



Optimal Control Theory — *The equivalence of Fourier-based and Wasserstein metrics on imaging problems*, by GENNARO AURICCHIO, ANDREA CODEGONI, STEFANO GUALANDI, GIUSEPPE TOSCANI and MARCO VENERONI, communicated on 24 June 2020.

ABSTRACT. — We investigate properties of some extensions of a class of Fourier-based probability metrics, originally introduced to study convergence to equilibrium for the solution to the spatially homogeneous Boltzmann equation. At difference with the original one, the new Fourier-based metrics are well-defined also for probability distributions with different centers of mass, and for discrete probability measures supported over a regular grid. Among other properties, it is shown that, in the discrete setting, these new Fourier-based metrics are equivalent either to the Euclidean–Wasserstein distance W_2 , or to the Kantorovich–Wasserstein distance W_1 , with explicit constants of equivalence. Numerical results then show that in benchmark problems of image processing, Fourier metrics provide a better runtime with respect to Wasserstein ones.

KEY WORDS: Fourier-based metrics, Wasserstein distance, Fourier transform, image analysis

MATHEMATICS SUBJECT CLASSIFICATION: 60A10, 60E15, 42A38

1. INTRODUCTION

In computational applied mathematics, numerical methods based on Wasserstein distances achieved a leading role over the last years. Examples include the comparison of histograms in higher dimensions [6, 9, 22], image retrieval [21], image registration [4, 11], or, more recently, the computations of barycenters among images [7, 15]. Surprisingly, the possibility to identify the cost function in a Wasserstein distance, together with the possibility of representing images as histograms, led to the definition of classifiers able to mimic the human eye [16, 21, 24].

More recently, metrics which are able to compare at best probability distributions were introduced and studied in connection with machine learning, where testing the efficiency of new classes of loss functions for neural networks training has become increasingly important. In this area, the Wasserstein distance often turns out to be the appropriate tool [1, 5, 18]. Its main drawback, though, is that it suffers from high computational complexity. For this reason, attempts to use other metrics, which require a lower computational cost while maintaining a good approximation, have been object of recent research [28]. There, the theory of approximation in the space of wavelets was the main mathematical tool.

Following the line of thought of [28], we consider here an alternative to the approximation in terms of wavelets, which is furnished by metrics based on the

Fourier transform. In terms of computational complexity, the price to pay for a dimension $N \gg 1$ of the data changes from a time $O(N)$ to the time $O(N \log N)$ required to evaluate the fast Fourier transform.

While this represents a worsening, with respect to the use of wavelets, in terms of computational complexity, there is an effective improvement with respect to the computational complexity required to evaluate Wasserstein-type metrics, which is of the order $O(N^3 \log N)$. Furthermore, from the point of view of the important questions related to the comparison of these metrics with Wasserstein metrics in problems motivated by real applications, we prove in this paper that in the case of probability measures supported on a bounded domain, one has a precise and explicit evaluation of the constants of equivalence among these Fourier-based metrics and the Wasserstein ones, a result which is not present in [28].

The Fourier-based metrics considered in this paper were introduced in [19], in connection with the study of the trend to equilibrium for solutions of the spatially homogeneous Boltzmann equation for Maxwell molecules. Since then, many applications of these metrics have followed in both kinetic theory and probability [10, 12–14, 20, 25, 30]. All these problems deal with functions supported on the whole space \mathbb{R}^d , with $d \geq 1$, that exhibit a suitable decay at infinity which guarantees the existence of a suitable number of moments.

Given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $d \geq 1$, and a real parameter $s > 0$, the Fourier-based metrics d_s considered in [19] are given by

$$(1.1) \quad d_s(\mu, \nu) := \sup_{\mathbf{k} \in \mathbb{R}^d \setminus \{0\}} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|}{|\mathbf{k}|^s},$$

where $\hat{\mu}$ and $\hat{\nu}$ are the Fourier transforms of the measures μ and ν , respectively. As usual, given a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$, the Fourier transform of μ is defined by

$$\hat{\mu}(\mathbf{k}) := \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} d\mu(\mathbf{x}).$$

These metrics, for $s \geq 1$, are well-defined under the further assumption of boundedness and equality of some moments of the probability measures. Indeed, a necessary condition for d_s to be finite, is that moments up to $[s]$ (the integer part of s) are equal for both measures [19].

In dimension $d = 1$, similar metrics were introduced a few years later by Baringhaus and Grübel in connection with the characterization of convex combinations of random variables [8]. Given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $d \geq 1$, and two real parameters $s > 0$ and $p \geq 1$, the multi-dimensional version of these Fourier-based metrics reads

$$(1.2) \quad D_{s,p}(\mu, \nu) := \left(\int_{\mathbb{R}^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{(ps+d)}} d\mathbf{k} \right)^{1/p}.$$

The metrics defined by (1.1) and (1.2) belong to the set of ideal metrics [32], and have been shown to be equivalent to other common probability distances [19, 30], including the Wasserstein distance $W_2(\mu, \nu)$ [14], given by

$$(1.3) \quad W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\pi(\mathbf{x}, \mathbf{y}) \right\}^{1/2},$$

where the infimum is taken on the set $\Pi(\mu, \nu)$ of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal densities μ and ν . However, in dimension $d > 1$ the constants of equivalence are not explicit [14], so that it is difficult to establish a comparison between these metrics' efficacy in applications.

An unpleasant aspect related to the application of the previous Fourier-based distances is related to its finiteness, that requires, for high values of s , a sufficiently high number of equal moments for the underlying probability measures. In the context of kinetic equations of Boltzmann type, where conservation of momentum and energy of the solution is a consequence of the microscopic conservation laws of binary interactions among particles, this requirement on d_s , with $2 < s < 3$, is clearly not restrictive. However, in order to apply the Fourier-based metrics outside of the context of kinetic equations, this requirement appears unnatural. To clarify this point, let us consider the case in which we want to compare the distance between two images. If we take two grey scale images and model them as probability distributions, there is no reason why these distributions possess the same expected value. The simplest example is furnished by two images consisting of a black dot, each one centered in a different point of the region, that can be modeled as two Dirac delta functions centered in two different points.

In this paper we improve the existing results concerning the evaluation of the constants in the equivalence relations between the Fourier-based metrics and the Wasserstein one, in a relevant setting with respect to applications. This equivalence is related to the comparison of two discrete measures and it is based on the properties of the Fourier transform in the discrete setting. To this extent, we consider a new version of these metrics, the *periodic Fourier-based metrics*, that play the role of the metrics (1.1) and (1.2) in the discrete setting. With our results, we show that the new family of Fourier-based metrics represents a fruitful alternative to the Wasserstein metrics, both from the theoretical and the computational points of view.

To weaken the restriction about moments, we further consider a variant of the Fourier metric d_2 that remains well-defined even for probability measures with different mean values.

The content of this paper is as follows. In Section 2 we introduce the notations and the basic concepts of measure theory and optimal transport. Furthermore, we define the Fourier-based metrics, we recall their main properties, and we introduce our extension. Then, in view of applications, in Section 3, we consider a discrete setting and we define and study the properties of the new family of periodic Fourier-based metrics, highlighting their explicit equivalence with the Wasserstein distance in various cases. Section 4 presents numerical results obtained

comparing our implementation of the *periodic Fourier-based metrics* with the Wasserstein metrics as implemented in the POT library [17]. The concluding remarks are contained in Section 5.

2. AN EXTENSION OF FOURIER-BASED METRICS

In what follows, we briefly review some basic notions of optimal transport, together with the definition and some properties of Wasserstein and Fourier-based metrics. The final goal is to extend the definition of the metrics (1.1) and (1.2) for the particular case $s = 2$, which allows for a direct and fruitful comparison between the Fourier-based metrics and the Wasserstein metric W_2 defined in (1.3). In what follows, we only present the notions that are necessary for our purpose. For a deeper insight on optimal transport, we refer the reader to [2, 3, 26, 31]. Likewise, we address the interested reader to [14] for an exhaustive review of the properties of the Fourier-based metrics and their connections with other metrics used in probability theory.

We work on the Euclidean space \mathbb{R}^d , endowed with the Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$. We use bold letters to denote vectors of \mathbb{R}^d . If $\mathbf{x} \in \mathbb{R}^d$, then x_i denotes its i -th coordinate. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^d x_i y_i$ is their scalar product and $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is the Euclidean norm (or modulus) of \mathbf{x} .

The set of probability measures on \mathbb{R}^d is denoted by $\mathcal{P}(\mathbb{R}^d)$. For all $m \in \mathbb{N}$ we denote by $\mathcal{P}_m(\mathbb{R}^d)$ the set of probability measures with finite moments up to order m

$$\mathcal{P}_m(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathbf{x}^\beta d\mu(\mathbf{x}) < +\infty, \forall \beta \in \mathbb{N}^d, |\beta| \leq m \right\}.$$

Given $\mu \in \mathcal{P}(\mathbb{R}^d)$ and a Borel map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, then the image measure (or push-forward) of μ by f is $f_{\#}\mu \in \mathcal{P}(\mathbb{R}^d)$, given by $f_{\#}\mu(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. Equivalently, for every continuous compactly supported function ϕ on \mathbb{R}^d , it holds

$$\int_{\mathbb{R}^d} \phi(y) d(f_{\#}\mu)(y) = \int_{\mathbb{R}^d} \phi(f(x)) d\mu(x).$$

Our first goal is to define the Fourier-based metrics d_s , in the range $1 < s \leq 2$, on $\mathcal{P}(\mathbb{R}^d)$.

DEFINITION 1. Given $\mu \in \mathcal{P}_1(\mathbb{R}^d)$, we say that

$$\mathbf{m}_\mu = \int_{\mathbb{R}^d} \mathbf{x} d\mu(\mathbf{x})$$

is the *center* of μ .

The center of a measure μ can be moved by resorting to a translation. Given $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ and $\boldsymbol{\tau} \in \mathbb{R}^d$, we define the translated measure $\mu_\tau \in \mathcal{P}_1(\mathbb{R}^d)$ by

$$\mu_\tau = S_{\#}^\tau \mu, \quad \text{where } S^\tau(\mathbf{x}) = \mathbf{x} + \boldsymbol{\tau}.$$

LEMMA 1. Given $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, there exists a unique vector $\tau \in \mathbb{R}^d$ such that

$$\mathbf{m}_\mu = \mathbf{m}_{\nu_\tau}.$$

PROOF. Let $\tau = \mathbf{m}_\mu - \mathbf{m}_\nu$, then

$$\mathbf{m}_{\nu_\tau} = \int_{\mathbb{R}^d} \mathbf{x} d\nu_\tau(\mathbf{x}) = \int_{\mathbb{R}^d} (\mathbf{x} + \tau) d\nu(\mathbf{x}) = \mathbf{m}_\nu + \tau = \mathbf{m}_\mu. \quad \square$$

Let us recall now the definition of transport plan, and the consequent definition of Wasserstein distance.

DEFINITION 2 (Transport plan). Given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, a vector $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is called a transport plan between μ and ν if its marginals coincide with μ, ν , that is

$$(2.4) \quad \pi(A \times \mathbb{R}^d) = \mu(A) \quad \forall A \in \mathcal{B}(\mathbb{R}^d),$$

$$(2.5) \quad \pi(\mathbb{R}^d \times B) = \nu(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^d).$$

We denote by $\Pi(\mu, \nu)$ the set of all transport plans between μ and ν .

DEFINITION 3 (Wasserstein distance). Given $p \in \mathbb{N}$ and $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, the Wasserstein distance of order p between μ and ν is defined as

$$(2.6) \quad W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^p d\pi(\mathbf{x}, \mathbf{y}) \right\}^{1/p},$$

where $|\cdot|$ is a norm defined in \mathbb{R}^d .

In this paper, we consider only the Euclidean norm, and we focus on Wasserstein distances with exponents $p = 1$ and $p = 2$, namely

$$(2.7) \quad W_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}| d\pi(\mathbf{x}, \mathbf{y}) \right\},$$

$$(2.8) \quad W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\pi(\mathbf{x}, \mathbf{y}) \right\}^{1/2}.$$

The W_2 metric satisfies an explicit translation property (Remark 2.19, [16]). We give below a short proof of this property.

LEMMA 2. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, with centers \mathbf{m}_μ and \mathbf{m}_ν , respectively. For any given pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ we have

$$(2.9) \quad W_2(\mu_{\mathbf{v}}, \nu_{\mathbf{w}})^2 = W_2(\mu, \nu)^2 + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{v} - \mathbf{w}, \mathbf{m}_\mu - \mathbf{m}_\nu \rangle.$$

In addition, if we choose $\mathbf{v} = -\mathbf{m}_\mu$ and $\mathbf{w} = -\mathbf{m}_\nu$, it holds

$$(2.10) \quad W_2(\mu_{-\mathbf{m}_\mu}, \nu_{-\mathbf{m}_\nu})^2 = W_2(\mu, \nu)^2 - |\mathbf{m}_\mu - \mathbf{m}_\nu|^2.$$

PROOF. Given a transport plan $\pi \in \Pi(\mu, \nu)$, we consider the transport plan

$$\tilde{\pi} := (S^v, S^w)_\# \pi,$$

where $S^v(\mathbf{x}) = \mathbf{x} + \mathbf{v}$, $S^w(\mathbf{y}) = \mathbf{y} + \mathbf{w}$. $\tilde{\pi}$ is a transport plan between the translated measures μ_v and ν_w . Then, by definition of push-forward, we get

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\tilde{\pi}(\mathbf{x}, \mathbf{y}) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |(\mathbf{x} + \mathbf{v}) - (\mathbf{y} + \mathbf{w})|^2 d\pi(\mathbf{x}, \mathbf{y}) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{x} - \mathbf{y}, \mathbf{v} - \mathbf{w} \rangle) d\pi(\mathbf{x}, \mathbf{y}) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\pi(\mathbf{x}, \mathbf{y}) + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{m}_\mu - \mathbf{m}_\nu, \mathbf{v} - \mathbf{w} \rangle. \end{aligned}$$

If π is an optimal transport plan between μ and ν , we have

$$\begin{aligned} W_2(\mu_v, \nu_w)^2 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\tilde{\pi}(\mathbf{x}, \mathbf{y}) \\ &= W_2(\mu, \nu)^2 + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{v} - \mathbf{w}, \mathbf{m}_\mu - \mathbf{m}_\nu \rangle. \end{aligned}$$

By repeating the previous argument with an optimal transport plan between μ_v, ν_w , we find

$$\begin{aligned} W_2(\mu_v, \nu_w)^2 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^2 d\pi(\mathbf{x}, \mathbf{y}) + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{v} - \mathbf{w}, \mathbf{m}_\mu - \mathbf{m}_\nu \rangle \\ &\geq W_2(\mu, \nu)^2 + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{v} - \mathbf{w}, \mathbf{m}_\mu - \mathbf{m}_\nu \rangle. \end{aligned}$$

Hence, we can conclude

$$W_2(\mu_v, \nu_w)^2 = W_2(\mu, \nu)^2 + |\mathbf{v} - \mathbf{w}|^2 + 2\langle \mathbf{v} - \mathbf{w}, \mathbf{m}_\mu - \mathbf{m}_\nu \rangle. \quad \square$$

The idea of using translation operators to compute the distance of probability measures with different centers can be used to properly modify the Fourier-based metrics d_s and $D_{s,p}$ defined in (1.1) and (1.2). Indeed, as briefly discussed in the introduction, the case $s \geq 1$ requires the probability measures to satisfy the further condition given below [19].

PROPOSITION 1 (Proposition 2.6, [14]). *Let $\lfloor s \rfloor$ denote the integer part of $s \in \mathbb{R}$, and assume that the densities $\mu, \nu \in \mathcal{P}_s(\mathbb{R}^d)$ possess equal moments up to $\lfloor s \rfloor$ if $s \notin \mathbb{N}$, or equal moments up to $s - 1$ if $s \in \mathbb{N}$. Then the Fourier-based distance $d_s(\mu, \nu)$ is well-defined. In particular, $d_2(\mu, \nu)$ is well-defined for two densities with the same center.*

The interest in the d_2 metric is related to its equivalence to the Euclidean Wasserstein distance W_2 . A detailed proof in dimension $d \geq 1$ can be found in the review paper [14].

THEOREM 1 (Proposition 2.12 and Corollary 2.17, [14]). *For any given pair of probability densities $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathbf{m}_\mu = \mathbf{m}_\nu$, the d_2 metric is equivalent to the Euclidean Wasserstein distance W_2 , that is, there exist two positive bounded constants $c < C$ such that*

$$(2.11) \quad cW_2(\mu, \nu) \leq d_2(\mu, \nu) \leq CW_2(\mu, \nu).$$

The proof in [14] does not provide in general the explicit expression of the two constants c and C . The value of these constants is quite involved, and it is strongly dependent on higher moments of the densities.

The equivalence result of Theorem 1 can easily be extended to cover the case of probability measures with different centers of mass. To this aim it is necessary, in analogy with the property of Wasserstein distance W_2 stated in Lemma 2, to modify the Fourier-based metrics d_2 and $D_{2,p}$ in such a way to allow for probability measures with different centers of mass. We start by considering the case of the metric d_2 .

DEFINITION 4 (Translated Fourier-based metric). We define the function $\mathcal{D}_2 : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ as:

$$(2.12) \quad \mathcal{D}_2(\mu, \nu) := \sqrt{d_2(\mu, \nu_{\mathbf{m}_\mu - \mathbf{m}_\nu})^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2}.$$

Owing to Remark 1 and Proposition 1, $\mathcal{D}_2(\mu, \nu)$ is well-defined for each pair of probability measures in $\mathcal{P}_2(\mathbb{R}^d)$, independently of their centers. Note that $\nu_{\mathbf{m}_\mu - \mathbf{m}_\nu}$, which is the translation of ν by $\mathbf{m}_\mu - \mathbf{m}_\nu$, has the same center as μ . One could give an equivalent definition of \mathcal{D}_2 by translating μ , instead of ν , or by translating both centers to $\mathbf{0}$.

LEMMA 3. *Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, then*

$$|\widehat{\mu}_{\mathbf{v}}(\mathbf{k}) - \widehat{\nu}_{\mathbf{w}}(\mathbf{k})| = |\widehat{\mu}(\mathbf{k}) - \widehat{\nu_{\mathbf{w}-\mathbf{v}}}(\mathbf{k})| = |\widehat{\mu_{\mathbf{v}-\mathbf{w}}}(\mathbf{k}) - \widehat{\nu}(\mathbf{k})|.$$

Therefore

$$d_2(\mu_{\mathbf{v}}, \nu_{\mathbf{w}}) = d_2(\mu, \nu_{\mathbf{w}-\mathbf{v}}) = d_2(\mu_{\mathbf{v}-\mathbf{w}}, \nu).$$

In particular, the function $(\mu, \nu) \rightarrow d_2(\mu, \nu_{\mathbf{m}_\mu - \mathbf{m}_\nu})$ is symmetric.

PROOF. By the translation property of the Fourier Transform, for all $\mathbf{v} \in \mathbb{R}^d$ we have the identity

$$\widehat{\mu}_{\mathbf{v}}(\mathbf{k}) = e^{-i\mathbf{v}\cdot\mathbf{k}}\widehat{\mu}(\mathbf{k}).$$

Therefore

$$\begin{aligned} |e^{-iv\cdot k}\hat{\mu}(k) - e^{-iw\cdot k}\hat{\nu}(k)| &= |e^{-iw\cdot k}(e^{-i(v-w)\cdot k}\hat{\mu}(k) - \hat{\nu}(k))| \\ &= |e^{-i(v-w)\cdot k}\hat{\mu}(k) - \hat{\nu}(k)|. \end{aligned}$$

This shows that

$$\sup_{k \in \mathbb{R}^d \setminus \{0\}} \frac{|e^{-iv\cdot k}\hat{\mu}(k) - e^{-iw\cdot k}\hat{\nu}(k)|}{|k|^2} = \sup_{k \in \mathbb{R}^d \setminus \{0\}} \frac{|e^{-i(v-w)\cdot k}\hat{\mu}(k) - \hat{\nu}(k)|}{|k|^2}. \quad \square$$

Lemma 3 implies the following theorem.

THEOREM 2. *The function \mathcal{D}_2 defined in (2.12) is a distance over $\mathcal{P}_2(\mathbb{R}^d)$.*

PROOF. Clearly $\mathcal{D}_2(\mu, \nu) \geq 0$, $\forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, and $\mathcal{D}_2(\mu, \nu) = 0$ if and only if $\mu = \nu$. Symmetry follows from Lemma 3. Finally, both $d_2(\mu, \nu)$, in reason of the fact that it is a distance, and $|\mathbf{m}_\mu - \mathbf{m}_\nu|$ satisfy the triangular inequality. \square

An analogous extension can be done for the metric $D_{2,p}$ defined in (1.2).

DEFINITION 5. Given $p \geq 1$, we define $\mathcal{D}_{2,p} : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$\mathcal{D}_{2,p}(\mu, \nu) := \sqrt{D_{2,p}(\mu, \nu_{\mathbf{m}_\mu - \mathbf{m}_\nu})^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2}.$$

$\mathcal{D}_{2,p}$ is a metric on $\mathcal{P}_2(\mathbb{R}^d)$.

It is remarkable that the result of Theorem 1 can be extended to the \mathcal{D}_2 metric.

THEOREM 3. *The function \mathcal{D}_2 defined in (2.12) is equivalent to the W_2 distance.*

PROOF. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let μ^*, ν^* denote the two corresponding translated measures centered in $\mathbf{0}$. By Lemma 2, we have

$$(2.13) \quad W_2^2(\mu, \nu) = W_2^2(\mu^*, \nu^*) + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2.$$

Owing to Theorem 1, there exist two constants $c, C \in (0, \infty)$ such that

$$(2.14) \quad cd_2(\mu^*, \nu^*) \leq W_2(\mu^*, \nu^*) \leq Cd_2(\mu^*, \nu^*).$$

Using (2.13) in (2.14), we get

$$cd_2(\mu^*, \nu^*)^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2 \leq W_2(\mu, \nu)^2 \leq Cd_2(\mu^*, \nu^*)^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2,$$

which can be rewritten as

$$\begin{aligned} \min\{c, 1\}(d_2(\mu^*, \nu^*)^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2) &\leq W_2(\mu, \nu)^2 \\ &\leq \max\{1, C\}(d_2(\mu^*, \nu^*)^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2). \end{aligned}$$

Finally

$$\min\{c, 1\} \mathcal{D}_2^2(\mu, \nu) \leq W_2^2(\mu, \nu) \leq \max\{1, C\} \mathcal{D}_2^2(\mu, \nu). \quad \square$$

3. THE PERIODIC FOURIER-BASED METRICS

In this section, we introduce a family of (Discrete) Periodic Fourier-based metrics suitable to measure the distance between discrete probability measures whose support is restricted to a given set of points, and we discuss their equivalence with the Wasserstein metrics. The main result is that in this case one obtains a precise estimation of the constants of equivalence.

DEFINITION 6 (Regular grid). For $N \in \mathbb{N} \setminus \{0\}$, we define the regular grid

$$G_N := \{\mathbf{x} \in \mathbb{R}^d : N\mathbf{x} \in \mathbb{Z}^d \cap [0, N]^d\}.$$

Note that $G_N \subset [0, 1]^d$.

DEFINITION 7 (Discrete measure over a grid). We say that μ is a discrete measure over G_N if its support is contained in G_N , that is, if μ has the form

$$(3.15) \quad \mu(\mathbf{x}) = \sum_{\mathbf{y} \in G_N} \mu_{\mathbf{y}} \delta(\mathbf{x} - \mathbf{y}),$$

where $\mu_{\mathbf{y}} \in \mathbb{R}$, $\mu_{\mathbf{y}} \geq 0$ for all $\mathbf{y} \in G_N$.

The Discrete Fourier transform of a discrete measure over G_N is given by

$$(3.16) \quad \hat{\mu}(\mathbf{k}) = \sum_{\mathbf{x} \in G_N} \mu_{\mathbf{x}} e^{-i\langle \mathbf{x}, \mathbf{k} \rangle}.$$

The periodicity of the complex exponential implies that $\hat{\mu}$ is $2\pi N$ -periodic over all directions, so that it is sufficient to study $\hat{\mu}$ over a strict subset of \mathbb{R}^d , e.g., over $[0, 2\pi N]^d$. For instance, the value of the Fourier-based metric (1.1) is achieved by searching for the “sup” operator on the bounded set $[0, 2\pi N]^d$. Since

$$\frac{1}{|\mathbf{k}|^2} \geq \frac{1}{|\mathbf{k}'|^2}, \quad \forall \mathbf{k} \in (0, 2\pi N]^d, \forall \mathbf{k}' \in \mathbb{R}_+^d \setminus [0, 2\pi N]^d$$

and the function

$$\mathbf{k} \rightarrow |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|$$

is $2\pi N$ -periodic, for any given constant $s > 0$ the Discrete Fourier-based metric can be defined as

$$(3.17) \quad d_s(\mu, \nu) = \sup_{\mathbf{k} \in [0, 2\pi N]^d \setminus \{0\}} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|}{|\mathbf{k}|^s}.$$

DEFINITION 8 (Dilated discrete measures). Given a discrete measure μ over G_N and $\gamma \in \mathbb{R}$ such that $\gamma > 0$, the γ -dilated measure μ_γ is

$$\mu_\gamma(\mathbf{x}) = \sum_{\mathbf{y} \in G_N} \mu_{\mathbf{y}} \delta(\gamma \mathbf{x} - \mathbf{y}).$$

The Fourier transform of μ_γ is

$$(3.18) \quad \hat{\mu}_\gamma(\mathbf{k}) = \sum_{\mathbf{x} \in G_N} \mu_{\mathbf{x}} e^{-\frac{i}{\gamma} \langle \mathbf{k}, \mathbf{x} \rangle} = \hat{\mu}\left(\frac{\mathbf{k}}{\gamma}\right).$$

Therefore, if $\hat{\mu}$ is T -periodic, then $\hat{\mu}_\gamma$ is γT -periodic. Like the original metrics (1.1) [14], the metric (3.17) satisfies the dilation property

$$(3.19) \quad d_s(\mu_\gamma, \nu_\gamma) = \frac{1}{\gamma^s} d_s(\mu, \nu).$$

In particular, if we consider μ of the form (3.15), the Fourier transform of its $\frac{1}{N}$ -dilation is 2π -periodic.

We recall the definition of the metrics (1.2):

$$D_{s,p}(\mu, \nu) := \left(\int_{\mathbb{R}^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{(sp+d)}} d\mathbf{k} \right)^{\frac{1}{p}},$$

where $s > 0$ and $p \geq 1$. As we did for the Fourier Based Metrics d_s , thanks to the periodicity of the Fourier transform, we can restrict the domain of integration to $[0, T]^d$. In this case, for any given choice of the parameters p and s , this distance is well-defined any time the integrand is integrable in a neighbourhood of the origin. This corresponds to requiring that $\frac{1}{|\mathbf{k}|^\gamma}$ is integrable on the d -dimensional ball $B_1(0) = \{\mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| \leq 1\}$, that is, if and only if $\gamma < d$. This consideration suggests the following definition.

DEFINITION 9 (The Periodic Fourier-based Metric). Let μ and ν be two probability measures over G_N . The (s, p, α) -Periodic Fourier-based Metric (or PFM) between μ and ν is defined as

$$(3.20) \quad f_{s,p}^{(\alpha)}(\mu, \nu) := \left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{sp+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}},$$

where $p, s, \alpha \in \mathbb{R}$ and T is the period of $\hat{\mu}$ and $\hat{\nu}$. When $\alpha = 0$ and $s \in \mathbb{N}$ we say that $f_{s,p}^{(\alpha)} := f_{s,p}^{(0)}$ is *pure*.

As discussed in the introduction, in dimension $d = 1$ the continuous version of the metrics (3.20) has been considered in [8]. Recently, these metrics have been

considered in relation with the problem of convergence toward equilibrium of a Fokker–Planck type equation modeling wealth distribution [29], where various properties of these metrics have been studied. As pointed out in [29], if μ and ν have equal r -moments, the function $|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|$ behaves like $|\mathbf{k}|^{r+1}$ as $\mathbf{k} \rightarrow 0$. As a consequence, the value of $f_{s,p}^{(\alpha)}(\mu, \nu)$ is finite only if the following condition is verified

$$(3.21) \quad p(s - r - 1) + \alpha < d.$$

If s, p and α satisfy (3.21), and thus $f_{s,p}^{(\alpha)} < +\infty$, we say that $f_{s,p}^{(\alpha)}$ is feasible.

PROPOSITION 2. *Let μ and ν be two probability measures over G_N . For any given constant $\gamma > 0$, the following dilation property holds*

$$f_{s,p}^{(\alpha)}(\mu_\gamma, \nu_\gamma) = \frac{1}{|\gamma|^{s+\frac{\alpha}{p}}} f_{s,p}^{(\alpha)}(\mu, \nu).$$

PROOF. Using relation (3.18) and the change of variables $\mathbf{k} = \gamma\mathbf{k}'$, we get

$$\begin{aligned} f_{s,p}^{(\alpha)}(\mu_\gamma, \nu_\gamma) &= \left(\frac{1}{|\gamma T|^d} \int_{[0, \gamma T]^d} \frac{|\hat{\mu}_\gamma(\mathbf{k}) - \hat{\nu}_\gamma(\mathbf{k})|^p}{|\mathbf{k}|^{sp+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|\gamma T|^d} \int_{[0, \gamma T]^d} \frac{|\hat{\mu}(\frac{\mathbf{k}}{\gamma}) - \hat{\nu}(\frac{\mathbf{k}}{\gamma})|^p}{|\mathbf{k}|^{sp+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|\gamma|^d} \frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}') - \hat{\nu}(\mathbf{k}')|^p}{|\gamma|^{sp+\alpha} |\mathbf{k}'|^{sp+\alpha}} |\gamma|^d d\mathbf{k}' \right)^{\frac{1}{p}} \\ &= \frac{1}{|\gamma|^{s+\frac{\alpha}{p}}} \left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}') - \hat{\nu}(\mathbf{k}')|^p}{|\mathbf{k}'|^{sp+\alpha}} d\mathbf{k}' \right)^{\frac{1}{p}} \\ &= \frac{1}{|\gamma|^{s+\frac{\alpha}{p}}} f_{s,p}^{(\alpha)}(\mu, \nu). \quad \square \end{aligned}$$

It is important to remark that, at difference with the metrics (1.2), the analogous of the dilation property (3.19) is true only for $\alpha = 0$, that is only for pure metrics. We show next that the $f_{s,p}^{(\alpha)}$ metrics satisfy various monotonicity properties with respect to the parameters p and s .

PROPOSITION 3. *Let μ and ν be two probability measures over G_N , with moments equal up to r . If $t \leq s$, then*

$$f_{t,p}^{(\alpha)}(\mu, \nu) \leq (\sqrt{d}|T|)^{(s-t)} f_{s,p}^{(\alpha)}(\mu, \nu),$$

for any p and α for which the metric is feasible, i.e., for $p(s - r - 1) + \alpha < d$.

PROOF. We compute

$$\begin{aligned}
 f_{t,p}^{(\alpha)}(\mu, \nu) &= \left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{tp+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}} \\
 &= \left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\mathbf{k}|^{p(s-t)} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{p(s-t)+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}} \\
 &= \left(\frac{1}{|T|^d} \int_{[0, T]^d} |\mathbf{k}|^{p(s-t)} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{sp+\alpha}} d\mathbf{k} \right)^{\frac{1}{p}} \\
 &\leq (\sqrt{d}|T|)^{(s-t)} f_{s,p}^{(\alpha)}(\mu, \nu).
 \end{aligned}$$

The last inequality is obtained resorting to the bound $|\mathbf{k}| \leq \sqrt{d}|T|$. □

PROPOSITION 4. *Let μ and ν be two probability measures over G_N . If $\alpha = 0$ and $p \leq q$, then*

$$f_{s,p}(\mu, \nu) \leq f_{s,q}(\mu, \nu).$$

PROOF. We have

$$\begin{aligned}
 f_{s,p}(\mu, \nu) &= \left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{sp}} d\mathbf{k} \right)^{\frac{1}{p}} \\
 &= \left(\left(\frac{1}{|T|^d} \int_{[0, T]^d} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{sp}} d\mathbf{k} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
 &\leq \left(\frac{1}{|T|^d} \int_{[0, T]^d} \left(\frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^p}{|\mathbf{k}|^{sp}} \right)^{\frac{q}{p}} d\mathbf{k} \right)^{\frac{1}{q}} \\
 &= f_{s,q}(\mu, \nu).
 \end{aligned}$$

The last inequality follows from Jensen’s inequality. □

REMARK 1. By letting $p \rightarrow +\infty$, we get

$$\lim_{p \rightarrow \infty} f_{s,p}(\mu, \nu) = f_{s,\infty}(\mu, \nu) := d_s(\mu, \nu).$$

Thanks to Hölder inequality, for all $p < +\infty$ we have the bound

$$(3.22) \quad f_{s,p}(\mu, \nu) \leq d_s(\mu, \nu).$$

The results of this Section are preliminary to our main result, which deals with the equivalence of the pure metrics, for $p = 2$, with the Wasserstein metrics. For the sake of simplicity, and without loss of generality, in the next subsection we consider measures in dimension $d = 2$.

3.1. *Equivalence with the Wasserstein metric W_1*

We consider the two cases $s = 1$ and $s = 2$, in dimension $d = 2$, and we show that $f_{1,2}$ and $f_{2,2}$ are equivalent to W_1 and W_2 , respectively.

We start with the case $s = 1$. For any $\mu, \nu \in \mathcal{P}(G_N)$, the PFM is

$$(3.23) \quad f_{1,2}(\mu, \nu) = \left(\frac{1}{|T|^2} \int_{[0, T]^2} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2}{|\mathbf{k}|^2} d\mathbf{k} \right)^{\frac{1}{2}}.$$

We have the following

THEOREM 4. *For any pair of measures $\mu, \nu \in \mathcal{P}(G_N)$, we have the inequality*

$$f_{1,2}(\mu, \nu) \leq W_1(\mu, \nu).$$

PROOF. Let π be a transport plan between μ and ν . It holds

$$\begin{aligned} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})| &= \left| \sum_{\mathbf{x}, \mathbf{y} \in G_N} e^{-i\mathbf{k} \cdot \mathbf{x}} \pi(\mathbf{x}, \mathbf{y}) - \sum_{\mathbf{x}, \mathbf{y} \in G_N} e^{-i\mathbf{k} \cdot \mathbf{y}} \pi(\mathbf{x}, \mathbf{y}) \right| \\ &= \left| \sum_{\mathbf{x}, \mathbf{y} \in G_N} (e^{-i\mathbf{k} \cdot \mathbf{x}} - e^{-i\mathbf{k} \cdot \mathbf{y}}) \pi(\mathbf{x}, \mathbf{y}) \right| \\ &\leq \sum_{\mathbf{x}, \mathbf{y} \in G_N} |e^{-i\mathbf{k} \cdot \mathbf{x}} - e^{-i\mathbf{k} \cdot \mathbf{y}}| \pi(\mathbf{x}, \mathbf{y}) \\ &= \sum_{\mathbf{x}, \mathbf{y} \in G_N} |1 - e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}| \pi(\mathbf{x}, \mathbf{y}) \\ &\leq \sum_{\mathbf{x}, \mathbf{y} \in G_N} |\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})| \pi(\mathbf{x}, \mathbf{y}) \\ &\leq |\mathbf{k}| \sum_{\mathbf{x}, \mathbf{y} \in G_N} |\mathbf{x} - \mathbf{y}| \pi(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Hence, if π is the optimal transport plan, we conclude with the inequality

$$(3.24) \quad |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})| \leq |\mathbf{k}| W_1(\mu, \nu).$$

Using inequality (3.24) into definition (3.23), we finally obtain the bound

$$(3.25) \quad f_{1,2}(\mu, \nu) \leq \left(\frac{1}{|T|^2} \int_{[0, T]^2} \frac{(|\mathbf{k}| W_1(\mu, \nu))^2}{|\mathbf{k}|^2} d\mathbf{k} \right)^{\frac{1}{2}} = W_1(\mu, \nu). \quad \square$$

Since $W_1(\mu, \nu) < +\infty$ for every $\mu, \nu \in \mathcal{P}(G_N)$, inequality (3.25) implies that $f_{1,2}$ is bounded in correspondence to any pair of probability measures over the grid G_N .

We now show that $f_{1,2}$ and W_1 satisfy a reverse inequality, thus concluding that the two metrics are equivalent.

THEOREM 5. *For any pair of measures $\mu, \nu \in \mathcal{P}(G_N)$ it holds*

$$(3.26) \quad W_1(\mu, \nu) \leq \frac{T^2}{2\pi} f_{1,2}(\mu, \nu).$$

PROOF. Owing to the dual characterization of the W_1 distance (see [31], Chapter 5), there exists a 1-Lipschitz function ϕ such that

$$W_1(\mu, \nu) = \int_{\mathbb{R}^2} \phi(\mathbf{x}) d\mu(\mathbf{x}) - \int_{\mathbb{R}^2} \phi(\mathbf{x}) d\nu(\mathbf{x}).$$

Since μ and ν are discrete and supported on a subset of $[0, 1]^2$, we can write

$$W_1(\mu, \nu) = \sum_{\mathbf{x} \in G_N} \phi(\mathbf{x})(\mu_{\mathbf{x}} - \nu_{\mathbf{x}}).$$

Therefore, resorting to the fact that both the measures have the same mass, for any given constant $c \in \mathbb{R}$ we have

$$W_1(\mu, \nu) = \sum_{\mathbf{x} \in G_N} (\phi(\mathbf{x}) + c)(\mu_{\mathbf{x}} - \nu_{\mathbf{x}}).$$

The last identity permits to choose ϕ such that $\phi(\frac{N}{2}, \frac{N}{2}) = 0$. Since ϕ is 1-Lipschitz, we conclude that

$$(3.27) \quad |\phi(\mathbf{x})| \leq \frac{\sqrt{2}}{2}, \quad \forall \mathbf{x} \in G_N.$$

By Hölder inequality we obtain

$$W_1(\mu, \nu) \leq \left(\sum_{\mathbf{x} \in G_N} |\phi(\mathbf{x})|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathbf{x} \in G_N} |\mu_{\mathbf{x}} - \nu_{\mathbf{x}}|^2 \right)^{\frac{1}{2}}.$$

Since

$$\sum_{\mathbf{x} \in G_N} |\mu_{\mathbf{x}} - \nu_{\mathbf{x}}|^2 = \frac{1}{|T|^2} \int_{[0, T]^2} A(\mathbf{k}) B(\mathbf{k}) d\mathbf{k}$$

where

$$A(\mathbf{k}) = \sum_{\mathbf{x} \in G_N} (\mu_{\mathbf{x}} - \nu_{\mathbf{x}}) e^{-i\langle \mathbf{x}, \mathbf{k} \rangle}$$

$$B(\mathbf{k}) = \sum_{\mathbf{y} \in G_N} (\mu_{\mathbf{y}} - \nu_{\mathbf{y}}) e^{+i\langle \mathbf{y}, \mathbf{k} \rangle}$$

we have

$$\sum_{x \in G_N} |\mu_x - \nu_x|^2 = \frac{1}{|T|^2} \int_{[0, T]^2} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2 d\mathbf{k}.$$

Now using (3.27) we obtain

$$\begin{aligned} W_1(\mu, \nu) &\leq \frac{\sqrt{2}N}{2} \left(\frac{1}{|T|^2} \int_{[0, T]^2} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2 d\mathbf{k} \right)^{\frac{1}{2}} \\ &= \frac{\sqrt{2}N}{2} \left(\frac{1}{|T|^2} \int_{[0, T]^2} |\mathbf{k}|^2 \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2}{|\mathbf{k}|^2} d\mathbf{k} \right)^{\frac{1}{2}}. \end{aligned}$$

Since $|\mathbf{k}|^2 \leq 2T^2$ and $T = 2\pi N$, we can finally conclude that

$$W_1(\mu, \nu) \leq \frac{T^2}{2\pi} \left(\frac{1}{|T|^2} \int_{[0, T]^2} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2}{|\mathbf{k}|^2} d\mathbf{k} \right)^{\frac{1}{2}} = \frac{T^2}{2\pi} f_{1,2}(\mu, \nu). \quad \square$$

In consequence of the previous estimates, it is immediate to show that the metrics d_s and W_1 are equivalent. This is proven in the following

COROLLARY 1. *For any pair of measures $\mu, \nu \in \mathcal{P}(G_N)$*

$$d_1(\mu, \nu) \leq W_1(\mu, \nu) \leq \frac{T^2}{2\pi} d_1(\mu, \nu).$$

PROOF. The first inequality is a consequence of bound (3.24). The second one follows from inequality (3.22). \square

3.2. Equivalence with the Wasserstein metric W_2

The aim of this Section is to show the equivalence of the Fourier-based metric $f_{2,2}$ and the Wasserstein metric W_2 . Let $s = 2$. In this case, the PFM takes the form

$$f_{2,2}(\mu, \nu) = \left(\frac{1}{|T|^2} \int_{[0, T]^2} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2}{|\mathbf{k}|^4} d\mathbf{k} \right)^{\frac{1}{2}}.$$

Clearly, the distance between the two probability measures is well-defined only when μ and ν possess the same expected value. Since, in general this is not the case, we start by translating the measures, as done in Section 2, in order to satisfy this condition. The following proposition shows that, for probability measures with the same center, the topology induced by $f_{2,2}$ is not stronger than the topology induced by W_2 .

THEOREM 6. For any pair of measures $\mu, \nu \in \mathcal{P}(G_N)$ such that $\mathbf{m}_\mu = \mathbf{m}_\nu$, it holds

$$(3.28) \quad f_{2,2}(\mu, \nu) \leq 2\sqrt{2}W_2(\mu, \nu).$$

In particular, $f_{2,2}(\mu, \nu) < \infty$.

PROOF. For any given pair of probability measures μ and ν in $\mathcal{P}(G_N)$, with centers $\mathbf{m}_\mu = \mathbf{m}_\nu$, we have

$$i\mathbf{k} \sum_{x \in G_N} x\mu_x = i\mathbf{k} \sum_{y \in G_N} y\nu_y.$$

For any transport plan π between μ and ν , we can rewrite the previous relations in the form

$$(3.29) \quad i\mathbf{k} \sum_{x,y \in G_N} (\mathbf{x} - \mathbf{y})\pi_{x,y} = 0.$$

Using identity (3.29) we obtain

$$\begin{aligned} \hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k}) &= \sum_{x \in G_N} \mu_x e^{-i\mathbf{k} \cdot \mathbf{x}} - \sum_{y \in G_N} \nu_y e^{-i\mathbf{k} \cdot \mathbf{y}} \\ &= \sum_{x,y \in G_N} (e^{-i\mathbf{k} \cdot \mathbf{x}} - e^{-i\mathbf{k} \cdot \mathbf{y}} - i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))\pi_{x,y} \\ &= \sum_{x,y \in G_N} e^{-i\mathbf{k} \cdot \mathbf{y}} (e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} - 1 - i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))\pi_{x,y} \\ &\quad + \sum_{x,y \in G_N} i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})(e^{-i\mathbf{k} \cdot \mathbf{y}} - 1)\pi_{x,y}. \end{aligned}$$

Using that for all $\theta \in \mathbb{R}$

$$\begin{aligned} |e^{i\theta} - 1| &\leq |\theta|, \\ |e^{i\theta} - 1 - i\theta| &\leq \frac{\theta^2}{2} \end{aligned}$$

we obtain

$$\begin{aligned} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})| &\leq \frac{|\mathbf{k}|^2}{2} \sum_{x,y \in G_N} |\mathbf{x} - \mathbf{y}|^2 \pi_{x,y} + |\mathbf{k}|^2 \sum_{x,y \in G_N} |\mathbf{x} - \mathbf{y}| |\mathbf{y}| \pi_{x,y} \\ &\leq \frac{|\mathbf{k}|^2}{2} \sum_{x,y \in G_N} |\mathbf{x} - \mathbf{y}|^2 \pi_{x,y} \\ &\quad + |\mathbf{k}|^2 \left(\sum_{x,y \in G_N} |\mathbf{y}|^2 \pi_{x,y} \right)^{\frac{1}{2}} \left(\sum_{x,y \in G_N} |\mathbf{x} - \mathbf{y}|^2 \pi_{x,y} \right)^{\frac{1}{2}}. \end{aligned}$$

In particular, if we take π as the optimal transportation plan between μ and ν for the cost $|\mathbf{x} - \mathbf{y}|^2$ we get

$$\begin{aligned} \frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|}{|\mathbf{k}|^2} &\leq \frac{W_2^2(\mu, \nu)}{2} + \left(\sum_{\mathbf{y} \in G_N} |\mathbf{y}|^2 \nu_{\mathbf{y}} \right)^{\frac{1}{2}} W_2(\mu, \nu) \\ &= W_2(\mu, \nu) \left(\frac{W_2(\mu, \nu)}{2} + \left(\sum_{\mathbf{y} \in G_N} |\mathbf{y}|^2 \nu_{\mathbf{y}} \right)^{\frac{1}{2}} \right). \end{aligned}$$

Since

$$W_2(\mu, \nu) \leq W_2(\mu, \delta) + W_2(\delta, \nu) \leq \left(\sum_{\mathbf{x} \in G_N} |\mathbf{x}|^2 \mu_{\mathbf{x}} \right)^{\frac{1}{2}} + \left(\sum_{\mathbf{y} \in G_N} |\mathbf{y}|^2 \nu_{\mathbf{y}} \right)^{\frac{1}{2}},$$

and, as μ and ν are supported in $[0, 1]^2$,

$$\sqrt{\sum_{\mathbf{x} \in G_N} |\mathbf{x}|^2 \mu_{\mathbf{x}}} \leq \sqrt{2}, \quad \sqrt{\sum_{\mathbf{y} \in G_N} |\mathbf{y}|^2 \nu_{\mathbf{y}}} \leq \sqrt{2},$$

we obtain (3.28):

$$\frac{|\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|}{|\mathbf{k}|^2} \leq 2\sqrt{2} W_2(\mu, \nu). \quad \square$$

We conclude by showing the validity of a reverse inequality, thus proving the equivalence between $f_{2,2}$ and W_2 .

THEOREM 7. *For any pair of measures $\mu, \nu \in \mathcal{P}(G_N)$, we have the inequality*

$$W_2^2(\mu, \nu) \leq \frac{T^3}{\pi} f_{2,2}(\mu, \nu).$$

PROOF. Let π be the optimal transportation plan between μ and ν for the cost $|\mathbf{x} - \mathbf{y}|$, since $|\mathbf{x} - \mathbf{y}| \leq \sqrt{2}$ for all $\mathbf{x}, \mathbf{y} \in G_N \subset [0, 1]^2$, it holds

$$W_2^2(\mu, \nu) \leq \sum_{\mathbf{x}, \mathbf{y} \in G_N} |\mathbf{x} - \mathbf{y}|^2 \pi_{\mathbf{x}, \mathbf{y}} \leq \sum_{\mathbf{x}, \mathbf{y} \in G_N} \sqrt{2} |\mathbf{x} - \mathbf{y}| \pi_{\mathbf{x}, \mathbf{y}} = \sqrt{2} W_1(\mu, \nu).$$

Then, by Theorem 5 and Proposition 3 with $t = 1$ and $p = s = 2$, we get

$$\sqrt{2} W_1(\mu, \nu) \leq \frac{\sqrt{2} T^2}{2\pi} f_{1,2}(\mu, \nu) \leq \frac{T^3}{\pi} f_{2,2}(\mu, \nu),$$

which, together with the last inequality, concludes the proof. □

The previous bounds hold provided that μ and ν are centered in the same point. However, when $\mathbf{m}_\mu - \mathbf{m}_\nu \neq 0$, we can resort, as in Section 2, to the new metric

$$\mathcal{F}_{2,2}(\mu, \nu) := \sqrt{(f_{2,2}(\mu, \nu_{\mathbf{m}_\mu - \mathbf{m}_\nu})^2 + |\mathbf{m}_\mu - \mathbf{m}_\nu|^2)},$$

which is well-defined also for probability measures having different centers. This shows that we can generalize, similarly to Theorem 2 and Theorem 3, the equivalence of $\mathcal{F}_{2,2}$ and W_2 to measures which are not centered in the same point.

3.3. Connections with other distances

As discussed in [29], the case in which $s \leq 0$ leads to stronger metrics. In this case, we clearly lose relations like (3.28), that link from above the Wasserstein metric with the Fourier-based metric. An interesting case is furnished by choosing both $s = 0$ and $\alpha = 0$ into (3.20). The metric in this case is defined by

$$\begin{aligned} f_{0,2}(\mu, \nu) &= \left(\frac{1}{|T|^d} \int_{[0, T]^d} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2 d\mathbf{k} \right)^{\frac{1}{2}} \\ &= \left(\sum_{\mathbf{x} \in G} |\mu(\mathbf{x}) - \nu(\mathbf{x})|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which is the Total Variation distance between the probability measures μ and ν .

We remark that the distance above does not require the measures to possess the same mass. By fixing in definition (3.20) $s = 0$ and $\alpha \in [0, 2)$, one obtains a sequence of metrics that interpolate between the Total Variation distance and the W_1 distance, namely a family of measures that move from a strong metric to a weaker one. However, if $\alpha > 0$, the measures must have the same mass.

In the case $s < 0$ and $\alpha = 0$, the Fourier-based metric (3.20) becomes

$$f_{s,2}(\mu, \nu) = \left(\frac{1}{|T|^d} \int_{[0, T]^d} |\mathbf{k}|^{2|s|} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2 d\mathbf{k} \right)^{\frac{1}{2}}.$$

In particular, when $-s = n \in \mathbb{N}_+$, we find that

$$f_{-n,2}(\mu, \nu) = \left(\frac{1}{|T|^d} \int_{[0, T]^d} |\mathbf{k}|^{2n} |\hat{\mu}(\mathbf{k}) - \hat{\nu}(\mathbf{k})|^2 d\mathbf{k} \right)^{\frac{1}{2}}.$$

This metric, by Fourier identity, controls the n -th derivative of the measures μ and ν , and does not require the measures to have the same mass.

4. NUMERICAL RESULTS

We run extensive numerical tests to compare the Wasserstein metrics W_1 and W_2 with the corresponding Periodic Fourier-based Metrics $f_{1,2}^0$ and $f_{2,2}^0$.

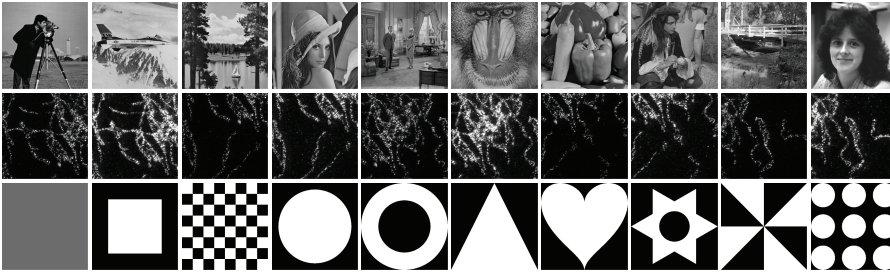


Figure 1. DOTmark benchmark: Classic, Microscopy, and Shapes images.

The goal of our tests is to compare empirically the distance values obtained with the different metrics, and to measure the runtime gain that we can achieve using the Fourier-based metrics. In the following paragraphs, we report the main conclusions of our tests.

Implementation details. We implemented our algorithms in Python 3.7, using the Fast Fourier Transform implemented in the Numpy library [23]. To compute the Wasserstein distances, we use the Python Optimal Transport (POT) library [17]. All the tests are executed on a MacBook Pro 13 equipped with a 2.5 GHz Intel Core i7 dual-core and 16 GB of Ram.

Dataset. As problem instances, we use the DOTmark benchmark [27], which contains 10 classes of gray scale images, each containing 10 different images. Every image is given in the data set at the following pixel resolutions: 32×32 , 64×64 , 128×128 , 256×256 , and 512×512 . Figure 1 shows the Classic, Microscopy, and Shapes images, respectively, at the highest pixel resolution (one class for each row).

Results. For each pair of images of the DOTmark dataset, the reciprocal distance values using the W_1 , W_2 , $f_{1,2}^0$ and $f_{2,2}^0$ metrics has been computed, and the corresponding runtime in seconds has been recorded.

The scatter plot in Figure 2 shows the relation between the W_2 and the $f_{2,2}^0$ distances for each pair of images at pixel resolution 32×32 . The plot shows that not only the two metrics are theoretically equivalent, as proved in Theorem 6 and 7, but also that they yield very similar values. A partial exception is present in the Shape class, which, however, contains artificial shape images. On the much more (application-wise) interesting Classic images, the two metrics return very close values.

Table 1 reports the averages and the standard deviations of the runtime, measured in seconds, at different image size. For each row and each metric, the averages are computed over 450 instances. The numerical results clearly show that the PFM metrics are orders of magnitude faster, and permit to compute the distance even for the largest 512×512 images in around 10 seconds. Note that using the

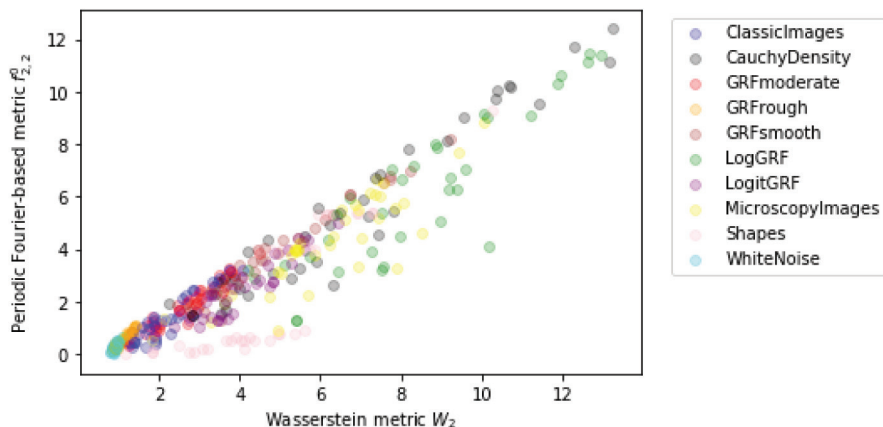


Figure 2. Wasserstein metric W_2 versus Periodic Fourier-based metric $f_{2,2}^0$: Comparison of distance values for 450 pair of images of size 32×32 .

Table 1. Runtime vs. image size for different metrics: The runtime is measured in seconds and reported as “*Mean (StdDev)*”. Each row gives the averages over 450 instances of pairwise distances.

Dimension	Averages runtime in seconds			
	W_1	W_2	$f_{1,2}^0$	$f_{2,2}^0$
32×32	0.84 (0.30)	1.06 (0.32)	0.002 (10^{-4})	0.006 (10^{-4})
64×64	21.9 (7.96)	23.4 (8.49)	0.01 (10^{-3})	0.02 (10^{-3})
128×128	205.0 (45.9)	199.0 (45.0)	0.28 (0.07)	0.63 (0.16)
256×256			1.21 (0.40)	2.96 (0.94)
512×512			4.74 (1.32)	11.55 (2.84)

POT library, we were unable to compute the W_1 and W_2 distances for images of size 256×256 and 512×512 , due to memory issues.

5. CONCLUSIONS

In this paper we showed that the class of Fourier-based metrics introduced in [19] and [8] are useful tools to measure the distance between pairs of probability distributions, which, in reason of their equivalence, represent an interesting alternative in problems where Wasserstein distances were already successfully employed.

The main result of this paper is that the constants in the equivalence relation can be precisely quantified if discrete probability measures are considered. In addition, preliminary computational results have shown that in image processing, at difference with Wasserstein metrics, Fourier metrics provide a noticeable performance with respect to time, even when dealing with very large images. Starting from these results, we believe it will be possible to design new numerical methods

in computer imaging, which combine theoretical convergence results with a low computational cost, at difference with Wasserstein metric, which, nowadays, has still a heavy computational load.

ACKNOWLEDGMENTS. This paper was written within the activities of the GNFM of INDAM. The research was partially supported by the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics “F. Casorati”, University of Pavia. The PhD scholarship of Andrea Codegani is founded by Sea Vision S.r.l.

REFERENCES

- [1] J. ADLER - S. LUNZ, Banach Wasserstein GAN. In *Advances in Neural Information Processing Systems*, pages 6754–6763, (2018).
- [2] L. AMBROSIO - N. GIGLI, A user’s guide to optimal transport. In *Modelling and Optimisation of Flows on Networks*, pages 1–155. Springer, (2013).
- [3] L. AMBROSIO - N. GIGLI - G. SAVARÉ, *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, (2008).
- [4] S. ANGENENT - S. HAKER - A. TANNENBAUM - L. ZHU, Optimal mass transport for registration and warping. *International Journal of Computer Vision*, 60(3):225–240, (2004).
- [5] M. ARJOVSKY - L. BOTTOU - S. CHINTALA, Wasserstein Generative Adversarial Networks. *Proceedings of the 34th International Conference on Machine Learning, PMLR*, 70:214–223, (2017).
- [6] G. AURICCHIO - F. BASSETTI - S. GUALANDI - M. VENERONI, Computing Kantorovich-Wasserstein distances on d -dimensional histograms using $(d + 1)$ -partite graphs. In *Advances in Neural Information Processing Systems*, pages 5793–5803, (2018).
- [7] G. AURICCHIO - F. BASSETTI - S. GUALANDI - M. VENERONI, Computing Wasserstein Barycenters via Linear Programming. In *International Conference on Integration of Constraint Programming, Artificial Intelligence, and Operations Research*, pages 355–363. Springer, (2019).
- [8] L. BARINGHAUS - R. GRÜBEL, On a class of characterization problems for random convex combinations. *Annals of the Institute of Statistical Mathematics*, 49(3):555–567, (1997).
- [9] F. BASSETTI - S. GUALANDI - M. VENERONI, On the computation of Kantorovich-Wasserstein distances between 2d-histograms by uncapacitated minimum cost flows. *arXiv preprint arXiv:1804.00445*, (2018).
- [10] M. BISI - J. A. CARRILLO - G. TOSCANI, Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model. *Journal of Statistical Physics*, 124(2–4):625–653, (2006).
- [11] N. BONNEEL - D. COEURJOLLY, SPOT: Sliced Partial Optimal Transport. *ACM Transactions on Graphics*, 38(4):1–13, (2019).
- [12] E. A. CARLEN - M. C. CARVALHO - E. GABETTA, Central limit theorem for Maxwellian molecules and truncation of the Wild expansion. *Communications on Pure and Applied Mathematics*, 53(3):370–397, (2000).
- [13] E. A. CARLEN - E. GABETTA - G. TOSCANI, Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas. *Communications in Mathematical Physics*, 199(3):521–546, (1999).

- [14] J. A. CARRILLO - G. TOSCANI, Contractive probability metrics and asymptotic behavior of dissipative kinetic equations. *Rivista Matematica Università di Parma*, 7(6):75–198, (2007).
- [15] M. CUTURI - A. DOUCET, Fast computation of Wasserstein barycenters. In *International Conference on Machine Learning*, pages 685–693, (2014).
- [16] M. CUTURI - G. PEYRÉ, Computational optimal transport. *Foundations and Trends in Machine Learning*, 11(5–6):355–607, (2019).
- [17] R. FLAMARY - N. COURTY, POT Python Optimal Transport library, 2017.
- [18] C. FROGNER - C. ZHANG - H. MOBAHI - M. ARAYA - T. POGGIO, Learning with a Wasserstein loss. In *Advances in Neural Information Processing Systems*, pages 2053–2061, (2015).
- [19] G. GABETTA - G. TOSCANI - B. WENNBURG, Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. *Journal of Statistical Physics*, 81(5–6):901–934, (1995).
- [20] T. GOUDON - S. JUNCA - G. TOSCANI, Fourier-based distances and Berry-Esseen like inequalities for smooth densities. *Monatshefte für Mathematik*, 135(2):115–136, (2002).
- [21] L. J. GUIBAS - Y. RUBNER - C. TOMASI, The earth mover’s distance as a metric for image retrieval. *International Journal of Computer Vision*, 40(2):99–121, (2000).
- [22] H. LING - K. OKADA, An efficient Earth Mover’s Distance algorithm for robust histogram comparison. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(5):840–853, (2007).
- [23] T. OLIPHANT, *A guide to NumPy, 2nd Ed.* Create Space Independent Publishing Platform, North Charleston, SC, USA, (2015)
- [24] O. PELE - M. WERMAN, Fast and robust earth mover’s distances. In *2009 IEEE 12th International Conference on Computer Vision*, pages 460–467. IEEE, (2009).
- [25] A. PULVIRENTI - G. TOSCANI, Asymptotic properties of the inelastic Kac model. *Journal of Statistical Physics*, 114(5–6):1453–1480, (2004).
- [26] F. SANTAMBROGIO, *Optimal transport for applied mathematicians*, volume 55. Springer, (2015).
- [27] J. SCHRIEBER - D. SCHUHMACHER - C. GOTTSCHLICH, Dotmark – A benchmark for Discrete Optimal Transport. *IEEE Access*, 5:271–282, 2017.
- [28] S. SHIRDHONKAR - D. W. JACOBS, Approximate earth mover’s distance in linear time. In *2008 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–8, 2008.
- [29] M. TORREGROSSA - G. TOSCANI, Wealth distribution in presence of debts. A Fokker–Planck description. *Communications Mathematical Sciences*, 16(2):537–560, (2018).
- [30] G. TOSCANI - C. VILLANI, Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *Journal of Statistical Physics*, 94(3–4):619–637, (1999).
- [31] C. VILLANI, *Optimal transport: old and new.* Springer Science & Business Media, (2008).
- [32] V. M. ZOLOTAREV, Probability metrics. *Theory of Probability & Its Applications*, 28(2):278–302, 1984.

Gennaro Auricchio
Dipartimento di Matematica “F. Casorati”
Università degli Studi di Pavia
Via Ferrata 5
27100 Pavia, Italy
gennaro.auricchio01@universitadipavia.it

Andrea Codegoni
Dipartimento di Matematica “F. Casorati”
Università degli Studi di Pavia
Via Ferrata 5
27100 Pavia, Italy
andrea.codegoni01@universitadipavia.it

Stefano Gualandi
Dipartimento di Matematica “F. Casorati”
Università degli Studi di Pavia
Via Ferrata 5
27100 Pavia, Italy
stefano.gualandi@unipv.it

Giuseppe Toscani
Dipartimento di Matematica “F. Casorati”
Università degli Studi di Pavia
Via Ferrata 5
27100 Pavia, Italy
giuseppe.toscani@unipv.it

Marco Veneroni
Dipartimento di Matematica “F. Casorati”
Università degli Studi di Pavia
Via Ferrata 5
27100 Pavia, Italy
marco.veneroni@unipv.it

