0} t \log \mathbb{E}\left[\left(S_t - \mathrm{e}^k\right)_+\right] = -\inf_{y\geq k} \mathrm{I}^X(g)|_{g(1)=y}, & \text{if } k > 0, \\ \lim_{t\downarrow 0} t \log \mathbb{E}\left[\left(\mathrm{e}^k - S_t\right)_+\right] = -\inf_{y\leq k} \mathrm{I}^X(g)|_{g(1)=y}, & \text{if } k < 0. \end{array}$$

Similarly to [60, Theorem 2.4], we finally deduce the behaviour of the short-time smile:

Corollary 4.3.15. *For $y_0 \in \mathfrak{D} = (y_\sigma, \infty)$, with $y_\sigma < 1$, the implied volatility behaves as*

$$\lim_{t\downarrow 0} \Sigma_t(k) = \begin{cases} \dfrac{k^2}{2}\left(\inf\limits_{y\geq k} \mathrm{I}^X(g)|_{g(1)=y}\right)^{-1}, & \text{if } k > 0, \\ \dfrac{k^2}{2}\left(\inf\limits_{y\leq k} \mathrm{I}^X(g)|_{g(1)=y}\right)^{-1}, & \text{if } k < 0. \end{cases}$$

Appendix

4.A Proof of Theorem 4.3.3

The proof below relies on the techniques developed in [106, Chapter 15, Section 6].

4.A.1 Proof for the domain $\mathfrak{D} = (y_\sigma, \infty)$

Left boundary y_σ

The classification of the left boundary y_σ follows from Lemma 4.A.1. Introduce, on the domain $(0, \infty)$, the process $Z := (Y - y_\sigma)$, satisfying the SDE

$$\mathrm{d}Z_t = \overline{b}(Z_t)\mathrm{d}t + \overline{\sigma}(Z_t)\mathrm{d}W_t, \qquad Z_0 := y_0 - y_\sigma > 0,$$

with $\overline{b}(z) := b(z + y_\sigma)$ and $\overline{\sigma}(z) := \widetilde{\sigma}(z + y_\sigma)$, for $z > 0$. Armed with Lemma 4.A.1, we attack the boundary classification of the origin for Z, which corresponds to the classification of the left boundary y_σ for the original process Y. The classification for the different cases, namely $y_\sigma > 1$, $y_\sigma = 1$ and $y_\sigma < 1$ follows from a careful inspection of [106, Table 6.1, Chapter 15] together with [106, Lemma 6.3, Chapter 15]. Introduce

$$\begin{aligned}
\overline{s}(x) &:= \exp\left\{\int_x^{\overline{a}} \frac{2\overline{b}(y)}{\overline{\sigma}^2(y)}\mathrm{d}y\right\}, & \overline{S}(0,\overline{a}] &:= \int_0^{\overline{a}} \overline{s}(y)\mathrm{d}y, & \overline{S}[x,\overline{a}] &:= \int_x^{\overline{a}} \overline{s}(y)\mathrm{d}y,\\
\overline{M}(0,\overline{a}] &:= \int_0^{\overline{a}} \frac{\mathrm{d}x}{\overline{\sigma}^2(x)\overline{s}(x)}, & \overline{\Sigma}(0) &:= \int_0^{\overline{a}} \frac{\overline{S}(0,x]}{\overline{\sigma}^2(x)\overline{s}(x)}\mathrm{d}x, & \overline{N}(0) &:= \int_0^{\overline{a}} \frac{\overline{S}[x,\overline{a}]\mathrm{d}x}{\overline{\sigma}^2(x)\overline{s}(x)},
\end{aligned} \tag{4.A.1}$$

for $\overline{a} > 0$ and $x \in (0, \overline{a}]$. We then deduce their behaviour.

Lemma 4.A.1. *The following hold:*

$$\begin{array}{llll}
\overline{S}(0,\overline{a}] < \infty, & \overline{M}(0,\overline{a}] = \infty, & \overline{\Sigma}(0) < \infty, & \text{if } y_\sigma > 1,\\
\overline{S}(0,\overline{a}] < \infty, & & \overline{\Sigma}(0) = \infty, & \text{if } y_\sigma = 1,\\
\overline{S}(0,\overline{a}] = \infty, & & \overline{N}(0) < \infty, & \text{if } y_\sigma < 1.
\end{array}$$

Remark 4.A.2. *We do not need $\overline{M}(0,\overline{a}]$ in the second and third cases because only a few combinations for boundedness/unboundedness of these quantities are possible. They are displayed in [106, Table 6.1, page 233]. In particular, in the second line, $\overline{S}(0,\overline{a}]$ can be finite and $\overline{\Sigma}(0)$ not if and only if $\overline{M}(0,\overline{a}]$ and $\overline{N}(0)$ are infinite. In the third line $\overline{N}(0)$ finite implies $\overline{M}(0,\overline{a}]$ finite and $\overline{S}(0,\overline{a}]$ infinite implies $\overline{\Sigma}(0)$ infinite by [106, Lemma 6.3, page 231]. Similar arguments motivate the form of the statements in Lemmas 4.A.3-4.A.4-4.A.5.*

Proof of Lemma 4.A.1. We start with the limiting behaviour of the function $\overline{s}$ and its integral, the scale measure. Since

$$\frac{2\overline{b}(y)}{\overline{\sigma}^2(y)} = \frac{2(1 - y_\sigma - y)}{h\gamma^2(y + y_\sigma)^{1-2\beta}\left(1 - (1 + \frac{y}{y_\sigma})^\beta\right)^2},$$

a straightforward Taylor expansion around the origin yields

$$\left(1-\left(1+\frac{y}{y_\sigma}\right)^\beta\right)^{-2} = \frac{y_\sigma^2}{\beta^2 y^2}\left[1-\chi_1\frac{y}{y_\sigma}+\chi_2\frac{y^2}{y_\sigma^2}-\chi_3\frac{y^3}{y_\sigma^3}+\mathcal{O}(y^4)\right], \tag{4.A.2}$$

with

$$\chi_1 := \beta-1, \qquad \chi_2 := \frac{(5\beta-1)(\beta-1)}{12}, \qquad \chi_3 := \frac{(\beta^2-1)\beta}{12}. \tag{4.A.3}$$

Introduce $K := \frac{2y_\sigma^{2\beta+1}(1-y_\sigma)}{h\beta^2\gamma^2}$, and $K_\beta^a := -\frac{1}{\overline{a}} + \overline{\chi}_1\log(\overline{a}) + \overline{\chi}_2\overline{a}$. Using (4.A.2), we obtain the asymptotic behaviour, as y approaches zero,

$$\frac{2\overline{b}(y)}{\overline{\sigma}^2(y)} = \frac{K}{y^2}\left(1+\overline{\chi}_1 y+\overline{\chi}_2 y^2-\overline{\chi}_3 y^3+\mathcal{O}\left(y^4\right)\right),$$

for some constants $\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3$ depending on χ_1, χ_2, χ_3, the values of which are not important[1]. Notice that $K<0$, as $y_\sigma > 1$. Since the expansion is uniform on $[x,\overline{a}]$, one obtains

$$\begin{aligned}\overline{s}(x) = \exp\left\{2\int_x^{\overline{a}}\frac{\overline{b}(y)}{\overline{\sigma}^2(y)}\mathrm{d}y\right\} &= \exp\left\{\frac{K}{x}\left(1-\overline{\chi_1}x\log x\right)\right\}\mathrm{e}^{KK_\beta^a}\exp\left\{-\overline{\chi_2}Kx+\mathcal{O}(x^2)\right\}\\ &= \exp\left\{\frac{K}{x}+KK_\beta^a\right\}x^{-K\overline{\chi_1}}(1+\mathcal{O}(x)).\end{aligned} \tag{4.A.4}$$

Since $K<0$, $\overline{s}(x)$ tends to zero as x tends to zero from above, and $\overline{S}(0,\overline{a}]$ is finite.

In the case $y_\sigma < 1$, the expansion (4.A.4) is still valid, albeit with $K>0$. Therefore, $\overline{s}$ explodes at the origin and $\overline{S}(0,\overline{a}] = \int_0^{\overline{a}}\overline{s}(y)\mathrm{d}y$ is infinite.

The case $y_\sigma = 1$ is slightly different and has to be studied separately. First of all, a Taylor expansion around the origin provides

$$\left[1-(1+y)^\beta\right]^{-2} = \frac{1}{\beta^2 y^2}\left[1-\chi_1 y+\chi_2 y^2-\chi_3 y^3+\mathcal{O}\left(y^4\right)\right], \tag{4.A.5}$$

with χ_1, χ_2 and χ_3 in (4.A.3). This implies, as y approaches zero,

$$\frac{2\overline{b}(y)}{\overline{\sigma}^2(y)} = -\frac{2y}{h\gamma^2(y+1)^{1-2\beta}}\frac{1}{(1-(1+y)^\beta)^2} = \overline{\overline{K}}\left(\frac{1}{y}+\overline{\chi}_1+\overline{\chi}_2 x+\overline{\chi}_3 y^2+\mathcal{O}\left(y^3\right)\right),$$

with $\overline{\overline{K}} := -\frac{2}{h\beta^2\gamma^2} < 0$, and $\overline{\overline{K}}_\beta^{\overline{a}} := \log(\overline{a}) + \overline{\chi}_1\overline{a} + \frac{\overline{\chi}_2}{2}\overline{a}^2 + \frac{\overline{\chi}_3}{3}\overline{a}^3$. Since the expansion is uniform on

[1] In the following the symbols $\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3$ will refer to different quantities whose specific values are not important for the convergence of the quantities we are interested in.

$[x, \overline{a}]$, one obtains

$$\begin{aligned}\overline{s}(x) &= \exp\left\{2\int_x^{\overline{a}} \frac{\overline{b}(y)}{\overline{\sigma}^2(y)}\mathrm{d}y\right\} = \exp\left\{\overline{\overline{K}}\left(-\log(x) - \overline{\chi}_1 x + \mathcal{O}(x^2)\right)\right\}\exp\left\{\overline{\overline{K}}\,\overline{\overline{K}}_\beta^{\overline{a}}\right\} \\ &= \exp\left\{-\overline{\overline{K}}\log(x) + \overline{\overline{K}}\,\overline{\overline{K}}_\beta^{\overline{a}}\right\}(1+\mathcal{O}(x)) = \exp\left\{\overline{\overline{K}}\,\overline{\overline{K}}_\beta^{\overline{a}}\right\} x^{-\overline{\overline{K}}}(1+\mathcal{O}(x)).\end{aligned} \tag{4.A.6}$$

Since $\overline{\overline{K}} < 0$, $\overline{s}(x)$ tends to zero as x tends to zero and therefore $\overline{S}(0, \overline{a}] = \int_0^{\overline{a}} \overline{s}(x)\mathrm{d}x$ is finite.

Now, for $y_\sigma > 1$, it is straightforward to see that

$$\overline{M}(0, \overline{a}] = \int_0^{\overline{a}} \frac{\mathrm{d}x}{\overline{s}(x)\overline{\sigma}^2(x)} = \int_0^{\overline{a}} \frac{\mathrm{d}x}{\overline{s}(x)\widetilde{\sigma}^2(x+y_\sigma)} \geq \int_0^{\overline{a}} \frac{\mathrm{d}x}{\widetilde{\sigma}^2(x+y_\sigma)},$$

which is clearly infinite, because $\overline{s}$ is bounded above by 1 on $(0, \overline{a}]$ and from the asymptotic behaviour of the integrand around the origin. Indeed, for $\varepsilon > 0$,

$$\int_0^\varepsilon \frac{1}{\widetilde{\sigma}^2(x+y_\sigma)}\mathrm{d}x \geq \frac{K_\varepsilon}{\gamma^2}\int_0^\varepsilon \left(1 - \left(1 + \frac{x}{y_\sigma}\right)^\beta\right)^{-2}\mathrm{d}x$$

with $K_\varepsilon := (y_\sigma + \varepsilon)^{-2(1-\beta)}$ for $\beta \in (0,1)$, $K_\varepsilon := y_\sigma^{2(\beta-1)}$ for $\beta > 1$ and $K_\varepsilon := 1$ for $\beta = 1$. Recalling the Taylor expansion in (4.A.2), we see that the integrand is not integrable around zero. We thus conclude about the right behaviour of y_σ by noting that the integral representation of $\overline{M}(0, \overline{a}]$ diverges.

We now prove the last statement of the lemma, and start with the case $y_\sigma > 1$. Using (4.A.4), we write the asymptotic behaviour of $\overline{S}(0, x]$ around the origin by integrating the asymptotic behaviour of $\overline{s}(\cdot)$ around zero. Classical asymptotic expansions for integrals [123, Chapter 3.3, pages 62 and 67] (note that the leading contribution arises at the right boundary of the integration domain) yields, after the change of variable $y \mapsto zx$,

$$\begin{aligned}\overline{S}(0, x] &= \int_0^x \overline{s}(y)\mathrm{d}y \\ &= \mathrm{e}^{KK_\beta^a} x^{-K\overline{\chi}_1+1}\int_0^1 \exp\left\{\frac{K}{zx}\right\} z^{-K\overline{\chi}_1}\mathrm{d}z = \mathrm{e}^{KK_\beta^a} x^{-K\overline{\chi}_1+1}\exp\left\{\frac{K}{x}\right\}\left(-\frac{x}{K} + \mathcal{O}(x^2)\right),\end{aligned}$$

as x tends to zero. Combining this with (4.A.2) and (4.A.4), we obtain

$$\frac{\overline{S}(0, x]}{\overline{s}(x)\overline{\sigma}^2(x)} = \frac{y_\sigma^{2\beta}}{\beta^2\gamma^2} x\left(-\frac{x}{K} + \mathcal{O}(x^2)\right)\frac{1}{x^2}(1+\mathcal{O}(x)) = -\frac{y_\sigma^{2\beta}}{K\beta^2\gamma^2}(1+\mathcal{O}(x)),$$

which is integrable on $(0, \overline{a}]$ and concludes the proof, using the fact that

$$\overline{\Sigma}(0) = \int_0^{\overline{a}} \frac{\overline{S}(0, x]}{\overline{s}(x)\overline{\sigma}^2(x)}\mathrm{d}x < \infty.$$

When $y_\sigma = 1$, using (4.A.6), we can write the asymptotic behaviour of $\overline{S}(0, \cdot]$ around the origin by integrating that of $\overline{s}(\cdot)$ around zero. This yields, after the change of variable $y \mapsto zx$,

$$\overline{S}(0,x] = \int_0^x \overline{s}(y)\mathrm{d}y = \mathrm{e}^{\overline{\overline{KK}}_\beta^{\overline{a}}} x^{-\overline{\overline{K}}+1} \int_0^1 z^{-\overline{\overline{K}}}\mathrm{d}z = \frac{\mathrm{e}^{\overline{\overline{KK}}_\beta^{\overline{a}}}}{1-\overline{\overline{K}}} x^{-\overline{\overline{K}}+1} \left(1 + O(x)\right),$$

as x tends to zero. Exploiting this together with (4.A.5) and (4.A.6), we obtain

$$\frac{\overline{S}(0,x]}{\overline{s}(x)\overline{\sigma}^2(x)} = \frac{1}{\beta^2\gamma^2(1-\overline{\overline{K}})}\frac{1}{x}\left(1 + O(x)\right),$$

which is not integrable on $(0, \overline{a}]$ and thus

$$\overline{\Sigma}(0) = \int_0^{\overline{a}} \frac{\overline{S}(0,x]}{\overline{s}(x)\overline{\sigma}^2(x)}\mathrm{d}x = \infty.$$

When $y_\sigma < 1$, we look at $\overline{N}(0) = \int_0^{\overline{a}} \frac{\overline{S}[x,\overline{a}]}{\overline{\sigma}^2(x)\overline{s}(x)}\mathrm{d}x$. Since both $\overline{S}[a,\cdot]$ and $\overline{s}(\cdot)\overline{\sigma}^2(\cdot)$ diverge to infinity at the origin, we study the behaviour of the integrand around zero. For $\delta > 0$ such that $x < \delta < \overline{a}$ and $x > 0$, $\overline{S}[x,\overline{a}] = \int_x^{\overline{a}} \overline{s}(y)\mathrm{d}y = \int_x^{\delta} \overline{s}(y)\mathrm{d}y + \int_\delta^{\overline{a}} \overline{s}(y)\mathrm{d}y$. The second integral exists since $\overline{s}$ is continuous on compacts in $\mathbb{R}_+$. Regarding the first one, classical asymptotic expansions for integrals and (4.A.4), yield, after the change of variable $y \mapsto zx$,

$$\int_x^{\delta} \overline{s}(y)\mathrm{d}y = \mathrm{e}^{KK_\beta^a} x^{1-K\overline{\chi}_1} \int_1^{\delta/x} \exp\left\{\frac{K}{xz}\right\} z^{-K\overline{\chi}_1}\mathrm{d}z = \mathrm{e}^{KK_\beta^a} x^{1-K\overline{\chi}_1}\mathrm{e}^{\frac{K}{x}}\left(\frac{x}{K} + O(x^2)\right),$$

and the asymptotic behaviour of the integrand around the origin becomes

$$\frac{\overline{S}[x,\overline{a}]}{\overline{s}(x)\overline{\sigma}^2(x)} = \frac{y_\sigma^{2\beta}}{\beta^2\gamma^2}\left(\frac{x^2}{K} + O(x^3)\right)\frac{1}{x^2}\left(1 + O(x)\right) = \frac{y_\sigma^{2\beta}}{\beta^2\gamma^2 K}\left(1 + O(x)\right),$$

which is integrable at the origin, and the claim is proved. □

Right boundary ∞

To end the boundary classification for the first domain we are left to study the behaviour at the right boundary ∞. We exploit the Lemma 4.A.3 and [106, Table 6.1, Chapter 15]. First, for $y > y_\sigma$, let $s(y) = \exp\left\{-\int_a^y \frac{2b(x)}{\overline{\sigma}^2(x)}\mathrm{d}x\right\}$, with $a > y_\sigma$ fixed. Then, recall the definitions of some quantities:

$$S[a,\infty) = \int_a^\infty s(x)\mathrm{d}x, \quad M[a,\infty) = \int_a^\infty \frac{\mathrm{d}x}{\overline{\sigma}^2(x)s(x)}, \quad N(\infty) = \int_a^\infty \frac{S[a,x]}{\overline{\sigma}^2(x)s(x)}\mathrm{d}x.$$

Lemma 4.A.3. *The following hold:*

$$S[a,\infty) = \infty, \qquad M[a,\infty) < \infty, \qquad N(\infty) < \infty.$$

Proof. As y tends to infinity,

$$\frac{2b(y)}{\widetilde{\sigma}^2(y)} = \frac{2(1-y)}{hs}\left(-\frac{\alpha}{\beta} + \gamma y^{-\beta}\right)^{-2} \sim -\frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2,$$

and therefore, as x tends to infinity,

$$-\int_a^x \frac{2b(y)}{\widetilde{\sigma}^2(y)}\mathrm{d}y \sim \frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2 x. \tag{4.A.7}$$

Then, $S[a,\infty) = \int_a^\infty \exp\left(-\int_a^x \frac{2b(y)}{\widetilde{\sigma}^2(y)}\mathrm{d}y\right)\mathrm{d}x \sim \int_a^\infty \exp\left(\frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2 x\right)\mathrm{d}x$ is infinite. Now $M[a,\infty)$ is finite since $\frac{1}{\widetilde{\sigma}^2(x)s(x)} \sim \frac{\exp\left(-\frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2 x\right)}{x^2\left(-\frac{\beta}{\alpha}+\gamma x^{-\beta}\right)}$ as $x \uparrow \infty$, which is integrable.

Finally, $S[a,x] \sim \left[\frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2\right]^{-1} \exp\left(\frac{2}{h}(\frac{\beta}{\alpha})^2 x\right)$ as $x \uparrow \infty$, and so $\frac{S[a,x]}{\widetilde{\sigma}^2(x)s(x)} \sim \frac{\alpha^3 h}{2\beta^3}\frac{1}{x^2}$, which is integrable at infinity and $N(\infty)$ is finite. □

4.A.2 Proof for the domain $\mathfrak{D} = (0, y_\sigma)$

Left boundary 0

Consider $\widetilde{s}(x) = \exp\left\{\int_x^{\widetilde{a}} \frac{2b(\xi)}{\widetilde{\sigma}^2(\xi)}\mathrm{d}\xi\right\}$ for $y_\sigma > \widetilde{a} > 0$, $x \in (0, \widetilde{a}]$, and

$$\widetilde{S}(0,\widetilde{a}] := \int_0^{\widetilde{a}} \widetilde{s}(y)\mathrm{d}y, \qquad \widetilde{M}(0,\widetilde{a}] := \int_0^{\widetilde{a}} \frac{\mathrm{d}x}{\widetilde{\sigma}^2(x)\widetilde{s}(x)}, \qquad \widetilde{\Sigma}(0) := \int_0^{\widetilde{a}} \frac{\widetilde{S}(0,x]}{\widetilde{\sigma}^2(x)s(x)}\mathrm{d}x.$$

The following lemma, together with [106, Table 6.1, Chapter 15], helps for the left boundary 0:

Lemma 4.A.4. *For any* $\widetilde{a} \in (0, y_\sigma)$,

$$\begin{aligned} &\widetilde{S}(0,\widetilde{a}] < \infty, \quad \widetilde{M}(0,\widetilde{a}] < \infty, \quad \widetilde{\Sigma}(0) < \infty, \qquad \text{if } \beta < \frac{1}{2}, \\ &\widetilde{S}(0,\widetilde{a}] < \infty, \quad \widetilde{M}(0,\widetilde{a}] = \infty, \quad \widetilde{\Sigma}(0) < \infty, \qquad \text{if } \beta \geq \frac{1}{2}. \end{aligned}$$

Proof of Lemma 4.A.4. We start by showing that $\widetilde{S}(0,\widetilde{a}]$ is always finite. The only possible issue for integrability is at zero, so we expand the integrand in a neighborhood of the origin:

$$\frac{2b(s)}{\widetilde{\sigma}^2(s)} = \frac{2(1-s)}{h\gamma^2 s^{1-2\beta}}\left[1 - \frac{\alpha}{\beta\gamma}s^\beta\right]^{-2} = \frac{2s^{2\beta-1}}{h\gamma^2}\left[1 + \frac{2\alpha}{\beta\gamma}s^\beta + 3\left(\frac{\alpha}{\beta\gamma}\right)^2 s^{2\beta} + O(s^{3\beta})\right].$$

Since the expansion is uniform on $(0, \widetilde{a})$,

$$-\int_{\widetilde{a}}^x \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s = -\frac{2}{\beta h\gamma^2}\left(\frac{x^{2\beta}}{2} + \frac{2\alpha}{3\beta\gamma}x^{3\beta} + \frac{3\alpha^2}{4\beta^2\gamma^2}x^{4\beta} + O(x^{5\beta}) + \overline{\overline{K}}_\beta^{\widetilde{a}}\right),$$

with $\overline{\overline{K}}_\beta^{\,\widetilde{a}} := -\frac{\widetilde{a}^{2\beta}}{2} - \frac{2\alpha}{3\beta\gamma}\widetilde{a}^{3\beta} - \frac{3\alpha^2}{4\beta^2\gamma^2}\widetilde{a}^{4\beta}$, we obtain

$$\widetilde{s}(x) = \exp\left\{-\int_{\widetilde{a}}^{x} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s\right\} = \exp\left\{-\frac{x^{2\beta} + \mathcal{O}(x^{3\beta})}{\beta h\gamma^2} - \frac{2\overline{\overline{K}}_\beta^{\,\widetilde{a}}}{\beta h\gamma^2}\right\} = \exp\left\{-\frac{2}{\beta h\gamma^2}\overline{\overline{K}}_\beta^{\,\widetilde{a}}\right\}\left(1 + \mathcal{O}\left(x^{2\beta}\right)\right),$$

and so $\widetilde{S}(0, \widetilde{a}]$ is always finite.

Now, around zero we have the Taylor expansions

$$\begin{aligned}\widetilde{s}(x) &= \exp\left\{-\frac{2\overline{\overline{K}}_\beta^{\,\widetilde{a}}}{\beta h\gamma^2}\right\}\left(1 + \mathcal{O}\left(x^{2\beta}\right)\right),\\ \widetilde{\sigma}^2(x) &= \frac{1}{\gamma^2 x^{2-2\beta}}\left(1 - \frac{\alpha}{\beta\gamma}x^\beta\right)^2 = \frac{1}{\gamma^2 x^{2-2\beta}}\left(1 + 2\frac{\alpha}{\beta\gamma}x^\beta + 3\frac{\alpha^2}{\beta^2\gamma^2}x^{2\beta} + \mathcal{O}(x^{3\beta})\right)\end{aligned} \tag{4.A.8}$$

and so $(\widetilde{\sigma}^2(x)\widetilde{s}(x))^{-1} = \exp\left\{-\frac{2}{\beta h\gamma^2}\overline{\overline{K}}_\beta^{\,\widetilde{a}}\right\}\frac{1}{\gamma^2 x^{2-2\beta}}\left(1 + 2\frac{\alpha}{\beta\gamma}x^\beta + \mathcal{O}(x^{2\beta})\right)$, which is integrable around zero, and so $\widetilde{M}(0, \widetilde{a}]$ finite, if and only if $\beta > \frac{1}{2}$.

Finally, we easily compute a Taylor expansion for $\widetilde{S}(0, x]$ around the origin by integration:

$$\widetilde{S}(0, x] = \exp\left\{-\frac{2}{\beta h\gamma^2}\overline{\overline{K}}_\beta^{\,\widetilde{a}}\right\} x\left(1 + \mathcal{O}\left(x^{2\beta}\right)\right),$$

hence using (4.A.8), the Taylor expansion around the origin for the integrand in $\widetilde{\Sigma}(0)$ reads

$$\frac{\widetilde{S}(0, x]}{\widetilde{\sigma}^2(x)\widetilde{s}(x)} = \frac{1 + \mathcal{O}(x^\beta)}{\gamma^2 x^{1-2\beta}}.$$

Since this is integrable around the origin if and only if $\beta > 0$, $\widetilde{\Sigma}(0)$ is finite for all $\beta > 0$. □

Right boundary y_σ

The strategy to prove the third table in the theorem is similar, albeit with different computations, to the first case. Similarly to before, introduce the process $\widehat{Z} := y_\sigma - Y$ satisfying the SDE

$$\mathrm{d}\widehat{Z}_t = \widehat{b}(Z_t)\mathrm{d}t + \widehat{\sigma}(\widehat{Z}_t)\mathrm{d}W_t, \qquad \widehat{Z}_0 := y_\sigma - y_0 > 0,$$

as well as the maps $\widehat{b}(x) := -b(y_\sigma - x)$ and $\widehat{\sigma}(x) := -\widetilde{\sigma}(y_\sigma - x)$ for $x > 0$. With Lemma 4.A.5, we obtain the boundary classification of the origin as left boundary for $\widehat{Z}$ on the domain $\mathfrak{D} = (0, y_\sigma)$ corresponding to the right boundary classification of y_σ for Y on the same domain. All the cases $y_\sigma < 1$, $y_\sigma = 1$ and $y_\sigma > 1$ follow [106, Table 6.1, Chapter 15] and the following lemma.

Introduce, for $y_\sigma > \widehat{a} > 0$ and $x \in (0, \widehat{a}]$,

$$\begin{aligned} \widehat{s}(x) &:= \exp\left\{\int_x^{\widehat{a}} \frac{2\widehat{b}(y)}{\widehat{\sigma}^2(y)}\mathrm{d}y\right\}, & \widehat{S}(0,\widehat{a}] &:= \int_0^{\widehat{a}} \widehat{s}(y)\mathrm{d}y, & \widehat{M}(0,\widehat{a}] &:= \int_0^{\widehat{a}} \frac{\mathrm{d}x}{\widehat{\sigma}^2(x)\widehat{s}(x)}, \\ \widehat{N}(0) &:= \int_0^{\widehat{a}} \frac{\widehat{S}[x,\widehat{a}]}{\widehat{\sigma}^2(x)\widehat{s}(x)}\mathrm{d}x, & \widehat{\Sigma}(0) &:= \int_0^{\widehat{a}} \frac{\widehat{S}(0,x]}{\widehat{\sigma}^2(x)\widehat{s}(x)}\mathrm{d}x, & S[x,\widehat{a}] &:= \int_x^{\widehat{a}} \widehat{s}(y)\mathrm{d}y. \end{aligned}$$

Lemma 4.A.5. *The following hold:*

$$\begin{array}{llll} \widehat{S}(0,\widehat{a}] = \infty, & & \widehat{N}(0) < \infty, & \text{if } y_\sigma > 1, \\ \widehat{S}(0,\widehat{a}] < \infty, & & \widehat{\Sigma}(0) = \infty, & \text{if } y_\sigma = 1, \\ \widehat{S}(0,\widehat{a}] < \infty, & \widehat{M}(0,\widehat{a}] = \infty, & \widehat{\Sigma}(0) < \infty, & \text{if } y_\sigma < 1. \end{array}$$

Proof of Lemma 4.A.5. A straightforward Taylor expansion around the origin yields

$$\left\{1 - \left(1 - \frac{y}{y_\sigma}\right)^\beta\right\}^{-2} = \frac{y_\sigma^2}{\beta^2 y^2}\left\{1 + \chi_1\frac{y}{y_\sigma} + \chi_2\frac{y^2}{y_\sigma^2} + \chi_3\frac{y^3}{y_\sigma^3} + \mathcal{O}(x^4)\right\}, \tag{4.A.9}$$

with χ_1, χ_2 and χ_3 defined in (4.A.3). We start with the behaviour of the function $\widehat{s}$ and its integrated version. Consider first the case $y_\sigma > 1$. We split the range of possibilities into two possible intervals for $\widehat{a}$:

(i) If $\widehat{a} < y_\sigma - 1$, then $y_\sigma - x \geq y_\sigma - \widehat{a} > 1$ and b is negative on $[y_\sigma - \widehat{a}, y_\sigma - x]$. Then, for $x \in (0, \widehat{a}]$,

$$\begin{aligned} \int_x^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)}\mathrm{d}y = \int_x^{\widehat{a}} \frac{-b(y_\sigma - y)}{\widehat{\sigma}^2(y_\sigma - y)}\mathrm{d}y &= -\int_x^{\widehat{a}} \frac{b(y_\sigma - y)\mathrm{d}y}{(y_\sigma - y)^{2(1-\beta)}\gamma^2\left(1 - (1 - \frac{y}{y_\sigma})^\beta\right)^2} \\ &= \frac{1}{h}\int_x^{\widehat{a}} \frac{(y_\sigma - y)^{2\beta-1}(y_\sigma - y - 1)}{\gamma^2\left(1 - (1 - \frac{y}{y_\sigma})^\beta\right)^2}\mathrm{d}y \\ &\geq \frac{\left(y_\sigma - \widehat{a}\right)^{2\beta-1}\left(y_\sigma - \widehat{a} - 1\right)}{\gamma^2 h}\int_x^{\widehat{a}}\left(1 - \left(1 - \frac{y}{y_\sigma}\right)^\beta\right)^{-2}\mathrm{d}y, \end{aligned}$$

as $\min_{y\in[x,\widehat{a}]}\left[(y_\sigma - y)^{2\beta-1}(y_\sigma - y - 1)\right] = \left(y_\sigma - \widehat{a}\right)^{2\beta-1}\left(y_\sigma - \widehat{a} - 1\right) > 0$. Indeed the map $y \mapsto y^{2\beta-1}(y-1)$ is increasing on $[y_\sigma - \widehat{a}, y_\sigma - x]$ because $y_\sigma - \widehat{a} > 1$. Noting that (4.A.9) is uniform on $[x, \widehat{a}]$, we obtain, as x approaches zero

$$\begin{aligned} &\exp\left(\frac{2\left(y_\sigma - \widehat{a}\right)^{2\beta-1}\left(y_\sigma - \widehat{a} - 1\right)}{\gamma^2 h}\int_x^{\widehat{a}} \frac{\mathrm{d}y}{\left(1 - (1 - \frac{y}{y_\sigma})^\beta\right)^2}\right) \\ &\quad = \exp\left(\frac{\widehat{K}}{x}\right)\mathrm{e}^{\widehat{K}\widetilde{\widehat{K}}_\beta^{\widehat{a}}} x^{-\frac{\chi_1\widehat{K}}{y_\sigma}}(1 + \mathcal{O}(x)), \end{aligned}$$

with $\widehat{K} := \frac{2(y_\sigma - \widehat{a})^{2\beta-1}(y_\sigma - \widehat{a} - 1)y_\sigma^2}{h\beta^2\gamma^2} > 0$ and $\widetilde{\widehat{K}}_\beta^{\widehat{a}} := -\frac{1}{\widehat{a}} + \frac{\chi_1}{y_\sigma}\log(\widehat{a}) + \frac{\chi_2}{y_\sigma^2}\widehat{a}$, and therefore $\lim_{x\downarrow 0}\widehat{s}(x) = \infty$ and $\widehat{S}(0,\widehat{a}] = \int_0^{\widehat{a}}\widehat{s}(x)\mathrm{d}x = \infty$.

(ii) If $y_\sigma - 1 \le \widehat{a} < y_\sigma$, then for $x \in (0, \widehat{a}]$,

$$\int_x^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y = \int_x^{y_\sigma - 1} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y + \int_{y_\sigma - 1}^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y.$$

Similarly to (i), one can prove that $\lim_{x\downarrow 0} \int_x^{y_\sigma - 1} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y = \infty$. Then, on $(y_\sigma - 1, \widehat{a}]$, $\widehat{\sigma}$ is not null and is continuous, thus bounded; similarly, $\widehat{b}$ is negative and continuous, hence bounded on $(y_\sigma - 1, \widehat{a}]$. Therefore $\int_{y_\sigma - 1}^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y < \infty$, for $x \in (0, \widehat{a}]$, and

$$\lim_{x\downarrow 0} \widehat{s}(x) = \exp\left\{ \int_{y_\sigma - 1}^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y \right\} \exp\left\{ \int_x^{y_\sigma - 1} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y \right\} = \infty.$$

Let us now describe the case $y_\sigma = 1$. Using the Taylor expansion around zero

$$\left[1 - (1-y)^\beta\right]^{-2} = \frac{1}{\beta^2 y^2} \left[1 + \chi_1 y + \chi_2 y^2 + \chi_3 y^3 + \mathcal{O}(y^4)\right], \tag{4.A.10}$$

with χ_1, χ_2 and χ_3 as in (4.A.3), we have, as y approaches zero,

$$\frac{2\widehat{b}(y)}{\widehat{\sigma}^2(y)} = \frac{\overline{\overline{K}}}{y} \left(1 + \overline{\chi}_1 y + \overline{\chi}_2 y^2 + \overline{\chi}_3 y^3 + \mathcal{O}(y^4)\right),$$

with $\overline{\overline{K}} = -\frac{2}{h\beta^2\gamma^2} < 0$ and $\widehat{\overline{\overline{K}}}_\beta^{\widehat{a}} := \log(\widehat{a}) + \overline{\chi_1 \widehat{a}} + \frac{\overline{\chi}_2}{2}\widehat{a}^2 + \frac{\overline{\chi}_3}{3}\widehat{a}^3$. Since the expansion is again uniform on $[x, \widehat{a}]$, we obtain

$$\begin{aligned} \widehat{s}(x) &= \exp\left\{ 2\int_x^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y \right\} = \exp\left\{ \overline{\overline{K}} \left(-\log(x) - \overline{\chi}_1 x + o(x) \right) \right\} \exp\left\{ \overline{\overline{K}} \widehat{\overline{\overline{K}}}_\beta^{\widehat{a}} \right\}, \\ &= \exp\left\{ \overline{\overline{K}} \widehat{\overline{\overline{K}}}_\beta^{\widehat{a}} \right\} x^{-\overline{\overline{K}}} (1 + \mathcal{O}(x)). \end{aligned} \tag{4.A.11}$$

Since $\overline{\overline{K}} < 0$, we easily deduce that $\lim_{x\downarrow 0} \widehat{s}(x) = 0$, and $\widehat{S}(0, \widehat{a}] = \int_0^{\widehat{a}} \widehat{s}(x) \mathrm{d}x$ is finite.

Consider now the case $y_\sigma < 1$. Using (4.A.9), we have, as y approaches zero,

$$\frac{2\widehat{b}(y)}{\widehat{\sigma}^2(y)} = -\frac{K}{y^2} \left(1 + \overline{\chi}_1 y + \overline{\chi}_2 y^2 + \overline{\chi}_3 y^3 + \mathcal{O}(y^4)\right),$$

with $K = \frac{2y_\sigma^{2\beta+1}(1-y_\sigma)}{h\beta^2\gamma^2} > 0$, as $y_\sigma < 1$. Since the expansion is uniform on $[x, \widehat{a}]$, we obtain

$$\begin{aligned} \widehat{s}(x) &= \exp\left\{ 2\int_x^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)} \mathrm{d}y \right\} = \exp\left\{ -\frac{K}{x} \left[1 - \overline{\chi}_1 x \log x\right] \right\} \exp\left\{ -K\widehat{K}_\beta^{\widehat{a}} + \overline{\chi}_2 K x + \mathcal{O}(x^2) \right\}, \\ &= \exp\left\{ -\frac{K}{x} - K\widehat{K}_\beta^{\widehat{a}} \right\} x^{K\overline{\chi}_1} (1 + \mathcal{O}(x)), \end{aligned} \tag{4.A.12}$$

where $\widehat{K}_\beta^{\widehat{a}} = -\frac{1}{\widehat{a}} + \overline{\chi}_1 \log(\widehat{a}) + \overline{\chi}_1 \widehat{a}$. Since $K > 0$, we easily deduce that $\lim_{x\downarrow 0} \widehat{s}(x) = 0$, and $\widehat{S}(0, \widehat{a}] = \int_0^{\widehat{a}} \widehat{s}(x) \mathrm{d}x$ is finite.

The middle statement (when $y_\sigma < 1$) in the lemma is straightforward. When $x \in (0, \widehat{a}]$,

$\int_x^{\widehat{a}} \frac{\widehat{b}(y)}{\widehat{\sigma}^2(y)}\mathrm{d}y = -\int_{y_\sigma-\widehat{a}}^{y_\sigma-x} \frac{b(y)}{\widehat{\sigma}^2(y)}\mathrm{d}y$. Since $0 < y_\sigma - \widehat{a} < y_\sigma - x < 1$, b is positive on $[y_\sigma - \widehat{a}, y_\sigma - x]$ and the above integral is therefore negative. Hence, $\widehat{s}$ is bounded by 1 on $(0, \widehat{a}]$, and

$$\widehat{M}(0,\widehat{a}] = \int_0^{\widehat{a}} \frac{1}{\widehat{s}(x)\widehat{\sigma}^2(x)}\mathrm{d}x \geq \int_0^{\widehat{a}} \frac{\mathrm{d}x}{\widehat{\sigma}^2(y_\sigma - x)} = \infty,$$

using (4.A.9), which concludes the proof.

The final integrals in the lemma are delicate. We start with the case $y_\sigma < 1$. Using (4.A.12), we obtain the asymptotic behaviour of $\widehat{S}(0,\cdot]$ around zero by integrating that of $\widehat{s}(\cdot)$ around 0. Classical asymptotic expansions for integrals (note that the leading contribution arises at the right boundary of the integration domain) yield, after the change of variable $y \mapsto xz$,

$$\begin{aligned}\widehat{S}(0,x] &= \int_0^x \widehat{s}(y)\mathrm{d}y = \mathrm{e}^{-K\widehat{K}_\beta^{\widehat{a}}} x^{1+K\overline{X}_1} \int_0^1 \exp\left\{-\frac{K}{xz}\right\} z^{K\overline{X}_1}\mathrm{d}z, \\ &= \mathrm{e}^{-K\widehat{K}_\beta^{\widehat{a}}} x^{1+K\overline{X}_1} \exp\left\{-\frac{K}{x}\right\}\left(\frac{x}{K} + \mathcal{O}(x^2)\right), \qquad \text{as } x \downarrow 0.\end{aligned} \tag{4.A.13}$$

Therefore, combining (4.A.9), (4.A.12) and (4.A.13), we obtain

$$\frac{\widehat{S}(0,x]}{\widehat{s}(x)\widehat{\sigma}^2(x)} = x\left(\frac{1}{K}x + \mathcal{O}(x^2)\right)\frac{y_\sigma^{2\beta}}{\beta^2\gamma^2}\frac{1}{x^2}(1+\mathcal{O}(x)) = \frac{y_\sigma^{2\beta}}{\beta^2\gamma^2 K}(1+\mathcal{O}(x)),$$

which is integrable on $(0, a]$ and concludes the proof.

In the case $y_\sigma = 1$, exploiting (4.A.11), we obtain the asymptotic behaviour of $\widehat{S}(0,\cdot]$ around zero by integrating that of $\widehat{s}(\cdot)$ around 0. Indeed, after the change of variable $y \mapsto xz$,

$$\widehat{S}(0,x] = \int_0^x \widehat{s}(y)\mathrm{d}y = \frac{\exp\left\{\overline{\overline{K}}\widehat{\widehat{K}}_\beta^{\widehat{a}}\right\}}{x^{\overline{\overline{K}}-1}} \int_0^1 z^{-\overline{\overline{K}}}\mathrm{d}z = \exp\left\{\overline{\overline{K}}\widehat{\widehat{K}}_\beta^{\widehat{a}}\right\} x^{1-\overline{\overline{K}}}(1+\mathcal{O}(x)), \tag{4.A.14}$$

as x tends to zero. Then, exploiting (4.A.10), (4.A.11) and (4.A.14), we obtain

$$\frac{\widehat{S}(0,x]}{\widehat{s}(x)\widehat{\sigma}^2(x)} = \frac{1}{\beta^2\gamma^2}\frac{1}{x}(1+\mathcal{O}(x)),$$

which is not integrable on $(0, \widehat{a}]$ and thus $\widehat{\Sigma}(0) = \infty$.

Finally, we move to the case $y_\sigma > 1$. Since $\lim_{x\downarrow 0} \widehat{S}[x, \widehat{a}] = \infty$ and $\lim_{x\downarrow 0} \widehat{s}(x)\widehat{\sigma}^2(x) = \infty$, one needs to study the behaviour of the integrand around zero to conclude. For $\delta > 0$ such that $x < \delta < \widehat{a}$ and $x > 0$, $\int_x^{\widehat{a}} \widehat{s}(y)\mathrm{d}y = \int_x^\delta \widehat{s}(y)\mathrm{d}y + \int_\delta^{\widehat{a}} \widehat{s}(y)\mathrm{d}y$. Note that the second integral is convergent as the integral of a continuous function over a closed interval of $\mathbb{R}$.

Classical asymptotic expansions for integrals [123, Chapter 3.3] and (4.A.12), yield, after mapping $y \mapsto zx$,

$$\int_x^\delta \widehat{s}(y)\mathrm{d}y = \mathrm{e}^{-K\widehat{K}_\beta^{\widehat{a}}} x^{1+K\overline{X}_1} \int_1^{\delta/x} \exp\left\{-\frac{K}{xz}\right\} z^{K\overline{X}_1}\mathrm{d}z = \mathrm{e}^{-K\widehat{K}_\beta^{\widehat{a}}} x^{1+K\overline{X}_1}\mathrm{e}^{-\frac{K}{x}}\left(-\frac{x}{K} + \mathcal{O}(x^2)\right),$$

and the asymptotic behaviour of the integrand around the origin is given by

$$\frac{\widehat{S}[x,\widehat{a}]}{\widehat{s}(x)\widehat{\sigma}^2(x)} = -\frac{y_\sigma^{2\beta}}{\beta^2\gamma^2}\left(\frac{x^2}{K} + O(x^3)\right)\frac{1+O(x)}{x^2} = -\frac{y_\sigma^{2\beta}}{\beta^2\gamma^2 K}(1+O(x)),$$

which is integrable at the origin, and concludes the proof since $\widehat{N}(0)$ is therefore finite. □

4.B Ergodicity proofs

4.B.1 Proof of Theorem 4.3.5

This study is based on [132, Theorem 1.1, Chapter 5.1]. Since $\widetilde{\sigma}$ is null at y_σ (and possibly at zero), we consider separately the two domains $\mathfrak{D}_1 := (0, y_\sigma)$ and $\mathfrak{D}_2 := (y_\sigma, \infty)$ so that Assumption A iii) in the aforementioned theorem is satisfied. We start with $y_0 \in \mathfrak{D}_2 = (y_\sigma, \infty)$. We have to check the finiteness of

$$\mathbf{A} := \int_{y_\sigma}^{y_0} \exp\left\{-\int_{y_0}^{y} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s\right\}\mathrm{d}y \quad \text{and} \quad \mathbf{B} := \int_{y_0}^{\infty} \exp\left\{-\int_{y_0}^{y} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s\right\}\mathrm{d}y.$$

Starting with $\mathbf{A}$, the changes of variables $y \longrightarrow x + y_\sigma$ and $s \longrightarrow v + y_\sigma$ yield

$$\mathbf{A} = \int_0^{y_0-y_\sigma} \exp\left\{\int_x^{y_0-y_\sigma} \frac{2b(v+y_\sigma)}{\widetilde{\sigma}^2(v+y_\sigma)}\mathrm{d}v\right\}\mathrm{d}x = \int_0^{y_0-y_\sigma} \exp\left\{\int_x^{y_0-y_\sigma} \frac{2\overline{b}(v)}{\overline{\sigma}^2(v)}\mathrm{d}v\right\}\mathrm{d}x,$$

with $\overline{b}(v) := b(v+y_\sigma)$ and $\overline{\sigma}(v) := \widetilde{\sigma}(v+y_\sigma)$, $v \geq 0$. Notice that $\mathbf{A} = \int_0^a \overline{s}(x)\mathrm{d}x$, with $\overline{s}$ as in (4.A.1) and $a := y_0 - y_\sigma \geq 0$. Thus, exploiting the proof of Lemma 4.A.1, $\mathbf{A}$ is finite if and only if $y_\sigma \geq 1$. Regarding $\mathbf{B}$, we need to study the integrand at infinity. Since, as $s \uparrow \infty$,

$$\frac{2b(s)}{\widetilde{\sigma}^2(s)} = \frac{2(1-s)}{hs\left(-\frac{\alpha}{\beta} + \gamma s^{-\beta}\right)^2} \sim -\frac{2}{h}\left(\frac{\beta}{\alpha}\right)^2,$$

then $\int_{y_0}^{y} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s \sim -\frac{2\beta^2}{h\alpha^2}y$, as $y \uparrow \infty$, so $\mathbf{B}$ is infinite, concluding the $y_0 \in \mathfrak{D}_2$ discussion. Consider now the domain $\mathfrak{D}_1 = (0, y_\sigma)$. We have to check the finiteness of

$$\mathbf{C} := \int_0^{y_0} \exp\left\{-\int_{y_0}^{y} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s\right\}\mathrm{d}y, \quad \text{and} \quad \mathbf{D} := \int_{y_0}^{y_\sigma} \exp\left\{-\int_{y_0}^{y} \frac{2b(s)}{\widetilde{\sigma}^2(s)}\mathrm{d}s\right\}\mathrm{d}y.$$

For $\mathbf{C}$, the only possible issue for integrability is at zero, so we expand the integrand in a neighborhood of the origin:

$$\frac{2b(s)}{\widetilde{\sigma}^2(s)} = \frac{2(1-s)}{h\gamma^2 s^{1-2\beta}}\left(1 - \frac{\alpha}{\beta\gamma}s^\beta\right)^{-2} = \frac{2s^{2\beta-1}}{h\gamma^2}\left[1 + \frac{2\alpha}{\beta\gamma}s^\beta + 3\left(\frac{\alpha}{\beta\gamma}\right)^2 s^{2\beta} + O\left(s^{3\beta}\right)\right].$$

Since the expansion is uniform on $(0, y_0)$,

$$\int_{y_0}^{x} \frac{2b(s)}{\widetilde{\sigma}^2(s)} \mathrm{d}s = \frac{2}{\beta h \gamma^2} \left(\frac{x^{2\beta}}{2} + \frac{2\alpha}{3\beta\gamma} x^{3\beta} + \frac{3\alpha^2}{4\beta^2\gamma^2} x^{4\beta} + o\left(x^{4\beta}\right) + \widetilde{K}_{\beta}^{y_0} \right),$$

with $\widetilde{K}_{\beta}^{y_0} := -\frac{y_0^{2\beta}}{2} - \frac{2\alpha}{3\beta\gamma} y_0^{3\beta} - \frac{3\alpha^2}{4\beta^2\gamma^2} y_0^{4\beta}$. We then obtain

$$\exp\left\{-\int_{y_0}^{x} \frac{2b(s)}{\widetilde{\sigma}^2(s)} \mathrm{d}s\right\} = \exp\left\{-\frac{x^{2\beta} + o\left(x^{2\beta}\right)}{\beta h \gamma^2} - \frac{2\widetilde{K}_{\beta}^{y_0}}{\beta h \gamma^2}\right\} = \exp\left\{-\frac{2\widetilde{K}_{\beta}^{y_0}}{\beta h \gamma^2}\right\} \left(1 + o\left(x^{2\beta}\right)\right),$$

and so **C** is always finite. Finally, regarding **D**, the following sequence of change of variables, as $x \downarrow y_\sigma - y$ and $s \downarrow y_\sigma - v$, yields

$$\begin{aligned}
\mathbf{D} &= \int_{y_0}^{y_\sigma} \exp\left\{-\int_{y_0}^{x} \frac{2b(s)}{\widetilde{\sigma}^2(s)} \mathrm{d}s\right\} \mathrm{d}x = \int_{0}^{y_\sigma - y_0} \exp\left\{-\int_{y}^{y_\sigma - y_0} \frac{2b(y_\sigma - v)}{\widetilde{\sigma}^2(y_\sigma - v)} \mathrm{d}v\right\} \mathrm{d}y \\
&= \int_{0}^{y_\sigma - y_0} \exp\left\{\int_{y}^{y_\sigma - y_0} \frac{2\widehat{b}(v)}{\widetilde{\sigma}^2(y_\sigma - v)} \mathrm{d}v\right\} \mathrm{d}y,
\end{aligned}$$

with $\widehat{b}(v) = -b(y_\sigma - v)$ and $\widehat{\sigma}(v) = -\widetilde{\sigma}(y_\sigma - v)$, for $v \in (0, y_\sigma)$. Now, notice that $\mathbf{D} = \int_0^a \widehat{s}(x)\mathrm{d}x$, with $\widehat{s}$ defined in (4.A.1) and $a = y_\sigma - y_0$. Thus, exploiting the proof of Lemma 4.A.5, the integral **D** is finite if and only if $y_\sigma \le 1$, and the theorem follows.

4.B.2 Proof of Proposition 4.3.7

Pursuing the analysis in [106, page 242], we can prove that

$$\begin{aligned}
S_\Pi(y_\sigma, x] &:= \int_{y_\sigma}^{x} s(\xi)\mathrm{d}\xi = \infty, \qquad & S_\Pi[x, \infty) &:= \int_{x}^{\infty} s(\xi)\mathrm{d}\xi = \infty, \\
M_\Pi(y_\sigma, x] &:= \int_{y_\sigma}^{x} \frac{\mathrm{d}\xi}{\widetilde{\sigma}^2(\xi)s(\xi)} < \infty, & M_\Pi[x, \infty) &:= \int_{x}^{\infty} \frac{\mathrm{d}\xi}{\widetilde{\sigma}^2(\xi)s(\xi)} < \infty,
\end{aligned}$$

where $s(\xi) := \exp\left\{-\int_a^\xi \frac{2b(\eta)}{\widetilde{\sigma}^2(\eta)} \mathrm{d}\eta\right\}$, with $a > y_\sigma$ fixed. Thus, the stationary probability measure is given by (4.3.3). Let us start by showing that $S_\Pi(y_\sigma, x]$ and $S_\Pi[x, \infty)$ are infinite. Exploiting the computations in the proof of Theorem 4.3.5, for the case $y_\sigma < 1$ and $\mathfrak{D} = (y_\sigma, \infty)$, we obtain the unboundedness of

$$S_\Pi(y_\sigma, x] = \int_{y_\sigma}^{x} s(\xi)\mathrm{d}\xi = \int_{0}^{x - y_\sigma} \exp\left\{-\int_{y}^{x - y_\sigma} \frac{2b(\eta + y_\sigma)}{\widetilde{\sigma}^2(\eta + y_\sigma)} \mathrm{d}\eta\right\} \mathrm{d}y = \int_{0}^{x - y_\sigma} \overline{s}(y)\mathrm{d}y,$$

and

$$S_\Pi[x, \infty) = \int_{x}^{\infty} s(\xi)\mathrm{d}\xi = \int_{x}^{\infty} \exp\left\{-\frac{2}{h}\int_{x}^{\xi} \frac{(1-\eta)\mathrm{d}\eta}{\eta(-\frac{\alpha}{\beta} + \gamma\eta^{-\beta})}\right\} \mathrm{d}y \sim \int_{x}^{\infty} \exp\left\{\frac{2}{h}\int_{x}^{\xi} \frac{\beta}{\alpha}\eta\mathrm{d}\eta\right\} \mathrm{d}y.$$

We are only left to prove that $M_\Pi(y_\sigma, x]$ and $M_\Pi[x, \infty)$ are finite. Exploiting the changes of variables $\xi = v + y_\sigma$ and $\eta = z + y_\sigma$, $M_\Pi(y_\sigma, x]$ can be rewritten as

$$\begin{aligned} M_\Pi(y_\sigma, x] &= \int_{y_\sigma}^{x} \frac{\mathrm{d}\xi}{\overline{\sigma}^2(\xi)s(\xi)} = \int_{y_\sigma}^{x} \frac{1}{\overline{\sigma}^2(\xi)} \exp\left\{\int_x^\xi \frac{2b(\eta)}{\overline{\sigma}^2(\eta)}\mathrm{d}\eta\right\}\mathrm{d}\xi \\ &= \int_0^{x-y_\sigma} \frac{1}{\overline{\sigma}^2(v+y_\sigma)} \exp\left\{\int_{x-y_\sigma}^{v} \frac{2b(z+y_\sigma)}{\overline{\sigma}^2(z+y_\sigma)}\mathrm{d}z\right\}\mathrm{d}v \\ &= \int_0^{x-y_\sigma} \frac{\mathrm{d}v}{\overline{\sigma}^2(v+y_\sigma)\overline{s}(v)} = \int_0^{x-y_\sigma} \frac{\mathrm{d}v}{\overline{\sigma}^2(v)\overline{s}(v)} = \overline{M}(0, x-y_\sigma]. \end{aligned}$$

Then, in a neighborhood of zero we have

$$\frac{1}{\overline{\sigma}^2(v)\overline{s}(v)} = \frac{y_\sigma^{2\beta}}{\beta^2\gamma^2}\mathrm{e}^{-KK_\beta^{x-y_\sigma}} \exp\left\{-\frac{K}{v}\right\} v^{K\overline{\chi}_1-2}(1+\mathcal{O}(v)),$$

which is integrable around zero since $K > 0$ and thus $M_\Pi(y_\sigma, x]$ is finite.

To conclude we have to study the finiteness of $M_\Pi[x, \infty)$, which means that we have to check the integrability of $\frac{1}{\overline{\sigma}^2(\xi)s(\xi)}$ at infinity. Since

$$\frac{1}{\overline{\sigma}^2(\xi)s(\xi)} = \frac{1}{\overline{\sigma}^2(\xi)} \exp\left\{\int_x^\xi \frac{2b(\eta)}{\overline{\sigma}^2(\eta)}\mathrm{d}\eta\right\} \sim \frac{\xi^{2\beta-2}}{\gamma} \exp\left\{\int_x^\xi -\frac{2}{h}(1+\mathcal{O}(\eta))\mathrm{d}\eta\right\} \sim \frac{\xi^{2\beta-2}}{\gamma}\mathrm{e}^{-\frac{2}{h}\xi}$$

is integrable at infinity, the result follows.

4.B.3 Proof of Proposition 4.3.9

The aim is to approximate the price of an option with smooth payoff h as $P^\varepsilon \approx Q^\varepsilon := P_0 + \sqrt{\varepsilon}P_1$, that is $P^\varepsilon = Q^\varepsilon + \mathcal{O}(\varepsilon)$. We show that both P_0 and P_1 in fact do not depend on y and provide a precise estimate for the error term. A Taylor expansion of P^ε around $\varepsilon = 0$ gives

$$P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon Q_2 + \varepsilon^{3/2}Q_3 + \mathcal{O}\left(\varepsilon^{3/2}\right) = Q^\varepsilon + \varepsilon Q_2 + \varepsilon^{3/2}Q_3 + \mathcal{O}\left(\varepsilon^{3/2}\right).$$

The pricing PDE (4.3.5) then reads

$$\begin{aligned} 0 &= \left(\frac{1}{\varepsilon}\mathcal{L}_Y + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}\right)P^\varepsilon \\ &= \left(\frac{1}{\varepsilon}\mathcal{L}_Y + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}\right)\left(P_0 + \sqrt{\varepsilon}P_1 + \varepsilon Q_2 + \varepsilon^{3/2}Q_3 + \mathcal{O}\left(\varepsilon^{3/2}\right)\right) \\ &= \frac{\mathcal{L}_Y P_0}{\varepsilon} + \frac{\mathcal{L}_Y P_1 + \mathcal{L}_1 P_0}{\sqrt{\varepsilon}} + \left[\mathcal{L}_Y Q_2 + \mathcal{L}_1 P_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_0\right] + \sqrt{\varepsilon}\left[\mathcal{L}_Y Q_3 + \mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_1\right] + \mathcal{O}(\varepsilon). \end{aligned}$$

Since this should be null for all (small) ε, each term should be equal to zero. More specifically,

a. $\mathcal{L}_Y P_0 = 0$. Since $\mathcal{L}_Y$ has no x-derivative, $P_0(t,x,y) = P_0(t,x)$ with $P_0(T,x) = h(x)$;

b. $\mathcal{L}_Y P_1 + \mathcal{L}_1 P_0 = 0 = \mathcal{L}_Y P_1$ using a.. Similarly $P_1(t,x,y) = P_1(t,x)$ with $P_1(T,x) = 0$;

c. $0 = \mathcal{L}_Y Q_2 + \mathcal{L}_1 P_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_0 = \mathcal{L}_Y Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_0$. This is a Poisson equation associated to $\mathcal{L}_Y$

and requires a suitable solvability condition: Similarly to [63], the Fredholm alternative[2] imposes the condition

$$0 = \langle \mathcal{L}_{\mathrm{BS}}^{\sigma} P_0, \Pi \rangle = \int_{y_\sigma}^{\infty} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma} P_0 \right) \Pi(\mathrm{d}y) = \int_{y_\sigma}^{\infty} \left[\partial_t + \frac{\sigma^2(y)}{2} \mathcal{D}_x \right] P_0(t,x) \Pi(\mathrm{d}y)$$
$$= \left[\partial_t + \frac{1}{2} \int_{y_\sigma}^{\infty} \sigma^2(y) \Pi(\mathrm{d}y) \right] \mathcal{D}_x P_0(t,x),$$

where Π is the unique stationary distribution of Y on $\mathfrak{D} = (y_\sigma, \infty)$ from Proposition 4.3.7, and with the operator $\mathcal{D}_x := \partial_x^2 - \partial_x$.

This last computation in particular reveals that

$$\left\langle \mathcal{L}_{\mathrm{BS}}^{\sigma}, \Pi \right\rangle = \mathcal{L}_{\mathrm{BS}}^{\varkappa},$$

so that P_0 in fact satisfies $\mathcal{L}_{\mathrm{BS}}^{\varkappa} P_0(t,x) = 0$, with boundary condition $P_0(T,x) = h(x)$, so that P_0 corresponds to the Black-Scholes option price with payoff h and variance $\varkappa^2 := \langle \sigma^2, \Pi \rangle = \int_{y_\sigma}^{\infty} \sigma^2(y) \Pi(\mathrm{d}y)$, as given in the proposition.

Remark 4.B.1. *The variance $\varkappa^2$ is clearly finite: using the asymptotic computations in the previous section, as $y \uparrow \infty$, the integrand behaves as $\exp\{-\frac{2}{h}(\frac{\alpha}{\beta})^2 y\} y^{-2}$ which is integrable at infinity (4.A.7). When $y \downarrow y_\sigma$ it behaves as $\exp\{-\frac{K}{(y-y_\sigma)} - K K_\beta^\alpha\}(y-y_\sigma)^{-\overline{x}_1 - 2}(1 + \mathcal{O}(y - y_\sigma))$, also integrable since $K > 0$ (4.A.4).*

Observe now from c. above that

$$Q_2 = -\mathcal{L}_Y^{-1} \left(\mathcal{L}_1 P_1 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_0 \right) = -\mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_0 \right) = -\mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} - \mathcal{L}_{\mathrm{BS}}^{\varkappa} \right) P_0. \tag{4.B.1}$$

Note that we do not formally need to invert $\mathcal{L}_Y$, but it makes the notations below clearer.

d. Regarding the $\sqrt{\varepsilon}$ term, $\mathcal{L}_Y Q_3 + \mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} P_1 = 0$. This is again a Poisson equation with solvability condition (using (4.B.1))

$$\mathcal{L}_{\mathrm{BS}}^{\varkappa} P_1 = \left\langle \mathcal{L}_{\mathrm{BS}}^{\sigma}, \Pi \right\rangle P_1 = -\langle \mathcal{L}_1 Q_2, \Pi \rangle = \left\langle \mathcal{L}_1 \mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma} - \mathcal{L}_{\mathrm{BS}}^{\varkappa} \right), \Pi \right\rangle P_0. \tag{4.B.2}$$

Combining this with the terminal condition on P_1 obtained in b., we obtain

$$\begin{cases} \mathcal{L}_{\mathrm{BS}}^{\varkappa} P_1(t,x) = \left\langle \mathcal{L}_1 \mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma} - \mathcal{L}_{\mathrm{BS}}^{\varkappa} \right), \Pi \right\rangle P_0, \\ P_1(T,x) = 0, \end{cases}$$

so that P_1 is the solution of a Black-Scholes system with variance $\varkappa^2$ and source

$$\left\langle \mathcal{L}_1 \mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma} - \mathcal{L}_{\mathrm{BS}}^{\varkappa} \right), \Pi \right\rangle P_0 = \frac{1}{2} \left\langle \mathcal{L}_1 \mathcal{L}_Y^{-1} \left(\sigma^2 - \varkappa^2 \right), \Pi \right\rangle \mathcal{D}_x P_0.$$

Setting ψ to be the solution to $\mathcal{L}_Y \psi(y) = \sigma^2(y) - \varkappa^2$ in (4.3.7), we obtain

$$\left\langle \mathcal{L}_1 \mathcal{L}_Y^{-1} \left(\mathcal{L}_{\mathrm{BS}}^{\sigma} - \mathcal{L}_{\mathrm{BS}}^{\varkappa} \right), \Pi \right\rangle P_0 = \frac{1}{2} \left\langle \mathcal{L}_1 \psi(\cdot), \Pi \right\rangle \mathcal{D}_x P_0 = \frac{1}{2} \langle \varpi, \Pi \rangle \partial_x \mathcal{D}_x P_0,$$

[2] As far as we know, there is no general Fredholm alternative for hypoelliptic operators. Numerical tests seem to clearly indicate the presence of a spectral gap in our case, which would be enough, but we leave this very lengthy and detailed analysis to further research.

by definition of $\mathcal{L}_1$ in (4.3.4) and of ϖ in (4.3.6). The last term on the right-hand side is well defined provided that (4.3.7) admits a unique solution such that $\langle\varpi,\Pi\rangle$ is finite. The existence of such a unique (up to some positive constant) solution is ensured by the validity of the corresponding solvability condition, consequence of $\boldsymbol{\varkappa}^2$ being finite (as proved in Remark 4.B.1). A similar argument shows that $\langle\varpi,\Pi\rangle$ is also finite once we prove polynomial growth at infinity and the boundedness of ψ around y_σ. Indeed, in that case, for $y\uparrow\infty$, the integrand behaves like $\exp\{-\frac{2}{h}(\frac{\alpha}{\beta})^2y\}y^{-1}\psi'(y)\sim\exp\{-\frac{2}{h}(\frac{\alpha}{\beta})^2y\}(1+y^{n-1})$, which is integrable. As $y\downarrow y_\sigma$, we have $\exp\{-\frac{K}{(y-y_\sigma)}-KK^{\alpha}_{\beta}\}(y-y_\sigma)^{-K\overline{\chi}_1-1}\psi'(y)(1+\mathcal{O}(y-y_\sigma))$, which is integrable since $K>0$. We thus conclude that

$$\sqrt{\varepsilon}P_1(t,x)=-(T-t)\Omega^\varepsilon\partial_x\mathcal{D}_xP_0(t,x),\tag{4.B.3}$$

with $\Omega^\varepsilon:=\frac{\sqrt{\varepsilon}}{2}\langle\varpi,\Pi\rangle$. This implies $P_1(T,x)=0$ and, since $\langle\mathcal{L}^{\sigma}_{\text{BS}},\Pi\rangle P_0=\mathcal{L}^{\boldsymbol{\varkappa}}_{\text{BS}}P_0=0$,

$$\begin{aligned}\mathcal{L}^{\boldsymbol{\varkappa}}_{\text{BS}}P_1&=\frac{1}{\sqrt{\varepsilon}}\mathcal{L}^{\boldsymbol{\varkappa}}_{\text{BS}}\Big(-(T-t)\Omega^\varepsilon\partial_x\mathcal{D}_xP_0\Big)\\&=\frac{\Omega^\varepsilon\partial_x\mathcal{D}_xP_0-(T-t)\Omega^\varepsilon\partial_x\mathcal{D}_x\mathcal{L}^{\boldsymbol{\varkappa}}_{\text{BS}}P_0}{\sqrt{\varepsilon}}=\frac{\Omega^\varepsilon}{\sqrt{\varepsilon}}\partial_x\mathcal{D}_xP_0=\left\langle\mathcal{L}_1\mathcal{L}_Y^{-1}\left(\mathcal{L}^{\sigma}_{\text{BS}}-\mathcal{L}^{\boldsymbol{\varkappa}}_{\text{BS}}\right),\Pi\right\rangle P_0,\end{aligned}$$

which corresponds precisely to (4.B.2).

We now move on to the proof of the error term, assuming a smooth payoff h. With $\mathcal{L}_\varepsilon:=\frac{1}{\varepsilon}\mathcal{L}_Y+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}^{\sigma(y)}_{\text{BS}}$ and

$$Z^\varepsilon:=\varepsilon Q_2+\varepsilon\sqrt{\varepsilon}Q_3-(P^\varepsilon-Q^\varepsilon)=\varepsilon Q_2+\varepsilon\sqrt{\varepsilon}Q_3-\left[P^\varepsilon-\left(P_0+P_1\sqrt{\varepsilon}\right)\right],$$

the pricing PDE (4.3.5) now yields

$$\begin{aligned}\mathcal{L}_\varepsilon Z^\varepsilon&=\mathcal{L}_\varepsilon\left(\varepsilon Q_2+\varepsilon\sqrt{\varepsilon}Q_3-\left[P^\varepsilon-\left(P_0+P_1\sqrt{\varepsilon}\right)\right]\right)=\mathcal{L}_\varepsilon\left(P_0+P_1\sqrt{\varepsilon}+\varepsilon Q_2+\varepsilon\sqrt{\varepsilon}Q_3-P^\varepsilon\right)\\&=\frac{\mathcal{L}_YP_0}{\varepsilon}+\frac{\mathcal{L}_YP_1+\mathcal{L}_1P_0}{\sqrt{\varepsilon}}+\left(\mathcal{L}_YQ_2+\mathcal{L}_1P_1+\mathcal{L}^{\sigma(y)}_{\text{BS}}P_0\right)+\sqrt{\varepsilon}\left(\mathcal{L}_YQ_3+\mathcal{L}_1Q_2+\mathcal{L}^{\sigma(y)}_{\text{BS}}P_1\right)\\&\quad+\varepsilon\left(\mathcal{L}_1Q_3+\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_2\right)+\varepsilon\sqrt{\varepsilon}\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_3-\mathcal{L}_\varepsilon P^\varepsilon\\&=\varepsilon\left(\mathcal{L}_1Q_3+\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_2+\sqrt{\varepsilon}\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_3\right).\end{aligned}$$

Setting

$$\begin{cases}F_\varepsilon(t,x,y)&:=\mathcal{L}_1Q_3+\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_2+\sqrt{\varepsilon}\mathcal{L}^{\sigma(y)}_{\text{BS}}Q_3\\G_\varepsilon(x,y)&:=Q_2(T,x,y)+\sqrt{\varepsilon}Q_3(T,x,y),\end{cases}\tag{4.B.4}$$

we write a parabolic PDE associated to Z^ε:

$$\mathcal{L}_\varepsilon Z^\varepsilon=\varepsilon F_\varepsilon,\qquad\text{with boundary condition}\qquad Z^\varepsilon(T,x,y)=\varepsilon G_\varepsilon(x,y).\tag{4.B.5}$$

The first is a consequence of the identities above, while the second follows from

$$\begin{aligned}Z^\varepsilon(T,x,y)&=\varepsilon Q_2(T,x,y)+\varepsilon\sqrt{\varepsilon}Q_3(T,x,y)-[P^\varepsilon(T,x,y)-(P_0(T,x,y)+P_1(T,x,y)\sqrt{\varepsilon})]\\&=\varepsilon Q_2(T,x,y)+\varepsilon\sqrt{\varepsilon}Q_3(T,x,y)-[h(x)-(h(x)+0)]\end{aligned}$$

$$= \varepsilon Q_2(T,x,y) + \varepsilon\sqrt{\varepsilon}Q_3(T,x,y) = \varepsilon G_\varepsilon(x,y)$$

We now investigate the form of Q_2, Q_3. From the third identity in (4.B.1), $Q_2 = -\frac{1}{2}\psi(y)\mathcal{D}_x P_0$, where ψ is the solution to (4.3.7), which implies (recall that P_0 does not depend on y)

$$\mathcal{L}_Y Q_2 = -\frac{1}{2}\mathcal{L}_Y\left[\psi(y)\mathcal{D}_x P_0\right] = -\frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x P_0 = -\left(\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} - \mathcal{L}_{\mathrm{BS}}^{\varkappa}\right)P_0 = -\mathcal{L}_{\mathrm{BS}}^{\sigma(y)}P_0$$

The core idea here is to rewrite F_ε and G_ε to obtain the order of convergence of the first-order price approximation Q^ε. In the following computations, P_0 has smooth derivatives since the payoff h is smooth by assumption. The identity

$$\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} = \partial_t + \frac{\sigma^2(y)}{2}\mathcal{D}_x = \mathcal{L}_{\mathrm{BS}}^{\varkappa} + \frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x \tag{4.B.6}$$

holds, yielding an explicit expression for the second term on the right-hand side of (4.B.4):

$$\begin{aligned}\mathcal{L}_{\mathrm{BS}}^{\sigma(y)}Q_2 = \left(\mathcal{L}_{\mathrm{BS}}^{\varkappa} + \frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x\right)\left(-\frac{\psi(y)}{2}\mathcal{D}_x P_0\right) &= -\frac{\psi(y)}{2}\left(\mathcal{L}_{\mathrm{BS}}^{\varkappa}\mathcal{D}_x P_0 + \frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x^2 P_0\right)\\ &= -\frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\mathcal{D}_x^2 P_0,\end{aligned} \tag{4.B.7}$$

since these differential operators commute. Now, Q_3 is solution to the Poisson equation $\mathcal{L}_Y Q_3 = -\left(\mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}P_1\right)$, and the validity of the centering condition for the Poisson equation is guaranteed by the choice of P_1. Equivalently,

$$\begin{aligned}Q_3 &= -\mathcal{L}_Y^{-1}\left(\mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}P_1\right)\\ &= -\mathcal{L}_Y^{-1}\left(\mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)}P_1 - \left\langle \mathcal{L}_1 Q_2 + \mathcal{L}_{\mathrm{BS}}^{\sigma}P_1, \Pi\right\rangle\right)\\ &= -\mathcal{L}_Y^{-1}\left(\mathcal{L}_1 Q_2 - \langle \mathcal{L}_1 Q_2, \Pi\rangle + \left(\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} - \mathcal{L}_{\mathrm{BS}}^{\varkappa}\right)P_1\right).\end{aligned} \tag{4.B.8}$$

We make the terms on the right more explicit

$$\begin{aligned}\mathcal{L}_1 Q_2 = -\left(y\sigma^2(y)\partial_{xy}\right)\left(\frac{\psi(y)}{2}\mathcal{D}_x P_0\right) &= -\frac{y\sigma^2(y)}{2}\partial_x\left(\psi'(y)\mathcal{D}_x P_0 + \psi(y)\mathcal{D}_x\partial_y P_0\right)\\ &= -\frac{\varpi(y)}{2}\partial_x\mathcal{D}_x P_0.\end{aligned} \tag{4.B.9}$$

Now, let ϑ be the solution to the Poisson equation

$$\mathcal{L}_Y\vartheta = \varpi(y) - \langle\varpi, \Pi\rangle\,, \tag{4.B.10}$$

and plug (4.B.6) and (4.B.9) into (4.B.8) to obtain

$$\begin{aligned}Q_3 &= -\mathcal{L}_Y^{-1}\left(-\frac{\varpi(y)}{2}\partial_x\mathcal{D}_x P_0 + \frac{\langle\varpi,\Pi\rangle}{2}\partial_x\mathcal{D}_x P_0 + \frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x P_1\right)\\ &= \frac{1}{2}\mathcal{L}_Y^{-1}\left[\left(\varpi(y) - \langle\varpi,\Pi\rangle\right)\partial_x\mathcal{D}_x P_0 - \left(\sigma^2(y)-\varkappa^2\right)\mathcal{D}_x P_1\right]\end{aligned} \tag{4.B.11}$$

$$= \frac{1}{2}\mathcal{L}_Y^{-1}\Big(\mathcal{L}_Y \vartheta \partial_x \mathcal{D}_x P_0 - \mathcal{L}_Y \psi \mathcal{D}_x P_1\Big) = \frac{1}{2}\Big(\vartheta \partial_x \mathcal{D}_x P_0 - \psi \mathcal{D}_x P_1\Big).$$

Exploiting the definition of $\mathcal{L}_1$, we obtain the first term in the expansion for F_ε:

$$\begin{aligned}\mathcal{L}_1 Q_3 &= \frac{y\sigma^2(y)}{2}\partial_{xy}\Big(\vartheta \partial_x \mathcal{D}_x P_0 - \psi \mathcal{D}_x P_1\Big)\\ &= \frac{y\sigma^2(y)}{2}\Big(\vartheta'(y)\partial_x^2 \mathcal{D}_x P_0 - \psi'(y)\partial_x \mathcal{D}_x P_1\Big).\end{aligned} \tag{4.B.12}$$

Finally, exploiting (4.B.6)-(4.B.11), together with (4.B.3), we write

$$\begin{aligned}\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} Q_3 &= \frac{1}{2}\left(\mathcal{L}_{\mathrm{BS}}^{\varkappa} + \frac{\sigma^2(y)-\varkappa^2}{2}\mathcal{D}_x\right)\left(\vartheta \partial_x \mathcal{D}_x P_0 - \psi \mathcal{D}_x P_1\right)\\ &= -\frac{\psi(y)}{2}\widetilde{\Omega}\partial_x \mathcal{D}_x^2 P_0 + \frac{\sigma^2(y)-\varkappa^2}{4}\vartheta(y)\partial_x \mathcal{D}_x^2 P_0 - \frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\mathcal{D}_x^2 P_1,\end{aligned} \tag{4.B.13}$$

where $\widetilde{\Omega} := \frac{1}{\sqrt{\varepsilon}}\Omega^\varepsilon = \frac{1}{2}\langle \varpi, \Pi\rangle$.

Placing (4.B.12)-(4.B.7)-(4.B.13) in (4.B.4), we then obtain

$$\begin{aligned}F_\varepsilon(t,x,y) &= \mathcal{L}_1 Q_3 + \mathcal{L}_{\mathrm{BS}}^{\sigma(y)} Q_2 + \sqrt{\varepsilon}\mathcal{L}_{\mathrm{BS}}^{\sigma(y)} Q_3\\ &= \frac{y\sigma^2(y)}{2}\Big(\vartheta'(y)\partial_x^2 \mathcal{D}_x P_0 - \psi'(y)\partial_x \mathcal{D}_x P_1\Big) - \frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\mathcal{D}_x^2 P_0\\ &\quad + \sqrt{\varepsilon}\left[-\frac{\psi(y)}{2}\widetilde{\Omega}\partial_x \mathcal{D}_x^2 P_0 + \frac{\sigma^2(y)-\varkappa^2}{4}\vartheta(y)\partial_x \mathcal{D}_x^2 P_0 - \frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\mathcal{D}_x^2 P_1\right]\\ &= \frac{y\sigma^2(y)}{2}\vartheta'(y)\left(\partial_x^4 - \partial_x^3\right)P_0 - \frac{\varpi(y)}{2}\left(\partial_x^3 - \partial_x^2\right)P_1 - \frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\left(\partial_x^4 - 2\partial_x^3 + \partial_x^2\right)P_0\\ &\quad + \frac{\sqrt{\varepsilon}}{2}\left[\left(\frac{\sigma^2(y)-\varkappa^2}{2}\vartheta(y) - \psi(y)\widetilde{\Omega}\right)\left(\partial_x^5 - 2\partial_x^4 + \partial_x^3\right)P_0 - \frac{\sigma^2(y)-\varkappa^2}{2}\psi(y)\left(\partial_x^4 - 2\partial_x^3 + \partial_x^2\right)P_1\right].\end{aligned}$$

Exploiting the fact that we chose $P_1 = -(T-t)\widetilde{\Omega}\left(\partial_x^3 - \partial_x^2\right)P_0$ (as in (4.B.3)), we obtain

$$\begin{aligned}F_\varepsilon(t,x,y) &= -\frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\partial_x^2 P_0 + \left(-\frac{y\sigma^2(y)}{2}\vartheta'(y) + \frac{\sigma^2(y)-\varkappa^2}{2}\psi(y)\right)\partial_x^3 P_0\\ &\quad + \left(\frac{y\sigma^2(y)}{2}\vartheta'(y) - \frac{\sigma^2(y)-\varkappa^2}{4}\psi(y)\right)\partial_x^4 P_0 + \frac{T-t}{2}\varpi(y)\widetilde{\Omega}\left(\partial_x^4 - 2\partial_x^5 + \partial_x^6\right)P_0\\ &\quad + \frac{\sqrt{\varepsilon}}{2}\left\{\left[\frac{\sigma^2(y)-\varkappa^2}{2}\vartheta(y) - \psi(y)\widetilde{\Omega}\right]\left(\partial_x^3 - 2\partial_x^4 + \partial_x^5\right)P_0\right.\\ &\quad \left. + (T-t)\left(\frac{\sigma^2(y)-\varkappa^2}{2}\psi(y)\widetilde{\Omega}\right)\left(-\partial_x^4 + 3\partial_x^5 - 3\partial_x^6 + \partial_x^7\right)P_0\right\}.\end{aligned} \tag{4.B.14}$$

Performing similar computations for G_ε, we obtain

$$\begin{aligned}G_\varepsilon(x,y) &= Q_2(T,x,y) + \sqrt{\varepsilon}Q_3(T,x,y)\\ &= -\frac{\psi(y)}{2}\mathcal{D}_x P_0 + \sqrt{\varepsilon}\left(\frac{\vartheta(y)}{2}\partial_x \mathcal{D}_x P_0 - \frac{\psi(y)}{2}\mathcal{D}_x P_1\right)\end{aligned}$$

$$= -\frac{\psi(y)}{2}\left(\partial_x^2 - \partial_x\right)P_0 + \frac{\sqrt{\varepsilon}}{2}\left[\vartheta(y)\left(\partial_x^3 - \partial_x^2\right) + \psi(y)(T-T)\widetilde{\Omega}\left(\partial_x^5 - 2\partial_x^4 + \partial_x^3\right)\right]P_0$$
$$= -\frac{\psi(y)}{2}\left(\partial_x^2 - \partial_x\right)P_0 + \frac{\sqrt{\varepsilon}}{2}\vartheta(y)\left(\partial_x^3 - \partial_x^2\right)P_0, \tag{4.B.15}$$

with P_0, P_1 evaluated at (T, x). The probabilistic representation of Z^ε as the solution of the Poisson equation in (4.B.5) reads

$$Z^\varepsilon(t,x,y) = \varepsilon\,\mathbb{E}_{t,x,y}\left[G_\varepsilon(X_T, Y_T) + \int_t^T F_\varepsilon(s, X_s, Y_s)\mathrm{d}s\right]. \tag{4.B.16}$$

To show that this is of order $\mathcal{O}(\varepsilon)$ as $\varepsilon \downarrow 0$, it is enough to bound F_ε and G_ε uniformly in ε. The key ingredients here are the following two lemmas. The proof of the first one, being long and technical, is postponed to Appendix 4.B.4.

Lemma 4.B.2. *Let ξ be a solution to the Poisson equation $\mathcal{L}_Y\xi = g$ on (y_σ, ∞), with*

$$\begin{cases} |g(y)| & \leq C, & \text{for } y \in (y_\sigma, \underline{y}), \\ |g(y)| & \leq C\left(1 + |y|^n\right), & \text{for } y \geq \overline{y}, \\ \langle g, \Pi\rangle & = 0, \end{cases}$$

for some $C > 0$, $n \in \mathbb{N}$, $\underline{y} \in (y_\sigma, \overline{y})$. Then there exist $C' > 0$, $n' \in \mathbb{N}$, $y_\sigma < \underline{y}' < \overline{y}'$ such that

$$\begin{cases} |\xi'(y)| & \leq C', & \text{for } y \in (y_\sigma, \underline{y}'), \\ |\xi'(y)| & \leq C'(1 + |y|^{n'}), & \text{for } y \geq \overline{y}, \end{cases}$$

and consequently

$$\begin{cases} |\xi(y)| & \leq C'', & \text{for } y \in (y_\sigma, \underline{y}'), \\ |\xi(y)| & \leq C''(1 + |y|^{n'+1}), & \text{for } y \geq \overline{y}, \end{cases}$$

with C'' suitable positive constant.

Lemma 4.B.3. *If h is smooth and bounded with bounded derivatives, then $\partial_x^n P_0$ exists and is bounded for any $n \in \mathbb{N}$.*

Proof. Since $P_0(t, x)$ is the BS price with constant volatility $\boldsymbol{\varkappa}^2$, denoting $f(\cdot)$ the density function of $\mathcal{N}\left(-\frac{1}{2}\boldsymbol{\varkappa}^2(T-t), \boldsymbol{\varkappa}^2\sqrt{T-t}\right)$ and assuming that the first n derivatives of the function h are uniformly bounded by $K > 0$, we have, for $n = 0$,

$$|P_0(t,x)| = \left|\int_{\mathbb{R}} h(\mathrm{e}^{x+z})f(z)\mathrm{d}z\right| \leq \int_{\mathbb{R}} |h(\mathrm{e}^{x+z})|\, f(z)\mathrm{d}z \leq K$$

and then, for any $n \geq 1$,

$$\partial_x^n P_0(t,x) = \partial_x^n\left(\int_{\mathbb{R}} h(\mathrm{e}^{x+z})f(z)\mathrm{d}z\right) = \partial_x^{n-1}\left(\int_{\mathbb{R}} h'(\mathrm{e}^{x+z})\mathrm{e}^{x+z}f(z)\mathrm{d}z\right)$$
$$= \partial_x^{n-2}\left(\int_{\mathbb{R}}(h''(\mathrm{e}^{x+z})\mathrm{e}^{2(x+z)} + h'(\mathrm{e}^{x+z})\mathrm{e}^{x+z})f(z)\mathrm{d}z\right)$$

$$= \cdots = \int_{\mathbb{R}} \sum_{k=1}^{n} \binom{n}{k} \partial_x^k h(\mathrm{e}^{x+z}) \mathrm{e}^{k(x+z)} f(z) \mathrm{d}z$$
$$= \int_{\mathbb{R}} \sum_{k=1}^{n} \binom{n}{k} \partial_x^k h(\mathrm{e}^{x+z}) \mathrm{e}^{k(x+z)} f(z) \mathrm{d}z,$$

and so

$$|\partial_x^n P_0(t,x)| \le \sum_{k=1}^{n} \binom{n}{k} \int_{\mathbb{R}} \left|\partial_x^k h(\mathrm{e}^{x+z})\right| \mathrm{e}^{k(x+z)} f(z) \mathrm{d}z \le K \int_{\mathbb{R}} \sum_{k=1}^{n} \binom{n}{k} \mathrm{e}^{k(x+z)} f(z) \mathrm{d}z$$
$$= K \sum_{k=1}^{n} \binom{n}{k} \mathrm{e}^{kx} \mathrm{e}^{-k\frac{1}{2}\boldsymbol{\varkappa}^2(T-t)} \mathrm{e}^{k^2\frac{1}{2}\boldsymbol{\varkappa}^2(T-t)} = K \sum_{k=1}^{n} \binom{n}{k} \mathrm{e}^{kx+(k^2-k)\frac{1}{2}\boldsymbol{\varkappa}^2(T-t)}.$$

Since this is clearly finite, the lemma follows. □

Now, ψ and ϑ are respectively the solutions to the Poisson equations (4.3.7)-(4.B.10) and satisfy the hypotheses in Lemma 4.B.2. Indeed, for ψ, the function $g : y \mapsto \sigma^2(y) - \boldsymbol{\varkappa}^2$ clearly satisfies $\langle g, \Pi\rangle = \langle \sigma^2(\cdot) - \boldsymbol{\varkappa}^2, \Pi\rangle = \langle \sigma^2(\cdot), \Pi\rangle - \boldsymbol{\varkappa}^2 = \boldsymbol{\varkappa}^2 - \boldsymbol{\varkappa}^2 = 0$. Furthermore, on (y_σ, ∞),

$$|g(y)| = |\sigma^2(y) - \boldsymbol{\varkappa}^2| \le \sigma^2(y) + \boldsymbol{\varkappa}^2 = \left(\frac{\alpha}{\beta} - \gamma y^{-\beta}\right)^2 + \boldsymbol{\varkappa}^2 \le \frac{\alpha^2}{\beta^2} + \boldsymbol{\varkappa}^2$$

is finite. Analogously, for ϑ, the function $g : y \mapsto \varpi(y) - \langle \varpi, \Pi\rangle$ clearly satisfies $\langle g, \Pi\rangle = \langle \varpi(y) - \langle \varpi, \Pi\rangle, \Pi\rangle = \langle \varpi, \Pi\rangle - \langle \varpi, \Pi\rangle = 0$. Clearly, $\langle \varpi, \Pi\rangle$ is a finite positive constant. Let us check the polynomial growth assumption on (y_σ, ∞). Since σ is bounded there and since ψ (and its first derivative) has polynomial growth, then

$$|g(y)| = |\varpi(y) - \langle \varpi, \Pi\rangle| = |\varpi(y)| + \langle \varpi, \Pi\rangle \le |y||\sigma^2(y)||\psi'(y)| + \langle \varpi, \Pi\rangle$$
$$\le |y|\frac{\alpha^2}{\beta^2} K' \left(1 + |y|^{n'}\right) + \langle \varpi, \Pi\rangle \le \left(\frac{\alpha^2}{\beta^2} K' + \langle \varpi, \Pi\rangle\right)\left(1 + |y|^{n'+1}\right),$$

which yields the desired growth condition. Thus, ψ and ϑ have at most polynomial growth at infinity, which we denote n_ψ and n_ϑ, and are bounded by a suitable constant when approaching y_σ. Plugging (4.B.14) and (4.B.15) in (4.B.16), we can write

$$Z^\varepsilon(t,x,y) = \varepsilon \mathbb{E}_{t,x,y}\left[G_\varepsilon(X_T, Y_T) + \int_t^T F_\varepsilon(s, X_s, Y_s) \mathrm{d}s\right]$$
$$= \varepsilon \mathbb{E}_{t,x,y}\Bigg[-\frac{1}{2}\psi(Y_T)(\partial_x^2 - \partial_x)P_0(T, X_T) + \frac{\sqrt{\varepsilon}}{2}\vartheta(Y_T)(\partial_x^3 - \partial_x^2)P_0(T, X_T)$$
$$+ \int_t^T \Bigg\{ -\frac{\sigma^2(Y_s) - \boldsymbol{\varkappa}^2}{4}\psi(Y_s)\partial_x^2 + \left(-\frac{\sigma^2(Y_s)}{2}Y_s\vartheta'(Y_s) + \frac{\sigma^2(Y_s) - \boldsymbol{\varkappa}^2}{2}\psi(Y_s)\right)\partial_x^3$$
$$+ \left[\frac{\sigma^2(Y_s)}{2}Y_s\vartheta'(Y_s) - \frac{\sigma^2(Y_s) - \boldsymbol{\varkappa}^2}{4}\psi(Y_s)\right]\partial_x^4 + \frac{T-s}{2}Y_s\sigma^2(Y_s)\psi'(Y_s)\widetilde{\Omega}\left(\partial_x^4 - 2\partial_x^5 + \partial_x^6\right)$$
$$+ \sqrt{\varepsilon}\Bigg\{\frac{1}{2}\left[\frac{\sigma^2(Y_s) - \boldsymbol{\varkappa}^2}{2}\vartheta(Y_s) - \psi(Y_s)\widetilde{\Omega}\right]\left(\partial_x^3 - 2\partial_x^4 + \partial_x^5\right)$$

$$+ (T-s)\frac{\sigma^2(Y_s)-\varkappa^2}{4}\psi(Y_s)\widetilde{\Omega}\left(-\partial_x^4+3\partial_x^5-3\partial_x^6+\partial_x^7\right)\Bigg\}P_0(s,X_s)\mathrm{d}s\Bigg].$$

Now, an application of Lemma 4.B.3 yields (with $\zeta := \frac{\alpha^2}{\beta^2}+\varkappa^2$)

$$\begin{aligned}|Z^\varepsilon(t,x,y)| &\leq \varepsilon\mathbb{E}_{t,x,y}\Bigg[|\psi(Y_T)|+\sqrt{\varepsilon}|\vartheta(Y_T)| \\ &+\int_t^T\Bigg\{\frac{\zeta}{4}|\psi(Y_s)|+\frac{\alpha^2}{\beta^2}|Y_s||\vartheta'(Y_s)|+|\psi(Y_s)|\frac{\zeta}{2}+\frac{\alpha^2}{2\beta^2}|Y_s||\vartheta'(Y_s)|+\frac{\zeta}{4}|\psi(Y_s)| \\ &+\frac{2\alpha^2(T-s)}{\beta^2}|Y_s||\psi'(Y_s)|\widetilde{\Omega}+\sqrt{\varepsilon}\Big[\zeta|\vartheta(Y_s)|+2|\psi(Y_s)|\widetilde{\Omega}+2(T-s)\zeta|\psi(Y_s)|\widetilde{\Omega}\Big]\Bigg\}\mathrm{d}s\Bigg] \\ &\lesssim \varepsilon\mathbb{E}_{t,x,y}\Bigg[|\psi(Y_T)|+\sqrt{\varepsilon}|\vartheta(Y_T)|+\int_t^T\Bigg\{|\psi(Y_s)|+|Y_s||\vartheta'(Y_s)|+(T-s)|Y_s||\psi'(Y_s)| \\ &+\sqrt{\varepsilon}\Big(|\vartheta(Y_s)|+|\psi(Y_s)|+(T-s)|\psi(Y_s)|\Big)\Bigg\}\mathrm{d}s\Bigg],\end{aligned}$$

where $\lesssim$ means less than modulo multiplication by some strictly positive constant. Finally, applying Lemma 4.B.2, we obtain

$$\begin{aligned}\left|Z^\varepsilon(t,x,y)\right| \lesssim \varepsilon\,\mathbb{E}_{t,x,y}\Bigg[&(1+|Y_T|^{n_\psi})+\sqrt{\varepsilon}\,(1+|Y_T|^{n_\vartheta}) \\ &+\int_t^T\left\{1+|Y_s|^{n_\psi}+|Y_s|\left\{\left[1+|Y_s|^{n_\vartheta-1}\right]+(T-s)\left[1+|Y_s|^{n_\psi-1}\right]\right\}\right. \\ &\left.+\sqrt{\varepsilon}\left\{1+|Y_s|^{n_\vartheta}+(1+(T-s))|Y_s|^{n_\psi}\right\}\right\}\mathrm{d}s\Bigg] \lesssim \varepsilon.\end{aligned}$$

The finiteness in the last line is a consequence of Appendix 4.B.5 on the uniform finiteness of the moments of Y, and the proposition thus follows.

4.B.4 Proof of Lemma 4.B.2

With the notations introduced in Section 4.3.2, the third assumption on g can be rewritten as

$$0=\langle g,\Pi\rangle=\int_{y_\sigma}^\infty g(y)\Pi(\mathrm{d}y)=\left(\int_{y_\sigma}^\infty\frac{\mathrm{d}\xi}{\widetilde{\sigma}^2(\xi)s(\xi)}\right)^{-1}\int_{y_\sigma}^\infty g(y)m(y)\mathrm{d}y,$$

and therefore

$$\int_{y_\sigma}^\infty g(y)m(y)\mathrm{d}y=0. \tag{4.B.17}$$

Recall that the equation $\mathcal{L}_Y\xi=g$ solved by ξ on (y_σ,∞) is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{\mathrm{d}}{\mathrm{d}S}\xi(y)\right)=g(y).$$

Integrating both sides yields

$$\xi'(y) = 2s(y)\int_{y_\sigma}^{y} g(z)m(z)\mathrm{d}z. \tag{4.B.18}$$

We first study the behaviour around y_σ. Consider $y \in (y_\sigma, \underline{y})$, for a sufficiently small $\underline{y}$. Since the function g is bounded by assumption, then

$$|\xi'(y)| = 2Cs(y)\int_{y_\sigma}^{y} m(z)\mathrm{d}z = 2Cs(y)M_\Pi(y_\sigma, y].$$

In the proof of the boundary classification of the left boundary point $y_\sigma < 1$ for the domain (y_σ, ∞), we have seen that (4.A.4)

$$s(y) = \exp\left(\frac{K}{y-y_\sigma} + KK_\beta^a\right)(y-y_\sigma)^{-K\overline{X}_1}(1+O(y-y_\sigma)), \qquad \text{for } y \in (y_\sigma, \underline{y}),$$

with $K = \frac{2y_\sigma^{2\beta+1}(1-y_\sigma)}{h\beta^2\gamma^2}$, positive constant, and $M_\Pi(y_\sigma, y] = \int_0^{y-y_\sigma} \frac{\mathrm{d}x}{\overline{\sigma}^2(x)\overline{s}(x)}$, with

$$\overline{s}(x) = \exp\left(\frac{K}{x} + KK_\beta^a\right)x^{-K\overline{X}_1}(1+O(x)), \qquad \text{for } x \in (0, \underline{y} - y_\sigma),$$

and $\overline{\sigma}(x)^{-2} = \frac{y_\sigma^{2+2\beta}}{\beta^2\gamma^2x^2}(1+O(x))$, for $x \in (y_\sigma, \underline{y})$. Thus, exploiting these two expansions, the change of variables $x = (y-y_\sigma)z$ and the asymptotic expansion for integrals in [123, Chapter 3.3, pages 62 and 67], we obtain

$$\begin{aligned}
M(y_\sigma, y] = \int_0^{y-y_\sigma} \frac{\mathrm{d}x}{\overline{\sigma}^2(x)\overline{s}(x)} &= \frac{y_\sigma^{2+2\beta}}{\beta^2\gamma^2}\int_0^{y-y_\sigma} \exp\left\{-\frac{K}{x} - KK_\beta^a\right\} x^{K\overline{X}_1-2}(1+O(x))\mathrm{d}x \\
&= \frac{y_\sigma^{2+2\beta}}{\beta^2\gamma^2}\mathrm{e}^{-KK_\beta^a}(y-y_\sigma)^{K\overline{X}_1-1}\int_0^1 \exp\left\{-\frac{K}{(y-y_\sigma)z}\right\} z^{K\overline{X}_1-2}(1+O(z))\mathrm{d}z \\
&= \frac{y_\sigma^{2+2\beta}}{\beta^2\gamma^2}\mathrm{e}^{-KK_\beta^a}(y-y_\sigma)^{K\overline{X}_1-1}\exp\left\{-\frac{K}{y-y_\sigma}\right\}\left(\frac{y-y_\sigma}{K} + O((y-y_\sigma)^2)\right) \\
&= \frac{y_\sigma^{2+2\beta}}{\beta^2\gamma^2 K}\mathrm{e}^{-KK_\beta^a}(y-y_\sigma)^{K\overline{X}_1}\exp\left\{-\frac{K}{y-y_\sigma}\right\}\left(1+O(y-y_\sigma)\right).
\end{aligned}$$

Thus, we conclude

$$|\xi'(y)| \leq 2Cs(y)M(y_\sigma, y] = \frac{2y_\sigma^{2+2\beta}}{\beta^2\gamma^2}(1+O(y-y_\sigma)),$$

which yields the boundedness of $\xi'(y)$ and of $\xi(y)$ itself as y approaches y_σ.

About the behaviour at infinity, applying the centering condition (4.B.17) to (4.B.18) yields

$$\xi'(y) = 2s(y)\int_{y_\sigma}^{y} g(z)m(z)\mathrm{d}z$$

$$= 2s(y)\left(\int_{y_\sigma}^{y} g(z)m(z)\mathrm{d}z + \int_{y}^{\infty} g(z)m(z)\mathrm{d}z - \int_{y}^{\infty} g(z)m(z)\mathrm{d}z\right)$$
$$= -2s(y)\int_{y}^{\infty} g(z)m(z)\mathrm{d}z$$

Since s and m are non-negative, the polynomial growth assumption in the statement of Lemma 4.B.2 and the definition of m give

$$|\xi'(y)| = 2s(y)\left|\int_{y}^{\infty} g(z)m(z)\mathrm{d}z\right| \leq 2s(y)\int_{y}^{\infty} |g(z)|m(z)\mathrm{d}z \tag{4.B.19}$$
$$\leq 2Cs(y)\int_{y}^{\infty} z^n m(z)\mathrm{d}z \leq 2Cs(y)\int_{y}^{\infty} \frac{z^{n-2}}{\sigma^2(z)s(z)}\mathrm{d}z.$$

Since $\overline{y}$ can be picked as $\overline{y} > 1$, then $|z|^{n-2} \leq 1$ for $n \in \{0, 1\}$, and we thus take 1 in place of z^{n-2}. We make a short digression to study $s(y)$, for $y \in (a, \infty)$ with $a > y_\sigma$. By definition,

$$s(y) = \exp\left\{-\int_{a}^{y} \frac{2b(\eta)}{\widetilde{\sigma}^2(\eta)}\mathrm{d}\eta\right\} = \mathrm{e}^{-f_a(y)},$$

with $f_a(y) := \int_a^y \frac{2b(\eta)}{\widetilde{\sigma}^2(\eta)}\mathrm{d}\eta$, which we can compute explicitly as

$$f_a(y) = \frac{2}{h}\left(\int_a^y \frac{\mathrm{d}\eta}{\eta\left(-\frac{\alpha}{\beta} + \gamma\eta^{-\beta}\right)^2} - \int_a^y \frac{\mathrm{d}\eta}{\left(-\frac{\alpha}{\beta} + \gamma\eta^{-\beta}\right)^2}\right) = \frac{2}{h}\Big(I_1(a, y) - I_2(a, y)\Big),$$

where

$$I_1(a, y) := \frac{\beta}{\alpha^2}\log\left(\frac{\beta\gamma - \alpha y^\beta}{\beta\gamma - \alpha a^\beta}\right) + \frac{\beta^2\gamma}{\alpha}\frac{a^{-\beta} - y^{-\beta}}{(\alpha - \beta\gamma a^{-\beta})(\alpha - \beta\gamma y^{-\beta})},$$
$$I_2(a, y) := \frac{1}{\gamma(2\beta + 1)}\left[y^{2\beta+1}{}_2F_1\left(2, 2 + \frac{1}{\beta}; 3 + \frac{1}{\beta}; \frac{\alpha y^\beta}{\beta\gamma}\right) - a^{2\beta+1}{}_2F_1\left(2, 2 + \frac{1}{\beta}; 3 + \frac{1}{\beta}; \frac{\alpha a^\beta}{\beta\gamma}\right)\right].$$

- Since $y > a$, then the first term in I_1 satisfies $\frac{\beta\gamma - \alpha y^\beta}{\beta\gamma - \alpha a^\beta} \in (0, 1]$ so that its logarithm is well-posed and negative.

- Likewise, the second term in I_1 is positive and (as a function of y) increasing and bounded by its ∞-limit equal to $\frac{a^{-\beta}}{(\alpha - \beta\gamma a^{-\beta})\alpha}$.

- The two terms in I_2 can be rewritten exploiting the following series representation of the hypergeometric function [118, Volume I, Chapter III, Section 3.6, Equation (1)], which holds for any $|z| > 1$ and $a - b \notin \mathbb{Z}$:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\frac{1}{(-z)^a}\sum_{k=0}^{\infty}\frac{(a)_k(a-c+1)_k}{k!(a-b+1)_k}\frac{1}{z^k}$$
$$+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\frac{1}{(-z)^b}\sum_{k=0}^{\infty}\frac{(b)_k(b-c+1)_k}{k!(b-a+1)_k}\frac{1}{z^k}.$$

In our specific case this reads

$$
{}_2F_1\left(2,2+\frac{1}{\beta};3+\frac{1}{\beta};z\right)=(2\beta+1)\sum_{k=0}^{\infty}\frac{k+1}{1-k\beta}z^{-(2+k)}+\frac{1}{2}\Gamma\left(-\frac{1}{\beta}\right)\Gamma\left(3+\frac{1}{\beta}\right)\left(-\frac{1}{z}\right)^{2+\frac{1}{\beta}},
$$

which implies $-I_2(a,y)=\sum_{k\geq 0}\daleth_k(a^{1-\beta k}-y^{1-\beta k})$, where we define $\daleth_k:=\frac{k+1}{1-k\beta}\gamma^{k+1}\left(\frac{\alpha}{\beta}\right)^{k+2}$ for convenience. We further introduce the useful quantities

$$
\underline{\underline{\sum}}^{z}_{n}:=\sum_{k=0}^{n-1}\daleth_k z^{1-\beta k}\qquad\text{and}\qquad\overline{\overline{\sum}}^{z}_{n}:=\sum_{k=n}^{\infty}\daleth_k z^{1-\beta k}.
$$

Then, for any $\beta\in(0,\frac{1}{2})$, there exists $n_\beta\in\mathbb{N}\setminus\{0,1,2\}$ such that $1-\beta n<0$, for $n\geq n_\beta$, and $1-\beta n\geq 0$, for $n<n_\beta$. Hence, for any $z\in(y,\infty)$, with $y>a>y_\sigma$,

$$
-I_2(a,z)=C_a-\sum_{k=0}^{\infty}\daleth_k z^{1-\beta k}=C_a-\underline{\underline{\sum}}^{z}_{n_\beta}-\overline{\overline{\sum}}^{z}_{n_\beta}\leq C_a-\underline{\underline{\sum}}^{z}_{n_\beta}-\overline{\overline{\sum}}^{y}_{n_\beta},
$$

where the constant $C_a:=\underline{\underline{\sum}}^{a}_{\infty}$ is finite.

As a consequence of these bullet points, we deduce

$$
\begin{aligned}
\frac{1}{s(z)}&=\mathrm{e}^{f_a(z)}=\left(\frac{\beta\gamma-\alpha z^\beta}{\beta\gamma-\alpha a^\beta}\right)^{\frac{2\beta}{h\alpha^2}}\exp\left\{\frac{2\beta^2\gamma}{h\alpha}\frac{a^{-\beta}-z^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})(\alpha-\beta\gamma z^{-\beta})}\right\}\mathrm{e}^{-\frac{2}{h}I_2(a,z)}\\
&\leq\exp\left\{\frac{2\beta^2\gamma}{h\alpha}\frac{a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})\alpha}\right\}\exp\left\{\frac{2}{h}\left[C_a-\underline{\underline{\sum}}^{z}_{n_\beta}-\overline{\overline{\sum}}^{y}_{n_\beta}\right]\right\}\\
&=\exp\left\{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})\alpha^2 h}+\frac{2C_a}{h}\right\}\exp\left\{-\frac{2}{h}\left(\underline{\underline{\sum}}^{z}_{n_\beta}+\overline{\overline{\sum}}^{y}_{n_\beta}\right)\right\}.
\end{aligned}
$$

Let us now go back to the starting problem and consider $y>a>y_\sigma$. Replacing the expression in (4.B.19), we have

$$
\begin{aligned}
|\xi'(y)|&\leq 2Cs(y)\int_y^\infty\frac{z^{n-2}}{\left(\frac{\alpha}{\beta}-\gamma z^{-\beta}\right)^2 s(z)}\mathrm{d}z\leq 2C\left(\frac{\alpha}{\beta}-\frac{\gamma}{y^\beta}\right)^{-2}s(y)\int_y^\infty\frac{z^{n-2}}{s(z)}\mathrm{d}z\\
&=2C\left(\frac{\alpha}{\beta}-\frac{\gamma}{y^\beta}\right)^{-2}\left(\frac{\beta\gamma-\alpha a^\beta}{\beta\gamma-\alpha y^\beta}\right)^{\frac{2\beta}{h\alpha^2}}\mathrm{e}^{\frac{\beta^2\gamma}{\alpha}\frac{y^{-\beta}-a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})(\alpha-\beta\gamma y^{-\beta})}}\mathrm{e}^{\frac{2}{h}I_2(a,y)}\int_y^\infty\frac{z^{n-2}}{s(z)}\mathrm{d}z\\
&\leq 2C\left(\frac{\alpha}{\beta}-\frac{\gamma}{y^\beta}\right)^{-2}\left(\frac{\beta\gamma-\alpha a^\beta}{\beta\gamma-\alpha y^\beta}\right)^{\frac{2\beta}{\alpha^2 h}}\exp\left\{-\frac{2}{h}\left[C_a-\underline{\underline{\sum}}^{y}_{n_\beta}-\underline{\underline{\sum}}^{y}_{n_\beta}\right]\right\}\times\\
&\times\int_y^\infty z^{n-2}\mathrm{e}^{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})\alpha^2 h}+\frac{2}{h}C_a}\exp\left\{-\frac{2}{h}\underline{\underline{\sum}}^{z}_{n_\beta}-\frac{2}{h}\overline{\overline{\sum}}^{y}_{n_\beta}\right\}\mathrm{d}z\\
&\leq 2C\mathrm{e}^{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})\alpha^2 h}}\left(\frac{\alpha}{\beta}-\frac{\gamma}{y^\beta}\right)^{-2}\left(\frac{\beta\gamma-\alpha a^\beta}{\beta\gamma-\alpha y^\beta}\right)^{\frac{2\beta}{h\alpha^2}}\exp\left\{\frac{2}{h}\underline{\underline{\sum}}^{y}_{n_\beta}\right\}\int_y^\infty z^{n-2}\exp\left\{-\frac{2}{h}\underline{\underline{\sum}}^{z}_{n_\beta}\right\}\mathrm{d}z.
\end{aligned}
$$

Now, suppose that the integral in the last line satisfies a bound of the form

$$
\int_y^\infty z^{n-2}\exp\left\{-\frac{2}{h}\underline{\underline{\sum}}^{z}_{n_\beta}\right\}\mathrm{d}z\leq K\sum_{j=0}^{N}\frac{N!}{j!}\left(\frac{2}{h}\underline{\underline{\sum}}^{y}_{n_\beta}\right)^j\exp\left\{-\frac{2}{h}\underline{\underline{\sum}}^{y}_{n_\beta}\right\},\tag{4.B.20}
$$

for some $K > 0$ and some integer $N > 0$. Plugging this in the equation above, we obtain

$$\begin{aligned}
|\xi'(y)| &\leq 2KC \exp\left\{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})h\alpha^2}\right\}\left(\frac{\alpha}{\beta}-\gamma y^{-\beta}\right)^{-2}\left(\frac{\beta\gamma-\alpha a^{\beta}}{\beta\gamma-\alpha y^{\beta}}\right)^{\frac{2\beta}{h\alpha^2}}\sum_{j=0}^{N}\frac{N!}{j!}\left(\frac{2}{h}\sum\nolimits_{n_\beta}^{y}\right)^j \\
&= 2KC \exp\left\{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})h\alpha^2}\right\}\frac{\left(\beta\gamma-\alpha a^{\beta}\right)^{\frac{2\beta}{h\alpha^2}}}{\beta^2}\frac{y^{2\beta}}{\left(\beta\gamma-\alpha y^{\beta}\right)^{2+\frac{2\beta}{\alpha^2 h}}}\sum_{j=0}^{N}\frac{N!}{j!}\left(\frac{2}{h}\sum\nolimits_{n_\beta}^{y}\right)^j \\
&\leq 2KC \exp\left\{\frac{2\beta^2\gamma a^{-\beta}}{(\alpha-\beta\gamma a^{-\beta})h\alpha^2}\right\}\frac{\left(\beta\gamma-\alpha a^{\beta}\right)^{-2}}{\beta^2}y^{2\beta}\sum_{j=0}^{N}\frac{N!}{j!}\left(\frac{2}{h}\sum\nolimits_{n_\beta}^{y}\right)^j \\
&\leq K_a\left(1+|y|^{2\beta+N(1-\beta n_\beta)}\right) = K_a\left(1+|y|^{N_\beta}\right),
\end{aligned}$$

where N_β and K_a are respectively a suitably chosen positive integer and a positive constant. Thus, this last inequality yields the desired polynomial growth for ξ' and ξ. Finally the inequality in (4.B.20) is a consequence of the following lemma.

Lemma 4.B.4. *Let $y > a > 1$, $k, N \in \mathbb{N}_0$ and $I := \int_y^\infty z^k \exp\{-\sum_{j=0}^N A_j z^{d_j}\}\mathrm{d}z$, with $(A_j)_{j\in\{0,\dots,N\}} \geq 0$ and $(d_j)_{j\in\{0,\dots,N\}} \in (0,1)$. Then, there exists $r \in \mathbb{N}$ and $C > 0$ such that*

$$I \leq C\exp\left\{-\sum_{j=0}^{N}A_j y^{d_j}\right\}\sum_{k=0}^{r}\frac{r!}{k!}\left(\sum_{j=0}^{N}A_j y^{d_j}\right)^k.$$

Proof. The function $g : (y,\infty) \to \mathbb{R}_+$, defined as $g(z) := \sum_{j=0}^N A_j z^{d_j}$, is positive and strictly increasing, hence invertible. Its inverse $g^{\leftarrow}$ is thus strictly increasing and $\lim_{z\uparrow\infty} g(z) = +\infty$. The change of variables $g(z) = u$ thus implies

$$I = \int_{g(y)}^{\infty} \mathrm{e}^{-u}\frac{g^{\leftarrow}(u)^k}{g'\left(g^{\leftarrow}(y)\right)}\mathrm{d}u.$$

Notice that the first derivative of g, given by $g'(z) = \sum_{j=0}^N A_j d_j z^{d_j-1}$, is clearly positive and strictly decreasing on (y,∞). Now, set $\alpha_0 = \min_{j\in\{0,\dots,N\}} A_j$ and $\delta_0 = \min_{j\in\{0,\dots,N\}} d_j$. Since $g(z) \geq \alpha_0 z^{\delta_0}$, then $g\left(\left(\frac{z}{\alpha_0}\right)^{\frac{1}{\delta_0}}\right) \geq z$, and so, by the monotonicity of $g^{\leftarrow}$, we have

$$g^{\leftarrow}(z) \leq g^{\leftarrow}\left(g\left(\left(\frac{z}{\alpha_0}\right)^{\frac{1}{\delta_0}}\right)\right) \leq \alpha_0^{-\frac{1}{\delta_0}} z^{\frac{1}{\delta_0}}, \tag{4.B.21}$$

as well as $g'(z) \geq \alpha_0\delta_0 z^{\delta_0-1}$. Applying this inequality and then (4.B.21) to the chain of inequalities for I, gives, for a suitably chosen positive integer r some constant $C > 0$,

$$\begin{aligned}
I &\leq \int_{g(y)}^{\infty} \mathrm{e}^{-u}\frac{g^{\leftarrow}(u)^k}{\alpha_0\delta_0 g^{\leftarrow}(u)^{\delta_0-1}}\mathrm{d}u = \frac{1}{\alpha_0\delta_0}\int_{g(y)}^{\infty}\mathrm{e}^{-u}g^{\leftarrow}(u)^{k+1-\delta_0}\mathrm{d}u \\
&\leq \frac{1}{\alpha_0\delta_0}\int_{g(y)}^{\infty}\mathrm{e}^{-u}\left(\frac{u}{\alpha_0}\right)^{\frac{1}{\delta_0}(k+1-\delta_0)}\mathrm{d}u = \alpha_0^{-\frac{k+1}{\delta_0}}\delta_0^{-1}\int_{g(y)}^{\infty}\mathrm{e}^{-u}u^{\frac{1}{\delta_0}(k+1-\delta_0)}\mathrm{d}u
\end{aligned}$$

$$\leq C \int_{g(y)}^{\infty} \mathrm{e}^{-u} u^r \mathrm{d}u = \mathrm{e}^{-g(y)} \sum_{j=0}^{r} \frac{r!}{j!} g(y)^j,$$

which ends the proof of the inequality in the statement of the theorem. □

4.B.5 Uniform bounds for the moments of Y

Because of Section 4.3.1, Theorem 4.3.5 and Proposition 4.3.7, we restrict our interest to the case with $\beta \in (0, \frac{1}{2}) \cup \{1\}$ and domain $\mathfrak{D} = (y_\sigma, \infty)$, with $y_\sigma := \left(\frac{\beta\gamma}{\alpha}\right)^{1/\beta} < 1$. We need to prove that, for any $n \in \mathbb{N}$, the uniform (in time) bound $\sup_{t\geq 0} \mathbb{E}[Y_t^n] \leq K$ holds. We shall use the following lemma, the proof of which is relegated below:

Lemma 4.B.5. *On any compact interval of the form $[0, T]$, any moment of Y is uniformly bounded and* $\lim_{t\to s} \mathbb{E}[(Y_t - Y_s)^{n+2}] = 0$.

This claim implies immediately that $\mathbb{E}[Y_t^{n+2}]$, $\mathbb{E}[Y_t^{n+1}]$, $\mathbb{E}[Y_t^{n}]$, $\mathbb{E}[Y_t^{n-\beta}]$ and $\mathbb{E}[Y_t^{n-2\beta}]$ are all continuous on any compact interval. Now, Itô's formula implies yields

$$\begin{aligned}
Y_t^n &= y_0^n + \int_0^t nY_s^{n-1}\mathrm{d}Y_s + \frac{1}{2}\int_0^t n(n-1)Y_s^{n-2}\mathrm{d}\langle Y\rangle_s^2 \\
&= y_0^n + \frac{n}{h}\int_0^t \left(Y_s^n - Y_s^{n+1}\right)\mathrm{d}s + \int_0^t \left(-\frac{\alpha n}{\beta}Y_s^n + \gamma n Y_s^{n-\beta}\right)\mathrm{d}W_s \\
&\quad + \frac{n(n-1)}{2}\int_0^t \left(\frac{\alpha^2}{\beta^2}Y_s^n + \gamma^2 Y_s^{n-2\beta} - \frac{2\alpha\gamma}{\beta}Y_s^{n-\beta}\right)\mathrm{d}s.
\end{aligned}$$

Taking expectations on both sides and exploiting the regularity of the processes involved (from the aforementioned claim) we obtain

$$\begin{aligned}
\mathbb{E}[Y_t^n] = y_0^n &+ \left(\frac{n}{h} + \frac{\alpha^2 n(n-1)}{2\beta^2}\right)\int_0^t \mathbb{E}[Y_s^n]\mathrm{d}s - \frac{n}{h}\int_0^t \mathbb{E}[Y_s^{n+1}]\mathrm{d}s + 0 + \frac{\gamma^2 n(n-1)}{2}\int_0^t \mathbb{E}[Y_s^{n-2\beta}]\mathrm{d}s \\
&- \frac{\alpha\gamma n(n-1)}{\beta}\int_0^t \mathbb{E}[Y_s^{n-\beta}]\mathrm{d}s.
\end{aligned}$$

Define now the function $t \mapsto \varphi(t) := \mathbb{E}[Y_t^n]$, which is differentiable since on any compact $[0, T]$, $|\partial_t f(t, Y)|$ is bounded in L^1, for $f(t, Y) := \int_0^t Y_s^n \mathrm{d}s$. Since the process Y is positive almost surely, differentiating the expression above and applying Hölder inequality yield

$$\begin{aligned}
\varphi'(t) &= \left(\frac{2}{h} - \frac{\alpha^2}{\beta^2}\right)\varphi(t) - \frac{2}{h}\mathbb{E}\left[Y_t^3\right] + \gamma^2\mathbb{E}\left[Y_t^{2-2\beta}\right] - \frac{2\alpha\gamma}{\beta}\mathbb{E}\left[Y_t^{2-\beta}\right] \\
&\leq \left(\frac{n}{h} + \frac{\alpha^2 n(n-1)}{2\beta^2}\right)\varphi(t) - \frac{n}{h}\mathbb{E}\left[Y_t^{n+1}\right] + \frac{\gamma^2 n(n-1)}{2}\mathbb{E}\left[Y_t^{n-2\beta}\right] \\
&\leq \left(\frac{n}{h} + \frac{\alpha^2 n(n-1)}{2\beta^2}\right)\varphi(t) - \frac{n}{h}\varphi(t)^{1+\frac{1}{n}} + \frac{\gamma^2 n(n-1)}{2}\varphi(t)^{1-\frac{2}{n}\beta} = \psi(\varphi(t)),
\end{aligned}$$

with $\psi(y) := \left(\frac{n}{h} + \frac{\alpha^2 n(n-1)}{2\beta^2}\right) y - \frac{n}{h} y^{1+\frac{1}{n}} + \frac{\gamma^2 n(n-1)}{2} y^{1-\frac{2}{n}\beta}$. Since $\lim_{y\uparrow\infty} \psi(y) = -\infty$, there exists y^* such that $\psi(y) \leq -1$ for all $y \geq y^*$.

This implies that $\varphi(\cdot)$ is uniformly bounded. First, without loss of generality we can assume $y^* \geq y_0^2$. Now, either the level y^* is never reached, so that that the function φ is uniformly bounded by y^*, or that y^* is actually attained at some time t^*, namely $\varphi(t^*) = y^*$. Let us show that in this last case the level $y^* + 1$ cannot be attained and consequently φ is uniformly bounded by $y^* + 1$. Assume by contradiction that there exists $\bar{t}$ such that $\varphi(\bar{t}) = y^* + 1$. Since φ is continuous, then $\bar{t} \geq t^*$. Set $\widehat{t} := \max\{0 \leq t \leq \bar{t} : \varphi(t) = y^*\}$. Clearly then $\psi(\varphi(t)) \leq -1$ for all $t \in [\widehat{t}, \bar{t}]$, and furthermore

$$y^*+1 = \varphi(\bar{t}) = \varphi(\widehat{t}) + \int_{\widehat{t}}^{\bar{t}} \varphi'(t)\mathrm{d}t = y^* + \int_{\widehat{t}}^{\bar{t}} \varphi'(t)\mathrm{d}t \leq y^* + \int_{\widehat{t}}^{\bar{t}} \varphi'(t)\mathrm{d}t \leq y^* + \int_{\widehat{t}}^{\bar{t}} \psi(\varphi(t))\mathrm{d}t \leq y^*,$$

which is obviously a contradiction and thus completes the proof.

We now prove Lemma 4.B.5.

Proof of Lemma 4.B.5. The finiteness of any moment of Y can be recovered proceeding as in [51]. Indeed, let $\tau_M := \inf\{t \geq 0 : Y_t \geq M\}$ for any $M > 0$, so that $Y_{t\wedge\tau_M} \leq M$ and hence is bounded almost surely. Consider a function $h \in \mathcal{C}^2([0,\infty))$ with the following properties:

$$\begin{cases} h(y) = 1, & y \leq \frac{1}{2}, \\ h(y) \geq y^k, & \text{everywhere}, \\ h(y) = y^k, & y \geq 2. \end{cases}$$

It is then easy to see that there exists a constant $\widetilde{C} > 0$ such that, for all $y \geq 0$,

$$\frac{\widetilde{\sigma}^2(y)}{2} h''(y) + bh'(y) \leq \widetilde{C}h(y).$$

Then, set $f(t) := \mathbb{E}_{y_0}[h(Y_{t\wedge\tau})]$. Itô's formula implies

$$\begin{aligned} f(t) &= h(y_0) + \mathbb{E}_{y_0}\left[\int_0^{t\wedge\tau} \frac{\widetilde{\sigma}^2(Y_s)}{2} h''(Y_s) + b(Y_s)h'(Y_s)\mathrm{d}s\right] \\ &= h(y_0) + \widetilde{C}\,\mathbb{E}_{y_0}\left[\int_0^{t\wedge\tau} h(Y_s)\mathrm{d}s\right] = h(y_0) + \widetilde{C}\mathbb{E}_{y_0}\left[\int_0^{t\wedge\tau} h(Y_{s\wedge\tau})\mathrm{d}s\right] \\ &\leq h(y_0) + \widetilde{C}\,\mathbb{E}_{y_0}\left[\int_0^{t} h(Y_{s\wedge\tau})\mathrm{d}s\right] = h(y_0) + \widetilde{C}\int_0^t f(s)\mathrm{d}s. \end{aligned}$$

Finally, an application of Gronwall's inequality yields

$$\mathbb{E}_{y_0}\left[Y_{t\wedge\tau}^k\right] \leq \mathbb{E}_{y_0}\left[h(Y_{t\wedge\tau})\right] \leq h(y_0)\mathrm{e}^{\widetilde{C}t} \leq C\left(1 + y_0^k\right),$$

which does not depend on M, proving the uniform finiteness of moments of Y on $[0,T]$.

Regarding the second item of the lemma, applying, in sequence, Hölder, BDG and Hölder inequalities, Fubini's Theorem and the previously boundedness of moments of Y, we obtain

$$\mathbb{E}[(Y_t - Y_s)^n] = \mathbb{E}\left[\sum_{k=0}^{n} \binom{n}{k} \left(\int_s^t b(Y_u)\mathrm{d}u\right)^{n-k} \left(\int_s^t \widetilde{\sigma}(Y_u)\mathrm{d}W_u\right)^k\right]$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left[\left(\int_s^t b(Y_u)\mathrm{d}u\right)^n\right]^{\frac{n-k}{n}}\mathbb{E}\left[\left(\int_s^t \widetilde{\sigma}(Y_u)\mathrm{d}W_u\right)^n\right]^{\frac{k}{n}}\\
&\leq \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left[\int_s^t b(Y_u)^n\mathrm{d}u\right]^{\frac{n-k}{n}}\mathbb{E}\left[\left(\int_s^t \widetilde{\sigma}(Y_u)^2\mathrm{d}u\right)^{\frac{n}{2}}\right]^{\frac{k}{n}}\\
&\leq \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left[\int_s^t b(Y_u)^n\mathrm{d}u\right]^{\frac{n-k}{n}}\mathbb{E}\left[\int_s^t \widetilde{\sigma}(Y_u)^n\mathrm{d}u\right]^{\frac{k}{n}}\\
&\leq \sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{h^n}\int_s^t \mathbb{E}[Y_u^n(1-Y_u)^n]\mathrm{d}u\right\}^{\frac{n-k}{n}}\left\{\int_s^t \mathbb{E}\left[Y_u^n\left(-\frac{\alpha}{\beta}+\gamma Y_u^{-\beta}\right)^n\right]\mathrm{d}u\right\}^{\frac{k}{n}}\\
&\leq \sum_{k=0}^{n}\binom{n}{k}\frac{1}{h^{n-k}}\left\{\int_s^t \mathbb{E}[Y_u^n+Y_u^{2n}]\mathrm{d}u\right\}^{\frac{n-k}{n}} 2^{\frac{k(n-1)}{n}}\left\{\int_s^t \mathbb{E}\left[\frac{\alpha^n}{\beta^n}Y_u^n+\gamma^n Y_u^{n(1-\beta)}\right]\mathrm{d}u\right\}^{\frac{k}{n}}.
\end{aligned}
$$

Since all moments of Y are uniform bounded over $[0,T]$, we obtain

$$
\lim_{t\to s}\mathbb{E}[(Y_t-Y_s)^n] \leq \lim_{t\to s} C(T,y_0,n)(t-s) = 0,
$$

completing the proof. □

4.C Large deviations proofs

4.C.1 Proof of Proposition 4.3.13

Since the process Y^ε lies in $\mathbb{R}_+^*$ instead of $\mathbb{R}$, we adapt the proof of [130, Theorem 2.9] to prove a large deviations principle with speed ε and rate function I^Y. Since $y_\sigma > 0$, and in both cases $y_0 \geq y_\sigma$ and $y_0 < y_\sigma$, the function $\widetilde{\sigma}$ is locally Lipschitz continuous on $\mathbb{R}_+^*$. Furthermore, for $f \in \overline{\mathcal{H}}$, the Picard-Lindelöf Theorem implies that the controlled ODE $\dot{g}_t = \widetilde{\sigma}(g_t)\dot{f}_t$, with $g_0 = y_0$ admits the solution

$$
\mathcal{S}_2^{y_0}(f)(t) = \left(\frac{\beta\gamma}{\alpha}\right)^{\frac{1}{\beta}}\left[\mathrm{e}^{-\alpha\int_0^t \dot{f}_u\mathrm{d}u}\left(y_0^\beta\frac{\alpha}{\beta\gamma}-1\right)+1\right]^{1/\beta}, \qquad \text{for } t\in[0,T], \quad y_0>0.
$$

This formulation requires the term $\left[\mathrm{e}^{-\alpha\int_0^t \dot{f}_u\mathrm{d}u}\left(y_0^\beta\frac{\alpha}{\beta\gamma}-1\right)+1\right]$ to be positive for all $y_0>0$:

- If $y_0 \geq y_\sigma$, then $y_0^\beta\frac{\alpha}{\beta\gamma}-1 \geq 0$ and $\mathcal{S}_2^{y_0}(f)$ is positive on $[0,T]$;
- If $y_0 < y_\sigma$, then $y_0^\beta\frac{\alpha}{\beta\gamma}-1 < 0$ and $\mathcal{S}_2^{y_0}(f)$ is positive on $[0,T]$ if and only if (4.3.9) holds.

The crucial step in [130, Theorem 2.9] is [130, Theorem 2.7], which states that if $\sqrt{\varepsilon}W$ is close to $f \in \overline{\mathcal{H}}$, then Y^ε should be close to $\mathcal{S}_2^{y_0}(f)$, the solution of the controlled ODE. The case of bounded and locally Lipschitz coefficients on the whole real line was done in [130, Theorem 2.7], but with such conditions on a domain, a new localisation argument is required. Given suitable $\eta > \delta > 0$, with δ sufficiently small, there exists $r \in (0,\eta)$ such that the δ-tube around $\mathcal{S}_2^{y_0}(f)$ is contained in $B_r(\eta)$. For this radius r to exist, one simply needs to make sure that the solution $\mathcal{S}_2^{y_0}(f)$ of the controlled ODE never reaches zero (explosion is impossible as infinity is recurrent), which is obvious when $y_0 \geq y_\sigma$, and guaranteed by Condition (4.3.9) when $y_0 < y_\sigma$. Then both functions

$$\mathfrak{b}(x) := \begin{cases} b(x), & x \in [\eta - r, \eta + r], \\ b\left(\frac{(\eta - r)x}{|x|}\right) = b(\eta - r), & x < \eta - r, \\ b\left(\frac{(\eta + r)x}{|x|}\right) = b(\eta + r), & x > \eta + r, \end{cases}$$

and

$$\mathfrak{s}(x) := \begin{cases} \widetilde{\sigma}(x), & x \in [\eta - r, \eta + r], \\ \widetilde{\sigma}\left(\frac{(\eta - r)x}{|x|}\right) = \widetilde{\sigma}(\eta - r), & x < \eta - r, \\ \widetilde{\sigma}\left(\frac{(\eta + r)x}{|x|}\right) = \widetilde{\sigma}(\eta + r), & x > \eta + r, \end{cases}$$

are bounded and globally Lipschitz continuous on $\mathbb{R}_+^*$, and clearly $\varepsilon\mathfrak{b}(\cdot)$ converges uniformly to zero on $\mathbb{R}_+^*$ as ε goes to zero.

Denote $\overline{Y}^\varepsilon$ the solution to $\mathrm{d}\overline{Y}_t^\varepsilon = \varepsilon\mathfrak{b}(\overline{Y}_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathfrak{s}(\overline{Y}_t^\varepsilon)\mathrm{d}W_t$ with $\overline{Y}_0^\varepsilon = y_0 > 0$. Then the two sequences $(\overline{Y}^\varepsilon)_{\varepsilon>0}$ and $(Y^\varepsilon)_{\varepsilon>0}$ are identical in $B_r(\eta)$. Thus, for each $0 < \delta < y_0$ (small enough) there exist $\xi > 0$ such that, for all $x \in B_\xi(y_0)$,

$$\mathbb{P}\left[\|Y^\varepsilon - \mathcal{S}_2^{y_0}(f)\|_\infty > \delta, \|\sqrt{\varepsilon}W - f\|_\infty \leq \zeta\right] = \mathbb{P}\left[\|\overline{Y}^\varepsilon - \mathcal{S}_2^{y_0}(f)\|_\infty > \delta, \|\sqrt{\varepsilon}W - f\|_\infty \leq \zeta\right],$$

for all $f \in \overline{\mathcal{H}}$ s.t. $\Lambda(f) \leq \lambda$, with $\zeta, \lambda > 0$ fixed. Hence, for each $R, \lambda > 0$ and $\delta > 0$ small enough, there exist $\zeta, \xi, \varepsilon_0 > 0$ such that, for all $f \in \overline{\mathcal{H}}$ with $\Lambda(f) \leq \lambda$, $x \in B_\xi(y_0)$, $\varepsilon \leq \varepsilon_0$,

$$\mathbb{P}\left[\|Y^\varepsilon - \mathcal{S}_2^{y_0}(f)\|_\infty > \delta, \|\sqrt{\varepsilon}W - f\|_\infty \leq \zeta\right] \leq \exp\left\{-\frac{R}{\varepsilon}\right\}$$

holds from [130, Proposition 2.15] and so [130, Theorem 2.7] is satisfied here as well. Finally, large deviations follow from the same reasoning as in the proof of [130, Theorem 2.9].

4.C.2 Proof of Theorem 4.3.12

To obtain a large deviations principle for X^ε, a large deviations principle for the rescaled process $\mathrm{X}^\varepsilon := (X^\varepsilon, Y^\varepsilon)$ needs to be proved. This is

$$\mathrm{dX}_t^\varepsilon = \varepsilon\mathrm{b}(\mathrm{X}_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{a}(\mathrm{X}_t^\varepsilon)\mathrm{d}W_t,$$

with initial condition $\mathrm{X}_0^\varepsilon := \mathrm{x}_0 = \begin{pmatrix} \log s_0 \\ y_0 \end{pmatrix}$ and the maps $\mathrm{b}, \mathrm{a} : \mathbb{R}_+^* \to \mathbb{R}^2$ defined as

$$\mathrm{b}(\mathrm{X}_t^\varepsilon) = \begin{pmatrix} -\frac{1}{2}\sigma^2(Y_t^\varepsilon) \\ b(Y_t^\varepsilon) \end{pmatrix} \quad \text{and} \quad \mathrm{a}(\mathrm{X}_t^\varepsilon) = \begin{pmatrix} \sigma(Y_t^\varepsilon) \\ \widetilde{\sigma}(Y_t^\varepsilon) \end{pmatrix}.$$

These two maps are both locally Lipschitz continuous on $\mathbb{R} \times \mathbb{R}_+^*$. Solving the controlled ODE for Y^ε is sufficient to solve the controlled ODE for the process X^ε. Using the proof of Proposition 4.3.13, for $\mathrm{f} := (f, f)$ with $f \in \overline{\mathcal{H}}$, the controlled ODE $\dot{\mathrm{g}}_t = \dot{f}_t \mathrm{a}(\mathrm{g}_t)$, with $\mathrm{g}_0 = \mathrm{x}_0$ has a solution $\mathrm{g} = \mathcal{S}^{\mathrm{x}_0}(f)$ on $[0, T]$. For $y_0 > y_\sigma$, the solution $\mathcal{S}_2^{y_0}$ is strictly positive and $\mathcal{S}^{\mathrm{x}_0}(f)$ exists on $[0, T]$ for all $f \in \overline{\mathcal{H}}$ and $\mathrm{x}_0 \in \mathbb{R} \times \mathbb{R}_+^*$. In this case, $\overline{\mathcal{H}}$ boils down to the Cameron-Martin space. For $y_0 < y_\sigma$, Condition (4.3.9) ensures that $\mathcal{S}_2^{y_0}$ is positive. Applying [130, Theorem 2.9], the sequence X^ε then satisfies a large deviations principle on $\mathcal{C}([0, T], \mathbb{R} \times \mathbb{R}_+^*)$ as ε tends to zero, with speed ε and rate function

$$\mathrm{I}^{Y,X}(\mathrm{g}) := \inf \left\{ \Lambda(f), f \in \overline{\mathcal{H}}, \mathcal{S}^{\mathrm{x}_0}(f) = \mathrm{g} \right\}.$$

To obtain a large deviations principle for the log-stock price X^ε, we apply the Contraction Principle [46, Theorem 4.2.1], since the projection on the first component is continuous.

4.C.3 Proof of Corollary 4.3.14

We prove the lower and upper bounds separately, which turn out to be equal. For simplicity, we introduce the following notation, for all $k \neq 0$:

$$\widetilde{\mathrm{I}}^X(k) = \begin{cases} \inf_{y \geq k} \mathrm{I}^X(g)|_{g(1)=y}, & \text{if } k > 0, \\ \inf_{y \leq k} \mathrm{I}^X(g)|_{g(1)=y}, & \text{if } k < 0. \end{cases}$$

Assuming that the rate function is continuous[3], $\lim_{t \downarrow 0} t \log \mathbb{P}\left[S_t > \mathrm{e}^k\right] = -\widetilde{\mathrm{I}}^X(k)$. We only consider $k > 0$, the other case being symmetric. The proof of this identity is similar to that of [61, Corollary 4.13, Appendix C].

- For any $\delta > 0$, the inequality $\mathbb{E}[(S_t - \mathrm{e}^k)_+] \geq k\mathrm{e}^k \delta \mathbb{P}[S_t > \mathrm{e}^{k(1+\delta)}]$ and Theorem 4.3.12, together with the continuity of the rate function, then imply
$$\begin{aligned}\liminf_{t \downarrow 0} t \log \mathbb{E}\left[\left(S_t - \mathrm{e}^k\right)_+\right] &\geq \liminf_{t \downarrow 0} \left\{t(k + \log k + \log \delta) + t \log \mathbb{P}\left[S_t > \mathrm{e}^{k(1+\delta)}\right]\right\} \\ &= -\widetilde{\mathrm{I}}^X(k(1+\delta)).\end{aligned}$$
Take $\delta \downarrow 0$, by continuity of $\widetilde{\mathrm{I}}^X(k)$, we obtain the desired lower bound.
- To establish the desired upper bound, we note that for any $q > 1$, we have
$$\mathbb{E}\left[\left(S_t - \mathrm{e}^k\right)_+\right] \leq \mathbb{E}\left[\left(S_t - \mathrm{e}^k\right)_+^q\right]^{1/q} \mathbb{P}\left[S_t \geq \mathrm{e}^k\right]^{1-1/q}.$$
and therefore $t \log \mathbb{E}[(S_t - \mathrm{e}^k)_+] \leq \frac{t}{q} \log \mathbb{E}[S_t^q] + t(1 - \frac{1}{q}) \log \mathbb{P}[S_t \geq \mathrm{e}^k]$. From Theorem 4.3.3, for $y_\sigma \leq \min\{y_0, 1\}$, the process $(Y_t)_{t \in [0,T]}$ remains in (y_σ, ∞). The map σ is bounded on (y_σ, ∞), in particular $0 \leq \sigma(y) \leq \alpha/\beta$, and thus adapting the arguments in [59, proof of Corollary 1.2], we have $\limsup_{t \downarrow 0} \frac{t}{q} \log \mathbb{E}[S_t^q] \leq 0$. Indeed, exploiting Hölder inequality and the closed-form formula for the exponential moments of a Gaussian random variable,

[3]Unless the rate function is available in closed form, it is hard to check for continuity. This was done directly for the Heston model in [60] and in [61, Corollary 4.10] for a simplified rough volatility model. The most general related statement is available in [68] based on non-degeneracy assumptions.

we have

$$\begin{aligned}\mathbb{E}[S_t^q] &= s_0^q \mathbb{E}\left[\mathrm{e}^{q(X_t-x_0)}\right] \\ &= s_0^q \mathbb{E}\left[\exp\left\{q\left(-\frac{1}{2}\int_0^t \sigma(Y_s)^2 \mathrm{d}s + \int_0^t \sigma(Y_s)\mathrm{d}W_s\right)\right\}\right] \\ &\leq s_0^q \mathbb{E}\left[\exp\left\{-q\int_0^t \sigma(Y_s)^2\mathrm{d}s\right\}\right]^{\frac{1}{2}} \mathbb{E}\left[\exp\left\{2q\int_0^t \sigma(Y_s)\mathrm{d}W_s\right\}\right]^{\frac{1}{2}} \\ &\leq s_0^q \exp\left\{\frac{4q^2}{4}\mathbb{V}\left[\int_0^t \sigma(Y_s)\mathrm{d}W_s\right]\right\} \leq s_0^q \exp\left\{q^2\int_0^t \mathbb{E}\left[\sigma(Y_s)^2\right]\mathrm{d}s\right\} \leq s_0^q \exp\left\{\frac{\alpha^2 q^2}{\beta^2}t\right\},\end{aligned}$$

which yields

$$\limsup_{t\downarrow 0} \frac{t}{q}\log \mathbb{E}[S_t^q] \leq \limsup_{t\downarrow 0} t \log\left(s_0^q \exp\left(\frac{\alpha^2 q^2}{\beta^2}t\right)\right) \leq \limsup_{t\downarrow 0} t\left\{q x_0 + \frac{\alpha^2 q^2}{\beta^2}t\right\} = 0.$$

Therefore, for fixed $q > 1$, we have $\limsup_{t\downarrow 0} t\log\mathbb{E}[(S_t - \mathrm{e}^k)_+] \leq -(1-\frac{1}{q})\widetilde{\mathrm{I}}^X(k)$. Taking q to infinity yields the desired upper bound.

References

[1] E. Abi Jaber and O. El Euch. "Multifactor approximation of rough volatility models". In: *SIAM Journal on Financial Mathematics* 10.2 (2019), pp. 309–349.

[2] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Vol. 80. Springer Monographs in Mathematics, Springer, 2007.

[3] R. Aïd, M. Basei, and H. Pham. "A McKean–Vlasov approach to distributed electricity generation development". In: *Mathematical Methods of Operations Research* 91.2 (2020), pp. 269–310.

[4] R. Aïd, O. Bonesini, G. Callegaro, and L. Campi. "A McKean-Vlasov Game of Commodity Production, Consumption and Trading". In: *Applied Mathematics & Optimization* 86.40 (2022), pp. 1–37.

[5] R. Aïd, G. Callegaro, and L. Campi. "No–arbitrage commodity option pricing with market manipulation". In: *Mathematics and Financial Economics* 14.3 (2020), pp. 577–603.

[6] B. Allaz. "Oligopoly, uncertainty and strategic forward transactions". In: *International Journal of Industrial Organization* 10.2 (1992), pp. 297–308.

[7] B. Allaz and J.-L. Vila. "Cournot competition, forward markets and efficiency". In: *Journal of Economic theory* 59.1 (1993), pp. 1–16.

[8] E. Alos, J. A. León, and J. Vives. "On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility". In: *Finance and Stochastics* 11.4 (2007), pp. 571–589.

[9] R. J. Aumann. "Correlated equilibrium as an expression of Bayesian rationality". In: *Econometrica: Journal of the Econometric Society* 55 (1987), pp. 1–18.

[10] R. J. Aumann. "Subjectivity and correlation in randomized strategies". In: *Journal of Mathematical Economics* 1.1 (1974), pp. 67–96.

[11] I. Bárány. "Fair distribution protocols or how the players replace fortune". In: *Mathematics of Operations Research* 17.2 (1992), pp. 327–340.

[12] M. Basei and H. Pham. "A weak martingale approach to linear-quadratic McKean–Vlasov stochastic control problems". In: *Journal of Optimization Theory and Applications* 181.2 (2019), pp. 347–382.

[13] C. Bayer, C. Ben Hammouda, and R. Tempone. "Hierarchical adaptive sparse grids and quasi-Monte Carlo for option pricing under the rough Bergomi model". In: *Quantitative Finance* 20.9 (2020), pp. 1457–1473.

[14] C. Bayer, P. Friz, and J. Gatheral. "Pricing under rough volatility". In: *Quantitative Finance* 16.6 (2016), pp. 887–904.

[15] C. Bayer, M. Fukasawa, and S. Nakahara. "Short Communication: On the Weak Convergence Rate in the Discretization of Rough Volatility Models". In: *SIAM Journal on Financial Mathematics* 13.2 (2022), pp. 66–73.

[16] C. Bayer, E. J. Hall, and R. Tempone. "Weak error rates for option pricing under linear rough volatility". Preprint available at arXiv. 2021.

[17] M. Bennedsen, A. Lunde, and M. S. Pakkanen. "Hybrid scheme for Brownian semistationary processes". In: *Finance and Stochastics* 21.4 (2017), pp. 931–965.

[18] A. Bensoussan, J. Frehse, and P. Yam. *Mean field games and mean field type control theory*. Vol. 101. SpringerBriefs in Mathematics, Springer, 2013.

[19] A. Bensoussan, K. Sung, S. C. P. Yam, and S.-P. Yung. "Linear-quadratic mean field games". In: *Journal of Optimization Theory and Applications* 169.2 (2016), pp. 496–529.

[20] F. E. Benth, Á. Cartea, and R. Kiesel. "Pricing forward contracts in power markets by the certainty equivalence principle: Explaining the sign of the market risk premium". In: *Journal of banking & finance* 32.10 (2008), pp. 2006–2021.

[21] L. Bergomi. "Smile dynamics II". In: *Risk, available at SSRN 1493302* (2005).

[22] P. Billingsley. *Convergence of probability measures*. 2nd ed. John Wiley & Sons, 1999.

[23] A. N. Bishop, P. Del Moral, K. Kamatani, and B. Remillard. "On one-dimensional Riccati diffusions". In: *The Annals of Applied Probability* 29.2 (2019), pp. 1127–1187.

[24] T. Bollerslev. "Generalized autoregressive conditional heteroskedasticity". In: *Journal of econometrics* 31.3 (1986), pp. 307–327.

[25] O. Bonesini, G. Callegaro, and A. Jacquier. "Functional quantization of rough volatility and applications to the VIX". Preprint available at arXiv.

[26] O. Bonesini, L. Campi, and M. Fischer. "Correlated equilibria for mean field games with progressive strategies". Preprint available at arXiv.

[27] O. Bonesini, A. Jacquier, and L. Chloé. "A theoretical analysis of Guyons toy volatility model". Preprint available at arXiv.

[28] P. E. Caines, M. Huang, and R. P. Malhamé. "Mean Field Games". In: *Handbook of Dynamic Game Theory*. Ed. by T. Baar and G. Zaccour. Springer International Publishing, 2018, pp. 345–372.

[29] G. Callegaro, M. Grasselli, and G. Pagès. "Fast hybrid schemes for fractional Riccati equations (rough is not so tough)". In: *Mathematics of Operations Research* 46.1 (2021), pp. 221–254.

[30] L. Campi and M. Fischer. "Correlated equilibria and mean field games: a simple model". In: *Mathematics of Operations Research* 47.3 (2022), pp. 2240–2259.

[31] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games:(ams-201)*. Princeton University Press, 2019.

[32] R. Carmona and F. Delarue. "Probabilistic analysis of mean-field games". In: *SIAM Journal on Control and Optimization* 51.4 (2013), pp. 2705–2734.

[33] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications I-II*. Springer, 2018.

[34] R. Carmona, F. Delarue, and A. Lachapelle. "Control of McKean–Vlasov dynamics versus mean field games". In: *Mathematics and Financial Economics* 7.2 (2013), pp. 131–166.

[35] P. Carr and D. B. Madan. "Joint modeling of VIX and SPX options at a single and common maturity with risk management applications". In: *IIE Transactions* 46.11 (2014), pp. 1125–1131.

[36] I.-H. Cheng and W. Xiong. "Why do hedgers trade so much?" In: *The Journal of Legal Studies* 43.S2 (2014), pp. 183–207.

[37] A. S. Cherny and H.-J. Engelbert. "Singular stochastic differential equations". In: *Lecture Notes in Mathematics* 1858 (2005), pp. 523–544.

[38] Y. S. Chow and H. Teicher. *Probability theory: independence, interchangeability, martingales*. 3rd ed. Springer, 2003.

[39] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. Tata McGraw-Hill Education, 1955.

[40] F. Comte and E. Renault. "Long memory continuous time models". In: *Journal of Econometrics* 73.1 (1996), pp. 101–149.

[41] S. Corlay. "Quelques aspects de la quantification optimale et applications à la finance". PhD thesis. Université Pierre et Marie Curie (Paris 6), 2011.

[42] A. Cosso and H. Pham. "Zero-sum stochastic differential games of generalized McKean–Vlasov type". In: *Journal de Mathématiques Pures et Appliquées* 129 (2019), pp. 180–212.

[43] C. Cuchiero, G. Gazzani, and S. Svaluto-Ferro. "Signature-based models: theory and calibration". Preprint available at arXiv.

[44] A. Dandapani, P. Jusselin, and M. Rosenbaum. "From quadratic Hawkes processes to super-Heston rough volatility models with Zumbach effect". In: *Quantitative Finance* 21.8 (2021), pp. 1235–1247.

[45] F. A. De Roon, T. E. Nijman, and C. Veld. "Hedging pressure effects in futures markets". In: *The Journal of Finance* 55.3 (2000), pp. 1437–1456.

[46] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*. Vol. 38. Stochastic Modelling and Applied Probability, Springer, 1998.

[47] X. Ding and R. Wu. "A new proof for comparison theorems for stochastic differential inequalities with respect to semimartingales". In: *Stochastic Processes and their applications* 78.2 (1998), pp. 155–171.

[48] J.-C. Duan. "The GARCH option pricing model". In: *Mathematical finance* 5.1 (1995), pp. 13–32.

[49] R. Dumitrescu, M. Leutscher, and P. Tankov. “Control and optimal stopping Mean Field Games: a linear programming approach”. In: *Electronic Journal of Probability* 26 (2021), pp. 1–49.

[50] B. Dupire. “Pricing with a smile”. In: *Risk* 7.1 (1994), pp. 18–20.

[51] H. Dyrssen, E. Ekström, and J. Tysk. “Pricing equations in jump-to-default models”. In: *International Journal of Theoretical and Applied Finance* 17.3 (2014), pp. 1–13.

[52] I. Ekeland, D. Lautier, and B. Villeneuve. “Hedging pressure and speculation in commodity markets”. In: *Economic Theory* 68.1 (2019), pp. 83–123.

[53] O. El Euch, M. Fukasawa, and M. Rosenbaum. “The microstructural foundations of leverage effect and rough volatility”. In: *Finance and Stochastics* 22.2 (2018), pp. 241–280.

[54] O. El Euch, J. Gatheral, R. Radoii, and M. Rosenbaum. “The Zumbach effect under rough Heston”. In: *Quantitative Finance* 20.2 (2020), pp. 235–241.

[55] O. El Euch and M. Rosenbaum. “The characteristic function of rough Heston models”. In: *Mathematical Finance* 29.1 (2019), pp. 3–38.

[56] R. F. Engle. “Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation”. In: *Econometrica: Journal of the Econometric Society* 50 (1982), pp. 987–1007.

[57] J. Feng, M. Forde, and J.-P. Fouque. “Short-maturity asymptotics for a fast mean-reverting Heston stochastic volatility model”. In: *SIAM Journal on Financial Mathematics* 1.1 (2010), pp. 126–141.

[58] J. Feng, J.-P. Fouque, and R. Kumar. “Small-time asymptotics for fast mean-reverting stochastic volatility models”. In: *The Annals of Applied Probability* 22.4 (2012), pp. 1541–1575.

[59] M. Forde and A. Jacquier. “Small-time asymptotics for an uncorrelated local-stochastic volatility model”. In: *Applied Mathematical Finance* 18.6 (2011), pp. 517–535.

[60] M. Forde and A. Jacquier. “Small-time asymptotics for implied volatility under the Heston model”. In: *International Journal of Theoretical and Applied Finance* 12.6 (2009), pp. 861–876.

[61] M. Forde and H. Zhang. “Asymptotics for rough stochastic volatility models”. In: *SIAM Journal on Financial Mathematics* 8.1 (2017), pp. 114–145.

[62] F. Forges. “Correlated equilibria and communication in games”. In: *Complex Social and Behavioral Systems: Game Theory and Agent-Based Models* (2020), pp. 107–118.

[63] J.-P. Fouque, M. Lorig, and R. Sircar. “Second order multiscale stochastic volatility asymptotics: stochastic terminal layer analysis and calibration”. In: *Finance and Stochastics* 20.3 (2016), pp. 543–588.

[64] J.-P. Fouque, G. Papanicolaou, and K. R. Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, 2000.

[65] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. “Multiscale stochastic volatility asymptotics”. In: *Multiscale Modeling & Simulation* 2.1 (2003), pp. 22–42.

[66] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. "Singular perturbations in option pricing". In: *SIAM Journal on Applied Mathematics* 63.5 (2003), pp. 1648–1665.

[67] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna. *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*. Cambridge University Press, 2011.

[68] P. K. Friz, P. Gassiat, and P. Pigato. "Precise asymptotics: robust stochastic volatility models". In: *The Annals of Applied Probability* 31.2 (2021), pp. 896–940.

[69] P. K. Friz, J. Gatheral, A. Gulisashvili, A. Jacquier, and J. Teichmann. *Large deviations and asymptotic methods in finance*. Vol. 110. Springer Proceedings in Mathematics & Statistics, Springer, 2015.

[70] G. Fu and U. Horst. "Mean-field leader-follower games with terminal state constraint". In: *SIAM Journal on Control and Optimization* 58.4 (2020), pp. 2078–2113.

[71] M. Fukasawa. "Asymptotic analysis for stochastic volatility: martingale expansion". In: *Finance and Stochastics* 15.4 (2011), pp. 635–654.

[72] M. Fukasawa. "Volatility has to be rough". In: *Quantitative Finance* 21.1 (2021), pp. 1–8.

[73] M. Fukasawa and J. Gatheral. "A rough SABR formula". In: *Frontiers of Mathematical Finance* 1.1 (2022), pp. 81–97.

[74] M. Fukasawa, T. Takabatake, and R. Westphal. "Consistent estimation for fractional stochastic volatility model under high-frequency asymptotics (Is volatility rough?)" In: *Mathematical Finance* 32.4 (2022), pp. 1086–1132.

[75] P. Gassiat. "Weak error rates of numerical schemes for rough volatility". Preprint available at arXiv. 2022.

[76] J. Gatheral. "Consistent modeling of SPX and VIX options". In: *Bachelier congress*. Vol. 37. 2008, pp. 39–51.

[77] J. Gatheral, T. Jaisson, and M. Rosenbaum. "Volatility is rough". In: *Quantitative finance* 18.6 (2018), pp. 933–949.

[78] J. Gatheral, P. Jusselin, and M. Rosenbaum. "The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem". In: *Risk* (2020).

[79] A. Gersho and R. M. Gray. *Vector quantization and signal compression*. Vol. 159. The Springer International Series in Engineering and Computer Science, Springer, 1992.

[80] I. Gilboa and E. Zemel. "Nash and correlated equilibria: Some complexity considerations". In: *Games and Economic Behavior* 1.1 (1989), pp. 80–93.

[81] D. A. Gomes, J. Mohr, and R. R. Souza. "Continuous time finite state mean field games". In: *Applied Mathematics & Optimization* 68.1 (2013), pp. 99–143.

[82] A. D. Gottlieb. "Markov Transitions and the Propagation of Chaos". PhD thesis. Ernest Orlando Lawrence Berkeley National Laboratory, University of California, Berkeley, 1998.

[83] P. J. Graber. "Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource". In: *Applied Mathematics & Optimization* 74.3 (2016), pp. 459–486.

[84] S. Graf and H. Luschgy. *Foundations of quantization for probability distributions*. Vol. 1730. Lecture Notes in Mathematics, Springer, 2007.

[85] H. Guennoun, A. Jacquier, P. Roome, and F. Shi. "Asymptotic behavior of the fractional Heston model". In: *SIAM Journal on Financial Mathematics* 9.3 (2018), pp. 1017–1045.

[86] J. Guyon. "Path-dependent volatility". In: *Risk* (2014).

[87] J. Guyon. "Path-dependent volatility: practical examples". In: *Global Derivatives Conference*. 2017.

[88] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward. "Managing smile risk". In: *The Best of Wilmott* 1 (2002), pp. 249–296.

[89] S. Hart. "Adaptive heuristics". In: *Econometrica* 73.5 (2005), pp. 1401–1430.

[90] S. Hart and D. Schmeidler. "Existence of correlated equilibria". In: *Mathematics of Operations Research* 14.1 (1989), pp. 18–25.

[91] V. Henderson and D. Hobson. "Chapter Two. Utility Indifference Pricing: An Overview". In: *Theory and Applications*. Ed. by R. Carmona. Princeton University Press, 2009, pp. 44–74.

[92] S. L. Heston. "A closed-form solution for options with stochastic volatility with applications to bond and currency options". In: *The review of financial studies* 6.2 (1993), pp. 327–343.

[93] D. Hirshleifer. "Hedging pressure and futures price movements in a general equilibrium model". In: *Econometrica: Journal of the Econometric Society* 58.2 (1990), pp. 411–428.

[94] D. Hirshleifer. "Residual risk, trading costs, and commodity futures risk premia". In: *The Review of Financial Studies* 1.2 (1988), pp. 173–193.

[95] D. Hirshleifer. "Risk, futures pricing, and the organization of production in commodity markets". In: *Journal of Political Economy* 96.6 (1988), pp. 1206–1220.

[96] D. G. Hobson and L. C. Rogers. "Complete models with stochastic volatility". In: *Mathematical Finance* 8.1 (1998), pp. 27–48.

[97] B. Horvath, A. Jacquier, and A. Muguruza. "Functional central limit theorems for rough volatility". Preprint available at SSRN. 2019.

[98] B. Horvath, A. Jacquier, and P. Tankov. "Volatility options in rough volatility models". In: *SIAM Journal on Financial Mathematics* 11.2 (2020), pp. 437–469.

[99] M. Huang, R. P. Malhamé, and P. E. Caines. "Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle". In: *Communications in Information & Systems* 6.3 (2006), pp. 221–252.

[100] J. Huh, J. Jeon, and J.-H. Kim. "A scaled version of the double-mean-reverting model for VIX derivatives". In: *Mathematics and Financial Economics* 12.4 (2018), pp. 495–515.

[101] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. 2nd ed. North-Holland Mathematical Library, Elsevier, 2014.

[102] A. Ismail and H. Pham. "Robust Markowitz mean-variance portfolio selection under ambiguous covariance matrix". In: *Mathematical Finance* 29.1 (2019), pp. 174–207.

[103] A. Jacquier, C. Martini, and A. Muguruza. "On VIX futures in the rough Bergomi model". In: *Quantitative Finance* 18.1 (2018), pp. 45–61.

[104] A. Jacquier, M. S. Pakkanen, and H. Stone. "Pathwise large deviations for the rough Bergomi model". In: *Journal of Applied Probability* 55.4 (2018), pp. 1078–1092.

[105] O. Kallenberg. *Foundations of Modern Probability*. 2nd ed. Probability and Its Applications, Springer, 2001.

[106] S. Karlin and H. E. Taylor. *A second course in stochastic processes*. Elsevier, 1981.

[107] D. B. Karp. "Representations and inequalities for generalized hypergeometric functions". In: *Journal of Mathematical Sciences* 207.6 (2015), pp. 885–897.

[108] T. Kokholm and M. Stisen. "Joint pricing of VIX and SPX options with stochastic volatility and jump models". In: *The Journal of Risk Finance* (2015), pp. 27–48.

[109] D. Lacker. "Limit theory for controlled McKean–Vlasov dynamics". In: *SIAM Journal on Control and Optimization* 55.3 (2017), pp. 1641–1672.

[110] D. Lacker. "Mean field games via controlled martingale problems: existence of Markovian equilibria". In: *Stochastic Processes and their Applications* 125.7 (2015), pp. 2856–2894.

[111] D. Lacker. "On the convergence of closed-loop Nash equilibria to the mean field game limit". In: *The Annals of Applied Probability* 30.4 (2020), pp. 1693–1761.

[112] D. Lacker and L. Le Flem. "Closed-loop convergence for mean field games with common noise". To appear in *The Annals of Applied Probability*. 2022.

[113] J.-M. Lasry and P.-L. Lions. "Jeux à champ moyen. I–Le cas stationnaire". In: *Comptes Rendus Mathématique* 343.9 (2006), pp. 619–625.

[114] J.-M. Lasry and P.-L. Lions. "Jeux à champ moyen. II–Horizon fini et contrôle optimal". In: *Comptes Rendus Mathématique* 343.10 (2006), pp. 679–684.

[115] J.-M. Lasry and P.-L. Lions. "Mean field games". In: *Japanese Journal of Mathematics* 2.1 (2007), pp. 229–260.

[116] W. Lefebvre, G. Loeper, and H. Pham. "Mean-variance portfolio selection with tracking error penalization". In: *Mathematics* 8.11 (1915 2020), pp. 1–23.

[117] X. Li, J. Sun, and J. Xiong. "Linear quadratic optimal control problems for mean-field backward stochastic differential equations". In: *Applied Mathematics & Optimization* 80.1 (2019), pp. 223–250.

[118] Y. L. Luke. *The special functions and their approximations*. Vol. 2. Academic Press, 1969.

[119] H. Luschgy and G. Pagès. "Functional quantization of Gaussian processes". In: *Journal of Functional Analysis* 196.2 (2002), pp. 486–531.

[120] H. Luschgy and G. Pagès. "High-resolution product quantization for Gaussian processes under sup-norm distortion". In: *Bernoulli* 13.3 (2007), pp. 653–671.

[121] R. McCrickerd and M. S. Pakkanen. "Turbocharging Monte Carlo pricing for the rough Bergomi model". In: *Quantitative Finance* 18.11 (2018), pp. 1877–1886.

[122] E. Miller and H. Pham. "Linear-quadratic McKean-Vlasov stochastic differential games". In: *Modeling, Stochastic Control, Optimization, and Applications*. Ed. by G. Yin and Q. Zhang. Vol. 164. Springer, 2019, pp. 451–481.

[123] P. D. Miller. *Applied asymptotic analysis*. Vol. 75. Graduate Studies in Mathematics, American Mathematical Society, 2006.

[124] S. Mohammed and T. Zhang. "Anticipating stochastic differential systems with memory". In: *Stochastic processes and their applications* 119.9 (2009), pp. 2773–2802.

[125] P. Muller, R. Elie, M. Rowland, M. Lauriere, J. Perolat, S. Perrin, M. Geist, G. Piliouras, O. Pietquin, and K. Tuyls. "Learning Correlated Equilibria in Mean-Field Games:" Preprint available at arXiv. 2022.

[126] P. Muller, M. Rowland, R. Elie, G. Piliouras, J. Perolat, M. Lauriere, R. Marinier, O. Pietquin, and K. Tuyls. "Learning Equilibria in Mean-Field Games: Introducing Mean-Field PSRO". Preprint available at arXiv. 2021.

[127] F. Olver. *Asymptotics and special functions*. 1st ed. AK Peters/CRC Press, 1997.

[128] G. Pagès. "Quadratic optimal functional quantization of stochastic processes and numerical applications". In: *Monte Carlo and Quasi-Monte Carlo Methods 2006*. Springer. 2008, pp. 101–142.

[129] G. Pagès and J. Printems. "Functional quantization for numerics with an application to option pricing". In: *Monte Carlo Methods and Applications* 11.4 (2005), pp. 407–446.

[130] D. Peithmann. "Large deviations and exit time asymptotics for diffusions and stochastic resonance". PhD thesis. Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II, 2007.

[131] J. Picard. "Representation formulae for the fractional Brownian motion". In: *Séminaire de Probabilités XLIII*. Springer, 2011, pp. 3–70.

[132] R. G. Pinsky and T. T. Dieck. *Positive Harmonic Functions and Diffusion*. Vol. 45. Cambridge studies in advanced mathematics, Cambridge University Press, 1995.

[133] T. Roughgarden. *Twenty lectures on algorithmic game theory*. Cambridge University Press, 2016.

[134] W. F. Sheppard. "On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Division of a Scale". In: vol. 1. 1. Proceedings of the London Mathematical Society, Wiley Online Library, 1897, pp. 353–380.

[135] E. Solan. "Characterization of correlated equilibria in stochastic games". In: *International Journal of Game Theory* 30.2 (2001), pp. 259–277.

[136] E. Solan. *Rationality and Extensive Form Correlated Equilibria in Stochastic Games*. Tech. rep. D.P. 1298. The Center for Mathematical Studies in Economics and Management Science, Northwestern University, 2000.

[137] E. Solan and N. Vieille. “Correlated equilibrium in stochastic games”. In: *Games and Economic Behavior* 38.2 (2002), pp. 362–399.

[138] J. M. Steele. *The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities*. Cambridge University Press, 2004.

[139] E. M. Stein and J. C. Stein. “Stock price distributions with stochastic volatility: an analytic approach”. In: *The review of financial studies* 4.4 (1991), pp. 727–752.

[140] X. Y. Zhou and D. Li. “Continuous-time mean-variance portfolio selection: A stochastic LQ framework”. In: *Applied Mathematics & Optimization* 42.1 (2000), pp. 19–33.

[141] Q. Zhu, G. Loeper, W. Chen, and N. Langrené. “Markovian approximation of the rough Bergomi model for Monte Carlo option pricing”. In: *Mathematics* 9.5 (2021), p. 528.

[142] G. Zumbach. “Time reversal invariance in finance”. In: *Quantitative Finance* 9.5 (2009), pp. 505–515.

[143] G. Zumbach. “Volatility conditional on price trends”. In: *Quantitative Finance* 10.4 (2010), pp. 431–442.

Acknowledgments

These acknowledgments are, as usual, placed at the end of the manuscript. This is slightly unfair. They would deserve to be in the first pages given the fact that nothing would have been possible without the names that you are going to read in the following lines. I am proud of the work of these years not just for this paper-and-ink-made creature resulting from my efforts but rather for teaching me that mathematical research is not just a job, but it is a shared experience within a community, and it somehow involves all the aspects of one's life. I am extremely thankful for the mathematical, but also emotional, support I have received during this incredible experience, and I apologise in advance if I will accidentally forget someone in the following.

My first and most significant thank is due to my supervisors Prof. Markus Fischer and Prof. Giorgia Callegaro. I am extremely grateful to Markus, for guiding me carefully and supporting me in all the phases of my PhD. He trusted the project when I did not even believe in myself and pushed myself towards my limits. He helped to grow as an autonomous researcher but, for the past three years (and even during my Masters thesis), I always knew I could count on him, if in (mathematical) troubles. I feel tremendously blessed for having Giorgia as supervisor. She has provided me with the best example of how a gorgeous woman can balance all the aspects of her life: she has been a role model more than a supervisor. I am forever indebted to her at professional level, for his infinite patience and his priceless expertise, and at personal level, for holding my hand even when completing the most basic tasks. She made me realise that being collaborators relies on sharing successes as well as failures, doubts, frustrations, and anxieties and that it is exactly in these last ones that we challenge ourselves the most and prove our value as researchers.

I would like to show my gratitude to Christa Cuchiero and Peter Tankov, who are the reviewers of my PhD thesis, for accepting the burden of reading a so heterogeneous thesis, showing their interest in my work, for taking the time to read it and, finally, for their useful comments which helped to grow as a researcher. An acknowledgment is due also to the members of the commission of the Ph.D. defense.

A huge thank to Antoine Jacquier for almost being a third supervisor. I am grateful for the care and time he devoted to me despite being so busy and for all the useful advice he gave me during our numerous zoom calls and the time spent in London. I consider myself privileged to continue my career in your research group at Imperial College.
In a broader sense, my gratitude goes to all my coauthors Luciano Campi, René Aïd and Chloé

Lacombe, for showing what working in a team means even if you can only communicate via Zoom. In particular, Luciano and René deserve a special mention for being fantastic researchers but even better as human beings. The never-ending Zoom calls during the Pandemic were a nice parenthesis in the boredom of my routine, but the time spent together in Padova will always have a room in my heart.

During my studies, many people had an influence on me, but none was as deep as the one of Prof. Wolfgang Runggaldier, so I would like to thank him. He introduced me to research first, sharing his passion for mathematics and enthusiasm for life and learning in a broader sense, and he was the first one to suggest me to pursue this path.

I want to sincerely thank each person who has joined (even just for a short period) the MAT/06 and SECS-S/06 research group in Padova for any stimulating discussion, mathematical or not, as well as my PhD companions in the 35th cohort, Mattia, Daniele, Rita, Piero, Luca, Alessandro and Giovanni who were loyal lunch buddies and a priceless support during this journey, especially when Covid was winning over us. In particular, I am grateful to Mattia for all the laughs, the mourning, and the tears we shared: in a nutshell, for being the best desk-neighbour one can ask for. Then, a well-deserved thank is due to Daniele and Luca, for always being here to help me with any small technical issue and for sharing this experience with me from the very first day of the Bachelors Degree till the moment in which we pushed together the button to submit our PhD theses. More broadly, an enormous thank to all the PhD students for the fabulous time shared in Padova in these years, with a special mention to Karim for being such a lovely and supporting friend. Thank you to Giacomo, who happened to be also my partner in crime for (not as) many conferences (as we wanted).

I am extremely grateful to have had more than a few chances to present my research in front of experts in the field, so I repeat my appreciation for all the conference organisers who allowed me to do this and to all the PhD students I have met there for the peer support.

I want to express my endless gratitude to Loretta and Marinella and all the administrative staff members for the help and unlimited kindness of these years.

I will be forever indebted with all my friends from Verona: Marta, Andrea, Simone, Giacomo and Giovanni, as well as my friends from Padova: Marica, Elsa, Riccardo, Laura and Luca, for their abiding consolation and everlasting confidence.

Last but not least, I want to thank my family. My mother and sister in the first place for their limitless patience and comfort in the hardest times and for keeping my pieces together when I was falling apart under the weight of problems. Without their exceptional understanding and support in the past few years, it would have been impossible for me to complete this research. I will be forever grateful as they always believed in me and set me off on the road to this PhD a long time ago.